Bosonic ghost correlators: a case study

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[Joint with Xueting Li and Damodar Rajbhandari]

University of Melbourne

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[Challenges in Integrability, Wuhan]

Outline

Logarithmic CFTs

I want to understand conformal field theory (CFT)...

rational — factorisable — non-rational

[compactified bosons]
[free fermions]
[bc-ghosts]
[minimal models]
[compact WZW models]
[a few W-algebras]

[non-compact bosons]
[Liouville/Toda]
[non-compact WZW?]

discrete - - - continuous

[symplectic fermions]
[triplet models]
[polymers, percolation?]
[log minimal models?]
[SLE/CLE?]

[bosonic ghosts]
[supergroup WZW]
[Nappi–Witten]
[fractional-level WZW]
[most W-algebras]
[$O(n)$ models]

log-rational — logarithmic — generic

CFTs are built from reps of its chiral algebra, *aka*. vertex operator algebra (VOA).

A rational CFT has a VOA module category that is

- semisimple: modules are completely reducible,
- finite: there are finitely many irreducibles (up to $\cong$),
- q-finite: modules have q-characters ($\operatorname{tr} \mathsf{q}^{L_0 - \mathsf{c}/24}$ exists).

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Log-rational means non-semisimple but finite and q-finite.
[But few accessible examples.]
Non-rational means semisimple but not finite (but can be q-finite).
[Usually notoriously difficult.]

Generically, we lose all three conditions. But here we have surprisingly many accessible (and important!) examples... this is log CFT.

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But good news everyone: this means that there's still an awful lot to do...

Bosonic ghosts

The bosonic ghost system was introduced in [Friedan–Martinec–Shenker'86] to study gauge fixing for superstrings.

- It is generated by two fields β and γ satisfying

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- There is a current $J(z) = {:}\beta(z)\gamma(z){:}$ that assigns β and γ charges $j_\beta = 1$ and $j_\gamma = -1$, respectively.
- There is also a 1-parameter family of energy-momentum tensors. We choose

$$T(z) = -{:}\beta(z)\partial\gamma(z){:},$$

so that $h_\beta = 1$, $h_\gamma = 0$ and $\mathsf{c} = 2$.

[This is the right setup for, *eg.*, Wakimoto free-field realisations.]

The mode algebra has commutation relations

$$[\beta_m, \beta_n] = 0 = [\gamma_m, \gamma_n], \quad [\beta_m, \gamma_n] = -\delta_{m+n=0}, \qquad m, n \in \mathbb{Z}.$$

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It therefore admits spectral flow automorphisms

$$\begin{aligned} \sigma^\ell(\beta_n) &= \beta_{n-\ell}, \\ \sigma^\ell(\gamma_n) &= \gamma_{n+\ell}, \end{aligned} \quad \Rightarrow \quad \begin{aligned} \sigma^\ell(J_n) &= J_n + \ell\delta_{n=0}, \\ \sigma^\ell(L_n) &= L_n - \ell J_n - \tfrac{1}{2}\ell(\ell-1)\delta_{n=0}. \end{aligned}$$

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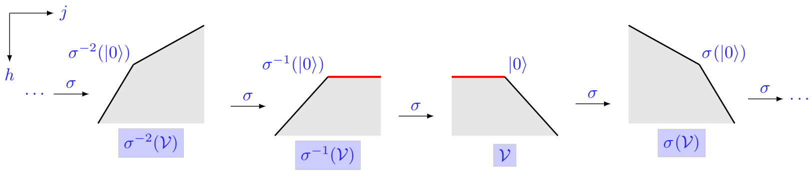
These can be used to twist the action of the ghost algebra on a module $\mathcal{M}$, producing new (non-isomorphic) modules $\sigma^\ell(\mathcal{M})$.



