A new (?) class of irreducible weight $\mathfrak{sl}_3$-modules

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Outline

1. Irreducible weight modules for $\mathfrak{sl}_3$
2. Vertex-algebraic motivations
3. Dense $\mathfrak{sl}_3$-modules with infinite multiplicities
4. Degenerations
5. Outlook

Weight modules for $\mathfrak{sl}_3$

Recall the following basis of $\mathfrak{sl}_3 = \mathfrak{sl}_3(\mathbb{C})$:

$$f^3 = E_{31}, \qquad \begin{matrix} f^1 = E_{21}, & h^1 = E_{11} - E_{22}, & e^1 = E_{12}, \\ f^2 = E_{32}, & h^2 = E_{22} - E_{33}, & e^2 = E_{23}, \end{matrix} \qquad e^3 = E_{13}.$$

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- a weight module: a module that decomposes as the direct sum of its weight spaces;
- the weight support: the set of weights of a module.

["Weight" also makes sense more generally as long as one has something like a Cartan subalgebra.]

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These lift to automorphisms of $\mathfrak{sl}_3$, also forming a copy of D_6:

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$\mathsf{d}(x)$	h^2	h^1	e^2	e^1	$-e^3$	f^2	f^1	$-f^3$
$\mathsf{w}_1(x)$	$-h^1$	h^3	f^1	e^3	e^2	e^1	f^3	f^2
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$\mathsf{Z}(\mathfrak{sl}_3)$ is generated by

$$C_2 = \frac{1}{3}(h^1h^1 + h^2h^2 + h^3h^3) + h^1 + h^2 + h^3 + 2(f^1e^1 + f^2e^2 + f^3e^3)$$

and $$C_3 = (g^1+1)(g^2+1)g^3 + f^1e^1(g^2+1) - f^2e^2(g^1+1) + f^3e^3g^3 + f^1f^2e^3 + f^3e^1e^2$$

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These Casimirs are almost D_6-invariant:

$$\mathsf{w}_i(C_2) = C_2, \quad \mathsf{d}(C_2) = C_2, \qquad \mathsf{w}_i(C_3) = C_2, \quad \mathsf{d}(C_3) = -C_3.$$

Weight modules for $\mathfrak{sl}_3$

Irreducible weight $\mathfrak{sl}_3$-modules come in three types, namely
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- highest weight: generated by a highest-weight vector, *ie.* a weight vector annihilated by e^α and e^β, where $\langle \alpha, \beta \rangle = -1$.

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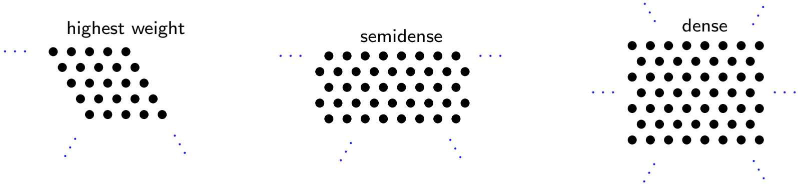

The multiplicities of a highest-weight or semidense module are all finite, but those of a dense module are either all finite or all infinite.

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- If the multiplicities are infinite, then nobody knows what to do.

Guess which case we're going to be looking at...

From vertex algebras to $\mathfrak{sl}_3$-modules

We are motivated to study weight modules because they are relevant to the representation theory of affine vertex-operator algebras.

When the level is integral, *ie.* $\mathsf{k} \in \mathbb{N}$, then the module category of the simple affine VOA $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ is finite, semisimple and modular [Gepner–Witten'86, Verlinde'88, Moore-Seiberg'89, Huang'05].

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- The weight-module category of the simple affine VOA for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ is modular [Creutzig–DR'13].
- Also true for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$ if k has denominator $\mathsf{v} = 2$ [Kawasetsu–DR–Wood'21, Fasquel–Raymond–DR'24].

Here, "weight" allows finite-rank Jordan blocks in the action of L_0 and "modular" means in the sense of [Creutzig–DR'13, DR–Wood'14].

The conjecture is that the weight-module category is always modular. It is also the smallest modular category in a sense to be made precise.

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The issue boils down to a question of practical classification theorems:

- Irreducible weight modules are "spectral flows" of relaxed highest-weight modules [Futorny–Tsylke'01, Adamović–Kawasetsu–DR'23].
- The latter are recognised by their "Zhu images", which are irreducible weight $\mathfrak{g}$-modules [Zhu'96, Frenkel–Zhu'92].

