Holonomies in quantum systems under adiabatic processes

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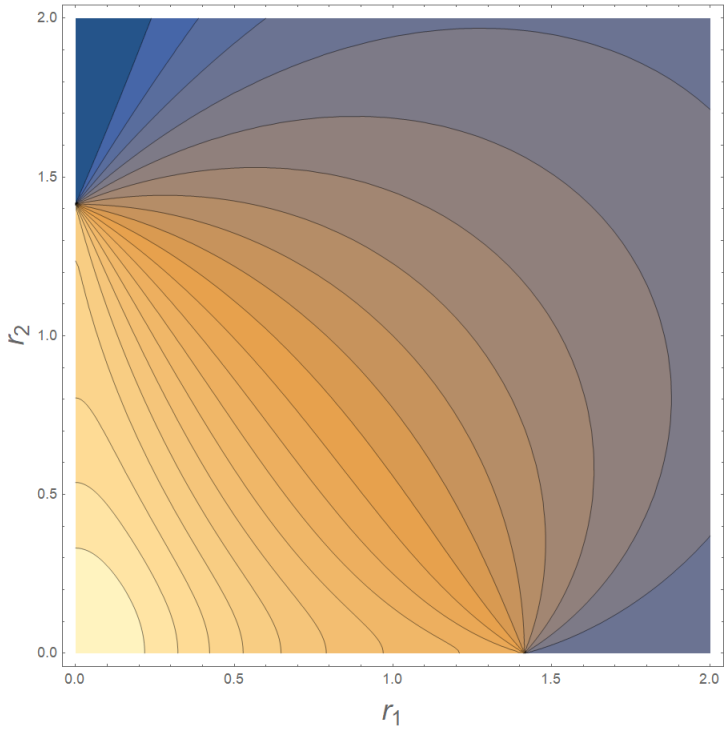

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Abstract

We develop a model for geometric phase in a finite dimensional quantum system, using the adiabatic theorem, and the language of vector bundles. We show how this model can be constructed using only local Hamiltonians, and establish how to use the energy eigenstates as local coordinates on our fibre. The base manifold, or parameter space, is taken to be the set of all parameters such that the local Hamiltonians are non-degenerate. This gives a direct correspondence between an experimental setup and our theory. In our model, geometric phase is calculated by parallel transporting a quantum state around a path in its parameter space. This means that its holonomy describes the system's Berry phase. We apply the model to a spin-1/2 system and the much richer nitrogen vacancy (NV) center. We find that the spin-1/2 system has a $U(1)$ holonomy group and can effectively be described by a S^2 vector bundle. The NV center exhibits a far more interesting and complicated topological structure, with holonomy group $U(1) \otimes U(1)$, observable discontinuities in the Berry phase, and a non-vanishing Chern class.

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Contents

Chapter 1

Introduction

Geometry plays a fundamental role in the field of mathematical physics. It is used to model classical mechanics on manifolds [26] and lays the groundwork for the two current pillars in theoretical physics: general relativity and quantum field theory.

A particularly interesting phenomenon in quantum mechanics is the appearance of geometric phase. When a quantum system evolves in time it will naturally accumulate a "dynamic" phase characterised by the time evolution operator, But it may also accumulate a phase due to the geometry describing the system. This was first observed in 1956 by Pancharatnam [39]. In 1959 Aharonov and Bohm published their famous paper [2] demonstrating how an electrically charged particle can be altered without any forces acting on it. They found that if a beam of electrons is split into two and made to interfere by passing around a solenoid, then the interference pattern can be altered by introducing a non-zero magnetic field inside the solenoid that vanishes outside and thus does not apply a force on the beams.

In 1984, Berry studied geometric phases for quantum systems under adiabatic changes in their Hamiltonian [4]. Berry found that an energy eigenstate of an adiabatically evolving system will smoothly change to remain as an eigenstate of the Hamiltonian, if there is a gap between its eigenvalue and the rest of the Hamiltonian's spectrum. Consequently, after slowly evolving a quantum system about a closed path in its Hamiltonian's parameters, states can return with a discrepancy in their phase known as Berry phase. In 1987 Aharonov and Anandan furthered the study of geometric phase for general cyclic processes [1]. In which, they made clear the distinction between "dynamic" and "geometric" phase in quantum systems.

Since then many examples of geometric phases in quantum systems have been experimentally demonstrated. In 2003 Jiangfeng made an observation of geometric phases for mixed states [15]. Later in 2019, Karnieli reported the first experimental verification of adiabatic geometric phase in nonlinear frequency conversion [23]. Quantum geometric phases have also been demonstrated in the adiabatic dynamics of chemical reactions [49, 54].

Geometric phases are of great importance to an experimenter because it can lead to different observations. Failing to account for it can mean that results wont match prediction. As progress is made in the fields of quantum computation and communication, where precise and accurate measurements of states are needed, it is critical to understand a system's geometric phases [41, 42].

It is apparent that a model for geometric phase is required to completely understand the dynamics of quantum systems. Many methods for calculating geometric phase do exist such as deriving the Bures metric (also known as the quantum geometric tensor) from infinitesimal perturbations to the fidelity of pure states [12, 27]. Another method is to relate the Berry curvature to the quantum metric tensor [8, 38]. This method uses the coordinate independence of the system's curvature 2-form to make this relation. However, this seems to only be useful for systems of constant Euler curvature such as the spin-1/2 system and

the nitrogen vacancy center under particular conditions.

A much more diverse way of modelling adiabatic geometric phase is to describe it as holonomy. The reason for this is that adiabatic processes can be described by smooth paths on a manifold. In 1992 Anandan used the classical method of parallel transport to calculate the Berry phase of the spin-1/2 system [3]. The use of parallel transport to describe Berry phases has appeared in many different forms, from Anandan's simple heuristic model to principal bundles with the base manifold taken as a projective Hilbert space, and fibre given by the set of normalised states [20]. The model that most resembles ours is the frame bundle developed by Tanaka and Cheon in 2009 [48]. There they extend the use of parallel transport to describe the accumulation of geometric phases for non-adiabatic processes. This suggests that our model could also be used for non-adiabatic general geometric phase.

Chapter 2 of this thesis will introduce the important concepts in differential geometry that will be required to fully understand how our model works. We start by defining coordinate manifolds and fibre bundles as well as explaining how to construct a fibre bundle, which is essential in establishing our model from a given Hamiltonian. We will then study examples of fibre bundles, including vector bundles, dual bundles and tensor product bundles. From there we will develop the notion of connections on manifolds with particular care taken to explain Ehresmann connections and horizontal lifts. These will provide us with ordinary differential equations that describe our system's holonomy. Chapter 2 will conclude with a discussion on what parallel transport is as well as a definition for holonomy, and finally a method for recovering a connection given information on how geometric objects change as they are parallel transported.

Chapter 3 is a set of instructions on how to build our geometric model for any given finite dimensional Hamiltonian. It starts off by establishing what geometric and Berry phases are as well as introducing the adiabatic theorem [5]. We will also walk through an example of calculating the Berry phase, by solving the Schrödinger equation. We then show how one can extract all information needed to construct a vector bundle from a family of locally defined n-dimensional Hamiltonians. We define the base manifold (or parameter space) as all points in which our local Hamiltonians are non-degenerate, our fibre is defined as the n-dimensional Hilbert space that our Hamiltonians act on, the structure group is $U(n)$, and the transition maps are taken to be outer products of energy eigenstates.

We then discuss our chosen local fibre coodinates which we have taken to be the energy eigenstates. This means that our local coordinates will vary from point to point over our base manifold. Using this variable coordinate system we show that horizontal lifts behave in the exact same way they do for fixed a coordinate system. Then using the results from the Berry phase example, we will derive the Berry connection using the connection recovery method discussed at the end of Chapter 2. Finally, we determine the global curvature 2-form from the local Berry connection. We then use the curvature to define the characteristic Chern class of our vector bundle. We are interested in the Chern class because it can be used to distinguish our bundle, but also it can be used to determine the Chern number. The curvature and Chern number are important quantities in physics, because they are observable [12, 27, 38]. We find that a quantum system will have a zero Chern number if its base manifold has dimension less than 4.

Chapter 4 is the first example of our model in use. We build the Berry bundle for a spin-1/2 system which we find has a parameter space given by $\mathbb{R}^3\backslash\{0\}$. However, the spin-1/2 holonomy is completely described by paths on S^2. Experimentally, this predicts that the magnitude of the magnetic field across the spin-1/2 particle, will have no influence on the accumulation of Berry phase. We also found that our model's holonomy agrees with experimentation [16, 22, 45], and that the Chern classes and Chern number vanish.

The aim of this thesis is to build a model for Berry phase in finite dimensional quantum systems. In particular we want to study the holonomy of the nitrogen vacancy (NV) center which is a spin 1-system with many attractive features that make it ideal for experimentation in nuclear magnetic resonance [7, 10], and a candidate for a qubit in quantum computation [47, 50]. Because of the many uses of the NV center it is important to have a solid theoretical

framework to describe its behaviour. Not only could our model be used to understand the NV center's Berry phase, but the methods discussed in Chapter 3 can be applied to any finite dimensional quantum system. This model was specifically designed such that smooth paths in the parameter space would physically correspond to an experimenter tuning parameters of the Hamiltonian, making it easier to translate between theory and practice. Furthermore, the model is adaptable if one was interested in studying non-adiabatic processes. As in [48], one could replace the Berry connection with a connection that mixes eigenstates and thus breaks the adiabatic theorem.

In Chapter 5 we study the nitrogen vacancy center. First, we establish the Hamiltonian for a spin-1 system in a magnetic field and with two incident microwaves. Then, by applying the rotating wave approximation, we simplify our system and derive our local Hamiltonian. We will use the characteristic polynomial of this Hamiltonian to determine that the parameter space of the NV center is $\mathbb{R}^5$ excluding two 2-dimensional cones (5.13). From there we determine the general energy eigenvalues, eigenstates and the local Berry connections. With just the parameter space and Berry connection, one can calculate the Berry phase for any piecewise smooth path in the parameter space. After establishing the NV center's Berry bundle we study some of its observable features: Because the Berry connection is a function of ratios in the field strengths, our parameter space can effectively be described by a real 4-dimensional manifold. Experimentally, this means that large magnetic fields are not needed to explore the NV center's Berry phase. The cones (5.13) produce observable, path dependent, discontinuities in the Berry phase. We also show that the holonomy of the NV center is $U(1) \otimes U(1)$, by giving a method for inducing any Berry phase. Finally, we determine the Chern class and show that it does not vanish.

Note that the material in Chapter 2 is presented in a way approachable for a Master's level student, who has some experience with geometry. Further information about the topics covered in Chapter 2 can be found in the following textbooks [17, 21, 35]. It is recommended that those unfamiliar with the subjects covered in this thesis, work through these texts alongside Chapter 2. In section 2.5 we will introduce the notion of local horizontal lifts for vector bundles. Because the derivation of a local horizontal lift cannot be found in the cited textbooks, we will provide a full derivation using only projection and linearity arguments. This is a fundamental piece of our model.

In Chapter 3 we will extend upon the topics in Chapter 2 beyond what is covered in the textbooks [17, 21, 35]. In particular, we will show how to extract the necessary information from a practical setup, to build a vector bundle. We will also take local energy eigenstates as our basis on our fibre. This means that we will be using local coordinates that smoothly vary from point to point in the manifold. This smooth variation in our local coordinates will need to be properly accounted for when considering the tangent vectors of our bundle.

In the conclusion we will summarise the topics and ideas covered throughout this thesis as well as reiterating the important aspects of our model. We will also review our results from the spin-1/2 system and the NV center.

Chapter 2

Background

This chapter will cover the necessary background in differential geometry to understand the later chapters. We will study fibre bundles and how to construct them. Then move on to connections, in particular the Ehresmann connection, and local horizontal lifts. This will lead into a short digression on Lie algebras, as well as a discussion on parallel transport. Finally, we will define holonomy, and introduce a method for recovering a connection from information on how objects are parallel transported. For further information on the topics covered in this chapter, see the following textbooks [35, Chapters 5, 7, 9 and 10], [17, Chapters 16, 17 and 18] and [21, Chapters 5 and 6].

2.1 Coordinate fibre bundles

Complicated structures often appear in geometry and physics. A common method for tackling problems on these structures is to break them apart into simpler spaces. This is the idea of a *manifold*: An $m-$dimensional manifold, M, is defined as a topological space with an open cover $\{U_\alpha\}_\alpha$ and homeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{R}^m$. We call the tuple (U_α, ϕ_α) a chart of M with domain U_α, and we call the collection of charts $\{(U_\alpha, \phi_\alpha)\}_\alpha$ an atlas of M.

We are interested in *smooth manifolds* which have the added condition that on overlaps $U_\alpha \cap U_\beta \neq \emptyset$ the maps

$$\begin{aligned} \phi_\beta \circ \phi_\alpha^{-1}\big|_{\phi_\alpha(U_\alpha \cap U_\beta)} &: \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^m, \\ \phi_\alpha \circ \phi_\beta^{-1}\Big|_{\phi_\beta(U_\alpha \cap U_\beta)} &: \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^m, \end{aligned} \tag{2.1}$$

are smooth. In this work we will consider all manifolds to be smooth unless stated otherwise.

If we have a function $f : M \to \mathbb{R}^k$ for some $k \in \mathbb{Z}_{>0}$, then we say f is smooth if

$$f|_{U_\alpha} \circ \phi_\alpha^{-1}\big|_{\phi_\alpha(U_\alpha)} : \phi_\alpha(U_\alpha) \to \mathbb{R}^k, \tag{2.2}$$

is smooth for all U_α in our cover of M. Furthermore, if a function f on M is a homeomorphism and both f and f^{-1} are smooth, then we call f a *diffeomorphism*.

In some case, we have two spaces that we want to stitch together. For example the surface of the Earth and space of all wind velocities. To attach complicated structures like these together we require a definition of *coordinate fibre bundles*. A *coordinate fibre bundle* (E,π,M,F,G) consists of the following components and properties:

- Differentiable manifolds E, M and F called the *total space*, *base space* and *fibre*, respectively.
- A surjective map $\pi : E \to M$ called the projection. We call the preimage $\pi^{-1}(p) = F_p$ the fibre at p.

- The maps $\omega_{\alpha,p} : F \to F_p : f \mapsto \omega_\alpha(p, f)$ are diffeomorphisms.
- A Lie group G which has a left action on F. We call G the structure group.
- An open covering $\{U_\alpha\}_{\alpha\in I}$ of M with diffeomorphisms $\omega_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ such that $\pi(\omega_\alpha(p, f)) = p$. We call ω_α a local trivialisation, and together with the open cover of M and the fibre F, they will form an atlas $\{U_\alpha \times F, \omega_\alpha\}_{\alpha\in I}$ for E.
- Furthermore, on nonempty overlaps, $U_\alpha \cap U_\beta$, we require $t_{\alpha\beta}(p) = \omega^{-1}_{\alpha,p} \circ \omega_{\beta,p} : F \to F$ be an element of G, and $\omega_\beta(p, f) = \omega_\alpha(p, t_{\alpha\beta}(p)f)$ for all $f \in F$. Note that it is clear by the definition, that $t_{\alpha\beta}$ is a smooth map. We call these the transition maps.

In our example above, the surface of the Earth is our base manifold and the space of wind velocities is our fibre. We will only be interested in coordinate fibre bundles, so we will refer to these as just fibre bundles.

Convention. We will use Greek letters (μ, ν, κ, η) to be the indices of our base space M and Latin letters (i, j, k, ℓ) to be the indices of our fibre F.

By definition, the transition functions satisfy the following consistency conditions,

$$\begin{aligned} t_{\alpha\alpha}(p) &= \text{identity map } (p \in U_\alpha), \\ t_{\alpha\beta}(p) &= t^{-1}_{\beta\alpha}(p) \quad (p \in U_\alpha \cap U_\beta), \\ t_{\alpha\beta}(p)t_{\beta\kappa}(p) &= t_{\alpha\kappa}(p) \quad (p \in U_\alpha \cap U_\beta \cap U_\kappa). \end{aligned} \tag{2.3}$$

The second condition ensures the local pieces of a fibre bundle, $\pi^{-1}(U_\alpha)$, are glued together in a way such that our trivialisations agree on overlaps. The third condition is an associativity condition; it tells us that for triple overlaps $U_\alpha \cap U_\beta \cap U_\kappa \neq \emptyset$ the ordering of gluing $U_\alpha \times F$, $U_\beta \times F$ and $U_\kappa \times F$ does not matter. If all transition functions can be taken as identity maps, then the fibre bundle is called a trivial bundle and is diffeomorphic to $M \times F$. As we will soon see in section 2.2 the transition functions encode all the information on how our fibre twists and changes from chart to chart.

It is important to note that the set of transition functions on a fibre bundle is not necessarily unique. Let $\{U_\alpha\}$ be an open covering of M and $\{\omega_\alpha\}, \{\hat{\omega}_\alpha\}$ be two sets of local trivialisations. Define the map for each $p \in M$

$$g_\alpha(p) = \omega^{-1}_{\alpha,p} \circ \hat{\omega}_{\alpha,p} : F \to F \tag{2.4}$$

These $g_\alpha(p)$ maps take values in the structure group G, and since $\omega_{\alpha,p}$ and $\hat{\omega}_{\alpha,p}$ are diffeomorphisms, then so is $g_\alpha(p)$.

The $g_\alpha(p)$'s provide a way to change our coordinates from one trivialisation into another,

$$\begin{aligned} g_\alpha(p)^{-1} \circ t_{\alpha\beta}(p) \circ g_\beta(p) &= \hat{\omega}^{-1}_{\alpha,p} \circ \omega_{\alpha,p} \circ \omega^{-1}_{\alpha,p} \circ \omega_{\beta,p} \circ \omega^{-1}_{\beta,p} \circ \hat{\omega}_{\beta,p} = \hat{\omega}^{-1}_{\alpha,p} \circ \hat{\omega}_{\beta,p}, \\ \implies g_\alpha(p)^{-1} \circ t_{\alpha\beta}(p) \circ g_\beta(p) &= \hat{t}_{\alpha\beta}(p). \end{aligned} \tag{2.5}$$

Physically, the transition maps $t_{\alpha\beta}$ are the gauge transformations of our system, while g_α corresponds to gauge degrees of freedom in the chart U_α [35, Section 9.2].

For a fibre bundle (E, π, M, F, G) we may want to use our base manifold as the domain of a function, whose codomain is the fibre. Consider again, our fibre bundle example with base manifold given by the surface of the Earth and fibre given by the space of wind velocities. We may wish to know how windy it is in a particular town, the function that takes the town's coordinates and returns the wind velocity at those coordinates describes what we call a *section*.

We define a *section* as follows: Given a fibre bundle (E, π, M, F, G), a section $s : M \to E$ is a smooth map such that $\pi \circ s = \mathrm{id}_M$. We denote by $\Gamma(M, F)$ the set of sections on M. We may also be interested in local sections; if $U \subseteq M$ then a section local to U is one that

is defined only on U or restricted to U. For some domain $\pi^{-1}(U_\alpha) \subset E$ with trivialisation ω_α, a local section $\sigma|_{U_\alpha}$ takes the form

$$\sigma|_{U_\alpha}(p) = \omega_\alpha(p; \sigma^1(p), \ldots, \sigma^n(p)), \tag{2.6}$$

where each $\sigma^i : M \to F$ is a smooth map. It is also important to note that in order for a section to be consistent over multiple domains in the open cover of M, we require for all $p \in U_\alpha \cap U_\beta$

$$\sigma|_{U_\alpha}(p) = t_{\alpha\beta}(p)\sigma|_{U_\beta}(p). \tag{2.7}$$

Notice in (2.6) that we locally pair our input value p with smooth maps σ^i on our fibre. This is based on how we visualise functions as graphs. For example a parabola has the graph $\{(x, x^2) : x \in \mathbb{R}\}$.

2.2 Constructing fibre bundles

When constructing a fibre bundle, the least amount of information we need is a base space M (which comes with an atlas and thus an open cover $\{U_\alpha\}$), a fibre F, a structure group G, and finally transition functions $t_{\alpha\beta}(p) : F \to F$ obeying the consistency conditions (2.3). Once we have these pieces we attach a fibre to each open set in our cover of M

$$X = \coprod_\alpha (U_\alpha \times F). \tag{2.8}$$

We then identify all the points we want to glue together by introducing the equivalence class on X: $(p, f) \sim (q, h)$ if and only if $p = q$ and there exists a transition function $t_{\alpha\beta}$ such that $h = t_{\alpha\beta}(p)f$.

It is a good exercise in using the consistency equations (2.3) to show that $\sim$ is an equivalence relation.

$(p, f) \sim (p, f)$ because $t_{\alpha\alpha} = \mathrm{id}$.
$(p, f) \sim (q, h) \implies p = q$ and $h = t_{\alpha\beta}(p)f$, thus we have,
$f = t_{\alpha\beta}^{-1}(p)h = t_{\beta\alpha}(p)h \implies (q, h) \sim (p, f)$.
$(p, f) \sim (q, h)$ and $(q, h) \sim (r, k) \implies p = q = r$,
$h = t_{\alpha\beta}(p)f$ and $k = t_{\kappa\alpha}(p)h$. Thus $k = t_{\kappa\alpha}(p)t_{\alpha\beta}(p)f = t_{\kappa\beta}(p)f \implies (p, f) \sim (r, k)$.

We then glue all equivalent points together by taking the quotient

$$E = X/\sim . \tag{2.9}$$

Finally, we define a projection and local trivialisations

$$\begin{aligned} &\pi : E \to M, \quad [(p, f)] \mapsto p, \\ &\omega_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha) = (U_\alpha \times F)/\sim, \ \ (p, f) \mapsto [(p, f)]. \end{aligned} \tag{2.10}$$

Thus, we have the fibre bundle (E, π, M, F, G).

Let us now show that π and our trivialisations satisfy the conditions in our definition of a fibre bundle given in section 2.1. That is to say, we need to show the following three conditions hold:

- For all $p \in M$ and for all α, $\omega_{\alpha,p} : F \to F_p$ is a diffeomorphism,
- π is a surjection with $\pi(\omega_\alpha(p, f)) = p$,
- E is a smooth manifold,

First note that if $f, h \in F$ such that there exists transition map $t_{\alpha\beta}$ with $f = t_{\alpha\beta}(p)h$ then $\omega_\alpha(p, f) = [(p, f)]$ implies that $\omega_\beta(p, h) = [(p, h)] = [(p, f)]$. However, this does not imply that $\omega_\alpha(p, h) = [(p, f)]$ unless $t_{\alpha\beta}(p) = id$. We will use this fact to show that ω_α is bijective: Take $p \in U_\alpha$ and $f \in F$. Then,

$$\begin{aligned}\omega_\alpha^{-1}\Big([(p, f)]\Big) &= \Big\{(p, h) \in M \times F : \exists\ t_{\alpha\beta}(p) \text{ such that } f = t_{\alpha\beta}(p)h \\ &\qquad\qquad \text{and } \omega_\alpha(p, h) = [(p, h)]\Big\}, \\ &= \{(p, f) \in E\}.\end{aligned}$$

Furthermore, for all $(p, f) \in U_\alpha \cap U_\beta \neq \emptyset$

$$\omega_\beta^{-1} \circ \omega_\alpha(p, f) = (p, t_{\alpha\beta}(p)f).$$

This is a smooth map because the identity and transition maps are smooth. Thus, for all $p \in U_\alpha$ the localisation $\omega_{\alpha,p}$ is a diffeomorphisms as per the first bullet.

The second bullet is true by definition of π, and $\pi\big(\omega_\alpha(p, f)\big) = \pi\big([(p, f)]\big) = p$. The last bullet point is covered by Steenrod in the text book, The topology of fibre bundles [43], for the case of continuous transition maps. This proof can be extended to smooth manifolds by enforcing that the $\{t_{\alpha\beta}\}$ are smooth. Hence, (E, π, M, F, G) with trivialisations $\{\omega_\alpha\}_\alpha$ forms a fibre bundle.

2.3 Examples of fibre bundles

Recall our example of a fibre bundle with the surface of the Earth for a base manifold and fibre given by the space of all wind velocities. This fibre is a vector space, so we call this type of fibre bundle a *vector bundle*. Let us make a proper definition: A *vector bundle* is a fibre bundle whose fibre is a vector space V with transition functions in $GL(V)$. For example $V = \mathbb{R}^k$ has transition functions in $GL(k, \mathbb{R})$. When the fibre of our vector bundle E, is $1-$dimensional, then we call E a line bundle.

Note: Consider a vector bundle E, with base space M and fibre V of dimension k. If $p \in M$, then $\pi^{-1}(p)$ is the $k-$dimensional vector space, $\{p\} \times V$. However, we do not identify $\pi^{-1}(p)$ with the standard $\mathbb{R}^k$ until the domain of M containing p, is specified. When a domain U_α that contains p is specified we can then use that domain's corresponding local trivialisation to identify a basis of V, which gives us a particular isomorphism $\pi^{-1}(p) \simeq \mathbb{R}^k$.

Example 2.3.1. A vector bundle with $m-$dimensional base manifold M, embedded in $\mathbb{R}^n$, and with fibre above any point $p \in M$ described by

$$\pi^{-1}(p) = \{v \in \mathbb{R}^n : v \text{ is tangential to } M \text{ at } p\}, \tag{2.11}$$

is called a tangent bundle and is denoted by TM.
The sections of TM are tangential vector fields on M. Take a vector field $X : M \to TM$, and domain U_α with local coordinates $\{x^\mu\}$. We can expand $X|_{U_\alpha}$ into its basis components,

$$X|_{U_\alpha}(x^1, \ldots, x^m) = X^\mu(x^1, \ldots, x^m)\ \frac{\partial}{\partial x^\mu}\bigg|_{(x^1, \ldots, x^m)}. \tag{2.12}$$

Now say that we have an overlapping domain U_β with local coordinates $\{y^\mu\}$. This means that $X|_{U_\alpha}$ and $X|_{U_\beta}$ describe the same local vector field on $U_\alpha \cap U_\beta \neq \emptyset$ and

therefore they must agree for all $p \in U_\alpha \cap U_\beta$. That is to say

$$\begin{aligned} (X|_{U_\alpha}(p))^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p &= (X|_{U_\beta}(p))^\nu \left.\frac{\partial}{\partial y^\nu}\right|_p, \\ \implies (X|_{U_\beta}(p))^\nu &= (X|_{U_\alpha}(p))^\mu \left.\frac{\partial y^\nu}{\partial x^\mu}\right|_p. \end{aligned} \tag{2.13}$$

This implies that our transition functions are Jacobians $t_{\alpha\beta}(p) = \left.\frac{\partial y}{\partial x}\right|_p$.
We can also consider the cotangent bundle T^*M, whose local sections are expanded as $1-$forms $\sigma|_{U_\alpha} = \sigma_\mu|_{U_\alpha} dx^\mu$. Thus, on $U_\alpha \cap U_\beta$

$$\begin{aligned} &(\sigma|_{U_\alpha})_\mu dx^\mu = (\sigma|_{U_\beta})_\nu dy^\nu, \\ \implies &(\sigma|_{U_\alpha})_\nu = (\sigma|_{U_\beta})_\mu \frac{\partial x^\nu}{\partial y^\mu} = (\sigma|_{U_\beta})_\mu \left(\left(\frac{\partial x}{\partial y}\right)^T\right)^\mu_{\ \nu}. \end{aligned} \tag{2.14}$$

So we find that $t_{\alpha\beta}(p) = \left.(\frac{\partial x}{\partial y})^T\right|_p = \left.([\frac{\partial y}{\partial x}]^{-1})^T\right|_p$.

Because we are interested in describing our model with a vector bundle, we will want to make use of the duals to our vectors. This means we will need *dual bundles* in our model, so it is good idea to make a proper definition. Let (E, π, M, F, G) be a vector bundle with vector space fibre V, base manifold M with open cover $\{U_\alpha\}$, and transition maps $\{t_{\alpha\beta} : V \to V\}_{\alpha,\beta}$. We define E's *dual bundle* (E^*, π^*, M, V^*, G) with

- fibre given by the dual of V; V^*,
- base manifold M with open cover $\{U_\alpha\}$,
- transition maps $\{t^*_{\alpha\beta} = t^T_{\beta\alpha} : V^* \to V^*\}_{\alpha,\beta}$.

Here the construction method in section 2.2 is used to build the dual bundle from this information.

Another set we are interested in when dealing with a vector space V, is the set of endomorphisms $\mathrm{End}(V)$. To understand where the endomorphisms in our model live, we will need to define *tensor product bundles*. Let $(E, \pi, M, V, GL(G))$ and $(E', \pi', M, V', GL(G'))$ be vector bundles. Choose the same open cover $\{U_\alpha\}_\alpha$ of the base manifold M for both bundles. Then let $\{t_{\alpha\beta} : V \to V\}_{\alpha,\beta}$ and $\{t'_{\alpha\beta} : V' \to V'\}_{\alpha,\beta}$ be the set of transition maps for E and E' respectively.

We define the *tensor product bundle* $\hat{\pi} : E \otimes E' \to M$ as a fibre bundle with

- fibre $V \otimes V'$,
- base manifold M with open cover $\{U_\alpha\}$,
- structure group $GL(G) \otimes GL(G')$,
- transition maps $\{t_{\alpha\beta} \otimes t'_{\beta\alpha} : V \otimes V' \to V \otimes V'\}_{\alpha,\beta}$.

Again, the construction method of section 2.2 is used to build the tensor product bundle from this information. By taking $V' = V^*$, this will form a bundle whose fibre elements are endomorphisms on V.

Tensor product bundles can also be used to define an inner product on a vector bundle: Given a vector bundle (E, π, M, V, G) with an $n-$dimensional fibre V, we construct the tensor product bundle $(E \otimes E, \hat{\pi}, M, V \otimes V, G)$. Let $\widehat{\omega}_\alpha : U_\alpha \times V \otimes V \to \hat{\pi}^{-1}(U_\alpha)$ be a

trivialisation on our tensor product bundle. Take a basis $\{e_1, \ldots, e_n\}$ of V and define the linear map $\Psi : V \to V^*$ by

$$\Psi(e_i)(e_j) = \delta_{ij}, \tag{2.15}$$

for all $i, j \in \{1, \ldots, n\}$. Finally, we introduce our inner product on E by

$$\langle \cdot, \cdot \rangle : E \otimes E \to M \times \mathbb{C}, \;\; \widehat{\omega}(p, v \otimes w) \mapsto (p, \Psi(v)(w)), \tag{2.16}$$

for all $p \in M$.

