VOA module categories

Conjecture: Every VOA can be used to build a consistent CFT (the diagonal invariant).

Consistency: Every CFT has a modular-invariant partition function.

$\therefore$ **Conjecture:** Every VOA is modular: it has a "physical" category of modules that is a "modular tensor category".

Theorem: [Moore-Seiberg '89, Huang '04]

The module category of a strongly rational VOA is a modular fusion category.

In particular:

- The characters of the irreducibles form a vector-valued modular function.
- The fusion coefficients and the S-matrix are related by the Verlinde formula

$$N_{ij}^{\;k} = \sum_{l} \frac{S_{il} S_{jl} S^*_{kl}}{S_{0l}},$$

where 0 denotes the VOA itself.

$SC = S^3 = CS$, $(TS)^3 = S^{-1}STSTSTS = S^{-1}CS = C$
$\Rightarrow TC = T(ST)^3 = (TS)^3T = CT$.

Defs:

- Strongly rational = rational + C_2-cofinite + $\mathbb{N}$-graded + self-dual + ...
- Modular group is $SL(2;\mathbb{Z}) = \langle S, T : S^2 = (ST)^3 = C, C^2 = 1 \rangle$.
- A <u>vvmf</u> is a vector $(\chi_i)_{i\in I}$ where the χ_i belong to a unitary $SL(2;\mathbb{Z})$-module in which
    - T is represented by a diagonal operator;
    - C ——— permutation;
    - S ——— symmetric operator.
- These reps are the <u>T-matrix</u>, <u>C-matrix</u> and <u>S-matrix</u>, resp. (Might not be matrices!)
- $\exists$ a tensor product on VOA-mods called the <u>fusion product</u> $\boxtimes$. An explicit decomposition of a fusion product into indecomposables is a <u>fusion rule</u>. The multiplicity of a given indecomposable is its <u>fusion multiplicity</u>:

$$V_i \boxtimes V_j = \bigoplus_k N_{ij}{}^k V_k \left(\equiv \bigoplus_k V_k^{\oplus N_{ij}{}^k} \right).$$

Important note: Because the modularity given above is defined through characters (more generally 1-pt torus amplitudes), it cannot see non-semisimple structures. If the irreducible characters χ_i are linearly independent, then modularity makes contact at best with the Grothendieck fusion ring.

(This assumes that the fusion product descends to a well-defined product on its Grothendieck group.)

Of course, MTCs are about categorifying this notion of modularity!

Pros: Stronger!

Cons: Not so easy to see the physical relevance.
Very hard to do examples...

But, $\exists$ some relatively accessible examples...

- Heisenberg (free boson)
- Lattice VOAs (compactified boson, eg. $L_1(sl_2)$)
- Orbifolds thereof (eg. Ising)

We start with Heisenberg...

Let $\mathfrak{gl}_1 = \text{span}\{a\}$ and $\hat{\mathfrak{gl}}_1 = \mathfrak{gl}_1[[x, x^{-1}]] \oplus \mathbb{C}K$

be its affinisation:

$$0 \to \mathbb{C} \to \hat{\mathfrak{gl}}_1 \longrightarrow \mathfrak{gl}_1[[x,x^{-1}]] \to 0$$

$$\begin{array}{ccc} \mathbb{C} & & \hat{\mathfrak{gl}}_1 \\ \uplus & & \uplus \\ K & & a_m \longmapsto ax^m \qquad (m \in \mathbb{Z}) \end{array}$$

$$[a_m, a_n] = m\,\delta_{m+n=0} K, \quad [a_m, K] = 0.$$

$\hat{\mathfrak{gl}}_1$ has a Δ-decomp., hence Verma modules.

- Let $\mathbb{C}\Omega_p$, $p \in \mathbb{R}$, be a 1D $\mathfrak{gl}_1$-module:

$$a\Omega_p = p\,\Omega_p.$$

- $\mathbb{C}\Omega_p$ becomes a $\hat{\mathfrak{gl}}_1^{\geq}$-module, where

$$\hat{\mathfrak{gl}}_1^{\geq} = \text{span}\{a_n, K : n \geq 0\},$$

by setting $a_0\Omega_p = a\Omega_p$, $K\Omega_p = \underset{\text{level}}{k}\,\Omega_p$ (for some $k \in \mathbb{C}$)

and $a_m\Omega_p = 0 \;\; \forall\, m > 0$.

- The corresponding Verma module (Fock space) is

$$\mathcal{F}_p = \text{Ind}_{\hat{\mathfrak{gl}}_1^{\geq}}^{\hat{\mathfrak{gl}}_1} \mathbb{C}\Omega_p.$$

PBW basis:

$$\{a_{-m_1} \cdots a_{-m_r}\Omega_p : r \in \mathbb{N};\ m_1 \geq \cdots \geq m_r > 0\}.$$

Thm: • $\mathcal{F}_0$ admits the structure of a vertex algebra, strongly generated by the field

$$a(z) = \sum_{m\in\mathbb{Z}} a_m z^{-m-1}.$$

• If $k\neq 0$, $\mathcal{F}_0$ is a VOA of central charge 1 and EMT

$$L(z) = \frac{1}{2k} :a(z)a(z): = \sum_{m\in\mathbb{Z}} L_m z^{-n-2}.$$

As VOAs, the $\mathcal{F}_0$ with different $k\in\mathbb{C}\setminus\{0\}$ are $\cong$.

$\mathcal{F}_0$ is called the Heisenberg VOA (the free boson) and often denoted by H. The $\mathcal{F}_p$ are H-mods with $L_0\Omega_p = \frac{1}{2}p^2\Omega_p$.

Prop: The $\mathcal{F}_p$ are irreducible as H-mods.