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- When $\mathsf{v} > 2$, there are irreducible weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$-modules whose Zhu images have infinite multiplicities [Arakawa–Futorny–Ramirez'16].

No known classification, but existence follows by comparing with a subclass for which identification is possible (Gelfand–Tsetlin modules) [Futorny–Grantcharov–Ramirez'18].

Inverse quantum hamiltonian reduction

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Moreover, IQHR knows about denominators.

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\begin{array}{ccc}
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\uparrow & \uparrow & \uparrow \\
\mathsf{Vir}_\mathsf{k}\text{-wtmod} & \mathsf{BP}_\mathsf{k}\text{-wtmod} & \mathsf{BP}_\mathsf{k}\text{-wtmod} \\
(\mathsf{v} > 1) & (\mathsf{v} = 2) & \uparrow \\
 & & \mathsf{W}_{3,\mathsf{k}}\text{-wtmod} \\
 & & (\mathsf{v} > 2)
\end{array}
$$

[Adamović'17, Adamović–Kawasetsu–DR'20]
[Adamović–Creutzig–Genra'21, Fasquel–Raymond–DR'24]

Ingredients

Semikhatov embeddings involve VOAs like

- The bosonic ghost system G.
  The irreducible relaxed highest-weight modules have Zhu images that are irreducible weight modules for the Weyl algebra:

$$\mathbb{C}[z^{-1}], \qquad z\mathbb{C}[z], \qquad z^{\lambda}\mathbb{C}[z, z^{-1}] \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}).$$

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- The Zamolodchikov algebra $\mathsf{W}_3^{\mathsf{k}}$.
  The Zhu images of the irreducible relaxed highest-weight modules are
  $$\mathbb{C}u^{\Delta,w} \quad (\Delta, w \in \mathbb{C}).$$

Our main example is a “composite” Semikhatov embedding:

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \hookrightarrow \mathsf{W}_3^{\mathsf{k}} \otimes \mathsf{G} \otimes \Pi \otimes \Pi$$

[Adamović–Kawasetsu–DR'20, Adamović–Creutzig–Genra'21, Fasquel–Raymond–DR'24].
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The corresponding Adamović restriction functors construct dense $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-modules whose Zhu images have infinite multiplicities:

$$\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]} = \mathbb{C}u^{\Delta,w} \otimes x^r\mathbb{C}[x,x^{-1}] \otimes y^s\mathbb{C}[y,y^{-1}] \otimes z^t\mathbb{C}[z,z^{-1}].$$

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These images are weight $\mathfrak{sl}_3$-modules with basis

$$|r,s,t\rangle = u^{\Delta,w} \otimes x^r \otimes y^s \otimes z^t, \quad r \in [\rho],\ s \in [\sigma],\ t \in [\tau].$$

They were first constructed in [Adamović–Creutzig–Genra'21].

A new (?) family of $\mathfrak{sl}_3$-modules

The action on the $\mathfrak{sl}_3$-module $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is given by

$$\begin{aligned}
e^1|r,s,t\rangle &= |r-1,s,t+1\rangle, \quad e^2|r,s,t\rangle = r|r+1,s,t\rangle, \quad e^3|r,s,t\rangle = |r,s,t+1\rangle, \\
h^1|r,s,t\rangle &= -(r+s-t-1)|r,s,t\rangle, \quad h^2|r,s,t\rangle = (2r+2s+t-2)|r,s,t\rangle, \\
f^1|r,s,t\rangle &= P(s)|r,s+1,t-1\rangle - r(s-t)|r+1,s,t-1\rangle, \\
f^2|r,s,t\rangle &= |r,s-1,t\rangle - (r+2s+t-2)|r-1,s,t\rangle, \\
f^3|r,s,t\rangle &= -r|r+1,s-1,t-1\rangle - P(s)|r-1,s+1,t-1\rangle \\
&\quad + \big(P(s) - P(s-1) + (r+2s+t-2)(s-t)\big)|r,s,t-1\rangle,
\end{aligned}$$

where $P(s) = w + \big((\mathsf{k}+3)\Delta + (\mathsf{k}+2)^2\big)s - s^3$.