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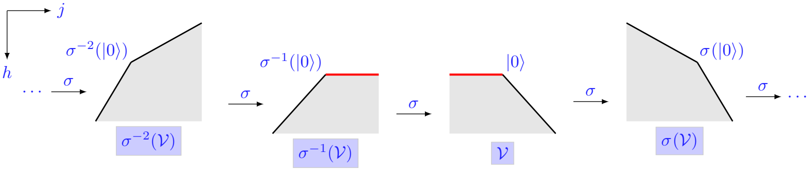

Only two spectral flows of the vacuum module $\mathcal{V}$ are lower bounded (meaning their conformal weights are bounded below).

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$$\mathcal{W}_{[j]}, \quad [j] \in \mathbb{C}/\mathbb{Z}.$$

Here, the top space (space of lowest conformal weight) has a basis $\{|\phi_i\rangle \ : \ i \in [j]\}$ satisfying

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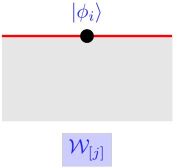
We take $[j] \neq [0]$ for irreducibility.



When $[j] = [0]$, there are two reducible lower-bounded modules $\mathcal{W}_{[0]}^{\pm}$.

- $\mathcal{W}_{[0]}^{+}$ has a submodule $\cong$ to $\mathcal{V}$ and $\mathcal{W}_{[0]}^{+}/\mathcal{V}$ is $\cong$ to $\sigma^{-1}(\mathcal{V})$.
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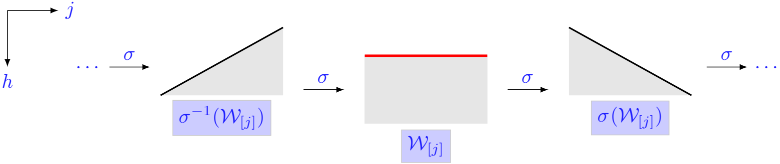

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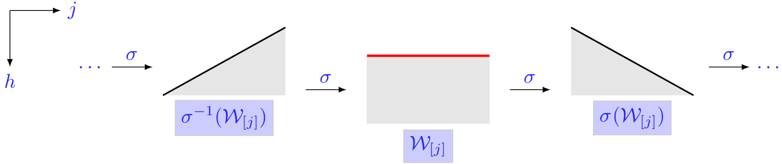

The characters of the $\sigma^{\ell}(\mathcal{W}_{[j]})$ can be computed explicitly. They transform under $\mathsf{SL}(2;\mathbb{Z})$ so should form a consistent CFT spectrum. [DR–Wood'14]

Fusion

The "standard Verlinde formula" of [Creutzig–DR'13] implies the generic fusion rules [DR–Wood'14, Adamović–Pedić'19]

$$\sigma^{\ell}(\mathcal{W}_{[j]}) \times \sigma^{\ell'}(\mathcal{W}_{[j']}) \cong \sigma^{\ell+\ell'}(\mathcal{W}_{[j+j']}) \oplus \sigma^{\ell+\ell'-1}(\mathcal{W}_{[j+j']}),$$

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$\mathcal{P}$ is a logarithmic module: reducible but indecomposable with a non-diagonalisable L_0-action [Lesage–Mathieu–Rasmussen–Saleur'03, DR'10].

It follows [Gurarie'93] that the bosonic ghost system has correlators with logarithmic singularities.

- $\mathcal{P}$ has a submodule $\cong$ to $\mathcal{W}_{[0]}^{+}$ and $\mathcal{P}/\mathcal{W}_{[0]}^{+}$ is $\cong$ to $\sigma(\mathcal{W}_{[0]}^{+})$.
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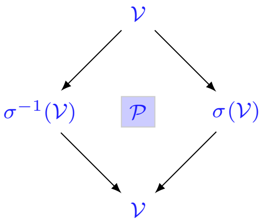