Pf: Any submodule of $\mathcal{F}_p$ is graded by L_0-evals (exercise). $\therefore$ any non-zero submod. of $\mathcal{F}_p$ has an L_0-evect of min. eval. This evect. is thus a hwv. Since $a_0 = p\cdot\mathrm{id}$ on $\mathcal{F}_p$, this evect has eval $\frac{1}{2}p^2$, hence it is $\propto \Omega_p$ and so the non-zero submod. is $\mathcal{F}_p$. $\blacksquare$

Since a_0 is central, $\mathrm{Ext}^1(\mathcal{F}_p, \mathcal{F}_q) = 0 \quad \forall\, p \neq q$.

There <u>are</u> self-extensions, since $\mathbb{C}\Omega_p$ has a self-extension, but we <u>don't</u> want to consider them (unless forced to).

<u>Def</u>: Let $\mathbb{W}_k$ denote the category of finite direct sums of level-k Fock spaces $\mathcal{F}_p$ with $p \in \mathbb{R}$. We will <u>assume</u> that $k > 0$.

<u>Thm</u>: (Futorny '96) $\mathbb{W}_k$ is in fact the Harish-Chandra category of weight $\widehat{\mathfrak{gl}}_1$-mods w/ real weights and finite-dim. wt. spaces.

The irreducible characters of $\mathbb{W}_k$ are the

$$\mathrm{ch}[\mathcal{F}_p](q) = \mathrm{tr}_{\mathcal{F}_p}\, q^{L_0 - c/24} = \frac{q^{p^2/2k - 1/24}}{\prod_{n=1}^{\infty}(1-q^n)} = \frac{q^{p^2/2k}}{\eta(q)}, \quad p \in \mathbb{R}.$$

$\uparrow$ Dedekind

Convergent when $|q| < 1$.

<u>Problem</u>: The irred. chars are <u>not</u> linear independent:

$$\mathrm{ch}[\mathcal{F}_p] = \mathrm{ch}[\mathcal{F}_{-p}].$$

The chars $\therefore$ do not faithfully represent the Grothendieck group $K(\mathbb{W}_k)$.

Solution: Upgrade the characters. Define

$$\mathrm{ch}[\mathcal{F}_p](z;q) = \mathrm{tr}_{\mathcal{F}_p}\, z^{a_0} q^{L_0 - c/24} = \frac{z^p q^{p^2/2k}}{\eta(q)}.$$

Convergent for $|q|<1$ and $z \neq 0$.

We now investigate the modularity of these chars, ie of $K(\mathbb{W}_k)$. Define ζ and τ by

$$z = e^{2\pi i \zeta} \quad \text{and} \quad q = e^{2\pi i \tau}.$$

The (physically correct) action of $SL(2;\mathbb{Z})$ on chars is then generated by

$$S: (\zeta|\tau) \mapsto (\zeta/\tau \,|\, -1/\tau) \quad \text{and} \quad T: (\zeta|\tau) \mapsto (\zeta|\tau+1).$$

T-transform:

$$\mathrm{ch}[\mathcal{F}_p](\zeta|\tau+1) = \frac{e^{2\pi i \zeta p}}{\prod_{n=1}^{\infty}(1-e^{2\pi i(\tau+1)n})}\, e^{2\pi i(\tau+1)(p^2/2k - 1/24)}$$

$$= e^{2\pi i(p^2/2k - 1/24)}\, \mathrm{ch}[\mathcal{F}_p](\zeta|\tau).$$

(This is diagonal!)

S-transform: $\eta(\tau)$ means $\eta(e^{2\pi i\tau})$!

Then, $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$.

Prop: $e^{-\pi i k \zeta^2/\tau}\, \mathrm{ch}[\mathcal{F}_p](\zeta/\tau \mid -1/\tau)$

$$= \int_{-\infty}^{\infty} e^{-2\pi i p q/k}\, \mathrm{ch}[\mathcal{F}_q](\zeta \mid \tau)\, \frac{dq}{\sqrt{k}}.$$

Pf: $$\mathrm{RHS} = \frac{1}{\eta(\tau)} \int_{-\infty}^{\infty} \underbrace{e^{-2\pi i p q/k}\, e^{2\pi i(\zeta q + \tau q^2/2k)}}_{e^{(\pi i \tau/k) q^2 + (2\pi i(\zeta - p/k)) q}} \frac{dq}{\sqrt{k}}$$

$$= \frac{1}{\sqrt{-i\tau}\,\eta(\tau)}\, e^{-\pi i k(\zeta^2/\tau)}\, e^{2\pi i (\zeta/\tau) p}\, e^{2\pi i (-1/\tau) p^2/2k}$$

$$= \mathrm{LHS}. \qquad \blacksquare$$

Problem: S-transform has an unwanted factor.

Solution: Use the full Cartan subalgebra of $\hat{\mathfrak{gl}}_1$.

Def: $$\mathrm{ch}[\mathcal{F}_p](y, z; q) = \mathrm{tr}_{\mathcal{F}_p}\, y^K z^{a_0} q^{L_0 - c/24} = \frac{y^k z^p q^{p^2/2k}}{\eta(q)}.$$

Def: For $y = e^{2\pi i \theta}$, let the $SL(2;\mathbb{Z})$-action be

$$S: (\theta \mid \zeta \mid \tau) \longmapsto (\theta - \zeta^2/2\tau \mid \zeta/\tau \mid -1/\tau)$$

and $$T: (\theta \mid \zeta \mid \tau) \longmapsto (\theta \mid \zeta \mid \tau + 1).$$

Exercise: Check that this is an $SL(2;\mathbb{Z})$-action!

