A new (?) class of irreducible weight $\mathfrak{sl}_3$-modules

David Ridout

[Joint with Chris Raymond (Hamburg)]

University of Melbourne

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Outline

1. Irreducible weight modules for $\mathfrak{sl}_3$
2. Vertex-algebraic motivations
3. Dense $\mathfrak{sl}_3$-modules with infinite multiplicities
4. Degenerations
5. Outlook

Weight modules for $\mathfrak{sl}_3$

Recall the following basis of $\mathfrak{sl}_3 = \mathfrak{sl}_3(\mathbb{C})$:

$$f^3 = E_{31}, \qquad \begin{matrix} f^1 = E_{21}, & h^1 = E_{11} - E_{22}, & e^1 = E_{12}, \\ f^2 = E_{32}, & h^2 = E_{22} - E_{33}, & e^2 = E_{23}, \end{matrix} \qquad e^3 = E_{13}.$$

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- a weight module: a module that decomposes as the direct sum of its weight spaces;
- the weight support: the set of weights of a module.

["Weight" also makes sense more generally as long as one has something like a Cartan subalgebra.]

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These lift to automorphisms of $\mathfrak{sl}_3$, also forming a copy of D_6:

x	h^1	h^2	e^1	e^2	e^3	f^1	f^2	f^3
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$\mathsf{w}_1(x)$	$-h^1$	h^3	f^1	e^3	e^2	e^1	f^3	f^2
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$\mathsf{Z}(\mathfrak{sl}_3)$ is generated by

$$C_2 = \frac{1}{3}(h^1h^1 + h^2h^2 + h^3h^3) + h^1 + h^2 + h^3 + 2(f^1e^1 + f^2e^2 + f^3e^3)$$

and

$$C_3 = (g^1+1)(g^2+1)g^3 + f^1e^1(g^2+1) - f^2e^2(g^1+1) + f^3e^3g^3 + f^1f^2e^3 + f^3e^1e^2$$

$$(h^3 = h^1 + h^2, \quad g^1 = \tfrac{2}{3}h^1 + \tfrac{1}{3}h^2, \quad g^2 = \tfrac{1}{3}h^1 + \tfrac{2}{3}h^2, \quad g^3 = g^1 - g^2).$$

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These Casimirs are almost D_6-invariant:

$$\mathsf{w}_i(C_2) = C_2, \quad \mathsf{d}(C_2) = C_2, \qquad \mathsf{w}_i(C_3) = C_2, \quad \mathsf{d}(C_3) = -C_3.$$

Weight modules for $\mathfrak{sl}_3$

Irreducible weight $\mathfrak{sl}_3$-modules come in three types, namely [Fernando'90, Britten–Lemire–Futorny'95, Mathieu'00]

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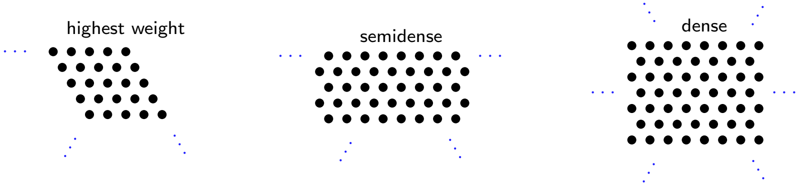

The multiplicities of a highest-weight or semidense module are all finite, but those of a dense module are either all finite or all infinite.

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- In the semidense case, we need to also include the eigenvalue of C_2 to determine the module up to $\cong$.

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- If the multiplicities are infinite, then nobody knows what to do.

Guess which case we're going to be looking at...

From vertex algebras to $\mathfrak{sl}_3$-modules

We are motivated to study weight modules because they are relevant to the representation theory of affine vertex-operator algebras.

When the level is integral, *ie.* $\mathsf{k} \in \mathbb{N}$, then the module category of the simple affine VOA $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ is finite, semisimple and modular [Gepner–Witten'86, Verlinde'88, Moore-Seiberg'89, Huang'05].