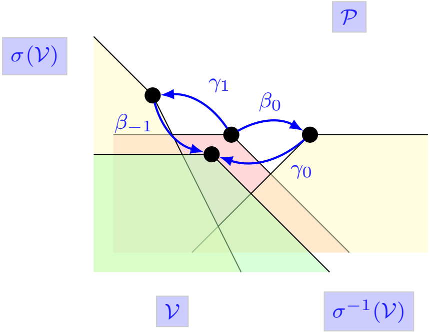

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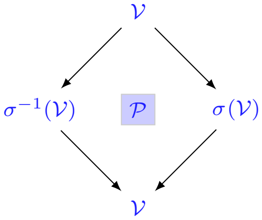

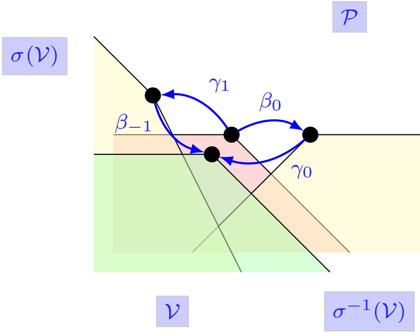

$\mathcal{P}$ is the projective cover of the vacuum module $\mathcal{V}$ [Allen–Wood'20].

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The Ward identity for L_1 is therefore modified and the solutions are

$$\begin{aligned}
\big\langle \psi_1(w_1) \big\rangle &= \delta_{h_1=0}\delta_{j=0}C_1, \\
\big\langle \psi_1(w_1)\psi_2(w_2) \big\rangle &= \delta_{2h_1-j_1=2h_2-j_2}\delta_{j=0}C_{12}w_{12}^{-(2h_1-j_1)}, \\
\big\langle \psi_1(w_1)\psi_2(w_2)\psi_3(w_3) \big\rangle &= \delta_{j=0}C_{123}\prod_{1\leqslant a<b\leqslant 3} w_{ab}^{h-2h_{ab}+j_{ab}}, \\
\big\langle \psi_1(w_1)\psi_2(w_2)\psi_3(w_3)\psi_4(w_4) \big\rangle &= \delta_{j=0}H(\eta)\prod_{1\leqslant a<b\leqslant 4} w_{ab}^{h/3-h_{ab}+j_{ab}/2}
\end{aligned}$$

($w_{ab} = w_a - w_b$, $\eta = \frac{w_{12}w_{34}}{w_{13}w_{24}}$, $j = \sum_i j_i$, $h = \sum_i h_i$, $h_{ab} = h_a + h_b$, $j_{ab} = j_a + j_b$).

Conjugates

The ghost primaries correspond to the $|\phi_i\rangle \in \mathcal{W}_{[i]}$ and their spectral flows

$$|\phi_i^\ell\rangle = \sigma^\ell(|\phi_i\rangle) \in \sigma^\ell(\mathcal{W}_{[i]}) \quad \Rightarrow \quad j_i^\ell = i - \ell, \quad h_i^\ell = i\ell - \frac{1}{2}\ell(\ell+1).$$

Their 2-point functions have the form

$$\left\langle \phi_i^\ell(w_1)\phi_j^m(w_2)\right\rangle = \delta_{i+j=1}^{\ell+m=1}\left[\begin{smallmatrix}\ell & m\\ i & j\end{smallmatrix}\right] w_{12}^{\ell^2-(2\ell-1)i}.$$

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This is consistent with the fusion rules because

$$\sigma^\ell(\mathcal{W}_{[i]}) \times \sigma^{1-\ell}(\mathcal{W}_{[-i]}) \cong \mathcal{P}.$$

(It is the log-partner of the vacuum that has the non-zero 1-point function — that of the vacuum vanishes.)

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Unfortunately, non-primary fields are not always descendants of primaries. *eg.*, $\gamma_n |\phi_i^\ell\rangle$ is not primary nor a descendant for $\ell > 1$ and $0 < n < \ell$.

Correlators with such non-descendant fields are not obviously expressible in terms of primary correlators. This makes life hard...

