The principal W-algebra of $psl(2|2)$

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$psl(2|2)$ is a weird Lie superalgebra, but it's important in physics:

- min. reduction is the (small) $N=4$ SCA
- $AdS_3 \times S^3 \times T^4 \simeq \underbrace{SU(1,1) \times S^3}_{} \times T^4$

  supersymmetrise : $PSU(1,1|2) \times \mathbb{T}^4$

Understanding the rep thy of the SVOA is a lofty goal, but an important one.

Possible path: IQHR.

unexplored?

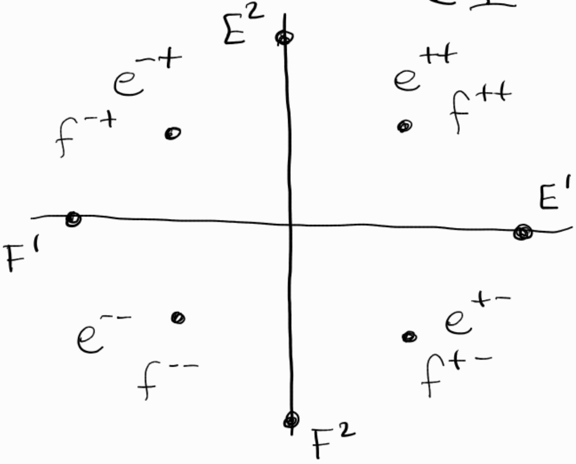
$$V^k(psl(2|2)) \longleftarrow (N=4)^k \longleftarrow W^k_{pr}$$

IQHR dichotomy: $\begin{cases} \text{Lift reps to } \underline{\text{relaxed}} \text{ reps, } \boxed{\text{OR}} \\ \exists \text{ easily found generating SV.} \end{cases}$

Probable obstacle: W^k_{pr} rep thy is hard...

1. $psl(2|2) = \{ A \in gl(2|2) : \mathrm{str}\, A = 0 \} / \mathbb{C} I$.

$$\left(\begin{array}{cc|cc} H^1 & E^1 & e^{+-} & e^{++} \\ F^1 & & e^{--} & e^{-+} \\ \hline f^{-+} & f^{++} & H^2 & E^2 \\ f^{--} & f^{+-} & F^2 & \end{array}\right)$$



- even subalg: $sl(2) \oplus sl(2)$
- $\mathrm{span}\{e^{\pm\pm}\} \simeq \mathcal{L}_1 \otimes \mathcal{L}_1 \simeq \mathrm{span}\{f^{\pm\pm}\}$
- Killing form $= 0$ (so $h^\vee = 0$) but str form non-deg.
- Weyl gp $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, but $\mathrm{Aut}\, \Delta \cong D_4$.
- $\exists\, \omega \in \mathrm{Aut}\, \Delta \setminus W$ that swaps the two $sl(2)$ subalgs.

But, $\mathrm{str}[\omega(A)\omega(B)] = -\mathrm{str}[AB]$!

2. $V^k(psl(2|2))$ and QHRs

$\widehat{psl}(2|2) = psl(2|2)[t, t^{-1}] \oplus \mathbb{C}K$ as usual, but

$$\hat{\omega}(K) = -K.$$

$\therefore$ $V^k(psl(2|2))$ admits the usual Sugawara construction (using str-form), ie get SVOA $\forall\, k \neq -h^\vee = 0$.

- $c = \dfrac{k \operatorname{sdim} psl(2|2)}{k+h^\vee} = \operatorname{sdim} psl(2|2) = -2$
- $V^k(psl(2|2)) \simeq V^{-k}(psl(2|2))$.

QHRs $\longleftrightarrow$ even nilp. orbits:

representative	0	F^1	F^2	F^1+F^2
SVOA	$V^k(psl(2\|2))$	$(N=4)^k$	$(N=4)^{-k}$	W^k_{pr}
c	-2	$-6(k+1)$	$6(k-1)$	-2
$\hat{\omega}$-action	$\circlearrowleft$	$\longleftrightarrow$		$\circlearrowleft$

3. W^k_{pr} has type $(2,2\,|\,1,1,2,2)$.

- $SF = V(psl(1|1))$ is a subalg of W^k_{pr}:

$$x^i(z)\,x^j(w) \sim \frac{2J^{ij}}{(z-w)^2}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- wt 2 generators span the proj. mod. of $psl(1|1)$:

$$\begin{tikzcd} & H \arrow[dl, "x_0^-"'] \arrow[dr, "x_0^+"] & \\ \psi^- \arrow[dr, "-x_0^+"'] & & \psi^+ \arrow[dl, "x_0^-"] \\ & S & \end{tikzcd}$$

- conf. vector is $L = S - \frac{1}{2} :x^+x^-:$.