A new (?) family of $\mathfrak{sl}_3$-modules

The action on the $\mathfrak{sl}_3$-module $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is given by

$$\begin{aligned}
e^1|r,s,t\rangle &= |r-1,s,t+1\rangle, \quad e^2|r,s,t\rangle = r|r+1,s,t\rangle, \quad e^3|r,s,t\rangle = |r,s,t+1\rangle, \\
h^1|r,s,t\rangle &= -(r+s-t-1)|r,s,t\rangle, \quad h^2|r,s,t\rangle = (2r+2s+t-2)|r,s,t\rangle, \\
f^1|r,s,t\rangle &= P(s)|r,s+1,t-1\rangle - r(s-t)|r+1,s,t-1\rangle, \\
f^2|r,s,t\rangle &= |r,s-1,t\rangle - (r+2s+t-2)|r-1,s,t\rangle, \\
f^3|r,s,t\rangle &= -r|r+1,s-1,t-1\rangle - P(s)|r-1,s+1,t-1\rangle \\
&\qquad + \big(P(s)-P(s-1)+(r+2s+t-2)(s-t)\big)|r,s,t-1\rangle,
\end{aligned}$$

where $P(s) = w + \big((\mathsf{k}+3)\Delta + (\mathsf{k}+2)^2\big)s - s^3$.

- The weight of $|r,s,t\rangle$ is $(r+s-1)\alpha_2 + t\alpha_3$.
- The C_2-eigenvalue is $2(\mathsf{k}+3)(\Delta+\mathsf{k}+1)$.
- The C_3-eigenvalue is w.

Generic irreducibility

[Adamović–Creutzig–Genra'21] conjectured that these $\mathfrak{sl}_3$-modules are generically irreducible. Settling this is our first result.

Theorem

- *$\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is irreducible unless*

$$[\rho] = [0], \quad P(s) = 0, \quad P(t) = 0 \quad \textit{or} \quad P(1 - r - s - t) = 0,$$

  for some $r \in [\rho]$, $s \in [\sigma]$ and $t \in [\tau]$.

- *When irreducible, $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is torsion free.*

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- *When irreducible,* $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ *is torsion free.*

- If $[\rho] = [0]$, then $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is reducible.
- If $P(s) = 0$ for some $s \in [\sigma]$, then $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is reducible.
- In the other two cases, reducibility is subtle and still being checked.

Quick aside: Gelfand–Tsetlin modules

There is a well studied class of generically irreducible dense $\mathfrak{sl}_3$-modules with infinite multiplicities: the Gelfand–Tsetlin modules of [Drozd–Futorny–Ovsienko'94, Futorny–Grantcharov–Ramirez'14].

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \mathfrak{sl}_3 \qquad \Rightarrow \qquad \underset{g^1}{\mathsf{Z}\big(\mathfrak{gl}_1^{(1)}\big)} \subset \underset{g^2,\, C_2^{(1)}}{\mathsf{Z}\big(\mathfrak{gl}_2^{(1)}\big)} \subset \underset{C_2,\, C_3}{\mathsf{Z}\big(\mathfrak{sl}_3\big)}$$

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These modules have several nice properties:

- They have a basis of simultaneous eigenvectors (Gelfand–Tsetlin patterns) of $g^1, g^2 \in \mathfrak{h}$, C_2, C_3 and the $\mathfrak{sl}_2$-Casimir $C_2^{(1)}$.

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Twisting by D_6 gives three distinct categories $\mathsf{GT}^{(i)}$, $i = 1, 2, 3$, parametrised by the positive root α_i defining the $\mathfrak{sl}_2$-Casimir $C_2^{(i)}$.

And back to inverse QHR

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$\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ *is never in* $\mathsf{GT}^{(1)}$, $\mathsf{GT}^{(2)}$ *or* $\mathsf{GT}^{(3)}$.

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- When $[\rho] \neq [0]$ or $P(s) \neq 0$ for all $s \in [\sigma]$, this follows because the $\mathfrak{sl}_2$-Casimirs $C_2^{(i)}$, $i = 1, 2, 3$, then have no eigenvectors on $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$.

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- Otherwise, $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is reducible and we have to work harder.

Of course, the reducible cases are where things get really interesting!

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Thus, $\mathcal{R}^{\Delta,w}_{[0,\sigma,\tau]}$ is (generically) a non-split extension of two Gelfand–Tsetlin modules from different Gelfand–Tsetlin categories.