KZ equation(s)

Our main aim is to compute some correlators with logarithmic singularities. For this, we restrict to the following case:

$$\left\langle \phi_{j_1}(w_1) \cdots \phi_{j_{N-1}}(w_{N-1}) \phi^{\ell}_{j_N}(w_N) \right\rangle, \quad \ell \in \mathbb{Z}_{\geqslant 1}.$$

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Ghost correlators satisfy KZ equations because $T(z)$ is composite.

- Inserting L_{-1} at the i-th coordinate ($i \neq N$) gives

$$\partial_i \langle \phi_{j_1} \cdots \phi_{j_{N-1}} \phi^{\ell}_{j_N} \rangle = j_i \sum_{k=1}^{\ell} k w_{iN}^{-k-1} \langle \phi_{j_1} \cdots \phi_{j_i+1} \cdots \phi_{j_{N-1}} (\gamma_k \phi^{\ell}_{j_N}) \rangle.$$

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  This involves non-descendants if $\ell > 1$.
- Inserting L_{-1} at the N-th coordinate similarly fails for $\ell > 1$. Instead, we insert $L_{-1} - (\ell - 1) J_{-1}$ to get

$$\Big(\partial_N + (\ell - 1) \sum_{i \neq N} j_i w_{iN}^{-1} \Big) \langle \phi_{j_1} \cdots \phi_{j_{N-1}} \phi^{\ell}_{j_N} \rangle = - \sum_{i \neq N} j_i w_{iN}^{-\ell-1} \langle \phi_{j_1} \cdots \phi_{j_i+1} \cdots \phi_{j_{N-1}} \phi^{\ell}_{j_N-1} \rangle.$$

Primary 2-point functions

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Substituting into (either) $\ell = 1$ KZ equation gives a recursion relation for the 2-point constants:

$$\left[\begin{smallmatrix} 0 & 1 \\ j_1 & j_2 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 1 \\ j_1-1 & j_2+1 \end{smallmatrix} \right], \quad j_1 + j_2 = 1.$$

Since there is no upper or lower bound on the charges, we have no boundary conditions.

Primary 2-point functions

The 2-point functions have the form

$$\langle \phi_{j_1} \phi^{\ell}_{j_2} \rangle = \delta^{\ell=1}_{j_1+j_2=1} \begin{bmatrix} 0 & 1 \\ j_1 & j_2 \end{bmatrix} w_{12}^{j_1}.$$

Substituting into (either) $\ell = 1$ KZ equation gives a recursion relation for the 2-point constants:

$$\begin{bmatrix} 0 & 1 \\ j_1 & j_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ j_1-1 & j_2+1 \end{bmatrix}, \quad j_1 + j_2 = 1.$$

Since there is no upper or lower bound on the charges, we have no boundary conditions.

But, this recursion is consistent with the expectation that we may normalise conjugate fields so that

$$\begin{bmatrix} 0 & 1 \\ j & 1-j \end{bmatrix} = 1.$$

Primary 3-point functions

The 3-point functions have the form

$$\left\langle \phi_{j_1} \phi_{j_2} \phi^{\ell}_{j_3} \right\rangle = \delta_{j_1+j_2+j_3=\ell} \left[\begin{smallmatrix} 0 & 0 & \ell \\ j_1 & j_2 & j_3 \end{smallmatrix} \right] \cdot w_{12}^{(j_3-\ell/2)(\ell-1)} w_{13}^{-j_2-(j_3-(\ell+1)/2)\ell} w_{23}^{-j_1-(j_3-(\ell+1)/2)\ell}.$$

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Now, the second KZ equation gives a polynomial identity in w_{13} and w_{23} relating 3-point constants. Assuming $j_1 + j_2 + j_3 = \ell$, it gives:

$\ell = 1$: $$\left[\begin{smallmatrix} 0 & 0 & 1 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 0 & 1 \\ j_1+1 & j_2 & j_3-1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 & 0 & 1 \\ j_1 & j_2+1 & j_3-1 \end{smallmatrix} \right].$$

