# Weight modules for affine vertex operator algebras with finite and infinite multiplicities

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[Vertex algebras and related topics]

# Outline

1. Affine VOAs and modules
2. $\mathfrak{sl}_2$ minimal models
3. $\mathfrak{sl}_3$ minimal models I — finite multiplicities
4. $\mathfrak{sl}_3$ minimal models II — infinite multiplicities
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# Affine VOAs and their modules

**Input:** simple Lie algebra $\mathfrak{g}$, complex number $k \neq -h^\vee$.

**Construction:** induce the trivial $\mathfrak{g}$-module to a level-$k$ $\widehat{\mathfrak{g}}$-module.

**Result:** the universal affine VOA $V^k(\mathfrak{g})$.

**Theorem** [Gorelik–Kac'06]: $V^k(\mathfrak{g})$ is not simple iff

$$k + h^\vee = \frac{u}{v}, \quad u \in \mathbb{Z}_{\geqslant 2},\ v \in \mathbb{Z}_{\geqslant 1},\ \gcd\{u, v\} = 1.$$

The simple quotient VOA is denoted by $L_k(\mathfrak{g})$, but I will denote it by $G(u, v)$ where $G$ is the type of $\mathfrak{g}$, *eg.* $\mathfrak{sl}_2$ has type $A_1$.

The representation theory of $V^k(\mathfrak{g})$ is essentially unconstrained:

$$V^k(\mathfrak{g})\text{-module} \equiv \text{“smooth” level-}k\ \widehat{\mathfrak{g}}\text{-module.}$$

That of $G(u, v)$ is much more interesting.

**Motivation:** $\mathsf{G}(\mathsf{u},\mathsf{v})$-modules are used to build (hopefully consistent) physical models, *eg.* conformal field theories.

**Require:** the physically relevant category of $\mathsf{G}(\mathsf{u},\mathsf{v})$-modules should be braided, tensor and modular (may be nonfinite and nonsemisimple).

**Expect:** this category is finite-length, rigid and has enough projectives satisfying BGG reciprocity (*ie.* it's a highest-weight category).

**Know:** very little, unless $\mathfrak{g} = \mathfrak{sl}_2$.

**Conjecture:** the weight category of (finitely generated) weight $\mathsf{G}(\mathsf{u},\mathsf{v})$-modules satisfies these expectations and requirements.

Define a weight space for $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ and $\mathsf{G}(\mathsf{u},\mathsf{v})$ to be the intersection of a weight space for $\mathfrak{g}$ and a generalised eigenspace of the Virasoro zero mode $L_0$ with finite-rank Jordan blocks.

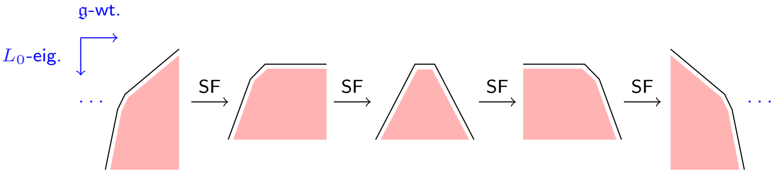
A weight module is one that is a direct sum of its weight spaces.

# Weight modules

The automorphisms of $\widehat{\mathfrak{g}}$ include the extended affine Weyl group $\mathsf{W} \ltimes \mathsf{P}^\vee$. The translations are called spectral flows. Twisting by these induces autoequivalences on the weight category of $\mathsf{G}(\mathsf{u}, \mathsf{v})$-modules. [Li'97]

A lower-bounded module is one on which $L_0$ has a minimal eigenvalue. The generalised eigenspace of minimal $L_0$-eigenvalue is the top space.

An irreducible weight $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$-module with finite multiplicities is a spectral flow of a lower-bounded one. [Futorny–Tsylke'01, Adamović–Kawasetsu–DR'23]



A lower-bounded module is almost irreducible if

- it is generated by its top space and
- its nonzero submodules have nonzero intersection with the top space.

