Modularity of weight categories over admissible-level affine VOAs

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Vertex Algebras & Related Topics

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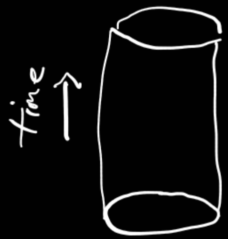
$\mathfrak{sl}_2$:
- 0810.3532 [hep-th]
- 1205.6513 [hep-th] w/ Creutzig
- 1306.4388 [hep-th] w/ Creutzig
- 1803.01989 [math.RT] w/ Kawasetsu
- 1906.02935 [math.RT] w/ Kawasetsu

- 2107.13204 [math.QA] w/ Kawasetsu & Wood

$\mathfrak{sl}_3$:
- 2007.00396 [math.QA] w/ Adamović & Kawasetsu
- 2303.03713 [math.QA] w/ Adamović & Kawasetsu
- 2406.10646 [math.QA] w/ Fasquel & Raymond

CFT describes the dynamics of 1D objects.



Conf. symm.

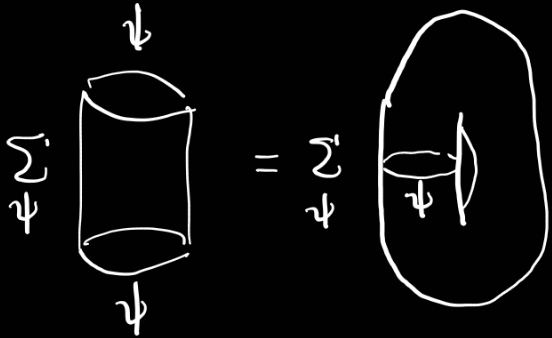
$\rightsquigarrow$ quantum state space $\mathcal{H}$ is a $(\mathrm{Vir} \oplus \mathrm{Vir})$-module

$\rightsquigarrow$ $(V \otimes V)$-module.

(some VOA)

anti holo. holomorphic

Partition function $(= \mathrm{ch}[\mathcal{H}])$ is a trace of the time-evolution operator.



It depends on the evolution

$\Rightarrow$ depends on the complex structure of the torus!

$\therefore$ it is <u>invariant</u> under the action of $SL(2;\mathbb{Z})$.

Conclusion: VOA characters should transform nicely under this $SL(2;\mathbb{Z})$-action.

And in many cases they do...

Thm: [Moore-Seiberg '89, Huang '04] Strongly* rational VOAs are modular:
$\mathbb{N}$-graded + ...

- Irred. chars span an $SL(2;\mathbb{Z})$-mod.
- T is represented by a unitary diagonal matrix.
- S is represented by a unitary symmetric matrix whose square is a permutation (conjugation).
- The S-matrix vacuum entries $S_{0\ell} \neq 0 \ \forall \ell$.
- The Verlinde formula for the fusion coefficients holds:

$$\boxed{N_{ij}{}^{k} = \sum_{\ell} \frac{S_{i\ell} S_{j\ell} S^*_{k\ell}}{S_{0\ell}}}.$$

But how does modularity work in non-rational cases?

Ex: $V = V^k(\mathfrak{gl}_1)$ (Heisenberg) is modular if we restrict to the (physical) category of weight modules with REAL weights & FINITE multiplicities.

The irreps are the Fock spaces $\mathcal{F}_p$, $p \in \mathbb{R}$. (Continuous spectrum!)

$\therefore$ S & T are not represented by matrices but by integral operators:

$$\mathcal{U}\{\mathrm{ch}[\mathcal{F}_p]\} = \int_{\mathbb{R}} \mathcal{U}_{pq}\, \mathrm{ch}[\mathcal{F}_q]\, dq, \qquad \mathcal{U} \in SL(2;\mathbb{Z}).$$

- T diagonal $\Rightarrow$ kernel is Dirac delta: $T_{pq} = e^{\pi i (p^2 - 1/12)} \delta(p-q)$.
- S is Fourier transform! : $S_{pq} = e^{-2\pi i pq}$.
- Verlinde with $\Sigma \to \int$ works! :

$$N_{pq}^{\;r} = \int_{\mathbb{R}} \frac{S_{ps} S_{qs} S_{rs}^*}{S_{0s}}\, ds = \delta(p+q-r) \Rightarrow \mathcal{F}_p \boxtimes \mathcal{F}_q = \int_{\mathbb{R}} N_{pq}^{\;r}\, \mathcal{F}_r\, dr = \mathcal{F}_{p+q}.$$

Today: $V = L_k(\mathfrak{sl}_2)$ for k fractional (= admissible & non-integral):

$$k+2 = \frac{u}{v}, \quad u, v \in \mathbb{Z}_{\geq 2}, \quad (u,v) = 1.$$

History:

'86: Fractional-level models proposed by Kent to extend GKO construction to non-unitary Vir min models.

'88: Kac-Wakimoto found a finite set of level-k hw $\hat{\mathfrak{sl}}_2$-irreps whose characters* spanned an $SL(2,\mathbb{Z})$-mod. (Suggesting rationality?)