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$\ell = 2$:

$$j_1 \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1+1 & j_2 & j_3-1 \end{smallmatrix} \right] = (j_3-1) \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right],$$

$$-j_2 \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1 & j_2+1 & j_3-1 \end{smallmatrix} \right] = (j_3-1) \left[\begin{smallmatrix} 0 & 0 & 2 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right].$$

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$\ell = 2$: $$j_1 \begin{bmatrix} 0 & 0 & 2 \\ j_1+1 & j_2 & j_3-1 \end{bmatrix} = (j_3 - 1) \begin{bmatrix} 0 & 0 & 2 \\ j_1 & j_2 & j_3 \end{bmatrix},$$ $$-j_2 \begin{bmatrix} 0 & 0 & 2 \\ j_1 & j_2+1 & j_3-1 \end{bmatrix} = (j_3 - 1) \begin{bmatrix} 0 & 0 & 2 \\ j_1 & j_2 & j_3 \end{bmatrix}.$$

$\ell \geqslant 3$: $$\begin{bmatrix} 0 & 0 & \ell \\ j_1 & j_2 & j_3 \end{bmatrix} = 0.$$

This last result is consistent with the (generic) fusion rules. The $\ell = 2$ result suggests that something happens when a $j_i \in \mathbb{Z}$.

Primary 4-point functions

The 4-point functions have the specialised form

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1) \phi_{j_3}(\eta) \middle| \phi^{\ell}_{j_4} \right\rangle = \delta_{j=\ell} \eta^{-2h/3-(j_1+j_2)/2} (1-\eta)^{h/3+(j_2+j_3)/2} H(\eta),$$

where $j = j_1 + j_2 + j_3 + j_4$ and $h = j_4 \ell - \frac{1}{2} \ell (\ell + 1)$.

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For $\ell = 1$, the KZ equation with $i = 3$ gives an infinite family of order-1 recursion relations:

$$\partial_\eta \langle \phi_{j_1} \big| \phi_{j_2}(1) \phi_{j_3}(\eta) \big| \phi^{1}_{j_4} \rangle = j_3 \eta^{-2} \langle \phi_{j_1} \big| \phi_{j_2}(1) \phi_{j_3+1}(\eta) \big| \phi^{1}_{j_4-1} \rangle.$$

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To solve this family of coupled ODEs, we derive another such family by inserting J_0 instead of L_{-1}:

$$\left\langle \phi_{j_1} \cdots \phi_{j_{N-1}} \phi^{\ell}_{j_N} \right\rangle = \sum_{k=1}^{\ell} w_{iN}^{-k} \left\langle \phi_{j_1} \cdots \phi_{j_i+1} \cdots \phi_{j_{N-1}} (\gamma_k \phi^{\ell}_{j_N}) \right\rangle.$$

This is thus an algebraic "KZ-like" equation.

Specialising, the J_0 KZ-like equation becomes

$$\left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3}(\eta) \middle| \phi^1_{j_4} \right\rangle = \eta^{-1} \left\langle \phi_{j_1} \middle| \phi_{j_2}(1)\phi_{j_3+1}(\eta) \middle| \phi^1_{j_4-1} \right\rangle.$$

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Substituting back into the L_{-1} KZ equation then uncouples the ODEs:

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We thus arrive at a very simple solution:

$$\langle \phi_{j_1} | \phi_{j_2}(1) \phi_{j_3}(\eta) | \phi^1_{j_4} \rangle = \delta_{j_1+j_2+j_3+j_4=1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} \eta^{j_3},$$
$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1+1 & j_2 & j_3 & j_4-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2+1 & j_3 & j_4-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ j_1 & j_2 & j_3+1 & j_4-1 \end{bmatrix}.$$

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To see such singularities, we will need to solve the $\ell = 2$ case. This will require more than just KZ (and KZ-like) technology.