The top space is naturally a module of a unital associative algebra called the Zhu algebra. [Zhu'96, Frenkel–Zhu'92]

$$\mathsf{Zhu}[\mathsf{V}^{\mathsf{k}}(\mathfrak{g})] \cong \mathsf{U}(\mathfrak{g}) \quad \Rightarrow \quad \mathsf{Zhu}[\mathsf{G}(\mathsf{u},\mathsf{v})] \cong \frac{\mathsf{U}(\mathfrak{g})}{\mathsf{I}_{\mathsf{k}}}.$$

The almost-irreducible $\mathsf{G}(\mathsf{u},\mathsf{v})$-modules are in bijection with their top spaces (regarded as $\mathsf{Zhu}[\mathsf{G}(\mathsf{u},\mathsf{v})]$-modules). [De Sole–Kac'05]

We thus have a path to the (weight) representation theory of $\mathsf{G}(\mathsf{u},\mathsf{v})$:

- Classify irreducible weight $\mathsf{Zhu}[\mathsf{G}(\mathsf{u},\mathsf{v})]$-modules ($\subset \mathsf{U}(\mathfrak{g})$-modules).
- Classify irreducible weight $\mathsf{G}(\mathsf{u},\mathsf{v})$-modules with finite multiplicities.
- Worry about the existence of irreducibles with infinite multiplicities.
- Repeat for appropriate reducible-but-indecomposable modules.
- Try to construct projectives and injectives...

# $\mathfrak{sl}_2$ minimal models

The VOA is $A_1(u, v)$, where $u, v \geqslant 2$ are coprime.

We study the almost-irreducible lower-bounded weight $A_1(u, v)$-modules by identifying their top spaces as $\mathfrak{sl}_2$-modules.

This can be done directly [Adamović–Milas'95, DR-Wood'14] but we will explain a new, more general, method: inverse quantum hamiltonian reduction.

It starts with the Virasoro minimal model VOA $\mathsf{Vir}(u, v)$, the quantum hamiltonian reduction of $A_1(u, v)$.

- Start with the free-field realisations $\mathsf{Vir}^k \hookrightarrow \mathsf{H}$ [Feigin–Fuchs'82]
- and $\mathsf{V}^k(\mathfrak{sl}_2) \hookrightarrow \mathsf{H} \otimes \beta\gamma$. [Wakimoto'86]
- Bosonise the ghosts: $\beta\gamma \hookrightarrow \Pi$. [Friedan–Martinec–Shenker'86]
- Trade FF for FMS: $\mathsf{V}^k(\mathfrak{sl}_2) \hookrightarrow \mathsf{Vir}^k \otimes \Pi$. [Semikhatov'94]
- Prove that $A_1(u, v) \hookrightarrow \mathsf{Vir}(u, v) \otimes \Pi$ iff $v \neq 1$. [Adamović'17]

Thus, every $\mathsf{Vir}(\mathsf{u},\mathsf{v})$-module $\mathcal{M}$ and every $\Pi$-module $\mathcal{N}$ yield an $\mathsf{A}_1(\mathsf{u},\mathsf{v})$-module $\mathcal{M}\otimes\mathcal{N}$, by restriction.

If $\mathcal{M}$ and $\mathcal{N}$ are almost irreducible, then $\mathcal{M}\otimes\mathcal{N}$ is almost irreducible. [Adamović–Kawasetsu–DR'20]

Every irreducible lower-bounded $\mathsf{A}_1(\mathsf{u},\mathsf{v})$-module is a subquotient of some $\mathcal{M}\otimes\mathcal{N}$. [Adamović–Kawasetsu–DR'23]

$\mathsf{Zhu}[\mathsf{Vir}(\mathsf{u},\mathsf{v})]$ is a (f-dim semisimple) quotient of $\mathbb{C}[L]$ and $\mathsf{Zhu}[\Pi]$ is generated by $c$, $d$ and the $\mathsf{e}^{nc}$ with $n\in\mathbb{Z}$, $c$ central and

$$\mathsf{e}^{mc}\mathsf{e}^{nc}=\mathsf{e}^{(m+n)c},\quad [d,\mathsf{e}^{nc}]=2n\mathsf{e}^{nc}.$$

An (almost) irreducible weight $\mathsf{Zhu}[\Pi]$-module is thus dense: its weights comprise some $[\lambda]\in\mathbb{C}/2\mathbb{Z}$. Its multiplicities are all $1$.