Note: $\mathrm{ch}[\mathcal{F}_p](\theta - \zeta^2/2\tau \mid \zeta/\tau \mid -1/\tau)$

$$= \frac{e^{2\pi i \theta k}\, e^{-\pi i k(\zeta^2/\tau)}\, e^{2\pi i(\zeta/\tau) p}\, e^{2\pi i(-1/\tau) p^2/2k}}{\sqrt{-i\tau}\,\eta(\tau)}.$$

Thm: (Modularity of $K(\mathbb{W}_k)$)

The irred. chars of $\mathbb{W}_k$ satisfy

$$S\{\mathrm{ch}[\mathcal{F}_p]\} = \int_{-\infty}^{\infty} S_{pq}\, \mathrm{ch}[\mathcal{F}_q] \frac{dq}{\sqrt{k}}$$

$$\& \quad T\{\mathrm{ch}[\mathcal{F}_p]\} = \int_{-\infty}^{\infty} T_{pq}\, \mathrm{ch}[\mathcal{F}_q] \frac{dq}{\sqrt{k}},$$

where $\quad S_{pq} = e^{-2\pi i pq/k}$

and $\quad T_{pq} = e^{2\pi i (p^2/2k - 1/24)} \delta\left(\frac{p}{\sqrt{k}} = \frac{q}{\sqrt{k}}\right).$

Note that S is symmetric $[S_{pq} = S_{qp}]$ and unitary:

$$\int_{-\infty}^{\infty} S_{pq} S_{qr}^{*} \frac{dq}{\sqrt{k}} = \int_{-\infty}^{\infty} e^{-2\pi i (p-r) q/k} \frac{dq}{\sqrt{k}}$$

$$= \delta\left(\frac{p}{\sqrt{k}} - \frac{r}{\sqrt{k}}\right).$$

Exercise: Check that $S^2 = C$ is a permutation...

Finally, we check the Verlinde formula for the Grothendieck fusion coefficients:

$$N_{pq}{}^{r} = \int_{-\infty}^{\infty} \frac{S_{ps} S_{qs} S_{rs}^{*}}{S_{0s}} \frac{ds}{\sqrt{k}} = \int_{-\infty}^{\infty} e^{-2\pi i (p+q-r)s/k} \frac{ds}{\sqrt{k}}$$

$$= \delta\left(\frac{p}{\sqrt{k}} + \frac{q}{\sqrt{k}} = \frac{r}{\sqrt{k}}\right).$$

These are non-neg. int. multiples of deltas!

$$\Rightarrow [\mathcal{F}_p] \boxtimes [\mathcal{F}_q] = \int_{-\infty}^{\infty} N_{pq}{}^{r} [\mathcal{F}_r] \frac{dr}{\sqrt{k}}$$

$$= \int_{-\infty}^{\infty} \delta\left(\frac{p+q}{\sqrt{k}} = \frac{r}{\sqrt{k}}\right) [\mathcal{F}_r] \frac{dr}{\sqrt{k}}$$

$$= \int_{-\infty}^{\infty} \delta(p+q=r) [\mathcal{F}_r]\, dr = [\mathcal{F}_{p+q}].$$

Since $\mathbb{W}_k$ is semisimple, the Grothendieck fusion rules <u>are</u> the fusion rules:

$$\mathcal{F}_p \boxtimes \mathcal{F}_q \cong \mathcal{F}_{p+q}.$$

[Of course, we have no theorem guaranteeing that the Verlinde computation gives the <u>correct</u> Grothendieck fusion coefficients...]

<u>Next task</u>: Upgrade to modularity of $\mathbb{W}_k$.

What does this even mean?

Recall that the objects of $\mathbb{W}_k$ are finite $\oplus$'s of the $\mathcal{F}_p$, with $p \in \mathbb{R}$. $\mathbb{W}_k$ is therefore:

abelian, $\mathbb{C}$-linear, locally finite and semisimple,

It should also be monoidal, with tensor product $\boxtimes$.

How to verify this?

Defining fusion rigorously takes forever... We cheat:

<u>Def:</u> • An <u>intertwining operator</u> is a "field" corresponding to a state in a VOA-module.

• A <u>primary field</u> is an intertwining operator that corresponds to a highest-weight vector.

In $\mathbb{W}_k$, the hwvs are the $\Omega_p \in \mathcal{F}_p$. Denote the corresponding primary fields by $\Omega_p(z)$. Because the state-field correspondence works for intertwining ops, we have

$$a(z)\Omega_p(w) = \sum_{n\in\mathbb{Z}} A^{(n)}(w)(z-w)^{-n-1}$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} a_n \Omega_p z^{-n-1} = a(z)\Omega_p = \sum_{n\in\mathbb{Z}}' A^{(n)} z^{-n-1} \Rightarrow A^{(n)} = a_n\Omega_p.$$

ie $A^{(n)} = 0 \ \forall n>0$, $A^{(0)} = a_0\Omega_p = p\Omega_p$, $A^{(-n)} = a_{-n}\Omega_p \ \forall n>0$

$\Rightarrow A^{(n)}(w) = 0$ " , $A^{(0)}(w) = p\Omega_p(w)$, $A^{(-n)}(w) = \dfrac{:\partial^{n-1}a(w)\,\Omega_p(w):}{(n-1)!}$

$$\Rightarrow a(z)\Omega_p(w) = \frac{p\,\Omega_p(w)}{z-w} + :a(z)\Omega_p(w):.$$

Similarly (exercise):

$$L(z)\Omega_p(w) = \frac{p^2}{2k}\frac{\Omega_p(w)}{(z-w)^2} + \frac{\partial\Omega_p(w)}{z-w} + :L(z)\Omega_p(w):.$$

The key fact that makes Heisenberg categorically tractable is that we can compute the $\Omega_k(w)$ explicitly!

For this, note that $L(z) = \frac{1}{2k} :a(z)a(z):$ gives

$$L_{-1} = \frac{1}{2k}\left[\sum_{r\leq -1} a_r a_{-1-r} + \sum_{r\geq 0} a_{-1-r} a_r\right] = \frac{1}{k}\left[a_{-1}a_0 + a_{-2}a_1 + a_{-3}a_2 + \dots\right]$$

$$\Rightarrow \quad L_{-1}\Omega_p = \frac{1}{k} a_{-1} a_0 \Omega_p = \frac{p}{k} a_{-1}\Omega_p$$

$$\Rightarrow \quad \partial\Omega_p(w) = \frac{p}{k} :a(w)\Omega_p(w):.$$

Ignore the normal ordering and this is an ODE for $\Omega_p(w)$:

$$\frac{\partial\Omega_p(w)}{\Omega_p(w)} = \frac{p}{k} a(w) \quad \Rightarrow \quad \Omega_p(w) = e^{p\int a(w)\,dw/k}.$$

Note: $\int a(w)\,dw = \sum_{n\in\mathbb{Z}} a_n \int w^{-n-1}\,dw = \hat{a} + a_0 \log w - \sum_{n\neq 0} \frac{a_n}{n} w^{-n}$.