Fake BPZ equations

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We can therefore find a modified energy-momentum tensor with this central charge that has regular OPE with $J(w)$:

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The ghost primary $\phi_{1/2}$ has modified conformal weight $-\frac{1}{8}$, so

$$\left(\widetilde{L}_{-1}^2 - \frac{1}{2}\widetilde{L}_{-2}\right)|\phi_{1/2}\rangle = 0,$$

$$\textit{ie.} \quad \left(L_{-1}^2 - \frac{1}{2}L_{-2} + J_{-1}L_{-1}\right)|\phi_{1/2}\rangle = 0.$$

This is a "fake" null vector — the LHS vanishes identically when written in terms of modes of β and γ.

This null vector nevertheless gives rise to fake (but still useful) BPZ equations. *eg.*, if $i \neq N$, we have

$$\left[\partial_i^2 + \sum_{k \neq i} \frac{\partial_k - 2 j_k \partial_i}{2 w_{ki}} - \frac{j_N \ell - \frac{1}{2}\ell(\ell+1)}{2 w_{Ni}^2}\right] \cdot \left\langle \phi_{j_1}(w_1) \cdots \phi_{1/2}(w_i) \cdots \phi_{j_{N-1}}(w_{N-1}) \phi_{j_N}^{\ell}(w_N) \right\rangle = 0.$$

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- For $N = 3$, the solutions of the Ward identities are solutions of fake BPZ if and only if $\ell = 1, 2$.
- For $N = 4$ and $\ell = 1$, fake BPZ becomes a second-order ODE with conformal blocks

$$\eta^{1/2} \quad \text{and} \quad \eta^{1/2} B\left(\eta; \frac{1}{2} - j_4, \frac{1}{2} - j_2\right).$$

Single-valuedness of the bulk correlator rules out the second block.

- For $N = 4$ and $\ell = 3$, the conformal blocks are

$$\eta^{2-j_4}(1-\eta)^{1/2-j_2} \quad \text{and} \quad \eta^{2-j_4}(1-\eta)^{1/2-j_2} B\left(\eta; j_4 - \frac{1}{2}, j_2 - \frac{1}{2}\right)$$

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$$\eta\,{}_2F_1\left(\frac{1}{2}, j_2 + j_4 - \frac{1}{2}; j_4 + \frac{1}{2}; \eta\right) \quad \text{and} \quad \eta^{3/2-j_4}\,{}_2F_1\left(j_2, 1-j_4; \frac{3}{2} - j_4; \eta\right).$$

  This time, single-valuedness does not rule out either block.

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Similarly, expanding in $1 - \eta$ shows that there is a logarithmic singularity at $\eta = 1$ when $j_2 \in \mathbb{Z} + \frac{1}{2}$.

This is precisely the behaviour expected given the fusion rules.

Conclusions

The representation theory of bosonic ghosts is considered to be well understood: we know the irreducibles, projectives and their fusion rules.

However, the primary correlators are perhaps not so easy to compute with standard methods.

KZ allows us to reproduce the generic fusion rules, but only gives recursion relations in general (because the irreducibles have infinitely many independent primaries).

One case ($\ell = 1$) could nevertheless be solved using a second KZ equation. Another case ($j_3 = \frac{1}{2}$) was solved using a coset BPZ equation.

The latter manifested the logarithmic nature of the theory, confirming the non-generic fusion rules.

Alas, neither method appears to be available in more general theories.

Outlook

There are still many avenues to explore:

- Do the $j_3 = \frac{1}{2}$ 4-point functions constrain the 3-point constants?
- Is it possible to interpolate the solutions for N-point constants using, *eg*., twisted localisation [Mathieu'00]?
- Are the remaining primary 4-point functions computable?
  - Try a conformal bootstrap approach [*cf*., Jesper's talk].
  - Try a Dotsenko–Fateev approach using FMS bosonisation.

The overarching aim is to determine if, for log CFTs like bosonic ghosts, the correlation functions are entirely fixed by mathematical consistency.

If not, what physical input is required? Either way, there is some surely beautiful new mathematical physics to explore!

"*Only those who attempt the absurd will achieve the impossible.*"

— Miguel de Unamuno