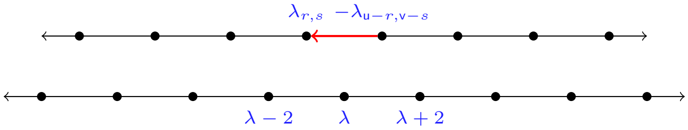
They form (irreducible) coherent families, hence so do the corresponding $\mathsf{Zhu}[\mathsf{A}_1(\mathsf{u},\mathsf{v})]$-modules. [Mathieu'00, Kawasetsu–DR'19]

Inverse QHR thus constructs almost-irreducible lower-bounded $\mathsf{A}_1(\mathsf{u},\mathsf{v})$-modules $\mathcal{R}_{[\lambda];r,s}$ whose top spaces are dense $\mathfrak{sl}_2$-modules.

The weights lie in $[\lambda] \in \mathbb{C}/2\mathbb{Z}$, while $1 \leqslant r \leqslant \mathsf{u}-1$ and $1 \leqslant s \leqslant \mathsf{v}-1$ (with $(r,s) \sim (\mathsf{u}-r, \mathsf{v}-s)$).

$\mathcal{R}_{[\lambda];r,s}$ is irreducible unless $[\lambda] = [\lambda_{r,s}], [\lambda_{\mathsf{u}-r,\mathsf{v}-s}]$. Then, it degenerates into highest-weight modules with infinite-dimensional top spaces:

$$0 \to \mathcal{H}_{r,s} \to \mathcal{R}_{[\lambda];r,s} \to \mathrm{w}(\mathcal{H}_{\mathsf{u}-r,\mathsf{v}-s}) \to 0.$$




All such highest-weight modules are constructed in this way. Spectral flow now gives the highest-weight modules with finite-dimensional top spaces.

Moreover:

- The weight category of $\mathsf{A}_1(\mathsf{u},\mathsf{v})$-modules is modular. [Creutzig–DR'13]
- The irreducible $\mathcal{R}_{[\lambda];r,s}$ are projective and injective. [Arakawa–Creutzig–Kawasetsu'23]
- The projective covers/injective hulls of the $\mathcal{H}_{r,s}$ are indecomposable sums of spectral flows of two of the $\mathcal{R}_{[\lambda_{r,s}];r,s}$. [Adamović'17]
- $L_0$ acts with rank-$2$ Jordan blocks on the reducible projectives. [Gaberdiel'01, Adamović–Milas'09, DR'10, Adamović'17]
- If the spectral flows of the $\widehat{\mathcal{R}}_{[\lambda_{r,s}];r,s}$ are (co)standard modules, then the projectives become tilting and they satisfy BGG reciprocity.
- The weight category of $\mathsf{A}_1(\mathsf{u},\mathsf{v})$ is blockwise abelian-equivalent to the category of finite-dimensional weight modules of the unrolled small quantum group of $\mathfrak{sl}_2$ at $\mathsf{q} = \mathsf{e}^{\pi \mathsf{i}\mathsf{u}/\mathsf{v}}$. [Costantino–Geer–Patureau-Mirand'14, Arakawa–Creutzig–Kawasetsu'23]

# A rank-2 story

A truism of Lie theory is that rank $1$ is too easy and one should master rank $2$ before pretending that the generalisation to all cases is clear.