A common method for modelling Berry phase is through the use of principal bundles [35, Section 10.6] [13, 20]. We will not use this method in our model, but an introduction on principal bundles will be provided in appendix A to help the reader translate between the methods in the cited texts and those in this thesis.

2.4 Connections, connections, and connections

A connection on a fibre bundle is a way of "connecting" fibres locally and thus allows us to transport objects like tangent vectors and tensors along a smooth curve in a consistent manner. This is a little bit mysterious now but we will make this clear in this section.

Why do we need a connection? Let us attempt to take the derivative of a vector field $\mathbf{V}(\mathbf{x}) = V^\mu(\mathbf{x})\mathbf{e}_\mu$ in $\mathbb{R}^m$ with respect to the ν^{th} coordinate, x^ν:

$$\lim_{\Delta x \to 0} \frac{V^\mu(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m) - V^\mu(\mathbf{x})}{\Delta x} e^\mu. \tag{2.17}$$

The first term in the numerator is a vector defined at the point $(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m)$, while second is a vector defined at $(x^1, \ldots, x^\nu, \ldots, x^m)$. In Euclidean space it is okay to just take the difference of these vector components because the tangent spaces are canonically identified. Over a manifold this is not always true. To subtract two vectors in different tangent spaces we need to transport one to the other. So in the above case we need to transport the second vector to $(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m)$ before we can take their difference. We call this transport of our vector, parallel transport.

On a manifold there is no natural way to transport tangent vectors from point to point, so we need to come up with rules based on what we are trying to transport. Take M to be an $m-$dimensional manifold and let $\mathbf{V}$ be a vector field on M. Let $\widehat{\mathbf{V}}(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m)$ denote our vector $\mathbf{V}(\mathbf{x})$ parallel transported from $\mathbf{x} = (x^1, \ldots, x^m)$ to $(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m)$. We should aim to have our parallel transport preserve vector structure. To do this we enforce that our parallel transport is linear with respect to its displacement and that it preserves vector addition;

$$\widehat{V}^\nu(x^1, \ldots, x^\nu + \delta x, \ldots, x^m) - V^\nu(x^1, \ldots, x^\nu + \delta x, \ldots, x^m) = \delta x,$$
$$\widehat{(\mathbf{V} + \mathbf{W})}(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m) = \widehat{\mathbf{V}}(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m) + \widehat{\mathbf{W}}(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m).$$

As we show in section 2.5, these conditions are met if

$$\hat{V}^\mu(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m) = V^\mu(\mathbf{x}) - V^\lambda(\mathbf{x}) A_\lambda{}^\mu{}_\nu(\mathbf{x}) \Delta \mathbf{x}^\nu, \tag{2.18}$$

where $\Delta \mathbf{x} = (0, \ldots, \Delta x, \ldots, 0)$ in this example, and $A_\lambda{}^\mu{}_\nu(\mathbf{x})$ are known as connection coefficients.

Thus a derivative of the vector field V in the $\frac{\partial}{\partial x^\nu}$ direction, better known as a covariant derivative of V with respect to $\frac{\partial}{\partial x^\nu}$, is defined by

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{V^\mu(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m) - \hat{V}^\mu(x^1, \ldots, x^\nu + \Delta x, \ldots, x^m)}{\Delta x} \frac{\partial}{\partial x^\mu} \\ = \left(\frac{\partial V^\mu}{\partial x^\nu}\bigg|_{\mathbf{x}} + V^\lambda(\mathbf{x}) A_\lambda{}^\mu{}_\nu(\mathbf{x}) \right) \frac{\partial}{\partial x^\mu}\bigg|_{\mathbf{x}}. \end{aligned} \tag{2.19}$$

We need a connection because there is no way to naturally move geometric objects from fibre to fibre on a bundle. If we define two different sets of trivialisations for our bundle and then use them to naively take derivatives of a vector field as we would in Euclidean space, then in general we will find contradictory results. This has physical ramifications, for example on a tangent bundle this would mean that two inertial observers would measure different velocities even when taking relativity into account.

The best we can do is specify rules we want satisfied when moving these objects around our manifold, and create a connection A to enforce them. Two examples of rules we could have our connection enforce are preservation of norms, and preservation of angles.

There are a lot of different types of connections. *Affine connections*, or covariant derivatives, hold great importance in general relativity and are often the first type of connection that one will encounter. Let M be a manifold and take $f \in C^\infty(M)$ and $X, Y \in \Gamma(M, TM)$. An *affine connection* ∇ is a map $\nabla : \Gamma(M,TM) \times \Gamma(M,TM) \to \Gamma(M,TM) : (X,Y) \mapsto \nabla_X(Y)$ satisfying

$$\nabla_X(Y+W) = \nabla_X(Y) + \nabla_X(W), \tag{2.20}$$

$$\nabla_{X+Y}(W) = \nabla_X(W) + \nabla_Y(W), \tag{2.21}$$

$$\nabla_{fX}(Y) = f\nabla_X(Y), \tag{2.22}$$

$$\nabla_X(fY) = (X(f))Y + f\nabla_X(Y). \tag{2.23}$$

We can also study an affine connection locally by defining *affine connection coefficients*. For any chart (U,ω) of an $m-$dimensional manifold M, with local coordinates $\{x^\mu\}$, we define the m^3 *connection coefficients* ${A_\mu}^\lambda{}_\nu$ by

$$\nabla_{\frac{\partial}{\partial x^\nu}} \left(\frac{\partial}{\partial x^\mu}\bigg|_p \right) = \nabla_\nu \left(\frac{\partial}{\partial x^\mu}\bigg|_p \right) = {A_\nu}^\lambda{}_\mu(p) \, \frac{\partial}{\partial x^\lambda}\bigg|_p , \tag{2.24}$$

for all $p \in U$.

These connection coefficients are examples of what we were talking about earlier in (2.19). They describe how our basis vectors in each T_pM rotate and change as they are transported an infinitesimal distance. By defining how we want our basis vectors in T_pM to change as they move from p to $q \in M$, we define the action of our affine connection on general vector fields $X, Y \in \Gamma(M,TM)$. Restricting to a domain U and using (2.20)-(2.23) we get

$$\nabla_X(Y)(p) = X^\nu \left(\frac{\partial Y^\lambda}{\partial x^\nu}\bigg|_p + {A_\nu}^\lambda{}_\mu(p) Y^\mu \right) \frac{\partial}{\partial x^\lambda}\bigg|_p , \tag{2.25}$$

for all $p \in U$.

Affine connections are used to locally connect tangent spaces. This allows us to generalise directional derivatives to covariant derivatives which act on tangent vectors. It is precisely this that allows us to parallel transport a tangent vector around our manifold. We will explore this further in section 2.8, but first we require a more general notion of a connection if we are to prescribe one to a general fibre bundle.

2.5 Ehresmann connection and the horizontal lift

Affine connections are useful, but we need something a bit more general that will work for more than just the tangent bundle. We need a notion of connections and parallel transport that works for a general fibre bundles, in particular, vector bundles.

First we define *vertical* and *horizontal subspaces* of the tangent spaces belonging to a fibre bundle. Let (E,π,M,F,G) be a fibre bundle and let $b \in E$. The *vertical subspace* at b, $V_bE \subset T_bE$, is set of vectors tangent to the fibre $F_{\pi(b)}$. A *horizontal subspace* at b, $H_bE \subset T_bE$, is defined as a complement of V_bE and therefore the *vertical* and *horizontal subspaces* of T_bE satisfy the property $V_bE \bigoplus H_bE \simeq T_bE$.

Furthermore, we define the equivalence relation on $\coprod_{b\in E}\{b\}\times T_bE$ given by

$$(b, X) \sim (b', Y),$$

if and only if $b = b'$ and there exists a transition function $t_{ij}(b) : T_bE \to T_bE$ such that

$$X = t_{ij}(b)Y.$$

Note that the transition functions on TE are different to those on E and are usually represented by Jacobians. Using the construction method in section 2.2 we define the *vertical bundle*:

$$VE = \coprod_{b\in E}\{b\}\times V_bE/\sim . \tag{2.26}$$

Because VE inherits its fibre, structure group, transition maps, and trivialisations from TE, then VE is a sub-bundle of TE.

We now define an *Ehresmann connection* on a fibre bundle (E, π, M, F, G) as a smooth projection $\xi : TE \to VE$, to the vertical bundle. This means that an *Ehresmann connection* satisfies

$$\xi\circ\xi = \xi, \quad \implies \quad \xi|_{VE} = \mathrm{id}_{VE}. \tag{2.27}$$

With a prescribed Ehresmann connection we have a unique horizontal bundle,

$$HE := \ker(\xi). \tag{2.28}$$

By this definition, H_bE depends smoothly on $b \in E$ and thus forms a sub-bundle of TE. Furthermore, because the fibres V_bE and H_bE are smooth in b, then for all vector fields $X \in TE$ there exists $X^H \in HE$ and $X^V \in VE$ with $X = X^V + X^H$.

An Ehresmann connection ξ provides us a way to parallel transport vectors in E about smooth paths $\gamma : [0, 1] \to M$. We do this by finding a new curve, $\gamma^\#$ in our bundle E, that projects to γ and whose tangent vector field $X^\#(t)$, is horizontal for all $t \in [0, 1]$. That is to say $\xi\big(X^\#(t)\big) = 0$ for all $t \in [0, 1]$. Imposing these conditions will provide us with a system linear ODEs for the components of $\gamma^\#$. If we also impose an initial condition, $\gamma^\#(0) = b$, we can make use of the Picard–Lindelöf theorem [9, Sections 1.1 and 1.2] to guarantee the existence and uniqueness of $\gamma^\#$. We call this the horizontal lift of γ to E starting at b.

The Picard–Lindelöf theorem provides us with a set of conditions that an initial value problem must satisfy in order to have a unique solution: Let $D \subseteq \mathbb{R}\times\mathbb{R}^n$ be a closed rectangle with (t_0, y_0) in the interior of D. Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \tag{2.29}$$

with

$$f : D \to \mathbb{R}^n. \tag{2.30}$$

If f is continuous in t and Lipschitz continuous in y, then there exists some $\epsilon > 0$ such that the initial value problem (2.29) has a unique solution

$$y(t) \quad \text{for all } t \in [t_0 - \epsilon, t_0 + \epsilon]. \tag{2.31}$$

Note that Lipschitz continuous is implied by smoothness. This means that the Picard–Lindelöf theorem will always hold for what we are interested in.

As before with an affine connection, a horizontal sub-bundle is not unique, so we define an Ehresmann connection, ξ, by forming a set of rules we wish parallel transport to obey. The condition $\xi\big(X^\#(t)\big) = 0$ enforces these rules onto $\gamma^\#$. Another useful property of an affine connection was the ability to localise it by obtaining the local connection coefficients. This also translates to an Ehresmann connection. For a fibre bundle (E, π, M, G, F) with atlas $\Big\{\big(U_\alpha\times F, \omega_\alpha\big)\Big\}$, and Ehresmann connection ξ, we define a *local connection* 1*-form* as

$$A^\alpha = d\omega_\alpha^{-1}\circ\omega_\alpha^*\big(\xi\big) : T(U_\alpha\times F) \to T(U_\alpha\times F) \tag{2.32}$$

The exact reasons why an Ehresmann connection defines local connection 1-forms will be explained at the end of the section.

In physics, we often prefer a local picture of differential geometry where local connection $1-$forms are better known as gauge fields. For this reason we will now establish a set of local coordinates for a vector bundle. In the definition of a vector bundle E, we choose an atlas $\{V_\alpha \times F, \omega_\alpha\}$ for E, where $\{V_\alpha\}$ provides an open covering of our base space M. Because M is a manifold, a very natural choice for this covering is the domains in the atlas we chose for M, $\{(U_\alpha, \phi_\alpha)\}$. We can use the trivialisations

$$\omega_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha),$$

to provide local coordinates on our fibre: Define local basis sections on E at $p \in U_\alpha$ as,

$$\iota_{\alpha,i}(p) = \omega_{\alpha,p}^{-1}(e_i) = \omega_\alpha^{-1}(p, e_i), \tag{2.33}$$

where $\omega_{\alpha,p} = \omega_\alpha|_{\pi^{-1}(p)}$ and e_i is a basis on F. This means that for all $p \in U_\alpha$, any vector $v \in E_p$ can be expanded as

$$v = v^i \iota_{\alpha,i}(p). \tag{2.34}$$

If we choose our covering of E to be $\{U_\alpha \times F, \omega_\alpha\}$ then we may also use the trivialisations of U_α,

$$\begin{aligned} &\hat{\phi}_\alpha : \mathbb{R}^m \times F \to U_\alpha \times F, \;\; (x, v) \mapsto (\phi_\alpha(x), v), \\ &\omega_\alpha \circ \hat{\phi}_\alpha : \mathbb{R}^m \times F \to \pi^{-1}(U_\alpha). \end{aligned} \tag{2.35}$$

Then for all $p \in U_\alpha$, this gives us a set local coordinates $(x^\mu; v^i)$ for E_p, and local coordinates $(\frac{\partial}{\partial x^\mu}\big|_p ; \frac{\partial}{\partial v^i}\big|_p)$ for T_pE.

Using our chosen local coordinates, we will develop a local horizontal lift for vector bundles. Our goal is to take some smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = p$, and lift it to a smooth curve $\gamma^\# : [0,1] \to E$. We also impose that the curve $\gamma^\#$, must satisfy $\gamma^\#(0) = b$, and has tangent field $X^\#(t) \in HE$ for all $t \in [0,1]$. This can be done uniquely with the use of the Picard–Lindelöf theorem [9, Sections 1.1 and 1.2] and by imposing a first order linear ODE with an initial condition

$$\dot{\gamma}^\#(t) = \mathcal{G}\Big(\gamma^\#(t), \dot{\gamma}(t)\Big), \qquad \gamma^\#(0) = b. \tag{2.36}$$

What we will find at the end of this derivation, is that $\mathcal{G}$ provides the local IVPs (2.66).

Notice that the map $\mathcal{G}$ takes elements of $F_p \times T_pM$ to T_bE. Hence, we define the horizontal lift of the vector $X \in T_pM$ tangent to $\gamma^\#(t)$ at $t = 0$ by

$$\mathcal{G}\Big(\gamma^\#(0), X\Big) = \mathcal{G}(b, X) = X_b^\#. \tag{2.37}$$

The horizontal lift of vectors tangent to our base manifold, gives us another way of defining horizontal subspaces:

$$H_bE = \Big\{X_b^\# : X \in T_{\pi(b)}M\Big\}. \tag{2.38}$$

This means that we can separate a tangent vector $Y \in T_bE$ into the horizontal lift $X_b^\#$, of a tangent vector of our base manifold M, and a vertical vector, $Y - X_b^\#$. By the end of this section, we will show that the definition (2.38), is consistent with the kernel of an Ehresmann connection

Using our local coordinates we can expand any tangent vector of $\pi^{-1}(U_\alpha) \subset E$ in terms of our local basis vectors $\big\{\frac{\partial}{\partial x^\mu}; \frac{\partial}{\partial v^i}\big\}$. Therefore for any $p \in U_\alpha$ the horizontal lift of $X \in T_pM$ at $b \in \pi^{-1}(p)$ takes the local form

$$d\omega_\alpha^{-1} \circ \mathcal{G}\big(b, X\big) = Y^\mu(b, X) \left.\frac{\partial}{\partial x^\mu}\right|_b + \mathcal{Z}^i(b, X) \left.\frac{\partial}{\partial v^i}\right|_b, \tag{2.39}$$

where $Y^\mu(b,X)$ and $\mathcal{Z}^i(b,X)$ are scalar functions $\pi^{-1}(q)\times T_qM\to\mathbb{C}$ for all $q\in U_\alpha$. Thus, for $\gamma^\#(t)=\omega_\alpha\big(\gamma(t);\lambda^i(t)e_i\big)$, the IVP (2.36) can be expressed locally as

$$\begin{aligned}&\frac{d\gamma^\mu}{dt}\frac{\partial}{\partial x^\mu}\bigg|_{\gamma^\#(t)}+\frac{d\lambda^i}{dt}\frac{\partial}{\partial v^i}\bigg|_{\gamma^\#(t)}=d\omega_\alpha^{-1}\mathcal{G}\big(\gamma^\#(t),\dot\gamma(t)\big),\\ \implies&\frac{d\gamma^\mu}{dt}=Y^\mu(\gamma^\#(t),\dot\gamma(t)),\quad \frac{d\lambda^i}{dt}=\mathcal{Z}^i(\gamma^\#(t),\dot\gamma(t)),\quad \gamma^\mu(0)=p^\mu,\quad \lambda^i(0)=b^i.\end{aligned}\tag{2.40}$$

Where $\mu=0,\dots,m$ and $i=1,\dots,n$.

To simplify the local IVPs (2.40) we will enforce four conditions onto $\mathcal{G}$. First we impose that $\gamma^\#$ projects to γ, that is

$$\pi\circ\gamma^\#=\gamma.\tag{2.41}$$

Then, for all $f\in C^\infty(M)$, (2.41) implies

$$\begin{aligned}&\frac{d}{dt}f\Big(\pi\circ\gamma^\#(t)\Big)\Big|_0=\frac{d}{dt}f\big(\gamma(t)\big)|_0,\\ \implies& X^\#(f\circ\pi)=X(f),\end{aligned}$$

which gives our first condition on a horizontal lift

$$d\pi\Big(\mathcal{G}(b,X)\Big)=X.\tag{2.42}$$

Notice that for all $p\in M$ and $b\in\pi^{-1}(p)$, the initial value problem (2.36) provides a map between vector spaces

$$\mathcal{G}_b:T_pM\to H_bE,\ \ \mathcal{G}_b(V)=\mathcal{G}(b,V).\tag{2.43}$$

We will enforce that this map is linear. This gives our second condition,

$$X_b^\#=\mathcal{G}_b(X)\text{ depends linearly on }X.\tag{2.44}$$

Immediately, conditions (2.44) and (2.42) tell us that if $b\in F_p$ and $X\in T_pM$ satisfy $\mathcal{G}_b(X)=X_b^\#=0$, then $X=d\pi\Big(X_b^\#\Big)=0$. Furthermore, if T_pM and H_bE are both finite dimensional vector spaces, then $\mathcal{G}_b$ is an isomorphism.

Now, for a given $X\in T_pM$, $\mathcal{G}(-,X)$ describes a map between two vector spaces. For this reason we enforce that $\mathcal{G}(b,X)$ is also linear in b. That is to say, we want the horizontal lift of a vector to be compatible with,

- Scalar multiplication: For $c\in\mathbb{C}$ and $b\in\pi^{-1}(p)\subset\pi^{-1}(U_\alpha)$, let

  $$m_c(b):=cb^i\iota_{\alpha,i}(p).$$

  So for all $c\in\mathbb{C}$, $b\in F_p$, $p\in M$ and $X\in T_pM$, we impose

  $$\mathcal{G}\big(m_c(b),X\big)=dm_c\Big(\mathcal{G}(b,X)\Big).\tag{2.45}$$

- Vector addition: For $\sigma\in\Gamma(M,E)$ and $b\in\pi^{-1}(p)\subset\pi^{-1}(U_\alpha)$, let

  $$S_\sigma(v):=(b^i+\sigma^i(p))\iota_{\alpha,i}(p).$$

  So for all $\sigma\in\Gamma(M,E)$, $b\in F_p\subset\pi^{-1}(U_\alpha)$ and $X\in T_pM$, we impose

  $$\mathcal{G}\big(S_\sigma(v),X\big)-dS_\sigma\Big(\mathcal{G}(b,X)\Big)=\mathcal{G}\big(\sigma(p),X\big)-d\sigma\big(X\big).\tag{2.46}$$

The equations (2.45) and (2.46) require some care to produce. For this reason we have provided full derivations in examples 2.5.1 and 2.5.2 respectively.

Consider the vector addition compatibility condition (2.46). Recall that $d\sigma : TM \to TE$ takes a vector that is tangential to some curve γ at p on our base space, and maps it to a vector of TE, tangential to the curve $\sigma \circ \gamma$ at $\sigma(p)$. Thus in general, $d\sigma(X)$ will have a vertical and horizontal component to it. Since σ is a section, then $d\pi \circ d\sigma(X) = X$. That is, the horizontal component of $d\sigma(X)$ is exactly $X^{\#}_{\sigma(p)}$. We will discuss how this leads to a covariant derivative on TE in section 2.8.

The conditions (2.45) and (2.46) are not very apparent, so let us spend some time on deriving them, before we proceed with constructing a horizontal lift on a vector bundle.

Example 2.5.1. Compatibility with scalar multiplication:
Take a smooth path $\gamma : [0,1] \to M$ with some starting point $\gamma(0) = p$. We want to find a curve $\gamma^{\#} : [0,1] \to E$, with $\gamma^{\#}(0) = b$ that projects to γ.
To do this uniquely, we develop a horizontal lift of a vector field, in order to define a linear IVP for $\gamma^{\#}$

$$\dot{\gamma}^{\#}(t) = X^{\#}_v = \mathcal{G}(v, X), \qquad \gamma^{\#}(0) = b, \tag{2.47}$$

where $v = \gamma^{\#}(t)$ and $X = \dot{\gamma}(t)$. We will assume that our path $\gamma(t)$ lies entirely within a domain $U_\alpha \subset M$, and thus we have local coordinates $(x^\mu; v^i)$.
We want to show that compatibility with scalar multiplication, gives us the equation (2.45). In this equation we have two horizontal vectors, so let us start by carefully defining the paths they are tangent to

$$\begin{aligned} X^{\#}_v &= \frac{d}{dt}(\gamma^{\#}(t)), \\ X^{\#}_{m_c(v)} &= \frac{d}{dt}(\gamma^{\#}_1(t)), \end{aligned}$$

which are described locally by

$$\begin{aligned} \gamma^{\#}(t) &= \omega_\alpha(\gamma(t);\, u^i(t)e_i), \\ \gamma^{\#}_1(t) &= \omega_\alpha(\gamma(t);\, u^i_1(t)e_i). \end{aligned}$$

Furthermore, each path satisfies the following properties

$$\begin{aligned} \gamma^{\#}(0) &= b, \\ \gamma^{\#}_1(0) &= m_c(b), \\ \pi \circ \gamma^{\#} &= \pi \circ \gamma^{\#}_1 = \gamma. \end{aligned}$$

Now take $f \in C^\infty(M \times F)$. Then for all $t \in [0,1]$

$$\begin{aligned} m_c(\gamma^{\#}(t)) &= \omega_\alpha\Big(\gamma(t);\, cu^i(t)e_i\Big), \\ \frac{d}{dt} f\left(\omega_\alpha^{-1} \circ m_c\Big(\gamma^{\#}(t)\Big)\right)\Big|_{t=0} &= \frac{d}{dt} f\Big(\gamma(t);\, cu^i(t)e_i\Big)\Big)|_{t=0}, \\ dm_c\big(X^{\#}_v\big) \cdot f &= \frac{d\gamma^\mu}{dt}\Big|_0 \frac{\partial f}{\partial x^\mu}\Big|_{m_c(b)} + \frac{d}{dt}\big(cu^i(t)\big)\Big|_0 \frac{\partial f}{\partial v^i}\Big|_{m_c(b)}. \end{aligned}$$

Let us take a closer look at the second term on the right hand side. Notice that

$$cu^i(0) = cb^i = u^i_1(0),$$

for all $i \in \{1, \dots, n\}$.
Recall that the coefficients of the fibre component of a horizontal lift, are determined via a linear ODE, $\mathcal{G}$. Further, $u^i(t)$ and $u_1^i(t)$ are solutions to our particular linear ODE but with different initial conditions. This implies that $cu^i(t)$ is also a solution, and by the Picard Lindelöf theorem [9, Sections 1.1 and 1.2]

$$cu^i(t) = u_1^i(t).$$

Thus, locally we have

$$dm_c\Big(X_b^{\#}\Big) \cdot f = \frac{d\gamma^\mu}{dt}\Big|_0 \frac{\partial f}{\partial x^\mu}\Big|_{m_c(b)} + \frac{d}{dt}\big(u_1^i(t)\big)\big|_0 \frac{\partial f}{\partial v^i}\Big|_{m_c(b)},$$

and since f was taken to be general we have the equality in horizontal vectors

$$dm_c\left(X_b^{\#}\right) = X_{m_c(b)}^{\#}.$$

Finally, this implies

$$\mathcal{G}\big(m_c(v), X\big) = dm_c\Big(\mathcal{G}(v, X)\Big). \tag{2.48}$$

Now based on the same set up we will use the same method to derive the vector addition condition (2.46).

Example 2.5.2. We want to show that compatibility with vector addition, gives us the equation (2.46). In this equation we have three horizontal vectors, so let us start by carefully defining the paths they are tangent to

$$\begin{aligned} X_b^{\#} &= \frac{d}{dt}\Big(\gamma^{\#}(t)\Big), \\ X_{S_\sigma(b)}^{\#} &= \frac{d}{dt}\Big(\gamma_1^{\#}(t)\Big), \\ X_{\sigma(p)}^{\#} &= \frac{d}{dt}\Big(\gamma_2^{\#}(t)\Big), \end{aligned}$$

which are described locally by

$$\begin{aligned} \gamma^{\#}(t) &= \omega_\alpha(\gamma(t);\, u^i(t)e_i), \\ \gamma_1^{\#}(t) &= \omega_\alpha(\gamma(t);\, u_1^i(t)e_i), \\ \gamma_2^{\#}(t) &= \omega_\alpha(\gamma(t);\, u_2^i(t)e_i). \end{aligned}$$

Furthermore, each path satisfies the following properties

$$\begin{aligned} \gamma^{\#}(0) &= b, \\ \gamma_1^{\#}(0) &= S_\sigma(b), \\ \gamma_2^{\#}(0) &= \sigma(p), \\ \pi \circ \gamma^{\#} &= \pi \circ \gamma_1^{\#} = \pi \circ \gamma_2^{\#} = \gamma. \end{aligned}$$

Now take $f \in C^\infty(M \times F)$. Then for all $t \in [0,1]$

$$
\begin{aligned}
&S_\sigma(\gamma^\#(t)) = \omega_\alpha\Big(\gamma(t);\, \big[u^i(t) + \sigma^i\big(\gamma(t)\big)\big]e_i\Big),\\
&\frac{d}{dt} f\bigg(\omega_\alpha^{-1} \circ S_\sigma\Big(\gamma^\#(t)\Big)\bigg)\bigg|_{t=0} = \frac{d}{dt} f\Big(\gamma(t);\, \big[u^i(t) + \sigma^i\big(\gamma(t)\big)\big]e_i\Big)|_{t=0},\\
&d\omega_\alpha^{-1} \circ dS_\sigma\big(X_v^\#\big)(f) = \left.\frac{d\gamma^\mu}{dt}\right|_0 \left.\frac{\partial f}{\partial x^\mu}\right|_{S_\sigma(b)} + \frac{d}{dt}\Big(u^i(t) + \sigma^i\big(\gamma(t)\big)\Big)\Big|_0 \left.\frac{\partial f}{\partial v^i}\right|_{S_\sigma(b)},\\
&= X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_{S_\sigma(b)} + \frac{d}{dt}\Big(u^i(t) + u_2^i(t) - u_2^i(t) + \sigma^i\big(\gamma(t)\big)\Big)\Big|_0 \left.\frac{\partial f}{\partial v^i}\right|_{S_\sigma(b)},\\
&= X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_{S_\sigma(b)} + \frac{d}{dt}\left(u^i(t) + u_2^i(t)\right)\Big|_0 \left.\frac{\partial f}{\partial v^i}\right|_{S_\sigma(b)}\\
&\qquad\qquad - \frac{d}{dt}\Big(u_2^i(t) - \sigma^i\big(\gamma(t)\big)\Big)\Big|_0 \left.\frac{\partial f}{\partial v^i}\right|_{S_\sigma(b)}.
\end{aligned}
$$

Let us take a closer look at the second term on the right hand side. Notice that

$$u^i(0) + u_2^i(0) = b^i + \sigma^i(p) = u_1^i(0).$$

for all $i \in \{1, \dots, n\}$. Recall that the coefficients of the fibre component of a horizontal lift, are determined via a linear ODE, $\mathcal{G}$. Further, $u^i(t), u_2^i(t)$ and $u_1^i(t)$ are all solutions to our particular linear ODE but with different initial conditions. This implies that $u^i(t) + u_2^i(t)$ is also a solution with the same initial condition as $u_1(t)$. Hence, by the Picard Lindelöf theorem [9, Sections 1.1 and 1.2]

$$u^i(t) + u_2^i(t) = u_1^i(t).$$

Thus we have

$$
\begin{aligned}
d\omega_\alpha^{-1} \circ dS_\sigma\Big(X_v^\#\Big)(f) &= X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_{S_\sigma(b)} + \left.\frac{du_1^i}{dt}\right|_0 \left.\frac{\partial f}{\partial v^i}\right|_{S_\sigma(b)}\\
&\qquad - \frac{d}{dt}\Big(u_2^i(t) - \sigma^i\big(\gamma(t)\big)\Big)\Big|_0 \left.\frac{\partial f}{\partial v^i}\right|_{S_\sigma(b)},\\
&= d\omega_\alpha^{-1} \circ X^\#_{S_\sigma(b)}(f) - \frac{d}{dt}\Big(u_2^i(t) - \sigma^i\big(\gamma(t)\big)\Big)\Big|_0 \left.\frac{\partial f}{\partial v^i}\right|_{S_\sigma(b)}.
\end{aligned}
$$

Now we need to use the fact that $T_{\sigma(p)}E$ and $T_{S_\sigma(b)}E$ are isomorphic vector spaces with isomorphism $\varrho : T_{\sigma(p)}E \to T_{S_\sigma(b)}E$ defined by

$$\varrho\left(\left.\frac{\partial}{\partial x^\mu}\right|_{\sigma(p)}\right) = \left.\frac{\partial}{\partial x^\mu}\right|_{S_\sigma(b)}, \qquad \varrho\left(\left.\frac{\partial}{\partial v^i}\right|_{\sigma(p)}\right) = \left.\frac{\partial}{\partial v^i}\right|_{S_\sigma(b)}. \tag{2.49}$$

This gives us the following relation:

$$
\begin{aligned}
&d\omega_\alpha^{-1} \circ dS_\sigma(X_v^\#) = d\omega_\alpha^{-1} \circ X^\#_{S_\sigma(b)} - \varrho\left(\frac{d}{dt}\left(u_2^i(t) - \sigma^i(\gamma(t))\right)\Big|_0 \left.\frac{\partial}{\partial v^i}\right|_{\sigma(p)}\right),\\
&= d\omega_\alpha^{-1} \circ X^\#_{S_\sigma(b)} - \varrho\bigg(X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{\sigma(p)} + \left.\frac{du_2^i}{dt}\right|_0 \left.\frac{\partial}{\partial v^i}\right|_{\sigma(p)} - X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{\sigma(p)}\\
&\qquad\qquad - \frac{d}{dt}\,\sigma^i(\gamma(t))\Big|_0 \left.\frac{\partial}{\partial v^i}\right|_{\sigma(p)}\bigg),\\
&= d\omega_\alpha^{-1} \circ X^\#_{S_\sigma(b)} - \varrho\Big(d\omega_\alpha^{-1} \circ X^\#_{\sigma(p)} - d\omega_\alpha^{-1} \circ d\sigma(X)\Big).
\end{aligned}
$$

Finally we arrive at

$$X^{\#}_{S_\sigma(b)} - dS_\sigma\left(X^{\#}_b\right) = \varrho\Big(X^{\#}_{\sigma(p)} - d\sigma\left(X\right)\Big), \tag{2.50}$$

which can be expressed in terms of $\mathcal{G}$

$$\mathcal{G}\big(S_\sigma(v), X\big) - dS_\sigma\Big(\mathcal{G}(v, X)\Big) = \varrho\Big(\mathcal{G}\big(\sigma(p), X\big) - d\sigma\big(X\big)\Big). \tag{2.51}$$

For the sake of simplifying this expression, we will drop ϱ and use context clues to determine when it is present.