'88: Koh-Sorba checked the Verlinde formula — FAIL!

'95: Adamović-Milas & Dong-Li-Mason showed that these irreps are the irred. hw $L_k(\mathfrak{sl}_2)$-mods. But, $\exists$ non-hw $L_k(\mathfrak{sl}_2)$-mods too!

'97: Feigin-Semikhatov-Tipunin noticed Kazama-Suzuki required relaxed hw mods.

'01: Gaberdiel showed that the hw $L_{-4/3}(\mathfrak{sl}_2)$-mods are not closed under fusion

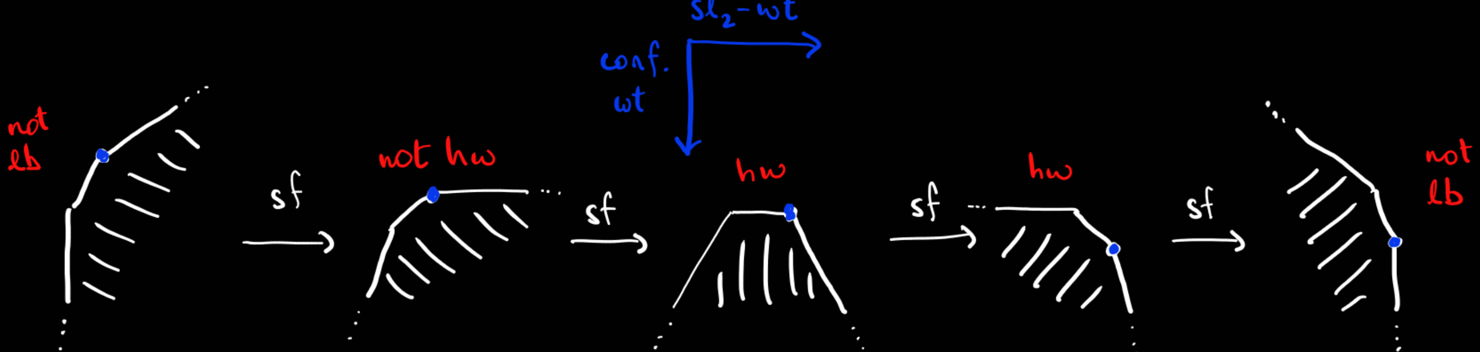
In fact, fusing hw mods results in relaxed hw mods, spectral flows and even logarithmic modules!

Spectral flow = translations in the <u>extended</u> affine Weyl group of $\hat{sl}_2$

$$\hat{W}^{ext} = W \ltimes P^\vee = \mathbb{Z}_2 \ltimes \mathbb{Z}.$$

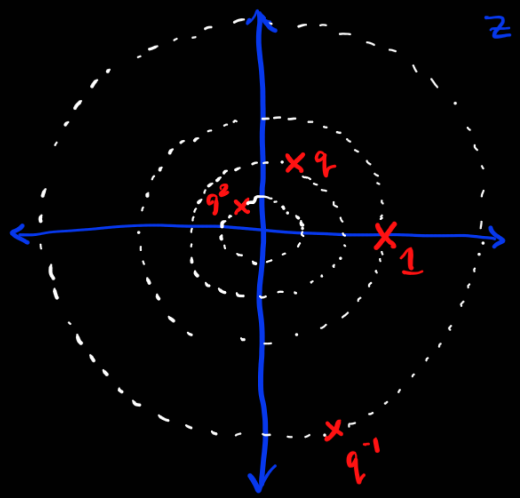
They do not preserve the Virasoro modes.

$\therefore$ twisting by spectral flow need not preserve being lower bounded!



But, twisting preserves being an $L_k(sl_2)$-module ... [Li '97]

- The Kac-Wakimoto character formula for the hw irred. $L_k(sl_2)$-mods is a ratio of theta functions.
- Theta functions have infinitely many zeroes.
- In the Weyl-Kac case ($k \in \mathbb{N}$), these zeroes <u>cancel</u> and the characters are holomorphic.
- <u>Not</u> true for fractional levels: hw chars have <u>poles</u>!
- This ratio of theta functions thus has <u>infinitely</u> many distinct Laurent expansions into characters.
- These <u>different</u> expansions correspond to <u>different</u> spectral flow twists!
- Modules $\to$ Characters is ∞-to-1.



This explains why Verlinde fails: the Kac-Wakimoto characters are <u>not</u> modular.

- S-transform does not respect the convergence regions.
- Ignoring convergence means that spectral flows of hw char are mistaken for $\pm$ hw char. The $\pm$ is analogous to

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x} \overset{!}{=} -\frac{x^{-1}}{1-x^{-1}} = -\sum_{j=-\infty}^{-1} x^j .$$

This means that Verlinde is computing the structure constants of a <u>quotient</u> of the fusion ring (by the spectral flow action).

How to repair this and get the "full" fusion ring???

The answer is suggested by the "identity" $\sum_{j\geq 0} x^j \overset{!}{=} -\sum_{j<0} x^j$.