This is nearly correct — we just have to normally order the resulting exponentials:

$$\Omega_p(w) = e^{p\hat{a}/k} e^{pa_0 \log w/k} e^{p\sum_{n>0} a_{-n} w^n/nk} e^{-p\sum_{n>0} a_n w^{-n}/nk}$$

$$= \left[e^{p\hat{a}/k} w^{pa_0/k}\right] \prod_{n\geq 1}\left[e^{(p/k)a_{-n}w^n/n} e^{-(p/k)a_n w^{-n}/n}\right].$$

Here, we note that $[a_n, a_{-n}] = nk$, while

$$a(z)\int a(w)\,dw \sim \int \frac{k\,dw}{(z-w)^2} \sim \frac{k}{z-w} \quad \text{(const. is regular!)}$$

$$\Rightarrow \quad [a_m, \hat{a}] = \delta_{m=0} k \quad \text{(exercise!)}.$$

Check: $\Omega_p(w)\Omega_0\Big|_{w=0} = e^{p\hat{a}/k}\, \cancel{w^{pa_0/k}} \prod_{n\geq 1}\left[e^{(p/k)a_{-n}w^n/n}\, e^{-(p/k)\cancel{a_n w^{-n}/n}}\right]\Omega_0\Big|_{w=0}$

$$= e^{p\hat{a}/k}\prod_{n\geq 1}\left[e^{(p/k)a_{-n}\cancel{w^n}/n}\right]\cdot\Omega_0\Big|_{w=0}$$

$$= e^{p\hat{a}/k}\,\Omega_0 .$$

Moreover, $\quad a_n e^{p\hat{a}/k}\Omega_0 = 0 \;\; \forall\, n>0 \qquad (\text{since } [a_n,\hat{a}]=0)$

and $\quad a_0 e^{p\hat{a}/k}\Omega_0 = p\, e^{p\hat{a}/k}\Omega_0 \qquad (\text{since } [a_0,\hat{a}]=k)$.

$$\therefore\; e^{p\hat{a}/k}\Omega_0 = \Omega_p .$$

Exercise: $[a_0, e^{\beta\hat{a}}] = \beta k e^{\beta\hat{a}}, \quad [a_n, e^{\beta a_{-n}}] = \beta k n\, e^{\beta a_{-n}},$

$$e^{\alpha a_0}e^{\beta\hat{a}} = e^{\alpha\beta k}e^{\beta\hat{a}}e^{\alpha a_0}, \quad e^{\alpha a_n}e^{\beta a_{-n}} = e^{\alpha\beta k n}e^{\beta a_{-n}}e^{\alpha a_n}.$$

Application: We compute the OPE for primary fields!

$$\Omega_p(z)\Omega_q = \prod_{n\geq 1} e^{(p/k)a_{-n}z^n/n}\, e^{-(p/k)\cancel{a_n} z^{-n}/n}\cdot e^{p\hat{a}/k} z^{pa_0/k}\Omega_q$$

$$= z^{pq/k}\prod_{n\geq 1} e^{(p/k)a_{-n}z^n/n}\cdot \underbrace{e^{p\hat{a}/k}e^{q\hat{a}/k}\Omega_0}_{=\Omega_{p+q}}$$

$$= z^{pq/k}\left(1+\frac{p}{k}a_{-1}z + \frac{1}{2}\left(\frac{p^2}{k^2}a_{-1}^2 + \frac{p}{k}a_{-2}\right)z^2 + \dots\right)\Omega_{p+q}$$

$$\Rightarrow \Omega_p(z)\Omega_q(w) = \Omega_{p+q}(w)(z-w)^{pq/k}$$

$$+ \frac{p}{k} :a(w)\Omega_{p+q}(w): (z-w)^{pq/k+1}$$

$$+ \frac{1}{2}\left[\frac{p^2}{k^2} :a(w)a(w)\Omega_{p+q}(w): + \frac{p}{k} :\partial a(w)\Omega_{p+q}(w):\right](z-w)^{pq/k+2}$$

$$+ \dots$$

So what? This gives us the fusion product!

"Def": The fusion product of two VOA modules V_i and V_j is the VOA module whose elements correspond to the intertwiners appearing in the OPEs $A_i(z)B_j(w)$, where A_i and B_j are intertwiners for V_i and V_j, respectively.

Prop: For affine VOAs, fusion products may be computed by restricting to primary fields.

We compute $\mathcal{F}_p \boxtimes \mathcal{F}_q$. The primary OPE to calculate is

$$\Omega_p(z)\Omega_q(w) = \underbrace{\Omega_{p+q}(w)}_{\text{primary for } \mathcal{F}_{p+q}!}(z-w)^{pq/k} + \begin{array}{c}\text{terms involving}\\ \text{non-primary fields.}\end{array}$$

$$\therefore \mathcal{F}_p \boxtimes \mathcal{F}_q \cong \mathcal{F}_{p+q}.$$

cf. Verlinde — complete agreement.

We thus equip $\mathbb{W}_k$ with $\boxtimes$.

Let us <u>assume</u> that this makes $\mathbb{W}_k$ into a mon. cat.

- $\boxtimes : \mathbb{W}_k \times \mathbb{W}_k \to \mathbb{W}_k$ is the <u>bifunctor</u>.

- $\mathcal{F}_0 = H$ is the monoidal <u>unit</u>: $\mathcal{F}_p \boxtimes \mathcal{F}_0 \cong \mathcal{F}_p \cong \mathcal{F}_0 \boxtimes \mathcal{F}_p$.