OPEs are a bit messy, but:

- $\mathrm{gr}\, H = \mathrm{gr}\, S = 0$, $\mathrm{gr}\, \chi^{\pm} = \mathrm{gr}\, \psi^{\pm} = \pm 1$ defines a horizontal grading on W^k_{pr}.

- S is a conf. vect. with $\underline{c=0}$ commuting with SF. $\mathrm{Com}(SF, W^k_{pr})$ has type $(2,4,6,\dots \mid 5^2,\dots)$.

  Is it something we know?

- Shap calculations suggest that W^k_{pr} is simple unless $k \in \mathbb{Q} \setminus \mathbb{Z}$.

- $k = \pm\frac{1}{2}$ is collapsing: $W^{pr}_{\pm 1/2} \simeq SF$.

$$L_{1/2}(psl(2|2)) \leftarrow ?$$

well studied $\rightarrow (N=4)_{1/2}$ $\qquad\qquad$ $(N=4)_{-1/2} \leftarrow ?$

$$\frac{SF}{c=-9 \qquad c=-2 \qquad c=-3}$$

This probably explains why $(N=4)_k$ ~~for~~ $c=-9$ is so nice. [IN PROGRESS!]

4. Rep thy

$\mathrm{Zhu}[W^k_{pr}]$ is gen. by $H, L; x^\pm, \psi^\pm$ subject to

- L is central, $(x^\pm)^2 = 0$, $(\psi^\pm)^2 = \frac{1}{2} x^\pm \psi^\pm$.
- $\{x^+, x^-\} = 0$, $\{x^\pm, \psi^\pm\} = 0$, $\{x^\pm, \psi^\mp\} = \pm(L + \frac{1}{2}x^+x^-)$
  $[x^\pm, H] = \psi^\pm$.
- $\{\psi^+, \psi^-\} = \frac{1}{2}(x^+\psi^- + x^-\psi^+)$,
  $[\psi^\pm, H\} = x^\pm(H + \frac{3}{2}k^2 L + \frac{1}{4}(k^2-1)) - \frac{1}{2}\psi^\pm$.

Note: $(\psi^\pm)^2 \neq 0$ but $(\psi^\pm)^3 = \frac{1}{2} x^\pm (\psi^\pm)^2 = \frac{1}{4}(x^\pm)^2 \psi^\pm = 0$.

The Zhu alg. is PBW: $\quad \mathrm{Odd}^- \cdot \mathrm{Even} \cdot \mathrm{Odd}^+$

Def: A weight vector of $\mathrm{Zhu}[W^k_{pr}]$ is a simultaneous eigenvector of H and L.

A hw vector is a wt vec. ann. by x^+ and ψ^+.

The Verma mod $V_{h,\Delta}$ is $\mathrm{Ind}\, \mathbb{C}_{h,\Delta}$ where h, Δ are eigs of H, L and x^+, ψ^+ ann. $\mathbb{C}_{h,\Delta}$.

Lem: If a $\mathrm{Zhu}[W^k_{pr}]$-mod has a wt vect, then it has a hw vect.

$\therefore$ all f-dim mods have a hw vect

Denote irred. quot. of $\mathcal{V}_{h,\Delta}$ by $\mathcal{L}_{h,\Delta}$.

Prop:
- $\dim \mathcal{V}_{h,\Delta} = 4$.
- $\mathcal{V}_{h,\Delta}$ is irreducible iff $\Delta \neq 0$.
- $\dim \mathcal{L}_{h,0} = 1$.

$$\begin{tikzcd}
 & v \arrow[dl, "\chi^-"'] \arrow[dr, "\psi^-"] & \\
\chi^- v \arrow[dr, "\psi^-"'] & & \psi^- v \arrow[dl, "{\chi^-,\ \psi^-}"] \\
 & \chi^-\psi^- v &
\end{tikzcd}$$

- $S = L + \frac{1}{2}\chi^+\chi^-$ acts non-diag:
  $S\psi^- v = \Delta\psi^- v - \frac{1}{2}\Delta\chi^- v$.
- H acts non-diag. iff
  $h = -\frac{1}{4}(6\Delta+1)k^2$.

Weird, but same for other principal W-algebras...

Prop: $\exists$ IQHR $(N=4)^k \hookrightarrow W^k_{pr} \otimes \Pi$ with
$\cup$
$V^{-k-1}(sl_2)$

$E \mapsto e^{2c}$,

$H \mapsto 2b$, $\quad F \longmapsto (H + \alpha L + \beta :\psi^+\psi^-: - :aa: + k\partial a)e^{-2c}$

$G^+ \mapsto \chi^+ e^{c}$, $\quad \bar{G}^+ \mapsto \chi^- e^{c}$, $\quad T \longmapsto L + \frac{1}{2}:cd: - \partial a$

$G^- \mapsto -\psi^+ e^{-c} + \dots$, $\quad \bar{G}^- \mapsto \psi^- e^{-c} + \dots$.

(Should change labels $\pm$ on W^k_{pr} fermions...)