We study the category of (finitely generated) weight $\mathsf{A}_2(\mathsf{u},\mathsf{v})$-modules when $\mathsf{u} \geqslant 3$ (the admissible levels).

This splits into three cases:

- $\mathsf{v} = 1$**:** The category is finite and semisimple. Every irreducible is highest-weight with a finite-dimensional top space. [zero]
- $\mathsf{v} = 2$**:** The category is infinite and nonsemisimple. There are irreducible highest-weight modules with infinite-dimensional top spaces but their multiplicities are bounded. [minimal]
- $\mathsf{v} \geqslant 3$**:** The category is infinite and nonsemisimple. There are irreducible highest-weight modules with infinite-dimensional top spaces and unbounded multiplicities. [principal]

Nilpotent orbits control the highest-weight modules and thus the weight modules with finite multiplicities. [Arakawa'12, Kawasetsu–DR'19]

There are two different inverse QHRs because there are two different (nontrivial) W-algebras associated to $\mathfrak{sl}_3$:

- The minimal one $\mathsf{BP}(\mathsf{u},\mathsf{v})$, *aka.* the Bershadsky–Polyakov algebra.
- The principal one $\mathsf{W}_3(\mathsf{u},\mathsf{v})$, *aka.* the Zamolodchikov algebra.

$$\mathsf{BP}(\mathsf{u},\mathsf{v}) \xhookrightarrow{\mathsf{v}\geqslant 3} \mathsf{W}_3(\mathsf{u},\mathsf{v}) \otimes \Pi, \quad \mathsf{A}_2(\mathsf{u},\mathsf{v}) \xhookrightarrow{\mathsf{v}\geqslant 2} \mathsf{BP}(\mathsf{u},\mathsf{v}) \otimes \beta\gamma \otimes \Pi.$$

The (new) Zhu algebras are as follows:

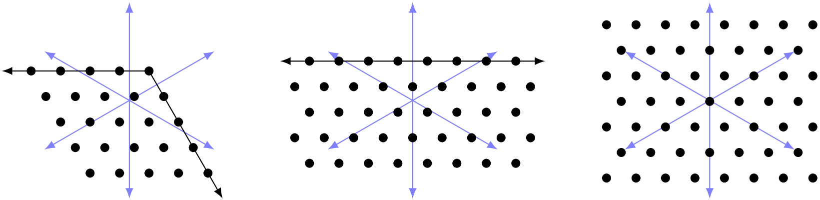
- $\mathsf{Zhu}[\beta\gamma]$ is the Weyl algebra: $[\gamma,\beta] = \mathbf{1}$. It has dense weight modules like $\mathsf{Zhu}[\Pi]$, but also has a single highest-weight module and a single lowest-weight module (both infinite-dimensional).
- $\mathsf{Zhu}[\mathsf{W}_3(\mathsf{u},\mathsf{v})]$ is a quotient of $\mathsf{Z}(\mathsf{U}(\mathfrak{sl}_3)) \cong \mathbb{C}[T,W]$.
- $\mathsf{Zhu}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$ is a quotient of a (centrally extended) Smith algebra. Its weight modules are much like those of $\mathfrak{sl}_2$.

Inverse QHR thus constructs almost-irreducible $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules ($\mathsf{v} \geqslant 3$) with dense top spaces whose weights have multiplicity $1$.

The inverse QHR construction of $\mathsf{A}_2(\mathsf{u},\mathsf{v})$-modules is more interesting.

If $\mathsf{v}=2$, then $\mathsf{BP}(\mathsf{u},\mathsf{v})$ is rational and so the irreducibles have finite-dimensional top spaces.

Depending on whether the $\beta\gamma$-module has dense or highest-weight top, we arrive at $\mathsf{A}_2(\mathsf{u},2)$-modules with dense or semidense tops whose multiplicities are bounded. Both form coherent families.



Every irreducible with an infinite-dimensional top space appears as a composition factor of a dense one. (Again, those with finite-dimensional top spaces are obtained by spectral flow.)