- There are <u>left</u> and <u>right unitors</u>

$$\lambda_p : \mathcal{F}_0 \boxtimes \mathcal{F}_p \xrightarrow{\sim} \mathcal{F}_p \quad \text{and} \quad \rho_p : \mathcal{F}_p \boxtimes \mathcal{F}_0 \xrightarrow{\sim} \mathcal{F}_p.$$

They are determined by

$$\lambda_p(\Omega_0 \boxtimes \Omega_p) \quad \text{and} \quad \rho_p(\Omega_p \boxtimes \Omega_0).$$

Now,
$$\Omega_0(z)\Omega_p = \Omega_{0+p} z^{0.p/k} + 0 \cdot \text{other terms}$$
$$= \Omega_p,$$

the left unitor satisfies $\lambda_p(\Omega_0 \boxtimes \Omega_p) = \Omega_p$.

However,
$$\Omega_p(z)\Omega_0 = \prod_{n \geq 1} e^{(p/k) a_{-n} z^n / n} \cdot \Omega_p$$
$$= \Omega_p + \frac{p}{k} a_{-1} \Omega_p z + \cdots$$

which suggests that the right unitor ρ_p might be non-trivial.

- Computing associators is always hard! But,

$$\alpha_{pqr} : \mathcal{F}_p \boxtimes (\mathcal{F}_q \boxtimes \mathcal{F}_r) \longrightarrow (\mathcal{F}_p \boxtimes \mathcal{F}_q) \boxtimes \mathcal{F}_r$$

is an endomorphism of $\mathcal{F}_{p+q+r}$, hence prop. to id_{p+q+r}.

$\therefore$ we may compute on $\Omega_p \boxtimes (\Omega_q \boxtimes \Omega_r)$:

$$\Omega_p(z)\big(\Omega_q(w)\Omega_r\big) = \Omega_p(z)\,\Omega_{q+r}\, w^{qr/k} + O\big(w^{qr/k+1}\big)$$

$$= \Omega_{p+q+r}\, z^{p(q+r)/k}\, w^{qr/k} + O\big(z^{p(q+r)/k+1}\big) + O\big(w^{qr/k+1}\big)$$

$$\& \quad \big(\Omega_p(z)\Omega_q(w)\big)\Omega_r = \Omega_{p+q}(w)\,\Omega_r\,(z-w)^{pq/k} + O\big((z-w)^{pq/k+1}\big)$$

$$= \Omega_{p+q+r}\, w^{(p+q)r/k}(z-w)^{pq/k} + O\big(w^{(p+q)r/k+1}\big) + O\big((z-w)^{pq/k+1}\big).$$

To compare, we need to expand in the <u>same</u> region.

But, we can't: eg expanding $(z-w)^b$ in positive powers of w gives <u>negative</u> powers of z:

$$(z-w)^b = z^b\left(1-\frac{w}{z}\right)^b = z^b - b z^{b-1} w + \cdots$$

We have to work harder... ☹ LHS is

$$\Omega_p(z)\big(\Omega_q(w)\Omega_r\big) = w^{qr/k}\,\Omega_p(z) \prod_{n\geq 1} e^{(q/k)a_{-n}w^n/n}\cdot \Omega_{q+r}$$

$$= w^{qr/k} z^{p(q+r)/k} \prod_{n\geq 1} e^{(p/k)a_{-n}z^n/n} \underbrace{e^{-(p/k)a_n z^{-n}/n}\, e^{(q/k)a_{-n}w^n/n}}_{= e^{-(pq/k)(w/z)^n/n}\, e^{(q/k)a_{-n}w^n/n}\, e^{-(p/k)a_n z^{-n}/n}} \cdot \Omega_{p+q+r}$$

$$= w^{qr/k} z^{p(q+r)/k} \underbrace{\exp\Big[-\frac{pq}{k}\sum_{n\geq 1}\frac{(w/z)^n}{n}\Big]}_{= \exp\left[-\frac{pq}{k}\log\left(1-\frac{w}{z}\right)\right] = \left(1-\frac{w}{z}\right)^{pq/k}} \prod_{n\geq 1} e^{[(p/k)z^n + (q/k)w^n]a_{-n}/n} \cdot \Omega_{p+q+r}$$

$(|z|>|w|)$

$$= w^{qr/k} z^{pr/k} (z-w)^{pq/k} \Omega_{p+q+r} + [\text{states of higher conf. wt}].$$

OTOH, the RHS is <u>even harder</u>:

$$\Omega_p(z)\Omega_q = z^{pq/k} \exp\left[\frac{p}{k}\sum_{n\geq 1} \frac{a_{-n} z^n}{n}\right] \Omega_{p+q}$$

$$= z^{pq/k} \sum_{j\geq 0}' \frac{(p/k)^j}{j!} \sum_{n_1,\dots,n_j\geq 1}' (a_{-n_1}\cdots a_{-n_j}\Omega_{p+q}) \frac{z^{n_1+\dots+n_j}}{n_1\cdots n_j}$$

$$\Rightarrow \Omega_p(z)\Omega_q(w) = (z-w)^{pq/k} \sum_{j\geq 0}' \frac{(p/k)^j}{j!}$$

$$\cdot \sum_{n_1,\dots,n_j\geq 1} \frac{(z-w)^{n_1+\dots+n_j}}{n_1\cdots n_j} (a_{-n_1}\cdots a_{-n_j}\Omega_{p+q})(w)$$

$$\left(\partial^{(m)} = \frac{1}{m!}\partial^m\right) \qquad = (z-w)^{pq/k} \sum_{j\geq 0}' \frac{(p/k)^j}{j!}$$

conf. wt is $\frac{(p+q)^2}{2k} + n_1 + \dots + n_j$

$$\cdot \sum_{n_1,\dots,n_j\geq 1} \frac{(z-w)^{n_1+\dots+n_j}}{n_1\cdots n_j} :\partial^{(n_1-1)}a(w)\cdots\partial^{(n_j-1)}a(w)\,\Omega_{p+q}(w):.$$

Since $\Omega_{p+q}(w)\Omega_r = \Omega_{p+q+r}\, w^{(p+q)r} + \left[\begin{matrix}\text{states of higher}\\ \text{conformal weight}\end{matrix}\right]$,

$\exists N$ s.t. $(\Omega_{p+q})_N \Omega_r = \Omega_{p+q+r}$ & $(\Omega_{p+q})_n \Omega_r = 0 \ \forall n>N$.