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- The weight-module category of the simple affine VOA for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ is modular [Creutzig–DR'13].
- Also true for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$ if k has denominator $\mathsf{v} = 2$ [Kawasetsu–DR–Wood'21, Fasquel–Raymond–DR'24].

Here, "weight" allows finite-rank Jordan blocks in the action of L_0 and "modular" means in the sense of [Creutzig–DR'13, DR–Wood'14].

The conjecture is that for $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$, there is always a weight-module category that is modular.

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The issue boils down to a question of practical classification theorems:

- Irreducible weight $\mathsf{L}_\mathsf{k}(\mathfrak{g})$-modules are "spectral flows" of relaxed highest-weight modules [Futorny–Tsylke'01].
- The latter are recognised by their "Zhu images", which are irreducible weight $\mathfrak{g}$-modules [Zhu'96, Frenkel–Zhu'92].

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- When $\mathsf{v} > 2$, there are irreducible weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$-modules whose Zhu images have infinite multiplicities [Arakawa–Futorny–Ramirez'16].

No known classification, so the question becomes one of identifying a weight-module categories that is modular.

Inverse quantum hamiltonian reduction

A very promising path to affine VOA representation theory is the paradigm of inverse quantum hamiltonian reduction.

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Moreover, IQHR knows about denominators.

$$\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)\text{-wtmod} \longleftarrow \mathsf{Vir}_\mathsf{k}\text{-wtmod} \quad (\mathsf{v} > 1)$$

$$\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)\text{-wtmod} \longleftarrow \mathsf{BP}_\mathsf{k}\text{-wtmod} \quad (\mathsf{v} = 2)$$

$$\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)\text{-wtmod} \longleftarrow \mathsf{BP}_\mathsf{k}\text{-wtmod} \longleftarrow \mathsf{W}_{3,\mathsf{k}}\text{-wtmod} \quad (\mathsf{v} > 2)$$

[Adamović'17, Adamović–Kawasetsu–DR'20]
[Adamović–Creutzig–Genra'21, Fasquel–Raymond–DR'24]

Ingredients

Semikhatov embeddings involve VOAs like

- The bosonic ghost system G.
  The irreducible relaxed highest-weight modules have Zhu images that are irreducible weight modules for the Weyl algebra:

$$\mathbb{C}[z^{-1}], \qquad z\mathbb{C}[z], \qquad z^{\lambda}\mathbb{C}[z, z^{-1}] \quad (\lambda \in (\mathbb{C} \setminus \mathbb{Z})/\mathbb{Z}).$$

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- The Zamolodchikov algebra $\mathsf{W}_3^{\mathsf{k}}$.
  The Zhu images of the irreducible relaxed highest-weight modules are

$$\mathbb{C}u^{\Delta,w} \quad (\Delta, w \in \mathbb{C}).$$

Our main example is a "composite" Semikhatov embedding:

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \hookrightarrow \mathsf{W}_3^{\mathsf{k}} \otimes \mathsf{G} \otimes \Pi \otimes \Pi$$

[Adamović–Kawasetsu–DR'20, Adamović–Creutzig–Genra'21, Fasquel–Raymond–DR'24].
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The corresponding Adamović restriction functors construct dense $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-modules whose Zhu images have infinite multiplicities:

$$\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]} = \mathbb{C}u^{\Delta,w} \otimes x^r\mathbb{C}[x,x^{-1}] \otimes y^s\mathbb{C}[y,y^{-1}] \otimes z^t\mathbb{C}[z,z^{-1}].$$

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These images are weight $\mathfrak{sl}_3$-modules with basis

$$|r,s,t\rangle = u^{\Delta,w} \otimes x^r \otimes y^s \otimes z^t, \quad r \in [\rho],\ s \in [\sigma],\ t \in [\tau].$$

They were first constructed in [Adamović–Creutzig–Genra'21].