Our understanding of the weight category of $\mathsf{A}_2(\mathsf{u}, 2)$-modules is still quite limited:

- This category is modular. [Kawasetsu–DR–Wood'21, Fasquel–Raymond–DR'24]
- We believe that the irreducibles with dense top spaces are projective and injective.
- The projective covers/injective hulls of the irreducibles with semidense (highest-weight) tops are believed to be indecomposable sums of spectral flows of two (three or six) of the reducibles with dense top spaces. [Creutzig–DR–Rupert'21]
- $L_0$ acts with rank-$2$ (rank-$3$) Jordan blocks on the reducible projectives. [Adamović–DR'24]
- Setting the (co)standard modules to be spectral flows of the modules with dense tops, we (conjecturally) observe BGG reciprocity.
- We also conjecture a blockwise abelian equivalence with the category of finite-dimensional weight modules of the unrolled small quantum group of $\mathfrak{sl}_3$ at $\mathsf{q} = \pm\mathfrak{i}$.

# To boldly go...

If $\mathsf{v} \geqslant 3$, then $\mathsf{BP}(\mathsf{u},\mathsf{v})$ has irreducibles with infinite-dimensional top spaces. Inverse QHR thus constructs almost-irreducible $\mathsf{A}_2(\mathsf{u},\mathsf{v})$-modules with a much larger range of top spaces than when $\mathsf{v} = 2$:

| $\mathsf{BP}(\mathsf{u},\mathsf{v})$ | $\beta\gamma$ | $\Pi$ | $\mathsf{A}_2(\mathsf{u},\mathsf{v})$ |
|---|---|---|---|
| f-dim | $\infty$-dim hw | dense | bounded semidense |
| f-dim | dense | dense | f-mult dense |
| $\infty$-dim hw | $\infty$-dim hw | dense | unbounded semidense |
| $\infty$-dim hw | dense | dense | $\infty$-mult dense |
| dense | $\infty$-dim hw | dense | $\infty$-mult dense |
| dense | dense | dense | $\infty$-mult dense |

We don't know how to classify the dense $\mathsf{Zhu}[\mathsf{A}_2(\mathsf{u},\mathsf{v})]$-modules with infinite multiplicities.

Nevertheless, inverse QHR indicates that there is substructure within this class of weight $\mathfrak{sl}_3$-modules.

Dense $\mathfrak{sl}_3$-modules with infinite multiplicities are not new, *cf.* Futorny *et al.*'s Gelfand–Tsetlin constructions. But, there is no general theory.

*eg.*, [Arakawa–Futorny–Ramirez'16] construct three families of such $\mathsf{A}_2(\mathsf{u},\mathsf{v})$-modules, but do not claim completeness.

More examples: apply Mathieu's twisted localisation functors to highest-weight modules with unbounded multiplicities.

We expect that the inverse QHR construction with dense tensorands is "universal". It gives $5$-parameter infinite-multiplicity weight modules:

$$\mathsf{A}_2(\mathsf{u},\mathsf{v}) \xrightarrow{\mathsf{v}\geqslant 2} \mathsf{BP}(\mathsf{u},\mathsf{v}) \otimes \beta\gamma \otimes \Pi \xrightarrow{\mathsf{v}\geqslant 3} \underset{\Delta,w}{\mathsf{W}_3(\mathsf{u},\mathsf{v})} \otimes \underset{[r]}{\Pi} \otimes \underset{[s]}{\beta\gamma} \otimes \underset{[t]}{\Pi}.$$

$\Delta$ and $w$ specify the "coherent family", while the basis weight vectors have the form $|r',s',t'\rangle$ with $r' \in [r]$, $s' \in [s]$ and $t' \in [t]$ ($[q] \in \mathbb{C}/\mathbb{Z}$).