We want the coeff. of Ω_{p+q+r} in $\boxed{N = \frac{r^2-(p+q+r)^2}{2k}}$

$$:\partial^{(n_1-1)}a \cdots \partial^{(n_j-1)}a\,\Omega_{p+q}:_N \Omega_r$$

$>N$ so annihilates Ω_r

$$= \sum_{n\leq -n_1}' (\partial^{(n_1-1)}a)_n :\partial^{(n_2-1)}a\cdots\partial^{(n_j-1)}a\,\Omega_{p+q}:_{N-n}\Omega_r$$

ann. Ω_r if $n>0$

$$+ \sum_{n\geq 0}' :\partial^{(n_2-1)}a\cdots\partial^{(n_j-1)}a\,\Omega_{p+q}:_{N-n} (\partial^{(n_1-1)}a)_n \Omega_r$$

$$= \,:\partial^{(n_2-1)}a \cdots \partial^{(n_j-1)} a\, \Omega_{p+q}:_N \,(\partial^{(n_1-1)} a)_0\, \Omega_r$$

$$\vdots$$

$$= (\Omega_{p+q})_N\, (\partial^{(n_j-1)} a)_0 \cdots (\partial^{(n_1-1)} a)_0\, \Omega_r$$

$$= (-1)^{n_1+\cdots+n_j-j}\, (\Omega_{p+q})_N\, a_0^j\, \Omega_r$$

$$= (-1)^{n_1+\cdots+n_j-j}\, r^j\, \Omega_{p+q+r} + \begin{bmatrix}\text{states of higher}\\ \text{conf. wt}\end{bmatrix}.$$

$$\therefore \left(\Omega_p(z)\Omega_q(w)\right)\Omega_r = (z-w)^{pq/k} \sum_{j\geq 0} \frac{(p/k)^j}{j!}$$

$$\cdot \sum_{n_1,\dots,n_j\geq 1} \frac{(w-z)^{n_1+\cdots+n_j}}{n_1\cdots n_j} (-1)^j r^j w^{-N-(p+q)^2/2k-n_1-\cdots-n_j}\Omega_{p+q+r} + [\cdots]$$

$$= (z-w)^{pq/k} \sum_{j\geq 0}\frac{(pr/k)^j}{j!}$$

$$\left[-N-\frac{(p+q)^2}{2k} = \frac{(p+q+r)^2-r^2-(p+q)^2}{2k} = \frac{(p+q)r}{k}\right]$$

$$\cdot \left[-\sum_{n_1\geq 1}\frac{(w-z)^{n_1}}{n_1 w^{n_1}}\right]\cdots\left[-\sum_{n_j\geq 1}\frac{(w-z)^{n_j}}{n_j w^{n_j}}\right] w^{(p+q)r/k}\,\Omega_{p+q+r} + [\cdots]$$

$$= (z-w)^{pq/k} \underbrace{\sum_{j\geq 0}\frac{(pr/k)^j}{j!}\left[\log\left(1-\frac{w-z}{w}\right)\right]^j}_{=\exp\left[\frac{pr}{k}\log\frac{z}{w}\right] = \left(\frac{z}{w}\right)^{pr/k}} \cdot w^{(p+q)r/k}\,\Omega_{p+q+r} + [\cdots]$$

$$(|w| > |w-z|)$$

$$= (z-w)^{pq/k} z^{pr/k} w^{qr/k}\, \Omega_{p+q+r} + \begin{bmatrix}\text{states of higher}\\ \text{conformal wt}\end{bmatrix}.$$

So RHS = LHS, implying that

$$\alpha_{pqr} = \mathrm{id}_{p+q+r} \quad !!$$

The conclusion is that if $\mathbb{W}_k$ is monoidal, then the associator is trivial. This makes it easy to check the pentagon (and triangle) relations!

<u>Note</u>: The associator need not be trivial for a lattice VOA because of the cocycles that enter the definition...

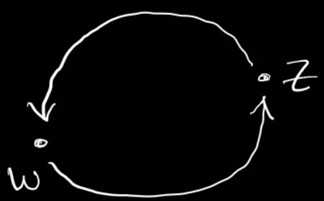
Consider next braidings.

- A braiding $R_{pq} : \mathcal{F}_p \boxtimes \mathcal{F}_q \to \mathcal{F}_q \boxtimes \mathcal{F}_p$ is an endomorphism of $\mathcal{F}_{p+q}$, hence it's a multiple of id_{p+q}.

Calculate it on $\Omega_p \boxtimes \Omega_q$. How?

We have to choose a means of swapping z and w:

$$z(t) = \frac{z+w}{2} + e^{\pi i t}\frac{z-w}{2} \qquad \qquad z(0)=z \quad z(1)=w,$$
$$\Rightarrow$$
$$w(t) = \frac{z+w}{2} - e^{\pi i t}\frac{z-w}{2} \qquad \qquad w(0)=w \quad w(1)=z.$$



Also $(z-w)(t) = e^{\pi i t}(z-w)$. Now compare:

$$t=0 : \quad \Omega_p(z)\,\Omega_q(w) = \Omega_{p+q}(w)\,(z-w)^{pq/k} + \dots$$

$$t=1 : \quad \Omega_q(w)\,\Omega_p(z) = \Omega_{q+p}(z)\,(w-z)^{qp/k} + \dots$$

$$= \Omega_{p+q}(z)\, e^{\pi i qp/k}\,(z-w)^{pq/k} + \dots$$

$$= e^{\pi i pq/k}\,\Omega_{p+q}(w)\,(z-w)^{pq/k} + \dots$$

$$\therefore \quad R_{pq}(\Omega_p \boxtimes \Omega_q) = e^{\pi i pq/k}\, \Omega_q \boxtimes \Omega_p .$$

