Modular tensor categories in an irrational world

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Outline

Rationality

Conformal field theory (CFT) is quantum field theory with invariance under conformal (angle-preserving) transformations.

In two dimensions, local conformal transformations are (anti)analytic. They give rise to two commuting copies of the Virasoro algebra.

The space of states H of the CFT is thus a $Vir \oplus Vir$-module.

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The space of states H of the CFT is thus a $Vir \oplus Vir$-module.

More generally, H is a module of two commuting copies of the symmetry algebra of the CFT, a vertex operator algebra (VOA) V.

Definition

A CFT is rational if H is

- *semisimple* as a $V \otimes V$-module; and
- decomposes into a *finite* number of simple $V \otimes V$-modules.

Examples and non-examples

1. The Ising model is rational with V being the simple Virasoro VOA of central charge $\frac{1}{2}$ and

$$\mathsf{H} = (\mathsf{L}_0 \otimes \mathsf{L}_0) \oplus (\mathsf{L}_{1/16} \otimes \mathsf{L}_{1/16}) \oplus (\mathsf{L}_{1/2} \otimes \mathsf{L}_{1/2}),$$

where L_h is the simple highest-weight Virasoro module of conformal weight h.

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where L_h is the simple highest-weight Virasoro module of conformal weight h.

2. The free boson is not rational with V being the Heisenberg VOA of central charge 1 and

$$\mathsf{H} = \int_{\mathbb{R}}^{\oplus} (\mathsf{F}_p \otimes \mathsf{F}_p) \, \mathrm{d}p,$$

where F_p is the Fock space of charge p. Whilst H is semisimple, it is composed of an uncountably infinite number of simples.

3. The triplet model is not rational. It has four simple V-modules — W_0, W_1, $\mathsf{W}_{-1/8}$ and $\mathsf{W}_{3/8}$ — and the state space decomposes as

$$\mathsf{H} = (\mathsf{W}_{-1/8} \otimes \mathsf{W}_{-1/8}) \oplus (\mathsf{W}_{3/8} \otimes \mathsf{W}_{3/8}) \oplus \begin{bmatrix} \text{an indecomposable} \\ \text{with } 8 \text{ composition} \\ \text{factors built from} \\ \mathsf{W}_0 \text{ and } \mathsf{W}_1 \end{bmatrix}.$$

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4. The singlet model is not rational either. It has an uncountable infinity of simple V-modules E_λ, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, and M_r, $r \in \mathbb{Z}$, with

$$\mathsf{H} = \int_{\mathbb{C}\setminus\mathbb{Z}}^{\oplus} (\mathsf{E}_\lambda \otimes \mathsf{E}_\lambda)\, \mathrm{d}\lambda \oplus \begin{bmatrix} \text{an indecomposable with} \\ \text{a countable infinity of} \\ \text{composition factors built} \\ \text{from the } \mathsf{M}_r \end{bmatrix}.$$

It is neither finite nor semisimple [Wang'97].

Message of the day

Highly opinionated message.

From the modular perspective, the difficulty of these four cases is

$$1. < 2. < 4. < 3.$$

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Starting with 3. is silly...

Modularity

Recall that the (unqualified) modular group is

$$\begin{aligned}\mathsf{SL}_2(\mathbb{Z}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\} \\ &= \langle S, T : S^2 = (ST)^3 = S^{-2} \rangle.\end{aligned}$$

It is common to identify S and T with the matrices

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A consistency condition of CFT is that the partition function (character of H) is invariant under the natural action of the modular group.

Verlinde noticed/conjectured that the fusion coefficients ${\mathcal{N}_{ij}}^{k}$, given by

$$\mathsf{M}_i \times \mathsf{M}_j \simeq \bigoplus_k {\mathcal{N}_{ij}}^{k} \, \mathsf{M}_k,$$

are determined by the modular S-transforms of the characters,

$$\mathrm{ch}\big[\mathsf{M}_i\big] \xrightarrow{\ \mathcal{S}\ } \sum_j S_{ij} \, \mathrm{ch}\big[\mathsf{M}_j\big],$$

via the celebrated (and highly non-trivial) Verlinde formula:

$${\mathcal{N}_{ij}}^{k} = \sum_\ell \frac{S_{i\ell} S_{j\ell} S^*_{k\ell}}{S_{1\ell}} \qquad (\mathsf{M}_1 \text{ is the VOA}).$$

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[Moore–Seiberg'89] explained that this (essentially) follows from other consistency conditions of CFTs.

This led [Turaev'94] to define the notion of a modular fusion category (MFC), which means (among other things) that:

- the module category is finite + semisimple (+ . . .);
- the module category admits a tensor product (fusion);
- there is a "nice" action of the modular group $\mathsf{SL}_2(\mathbb{Z})$; and
- the fusion product and the modular action are related by Verlinde.