To be able to apply the conditions (2.42), (2.44), (2.45) and (2.46) we require to know curves that generate the tangent basis vectors. Consider the following curves restricted to U_α that vary in only one local coordinate,

$$\gamma_\mu : (-\epsilon, \epsilon) \to U_\alpha \times F, \quad \gamma_\mu(t) = \Big(\big(p^\nu + \delta^\nu{}_\mu t\big)_\nu; b^i e_i\Big), \tag{2.52}$$

with $\mu \in \{1, 2, \ldots, m\}$ and

$$h_i : (-\epsilon, \epsilon) \to U_\alpha \times F, \quad h_i(t) = \big(p; \big[b^j + \delta^j{}_i t\big] e_j\big), \tag{2.53}$$

with $i \in \{1, 2, \ldots, n\}$ and $\epsilon > 0$. The path γ_μ varies only in the local coordinate x^μ on M, and similarly h_i only varies in the local e_i direction on F.

Now take $f \in C^\infty(M \times F)$ and compute the directional derivatives of f along each family of paths

$$\begin{aligned}
\frac{d}{dt} f\big(\gamma_\mu(t)\big)\big|_{t=0} &= \delta^\nu{}_\mu \left.\frac{\partial f}{\partial x^\nu}\right|_b = \left.\frac{\partial f}{\partial x^\mu}\right|_b, \\
\frac{d}{dt} f\big(h_i(t)\big)\big|_{t=0} &= \delta^j{}_i \left.\frac{\partial f}{\partial v^j}\right|_b = \left.\frac{\partial f}{\partial v^i}\right|_b.
\end{aligned}$$

Thus, we see that the tangents of the family of smooth paths $\{\gamma_\mu, h_i\}$ form our local tangent vector basis.

We may now finally begin enforcing the conditions (2.42), (2.44), (2.45) and (2.46) onto our local anstaz for a horizontal lift (2.39). The projection condition (2.42) tells us that for $f \in C^\infty(M)$

$$d\pi\Big(\mathcal{G}(b, X)\Big)(f) = \frac{d}{dt}\Big(f\big(\gamma(t)\big)\Big)\Big|_{t=0} = X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_p.$$

Now let us apply $d\pi$ to our local ansatz for $X^\#$ (2.39):

$$\begin{aligned}
d\pi\Big(\mathcal{G}(b, X)\Big)(f) &= d\pi \circ d\omega_\alpha \left(Y^\mu(b, X) \left.\frac{\partial}{\partial x^\mu}\right|_b + \mathcal{Z}^i(b, X) \left.\frac{\partial}{\partial v^i}\right|_b\right)(f), \\
&= Y^\mu(b, X) d\pi \left(\left.\frac{\partial}{\partial x^\mu}\right|_b\right)(f) + \mathcal{Z}^i(b, X) d\pi \left(\left.\frac{\partial}{\partial v^i}\right|_b\right)(f), \\
&= Y^\mu(b, X) \frac{d}{dt}\Big(f\big(\pi \circ \gamma_\mu(t)\big)\Big)\Big|_{t=0} + \mathcal{Z}^i(b, X) \frac{d}{dt}\Big(f\big(\pi \circ h_i(t)\big)\Big)\Big|_{t=0}, \\
&= Y^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_b + \mathcal{Z}^i(b, X, Y) \frac{d}{dt}\Big(f(p)\Big)\Big|_{t=0} = Y^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_b.
\end{aligned}$$

Thus condition (2.42) imposes

$$Y^\mu(b, X) = X^\mu, \qquad \text{for all } b \in E_p \tag{2.54}$$

Applying the condition that $\mathcal{G}(b,X)$ is linear in X (2.44), gives us the relation

$$\mathcal{Z}^i(b,X) = X^\mu Z_\mu{}^i(b), \tag{2.55}$$

for scalar functions $Z_\mu{}^i : \pi^{-1}(U_\alpha) \to \mathbb{C}$.

The next condition we impose is scalar multiplication compatibility (2.45). We will now apply dm_c to our local ansatz of $\mathcal{G}(b,X)$. Take $f \in C^\infty(M \times F)$ and consider the following

$$\begin{aligned}
d\omega_\alpha^{-1} \circ dm_c\big(\mathcal{G}(b,X)\big)(f) &= d\omega_\alpha^{-1} \circ dm_c \circ d\omega_\alpha \left(X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_b + X^\mu Z_\mu{}^i(b) \left.\frac{\partial}{\partial v^i}\right|_b \right)(f), \\
&= X^\mu \frac{d}{dt} f\Big(m_c \circ \gamma_\mu(t)\Big)\Big|_0 + X^\mu Z_\mu{}^i(b) f\Big(m_c \circ h_i(t)\Big)\Big|_0, \\
&= X^\mu \frac{d}{dt} f\Big(\big(p^\nu + \delta^\nu{}_\mu t\big)_\nu; cb^j e_j\Big)\Big|_0 + X^\mu Z_\mu{}^i(b) \frac{d}{dt} f\Big(p; \big[cb^j + c\delta^j{}_i t\big] e_j\Big)\Big|_0, \\
\implies d\omega_\alpha^{-1} \circ dm_c\left(\mathcal{G}(b,X)\right)(f) &= X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_{m_c(b)} + cX^\mu Z_\mu{}^i(b) \left.\frac{\partial f}{\partial v^i}\right|_{m_c(b)}.
\end{aligned} \tag{2.56}$$

Substituting (2.56) into (2.45) and expanding the left hand side in terms of our local basis vectors gives

$$X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_{m_c(b)} + X^\mu Z_\mu{}^i\big(m_c(b)\big) \left.\frac{\partial f}{\partial v^i}\right|_{m_c(b)} = X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_{m_c(b)} + cX^\mu Z_\mu{}^i(b) \left.\frac{\partial f}{\partial v^i}\right|_{m_c(b)},$$

which implies a condition on Z:

$$Z_\mu{}^i\big(m_c(b)\big) = Z_\mu{}^i\big(p^\nu; cb^j\big) = cZ_\mu{}^i\big(p^\nu; b^j\big) = cZ_\mu{}^i(b). \tag{2.57}$$

The final condition to enforce is vector addition compatibility (2.46). Let $\sigma \in \Gamma(M,E)$ and employ the same strategy we used on the scalar multiplication compatibility condition. First we will focus on the right hand side of (2.46). By equations (2.54) and (2.55) we know that $\mathcal{G}(\sigma(p),X)$ takes the local form

$$d\omega_\alpha^{-1} \circ \mathcal{G}(\sigma(p),X) = X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{\sigma(p)} + X^\mu Z_\mu{}^i\big(\sigma(p)\big) \left.\frac{\partial}{\partial v^i}\right|_{\sigma(p)}. \tag{2.58}$$

To find the local form of $d\sigma(X)$, we will apply an arbitrary function $f \in C^\infty(M \times F)$ to it

$$\begin{aligned}
\Big(d\omega_\alpha^{-1} \circ d\sigma(X)\Big)(f) &= X^\mu \frac{d}{dt} f\Big(\omega_\alpha^{-1} \circ \sigma\Big(\big(p^\nu + \delta^\nu{}_\mu t\big)_\nu\Big)\Big)\Big|_{t=0}, \\
&= X^\mu \delta^\nu{}_\mu \left.\frac{\partial f}{\partial x^\nu}\right|_{\sigma(p)} + X^\mu \delta^\nu{}_\mu \left.\frac{\partial \sigma^i}{\partial x^\nu}\right|_p \left.\frac{\partial f}{\partial v^i}\right|_{\sigma(p)},
\end{aligned}$$

which implies the local form of $d\sigma(X)$

$$d\omega_\alpha^{-1} \circ d\sigma(X) = X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{\sigma(p)} + X^\mu \left.\frac{\partial \sigma^i}{\partial x^\mu}\right|_p \left.\frac{\partial}{\partial v^i}\right|_{\sigma(p)}. \tag{2.59}$$

By equations (2.58) and (2.59), the right hand side of equation (2.46) takes the local form

$$d\omega_\alpha^{-1} \circ \mathcal{G}(\sigma(p),X) - d\omega_\alpha^{-1} \circ d\sigma(X) = X^\mu \left(Z_\mu{}^i\big(\sigma(p)\big) - \left.\frac{\partial \sigma^i}{\partial x^\mu}\right|_p \right) \left.\frac{\partial}{\partial v^i}\right|_{\sigma(p)}. \tag{2.60}$$

Now that we have the local vectors describing the right hand side of (2.46), we can tackle the left hand side. Again, by equations (2.54) and (2.55) we know that $\mathcal{G}(S_\sigma(b),X)$ takes the local form

$$d\omega_\alpha^{-1} \circ \mathcal{G}(S_\sigma(b),X) = X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{S_\sigma(b)} + X^\mu Z_\mu{}^i\big(S_\sigma(b)\big) \left.\frac{\partial}{\partial v^i}\right|_{S_\sigma(b)}. \tag{2.61}$$

To find the local form of $dS_\sigma\big(\mathcal{G}(b,X)\big)$ we will apply $f\in C^\infty(M\times F)$ to it

$$
\begin{aligned}
&d\omega_\alpha^{-1}\circ dS_\sigma\big(\mathcal{G}(b,X)\big)(f) = d\omega_\alpha^{-1}\circ dS_\sigma\circ d\omega_\alpha\left(X^\mu\left.\frac{\partial}{\partial x^\mu}\right|_b + X^\mu {Z_\mu}^i(b)\left.\frac{\partial}{\partial v^i}\right|_b\right)(f),\\
&= X^\mu\frac{d}{dt}f\Big(S_\sigma\circ\gamma_\mu(t)\Big)\Big|_0 + X^\mu {Z_\mu}^i(b)\frac{d}{dt}f\Big(S_\sigma\circ h_i(t)\Big)\Big|_0,\\
&= X^\mu\frac{d}{dt}f\Big(\big(p^\nu+\delta^\mu{}_\nu t\big)_\nu;\big[b^j+\sigma^j(p^\nu+\delta^\mu{}_\nu t)\big]e_j\Big)\Big|_0 + X^\mu {Z_\mu}^i(b)\frac{d}{dt}f\Big(p;\big[b^j+\delta^j{}_i t+\sigma^j(p)\big]e_j\Big)\Big|_0,\\
&= X^\mu\left(\left.\frac{\partial f}{\partial x^\mu}\right|_{S_\sigma(b)} + \left.\frac{\partial\sigma^j}{\partial x^\mu}\right|_p\left.\frac{\partial f}{\partial v^j}\right|_{S_\sigma(b)}\right) + X^\mu {Z_\mu}^i(b)\left.\frac{\partial f}{\partial v^i}\right|_{S_\sigma(b)}.
\end{aligned}
$$

Thus, locally we have

$$
d\omega_\alpha^{-1}\circ dS_\sigma\left(\mathcal{G}(b,X)\right) = X^\mu\left.\frac{\partial}{\partial x^\mu}\right|_{S_\sigma(b)} + X^\mu\left(\left.\frac{\partial\sigma^i}{\partial x^\mu}\right|_p + {Z_\mu}^i(b)\right)\left.\frac{\partial}{\partial v^i}\right|_{S_\sigma(b)}. \tag{2.62}
$$

By equations (2.61) and (2.62), the left hand side of equation (2.46) takes the local form

$$
\begin{aligned}
&d\omega_\alpha^{-1}\circ\mathcal{G}\big(S_\sigma(b),X\big) - d\omega_\alpha^{-1}\circ dS_\sigma\left(\mathcal{G}(b,X)\right)\\
&\qquad = X^\mu\left({Z_\mu}^i\big(S_\sigma(b)\big) - \left.\frac{\partial\sigma^i}{\partial x^\mu}\right|_p - {Z_\mu}^i(b)\right)\left.\frac{\partial}{\partial v^i}\right|_{S_\sigma(b)}.
\end{aligned} \tag{2.63}
$$

Recall the isomorphism $\varrho: T_{\sigma(b)}E\to T_{S_\sigma(b)}E$ defined in equation (2.49). We will now apply ϱ to equation (2.60) and compare to (2.63). Without explicitly writing $d\omega_\alpha^{-1}$, we have

$$
\begin{aligned}
&\mathcal{G}\big(S_\sigma(b),X\big) - dS_\sigma\left(\mathcal{G}(b,X)\right) = \varrho\Big(\mathcal{G}\big(\sigma(p),X\big) - d\sigma(X)\Big),\\
&X^\mu\left({Z_\mu}^i\big(S_\sigma(b)\big) - \left.\frac{\partial\sigma^i}{\partial x^\mu}\right|_p - {Z_\mu}^i(b)\right)\left.\frac{\partial}{\partial v^i}\right|_{S_\sigma(b)} = X^\mu\left({Z_\mu}^i\big(\sigma(p)\big) - \left.\frac{\partial\sigma^i}{\partial x^\mu}\right|_p\right)\left.\frac{\partial}{\partial v^i}\right|_{S_\sigma(b)},
\end{aligned}
$$

which simplifies to the final condition on $Z(b)$

$$
\begin{aligned}
{Z_\mu}^i\big(S_\sigma(b)\big) = {Z_\mu}^i\Big(\big(b^j+\sigma^j(p)\big)\iota_j(p)\Big) &= {Z_\mu}^i\big(b^j\iota_j(p)\big) + {Z_\mu}^i\big(\sigma^j(p)\iota_j(p)\big),\\
&= {Z_\mu}^i\big(\sigma(p)\big) + {Z_\mu}^i(b).
\end{aligned} \tag{2.64}
$$

That is to say ${Z_\mu}^i$ is a linear operator on our vector space fibre F.

Imposing our compatibility conditions (2.54), (2.55), (2.57) and (2.64) we find that the horizontal lift of $X\in T_pM$ to $b\in E_p$ must take the local form

$$
X_b^\# = \mathcal{G}\big(b,X\big) = X^\mu\left(\left.\frac{\partial}{\partial x^\mu}\right|_b + {A_\mu}^i{}_j(p)b^j\left.\frac{\partial}{\partial v^i}\right|_b\right), \tag{2.65}
$$

where $\big({A^i}_j\big) = \big({A_\mu}^i{}_j\big)dx^\mu$ is a local matrix valued connection 1−form. We will show that this local connection 1-from corresponds to that defined in (2.32). Hence the local IVPs (2.40), that uniquely define the horizontal lift of a smooth path $\gamma:[0,1]\to M$, are given by

$$
\frac{d\gamma^\mu}{dt} = X^\mu(t),\quad \frac{d\lambda^i}{dt} = X^\mu(t){A_\mu}^i{}_j\big(\gamma(t)\big)\lambda^j(t),\quad \gamma^\mu(0)=p^\mu,\quad \lambda^i(0)=b^i. \tag{2.66}
$$

Let us now use our horizontal lift to define an Ehresmann connection and vice versa. An Ehresmann connection is a global map that projects a vector tangent to E, to its vertical component. So for a given horizontal lift, we have a linear map $\xi':TE\to VE$ defined by

$$
\xi'(Y) = Y - d\pi(Y)_b^\#, \tag{2.67}
$$

for all $b \in E$, $Y \in T_bE$. This then implies that

$$\xi'|_{VE} = \mathrm{id}_{VE}, \tag{2.68}$$

$$\ker(\xi) = \left\{ X_b^{\#} \in T_bE : b \in E_p \text{ and } X \in T_pM \right\}. \tag{2.69}$$

Thus ξ' is by definition, an Ehresmann connection.

We will now explain why an Ehresmann connection ξ defines local connection 1-forms and thus induces a horizontal lift. Take $c_j = \omega_\alpha\left(p; \delta^i{}_j \iota_{\alpha,i}(p)\right) \in E_p$ where p is in a domain $U_\alpha \subset M$. Any tangent vector in $H_{c_j}E$ can be expressed locally using the basis $\{\frac{\partial}{\partial x^\mu}; \frac{\partial}{\partial v^i}\}$ where $\{\frac{\partial}{\partial v^i}\}$ is the basis for the vertical vectors. With this in mind consider $Y_\mu \in H_{c_j}E$ with $d\pi(Y_\mu) = \frac{\partial}{\partial x^\mu}\Big|_{c_j}$. Then locally we will have

$$Y_\mu = d\omega_\alpha \left(\frac{\partial}{\partial x^\mu}\bigg|_{c_j} + Y_\mu{}^i{}_j(p) \, \frac{\partial}{\partial v^i}\bigg|_{c_j} \right). \tag{2.70}$$

The reason why $Y_\mu{}^i{}_j$ is dependent on μ and j is because by fixing H_bE and enforcing that it must smoothly vary in b, we require the vertical component of Y_μ to smoothly depend on both $d\pi(Y_\mu)$ and where the vector is tangent, c_j.

Now let us apply ξ to Y_μ

$$\begin{aligned} \xi(Y_\mu) &= \xi \circ d\omega_\alpha \left(\frac{\partial}{\partial x^\mu}\bigg|_{c_j} + Y_\mu{}^i{}_j(p) \, \frac{\partial}{\partial v^i}\bigg|_{c_j} \right), \\ 0 &= \xi \circ d\omega_\alpha \left(\frac{\partial}{\partial x^\mu}\bigg|_{c_j} \right) + Y_\mu{}^i{}_j(p) d\omega_\alpha \left(\frac{\partial}{\partial v^i}\bigg|_{c_j} \right), \end{aligned}$$

which implies the local equation

$$d\omega_\alpha^{-1} \circ \xi \circ d\omega_\alpha \left(\frac{\partial}{\partial x^\mu}\bigg|_{c_j} \right) = -Y_\mu{}^i{}_j(p) \, \frac{\partial}{\partial v^i}\bigg|_{c_j}. \tag{2.71}$$

That is to say, we determine how an Ehresmann connection acts on the basis vectors $\{\frac{\partial}{\partial x^\mu}\big|_{c_j}\}$, and then use linearity in the fibre and tangent spaces to extend c_j to any $b \in \pi^{-1}(p)$, and Y_μ to any tangent vector. What we find is that conjugating ξ with the tangent map $d\omega_\alpha$ defines a local matrix valued $1-$form

$$d\omega_\alpha^{-1} \circ \left(\omega_\alpha\right)^*(\xi) = -Y_\mu{}^i{}_j(p) b^j dx^\mu \, \frac{\partial}{\partial v^i}\bigg|_b. \tag{2.72}$$

Hence, an Ehresmann connection defines a horizontal lift and a horizontal lift defines an Ehresmann connection. We are only interested in the local description of fibre bundles and so we will make use of the horizontal lift for all of our practical calculations.

2.6 Lie algebras and left invariant vector fields

Now that we know what connection $1-$forms are used for, we need to figure out what values they can take. It is important to note that a local connection $1-$form or gauge potential is not unique, but under specific conditions some gauge potentials become more useful than others, and this is exactly the case here.

We know that A is a matrix valued $1-$form on M that acts on the left of our fibre F. So one might think that A takes values in our structure group G, but recall that we defined a horizontal lift of path γ at b, as a unique path $\gamma^{\#}$ in E that projects to γ and starts

at b. The way we guaranteed the uniqueness of $\gamma^{\#}$ was to create an initial value problem (IVP) in the form of a horizontal vector, of which $\gamma^{\#}$ was the solution to. This means that A describes an infinitesimal action of our structure group on our fibre. Hence, A is a Lie algebra valued $1-$form on M.

When setting up our vector bundle, we ask how we want vectors in our fibre to transform. For a classical system we may only want our vectors to rotate in which case we use the structure group $SO(n)$. For a quantum system we restrict ourselves to unitary transformations, $U(n)$, in order to preserve hermiticity in operators and norms of states.

By definition, the Lie algebra of our structure group G is the first order approximation of G at the identity, T_eG. However, there is a set of vector fields on G, isomorphic to T_eG known as the *left-invariant vector fields*. Let G be a Lie group and take $h \in G$. Left multiplication $L_h : G \to G$, maps $g \in G$ to hg. We say that $X \in \Gamma(G, TG)$ is a *left-invariant vector field* of Lie group G if for all $g, h \in G$

$$L_{h*}\big(X\big)(g) = X(g). \tag{2.73}$$

To show T_eG is isomorphic to a set of left-invariant vector fields on G. Consider the following: Let $V \in T_eG$ and define vector field $X \in \Gamma(G, TG)$ by

$$X(g) = L_{g*}\big(V\big) = dL_g\big(V\big).$$

This vector field is left invariant

$$\begin{aligned} L_{h*}(X)(g) &= dL_h(X(h^{-1}g)) = dL_h(dL_{h^{-1}g}(V)) = (L_hL_{h^{-1}g})_*(V), \\ &= L_{g*}(V) = X(g). \end{aligned}$$

So vectors in T_eG define unique left-invariant vector fields. Conversely, for left-invariant vector fields X, we have the vector $X(e) \in T_eG$ and by definition

$$X(g) = dL_g\Big(X(e)\Big).$$

Thus we have shown that there exists a bijection between the set of left-invariant vector fields on G, $\chi(G)$, and T_eG:

$$\Phi : \chi(G) \to T_eG, \quad \Phi(X) = X(e). \tag{2.74}$$

Because this map is linear, then $\chi(G) \simeq T_eG$.

The reason the left invariant vector fields are so important to fibre bundles is because they generate a smooth Lie group action on our fibre. To see why, we need to look at the *integral curves* of these vector fields. An *integral curve* $\gamma : I \to G$ of a vector field $X \in \chi(G)$, is a solution to

$$\frac{d\gamma}{dt} = X(\gamma(t)), \qquad \text{for all } t \in I, \tag{2.75}$$

where I is an open connected subset of $\mathbb{R}$ containing 0.

Let $\gamma_p : I \to G$ be the integral curve of $X \in \chi(G)$ through the point $p \in U \subset G$, and let U have local coordinates $\{g^k\}$. Now take a smooth map $\phi : G \to G$ and note that $\phi \circ \gamma_p$ is the integral curve to $\phi_*(X) \in \chi(G)$ through the point $\phi(p)$. To show this we consider $f \in C^\infty(G)$ and apply the vector $\phi_*(X)\Big(\phi\big(\gamma_p(t)\big)\Big)$ to it: Locally we have

$$\phi_*(X)\Big(\phi\big(\gamma_p(t)\big)\Big)(f) = d\phi\Big(X\big(\gamma_p(t)\big)\Big)(f) = \frac{d}{dt} f\Big(\phi\big(\gamma_p(t)\big)\Big)\Big|_{t=0} = \frac{d}{dt}\Big(\phi\big(\gamma_p(t)\big)\Big)^k \frac{\partial f}{\partial g^k}.$$

Since this is true for all $f \in C^\infty(M)$ we arrive at

$$\phi_*(X)\Big(\phi\big(\gamma_p(t)\big)\Big) = \frac{d}{dt}\big(\phi \circ \gamma_p\big). \tag{2.76}$$

With this in mind consider a left-invariant vector field X on G. We aim to find its integral curve $\gamma_g : I \to G$ through the identity $e \in G$

$$X\big(\gamma_g(t)\big) = \frac{d\gamma_g}{dt}, \qquad \gamma_g(0) = e. \tag{2.77}$$

By equation (2.76), applying the push forward $L_{\gamma_g(s)*}$ to X for some fixed $s \in I$, has the solution

$$\gamma_g'(t,s) = L_{\gamma_g(s)}\big(\gamma_g(t)\big) = \gamma_g(s)\gamma_g(t)$$

with initial value

$$\gamma_g'(0,s) = \gamma_g(s).$$

Now we will use the fact that X is left-invariant to find another solution to our IVP. It is clear that $\gamma(t+s)$ satisfies our IVP and thus by the Picard–Lindelöf theorem [9, Sections 1.1 and 1.2]

$$\gamma_g(s)\gamma_g(t) = \gamma_g(s+t) \tag{2.78}$$

for all $s, t, s+t \in I$.

We have just shown that the integral curves through the identity of our left-invariant vector fields are something known as the *one-parameter subgroup of* G.

$$\big\{\psi_t : t \in I \text{ and for all } g \in G,\ \ \psi_t(g) = \gamma_g(t)\big\} \tag{2.79}$$

We can think about the *one-parameter subgroup* generated by a left-invariant vector field X as the general solution to

$$X(\gamma(t)) = \frac{d\gamma}{dt}.$$

Let $V \in T_eG$ and Φ_V be the one-parameter subgroup of G generated by the left-invariant field $X(g) = L_{g*}(V)$. We want to show $\exp(tV) = \Phi_V(t)$. To do this we will take $s \in \mathbb{R}\backslash\{0\}$, and differentiating $\Phi_V(st)$,

$$\frac{d}{dt}\big(\Phi_V(st)\big)\big|_{t=0} = s\frac{d}{d\tau}\big(\Phi_V(\tau)\big)|_{\tau=0} = sV,$$

where the last step is by the definition of an integral curve (2.75). Thus, $\Phi_V(st)$ is the one parameter subgroup of the vector field $X'(g) = L_{g*}(sV)$. But we also have X' generating $\Phi_{sV}(t)$, so by the Picard–Lindelöf theorem [9, Sections 1.1 and 1.2] we have $\Phi_{sV}(t) = \Phi_V(st)$. Hence,

$$\exp(sV) = \Phi_{sV}(1) = \Phi_V(s). \tag{2.80}$$

This proves that $\exp(tV) = \Phi_V(t)$.

In summary, the left invariant vector fields generate smooth Lie group actions. This is because their integral curves (the finite transformations T_eG exponentiates to) are the one-parameter subgroups of the structure group G. Thus, knowing the left-invariant vector fields, gives us all the information we need about the possible smooth infinitesimal group actions.

The Lie algebra and left invariant vector fields on G are precisely what we need to understand the local connection $1-$form A. For a vector bundle (E, π, M, F, G) with local connection $1-$forms $\{A^\alpha\}$, then for all $p \in U_\alpha$

$$A^\alpha_\mu(p) \in T_eG. \tag{2.81}$$

The left invariant vector fields on G tell us that $A^\alpha_\mu(p)$ is smooth in p, and acts smoothly on the fibre.

2.7 How a connection transforms

Given a fibre bundle (E, π, M, F, G) and atlas $\{(U_\alpha \times F, \omega_\alpha)\}$, we have local connection $1-$forms $A_\alpha \in T_eG \times \Omega^1(U_\alpha)$ on each domain U_α. On overlaps of domains $U_\alpha \cap U_\beta \neq \emptyset$ we will have two local connection $1-$forms, A_α and A_β. To have a consistent theory we will require compatibility conditions between $A_\alpha(p)$ and $A_\beta(p)$ for all $p \in U_\alpha \cap U_\beta$.

On a fibre bundle, we know that for overlapping domains U_α and U_β that $\omega_\beta(p, f) = \omega_\alpha(p, t_{\alpha\beta}(p)f) = t_{\alpha\beta}(p)\omega_\alpha(p, f)$ for all $p \in U_\alpha \cap U_\beta$. Thus we know that sections $\sigma \in \Gamma(M, E)$ transform from domain to domain as

$$\sigma_\alpha(p) = t_{\alpha\beta}(p)\sigma_\beta(p) \tag{2.82}$$

where $\sigma_\beta = \sigma|_{U_\beta}$ and $p \in U_\alpha \cap U_\beta$.

To understand how local connection $1-$forms transform on vector bundles, we need to consider how the conditions (2.42), (2.44), (2.45) and (2.46) transform from domain to domain. The only one of these conditions that transforms non-trivially, is the condition of vector addition compatibility (2.46). This is because for $p \in U_\alpha \cap U_\beta$, $X \in T_pM$ and $\sigma \in \Gamma(M, E)$ we require

$$X^\#_{\sigma_\alpha(p)} - d\sigma_\alpha(X) = X^\#_{\sigma_\beta(p)} - d\sigma_\beta(X). \tag{2.83}$$

Let $U_\alpha \times F$ have local basis sections $\{\iota_{k,\alpha}\}$ and $U_\beta \times F$ have local basis sections $\{\iota_{k,\beta}\}$. These provide respective local coordinates on the fibre, $\{v^k\}$ and $\{v'^k\}$. Then, by (2.60), equation (2.83) has the local form

$$X \cdot \Big(d\sigma_\alpha(p)^\ell - A^\alpha(p)^\ell{}_k \sigma_\alpha(p)^k\Big) \left.\frac{\partial}{\partial v^\ell}\right|_{\sigma(p)} = X \cdot \Big(d\sigma_\beta(p)^\ell - A^\beta(p)^\ell{}_k \sigma_\beta(p)^k\Big) \left.\frac{\partial}{\partial v'^\ell}\right|_{\sigma(p)}, \tag{2.84}$$

where $p \in U_\alpha \cap U_\beta$. Note that this holds for all $X \in T_pM$, so we will drop the X. It is important to keep in mind that α and β are fixed. There will appear instances of a raised and lowered β index, these do not denote a sum. The placement of the β index on the connection 1-forms, signifying that it is localised to U_β, was taken to be raised because of its existing lowered 1-form index.