But these weight modules are (generically) not Gelfand–Tsetlin. [Adamović–Creutzig–Genra'21]

# Some preliminary results [Raymond–DR'24]

- The "triply dense" $\mathsf{A}_2(\mathsf{u},\mathsf{v})$-modules, $\mathsf{v} \geqslant 3$, constructed by inverse QHR are generically irreducible.
- There is a codim-$1$ set of parameters at which these modules degenerate into two generically irreducible $\mathsf{A}_2(\mathsf{u},\mathsf{v})$-modules, both with infinite multiplicities. At least one is Gelfand–Tsetlin.
- There is a codim-$2$ set of parameters at which these modules degenerate into four generically irreducible $\mathsf{A}_2(\mathsf{u},\mathsf{v})$-modules. Two are semidense with unbounded multiplicities. The others are dense with infinite multiplicities; at least one is Gelfand–Tsetlin.
- There is a codim-$3$ set of parameters at which these modules degenerate into eight (?) irreducible $\mathsf{A}_2(\mathsf{u},\mathsf{v})$-modules. Four are highest-weight with unbounded multiplicities. The others have not yet been understood.

**Question**: Do the Gelfand–Tsetlin subquotients of these degenerations belong to the families constructed in [Arakawa–Futorny–Ramirez'16]?

Aside from completing the identification of the composition factors of these degenerations, our next task is to check that the category generated by these $A_2(u,v)$-modules is modular.

Their (generalised) characters certainly span an (uncountably infinite-dimensional) unitary module of the modular group $SL(2;\mathbb{Z})$.

The challenge is to demonstrate that the Verlinde formula gives (Grothendieck) fusion coefficients that are nonnegative integers.

If so, then we can predict the (Grothendieck) fusion rules of the weight category of $A_2(u,v)$.

Addressing this challenge first requires completing the explicit degeneration identifications.

If this is successful, we can then try to construct projective covers. This is expected to be extremely difficult... (*ie.*, fun).

# Conclusions

The consistency requirement that the physically relevant category be modular means that we have to consider (and understand) weight modules with infinite multiplicities.

Inverse QHR gives us powerful new methods to construct these modules.

We are now discovering how to analyse them and so verify modularity.

Philosophically, one might hope that every class of modules has a "purpose". A purpose for the inverse QHR modules is modularity.

A purpose for the previously known Gelfand–Tsetlin modules is to be subquotients of degenerate inverse QHR modules.

# Outlook

Inverse QHR seems to be the right way to analyse the representation theory of affine VOAs (and their associated W-algebras).

- Start with the regular W-algebra at an admissible (but nondegenerate) level. These are rational with known representation theories!
- Use inverse reduction to construct the relaxed modules of the subregular W-algebra. Get the other irreducibles as quotients.
- Repeat, working your way up the lattice of nilpotents until the representation theory of the desired W-algebra is known!

If the level is admissible but degenerate, don't despair: start instead with a rational exceptional W-algebra. [Arakawa–van Ekeren'19, McRae'21]

- When $\mathsf{v} = 1$, $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ is exceptional.
- For $\mathfrak{g} = \mathfrak{sl}_3$, $\mathsf{u} \geqslant 3$ and $\mathsf{v} = 2$, Bershadsky–Polyakov is exceptional.
- For $\mathfrak{g} = \mathfrak{sl}_n$, $\mathsf{u} \geqslant n$ and $\mathsf{v} = n - 1$, the subregular is exceptional.

[This needs generalising to the super case...]

# Outlook

In all but the simplest (*ie.*, rank-$1$) cases, inverse QHR will construct dense modules with infinite multiplicities.

Such modules might be profitably described by generalising Mathieu's coherent family technology.

There will also be qualitatively different types of infinite-multiplicity coherent families, some being related by degenerations and some perhaps being parametrised by nilpotent orbits.

The theory of Gelfand–Tsetlin modules with infinite multiplicities would then form an important special case of this theory.

Either way, there is some surely beautiful new mathematics to explore!

"*Only those who attempt the absurd will achieve the impossible.*"

— Miguel de Unamuno


Thank you for listening!