Ignoring convergence regions thus leads to $\sum_{j\in\mathbb{Z}} x^j = 0$, a silly result.

Instead, we transition from functions to distributions:

$\sum_{j\in\mathbb{Z}} x^j$ is a formal delta function as $P(x)\sum_{j\in\mathbb{Z}} x^j = P(1)\sum_{j\in\mathbb{Z}} x^j$.

Interpretation in $\mathfrak{sl}_2$ rep thy: $\sum_{j<0} x^j$ and $\sum_{j\geq 0} x^j$ are chars of hw and lw Vermas (resp.), while $\sum_{j\in\mathbb{Z}} x^j$ is dense

Interpretation in $\widehat{\mathfrak{sl}}_2$ rep thy:

Now compute the modular S-transforms of the chars of the relaxed $L_k(sl_2)$-mods (Adamović–Milas '95) and their spectral flows.

No convergence issues: relaxed characters are distributions supported at the (S-invariant!) poles of the Kac-Wakimoto character formula:

$$z = q^{\ell} \quad (\ell \in \mathbb{Z}) \quad \Rightarrow \quad e^{2\pi i \zeta} = e^{2\pi i \ell \tau} \quad \Rightarrow \zeta = \ell\tau + m \quad (m \in \mathbb{Z})$$

$$\overset{S}{\Rightarrow} \quad \zeta/\tau = -\ell/\tau + m \quad \Rightarrow \zeta = m\tau - \ell \quad \Rightarrow \quad z = q^{m}.$$

Relaxed $L_k(sl_2)$-modules are parameterised by $\lambda \in \mathbb{R}/2\mathbb{Z}$, $r = 1, \dots, u-1$ and $s = 1, \dots, v-1$ (with $(r,s) \sim (u-r, v-s)$). Including a spectral flow index $\ell \in \mathbb{Z}$ gives modular closure with

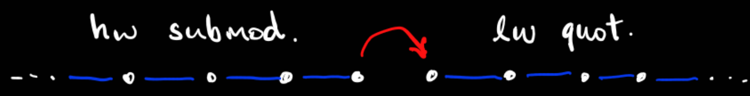
$$S^{\ell,\ell'}_{(\lambda;r,s)(\lambda';r',s')} = \frac{i}{2} \underbrace{e^{-\pi i (k\ell\ell' + \ell\lambda' + \ell'\lambda)}}_{\text{Fourier}} S^{Vir}_{(r,s)(r',s')}.$$

Vir min. model S-matrix!

But what about the hw mods and their spectral flows?

Can't check Verlinde without the vacuum char's S-transform!

Key: Dense modules come in "coherent families" that necessarily contain reducible members:



$\Rightarrow$ same is true for relaxed $\widehat{sl}_2$-mods & spec flows.

$\Rightarrow$ can <u>resolve</u> hw $L_k(sl_2)$-mods in terms of spec flows of relaxed $L_k(sl_2)$-mods:

$$\cdots \longrightarrow R^2 \longrightarrow R^1 \longrightarrow R_0 \longrightarrow H \to 0 .$$

$\Rightarrow$ hw chars and their S-transforms are infinite-linear combinations of relaxed ones!

- This bypasses the convergence issues that plague Kac-Wakimoto.
- Vacuum S-matrix is

$$S^{0,\ell'}_{vac}(\lambda';r',s') = \frac{i}{2}\,\frac{S^{Vir}_{(1,1)(r',s')}}{2\cos(\pi\lambda') + (-1)^{r'}2\cos(\pi k s')}.$$

- This has poles (!) precisely when the relaxed mod. $R_{\lambda';r',s'}$ is reducible. (Indicates non-semisimplicity!)
- Substituting into Verlinde ($\Sigma_\lambda \to \int d\lambda$) gives non-negative fusion coefficients!
- The real weight category of $L_k(sl_2)$ is therefore modular!

Problem solved!

- The appearance of Vir chars & S-matrices is beautifully explained by inverse quantum hamiltonian reduction [Adamović '17]. (cf. residues of hw chars [Mukhi-Panda '90].)
- The standard modules (= spec flows of relaxed mods) are key:
  - Their characters span an $SL(2;\mathbb{Z})$-module.
  - The irreducibles admit resolutions by standards.
  - The projectives & injectives admit filtrations where the quotients are standards or costandards (duals).

$\longrightarrow$ Tilting theory, BGG reciprocity, hw categories.

Higher ranks?? The calculations get hard quickly!

But, there are conceptual obstacles too...

- No obstacle for the real weight category of $L_{-3/2}(\mathfrak{sl}_3)$, though need coherent families [Mathieu '00]. ✓
- For other levels, characters of standards are not linearly indep.
- Inverse QHR suggests lin. indep. generalised chars for standards. They are modular for $k = \frac{1}{2}(-3+2m)$, $m \in \mathbb{N}$. ✓
- For other fractional k, inverse QHR suggests we need some standard modules with <u>infinite</u> multiplicities! ???
- For non-admissible levels, eg. $k = -7/3$, the affine rep thy is even more subtle, which stymies modularity checks... ☹