The transition map $t_{\alpha\beta}(p)$ acts on the local basis sections $\{\iota_{k,\alpha}(p), \iota_{k,\beta}(p)\}$, so we need to relate (2.84) back to these local basis sections. Luckily for us, we are interested in vector bundles, where $\pi^{-1}(p)$ and V_bE are n-dimensional vector spaces for all $p \in M$ and $b \in \pi^{-1}(p)$. Hence, we have an isomorphism

$$\varpi_b : V_bE \to \pi^{-1}(p), \qquad \varpi_b\left(\left.\frac{\partial}{\partial v^k}\right|_b\right) = \iota_{k,\alpha}\big(\pi(b)\big), \qquad \text{for all } b \in \pi^{-1}(U_\alpha). \tag{2.85}$$

Applying $\varpi_{\sigma(p)}$ to (2.84) leads to

$$\begin{aligned}\Big(d\sigma_\alpha(p)^\ell - A^\alpha(p)^\ell{}_k \sigma_\alpha(p)^k\Big)\iota_{\ell,\alpha}(p)& \\ = \Big(d\sigma_\beta(p)^\ell - A^\beta(p)^\ell{}_k \sigma_\beta(p)^k\Big) \left.\frac{\partial v^j}{\partial v'^\ell}\right|_p \iota_{j,\alpha}(p).&\end{aligned} \tag{2.86}$$

Note that the local coordinates satisfy $v^j = t_{\alpha\beta}(p)^j{}_\ell v'^\ell$. The right hand side of equation (2.86) then becomes

$$\begin{aligned}
\text{RHS} &= \Big(d\sigma_\beta(p)^\ell - A^\beta(p)^\ell{}_k \sigma_\beta(p)^k\Big) t_{\alpha\beta}(p)^j{}_\ell \iota_{j,\alpha}(p), \\
&= \Big(d\big(t_{\alpha\beta}(p)^j{}_\ell \sigma_\beta(p)^\ell\big) - d\big(t_{\alpha\beta}(p)^j{}_\ell\big)\sigma_\beta(p)^\ell - t_{\alpha\beta}(p)^j{}_\ell A^\beta(p)^\ell{}_k \sigma_\beta(p)^k\Big)\iota_{j,\alpha}(p), \\
&= \Big(d\big(\sigma_\alpha(p)^j\big) - d\big(t^{-1}_{\beta\alpha}(p)^j{}_\ell\big)\sigma_\beta(p)^\ell - t_{\alpha\beta}(p)^j{}_\ell A^\beta(p)^\ell{}_k t_{\beta\alpha}(p)^k{}_i \sigma_\alpha(p)^i\Big)\iota_{j,\alpha}(p), \\
&= \Big(d\big(\sigma_\alpha(p)^j\big) + t^{-1}_{\beta\alpha}(p)\big(dt_{\beta\alpha}(p)^j{}_\ell\big)t_{\alpha\beta}\sigma_\beta(p)^\ell + t_{\alpha\beta}(p)^j{}_\ell A^\beta(p)^\ell{}_k t_{\beta\alpha}(p)^k{}_i \sigma_\alpha(p)^i\Big)\iota_{j,\alpha}(p).
\end{aligned}$$

Thus, equation (2.86) reads

$$\Big(d - A^{\alpha}(p)\Big)\sigma_{\alpha}(p) = \Big(d + t^{-1}_{\beta\alpha}(p)dt_{\beta\alpha}(p) - t^{-1}_{\beta\alpha}(p)A^{\beta}(p)t_{\beta\alpha}(p)\Big)\sigma_{\alpha}(p), \tag{2.87}$$

Which implies

$$A^{\alpha}(p) = t^{-1}_{\beta\alpha}(p)A^{\beta}(p)t_{\beta\alpha}(p) - t^{-1}_{\beta\alpha}(p)dt_{\beta\alpha}(p). \tag{2.88}$$

To understand how A_{α} transforms in general, we must also take base manifold coordinate changes into account, seeing as A_i is a $1-$form on M. That is to say, if U_{α}, U_{β} are domains of our base manifold M with respective coordinates $\{x^{\mu}\}_{\mu=1}^{m}$ and $\{y^{\mu}\}_{\mu=1}^{m}$ and $p \in U_{\alpha} \cap U_{\beta}$, then

$$(A_{\alpha}(p))_{\nu}{}^{k}{}_{\ell} = \left(\frac{dx}{dy}\right)^{\mu}_{\ \nu}\left(\left(t^{-1}_{\beta\alpha}(p)dt_{\beta\alpha}(p)\right)_{\mu\ \ell}^{\ k} + \left(t^{-1}_{\beta\alpha}(p)\right)^{k}{}_{a}\left(A_{\beta}(p)\right)_{\mu}{}^{a}{}_{b}\left(t_{\beta\alpha}(p)\right)^{b}{}_{\ell}\right). \tag{2.89}$$

2.8 What is so parallel about parallel transport?

Because a connection is necessary to perform parallel transport, there are many different notions of what parallel transport is. However, they all have one thing in common and that is that, geometric objects undergoing parallel transport must be kept "parallel" with respect to the connection.

To gain a better understanding of what "parallel" means, we should look at an example. The usual example of parallel transport is that of a tangent vector on S^2. As we know from above, we cannot perform parallel transport immediately, we first need to set up rules we want it to obey and then form a connection to enforce those rules.

In the spirit of the word "parallel" let us make it so that our vector remains parallel with itself under an infinitesimal step. That is for two points infinitesimally close, we find the geodesic (in this case a great circle) passing through the points and we enforce that the angle between the vector and the tangent of the geodesic remains unchanged at the end of the step.

We could think about this example physically in the following way: Say we want to walk about some path on Earth with roughly the same latitude. We could do this by holding a compass in one hand and constantly checking that North is always pointing left or right. In the other hand we carry our trusty spear which we hold tangent to Earth's surface at all times. After walking forward in a straight line for some distance we notice that North has moved x degrees from our left. So we stop and align ourselves again by turning x degrees and to keep our spear parallel with itself one step ago we rotate the spear $-x$ degrees.

By doing this we parallel transport our spear along our path. When we return to our initial position, we will be standing in the same orientation in which we started, but our spear could be pointing in a completely different direction.

In this analogy the compass is our local connection, it tells us when we need to turn and by how much. The spear is of course the tangent vector of S^2 that we are parallel transporting.

The above explains the basic notion of what parallel transport it, but it can be generalised. Let us keep the theme of tangent vectors but upgrade our connection from a simple compass, to an affine connection. We want to parallel transport a tangent vector $v \in T_pM$ across some smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = p$. To do this we find the vector field $X(t)$ tangent to $\gamma(t)$ and take the covariant derivative of a general vector field $Y(t)$ defined on $\gamma(t)$, along $X(t)$, for all t

$$\nabla_X(Y) = X^{\nu}\left(\frac{\partial Y^{\lambda}}{\partial x^{\nu}} + A_{\nu}{}^{\lambda}{}_{\mu}Y^{\mu}\right)\frac{\partial}{\partial x^{\lambda}}.$$

We want to use this differential equation and the initial condition $Y(0) = v$, to find the parallel transport of v.

Recall, when parallel transporting our spear across a single step, our connection (the compass) told us how much we needed to rotate. This meant that our vector changed along our path by an amount described by the connection alone.

With this in mind we can see that in order to parallel transport v, we need to set the covariant derivative of Y in the direction X to zero

$$X^\nu(t)\frac{\partial Y^\lambda}{\partial x^\nu} + X^\nu(t)A_\nu{}^\lambda{}_\mu(\gamma(t))Y^\mu(t) = 0,$$

$$\implies \frac{dY^\lambda}{dt} + X^\nu(t)A_\nu{}^\lambda{}_\mu\big(\gamma(t)\big)Y^\mu(t) = 0 \qquad\qquad Y^\mu(0) = v^\mu, \tag{2.90}$$

which tells us that the only change allowed for our vector field Y across γ is that governed by our connection coefficients $A_\nu{}^\lambda{}_\mu(\gamma(t))$.

Solving this differential equation gives the parallel transport of v along γ as $Y(1)$. One can also think about the covariant derivative of Y with respect to X as the directional derivative of Y along the tangent of γ. Thus if $\nabla_X(Y) = 0$ then to first order $Y(0)$ does not change as it is moved along γ.

Now it's important to keep in mind that this parallel transport is not the same as our spear example where we kept our spear parallel relative to what our compass said. We now have a general affine connection, so when parallel transporting a tangent vector, we keep that vector parallel relative to our connection.

Ultimately, we want to parallel transport quantum states represented as vectors in a Hilbert space $\mathcal{H}$, across paths in an $m-$dimensional smooth manifold. So affine connections and arguments about spears will not cut it anymore. We upgrade our connection to an Ehresmann connection on a vector bundle E.

Under this prescription, we take some smooth path $\gamma : [0,1] \to M$ starting at p, and a vector $b \in \pi^{-1}(p)$ that we want to parallel transport. We then want to find the unique path in our bundle E that obeys the rules of our connection A, projects to γ, and starts at b. By design, this is the horizontal lift $\gamma^\#$. We use the horizontal lift of the tangent vector field $X(s)$ of $\gamma(s)$ as given by (2.65), to define an initial value problem with solution $\gamma^\#(s)$.

Assume $\gamma(s)$ is contained within a domain of our base manifold and let $\gamma^\#(s)$ take the local form

$$\gamma^\#(s) = (\gamma(s); \lambda^i(s)e_i). \tag{2.91}$$

Equation (2.66) gives us

$$\dot\gamma^\mu(t) = X^\mu(t), \qquad \gamma^\mu(0) = p^\mu, \tag{2.92}$$

$$\frac{d\lambda^i}{dt} = X^\mu(t)A_\mu{}^i{}_j\big(\gamma(t)\big)\lambda^j(t), \qquad u^i(0) = b^i. \tag{2.93}$$

with $\mu \in \{1,2,\dots,m\}$ and $i \in \{1,2,\dots,n\}$.

Equation (2.92) returns the definition of γ's tangent vector field. Equation (2.93) describes how the fibre component of $\gamma^\#$ varies. Note how similar it looks to (2.90). Solving (2.93) describes how the vector $b^i\iota_i(p)$ is parallel transported along γ.

What this means is that an object in our fibre is parallel transported along a smooth path $\gamma : [0,1] \to M$, if the tangent vector field of $\gamma^\# : [0,1] \to E$ is a horizontal vector for all $s \in [0,1]$. That is to say, objects in our fibre are kept parallel with respect to the horizontal subspace of tangent vectors.

Now recall the vector compatibility condition (2.46). We claimed that this can be used to define a covariant derivative on a section. This is possible when our fibre is a vector space because we have the isomorphism $\varpi_b : V_bE \to E_p$ with $p \in U_\alpha$ and $b \in E_p$, from equation 2.85. Applying $\varpi_{\sigma(p)} \circ d\omega_\alpha$ to the right hand side of (2.46) gives the following local expression

$$X^\mu \left.\frac{\partial\sigma^i}{\partial x^\mu}\right|_p \iota_{\alpha,i}(p) - X^\mu A_\mu{}^i{}_j\sigma^j(p)\iota_{i,\alpha}(p) = X\cdot(d-A)\sigma(p). \tag{2.94}$$

This equation describes a generalised covariant derivative of section σ along the direction X. Now consider a smooth path $\gamma : [0,1] \to U_\alpha$, with tangent vector field $X(t)$. Let $\sigma\big(\gamma(t)\big) = \lambda_i(t)\iota_{\alpha,i}\big(\gamma(t)\big)$ for unknown scalar functions $\lambda_i(t)$. Taking the generalised covariant derivative of σ along $X(t)$ locally gives

$$X(t)\cdot(d-A)\sigma\big(\gamma(t)\big) = \frac{d\lambda^i}{dt}\iota_{\alpha,i}\big(\gamma(t)\big) - X^\mu(t)A_\mu{}^i{}_j\big(\gamma(t)\big)\lambda^j(t)\iota_{\alpha,i}\big(\gamma(t)\big). \tag{2.95}$$

Thus if we set $X(t)\cdot(d-A)\sigma\big(\gamma(t)\big) = \omega_\alpha(p,0,\cdots,0)$ then we get the system of ODEs

$$\frac{d\lambda^i}{dt} - X^\mu(t)A_\mu{}^i{}_j\big(\gamma(t)\big)\lambda^j(t) = 0, \tag{2.96}$$

which are identical to (2.93). This means that we could also use this generalised covariant derivative to produce the IVPs that describes parallel transport.

2.9 Holonomy and recovering a connection from parallel transport.

Consider a vector bundle (E,π,M,F,G) where M has an open cover $C = \{U_\alpha\}$. Say this vector bundle has local connection $1-$forms A^α, defined on U_α for each U_α in C. Let, $\gamma : [0,1] \to M$ be a smooth closed path with $\gamma(0) = p$, and $b \in \pi^{-1}(p)$. Our local horizontal lift provides linear IVPs whose unique solution, $\gamma^\#(t)$, describes the parallel transport of $\gamma^\#(0) = b$ to $\gamma^\#(1) = b' \in \pi^{-1}(p)$ along γ. That is to say, this ODE solution describes an endomorphism of vector spaces

$$\Gamma(\gamma) : \pi^{-1}(p) \to \pi^{-1}(p), \tag{2.97}$$

that takes $b \in \pi^{-1}(p)$ as the initial condition to our parallel transport IVP (2.66), and returns the unique vector that b is parallel transported to. By the Picard–Lindelöf theorem [9, Sections 1.1 and 1.2], the map $\Gamma(\gamma)$ is invertible: For any smooth path $\gamma : [0,1] \to M$ define $\gamma' : [0,1] \to M$ such that $\gamma'(t) = \gamma(1-t)$ which implies that $\Gamma\big(\gamma'\big) = \Gamma(\gamma)^{-1}$.

This means that for any point $p \in M$, the set

$$\mathrm{hol}_p = \Big\{\Gamma(\gamma) : \gamma \text{ is a smooth closed path } [0,1] \to M \text{ and } \gamma(0) = p\Big\} \tag{2.98}$$

forms a group which we define as the *holonomy group* of the local connection $1-$forms $\{A_\alpha\}$, at p.

These linear maps $\Gamma(\gamma)$ can be generalised to smooth open paths; Consider the smooth path $\gamma : [0,1] \to M$ then we define the linear invertible map

$$\Gamma(\gamma)_t^\tau : \pi^{-1}\big(\gamma(t)\big) \to \pi^{-1}\big(\gamma(\tau)\big) \tag{2.99}$$

that parallel transports $b \in \pi^{-1}\big(\gamma(t)\big)$ to $b' \in \pi^{-1}\big(\gamma(\tau)\big)$ along γ, for any $t,\tau \in [0,1]$. By the Picard–Lindelöf theorem [9, Sections 1.1 and 1.2] we have the following conditions that $\Gamma(\gamma)_t^\tau$ satisfies

$$\Gamma(\gamma)_t^t = id : E_{\gamma(t)} \to E_{\gamma(t)}, \qquad \Gamma(\gamma)_s^\tau \circ \Gamma(\gamma)_t^s = \Gamma(\gamma)_t^\tau, \qquad \Gamma(\gamma)_t^\tau = \Big(\Gamma(\gamma)_\tau^t\Big)^{-1} \tag{2.100}$$

Now consider a smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = p \in U_\alpha$ and its horizontal lift $\gamma^\# : [0,1] \to E$ through the vector b. Let $X \in T_pM$ be the tangent vector of γ at the point p. Then by the definition of $\Gamma(\gamma)_0^t$ and the local horizontal lift (2.65), we locally have for all $f \in C^\infty(E)$

$$
\begin{aligned}
&d\omega_\alpha^{-1}\big(X_b^\#\big)(f) = \frac{d}{dt}\, f\Big(\Gamma(\gamma)_0^t \gamma^\#(0)\Big)\Big|_{t=0}\,,\\
&X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_b + b^i X\cdot A^j{}_i(p) \left.\frac{\partial f}{\partial v^i}\right|_b = \frac{d}{dt}\left(\Gamma(\gamma)_0^t b\right)^\mu\Big|_{t=0} \left.\frac{\partial f}{\partial x^\mu}\right|_b + \frac{d}{dt}\left(\Gamma(\gamma)_0^t b\right)^i\Big|_{t=0} \left.\frac{\partial f}{\partial v^i}\right|_b,\\
&X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_b + b^i X\cdot A^j{}_i(p) \left.\frac{\partial f}{\partial v^i}\right|_b = \left.\frac{d\gamma^\mu}{dt}\right|_{t=0} \left.\frac{\partial f}{\partial x^\mu}\right|_b + \frac{d}{dt}\left(\Gamma(\gamma)_0^t b\right)^i\Big|_{t=0} \left.\frac{\partial f}{\partial v^i}\right|_b,\\
&\qquad\qquad \implies b^i X\cdot A^j{}_i(p) = \frac{d}{dt}\left(\Gamma(\gamma)_0^t b\right)^j\Big|_{t=0}\,. \qquad (2.101)
\end{aligned}
$$

Where we have used the short hand $A^j{}_i(p) = A_\mu{}^j{}_i(p)dx^\mu$.

Equation (2.101) provides us a way of recovering the local connection 1-form from information on how objects in our fibre parallel transport. We will use this in chapter 3 to extract the Berry connection from the Schrödinger equation.

Chapter 3

Building the Berry bundle

In this chapter we introduce the notion of geometric phase in quantum systems and distinguish the special case of the Berry phase [4]. The model we introduce in this chapter is designed for the calculation of Berry phase, although it can be adapted for more general geometric phase [48]. The reason we focus on Berry phase is because it greatly simplifies our model, due to the adiabatic theorem [5]. We will show that for a given family of local Hamiltonians, that act on a finite dimensional Hilbert space, we can construct a fibre bundle. We will then equip this bundle with a particular set of local connection 1-forms known as local Berry connections. Under the influence of a Berry connection, quantum states being parallel transported around closed smooth paths in our base manifold, will accumulate a Berry phase. This reduces the problem of understanding a quantum system's Berry phase to understanding the holonomy of an associated fibre bundle. There exists several geometric models for Berry phase [8,12,20,24,27,38] but few making use of vector bundles. Our model most closely resembles the model introduced by Tanaka and Cheon in [48]. However, we use the local energy eigenstates as local basis sections for our model. This takes full advantage of the adiabatic theorem and shifts complexities from coefficients to derivatives.

3.1 Introduction to geometric phase

We are interested in geometric phases, in particular the Berry phase. A geometric phase is a phase difference that occurs when the parameters of a physical system are transported around a smooth closed path. It arises due to the geometrical properties of the parameter space on which our system's Hamiltonian is defined.

Geometric phases appear in both classical and quantum mechanics. A common method used to quantify geometric phase in classical mechanics is parallel transport, for example the Foucault pendulum [26] and the problem of polarized light in optical fibres [28].

Consider a smooth $m-$dimensional manifold $M \subseteq \mathbb{R}^m$, and a smooth Hamiltonian that takes a point in M to a self-adjoint operator on a Hilbert space $\mathcal{H}$

$$H : p \to H(p). \tag{3.1}$$

Let $H(p)$ have eigenvectors $\{|u_i(p)\rangle\}_{i=1}^n$ for all $p \in M$. These eigenvectors can be defined to smoothly vary over M. First, we prepare two identical states at $p \in M$, $|\psi_1(p)\rangle = |\psi_2(p)\rangle = \sum_{j=1}^n c^j |u_j(p)\rangle$. We then transport $|\psi_2(p)\rangle$ along some smooth closed path $\gamma : [0,1] \to M$ while keeping $|\psi_1(p)\rangle$ fixed at p.

The evolution of $|\psi_1(p)\rangle$ is completely described by the time evolution operator

$$|\psi_1(t,p)\rangle = U(t)|\psi_1(p)\rangle = e^{-iH(p)t}\sum_{j=1}^n c^j|u_j(p)\rangle = \sum_{j=1}^n e^{-iE_j(p)t}c^j|u_j(p)\rangle, \tag{3.2}$$

accumulating what we call a *dynamic phase*. We define *dynamic phases* as phases a quantum state will gain from the time evolution operator.

Naively applying the time evolution operator to $|\psi_2(p)\rangle$ leads to the result

$$|\psi_2(t,p)\rangle \stackrel{!}{=} e^{-i\int_0^t H(\gamma(\tau))d\tau}|\psi_2(p)\rangle. \tag{3.3}$$

However, this fails to agree with experimentation in general, with the usual example being the Aharonov–Bohm effect [2]. We call the phase accumulated by $|\psi_2(p)\rangle$ that is not explained by the time evolution operator a *geometric phase*: In terms of our example, let $|\psi_2(1,p)\rangle$ describe the second state after being transported along γ. Then

$$\exp(i\phi_{\text{geo}}) = \langle\psi_2(p)|e^{i\int_0^1 H(\gamma(\tau))d\tau}|\psi_2(1,p)\rangle. \tag{3.4}$$

For a given quantum system with Hamiltonian H, our goal is to use parallel transport to extract only the information about the geometric phase from the standard time dependent Schrödinger equation (TDSE). We will do this using a fibre bundle equipped with an Ehresmann connection.

To understand how our local connection 1-forms should behave, we will enforce that our quantum states obey the TDSE for all time. Thus, we can study an example of geometric phase using the TDSE. However, the notion of geometric phase is very general so we will restrict ourselves to the geometric phase accumulated along *adiabatic paths*.

An *adiabatic path* is a smooth path in our base manifold $\gamma : [0,1] \to M$, that evolves slowly enough that all energy eigenstates $\{|u_i\rangle\}_{i=1}^n$ remain as eigenstates over the course of the entire path:

$$H(\gamma(t))\,|u_i(\gamma(t))\rangle = E_i(\gamma(t))\,|u_i(\gamma(t))\rangle\,, \tag{3.5}$$

for all $i \in 1,\dots,n$ and $t \in [0,1]$.

We justify this by using the *adiabatic theorem* which states [5]: A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum. The gap in the eigenspectrum is necessary as it ensures the smoothness of the Hamiltonian's eigenvalue and eigenvector [25, Theorem 1.8].

Under an adiabatic path, an eigenstate will naturally accumulate a dynamic phase due to the time evolution operator, but may also accumulate a geometric phase.

Example 3.1.1. Consider a smooth $m-$dimensional manifold $M \subseteq \mathbb{R}^m$. Each dimension corresponds to a parameter that an experimentalist can in principle vary. For example the strength and direction of a magnetic field.
At any point $p \in M$ we attach an $n-$dimensional Hilbert space on which our smooth Hamiltonian $H(p)$ is defined. We will enforce that for every $p \in M$ our Hamiltonian, $H(p)$, is non-degenerate.
Let $\gamma : [0,1] \to M$ be a smooth adiabatic path with $\gamma(0) = p$. Furthermore, assume that along our entire path we can use the local coordinates $\{q^i\}_{i=1}^m$. We want to find the solution to the time dependent Schrödinger equation

$$i\frac{d}{d\tau}|\psi\big(\gamma(\tau)\big)\rangle = H\big(\gamma(\tau)\big)|\psi\big(\gamma(\tau)\big)\rangle, \tag{3.6}$$

with initial condition $\big|\psi\big(\gamma(0)\big)\big\rangle = \big|u_k(p)\big\rangle$. Because $|\psi\rangle$ starts as the k^{th} energy eigenstate, then by the adiabatic theorem (3.5) it will remain as the k^{th} energy eigenstate, up to some phase, for the entirety of the adiabatic process. This is because the energy eigenvalues never cross. That is to say for all $\tau \in [0,1]$ we have

$$H\big(\gamma(\tau)\big)\,\big|\psi\big(\gamma(\tau)\big)\big\rangle = E_k(\gamma(\tau))\,\big|\psi\big(\gamma(\tau)\big)\big\rangle\,. \tag{3.7}$$

Let us attempt to solve the TDSE (3.6) using an integrating factor.

$$\begin{aligned}
0 &= i\frac{d}{d\tau}\left|\psi\big(\gamma(\tau)\big)\right\rangle - H\big(\gamma(\tau)\big)\left|\psi\big(\gamma(\tau)\big)\right\rangle, \\
&= \frac{d}{d\tau}|\psi\big(\gamma(\tau)\big)\rangle + iE_k\big(\gamma(\tau)\big)\left|\psi\big(\gamma(\tau)\big)\right\rangle, \\
&\overset{!}{=} \frac{d}{d\tau}\left[\exp\left(i\int_0^\tau E_k\big(\gamma(\tau')\big)d\tau'\right)\left|\psi\big(\gamma(\tau)\big)\right\rangle\right], \\
\implies \left|\psi\big(\gamma(\tau)\big)\right\rangle &= \exp\left(-i\int_0^\tau E_k(\gamma(\tau'))d\tau'\right)\left|u_k(p)\right\rangle.
\end{aligned}$$

But this fails to be an eigenstate for our Hamiltonian at every point along γ: In general

$$H(\gamma(\tau))\left|\psi\big(\gamma(\tau)\big)\right\rangle = \exp\left(-i\int_0^\tau E_k(\gamma(\tau'))d\tau'\right)H\big(\gamma(\tau)\big)|u_k(p)\rangle$$

will not be equivalent to

$$E_k\big(\gamma(\tau)\big)\left|\psi\big(\gamma(\tau)\big)\right\rangle$$

Instead, take an Ansatz that accounts for the accumulation of a geometric phase

$$\left|\psi\big(\gamma(\tau)\big)\right\rangle = \exp\left(i\phi(\tau) - i\int_0^\tau E_k\big(\gamma(\tau')\big)d\tau'\right)\left|u_k\big(\gamma(\tau)\big)\right\rangle. \tag{3.8}$$

This state remains as an eigenstate of our Hamiltonian along the entirety of γ, satisfying condition (3.7). If we also let $\phi(0) = 0$, then our Ansatz will also satisfies our initial condition. Substituting our Ansatz into the TDSE equation (3.6) gives

$$\begin{aligned}
E_k\big(\gamma(\tau)\big)\left|\psi\big(\gamma(\tau)\big)\right\rangle &= i\frac{d}{d\tau}\exp\left(i\phi(\tau) - i\int_0^\tau E_k\big(\gamma(\tau')\big)d\tau'\right)\left|u_k\big(\gamma(\tau)\big)\right\rangle \\
&\qquad + \exp\left(i\phi(\tau) - i\int_0^\tau E_k\big(\gamma(\tau')\big)d\tau'\right)\frac{d}{d\tau}\left|u_k\big(\gamma(\tau)\big)\right\rangle, \\
&= -\left(\frac{d\phi}{d\tau} - E_k\big(\gamma(\tau)\big)\right)\exp\left(i\phi(\tau) - i\int_0^\tau E_k\big(\gamma(\tau')\big)d\tau'\right)\left|u_k\big(\gamma(\tau)\big)\right\rangle \\
&\qquad + \exp\left(i\phi(\tau) - i\int_0^\tau E_k\big(\gamma(\tau')\big)d\tau'\right)\left|\frac{d}{d\tau}u_k\big(\gamma(\tau)\big)\right\rangle, \\
\implies 0 &= -\frac{d\phi}{d\tau}\exp\left(i\phi(\tau) - i\int_0^\tau E_k\big(\gamma(\tau')\big)d\tau'\right)\left|u_k\big(\gamma(\tau)\big)\right\rangle \\
&\qquad + \exp\left(i\phi(\tau) - i\int_0^\tau E_k\big(\gamma(\tau')\big)d\tau'\right)\left|\frac{d}{d\tau}u_k\big(\gamma(\tau)\big)\right\rangle, \\
&= \exp\left(i\phi(\tau) - i\int_0^\tau E_k\big(\gamma(\tau')\big)d\tau'\right)\left(\left|\frac{d}{d\tau}u_k\big(\gamma(\tau)\big)\right\rangle - \frac{d\phi}{d\tau}\left|u_k\big(\gamma(\tau)\big)\right\rangle\right), \\
&\implies \frac{d\phi}{d\tau} = \left\langle u_k\big(\gamma(\tau)\big)\right|\left.\frac{d}{d\tau}u_k\big(\gamma(\tau)\big)\right\rangle.
\end{aligned} \tag{3.9}$$

Integrating both sides of (3.9) leads to

$$\phi(\tau) = \int_0^\tau \left\langle u_k(\gamma(\tau))\right|\left.\frac{d}{d\tau}u_k(\gamma(\tau))\right\rangle d\tau = \int_\gamma \left\langle u_k(\mathbf{s})\right|\left.\frac{\partial}{\partial q^i}u_k(\mathbf{s})\right\rangle dq^i, \tag{3.10}$$

where the q^i are the local coordinates we assumed we could take along the entire path γ. We have shown that the phase $\phi(\tau)$ is completely dependent on the adiabatic path $\gamma : [0,1] \to M$, and thus this phase is due to the geometry of the manifold M.

If γ is a closed adiabatic path, then we call the differences in geometric phase

$$\theta_{jk} = \oint_\gamma \left(\left\langle u_j(\mathbf{s}) \middle| \frac{\partial}{\partial q^i} u_j(\mathbf{s}) \right\rangle - \left\langle u_k(\mathbf{s}) \middle| \frac{\partial}{\partial q^i} u_k(\mathbf{s}) \right\rangle \right) dq^i, \tag{3.11}$$

a *Berry phase*.

Based on the above example 3.1.1, we aim to build a more intuitive and general geometric model, that makes use of vector bundles in order to calculate the accumulation of Berry phases for general pure states, and not just energy eigenstates. We will start by establishing our base manifold, the parameter space, corresponding to a given family of local Hamiltonians $\{H^\alpha\}$. We will then define our fibre above $p \in M$ to be the $n-$dimensional Hilbert space, that our Hamiltonians act on. In the spirit of quantum mechanics we will take energy eigenstates of our Hamiltonians to be the local basis sections of each fibre, giving the local view of our model a familiar and useful form. In doing this we introduce a variable coordinate system that moves complexities from coefficients, into derivatives. We will then test this model on the well studied spin$-\frac{1}{2}$ system in chapter 4, and finally use it to explore the holonomy of NV center in chapter 5.

3.2 Constructing a vector bundle from a family of local Hamiltonians

Recall that in section 2.2 we stated that the minimal amount of information needed to construct a fibre bundle is a base manifold M with open cover $\{U_\alpha\}$, a fibre F, a structure group G and transition functions $\{t_{\alpha\beta}\}$. We will now build each of these from a family of local Hamiltonians.