Since associativity is trivial, it's easy to check the "hexagon" relations, eg:

$$\begin{tikzcd}
\mathcal{F}_p \boxtimes \mathcal{F}_q \boxtimes \mathcal{F}_r \arrow[rr, "R_{p+q,r}"] \arrow[dr, "\mathrm{id}_p \boxtimes R_{qr}"'] & & \mathcal{F}_r \boxtimes \mathcal{F}_p \boxtimes \mathcal{F}_q \\
& \mathcal{F}_p \boxtimes \mathcal{F}_r \boxtimes \mathcal{F}_q \arrow[ur, "R_{pr} \boxtimes \mathrm{id}_q"'] &
\end{tikzcd} .$$

- This braiding is <u>not</u> symmetric:

$$\mathcal{F}_p \boxtimes \mathcal{F}_q \xrightarrow{R_{pq}} \mathcal{F}_q \boxtimes \mathcal{F}_p \xrightarrow{R_{qp}} \mathcal{F}_p \boxtimes \mathcal{F}_q$$

is $e^{2\pi i pq/k} \cdot \mathrm{id}_{p \boxtimes q}$, not $\mathrm{id}_{p\boxtimes q}$.

- It is instead <u>nondegenerate</u>: the Müger centre

$$\{ M \in \mathbb{W}_k : R_{NM} R_{MN} = \mathrm{id}_{M \boxtimes N} \ \forall N \in \mathbb{W}_k \}$$

contains only direct sums of $\mathcal{F}_0$, since

$$R_{qp} R_{pq} = e^{2\pi i pq/k}\, \mathrm{id}_{p\boxtimes q} = \mathrm{id}_{p \boxtimes q} \ \forall q \in \mathbb{R} \Rightarrow p = 0.$$

(ie the Müger centre is equiv. to Vec.) = whole cat. if symmetric

<u>Note</u>: We may now <u>take</u> the right unitor to be

$$\rho_p : \mathcal{F}_p \boxtimes \mathcal{F}_0 \xrightarrow{R_{p0}} \mathcal{F}_0 \boxtimes \mathcal{F}_p \xrightarrow{\lambda_p} \mathcal{F}_p ,$$
$$\Omega_p \boxtimes \Omega_0 \longmapsto \Omega_0 \boxtimes \Omega_p \longmapsto \Omega_p .$$

- To verify rigidity, we need to show that each $\mathcal{F}_p$ has a <u>dual</u> $\mathcal{F}_p^\vee$ and there are (non-zero!) morphisms

$$ev_p : \mathcal{F}_p^\vee \boxtimes \mathcal{F}_p \to \mathcal{F}_0 \quad \text{(evaluation)}$$

$$\& \quad coev_p : \mathcal{F}_0 \to \mathcal{F}_p \boxtimes \mathcal{F}_p^\vee \quad \text{(coevaluation)}$$

satisfying two identities:

$$\mathcal{F}_p \xrightarrow{\lambda_p^{-1}} \mathcal{F}_0 \boxtimes \mathcal{F}_p \xrightarrow{coev_p \boxtimes id_p} (\mathcal{F}_p \boxtimes \mathcal{F}_{-p}) \boxtimes \mathcal{F}_p$$

$$\xrightarrow{\alpha_{p,-p,p}^{-1}} \mathcal{F}_p \boxtimes (\mathcal{F}_{-p} \boxtimes \mathcal{F}_p) \xrightarrow{id_p \boxtimes ev_p} \mathcal{F}_p \boxtimes \mathcal{F}_0 \xrightarrow{\rho_p} \mathcal{F}_p$$

— AND —

$$\mathcal{F}_{-p} \xrightarrow{\rho_p^{-1}} \mathcal{F}_{-p} \boxtimes \mathcal{F}_0 \xrightarrow{id_{-p} \boxtimes coev_p} \mathcal{F}_{-p} \boxtimes (\mathcal{F}_p \boxtimes \mathcal{F}_{-p})$$

$$\xrightarrow{\alpha_{-p,p,-p}} (\mathcal{F}_{-p} \boxtimes \mathcal{F}_p) \boxtimes \mathcal{F}_{-p} \xrightarrow{ev_p \boxtimes id_{-p}} \mathcal{F}_0 \boxtimes \mathcal{F}_{-p} \xrightarrow{\lambda_p} \mathcal{F}_{-p}$$

are the identity maps (on $\mathcal{F}_p$ and $\mathcal{F}_{-p}$ resp.).

[This is the def. for a <u>left</u> dual; there is also a similar notion of a right dual. Because $\mathbb{W}_k$ is <u>braided</u>, existence of left duals $\Rightarrow$ that of right duals]

It's clear that the only candidate for $\mathcal{F}_p^\vee$ is $\mathcal{F}_{-p}$.

For ev_p, we act on $\Omega_{-p} \boxtimes \Omega_p$:

$$\Omega_{-p}(z)\Omega_p = z^{-p^2/k} \prod_{n \geq 1} e^{(-p/k) a_{-n} z^n/n} \cdot \Omega_0 .$$