The same is true when $P(s) = 0$ for some $s = \xi \in [\sigma]$, except that the submodule $\mathcal{N}^{\Delta,w}_{[\rho,\xi,\tau]}(2)$ lies in $\mathsf{GT}^{(1)}$ and the quotient in $\mathsf{GT}^{(2)}$ (and both are torsion free).

$$[\rho] = [0]: \begin{array}{c} \infty\text{-mult}^{(1)} \\ \big\downarrow \\ \infty\text{-mult}^{(2)} \end{array} \qquad\qquad \begin{array}{c} \infty\text{-mult}^{(2)} \\ \big\downarrow \\ \infty\text{-mult}^{(1)} \end{array} :P(s) = 0$$

Degenerations 2

Suppose that both $[\rho] = [0]$ and $P(s) = 0$ for some $s = \xi \in [\sigma]$. Then:

- $\mathcal{R}^{\Delta,w}_{[0,\xi,\tau]}$ has a semidense submodule and a semidense quotient

$$\mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(1) \cap \mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(2) \quad \text{and} \quad \frac{\mathcal{R}^{\Delta,w}_{[0,\xi,\tau]}}{\mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(1) + \mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(2)}.$$

  They are generically irreducible and belong to $\mathsf{GT}^{(i)}$ for all i.

- The other generically irreducible subquotients are

$$\frac{\mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(1)}{\mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(1) \cap \mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(2)} \quad \text{and} \quad \frac{\mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(2)}{\mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(1) \cap \mathcal{N}^{\Delta,w}_{[0,\xi,\tau]}(2)}.$$

  They are infinite-multiplicity in $\mathsf{GT}^{(2)}$ and $\mathsf{GT}^{(1)}$, respectively.

$[\rho] = [0]$ and $P(s) = 0$:

$$\begin{tikzcd} & \text{semidense} \arrow[dl] \arrow[dr] & \\ \infty\text{-mult}^{(1)} \arrow[dr] & & \infty\text{-mult}^{(2)} \arrow[dl] \\ & \text{semidense} & \end{tikzcd}$$

Degenerations 3

Because "double degeneration" resulted in semidense modules, we can analyse when they further degenerate into highest-weight modules.

It turns out that this happens when $P(t) = 0$ for some $t \in [\tau]$.

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The two semidense composition factors therefore degenerate into 4 (generically irreducible) highest-weight modules.

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The two semidense composition factors therefore degenerate into 4 (generically irreducible) highest-weight modules.

What happens to the 2 infinite-multiplicity factors? Each degenerates into an infinite-multiplicity submodule and a highest-weight quotient.

The picture is thus that we have 6 highest-weight composition factors and 2 infinite-multiplicity ones.

[These can further degenerate, *eg.* if we specialise the values of Δ and w.]

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- QHR suggests the following parametrisation of the $\mathsf{W}_3^{\mathsf{k}}$ eigenvalues:

$$\Delta = \frac{\xi_1^2 + \xi_2^2 + \xi_3^2 - 2(\mathsf{k}+2)^2}{2(\mathsf{k}+3)} \quad \text{and} \quad w = \xi_1\xi_2\xi_3,$$

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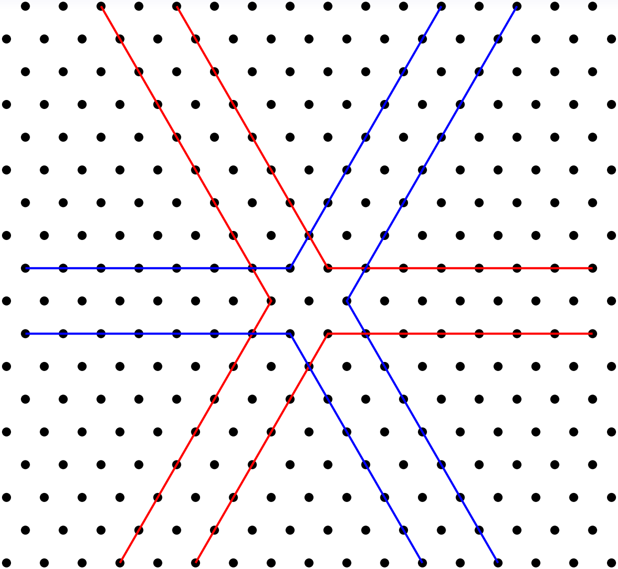
as well as the condition for highest-weight degenerations.

- The 6 highest-weight composition factors are (D_6-twists of) modules with highest weights of the form

$$\mathsf{w} \cdot \big(\lambda - (\mathsf{k}+3)\alpha_3\big), \quad \mathsf{w} \in \mathsf{S}_3.$$



Outlook

Our results demonstrate that the modular weight category needed for CFT applications is similar, but certainly different, to the Gelfand–Tsetlin categories. What is it precisely?

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In general, we expect a (partially ordered) filtration of tensor categories, parametrised by the nilpotent orbits of $\mathfrak{g}$:

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Either way, there is some surely beautiful new mathematics to explore!

"*Only those who attempt the absurd will achieve the impossible.*"