Firstly, we define a *family of local n-dimensional Hamiltonians* as a set of smooth n-dimensional Hamiltonians, $\{H^\alpha\}_\alpha$, that are related by unitary transformations: For Hamiltonians H and H', in a family of local n-dimensional Hamiltonians, we have

$$H' = UHU^\dagger, \qquad \text{for some } U \in U(n). \tag{3.12}$$

The term "local" should be understood in the same way that a gauge field is local. That is to say that these Hamiltonians will only be defined on the domains of our base manifold's charts.

Given a local Hamiltonian $H^\alpha(p)$ that is dependent on m parameters $p \in \mathbb{R}^m$, we are interested in how pure states accumulate Berry phase. In order to study this we need to avoid points in $\mathbb{R}^m$ where H^α is degenerate. We define the *parameter space* of the Hamiltonian H^α to be

$$M = \{p \in \mathbb{R}^m : H^\alpha(p) \text{ is non-degenerate}\}. \tag{3.13}$$

Because the transformations in equation (3.12) leave eigenvalues unchanged, then every member of our family of local Hamiltonians will be non-degenerate on M. We will take the parameter space of a local Hamiltonian H^α, as the base manifold of our vector bundle, and we will refer to this manifold as the parameter space.

We claim that the parameter space (3.13) is a smooth manifold. Let us verify this: First we state that we are only interested in smooth manifolds with metrizable topologies. Consider the smooth manifold $\mathbb{R}^m$ with atlas $\{(U_\alpha, \phi_\alpha)\}$. Say we have a smoothly varying local Hamiltonian on $\mathbb{R}^m$ that acts on some $n-$dimensional Hilbert space $\mathcal{H}$. Let

$$\{E_i : \mathbb{R}^m \to \mathbb{R}\}_{i=1}^n$$

be the set of energy eigenvalues of our Hamiltonian. We have degeneracy in $H(p)$ whenever $E_i(p) = E_j(p)$ for distinct $i, j \in \{1, \dots, n\}$. Consider the functions

$$\Phi_{ij} : \mathbb{R}^m \to \mathbb{R}, \quad \Phi_{ij}(p) = E_i(p) - E_j(p). \tag{3.14}$$

Since the energy eigenvalues are roots of the characteristic polynomial, then Φ_{ij} is continuous for all $i, j \in \{1, \dots, n\}$ [36].

The zero locus of Φ_{ij} gives us all the points in $\mathbb{R}^m$ in which $E_i(p) = E_j(p)$. Under a metrizable topology the set $\{0\}$ is closed, and therefore by the continuity of Φ_{ij}, the set $\Phi_{ij}^{-1}(0)$ is closed for all $i, j \in \{1, \dots, n\}$. This implies that $M = \mathbb{R}^m \backslash \bigcup_{i \neq j=1}^{n} \Phi_{ij}^{-1}(0) \subseteq \mathbb{R}^m$ is an open set. Finally, M inherits the subspace topology and a smooth structure by fixing the atlas $\{(M \cap U_\alpha, \omega_\alpha)\}$. This proves that the parameter space is a smooth manifold.

When choosing the atlas for our parameter space M we often want to use coordinate systems that take advantage of the symmetries in a local Hamiltonian and its energy eigenvalues.

Example 3.2.1. Here is an example of a 2-dimensional Hamiltonian

$$H^z(x, y, z) = x\sigma^x + y\sigma^y + z\sigma^z + = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}, \tag{3.15}$$

where σ^μ are the Pauli spin matrices. H^z has the energy eigenvalues

$$E_\pm(x, y, z) = \pm\sqrt{x^2 + y^2 + z^2}. \tag{3.16}$$

There is only one point of energy degeneracy at $x = y = z = 0$ and therefore our parameter space is given by

$$\mathbb{R}^3 \backslash \{0\}. \tag{3.17}$$

Seeing as the energy eigenvalues are symmetric under SO(3), a useful set of coordinates are spherical coordinates. Hence, we establish the domains

$$U_{z,1} = \mathbb{R}^3 \backslash \{(x, 0, z) : x, z \in \mathbb{R} \text{ and } x \geq 0\}, \quad U_{z,2} = \mathbb{R}^3 \backslash \{(x, 0, z) : x, z \in \mathbb{R} \text{ and } x \leq 0\}.$$

with trivialisations

$$\Phi_{z,1} : U_{z,1} \to \mathbb{R}^3, \quad (x, y, z) \mapsto (r, \theta, \phi_1), \tag{3.18}$$

$$\Phi_{z,2} : U_{z,2} \to \mathbb{R}^3, \quad (x, y, z) \mapsto (r, \theta, \phi_2), \tag{3.19}$$

where $(r, \theta, \phi_{1,2})$ are the standard spherical coordinates with $r > 0$, $\theta \in (0, \pi)$, $\phi_1 \in (0, 2\pi)$ and $\phi_2 \in (-\pi, \pi)$.
From here we define $H^z \circ \Phi_{z,j}$ as the Hamiltonian local to the domain $U_{z,j}$. That is for $(q_1, q_2, q_3) \in U_{z,j}$

$$H^z \circ \Phi_{z,j}(q_1, q_2, q_3) = r \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi_j} \\ \sin(\theta)e^{i\phi_j} & -\cos(\theta) \end{pmatrix} \tag{3.20}$$

Note that the domains (3.18) and (3.19) do not provide an open cover for (3.17). To fix this we introduce another set of domains

$$\begin{aligned} U_{x,1} &= \mathbb{R}^3 \backslash \{(x, y, 0) : x, y \in \mathbb{R} \text{ and } y \geq 0\}, \\ U_{x,2} &= \mathbb{R}^3 \backslash \{(x, y, 0) : x, y \in \mathbb{R} \text{ and } y \leq 0\}. \end{aligned} \tag{3.21}$$

with trivialisations

$$\begin{aligned} \Phi_{x,1} &: U_{x,1} \to \mathbb{R}^3, \quad (x, y, z) \mapsto (r, \vartheta, \varphi_1), \\ \Phi_{x,2} &: U_{x,2} \to \mathbb{R}^3, \quad (x, y, z) \mapsto (r, \vartheta, \varphi_2). \end{aligned} \tag{3.22}$$

However, under these local coordinates, H^z becomes difficult to use. Example 3.23 tackles this problem in further detail, but first we need to establish our transition maps.

To the parameter space M with open cover $\{U^\alpha\}$, we prescribe the fibre given by the $n-$dimensional Hilbert space $\mathcal{H}$ that our local Hamiltonians act on. For each $p \in U^\alpha$, the local Hamiltonian $H^\alpha \circ \Phi_\alpha(p)$ (with α fixed) has n non-degenerate energy eigenvectors $\{|u_i(p)\rangle\}_{i=1}^n$. These eigenvectors form an orthonormal basis for our Hilbert space for all $p \in U_\alpha$ and thus they will be used as our local basis vectors in each fibre $\mathcal{H}_p$. Furthermore, because the Hamiltonian is non-degenerate on M, the eigenvectors may be chosen to vary smoothly over the parameter space [25, Theorem 1.8].

Now that we have established our base manifold and fibre we need to consider the transition maps. Let $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ be an atlas for M and take $p \in U_\alpha \cap U_\beta \neq \emptyset$. Let $|u_i(p)\rangle_\alpha$ denote our i^{th} energy eigenstate of the local Hamiltonian $H^\alpha \circ \Phi_\alpha$, and let ${}_\alpha\langle u^i(p)|$ be its dual. We then define our transition maps as

$$t_{\beta,\alpha}(p) = |u_i(p)\rangle_{\beta\,\alpha}\langle u^i(p)|. \tag{3.23}$$

Thus, our transition maps describe a change of basis vectors.

These transition maps satisfy the consistency condition (2.3) because the non-degenerate eigenvectors of H form a complete set. Take $p \in U_\alpha$

$$t_{\alpha,\alpha}(p) = |u_i(p)\rangle_{\alpha\,\alpha}\langle u^i(p)| = id_{\mathcal{H}}, \tag{3.24}$$

where the last equality holds because the eigenstate basis is non-degenerate over all U_α. Now for the gluing condition, take $p \in U_\alpha \cap U_\beta \neq \emptyset$

$$\begin{aligned} t_{\alpha,\beta}(p)t_{\beta,\alpha}(p) &= |u_i(p)\rangle_{\alpha\ \beta}\langle u^i(p)|u_j(p)\rangle_{\beta\ \alpha}\langle u^j(p)| = |u_i(p)\rangle_\alpha \delta^i{}_{j\,\alpha}\langle u^j(p)| = id_{\mathcal{H}}, \\ t_{\beta,\alpha}(p)t_{\alpha,\beta}(p) &= |u_i(p)\rangle_{\beta\ \alpha}\langle u^i(p)|u_j(p)\rangle_{\alpha\ \beta}\langle u^j(p)| = |u_i(p)\rangle_{\beta\,\beta}\langle u^i(p)| = id_{\mathcal{H}}, \\ \implies t_{\alpha,\beta}(p) &= t^{-1}_{\beta,\alpha}(p). \end{aligned} \tag{3.25}$$

Finally, we have the associativity condition. Take $p \in U_\alpha \cap U_\beta \cap U_\kappa \neq \emptyset$

$$\begin{aligned} t_{\alpha,\beta}(p)t_{\beta,\kappa}(p) &= |u_i(p)\rangle_{\alpha\ \beta}\langle u^i(p)|u_j(p)\rangle_{\beta\ \kappa}|u^j(p)\rangle = |u_i(p)\rangle_\alpha \delta^i{}_{j\,\kappa}\langle u^j(p)|, \\ &= |u_i(p)\rangle_{\alpha\,\kappa}\langle u^i(p)| = t_{\alpha,\kappa}(p). \end{aligned} \tag{3.26}$$

Example 3.2.2. Consider the local Hamiltonian (3.15) again, with the atlas given in example 3.2.1. The Hamiltonian, H_z under the trivialisation $\Phi_{x,1}$ defined in (3.22), takes the form

$$H^z(\omega_{x,1}(q_1,q_2,q_3)) = r\begin{pmatrix} \sin(\vartheta)\sin(\varphi_1) & \cos(\vartheta) - i\sin(\vartheta)\cos(\varphi_1) \\ \cos(\vartheta) - i\sin(\vartheta)\cos(\varphi_1) & -\sin(\vartheta)\sin(\varphi_1) \end{pmatrix}. \tag{3.27}$$

Note that the local Hamiltonian (3.15) was defined on the basis of σ^z eigenstates. We could simplify (3.27) by changing its basis into σ^x eigenstates. We will call this local Hamiltonian, H^x.

Diagonalising σ^x gives us the relation for all $p \in U_x \cap U_z$

$$\begin{aligned} |u_1(p)\rangle_x &= \frac{1}{\sqrt{2}}\left(|u_1(p)\rangle_z + |u_2(p)\rangle_z\right), \\ |u_2(p)\rangle_x &= \frac{1}{\sqrt{2}}\left(|u_1(p)\rangle_z - |u_2(p)\rangle_z\right), \end{aligned}$$

and thus our transition map (3.23) is given by the unitary matrix

$$t_{x,z}(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This means that for all $p \in U_x \cap U_z$ we have

$$H^x(p) = t_{x,z}(p) H^z(p) t^{-1}_{x,z}(p),$$

and our local Hamiltonian on $U_{x,j}$ is given by

$$H^x\big(\Phi_{x,j}(q_1, q_2, q_3)\big) = r \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) e^{-i\varphi} \\ \sin(\vartheta) e^{i\varphi} & -\cos(\vartheta) \end{pmatrix}.$$

Lastly, our structure group is the largest Lie group that preserves the self adjoint condition of the Hamiltonian, and leaves observables invariant for every $p \in M$. Our structure group is $U(n)$.

We can now use the construction method outlined in section 2.2. We define

$$X = \coprod_{\alpha} (U_\alpha \times \mathcal{H})$$

and the equivalence relation on X: $(p, \psi_1(p)) \sim (q, \psi_2(q))$ if and only if $p = q$ and there exists a transition function $t_{\alpha\beta}(p)$ such that $\psi_2(q) = t_{\alpha\beta}(p)\psi_1(p)$. Thus our vector bundle is

$$E = X/\sim, \tag{3.28}$$

and our trivialisations are defined as

$$\omega_\alpha : U_\alpha \times \mathcal{H} \to \pi^{-1}(U_\alpha), \quad \omega_\alpha\Big(p, {}_\alpha\langle u^i(p) | \psi(p)\rangle | u_i(p)\rangle_\alpha\Big) = \Big[\big(p, \psi(p)\big)\Big]. \tag{3.29}$$

We can prescribe further structure by defining a dual bundle and a bundle of endomorphisms. We build our dual bundle E^* by using the method discussed in section 2.3. The bundle of endomorphisms is defined as the tensor product bundle $E \otimes E^*$. Thus, by our fibre bundle construction method in section 2.2 we construct a vector bundle $(E, \pi, M, \mathscr{H}, U(n))$, its dual $(E^*, \pi^*, M, \mathscr{H}^*, U(n))$ and its bundle of endomorphisms $(E \otimes E^*, \hat{\pi}, M, \mathscr{H}^* \otimes \mathscr{H}, U(n))$. The action of $U \in U(n)$ on the dual bundle is described by

$$U \cdot \Big[\big(p, \langle \psi(p)| \,\big)\Big] = \Big[\big(p, \langle \psi(p)| \, U^\dagger\big)\Big], \tag{3.30}$$

and the action of $U \in U(n)$ on the bundle of endomorphisms is described by

$$U \cdot \Big[\big(p, |\phi(p)\rangle\langle\psi(p)|\big)\Big] = \Big[\big(p, U|\phi(p)\rangle\langle\psi(p)|U^\dagger\big)\Big]. \tag{3.31}$$

In our model, the vector bundle E consists of all the physically relevant kets, while the dual bundle consists of all bras. The bundle of endomorphisms consists of all the linear operators that act on E. The self-adjoint ones are observables O:

$$\langle\psi(p)|\Big(O|\phi(p)\rangle\Big) = \Big(\langle\psi(p)|O^\dagger\Big)|\phi(p)\rangle, \tag{3.32}$$

for all $|\phi(p)\rangle, |\psi(p)\rangle \in \mathcal{H}$.

It is also worth noting how elements in the dual and endomorphism bundles transform locally. For $\Big[\big(p, \langle\psi(p)|\big)\Big] \in E^*_p$ with $p \in U_\alpha \cap U_\beta \neq \emptyset$ we have

$$\Big(p, {}_\alpha\langle\psi(p)|\Big) = \Big(p, {}_\beta\langle\psi(p)|t_{\beta,\alpha}(p)\Big). \tag{3.33}$$

For $\Big[\big(p, |\phi(p)\rangle\langle\psi(p)|\big)\Big] \in E \otimes E^*$ with $p \in U_\alpha \cap U_\beta \neq \emptyset$ we have

$$\Big(p, |\phi(p)\rangle_\alpha {}_\alpha\langle\psi(p)|\Big) = \Big(p, t_{\alpha,\beta}(p)|\phi(p)\rangle_\beta \, {}_\beta\langle\psi(p)|t_{\beta,\alpha}(p)\Big). \tag{3.34}$$

The vector bundle we have defined in this section is not yet a Berry bundle. We first need to prescribe variable basis sections and then local Berry connections.

3.3 Local coordinates and variable basis sections

Now that we have our vector bundle we want to establish local coordinates on our fibre, $\mathcal{H}$. So far we have hinted that we will take energy eigenstates of our local Hamiltonians to be the local coordinates of $\mathcal{H}_p$ for all $p \in M$. That is to say, for parameter space M with open cover $\{U_\alpha\}$ and $p \in U_\alpha$ the local Hamiltonian $H^\alpha(p)$ has n non-degenerate eigenstates that form an orthonormal basis on $\mathcal{H}_p$. Thus for $\Big[\big(p, |\psi\rangle\big)\Big] \in \pi^{-1}(p)$

$$\omega_\alpha^{-1}\bigg(\Big[\big(p, |\psi\rangle\big)\Big]\bigg) = \big(p, v^i |u_i(p)\rangle_\alpha\big) \tag{3.35}$$

where $v^i = {}_\alpha\big\langle u^i(p)\big|\,\psi\,\big\rangle$. This gives us a way to define local basis sections on our vector bundle

$$\iota_{\alpha,i}(p) = \omega_\alpha\big(p, |u_i(p)\rangle_\alpha\big) = \Big[\big(p, \big|u_i(p)\big\rangle_\alpha\big)\Big], \tag{3.36}$$

and thus for any state $|\psi\rangle \in E_p$ we have

$$\Big[\big(p, |\psi\rangle\big)\Big] = \bigg[\Big(p, v^i \big|u_i(p)\big\rangle_\alpha\Big)\bigg] = v^i\bigg[\Big(p, \big|u_i(p)\big\rangle_\alpha\Big)\bigg] = v^i \iota_{\alpha,i}(p). \tag{3.37}$$

Taking the local coordinates on U_α to be $\{x^\mu\}$, gives us the local coordinates in $\pi^{-1}(U_\alpha)$ of the form

$$\big(x^\mu; v^i\big). \tag{3.38}$$

However, this is no ordinary coordinate system. The local basis sections are described by energy eigenstates of our local Hamiltonians, and thus can be chosen to vary smoothly on the fibre. This variation in the coordinate system must be carefully considered as it has implications on how tangent vectors behave.

Example 3.3.1. Consider the problem of how tangent vectors behave locally if variations on the base manifold coordinates cause variations in the fibre coordinates: We want to define the vector tangent to the smooth path $\lambda : [0,1] \to \pi^{-1}(U_\alpha)$ with $\pi \circ \lambda = \gamma$, at the point $\lambda(0) = b \in \pi^{-1}(p)$. Locally, λ takes the form

$$\omega_\alpha^{-1}\big(\lambda(t)\big) = \bigg(\gamma(t); \lambda^i(t)\Big|u_i\big(\gamma(t)\big)\Big\rangle_\alpha\bigg). \tag{3.39}$$

Let Y be the vector that is tangent to $\lambda(t)$ at $t = 0$ and let $d\pi(Y) = X \in T_pM$. Then for all $f \in C^\infty(M \times \mathcal{H})$ we have

$$d\omega_\alpha^{-1}\big(Y\big)(f) = \frac{d}{dt} f\bigg(\gamma(t); \lambda^i(t)\Big|u_i\big(\gamma(t)\big)\Big\rangle_\alpha\bigg)\bigg|_{t=0}.$$

Now we see the impact of taking variable basis vectors on our Hilbert space $\mathcal{H}$. If our local basis vectors in our fibre are changing with the parameter t, then we will need to take their change into account. To continue from this point we will need to fix our local basis vectors. We can do this by expanding our energy eigenstates along our path in terms of energy eigenstates at some fixed point in M. While any point in M will work, the point $\gamma(0) = p$ is the natural choice because we want to find the

tangent vector at $b \in \pi^{-1}(p)$. Thus, we have

$$\begin{aligned} d\omega_\alpha^{-1}(Y)(f) &= \frac{d}{dt} f\left(\gamma(t); \lambda^i(t)\, {}_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma(t)\big)\Big\rangle_\alpha \Big|u_j(p)\Big\rangle_\alpha\right)\Big|_{t=0}, \\ &= \left.\frac{d\gamma^\mu}{dt}\right|_0 \left.\frac{\partial f}{\partial x^\mu}\right|_b + \frac{d}{dt}\left(\lambda^i(t)\, {}_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma(t)\big)\Big\rangle_\alpha\right)\Big|_{t=0} \left.\frac{\partial f}{\partial v^j}\right|_b, \\ &= X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_b + \left.\frac{d\lambda^i}{dt}\right|_0 {}_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma(0)\big)\Big\rangle_\alpha \left.\frac{\partial f}{\partial v^j}\right|_b \\ &\qquad + \lambda^i(0)\, {}_\alpha\left\langle u^j(p)\middle|\frac{d}{dt}\, u_i\big(\gamma(t)\big)\big|_0\right\rangle_\alpha \left.\frac{\partial f}{\partial v^j}\right|_b, \\ &= X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_b + \left(\left.\frac{d\lambda^i}{dt}\right|_0 \delta^j{}_i + b^i \left.\frac{d\gamma^\mu}{dt}\right|_0 {}_\alpha\left\langle u^j(p)\middle|\left.\frac{\partial u_i}{\partial x^\mu}\right|_p\right\rangle_\alpha\right) \left.\frac{\partial f}{\partial v^j}\right|_b. \end{aligned}$$

We find the vector tangent to λ at $t = 0$ takes the local form

$$\begin{aligned} d\omega_\alpha^{-1}(Y) &= X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_b + \left(\left.\frac{d\lambda^j}{dt}\right|_0 + b^i X \cdot {}_\alpha\big\langle u^j(p)\big|du_i(p)\big\rangle_\alpha\right) \left.\frac{\partial}{\partial v^j}\right|_b, \\ &= X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_b + \left(Y^j + b^i X \cdot {}_\alpha\big\langle u^j(p)\big|du_i(p)\big\rangle_\alpha\right) \left.\frac{\partial}{\partial v^j}\right|_b, \end{aligned} \tag{3.40}$$

where d is the exterior derivative $\Omega^0(M) \to \Omega^1(M)$.
Equation (3.40) describes a local tangent vector of E. What we have found is something that looks very much like a local horizontal lift (2.65) of a vector $X \in T_pM$. In fact, the only difference is the additional vertical vector

$$Y^i \frac{\partial}{\partial v^i}|_b.$$

The example 3.3.1 demonstrates that when we prescribe our local variable basis sections to our vector bundle we can take

$$\left(\left.\frac{\partial}{\partial x^\mu}\right|_b + b^i\, {}_\alpha\big\langle u^j(p)\big|du_i(p)\big\rangle_\alpha \left.\frac{\partial}{\partial v^j}\right|_b ; \left.\frac{\partial}{\partial v^i}\right|_b\right), \tag{3.41}$$

as our basis for local tangent vectors of T_bM with $\pi(b) = p$. These basis vectors are tangent to paths that only vary along a single local coordinate in E. However, for T_bE we will stick with the local basis

$$\left(\left.\frac{\partial}{\partial x^\mu}\right|_b ; \left.\frac{\partial}{\partial v^i}\right|_b\right). \tag{3.42}$$

This will allow us to translate a lot of our work from the earlier derivation of the horizontal lift in section 2.5. But, in order to use these basis vectors, we will need to understand what curves they are tangent to. In example 3.3.1 we found that a tangent vector to the local curve (3.39) takes the local form of (3.40). With this in mind consider the path $\eta_\mu : [0,1] \to \pi^{-1}(U_\alpha)$ along the x^μ direction. Let $\pi\big(\eta_\mu(t)\big) = \gamma_\mu(t)$ for all $t \in [0,1]$ and $\eta_\mu(0) = b \in \pi^{-1}(p)$. Locally, η_μ takes the form

$$\omega_\alpha^{-1}\big(\eta_\mu(t)\big) = \left(\gamma_\mu(t), \left(2b^j - b^i\, {}_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma(t)\big)\Big\rangle_\alpha\right)\Big|u_j\big(\gamma(t)\big)\Big\rangle_\alpha\right), \tag{3.43}$$

with γ_μ^ν defined in (2.52).

By example 3.3.1 the local tangent vector to η_μ at $t = 0$ is

$$\begin{aligned} d\omega_\alpha^{-1}(Y) &= \frac{d\gamma_\mu^\nu}{dt}\,\frac{\partial}{\partial x^\nu}\bigg|_b + \left(-b^i\,{}_\alpha\Big\langle u^j(p)\Big|\frac{d}{dt}\,u_i\big(\gamma(t)\big)\Big|_0\Big\rangle_\alpha + b^i X\cdot {}_\alpha\big\langle u^j(p)\big|du_i(p)\big\rangle_\alpha\right)\frac{\partial}{\partial v^j}\bigg|_b, \\ &= \delta_\mu^\nu\,\frac{\partial}{\partial x^\nu}\bigg|_b + \left(-b^i X\cdot {}_\alpha\big\langle u^j(p)\big|du_i(p)\big\rangle_\alpha + b^i X\cdot {}_\alpha\big\langle u^j(p)\big|du_i(p)\big\rangle_\alpha\right)\frac{\partial}{\partial v^j}\bigg|_b, \\ &= \frac{\partial}{\partial x^\mu}\bigg|_b. \end{aligned}$$

Similarly, by example 3.3.1 the path $h_i : [0,1] \to E$ described in equation (2.53) has local tangent vector at $t = 0$

$$\big(\omega_\alpha^{-1}\big)_*(Y) = \frac{dh_i}{dt}\bigg|_0\,\frac{\partial}{\partial v^j}\bigg|_b = \delta_i^j\,\frac{\partial}{\partial v^j}\bigg|_b = \frac{\partial}{\partial v^i}\bigg|_b.$$

The paths η_μ for $\mu \in \{1,\dots,m\}$ and h_i for $i \in \{1,\dots,n\}$ will be very useful in the derivation of the horizontal lift under our new local variable basis sections.

3.4 Horizontal lift

Now that we have well defined local basis sections described by the energy eigenstates of our local Hamiltonians, we can study parallel transport. In section 2.8 we discussed that the notion of parallel transporting a vector $v \in E$ over a smooth path $\gamma : [0,1] \to M$, is understood as finding the unique path in E that starts at v, projects to γ and whose tangent vector field is horizontal for all $t \in [0,1]$. That is to say the vector v is kept parallel with respect to an Ehresmann connection as it is transported. To find this unique path we built a local horizontal lift in section 2.5. However, that construction assumed the basis sections of E were constant on the fibre. We need to re-derive the horizontal lift with our local variable basis sections to see how it will change.

Let $(E,\pi,M,\mathcal{H},U(n))$ be our vector bundle with local Hamiltonians $\{H^\alpha\}$, and let $U_\alpha \subset M$ be a domain with local coordinates $\{x^\mu\}$. To $\pi^{-1}(U_\alpha)$ prescribe our local variable basis section $\{\iota_{\alpha,i}\}$. This gives $\pi^{-1}(U_\alpha)$ the set of local coordinates $\{(x^\mu;v^i)\}$. Now take $p \in U_\alpha$, $X \in T_pM$ tangent to some smooth path $\gamma : [0,1] \to M$ at $\gamma(0) = p$, and take some $b \in E_p$. We want to know what a local horizontal lift of X at b will look like.

Recall that a vector bundle's horizontal lift $\mathcal{G}$ obeys conditions (2.42), (2.44), (2.45) and (2.46).

Under our chosen local basis for T_bE (3.42) we have

$$d\omega_\alpha^{-1}\circ\mathcal{G}(b,X) = Y^\mu(b,X)\,\frac{\partial}{\partial x^\mu}\bigg|_b + \mathcal{Z}^i(b,X,Y)\,\frac{\partial}{\partial v^i}\bigg|_b, \tag{3.44}$$

for all $X \in T_pM$ and $b \in E_p$. Note that in example 3.3.1 we saw that tangent vectors with components in the $\{\frac{\partial}{\partial x^\mu}\big|_b\}$ directions, will have the coefficients of those components mix with the coefficients of vertical vectors. To account for this we need to include the vector Y in the argument of $\mathcal{Z}$.