A new (?) family of $\mathfrak{sl}_3$-modules

The action on the $\mathfrak{sl}_3$-module $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is given by [Fasquel–Raymond–DR'24]

$$\begin{aligned}
&e^1|r,s,t\rangle = |r-1,s,t+1\rangle, \quad e^2|r,s,t\rangle = r|r+1,s,t\rangle, \quad e^3|r,s,t\rangle = |r,s,t+1\rangle,\\
&h^1|r,s,t\rangle = -(r+s-t-1)|r,s,t\rangle, \quad h^2|r,s,t\rangle = (2r+2s+t-2)|r,s,t\rangle,\\
&f^1|r,s,t\rangle = P(s)|r,s+1,t-1\rangle - r(s-t)|r+1,s,t-1\rangle,\\
&f^2|r,s,t\rangle = |r,s-1,t\rangle - (r+2s+t-2)|r-1,s,t\rangle,\\
&f^3|r,s,t\rangle = -r|r+1,s-1,t-1\rangle - P(s)|r-1,s+1,t-1\rangle\\
&\qquad\qquad + \big(P(s) - P(s-1) + (r+2s+t-2)(s-t)\big)|r,s,t-1\rangle,
\end{aligned}$$

where $P(s) = w + \big((\mathsf{k}+3)\Delta + (\mathsf{k}+2)^2\big)s - s^3$.

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&e^1|r,s,t\rangle = |r-1,s,t+1\rangle, \quad e^2|r,s,t\rangle = r|r+1,s,t\rangle, \quad e^3|r,s,t\rangle = |r,s,t+1\rangle,\\
&h^1|r,s,t\rangle = -(r+s-t-1)|r,s,t\rangle, \quad h^2|r,s,t\rangle = (2r+2s+t-2)|r,s,t\rangle,\\
&f^1|r,s,t\rangle = P(s)|r,s+1,t-1\rangle - r(s-t)|r+1,s,t-1\rangle,\\
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&\qquad\qquad + \big(P(s)-P(s-1)+(r+2s+t-2)(s-t)\big)|r,s,t-1\rangle,
\end{aligned}$$

where $P(s) = w + \big((\mathsf{k}+3)\Delta + (\mathsf{k}+2)^2\big)s - s^3$.

- The weight of $|r,s,t\rangle$ is $(r+s-1)\alpha_2 + t\alpha_3$.
- The C_2-eigenvalue is $2(\mathsf{k}+3)(\Delta+\mathsf{k}+1)$.
- The C_3-eigenvalue is w.
- The form of P is fixed by the Serre relations.

Generic irreducibility

[Adamović–Creutzig–Genra'21] conjectured that these $\mathfrak{sl}_3$-modules are generically irreducible. Settling this is our first result.

Theorem

$\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is reducible if and only if

$$[\rho]=[0], \quad P(s)=0, \quad P(t)=0 \quad \textit{or} \quad P(1-r-s-t)=0,$$

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for some $r \in [\rho]$, $s \in [\sigma]$ *and* $t \in [\tau]$.

- Irreducibility follows by showing that every nonzero element generates.
- For $[\rho] = [0]$, ghost modules give an easy proper submodule.
- For $P(s) = 0$, Bershadsky–Polyakov modules give an easy proper submodule.
- In the other two cases, existence of a proper submodule is subtle.

Quick aside: Gelfand–Tsetlin modules

There is a well studied class of generically irreducible dense $\mathfrak{sl}_3$-modules with infinite multiplicities: the Gelfand–Tsetlin modules of [Drozd–Futorny–Ovsienko'94, Futorny–Grantcharov–Ramirez'14].

$$\mathfrak{gl}_1^{(1)} \subset \mathfrak{gl}_2^{(1)} \subset \mathfrak{sl}_3 \qquad \Rightarrow \qquad \begin{array}{ccc} \mathsf{Z}\big(\mathfrak{gl}_1^{(1)}\big) & \mathsf{Z}\big(\mathfrak{gl}_2^{(1)}\big) & \mathsf{Z}\big(\mathfrak{sl}_3\big) \\ g^1 & g^2, C_2^{(1)} & C_2, C_3 \end{array}$$

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These modules have several nice properties:

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Twisting by D_6 gives three distinct categories $\mathsf{GT}^{(i)}$, $i = 1, 2, 3$, parametrised by the positive root α_i defining the $\mathfrak{sl}_2$-Casimir $C_2^{(i)}$.