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A CFT is modular if its characters span a "nice" representation of $\mathsf{SL}_2(\mathbb{Z})$ and S reproduces the fusion coefficients via a "Verlinde-like formula".

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Huang proved that rational CFTs are modular. But, which generalisation of an MFC captures the modularity of non-rational CFTs?

Again with the examples...

1. The Ising model's three simples have characters given by

$$\mathrm{ch}\big[\mathsf{L}_0\big] = \frac{1}{2}\left(\sqrt{\frac{\vartheta_3(0,\tau)}{\eta(\tau)}} + \sqrt{\frac{\vartheta_4(0,\tau)}{\eta(\tau)}}\right), \qquad \mathrm{ch}\big[\mathsf{L}_{1/16}\big] = \sqrt{\frac{\vartheta_2(0,\tau)}{2\eta(\tau)}}.$$

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This CFT is modular with S- and T-matrices given by

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad \mathcal{T} = \mathsf{e}^{-\mathrm{i}\pi/24}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathsf{e}^{\mathrm{i}\pi/8} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

They generate a representation of $\mathsf{PSL}_2(\mathbb{Z})$: $\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = id$.

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The Verlinde formula gives non-negative integers that agree with the fusion coefficients, as required.

2. The free boson's simple characters are

$$\mathrm{ch}\big[\mathsf{F}_p\big] = \frac{\mathsf{e}^{2\pi \mathfrak{i} p \zeta} \mathsf{e}^{\pi \mathfrak{i} p^2 \tau}}{\eta(\tau)}, \quad p \in \mathbb{R}.$$

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This CFT is also modular with S and T being represented by integral operators with kernels

$$\mathcal{S}(p,p') = \mathsf{e}^{-2\pi \mathrm{i} pp'}, \quad \mathcal{T}(p,p') = \mathsf{e}^{-\pi \mathrm{i}/12} \mathsf{e}^{\pi \mathrm{i} p^2} \delta(p-p').$$

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Even though the free boson is not rational, the Verlinde-like formula (with $\sum \longrightarrow \int$) still gives the fusion coefficients.

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$$\mathrm{ch}\big[\mathsf{W}_0\big] = \tfrac{1}{2}\left(\frac{\vartheta_2(0,\tau)}{2\eta(\tau)} + \eta(\tau)^2\right), \quad \mathrm{ch}\big[\mathsf{W}_{-1/8}\big] = \frac{\vartheta_3(0,\tau)+\vartheta_4(0,\tau)}{2\eta(\tau)},$$

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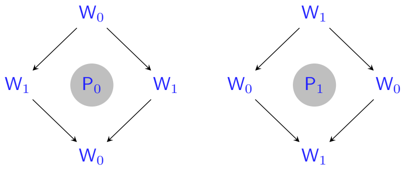
W_0 and W_1 harbour the non-semisimplicity of the CFT. In particular, their projective covers are reducible but indecomposable.



We do have an action of $\mathsf{SL}_2(\mathbb{Z})$ on the projective characters, but they are not linearly independent: $\mathrm{ch}\big[\mathsf{P}_0\big] = \mathrm{ch}\big[\mathsf{P}_1\big]$.

This gives a modular-invariant partition function:

$$\tfrac{1}{2}\Big|\mathrm{ch}\big[\mathsf{P}\big]\Big|^2 + \Big|\mathrm{ch}\big[\mathsf{W}_{-1/8}\big]\Big|^2 + \Big|\mathrm{ch}\big[\mathsf{W}_{3/8}\big]\Big|^2.$$

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However, the S-matrix is no good for Verlinde interpretations:

$$\mathcal{S} = \begin{pmatrix} 0 & 1 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

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On the other hand, the projectives carry a non-semisimple action of L_0, so one can introduce a pseudotrace to see the non-trivial Jordan blocks.

Here, this augments the characters by the pseudocharacter $-\mathfrak{i}\tau\,\eta(\tau)^2$. Together, they span a 5-dimensional $\mathsf{SL}_2(\mathbb{Z})$-module (but it is still no good for Verlinde games).

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   Moreover, the characters of the atypical modules are modular "almost everywhere" if we allow certain infinite-linear combinations:

$$\operatorname{ch}\left[\mathsf{M}_r\right] = \sum_{j\geqslant 0} (-1)^j \operatorname{ch}\left[\mathsf{E}_{r+j+1}\right]$$

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Most importantly (and surprisingly), the Verlinde formula works!

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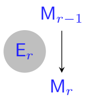
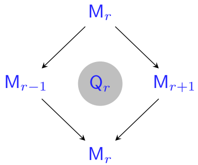
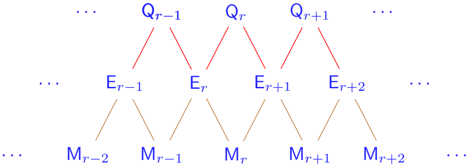
The standard modules occupy a central place in the representation theory.