By the projection condition (2.42) we have

$$d\pi\Big(\mathcal{G}(b,X)\Big)(f) = X^\mu\,\frac{\partial f}{\partial x^\mu}\bigg|_p.$$

Applying $d\pi$ to our ansatz (3.44) and using the paths (3.43), (2.53) , we find

$$\begin{aligned}
d\pi\Big(\mathcal{G}(b,X)\Big)(f) &= d\pi\circ d\omega_\alpha\left(Y^\mu\left.\frac{\partial}{\partial x^\mu}\right|_b+\mathcal{Z}^i(b,X,Y)\left.\frac{\partial}{\partial v^i}\right|_b\right)(f),\\
&= Y^\mu\frac{d}{dt}\Big(f\big(\pi\circ\eta_\mu(t)\big)\Big)\Big|_{t=0}+\mathcal{Z}^i(b,X,Y)\frac{d}{dt}\Big(f\big(\pi\circ h_i(t)\big)\Big)\Big|_{t=0},\\
&= Y^\mu\frac{d}{dt}\Big(f\big(\gamma_\mu(t)\big)\Big)\Big|_{t=0}+\mathcal{Z}^i(b,X,Y)\frac{d}{dt}\Big(f(p)\Big)\Big|_{t=0}=Y^\mu\left.\frac{d\gamma^\nu_\mu}{dt}\right|_0\left.\frac{\partial f}{\partial x^\nu}\right|_b,\\
&= Y^\mu\delta^\nu_\mu\left.\frac{\partial f}{\partial x^\nu}\right|_b=Y^\mu\left.\frac{\partial f}{\partial x^\mu}\right|_b.
\end{aligned}$$

Thus the projection condition (2.42) imposes

$$Y^\mu=X^\mu, \tag{3.45}$$

and therefore

$$\mathcal{Z}(b,X,Y)=\mathcal{Z}(b,X). \tag{3.46}$$

Similarly, because $\mathcal{G}$ is linear in X (2.44), we have

$$\mathcal{Z}^i(b,X)=Z_\mu{}^i(b)X^\mu \tag{3.47}$$

The last two conditions require a local form of $\gamma^\# : [0,1]\to E$. We will take this local form as

$$\omega_\alpha^{-1}\Big(\gamma^\#(t)\Big)=\omega_\alpha^{-1}\Big(\lambda^i(t)\iota_{\alpha,i}\big(\gamma(t)\big)\Big)=\Big(\gamma(t);\lambda^i(t)\left|u_i\big(\gamma(t)\big)\right\rangle_\alpha\Big). \tag{3.48}$$

The third condition on $\mathcal{G}$ is compatibility with scalar multiplication (2.45). Applying $f\in C^\infty(M\times\mathcal{H})$ to the right hand side of equation (2.45) gives

$$\begin{aligned}
d\omega_\alpha^{-1}\circ dm_c\Big(\big(G(b,X)\big)\Big)(f) &= d\omega_\alpha^{-1}\circ dm_c\circ d\omega_\alpha\left(X^\mu\left.\frac{\partial}{\partial x^\mu}\right|_b+Z^i(b,X)\left.\frac{\partial}{\partial v^i}\right|_b\right)(f),\\
&= X^\mu\frac{d}{dt}f\Big(m_c\circ\eta_\mu(t)\Big)\Big|_0+Z^i(b,X)f\Big(m_c\circ h_i(t)\Big)\Big|_0,\\
&= X^\mu\frac{d}{dt}f\Big(\gamma_\mu(t),c\Big(2b^j-b^i{}_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma(t)\big)\Big\rangle_\alpha\Big)\Big|u_j\big(\gamma(t)\big)\Big\rangle_\alpha\Big)\Big|_0\\
&\qquad+X^\mu Z_\mu{}^i(b)\frac{d}{dt}f\Big(c(b^j+\delta^j{}_i t)\iota_{\alpha,j}(p)\Big)\Big|_0.
\end{aligned}$$

Recall from example 3.3.1 that our local fibre coordinates vary along $\gamma^\#$. Thus, we choose to expand our energy eigenvectors $|u_i(\gamma(t))\rangle$ in terms of $|u_i(p)\rangle$. That is

$$\begin{aligned}
d\omega_\alpha^{-1}\circ dm_c\Big(\big(\mathcal{G}(b,X)\big)\Big)(f) &= Z^i(b,X)\frac{d}{dt}f\Big(c(b^j+\delta^j{}_i t)\iota_{\alpha,j}(p)\Big)\Big|_0\\
&\quad+X^\mu\frac{d}{dt}f\Big(\gamma_\mu(t),c\Big(2b^j-b^i{}_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma(t)\big)\Big\rangle_\alpha\Big)_\alpha\Big\langle u^k(p)\Big|u_j\big(\gamma_\mu(t)\big)\Big\rangle_\alpha\Big|u_k(p)\Big\rangle_\alpha\Big)\Big|_0,\\
&= cZ^i(b,X)\left.\frac{\partial f}{\partial v^i}\right|_{M_c(b)}\\
&\quad+X^\mu\left(\frac{d}{dt}\Big(c\Big(2b^j-b^i{}_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma(t)\big)\Big\rangle_\alpha\Big)_\alpha\Big\langle u^k(p)\Big|u_j\big(\gamma_\mu(t)\big)\Big\rangle_\alpha\Big)\Big|_0\left.\frac{\partial f}{\partial v^k}\right|_{M_c(b)}\right.\\
&\qquad\left.+\left.\frac{d\gamma^\nu_\mu(t)}{dt}\right|_0\left.\frac{\partial f}{\partial x^\nu}\right|_{M_c(b)}\right),\\
&= X^\mu\delta^\nu{}_\mu\left.\frac{\partial f}{\partial x^\nu}\right|_{M_c(b)}+cZ^i(b,X)\left.\frac{\partial f}{\partial v^i}\right|_{M_c(b)}\\
&\quad+X^\mu\left.\frac{d\gamma^\nu_\mu(t)}{dt}\right|_0\left(c\left(2b^j-b^i\delta^j{}_i\right)_\alpha\left\langle u^k(p)\left|\frac{\partial u_j}{\partial x^\nu}\right|_p\right\rangle_\alpha-cb^i{}_\alpha\left\langle u^j(p)\left|\frac{\partial u_i}{\partial x^\nu}\right|_p\right\rangle_\alpha\delta^k{}_j\right)\left.\frac{\partial f}{\partial v^k}\right|_{M_c(b)}
\end{aligned}$$

$$= X^\mu \Bigg(- cb^i \, {}_\alpha\Big\langle u^k(p) \Big| \frac{\partial u_i}{\partial x^\mu}\Big|_p \Big\rangle_\alpha + cb^j \, {}_\alpha\Big\langle u^k(p) \Big| \frac{\partial u_j}{\partial x^\mu}\Big|_p \Big\rangle_\alpha \Bigg) \frac{\partial f}{\partial v^k}\Big|_{m_c(b)} + X^\mu \frac{\partial f}{\partial x^\mu}\Big|_{m_c(b)} + cZ^i(b,X) \frac{\partial f}{\partial v^i}\Big|_{m_c(b)}. \tag{3.49}$$

This implies the right hand side of equation (2.45) takes the local form

$$d\omega_\alpha^{-1} \circ dm_c\big(G(b,X)\big) = X^\mu \frac{\partial}{\partial x^\mu}\Big|_{m_c(b)} + cX^\mu Z_\mu{}^i(b) \frac{\partial}{\partial v^i}\Big|_{m_c(b)}. \tag{3.50}$$

The left hand side of equation (2.45) is locally given by

$$d\omega_\alpha^{-1} \circ \mathcal{G}(m_c(b), X) = X^\mu \frac{\partial}{\partial x^\mu}\Big|_{m_c(b)} + X^\mu Z_\mu{}^i\big(cb^i \iota_i(p)\big) \frac{\partial}{\partial v^i}\Big|_{m_c(b)}. \tag{3.51}$$

Equating the left (3.51) and right (3.50) hand sides of (2.45) gives us the following relation

$$X^\mu Z_\mu{}^i\big(cb^j \iota_j(p)\big) = cX^\mu Z_\mu{}^i(b). \tag{3.52}$$

The fourth and last horizontal lift condition is vector addition compatibility (2.46). Let us employ the same strategy we used on the scalar multiplication compatibility condition. First we will focus on the right hand side of (2.46): By our ansatz (3.44) we know that $\mathcal{G}\big(\sigma(p), X\big)$ takes the local form

$$d\omega_\alpha^{-1} \circ \mathcal{G}\big(\sigma(p), X\big) = X^\mu \frac{\partial}{\partial x^\mu}\Big|_{\sigma(p)} + X^\mu Z_\mu{}^i\big(\sigma(p)\big) \frac{\partial}{\partial v^i}\Big|_{\sigma(p)}. \tag{3.53}$$

To find the local form of $d\sigma(X)$, we will apply an arbitrary function $f \in C^\infty(E)$ to it:

$$\begin{aligned}
\Big(d\omega_\alpha^{-1} \circ d\sigma(X)\Big)(f) &= X^\mu \frac{d}{dt} f\big(\sigma \circ \gamma_\mu(t)\big)\big|_{t=0} = X^\mu \frac{d}{dt} f\Big(\gamma_\mu(t), \sigma^i\big(\gamma_\mu(t)\big)\Big|u_i\big(\gamma_\mu(t)\big)\Big\rangle_\alpha\Big)\Big|_{t=0}, \\
&= X^\mu \frac{d}{dt} f\Big(\gamma_\mu(t), \sigma^i\big(\gamma_\mu(t)\big)_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma_\mu(t)\big)\Big\rangle_\alpha\Big|u_i(p)\Big\rangle)_\alpha\Big)\Big|_{t=0}, \\
&= X^\mu \frac{d\gamma_\mu^\nu}{dt}\Big|_0 \frac{\partial f}{\partial x^\nu}\Big|_{\sigma(p)} + X^\mu \frac{d}{dt}\Big(\sigma^i\big(\gamma(t)\big)_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma_\mu(t)\big)\Big\rangle_\alpha\Big)\Big|_0 \frac{\partial f}{\partial v^i}\Big|_{\sigma(p)},
\end{aligned}$$

which implies the local form of $d\sigma(X)$

$$d\sigma(X) = X^\mu \frac{\partial}{\partial x^\mu}\Big|_{\sigma(p)} + \Big(X^\mu \frac{\partial \sigma^i}{\partial x^\mu}\Big|_p + \sigma^i(p) X \cdot {}_\alpha\big\langle u^j(p)\big|du_i(p)\big\rangle_\alpha\Big) \frac{\partial}{\partial v^j}\Big|_{\sigma(p)}. \tag{3.54}$$

By equations (3.53) and (3.54), the vertical vector on the right hand side of equation (2.46) takes the local form

$$d\omega_\alpha^{-1}\Big(\mathcal{G}\big(\sigma(p), X\big) - d\sigma(X)\Big) = X^\mu \Bigg(Z_\mu{}^i\big(\sigma(p)\big) - \frac{\partial \sigma^i}{\partial x^\mu}\Big|_p - \sigma^i(p) \, {}_\alpha\Big\langle u^j(p)\Big| \frac{\partial u_i}{\partial x^\mu}\Big|_p \Big\rangle_\alpha\Bigg) \frac{\partial}{\partial v^j}\Big|_{\sigma(p)}. \tag{3.55}$$

Now that we have the local vectors describing the right hand side of (2.46) we can tackle the left hand side of (2.46): By our anzats 3.44 we have

$$d\omega_\alpha^{-1} \circ \mathcal{G}(S_\sigma(b), X) = X^\mu \frac{\partial}{\partial x^\mu}\Big|_{S_\sigma(b)} + X^\mu Z_\mu{}^i\big(S_\sigma(b)\big) \frac{\partial}{\partial v^i}\Big|_{S_\sigma(b)}. \tag{3.56}$$

To find the local form of $dS_\sigma\big(\mathcal{G}(b,X)\big)$ we will again make use of equation (3.44) and apply $f \in C^\infty(E)$ to it

$$\begin{aligned}
\Big(d\omega_\alpha^{-1} \circ dS_\sigma\big(\mathcal{G}(b,X)\big)\Big)(f) &= d\omega_\alpha^{-1} \circ dS_\sigma \circ d\omega_\alpha \left(X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_b + X^\mu {Z_\mu}^i(b) \left.\frac{\partial}{\partial v^i}\right|_b \right)(f), \\
&= X^\mu \frac{d}{dt} f\Big(S_\sigma \circ \eta_\mu(t)\Big)\Big|_0 + X^\mu {Z_\mu}^i(b) \frac{d}{dt} f\Big(S_\sigma \circ h_i(t)\Big)\Big|_0, \\
&= X^\mu \frac{d}{dt} f\left(\gamma_\mu(t), \Big(2b^j + \sigma\big(\gamma_\mu(t)\big) - b^i \,{}_\alpha\Big\langle u^j(p)\Big|u_i\big(\gamma(t)\big)\Big\rangle_\alpha\Big)\Big|u_j\big(\gamma(t)\big)\Big\rangle_\alpha\right)\Big|_0 \\
&\qquad + X^\mu {Z_\mu}^i(b) \frac{d}{dt} f\Big((b^j + \sigma\big(p\big) + {\delta^j}_i t)\iota_{\alpha,j}\big(p\big)\Big)\Big|_0,
\end{aligned}$$

$$\begin{aligned}
&= X^\mu {\delta^\nu}_\mu \left.\frac{\partial f}{\partial x^\nu}\right|_{S_\sigma(b)} + X^\mu \left.\frac{d\gamma_\mu^\nu}{dt}\right|_0 \left(\left.\frac{\partial \sigma^j}{\partial x^\nu}\right|_p - b^i \,{}_\alpha\left\langle u^j(p)\middle| \left.\frac{\partial u_i}{\partial x^\nu}\right|_p \right\rangle_\alpha \right) {\delta^k}_j \left.\frac{\partial f}{\partial v^k}\right|_{S_\sigma(b)} \\
&\quad + X^\mu \left.\frac{d\gamma_\mu^\nu}{dt}\right|_0 \Big(2b^j + \sigma^j\big(p\big) - b^i {\delta^j}_i\Big) \,{}_\alpha\left\langle u^k(p)\middle| \left.\frac{\partial u_j}{\partial x^\nu}\right|_p \right\rangle_\alpha \left.\frac{\partial f}{\partial v^k}\right|_{S_\sigma(b)} + X^\mu {Z_\mu}^i(b) {\delta^j}_i \left.\frac{\partial f}{\partial v^j}\right|_{S_\sigma(b)}.
\end{aligned}$$

Simplifying this, we find that locally we have

$$\begin{aligned}
d\omega^{-1} \circ dS_\sigma\big(\mathcal{G}(b,X)\big) =& X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{S_\sigma(b)} + X^\mu {Z_\mu}^i(b) \left.\frac{\partial}{\partial v^i}\right|_{S_\sigma(b)} \\
&+ \left(X(\sigma^j) + \sigma^i\big(p\big) X \cdot \,{}_\alpha\big\langle u^j(p)\big|du_i(p)\big\rangle_\alpha \right) \left.\frac{\partial}{\partial v^j}\right|_{S_\sigma(b)}.
\end{aligned} \tag{3.57}$$

By equations (3.56) and (3.57), the vertical vector on the left hand side of equation (2.46) takes the local form

$$\begin{aligned}
d\omega_\alpha^{-1}\Big(\mathcal{G}\big(S_\sigma(b),X\big) - dS_\sigma\Big(\mathcal{G}\big(b,X\big)\Big)\Big) = X^\mu \Bigg(& {Z_\mu}^i\big(S_\sigma(b)\big) - {Z_\mu}^i(b) - \left.\frac{\partial \sigma^j}{\partial x^\mu}\right|_p \\
& - \sigma^i\big(p\big) \,{}_\alpha\left\langle u^j(p)\middle| \left.\frac{\partial u_i}{\partial x^\mu}\right|_p \right\rangle_\alpha \Bigg) \left.\frac{\partial}{\partial v^j}\right|_{S_\sigma(b)}.
\end{aligned} \tag{3.58}$$

Now recall the vector space isomorphism $\varrho : V_{\sigma(p)}E \to V_{S_\sigma(b)}$ defined in equation (2.49). Applying ϱ to equation (3.55) and comparing to (3.58) gives us the relation

$${Z_\mu}^i\big(S_\sigma(b)\big) = {Z_\mu}^i\big(\sigma(p)\big) + {Z_\mu}^i(b). \tag{3.59}$$

In summary, for our vector bundle $(E,\pi,M,\mathcal{H},U(n))$ with family of local Hamiltonians $\{H^\alpha\}$, and domains $U_\alpha \subset M$, we prescribed our local variable basis section $\{\iota_{\alpha,i}\}_{i=1}^n$ to $\pi^{-1}(U_\alpha)$. This gives $\pi^{-1}(U_\alpha)$ the set of local coordinates $\{(x^\mu;v^i)\}$. The relations (3.45), (3.47), (3.52) and (3.59) imply that for $p \in U_\alpha$ and $X \in T_pM$ the horizontal lift of X at $b \in E_p$ takes the following local form

$$X_b^\# = X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_b + X \cdot \big(A^\alpha(p)\big)^i{}_j b^j \left.\frac{\partial}{\partial v^i}\right|_b. \tag{3.60}$$

The horizontal lift remains unchanged for variable basis sections and as before we will omit the local label α from local connections in future calculations unless it is necessary.

Even though the horizontal lift for local connection 1−forms $\{A^\alpha\}$ does not change with variable basis sections, we must still be careful to consider how their variation can influence derivatives and the connection.

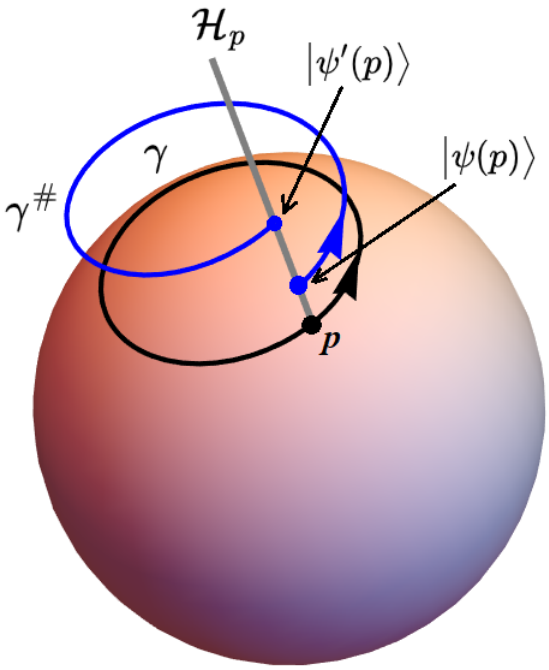

Figure 3.1: Diagram depicting a horizontal lift of the path γ (in black) to $\gamma^{\#}$ (in blue). The fibre is given by the grey line above the point p. Parallel transporting the vector $|\psi(p)\rangle$ along γ returns the vector $|\psi'(p)\rangle$

3.5 The Berry connection and parallel transporting states

Now that we have a horizontal lift together with our vector bundle $(E, \pi, M, U(n), \mathcal{H})$ we can form the initial value problem that is responsible for parallel transporting vectors in E. Let M have a open cover $\{U_\alpha\}$ and let $\pi^{-1}(U_\alpha)$ have the local coordinates $(x^\mu; v^i)$. Take a smooth path $\gamma : [0,1] \to U_\alpha$ starting at $\gamma(0) = p$, with tangent vector field $X(t) \in T_{\gamma(t)}M$. Lift γ to $\gamma^{\#} : [0,1] \to \pi^{-1}(U_\alpha)$ such that $\gamma^{\#}(0) = b$. Equation (3.60) tells us that the horizontal lift of $X(t)$ is locally given by

$$X^{\#}_{\gamma^{\#}(t)} = X^\mu(t) \left.\frac{\partial}{\partial x^\mu}\right|_{\gamma^{\#}(t)} + X(t) \cdot A^i{}_j(\gamma(t))\left(\gamma^{\#}(t)\right)^j \left.\frac{\partial}{\partial v^i}\right|_{\gamma^{\#}(t)}. \tag{3.61}$$

Let $\gamma^{\#}(t)$ take the form

$$\gamma^{\#}(t) = \lambda^i(t)\iota_{\alpha,i}(t) = \left[\left(\gamma(t), \lambda^i(t)\middle|u_i(\gamma(t))\right\rangle\right], \tag{3.62}$$

then applying $f \in C^\infty(E)$ to the left hand side of equation (3.61) gives us

$$\begin{aligned}
X^{\#}_{\gamma^{\#}(t)}(f) &= \frac{d}{d\tau} f\left(\gamma^{\#}(\tau)\right)\Big|_{\tau=t} = \frac{d}{dt} f\left(\gamma(\tau), \lambda^i(\tau)\middle|u_i(\gamma(\tau))\right\rangle\right)\Big|_{\tau=r}, \\
&= \frac{d}{d\tau} f\left(\gamma(\tau), \lambda^i(\tau)\left\langle u^j(\gamma(t))\middle|u_i(\gamma(\tau))\right\rangle\middle|u_j(\gamma(t))\right\rangle\right)\Big|_{\tau=t}, \\
&= \left.\frac{d\gamma^\mu}{d\tau}\right|_{\tau=t} \left.\frac{\partial f}{\partial x^\mu}\right|_{\gamma^{\#}(t)} + \left(\left.\frac{d\lambda^i}{d\tau}\right|_{\tau=t} \delta^j{}_i + \lambda^i(\tau)\left.\frac{d\gamma^\mu}{d\tau}\right|_{\tau=t} \left\langle u^j(\gamma(t))\middle|\left.\frac{\partial u_i}{\partial x^\mu}\right|_{\gamma(t)}\right\rangle\right) \left.\frac{\partial f}{\partial v^j}\right|_{\gamma^{\#}(t)}, \\
&= \left.\frac{d\gamma^\mu}{d\tau}\right|_{\tau=t} \left.\frac{\partial f}{\partial x^\mu}\right|_{\gamma^{\#}(t)} + \left(\left.\frac{d\lambda^j}{d\tau}\right|_{\tau=t} + \lambda^i(\tau) X \cdot \left\langle u^j(\gamma(t))\middle|du_i(\gamma(t))\right\rangle\right) \left.\frac{\partial f}{\partial v^j}\right|_{\gamma^{\#}(t)},
\end{aligned}$$

which implies that

$$X^{\#}_{\gamma^{\#}(t)} = \left.\frac{d\gamma^{\mu}}{d\tau}\right|_{\tau=t} \left.\frac{\partial}{\partial x^{\mu}}\right|_{\gamma^{\#}(t)} + \left(\left.\frac{d\lambda^{j}}{d\tau}\right|_{\tau=t} + \lambda^{i}(\tau) \left.\frac{d\gamma^{\mu}}{d\tau}\right|_{\tau=t} \left\langle u^{j}\big(\gamma(t)\big) \middle| \left.\frac{\partial u_i}{\partial x^{\mu}}\right|_{\gamma(t)} \right\rangle \right) \left.\frac{\partial}{\partial v^{j}}\right|_{\gamma^{\#}(t)}. \tag{3.63}$$

Substituting equation (3.63) into (3.61) and equating local vector coefficients gives us the IVPs

$$\frac{d\gamma^{\mu}}{dt} = X^{\mu}(t), \qquad \gamma^{\mu}(0) = p^{\mu}, \tag{3.64}$$

$$\frac{d\lambda^{j}}{dt} = X(t) \cdot \left(\Big(A^{\alpha}\big(\gamma(t)\big)\Big)^{j}{}_{i} - \big\langle u^{j}\big(\gamma(t)\big) \big| du_i\big(\gamma(t)\big) \big\rangle \right) \lambda^{i}(t), \qquad \lambda^{i}(0) = b^{i}. \tag{3.65}$$

These IVPs are our system's parallel transport equations under local connections $\{A^{\alpha}\}$.

We are interested in adiabatic paths, so to capture this with parallel transport we want our connection to enforce that all smooth paths are adiabatic. This means that based on example 3.1.1, specifically equation (3.10) a quantum state

$$|\psi(p)\rangle = \sum_{j=1}^{n} c^{j} |u_j(p)\rangle,$$

at $p \in M$, will change in the following way when it is parallel transported along a smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = p$

$$\Gamma(\gamma)_0^t : \Big[\big(p, |\psi(p)\rangle\big)\Big] \mapsto \left[\left(\gamma(t), \sum_{j=1}^{n} \exp(\phi_j(\gamma;t)) c^{j} \Big| u_j\big(\gamma(t)\big) \Big\rangle \right) \right],$$

where

$$\phi_i(\gamma;t) = \int_0^t \frac{d\gamma^{\mu}}{dt} \left\langle u_i\big(\gamma(t)\big) \middle| \left.\frac{\partial u_i}{\partial x^{\mu}}\right|_{\gamma(t)} \right\rangle dt.$$

This implies that our system's holonomy (2.98) at any point p is comprised of the diagonal matrices

$$\mathrm{hol}_p = \Big\{ \mathrm{diag}\big(\phi_1(\gamma;1), \phi_2(\gamma;1), \ldots, \phi_n(\gamma;1)\big) : \gamma \text{ is a smooth path } [0,1] \to M \Big\}. \tag{3.66}$$

We will now use the connection recovery method from equation (2.101). We find that transporting a state vector $b \in \pi^{-1}(p)$ over a smooth path $\gamma : [0,1] \to M$, with tangent vector X at $\gamma(0) = p$, gives

$$\begin{aligned}
b^{i} X \cdot A^{j}{}_{i}(p) &= \frac{d}{dt} \left(\Gamma(\gamma)_0^t b \right)^{i} \Big|_{t=0}, \\
&= \frac{d}{dt} \left(\Big(\gamma(t); \exp(\phi_i(\gamma;t)) b^{i} \Big| u_i\big(\gamma(t)\big) \Big\rangle \Big)^{i} \right) \Big|_{t=0}, \\
&= \frac{d}{dt} \left(\exp(\phi_i(\gamma;t)) b^{i} \Big\langle u^{j}(p) \Big| u_i\big(\gamma(t)\big) \Big\rangle \right) \Big|_{t=0}, \\
&= b^{i} \frac{d}{dt} \left(\int_0^t \frac{d\gamma^{\mu}}{dt} \left\langle u_i\big(\gamma(t)\big) \middle| \left.\frac{\partial u_i}{\partial x^{\mu}}\right|_{\gamma(t)} \right\rangle dt \right) \Big|_{t=0} \exp(\phi_i(\gamma;0)) \delta^{j}{}_{i} \\
&\qquad + b^{i} \frac{d}{dt} \left(\Big\langle u^{j}(p) \Big| u_i\big(\gamma(t)\big) \Big\rangle \right) \Big|_{t=0} \exp(\phi_i(\gamma;0)), \\
&= b^{i} X^{\mu} \left\langle u_i(p) \middle| \left.\frac{\partial u_i}{\partial x^{\mu}}\right|_{p} \right\rangle \delta^{j}{}_{i} + b^{i} X^{\mu} \left\langle u^{j}(p) \middle| \left.\frac{\partial u_i}{\partial x^{\mu}}\right|_{p} \right\rangle
\end{aligned}$$

Thus we find that for a chart (U_α, ϕ_α) and $p \in U_\alpha$

$$\big(A^\alpha(p)\big)^j{}_i = {}_\alpha\langle u_i(p)|du_i(p)\rangle_\alpha\, \delta^j{}_i + {}_\alpha\big\langle u^j(p)\big|du_i(p)\big\rangle_\alpha\,. \tag{3.67}$$

We call equation (3.67) a *local Berry connection* (or Berry connection for short). This connection enforces that all smooth paths be taken as adiabatic paths and thus satisfy the adiabatic theorem (3.5). Note, in the term $\langle u_i(p)|du_i(p)\rangle$, i is fixed and thus both indices are lowered.

An interesting and useful property of the Berry connection is that it is antihermitian by the following condition

$$\begin{aligned} 0 = \frac{\partial}{\partial x^\mu}\Big(\big\langle u^i(p)\big|u_j(p)\big\rangle\Big) &= \left\langle \frac{\partial u^i}{\partial x^\mu}\middle| u_j(p)\right\rangle + \left\langle u^i(p)\middle|\frac{\partial u_j}{\partial x^\mu}\right\rangle, \\ \implies \left\langle \frac{\partial u^i}{\partial x^\mu}\middle| u_j(p)\right\rangle &= -\left\langle u^i(p)\middle|\frac{\partial u_j}{\partial x^\mu}\right\rangle. \end{aligned} \tag{3.68}$$

Furthermore, the condition (3.68) implies that

$$\left\langle \frac{\partial u^j}{\partial x^\mu}\middle| u_j(p)\right\rangle \in i\mathbb{R} \quad \text{for all } p \in M \text{ and } j \in \{1,\dots,n\}. \tag{3.69}$$

Substituting the Berry connection into our parallel transport equation (3.65) we find that it simplifies into n decoupled linear IVPs

$$\frac{d\lambda^j}{dt} = X\big(t\big)\cdot\langle u_j(p)|du_j(p)\rangle\,\lambda^j(t), \qquad \lambda^j(0) = b^j, \tag{3.70}$$

with $j \in \{1,2,\dots,n\}$. The solution to these equations give rise to the Berry phases between eigenstates, whenever the state b is transported around a smooth closed path.

Recall from the discussion on connections in section 2.4. Connections correspond to gauge fields in physics. For this reason we note that it is not always possible to patch local connection 1-forms together in a consistent and smooth manner, over an entire manifold. A canonical example are the gauge fields belonging to a Dirac monopole. What this means is that we cannot guarantee that there exists a global connection in our model.

Finally, let M be a smooth base manifold with an open cover $\{U_\alpha\}$ and $\{t_{\alpha,\beta}\}$ a set transition maps. Let $\{H^\alpha\}$ be a given family of local Hamiltonians that take values from $\{U_\alpha\}$, operate on an $n-$dimensional Hilbert space $\mathcal{H}$, and agree on overlaps; for $p \in U_\alpha \cap U_\beta$

$$H^\alpha(p) = t_{\alpha,\beta}(p)H^\beta(p)t_{\beta,\alpha}(p). \tag{3.71}$$

We define the *Berry bundle* for our family of local Hamiltonians as the vector bundle from section 3.2, together with the variable basis sections (3.37), and the local Berry connection $1-$forms (3.67).

3.6 Curvature and Chern classes

Before we start this section it should be pointed out that the local Berry connection 1-forms (3.67), have a term of three lowered fixed indices

$$\langle u_i(p)|du_i(p)\rangle\,\delta^j{}_i. \tag{3.72}$$

For this reason, we will include explicit sums where necessary. Otherwise, repeated lowered indices are fixed.