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- Otherwise, $\mathcal{R}^{\Delta,w}_{[\rho,\sigma,\tau]}$ is reducible and we have to work harder.

Of course, the reducible cases are where things get really interesting!

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$[\rho] = [0]$:

- Generically, $\mathcal{R}^{\Delta,w}_{[0,\sigma,\tau]}$ has length 2.
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$P(s) = 0$ for some $s \in [\sigma]$:

- Again, generic length is 2 and the submodule and quotient are infinite-multiplicity.
- This time, the submodule is in $\mathsf{GT}^{(1)}$ while the quotient is in $\mathsf{GT}^{(2)}$.

$$[\rho] = [0]: \quad G2^{(1)} \longrightarrow G2^{(2)} \qquad\qquad G3^{(2)} \longrightarrow G3^{(1)} \quad : P(s) = 0$$

[G-labels follow classification in [Futorny–Grantcharov–Ramirez'18].]

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We can identify the semidense quotients explicitly, but not the submodule.

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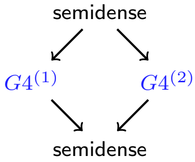

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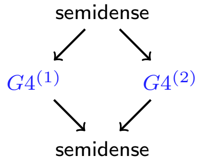

The other five double degenerations are still being worked out... the $P(t) = P(1 - r - s - t) = 0$ case is especially interesting (*ie.* hard).

Degenerations 3

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These can further degenerate, *eg.* if we specialise the values of Δ and w.

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$$\Delta = \frac{\xi_1^2 + \xi_2^2 + \xi_3^2 - 2(\mathsf{k}+2)^2}{2(\mathsf{k}+3)} \quad \text{and} \quad w = \xi_1\xi_2\xi_3,$$

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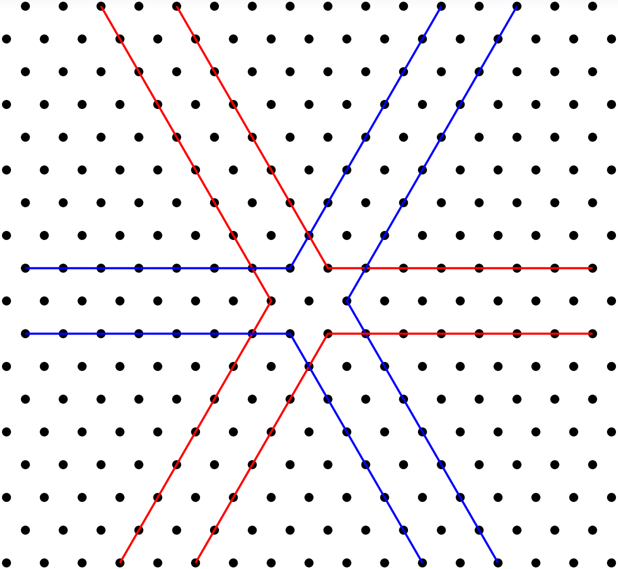
- The 6 highest-weight composition factors turn out to be (D_6-twists of) modules whose highest weights are

$$\mathsf{w} \cdot \big(\lambda - (\mathsf{k}+3)\alpha_3\big), \quad \mathsf{w} \in \mathsf{S}_3.$$



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Our results demonstrate that the modular weight category needed for CFT applications is not a Gelfand–Tsetlin category. What is it precisely?

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Either way, there is some surely beautiful new mathematics to explore!

"*Only those who attempt the absurd will achieve the impossible.*"

— Miguel de Unamuno