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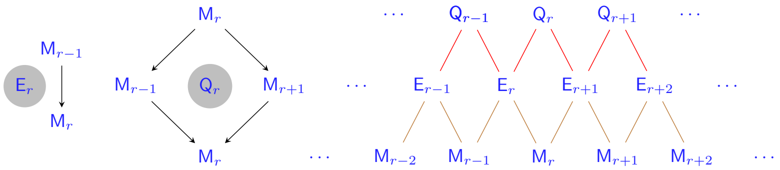

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Moreover, these modules exhibit BGG duality:

$$[Q_r : E_s] = [E_s : M_r].$$

In fancy terms, the singlet category is a BGG highest-weight category.

Why should we care?

These examples make manifest the claim that finite + non-semisimple is harder than non-finite + non-semisimple. We thus expect good modularity when the CFT is "semisimple almost everywhere".

Another controversial opinion.

Imposing finiteness on modularity is as silly as studying theta functions without Poisson resummation and Fourier analysis.

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This observation is the basis for the standard module formalism of [Creutzig–DR'13, DR–Wood'14].

- The trivial cases are the semisimple ones (1. & 2.): simple = standard = projective.
- In the non-finite + non-semisimple case (4.), all examples studied show a BGG/highest-weight structure like the singlet.
- It does not apply (directly) to the finite + non-semisimple case (3.), because the non-semisimplicity occupies a set of positive measure.

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Unfortunately, we're rubbish at constructing examples of type 3.

A better quest is thus to try to categorically characterise the standard modularity of non-finite + non-semisimple examples (*cf.* MFCs).

These are modular BGG categories.[1]

[1] [Please come up with a better name (and tell me)...]

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Part of this requires developing tools to compute correlation functions for these examples (and so extract categorical data).

One can then use this to inform the modular behaviour of finite + non-semisimple examples like the triplet.

We also need to construct more examples of this type, so as to determine which properties are natural/general.

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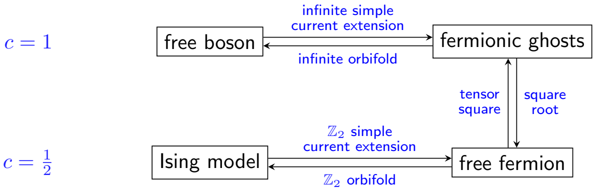

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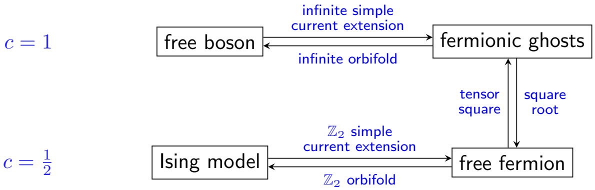

Many rational CFTs (*eg.* some W-algebras) are only known because they have been constructed from non-rational CFTs.

Fixing the triplet

There have been several articles proposing Verlinde-like formulae for the triplet model, *eg.* [Fuchs-Hwang-Semikhatov-Tipunin, Gaberdiel-Runkel, Gainutdinov-Runkel, Creutzig-Gannon].

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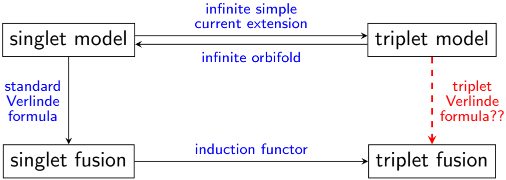
However, the triplet is related to the singlet by an infinite-order simple-current extension [Creutzig–DR'13]. The easiest approach to analysing triplet modularity exploits this fact [Melville–DR'15].



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This is delicate (but possible). The result is a modified Verlinde formula that takes into account the partition of simples into:

atypical: $\{\mathsf{W}_0, \mathsf{W}_1\}$ typical: $\{\mathsf{W}_{-1/8}, \mathsf{W}_{3/8}\}$.

The triplet Verlinde formula is:

$$\mathcal{N}_{ij}{}^{k} = \sum_{\ell \in \text{typ.}} \frac{S_{i\ell} S_{j\ell} S^{-1}_{\ell k}}{S_{1\ell}} + \delta_{ijk} \sum_{\ell \in \text{atyp.}} \frac{S_{i\ell} S_{j\ell} S^{-1}_{\ell k}}{S_{1\ell}},$$

where $\delta_{ijk} = 1$ if only $i \in$ atyp., only $j \in$ atyp. or $i, j, k \in$ atyp., and $\delta_{ijk} = 0$ otherwise.

Note that the S-matrix is that of the triplet with the factors of τ!

Conclusions

- Modularity is subtle for non-semisimple CFTs, *eg.* the triplet. We do not have an $\mathsf{SL}_2(\mathbb{Z})$-action on the characters in general and so the Verlinde formula does not seem to make sense.

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Final inflammatory opinion.

We should stop proposing finite generalisations of modular fusion categories and instead study modular BGG categories.