For a vector bundle (E,π,M,F,G) with local connection $1-$forms $\{A^\alpha\}$, the local curvature is given by $\mathcal{F}^\alpha = dA^\alpha + A^\alpha \wedge A^\alpha$. However, with our variable basis sections we need to be wary of derivatives, as shown in example 3.3.1:

$$\begin{aligned}
dA(p) &= \left.\frac{\partial A_\nu}{\partial x^\mu}\right|_{q=p} dx^\mu \wedge dx^\nu, \\
&= \frac{\partial}{\partial x^\mu}\Big(A_\nu{}^i{}_j(q)|u_i(q)\rangle\langle u^j(q)|\Big)\Big|_{q=p} dx^\mu \wedge dx^\nu, \\
&= \frac{\partial}{\partial x^\mu}\Big(A_\nu{}^i{}_j(q)\left\langle u^\ell(p)\middle|u_i(q)\right\rangle\left\langle u^j(q)\middle|u_k(p)\right\rangle\Big)\Big|_{q=p}|u_\ell(p)\rangle\langle u^k(p)|dx^\mu\wedge dx^\nu, \\
&= \left(\left.\frac{\partial A_\nu{}^i{}_j}{\partial x^\mu}\right|_p \delta^\ell{}_i\delta^j{}_k + A_\nu{}^i{}_j(p)\left\langle u^\ell(p)\middle|\left.\frac{\partial u_i}{\partial x^\mu}\right|_p\right\rangle\delta^j{}_k\right. \\
&\qquad\left. + A_\nu{}^i{}_j(p)\delta^\ell{}_i\left\langle\left.\frac{\partial u^j}{\partial x^\mu}\right|_p\middle|u_k(p)\right\rangle\right)|u_\ell(p)\rangle\langle u^k(p)|dx^\mu\wedge dx^\nu, \\
\implies dA(p) &= \left(\left.\frac{\partial A_\nu{}^\ell{}_k}{\partial x^\mu}\right|_p + A_\nu{}^i{}_k(p)\left\langle u^\ell(p)\middle|\left.\frac{\partial u_i}{\partial x^\mu}\right|_p\right\rangle\right. \\
&\qquad\left. + A_\nu{}^\ell{}_j(p)\left\langle\left.\frac{\partial u^j}{\partial x^\mu}\right|_p\middle|u_k(p)\right\rangle\right)|u_\ell(p)\rangle\langle u^k(p)|dx^\mu\wedge dx^\nu.
\end{aligned} \tag{3.73}$$

Thus, for the Berry bundle associated to a family of local Hamiltonians $\{H^\alpha\}$, the first term in (3.73) takes the form

$$\begin{aligned}
\left.\frac{\partial A_\nu{}^\ell{}_k}{\partial x^\mu}\right|_p dx^\mu\wedge dx^\nu &= \frac{\partial}{\partial x^\mu}\left(\left\langle u_\ell(q)\middle|\frac{\partial u_\ell}{\partial x^\nu}\right\rangle\delta^\ell{}_k + \left\langle u^\ell(q)\middle|\frac{\partial u_k}{\partial x^\nu}\right\rangle\right)\Big|_{q=p} dx^\mu\wedge dx^\nu, \\
&= \left(\left\langle\left.\frac{\partial u_\ell}{\partial x^\mu}\right|_p\middle|\left.\frac{\partial u_\ell}{\partial x^\nu}\right|_p\right\rangle\delta^\ell{}_k + \left\langle\left.\frac{\partial u^\ell}{\partial x^\mu}\right|_p\middle|\left.\frac{\partial u_k}{\partial x^\nu}\right|_p\right\rangle\right)dx^\mu\wedge dx^\nu,
\end{aligned} \tag{3.74}$$

where terms of the form $\left\langle u^\ell(q)\middle|\left.\frac{\partial^2 u_k}{\partial x^\mu\partial x^\nu}\right|_p\right\rangle dx^\mu\wedge dx^\nu = 0$ because $|u_k(q)\rangle$ is smooth for all $k\in\{1,\dots,n\}$. Let us move onto the second term in (3.73)

$$\begin{aligned}
A_\nu{}^i{}_k(p)\left\langle u^\ell(p)\middle|\left.\frac{\partial u_i}{\partial x^\mu}\right|_p\right\rangle dx^\mu\wedge dx^\nu &= \left(\left\langle u_i(p)\middle|\left.\frac{\partial u_i}{\partial x^\nu}\right|_p\right\rangle\delta^i{}_k\right. \\
&\qquad\left. + \left\langle u^i(p)\middle|\left.\frac{\partial u_k}{\partial x^\nu}\right|_p\right\rangle\right)\left\langle u^\ell(p)\middle|\left.\frac{\partial u_i}{\partial x^\mu}\right|_p\right\rangle dx^\mu\wedge dx^\nu.
\end{aligned}$$

This expression can be simplified using the relation (3.68) to

$$\left(\left\langle u_k(p)\middle|\left.\frac{\partial u_k}{\partial x^\nu}\right|_p\right\rangle\left\langle u^\ell(p)\middle|\left.\frac{\partial u_k}{\partial x^\mu}\right|_p\right\rangle - \left\langle\left.\frac{\partial u^\ell}{\partial x^\mu}\right|_p\middle|\left.\frac{\partial u_k}{\partial x^\nu}\right|_p\right\rangle\right)dx^\mu\wedge dx^\nu. \tag{3.75}$$

Similarly, the third term in (3.73) becomes

$$-\left(\left\langle u_\ell(p)\middle|\left.\frac{\partial u_\ell}{\partial x^\nu}\right|_p\right\rangle\left\langle u^\ell(p)\middle|\left.\frac{\partial u_k}{\partial x^\mu}\right|_p\right\rangle + \left\langle\left.\frac{\partial u^\ell}{\partial x^\mu}\right|_p\middle|\left.\frac{\partial u_k}{\partial x^\nu}\right|_p\right\rangle\right)dx^\mu\wedge dx^\nu. \tag{3.76}$$

The last part of the curvature is $A\wedge A$ which takes the form

$$\begin{aligned}
A\wedge A = A_\mu A_\nu dx^\mu\wedge dx^\nu &= \frac{1}{2}A_\mu A_\nu dx^\mu\wedge dx^\nu - \frac{1}{2}A_\mu A_\nu dx^\nu\wedge dx^\mu, \\
&= \frac{1}{2}A_\mu A_\nu dx^\mu\wedge dx^\nu - \frac{1}{2}A_\nu A_\mu dx^\mu\wedge dx^\nu = \frac{1}{2}[A_\mu,A_\nu]dx^\mu\wedge dx^\nu.
\end{aligned} \tag{3.77}$$

This can be simplified further by making use of the fact that our local energy eigenstates form a complete set. Thus,

$$A_\mu{}^i{}_k A_\nu{}^k{}_j = \left\langle u^i \middle| \frac{\partial u_k}{\partial x^\mu} \right\rangle \left\langle u^k \middle| \frac{\partial u_j}{\partial x^\nu} \right\rangle = -\left\langle \frac{\partial u^i}{\partial x^\mu} \middle| u_k \right\rangle \left\langle u^k \middle| \frac{\partial u_j}{\partial x^\nu} \right\rangle = -\left\langle \frac{\partial u^i}{\partial x^\mu} \middle| \frac{\partial u_j}{\partial x^\nu} \right\rangle.$$

This means that

$$\frac{1}{2}\Big[A_\sigma(p), A_\eta(p)\Big] dx^\sigma \wedge dx^\eta = \left\langle \frac{\partial u^i}{\partial x^\sigma} \middle| \frac{\partial u_j}{\partial x^\eta} \right\rangle |u_i(p)\rangle\langle u^j(p)| dx^\sigma \wedge dx^\eta. \tag{3.78}$$

Both (3.77) and (3.78) are useful expressions. It is not apparent in (3.78) that $A \wedge A$ is traceless, but it is clear in (3.77). Furthermore, these equations give us the relation

$$\left\langle \frac{\partial u^i}{\partial x^\sigma} \middle| \frac{\partial u_i}{\partial x^\eta} \right\rangle = 0, \tag{3.79}$$

for all σ and η.

combining together equations (3.74), (3.75), (3.76) and (3.78) we find that the local *Berry curvature* is

$$\begin{aligned} \mathcal{F}(p) = \Bigg\{ &\sum_{\ell=1}^{n} \left\langle \frac{\partial u_\ell}{\partial x^\mu}\bigg|_p \middle| \frac{\partial u_\ell}{\partial x^\nu}\bigg|_p \right\rangle |u_\ell(p)\rangle\langle u_\ell(p)| \\ &+ \sum_{\ell,k=1}^{n} \Bigg(\left\langle u_k(p) \middle| \frac{\partial u_k}{\partial x^\nu}\bigg|_p \right\rangle \left\langle u_\ell(p) \middle| \frac{\partial u_k}{\partial x^\mu}\bigg|_p \right\rangle \\ &- \left\langle u_\ell(p) \middle| \frac{\partial u_\ell}{\partial x^\nu}\bigg|_p \right\rangle \left\langle u_\ell(p) \middle| \frac{\partial u_k}{\partial x^\mu}\bigg|_p \right\rangle \Bigg) |u_\ell(p)\rangle\langle u_k(p)| \Bigg\} dx^\mu \wedge dx^\nu. \end{aligned} \tag{3.80}$$

Note that we have explicitly used only lowered indices for the energy eigenstates. This was done to avoid confusion in the summing over repeated indices as there are several places in which we have three or four repeated indices.

If a connection is a gauge field in physics, then the curvature must be the field strength tensor. What this means is that the Berry curvature is observable and therefore coordinate independent. Hence, the Berry curvature is a global $\mathfrak{u}(n)$ valued 2−form. Furthermore, because the curvature takes values in the Lie algebra $\mathfrak{u}(n)$, it is antihermitian.

A useful piece of information we can extract from the local Berry curvature is its associated Chern class. Chern classes are topological invariants associated with smooth vector bundles. That is to say that two vector bundles with different Chern classes are non diffeomorphic bundles. However, the converse is not true. We define the *total Chern class* of a local curvature as

$$c(\mathcal{F}) = \det\left(I + i\frac{\mathcal{F}}{2\pi}\right) = \prod_{k=1}^{n} (1 + \frac{i}{2\pi}\lambda_k). \tag{3.81}$$

It should be noted that we are using the same convention as that in [35, 11.2.1 Definitions].

Expanding the product in (3.81) gives us

$$\begin{aligned} c(\mathcal{F}) &= 1 + \frac{i}{2\pi}\sum_{k=1}^{n}\lambda_k - \frac{1}{4\pi^2}\sum_{\substack{j,k=1 \\ j\neq k}}^{n} \lambda_j\lambda_k + \dots, \\ &= 1 + \frac{i}{2\pi}\sum_{k=1}^{n}\lambda_k - \frac{1}{8\pi^2}\sum_{j,k=1}^{n} 2\lambda_j\lambda_k(1-\delta_{j,k}) + \dots, \\ &= 1 + \frac{i}{2\pi}\mathrm{tr}\big(\mathcal{F}(p)\big) + \frac{1}{8\pi^2}\Big(\mathrm{tr}\big(\mathcal{F}(p)\big) \wedge \mathrm{tr}\big(\mathcal{F}(p)\big) - \mathrm{tr}\big(\mathcal{F}(p) \wedge \mathcal{F}(p)\big)\Big) + \dots \end{aligned}$$

From this we define the *first Chern class*

$$c_1(\mathcal{F}) = \frac{i}{2\pi}\mathrm{tr}\big(\mathcal{F}(p)\big),$$

which consists of global $2-$forms, and the *second Chern class*

$$c_2(\mathcal{F}) = \frac{1}{8\pi^2}\Big(\mathrm{tr}\big(\mathcal{F}(p)\big) \wedge \mathrm{tr}\big(\mathcal{F}(p)\big) - \mathrm{tr}\big(\mathcal{F}(p) \wedge \mathcal{F}(p)\big)\Big),$$

which consists of local $4-$forms.

A useful property of the Berry curvature is that it is traceless. Thus, the first Chern class for any Berry bundle will always vanish everywhere and the second simplifies to

$$c_2(\mathcal{F}) = -\frac{1}{8\pi^2}\mathrm{tr}\big(\mathcal{F}(p) \wedge \mathcal{F}(p)\big), \tag{3.82}$$

This means that in order for a Berry bundle to have a non-trivial Chern class, it will need to have a base manifold of at least 4 real dimensions.

Chapter 4

Spin-1/2 system

Now that we have a way of constructing a Berry bundle for a given family of local Hamiltonians, it is time we apply it to a well studied system to see if it agrees with observations. The quantum system we will study is a single spin-1/2 fermion in a magnetic field. We will start by introducing local Hamiltonians, and use these to build the parameter space, transition maps, and energy eigenbasis for our fibre. Then, using an open cover for the parameter space, we will establish local Berry connection 1-forms, and use them to study the system's holonomy (all possible Berry phases). What we find is that all useful information in our parameter space can be extracted from S^2, effectively reducing complexity in adiabatic paths to their projection onto the sphere: We find that the Berry phase accumulated by a quantum state transported along a path in our parameter space is equal to the area enclosed by the path when projected onto S^2, which agrees with experimentation [16, 22, 45].

4.1 The spin-1/2 bundle

From now on we will refer to a system of the single spin-1/2 fermion in a magnetic field as the spin-1/2 system. This system is described by a $3-$dimensional real parameter space M, and a 2-dimensional complex Hilbert space fibre $\mathcal{H}$. It has a local Hamiltonian of the form

$$H(q_1, q_2, q_3) = q_1\sigma^x + q_2\sigma^y + q_3\sigma^z, \tag{4.1}$$

where $q_1, q_2, q_3 \in \mathbb{R}$ and σ^i are the 2×2 Pauli spin matrices. Physically, the parameters represent the magnetic field of the spin$-1/2$ system.

To understand this Hamiltonian better we will work in the basis of σ^z eigenstates $\{|u_1\rangle_z, |u_2\rangle_z\}$ so that

$$H^z(q_1, q_2, q_3) = \begin{pmatrix} q_3 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 \end{pmatrix}. \tag{4.2}$$

H^z has energy eigenvalues

$$E_\pm(q_1, q_2, q_3) = \pm\sqrt{q_1^2 + q_2^2 + q_3^2}. \tag{4.3}$$

The only point of degeneracy is at the origin. This means that our parameter space is

$$M = \mathbb{R}^3 \backslash \{0\}. \tag{4.4}$$

As in example 3.2.1 we note that the energy eigenvalues of $H^z(q_1, q_2, q_3)$ are symmetric under the $SO(3)$ action,

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \mapsto O \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \qquad O \in SO(3).$$

To take advantage of the system's spherical symmetry we prescribe the following domains and diffeomorphisms of M:

$$\begin{aligned} U_{z,1} &= \mathbb{R}^3 \backslash \{(x,0,z) : x,z \in \mathbb{R} \text{ and } x \geq 0\}, \\ U_{z,2} &= \mathbb{R}^3 \backslash \{(x,0,z) : x,z \in \mathbb{R} \text{ and } x \leq 0\}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \phi_{z,1} &: U_{z,1} \to \mathbb{R}^3, \quad (x,y,z) \mapsto \big(r,\theta,\phi_1\big), \\ \phi_{z,2} &: U_{z,2} \to \mathbb{R}^3, \quad (x,y,z) \mapsto \big(r,\theta,\phi_2\big). \end{aligned} \tag{4.6}$$

We define $\phi_1 \in (0,2\pi)$ and $\phi_2 \in (-\pi,\pi)$ as angles measured anticlockwise from positive x-axis in the xy-plane. The angle $\theta \in (0,\pi)$ is measured from the positive z-axis, and $r > 0$ is the radius parameter. Physically, these parameters correspond to the direction and magnitude of an incident magnetic field over the spin-1/2 sample. The reason for the removal of half planes in the definitions of domains (4.5) is to avoid discontinuities in ϕ_1 and ϕ_2.

Note that the domains in equation (4.5) do not cover M. Let us remedy this: First note that our local Hamiltonian with respect to a basis of σ^x eigenstates, $\big\{|u_1\rangle_x, |u_2\rangle_x\big\}$, takes the form

$$H^x(q_1,q_2,q_3) = \begin{pmatrix} q_1 & q_3 - iq_2 \\ q_3 + iq_2 & -q_1 \end{pmatrix}. \tag{4.7}$$

We will take advantage of the similarity between (4.7) and (4.2) by prescribing the domains and diffeomorphisms:

$$\begin{aligned} U_{x,1} &= \mathbb{R}^3 \backslash \{(x,y,0) : x,y \in \mathbb{R} \text{ and } y \geq 0\}, \\ U_{x,2} &= \mathbb{R}^3 \backslash \{(x,y,0) : x,y \in \mathbb{R} \text{ and } y \leq 0\} \to \mathbb{R}^3. \end{aligned} \tag{4.8}$$

$$\begin{aligned} \phi_{x,1} &: U_{x,1} \to \mathbb{R}^3, \quad (x,y,z) \mapsto \big(r,\vartheta,\varphi_1\big), \\ \phi_{x,2} &: U_{x,2} \to \mathbb{R}^3, \quad (x,y,z) \mapsto \big(r,\vartheta,\varphi_2\big), \end{aligned} \tag{4.9}$$

where $\varphi_1 \in (0,2\pi)$ and $\varphi_2 \in (-\pi,\pi)$ are angles measured anticlockwise from positive y-axis in the yz-plane. The angle $\vartheta \in (0,\pi)$ is measured from the positive x-axis, and $r > 0$ is the radius parameter.

On our fibre bundle E, we define the trivialisations

$$\begin{aligned} \omega_{z,1} &: U_{z,1} \times \mathcal{H} \to \pi^{-1}(U_{z,1}), \\ \omega_{z,2} &: U_{z,2} \times \mathcal{H} \to \pi^{-1}(U_{z,2}), \end{aligned} \tag{4.10}$$

where the local basis for our fibre is given by $\big\{|u_1\rangle_z, |u_2\rangle_z\big\}$. That is to say for $i = 1,2$, $p \in U_{z,i}$, and $\Big[\big(p, |\psi(p)\rangle\big)\Big] \in \pi^{-1}(p)$

$$\omega_{z,i}^{-1}\bigg(\Big[\big(p, |\psi(p)\rangle\big)\Big]\bigg) = \big(p,\ {}_z\langle u^j(p)|\psi(p)\rangle\ |u_j(p)\rangle_z\big). \tag{4.11}$$

Similarly, we define

$$\begin{aligned} \omega_{x,1} &: U_{x,1} \times \mathcal{H} \to \pi^{-1}(U_{x,1}), \\ \omega_{x,2} &: U_{x,2} \times \mathcal{H} \to \pi^{-1}(U_{x,2}), \end{aligned} \tag{4.12}$$

where the local basis for our fibre is given $\big\{|u_1\rangle_x, |u_2\rangle_x\big\}$. That is to say for $i = 1,2$, $p \in U_{x,i}$, and $\Big[\big(p, |\psi(p)\rangle\big)\Big] \in \pi^{-1}(p)$

$$\omega_{x,i}^{-1}\bigg(\Big[\big(p, |\psi(p)\rangle\big)\Big]\bigg) = \big(p,\ {}_x\langle u^j(p)|\psi(p)\rangle\ |u_j(p)\rangle_x\big). \tag{4.13}$$

The trivialisations in equation (4.10), and local coordinates in equation (4.6), give us the locally defined Hamiltonians

$$H^z \circ \phi_{z,i}(q_1, q_2, q_3) = H^z_i(r, \theta, \phi_i) = r \begin{pmatrix} \cos(\theta) & e^{-i\phi_i}\sin(\theta) \\ e^{i\phi_i}\sin(\theta) & -\sin(\theta) \end{pmatrix}, \tag{4.14}$$

for all $(q_1, q_2, q_3) \in U_{z,i}$. These local Hamiltonians have local normalised energy eigenvectors given by

$$\begin{aligned} |u_1(r, \theta, \phi_i)\rangle_z = |u_-(r, \theta, \phi_i)\rangle_z &= \begin{pmatrix} -\sin(\theta/2)e^{-i\phi_i} \\ \cos(\theta/2) \end{pmatrix}, \\ |u_2(r, \theta, \phi_i)\rangle_z = |u_+(r, \theta, \phi_i)\rangle_z &= \begin{pmatrix} \cos(\theta/2)e^{-i\phi_i} \\ \sin(\theta/2) \end{pmatrix}, \end{aligned} \tag{4.15}$$

with energy eigenvalues

$$H^z_i(r, \theta, \phi_i)|u_\pm(r, \theta, \phi_i)\rangle_z = \pm r|u_\pm(r, \theta, \phi_i)\rangle_z. \tag{4.16}$$

Similarly, the trivialisations in equation (4.12), and local coordinates in equation (4.9), give us the locally defined Hamiltonians

$$H^x \circ \phi_{x,i}(q_1, q_2, q_3) = H^x_i(r, \vartheta, \varphi_i) = r \begin{pmatrix} \cos(\vartheta) & e^{-i\varphi_i}\sin(\vartheta) \\ e^{i\varphi_i}\sin(\vartheta) & -\sin(\vartheta) \end{pmatrix}, \tag{4.17}$$

for all $(q_1, q_2, q_3) \in U_{x,i}$. These local Hamiltonians have local normalised energy eigenvectors given by

$$\begin{aligned} |u_1(r, \vartheta, \varphi_i)\rangle_x = |u_-(r, \vartheta, \varphi_i)\rangle_x &= \begin{pmatrix} -\sin(\vartheta/2)e^{-i\varphi_i} \\ \cos(\vartheta/2) \end{pmatrix}, \\ |u_2(r, \vartheta, \varphi_i)\rangle_x = |u_+(r, \vartheta, \varphi_i)\rangle_x &= \begin{pmatrix} \cos(\vartheta/2)e^{-i\varphi_i} \\ \sin(\vartheta/2) \end{pmatrix}, \end{aligned} \tag{4.18}$$

with energy eigenvalues

$$H^x_i(r, \vartheta, \varphi_i)|u_\pm(r, \vartheta, \varphi_i)\rangle_x = \pm r|u_\pm(r, \vartheta, \varphi_i)\rangle_x. \tag{4.19}$$

The transition maps in our model are given by outer products of eigenstates. In example 3.2.2 we found our only non-trivial transition maps were given by the projections

$$t_{x,z}(p) = |u_i(p)\rangle_{x\,z}\langle u^i(p)|, \quad \text{and} \quad t_{z,x}(p) = |u_i(p)\rangle_{z\,x}\langle u^i(p)|, \tag{4.20}$$

for all $p \in U_{x,i} \cap U_{z,j}$. Thus we have

$$t_{x,z}(p) = t_{z,x}(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.21}$$

Using our open cover $\{U_{x,1}, U_{x,2}, U_{z,1}, U_{z,2}\}$ for the parameter space, Hilbert space fibre $\mathcal{H}$, structure group $U(2)$, and transition maps (4.21), we construct a vector bundle $(E, \pi, M, \mathcal{H}, U(2))$. This vector bundle has trivialisations given by (4.10) and (4.12).

We can now use the local normalised energy eigenstates, (4.15) and (4.18), to determine our local Berry connections (3.67). Let us build each component starting with the exterior derivative of our energy eigenstates: Let us start with $U_{z,i}$ for $i = 1, 2$

$$\begin{aligned} |du_-(r, \theta, \phi_i)\rangle_z &= -\frac{1}{2}\begin{pmatrix} \cos(\theta/2)e^{-i\phi_i} \\ \sin(\theta/2) \end{pmatrix} d\theta + \begin{pmatrix} i\sin(\theta/2)e^{-i\phi_i} \\ 0 \end{pmatrix} d\phi_i, \\ |du_+(r, \theta, \phi_i)\rangle_z &= \frac{1}{2}\begin{pmatrix} -\sin(\theta/2)e^{-i\phi_i} \\ \cos(\theta/2) \end{pmatrix} d\theta + \begin{pmatrix} -i\cos(\theta/2)e^{-i\phi_i} \\ 0 \end{pmatrix} d\phi_i. \end{aligned}$$

Thus our Berry connection over $U_{z,i}$ is given by

$$\begin{aligned} A^z(r,\theta,\phi_i) &= \begin{pmatrix} 2\,{}_z\langle u_1(r,\theta,\phi_i)|du_1(r,\theta,\phi_i)\rangle_z & {}_z\langle u_1(r,\theta,\phi_i)|du_2(r,\theta,\phi_i)\rangle_z \\ {}_z\langle u_2(r,\theta,\phi_i)|du_1(r,\theta,\phi_i)\rangle_z & 2\,{}_z\langle u_2(r,\theta,\phi_i)|du_2(r,\theta,\phi_i)\rangle_z \end{pmatrix}, \\ &= \begin{pmatrix} -2i\sin^2(\theta/2)d\phi_i & \frac{1}{2}d\theta + i\cos(\theta/2)\sin(\theta/2)d\phi_i \\ -\frac{1}{2}d\theta + i\cos(\theta/2)\sin(\theta/2)d\phi_i & -2i\cos^2(\theta/2)d\phi_i \end{pmatrix}. \end{aligned} \tag{4.22}$$

Similarly, by our choice of local coordinates for $U_{x,i}$ we have a local Berry connection over $U_{x,i}$ defined by

$$A^x(r,\vartheta,\varphi_i) = \begin{pmatrix} -2i\sin^2(\vartheta/2)d\varphi_i & \frac{1}{2}d\vartheta + i\cos(\vartheta/2)\sin(\vartheta/2)d\varphi_i \\ -\frac{1}{2}d\vartheta + i\cos(\vartheta/2)\sin(\vartheta/2)d\varphi_i & -2i\cos^2(\vartheta/2)d\varphi_i \end{pmatrix}. \tag{4.23}$$

Now that we have our local Berry connections we can parallel transport quantum states along adiabatic paths in our parameter space M. This will allow us to deduce the holonomy (all possible Berry phases) of the system, as well as study the Chern class of our spin$-1/2$ Berry bundle.

4.2 Results

First we should note that the local coordinate r appears nowhere in our Berry connections, (4.22) and (4.23). This means that we can expect that under parallel transport, any radial component of an adiabatic path $\gamma : [0,1] \to M$, will not contribute to the accumulation of Berry phase. We can see this by finding the horizontal lift of γ through some initial state $|\psi_i\rangle$: Let $\gamma : [0,1] \to U_{z,i}$ be a smooth closed path with $\gamma(0) = \gamma(1) = p$ and $X : [0,1] \to TM$ the tangent vector field to γ with local form

$$d\omega_{z,i}^{-1}\big(X(t)\big) = X^r(t)\left.\frac{\partial}{\partial r}\right|_{\gamma(t)} + X^\theta(t)\left.\frac{\partial}{\partial \theta}\right|_{\gamma(t)} + X^\phi(t)\left.\frac{\partial}{\partial \phi}\right|_{\gamma(t)}. \tag{4.24}$$

Say we want to parallel transport the quantum state

$$|\psi_i(p)\rangle = b^1|u_1(p)\rangle_z + b^2|u_2(p)\rangle_z, \tag{4.25}$$

along γ. To do this we need to find the horizontal lift $\gamma^\# : [0,1] \to E$,

$$\gamma^\#(t) = \omega_{z,i}\Big(\gamma(t); \lambda^j(t)\Big|u_j\big(\gamma(t)\big)\Big\rangle\Big) = \omega_{z,i}\Big(\gamma(t); \Big|\psi(t)\Big\rangle\Big). \tag{4.26}$$

$\gamma^\#$ can be found by solving the IVPs provided by the horizontal lift of our vector field (3.64),(3.65). Our ansatz for $\gamma^\#$ (4.26) satisfies (3.64). That is to say, we only need to solve for the coefficients of our quantum state $|\psi(t)\rangle$, which are given by the IVP

$$\frac{d\lambda^j}{dt} = X(t)\cdot\left(\Big(A^\alpha\big(\gamma(t)\big)\Big)^j{}_i - \big\langle u^j\big(\gamma(t)\big)\big|du_i\big(\gamma(t)\big)\big\rangle\right)\lambda^i(t), \quad \lambda^j(0) = b^j. \tag{4.27}$$

For the spin-1/2 system, (4.27) immediately provides two decoupled IVPs

$$\begin{aligned} \frac{d\lambda^1}{dt} &= -i\sin^2\Big(\gamma^\theta(t)/2\Big)X^\phi(t)\lambda^1(t), \qquad \lambda^1(0) = b^1, \\ \frac{d\lambda^2}{dt} &= -i\cos^2\Big(\gamma^\theta(t)/2\Big)X^\phi(t)\lambda^2(t), \qquad \lambda^2(0) = b^2. \end{aligned} \tag{4.28}$$

As we can see, the local coordinate r appears nowhere in our ODEs and thus any radial component of an adiabatic path can be safely ignored. This can be used to reduce our parameter space to S^2. Physically, this means that an experiment can have the magnetic field strength fixed, while still having full access to the system's possible Berry phases.

The solution to the above IVPs (4.28) is given by

$$\lambda^1(t) = b^1 \exp\Big(-i \int_\gamma \sin^2\big(\theta/2\big) d\phi\Big), \quad \lambda^2(t) = b^2 \exp\Big(-i \int_\gamma \cos^2\big(\theta/2\big) d\phi\Big). \tag{4.29}$$

This means that after being transported around an adiabatic path $\gamma : [0,1] \to M$, our quantum state is mapped to

$$|\psi(p)\rangle \mapsto b^1 \exp\Big(-i \int_\gamma \sin^2\big(\theta/2\big) d\phi\Big)|u_1(p)\rangle + b^2 \exp\Big(-i \int_\gamma \cos^2\big(\theta/2\big) d\phi\Big)|u_2(p)\rangle. \tag{4.30}$$

Thus, the state has accumulated a Berry phase (3.11) of

$$\Phi_{2,1} = -\oint_\gamma \Big(\cos^2\big(\theta/2\big) - \sin^2\big(\theta/2\big)\Big) d\phi = -\oint_\gamma \cos\big(\theta\big) d\phi. \tag{4.31}$$

From here we can employ the generalised Stokes' theorem: Let $\Omega \subset U_{z,i}$ be a smooth orientable sub-manifold of the parameter space with boundary given by γ, then

$$\Phi_{2,1} = -\iint_\Omega d\Big(\cos\big(\theta\big) d\phi\Big) = \iint_\Omega \sin\big(\theta\big) d\theta d\phi. \tag{4.32}$$

Notice that the integrand of (4.32) is the jacobian for spherical coordinates with radius set to $r = 1$. To take advantage of this, we will project our adiabatic paths in M to S^2 by removing the radial component. Hence, the Berry phase accumulated by a quantum state in the spin$-1/2$ system as it is transported around an adiabatic process $\gamma : [0,1] \to M$, is completely described by the smaller area enclosed by the path that γ projects onto S^2. This gives a holonomy group at any point $p \in M$ of the form

$$\mathrm{hol}_p = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Phi} \end{pmatrix} : \Phi \in [0, 2\pi) \right\} \simeq U(1). \tag{4.33}$$

An important question to ask is why the Berry phase is described by the smaller area enclosed by the projection of an adiabatic path onto S^2 rather than the larger area. If D is the smaller area then $4\pi - D$ is the larger area. So under exponentiation we find that the smaller area describes the difference between the second eigenstate's geometric phase and the first's, $\Phi_{2,1}$, while the larger area is the reverse, $\Phi_{1,2}$.

The last thing we can study is the Chern class, but as we said in section 3.6, a Berry bundle with a parameter space of dimension less than 4 will always have a trivial Chern class and that is exactly what we see in the spin-$1/2$ system.