For $coev_p$, we "invert" this (with $p \to -p$) to act on Ω_0:

$$\Omega_0 = \oint_0 \Omega_p(z)\Omega_{-p} z^{p^2/k - 1} \frac{dz}{2\pi i}$$

Let's check the first identity. As usual, it's enough to check it on $\Omega_p \in \mathcal{F}_p$:

$$\Omega_p \xmapsto{\lambda_p^{-1}} \Omega_0 \boxtimes \Omega_p$$

$$\xmapsto{coev_p \boxtimes id_p} \oint_w \left[\Omega_p(z)\Omega_{-p}(w)\right](z-w)^{p^2/k - 1} \frac{dz}{2\pi i} \Omega_p$$

$$\xmapsto{\alpha^{-1}_{p,-p,p}} \oint_w \Omega_p(z)\left[\Omega_{-p}(w)\Omega_p\right](z-w)^{p^2/k-1} \frac{dz}{2\pi i}$$

$$\xmapsto{id_p \boxtimes ev_p} \oint_w \underbrace{\Omega_p(z)}_{\left[= \cdots \prod_{n \geq 1} e^{-(p/k) a_n z^{-n}/n}\right]} w^{-p^2/k} \prod_{n \geq 1} e^{(-p/k) a_{-n} w^n/n} \cdot \Omega_0 \cdot (z-w)^{p^2/k-1} \frac{dz}{2\pi i}$$

$$= w^{-p^2/k} \oint_w \underbrace{\prod_{n \geq 1} e^{(p^2/k)(w/z)^n/n}}_{\left[(1-w/z)^{-p^2/k} = (z-w)^{-p^2/k} z^{p^2/k}\right]} \cdot \prod_{n \geq 1} e^{(-p/k) a_{-n} w^n/n} \cdot \Omega_p(z)\Omega_0 \cdot (z-w)^{p^2/k-1} \frac{dz}{2\pi i}$$

$$= w^{-p^2/k} \prod_{n \geq 1} e^{(-p/k) a_{-n} w^n/n} \cdot \oint_w z^{p^2/k} \Omega_p(z)\Omega_0 (z-w)^{-1} \frac{dz}{2\pi i}$$

$$= \prod_{n\geq 1} e^{-(p/k)a_{-n}w^n/n} \cdot \Omega_p(w)\Omega_0$$

$$\xmapsto{\rho_p} \prod_{n\geq 1} e^{-(p/k)a_{-n}w^n/n} \cdot \prod_{n\geq 1} e^{(p/k)a_{-n}w^n/n} \cdot \Omega_p$$

$$= \Omega_p \qquad !$$

Second identity is similar! $\mathbb{W}_k$ is rigid.

- Finally, $\mathbb{W}_k$ is <u>ribbon</u> if $\exists\, \theta_p \in \mathrm{End}(\mathcal{F}_p)$ (the <u>twist</u>) s.t. $(\theta_p)^\vee = \theta_{-p}$ and

$$\theta_{p\boxtimes q} = (\theta_p \boxtimes \theta_q) R_{qp} R_{pq}.$$

For a VOA-module category, a twist is <u>always</u> given by

$$\theta_p = e^{2\pi i L_0}\Big|_{\mathcal{F}_p} = e^{\pi i p^2/k}\, \mathrm{id}_p.$$

LHS: $\theta_{p\boxtimes q}(\Omega_p \boxtimes \Omega_q) = \theta_{p+q}(\Omega_p \boxtimes \Omega_q)$

$$= e^{\pi i (p+q)^2/k}\, \Omega_p \boxtimes \Omega_q.$$

RHS: $(\theta_p \boxtimes \theta_q) R_{qp} R_{pq} (\Omega_p \boxtimes \Omega_q)$

$$= e^{2\pi i pq/k}\, \theta_p(\Omega_p) \boxtimes \theta_q(\Omega_q)$$

$$= e^{2\pi i pq/k} e^{\pi i p^2/k} e^{\pi i q^2/k}\, \Omega_p \boxtimes \Omega_q = \mathrm{LHS}.$$

We omit the verification that $\theta_p^\vee = \theta_{-p}$ (because we didn't discuss dual morphisms).

So: $\mathbb{W}_k$ is rigid & monoidal (hence tensor since $\mathrm{End}(\mathcal{F}_0) \simeq \mathbb{C}$), non-degenerately braided and ribbon.

It would be an MFC exept that it is not finite, but only locally finite.

From this, we should compute:

- the canonical pivot $a_p = u_p \theta_p : \mathcal{F}_p \to \mathcal{F}_p^{\vee\vee} \simeq \mathcal{F}_p$

  where u_p is

$$\mathcal{F}_p \xrightarrow{\rho_p^{-1}} \mathcal{F}_p \boxtimes \mathcal{F}_0 \xrightarrow{\mathrm{id}_p \boxtimes \mathrm{wev}_{-p}} \mathcal{F}_p \boxtimes (\mathcal{F}_{-p}^{\vee} \boxtimes \mathcal{F}_p^{\vee\vee})$$

$$\xrightarrow{\alpha_{p,-p,p}} (\mathcal{F}_p \boxtimes \mathcal{F}_{-p}^{\vee}) \boxtimes \mathcal{F}_p^{\vee\vee} \xrightarrow{R_{p,-p} \boxtimes \mathrm{id}_p} (\mathcal{F}_{-p}^{\vee} \boxtimes \mathcal{F}_p) \boxtimes \mathcal{F}_p^{\vee\vee}$$

$$\xrightarrow{\mathrm{ev}_p \boxtimes \mathrm{id}_p} \mathcal{F}_0 \boxtimes \mathcal{F}_p^{\vee\vee} \xrightarrow{\lambda_p} \mathcal{F}_p \simeq \mathcal{F}_p^{\vee\vee}$$

- the categorical trace $\mathrm{tr}_a(f)$ of $f \in \mathrm{End}(\mathcal{F}_p)$:

$$\mathcal{F}_0 \xrightarrow{\mathrm{wev}_p} \mathcal{F}_p \boxtimes \mathcal{F}_{-p} \xrightarrow{f \boxtimes \mathrm{id}_{-p}} \mathcal{F}_p \boxtimes \mathcal{F}_{-p} \xrightarrow{a_p \boxtimes \mathrm{id}_{-p}}$$

$$\mathcal{F}_p^{\vee\vee} \boxtimes \mathcal{F}_{-p}^{\vee} \xrightarrow{\mathrm{ev}_{-p}} \mathcal{F}_0 .$$

(Since $\mathrm{End}(\mathcal{F}_0) \simeq \mathbb{C}$, this "is" a number...)

- The S-matrix $S_{pq} = \mathrm{tr}_a(R_{qp} R_{pq})$.

  It should be (prop. to) the S-matrix of $K(\mathbb{W}_k)$.