Example 4.2.1. To end this chapter we will include an example of a full calculation of the parallel transport of

$$|\psi(p)\rangle = b^1|u_1(p)\rangle + b^2|u_2(p)\rangle, \tag{4.34}$$

about the smooth path $\gamma : [0,1] \to M$, where $\gamma(t)$ is a line of fixed latitude, of radius $R > 0$ starting and finishing at $(q_1, q_2, q_3) = (0, R, 0) \in M$.
Firstly we need to split this path into two pieces to express it in the local coordinates of $U_{z,1}$ and $U_{z,2}$. Consider the following parametrisation of γ

$$\gamma : [0,1] \to M, \quad \gamma(t) = \begin{cases} \gamma_1(t), & t \in [0, 1/2], \\ \gamma_2(t), & t \in (1/2, 1], \end{cases} \tag{4.35}$$

with

$$\begin{aligned}\gamma_1 : [0,1/2] \to U_{z,1}, \quad \omega_{z,1}\big(\gamma_1(t)\big) &= \bigg(R, \theta, \frac{\pi}{2} + 2\pi t\bigg), \\ \gamma_2 : (1/2,1] \to U_{z,2}, \quad \omega_{z,2}\big(\gamma_2(t)\big) &= \bigg(R, \theta, -\frac{\pi}{2} + 2\pi\bigg(t - \frac{1}{2}\bigg)\bigg).\end{aligned} \tag{4.36}$$

The local vector fields to these paths are given as

$$X_1(t) = 2\pi \left.\frac{\partial}{\partial\phi}\right|_{\gamma_1(t)}, \quad X_2(t) = 2\pi \left.\frac{\partial}{\partial\phi}\right|_{\gamma_2(t)}. \tag{4.37}$$

To parallel transport (4.34) about γ we will solve for the horizontal lift of γ_1 through $|\psi(p)\rangle$, and then find the horizontal lift of γ_2 through $\gamma_1^{\#}(1/2)$. Let the horizontal lifts of $\gamma_1^{\#}$ and $\gamma_2^{\#}$, be given by

$$\gamma_1^{\#}(t) = \omega_{z,1}\bigg(\gamma_1(t); \lambda_1^i(t)\Big|u_i\big(\gamma(t)\big)\Big\rangle_z\bigg), \quad \gamma_2^{\#}(t) = \omega_{z,2}\bigg(\gamma_2(t); \lambda_2^i(t)\Big|u_i\big(\gamma(t)\big)\Big\rangle_z\bigg).$$

Then, by the fibre IVPs in (4.28) we have

$$\frac{d\lambda_1^1}{dt} = -2\pi i \sin^2\left(\frac{\theta}{2}\right)\lambda_1^1(t), \quad \frac{d\lambda_1^2}{dt} = -2\pi i \cos^2\left(\frac{\theta}{2}\right)\lambda_1^2(t), \tag{4.38}$$

with $\lambda_1^i(0) = b^i$, and

$$\frac{d\lambda_2^1}{dt} = -2\pi i \sin^2\left(\frac{\theta}{2}\right)\lambda_2^1(t), \quad \frac{d\lambda_2^2}{dt} = -2\pi i \cos^2\left(\frac{\theta}{2}\right)\lambda_2^2(t), \tag{4.39}$$

with $\lambda_2^i(1/2) = \lambda_1^i(1/2)$.
Solving (4.38) gives

$$\begin{aligned}\lambda_1^1(t) &= b^1 \exp\bigg(-2\pi i \sin^2\left(\frac{\theta}{2}\right)t\bigg), \quad \lambda_1^1(1/2) = b^1 \exp\bigg(-\pi i \sin^2\left(\frac{\theta}{2}\right)\bigg), \\ \lambda_1^2(t) &= b^2 \exp\bigg(-2\pi i \cos^2\left(\frac{\theta}{2}\right)t\bigg), \quad \lambda_1^2(1/2) = b^2 \exp\bigg(-\pi i \cos^2\left(\frac{\theta}{2}\right)\bigg).\end{aligned} \tag{4.40}$$

Solving (4.39) gives

$$\lambda_2^1(t) = b^1 \exp\bigg(-2\pi i \sin^2\left(\frac{\theta}{2}\right)t\bigg), \quad \lambda_2^2(t) = b^2 \exp\bigg(-2\pi i \cos^2\left(\frac{\theta}{2}\right)t\bigg). \tag{4.41}$$

Hence, under the adiabatic path γ, the state (4.34) gains geometric phase

$$|\psi(p)\rangle \mapsto b^1 \exp\bigg(-2\pi i \sin^2\left(\frac{\theta}{2}\right)\bigg)|u_1(p)\rangle + \exp\bigg(-2\pi i \cos^2\left(\frac{\theta}{2}\right)\bigg)b^2|u_2(p)\rangle, \tag{4.42}$$

and thus has returns with a Berry phase of

$$\exp\bigg(-2\pi i \cos^2\left(\frac{\theta}{2}\right) + 2\pi i \sin^2\left(\frac{\theta}{2}\right)\bigg) = \exp(-2\pi i \cos(\theta)) = \exp\big(2\pi i\big(1 - \cos(\theta)\big)\big). \tag{4.43}$$

Physically, example 4.2.1 corresponds to an experimenter preparing the state (4.34) and then slowly sweeping the direction of the magnetic field in the xy-plane a full 2π radians.

A simple way to achieve this is to have spin-1/2 sample (such as a nitrogen vacancy center) attached to an electric motor and with a fixed magnetic field applied across the sample. The paper [52] uses a similar set up for a spin-1 system. The Berry phase can be measured by observing the interference pattern between the transported state, and the initial state (4.34).

In Figure 4.2 we demonstrate how a geometric phase accumulates when a quantum state is parallel transported anticlockwise along one of the four circular adiabatic paths on S^2 shown in Figure 4.1. Each path as equal radius, and the value β represents the angle between the center of the circular paths and the positive B_z axis. While each path gives us a distinct phase difference plot, they all finish at the same end point, $\Big(1, 2\pi\big(1-\cos(\theta)\big)\Big)$ where $\cos(\theta) = 1/\sqrt{1+r^2}$.

It is important to note that one cannot expect to be able to reproduce the plots Figure 4.2 in an experiment. These plots depict the phase difference between the first and second eigenstates as they are parallel transported about γ. This means that for all $t \in (0,1)$, our eigenstates are away from their starting parameters, $\gamma(0)$. Thus, a meaningful comparison between the initial state and the state transported to $\gamma(t)$ cannot be made unless $t = 0$ or 1.

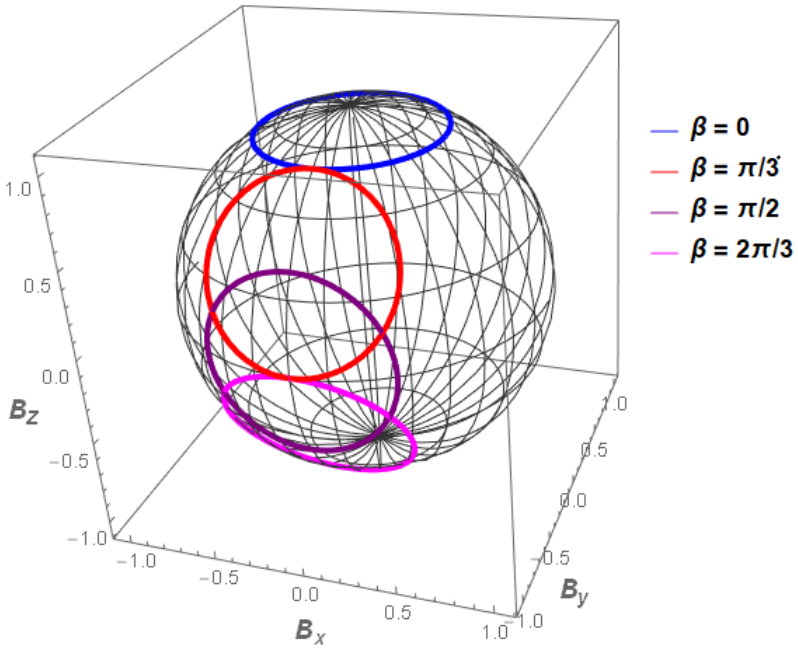

Figure 4.1: Plot of four circular paths of equal radius, each with center at $\beta = 0, \pi/3, \pi/2$ and $\beta = 2\pi/3$ radians from the positive $z-$axis in the yz-plane measured clockwise.

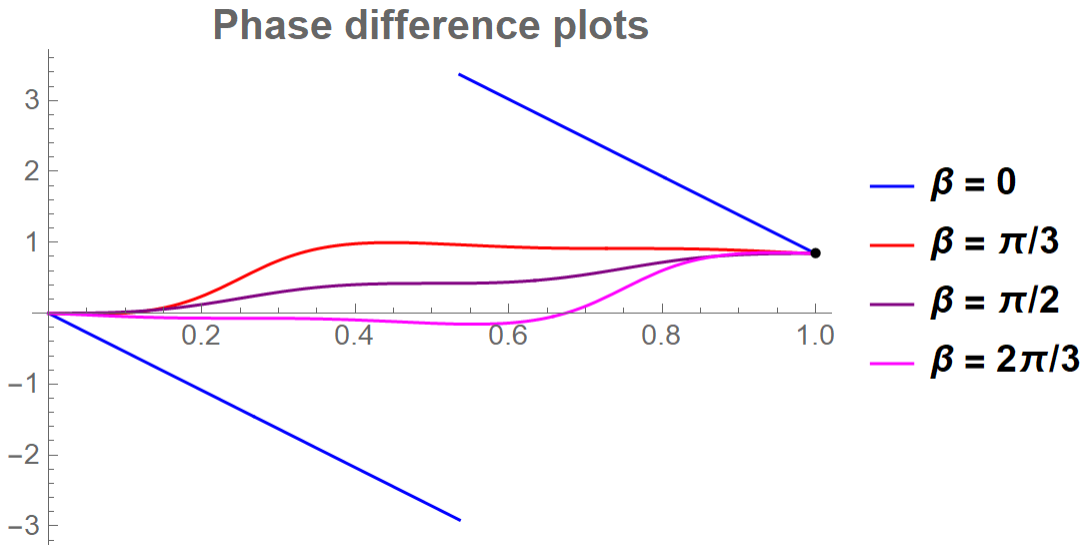

Figure 4.2: Plots of geometric phase difference as a quantum state is parallel transported about each of the respective circular paths in Figure 4.1. Notice that each path ends at the same point.

Chapter 5

The Nitrogen vacancy center

The next model we are interested in is the nitrogen vacancy center (NV center). We will start by introducing what the NV center is, why we are interested in it, and how to derive the local Hamiltonian we will be using to describe it. From there we will develop the NV center's Berry bundle which consists of a $5-$dimensional real parameter space, and a $3-$dimensional complex Hilbert space fibre. Once our local Berry connection 1-forms are known, we will move on to studying various observable features of the NV center. This will include a reduction in the dimension of its parameter space, general behaviour as well as special cases for the Berry phases, a procedure to induce any Berry phase, and finally the Chern class. As we will see in section 5.4, the Berry connection is a function of the ratios in the field strengths, effectively reducing its number of variables from five to two. This allows us to get a general idea for how Berry phase behaves in this system. We also find that Berry connection is discontinuous over the cones of degeneracy (5.13), leading to observable discontinuities in the Berry phase. Then we show that the holonomy for the NV system is $U(1) \otimes U(1)$, by providing a procedure to induce any Berry phase. Finally, we study the Chern class and show that it does not vanish, meaning that one could possibly calculate a Chern number in an experiment [30, 32, 56].

5.1 The NV center system

The NV center is a point defect in a diamond lattice consisting of an embedded nitrogen site with one of its nearest neighbours being a vacancy (a missing site in the lattice). Figure 5.1 depicts an NV center. It has applications in quantum computation [47, 50], NMR [7, 10], and quantum communications [19, 51]. Therefore, a theory of the NV center's Berry phase is essential in the development of these projects.

The NV center has two stable charge states, a neutral state NV^0 and a negatively charged state NV^-. We are only interested in the latter because it describes a spin-1 system, for this reason we will refer to it as the NV system. The NV system has two electrons located in the vacancy with quantum states expressed by the sum of their spins,

$$|0\rangle, |1\rangle, |-1\rangle. \tag{5.1}$$

The spin states $|1\rangle$ and $|-1\rangle$ are energy degenerate, but this is not a significant problem because we can induce a Zeeman splitting by applying a magnetic field along the defect axis of the NV center. To manipulate the quantum states we fire two independent microwaves at the NV, as discussed in [52].

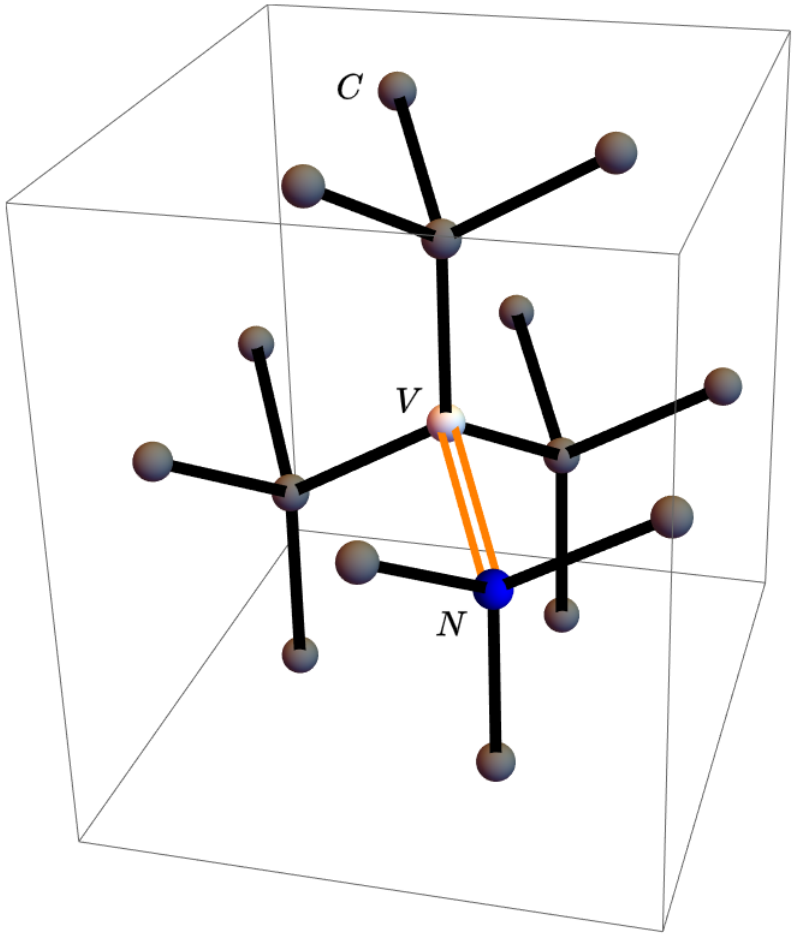

Figure 5.1: Diagram of an NV center made in Mathematica. The grey atoms represent carbon, the blue represents Nitrogen, and the white represents the vacancy. As we can see the nitrogen and the vacancy have two "covalent" bonds (in orange) meaning that they share two electrons.

The Hamiltonian for the NV center can be constructed from the 3×3 spin-1 matrices

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5.2}$$

In particular we have a magnetic field B along the defect axis ($z-$axis), and two microwaves with magnetic field components $\vec{b}_1 \cos(\omega_1 t + \phi_1)$, $\vec{b}_2 \cos(\omega_2 t + \phi_2)$ applied across the NV center. For ease of calculation we will assume that $\vec{b}_1$ and $\vec{b}_2$ have no z component. Then our Hamiltonian is given by

$$\begin{aligned} H(t) &= \hbar D S_z^2 + \gamma_{nv} B S_z + \gamma_{nv} \vec{b}_1 \cdot \vec{S} \cos(\omega_1 t + \phi_1) + \gamma_{nv} \vec{b}_2 \cdot \vec{S} \cos(\omega_2 t + \phi_2), \\ &= \hbar D S_z^2 + \gamma_{nv} B S_z + \gamma_{nv} \frac{b_{1,x} \cos(\omega_1 t + \phi_1) + \gamma_{nv} b_{2,x} \cos(\omega_2 t + \phi_2)}{\sqrt{2}} S_x \\ &\qquad + \gamma_{nv} \frac{b_{1,y} \cos(\omega_1 t + \phi_1) + \gamma_{nv} b_{2,y} \cos(\omega_2 t + \phi_2)}{\sqrt{2}} S_y, \end{aligned}$$

where $\hbar D S_z^2$ is a zero field term that comes from the intrinsic properties of the lattice, and

the terms γ_{nv} and D are real scalars.

Now we will change into a rotating frame by applying the transformation

$$H'(t) = U(t) \cdot H(t) \cdot U^\dagger(t) + iU^\dagger(t)\frac{d}{dt}\left(U(t)\right), \tag{5.3}$$

where $U(t)$ is given by

$$U(t) = \begin{pmatrix} \exp(i\omega_1 t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(i\omega_2 t) \end{pmatrix}. \tag{5.4}$$

Let us calculate each component of $H'(t)$. First the diagonal terms

$$\begin{aligned} &\hbar D U S_z^2 U^\dagger + \gamma_{nv} B U S_z U^\dagger + iU^\dagger \frac{d}{dt}\big(U(t)\big), \\ &= \begin{pmatrix} \hbar D + \gamma_{nv} B - \omega_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar D - \gamma_{nv} B - \omega_2 \end{pmatrix}. \end{aligned}$$

For the off diagonal terms we have

$$\begin{aligned} US_xU^\dagger &= \begin{pmatrix} 0 & e^{i\omega_1 t} & 0 \\ e^{-i\omega_1 t} & 0 & e^{-i\omega_2 t} \\ 0 & e^{i\omega_2 t} & 0 \end{pmatrix}, \\ US_yU^\dagger &= \begin{pmatrix} 0 & ie^{i\omega_1 t} & 0 \\ -ie^{-i\omega_1 t} & 0 & -ie^{-i\omega_2 t} \\ 0 & ie^{i\omega_2 t} & 0 \end{pmatrix}. \end{aligned}$$

If we then define E_{ij} to be the matrix with components

$$\Big(E_{ij}\Big)^k{}_\ell = \delta^k{}_i \delta^j{}_\ell, \tag{5.5}$$

then we can express the Hamiltonian in the rotating frame as

$$\begin{aligned} H'(t) = (\hbar D + \gamma_{nv} B - \omega_1)E_{11} + (\hbar D - \gamma_{nv} B - \omega_2)E_{33} \\ + \frac{\gamma_{nv}}{\sqrt{2}}\Big(b_{1,x}\cos(\omega_1 t + \phi_1) + b_{2,x}\cos(\omega_2 t + \phi_2)\Big)\Big(e^{i\omega_1 t}E_{12} + e^{-i\omega_1 t}E_{21}\Big) \\ + \frac{\gamma_{nv}}{\sqrt{2}}\Big(b_{1,x}\cos(\omega_1 t + \phi_1) + b_{2,x}\cos(\omega_2 t + \phi_2)\Big)\Big(e^{i\omega_2 t}E_{32} + e^{-i\omega_2 t}E_{23}\Big) \\ + \frac{\gamma_{nv}}{\sqrt{2}}\Big(b_{1,y}\cos(\omega_1 t + \phi_1) + b_{2,y}\cos(\omega_2 t + \phi_2)\Big)\Big(ie^{i\omega_1 t}E_{12} - ie^{-i\omega_1 t}E_{21}\Big) \\ + \frac{\gamma_{nv}}{\sqrt{2}}\Big(b_{1,y}\cos(\omega_1 t + \phi_1) + b_{2,y}\cos(\omega_2 t + \phi_2)\Big)\Big(ie^{i\omega_2 t}E_{32} - ie^{-i\omega_2 t}E_{23}\Big). \end{aligned}$$

These expressions are rather complicated, so to simplify them we will apply the rotating wave approximation (RWA). The rotating wave approximation is used in magnetic resonance to neglect terms in a Hamiltonian that oscillate rapidly relative to other terms. This approximation is valid when the applied electromagnetic radiation has a low enough intensity and when its frequency is near resonance with an atomic transition [53]. The idea is that rapidly oscillating terms will average out over time. An experimenter would take measurements over many identically run experiments, and average over the results in order to minimise random error. Because fast rotating terms have a period much smaller then the period of time it takes make a measurement, then averaging over a sufficiently large enough sample of experiments will cause any contribution these fast terms make, negligible. Under the RWA, not only will our local Hamiltonian be simplified, but it is still expected to match experimental results.

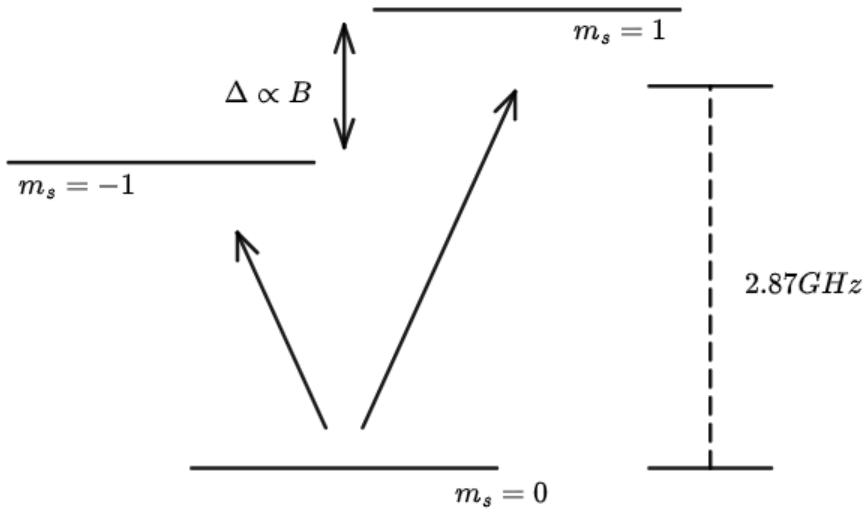

Figure 5.2: Diagram of the spin states of the NV-center in a magnetic field. The difference in the energy between the $m_s = 1$ and $m_s = -1$ states is proportional to the magnetic field across the defect axis.

In terms of the NV center's Hamiltonian, we will find fast rotating terms of the form

$$\exp(2i\omega_1 t), \qquad \exp(2i\omega_2 t), \qquad \exp\big(i(\omega_1+\omega_2)t\big), \tag{5.6}$$

all of which can be neglected under the RWA.

For example under the rotating wave approximation we get

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\Big(b_{1,x}\cos(\omega_1 t+\phi_1)+b_{2,x}\cos(\omega_2 t+\phi_2)\Big)\Big(e^{i\omega_1 t}E_{12}+e^{-i\omega_1 t}E_{21}\Big)\\
&\approx \frac{1}{2\sqrt{2}}\Big(b_{1,x}e^{-i\phi_1}E_{12}+b_{2,x}e^{i(\omega_1-\omega_2)t-i\phi_1}E_{12}\\
&\qquad\qquad + b_{1,x}e^{i\phi_1}E_{12}+b_{2,x}e^{i(\omega_2-\omega_1)t+i\phi_2}E_{12}\Big).
\end{aligned}$$

Notice that we cannot ensure that RWA will hold for terms of the form $\exp(i(\omega_2-\omega_1)t)$. However, these terms can be neglected, but only if the Zeeman splitting is large enough to prevent states from hoping between $m_s = 1$ and $m_s = -1$. In relation to Figure 5.2, we require Δ to be of the order of MHz. This will provide a sufficient energy gap between the spin 1 and -1 states, while keeping the spin 0 state isolated. With this in mind, set the frequencies of the incident microwaves such that

$$\begin{aligned}
\omega_1 &= \hbar D+\gamma_{nv}B-q_5,\\
\omega_2 &= \hbar D-\gamma_{nv}B+q_5,\\
\implies \omega_1-\omega_2 &= 2(\gamma_{nv}B-q_5),
\end{aligned} \tag{5.7}$$

where $|q_5| < \gamma_{nv}B$. Then, as long as the magnetic field B along the defect axis is large enough we can use the RWA to neglect terms of the form, $e^{i(\omega_1-\omega_2)t}$ [8]. This simplifies our Hamiltonian even further to

$$\begin{aligned}
H'(t) \approx{}& (\hbar D+\gamma_{nv}B-\omega_1)E_{11}+(\hbar D-\gamma_{nv}B-\omega_2)E_{33}\\
&+\frac{\gamma_{nv}}{\sqrt{2}}\Big(\big(b_{1,x}+ib_{1,y}\big)e^{-i\phi_1}E_{12}+\big(b_{1,x}-ib_{1,y}\big)e^{i\phi_1}E_{21},\\
&\quad+\big(b_{2,x}-ib_{2,y}\big)e^{-i\phi_2}E_{23}+\big(b_{2,x}+ib_{2,y}\big)e^{i\phi_2}E_{32}\Big).
\end{aligned}$$

Using the real parameters

$$\begin{aligned} q_5 &= \hbar D + \gamma_{nv} B - \omega_1 = -\hbar D + \gamma_{nv} B + \omega_2, \\ q_1 &= \mathrm{Re}\Big((b_{1,x} + ib_{1,y})e^{-i\phi_1}\Big) \quad q_2 = \mathrm{Im}\Big((b_{1,x} + ib_{1,y})e^{-i\phi_1}\Big), \\ q_3 &= \mathrm{Re}\Big((b_{2,x} + ib_{2,y})e^{-i\phi_2}\Big) \quad q_4 = \mathrm{Im}\Big((b_{2,x} + ib_{2,y})e^{-i\phi_2}\Big), \end{aligned} \tag{5.8}$$

simplifies our local Hamiltonian to

$$H(q_1, q_2, q_3, q_4, q_5) = \begin{pmatrix} q_5 & q_1 + iq_2 & 0 \\ q_1 - iq_2 & 0 & q_3 - iq_4 \\ 0 & q_3 + iq_4 & -q_5 \end{pmatrix}. \tag{5.9}$$

Each of these parameters does have an upper limit in its magnitude, in order for the rotating wave approximation to work. As long as the ratio of the system's Rabi oscillation period to Larmor precession period is less than ≈ 10, then the RWA should still be valid [18,29,31,55].

The Hamiltonian in equation (5.9) will serve as the local Hamiltonian for our Berry bundle. We will use the characteristic polynomial of (5.9) to determine the parameter space of the NV center and derive the energy eigenvalues and their respective eigenstates as functions of q_1, q_2, q_3, q_4 and q_5. From there we will choose an open cover for our parameter space, establish transition maps and finally calculate our local Berry connection 1-forms.

5.2 The NV parameter space

The characteristic polynomial, $\chi\big[H(q)\big]$ of our Hamiltonian's matrix representation, is a very useful tool in building the Berry bundle. Aside from using it to calculate the energy eigenvalues, it can be used to completely determine the parameter space, the eigenvectors, and to simplify many complicated expressions as we will see throughout this section.

Let $q = (q_1, q_2, q_3, q_4, q_5)$, then we have

$$\chi[H(q)](\lambda) = \lambda^3 - (q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2)\lambda - (q_1^2 + q_2^2 - q_3^2 - q_4^2)q_5. \tag{5.10}$$

Our characteristic polynomial is a special kind of cubic known as a depressed cubic with coefficients $r^2 = (q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2)$ and $c = (q_1^2 + q_2^2 - q_3^2 - q_4^2)q_5$. Luckily, the roots of a depressed cubic are very easy to calculate via the Cardano's formula. But before we find the energy eigenvalues, we can study the charateristic polynomial to determine for which parameters our local Hamiltonian is degenerate. The energy eigenvalues must be real and non-degenerate, meaning that we need the polynomial in equation (5.10) to have three distinct real roots. To determine when this is true we will simply find the critical points of $\chi[H(q)](\lambda)$, and find bounds for c such that one critical point is positive and the other is negative. Thus $\chi[H(q)](\lambda)$ has three distinct real roots when

$$27(q_1^2 + q_2^2 - q_3^2 - q_4^2)^2 q_5^2 < 4(q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2)^3. \tag{5.11}$$

We want to show that the inequality (5.11) holds for almost all $q_1, q_2, q_3, q_4, q_5 \in \mathbb{R}$ except on a set of cones (5.13) through the origin where the inequality becomes an equality. Set $q_1^2 + q_2^2 = x$, $q_3^2 + q_4^2 = y$, and $q_5^2 = z$. Now consider the polynomial in z

$$f : [0, \infty) \to \mathbb{R}, \quad f(z) = 4(x + y + z)^3 - 27(x - y)^2 z,$$

with $x, y \geq 0$. If the minimum of f is greater than or equal to zero then $f(z) \geq 0$ for all $x, y, z \geq 0$. We find that

$$\min_{z \in [0,\infty)} \big(f(z)\big) = \begin{cases} 54y\,(x - y)^2, & y < x \\ 54x\,(x - y)^2, & x \leq y \end{cases} \tag{5.12}$$

Thus, $f(z) \geq 0$ for all $x, y, z \geq 0$ with equality only for

$$\Big\{ (2z, 0, z) \in \mathbb{R}^3 : z \geq 0 \Big\}, \quad \text{and} \quad \Big\{ (0, 2z, z) \in \mathbb{R}^3 : z \geq 0 \Big\}$$

This proves that the inequality (5.11) holds for almost all $(q_1, q_2, q_3, q_4, q_5) \in \mathbb{R}^5$ and that our local Hamiltonian is energy degenerate only on the surfaces

$$\begin{aligned} C_{12} &= \left\{ (q_1, q_2, q_3, q_4, q_5) \in \mathbb{R}^5 : q_3 = q_4 = 0 \text{ and } 2q_5^2 = q_1^2 + q_2^2 \right\}, \\ C_{13} &= \left\{ (q_1, q_2, q_3, q_4, q_5) \in \mathbb{R}^5 : q_1 = q_2 = 0 \text{ and } 2q_5^2 = q_3^2 + q_4^2 \right\}. \end{aligned} \tag{5.13}$$

Thus, the parameter space for the NV center is

$$M = \mathbb{R}^5 \backslash \Big(C_{12} \cup C_{13} \Big), \tag{5.14}$$

and we call C_{12} and C_{34} the cones of degeneracy.

Now we will go ahead and solve the Hamiltonian's charateristic polynomial to find the energy eigenvalues. Because $\chi[H(q)](\lambda)$ is a depressed cubic we can use Cardano's formula to solve for its roots: Cardano's formula states that if

$$x^3 + px + q = 0 \tag{5.15}$$

is a cubic equation such that $p, q \in \mathbb{R}$ then it has the following solutions

$$\begin{aligned} x_1 &= e^{\frac{-2\pi i}{3}} u_1^{1/3} + e^{\frac{2\pi i}{3}} u_2^{1/3}, & u_1 &= -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \\ x_2 &= u_1^{1/3} + u_2^{1/3}, & u_2 &= -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \\ x_3 &= e^{\frac{2\pi i}{3}} u_1^{1/3} + e^{\frac{-2\pi i}{3}} u_2^{1/3}. \end{aligned} \tag{5.16}$$

Thus the energy eigenvalues are

$$\begin{aligned} E_1(q) &= e^{\frac{-2\pi i}{3}} \left(\frac{c}{2} + i\sqrt{\frac{r^6}{27} - \frac{c^2}{4}} \right)^{1/3} + e^{\frac{2\pi i}{3}} \left(\frac{c}{2} - i\sqrt{\frac{r^6}{27} - \frac{c^2}{4}} \right)^{1/3}, \\ E_2(q) &= \left(\frac{c}{2} + i\sqrt{\frac{r^6}{27} - \frac{c^2}{4}} \right)^{1/3} + \left(\frac{c}{2} - i\sqrt{\frac{r^6}{27} - \frac{c^2}{4}} \right)^{1/3}, \\ E_3(q) &= e^{\frac{2\pi i}{3}} \left(\frac{c}{2} + i\sqrt{\frac{r^6}{27} - \frac{c^2}{4}} \right)^{1/3} + ie^{\frac{-2\pi i}{3}} \left(\frac{c}{2} - i\sqrt{\frac{r^6}{27} - \frac{c^2}{4}} \right)^{1/3}. \end{aligned} \tag{5.17}$$

Recall $c = (q_1^2 + q_2^2 - q_3^2 - q_4^2)q_5$ and $r^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2$. The reason for ordering the energy eigenvalues is because E_1 belongs to the spin 0 state state while E_2 and E_3 belong to the spin 1 and -1 states respectively.

In relation to the cones of degeneracy (5.13) we find that for $(q_1, \ldots, q_5) \in C_{12}$ we have

$$\begin{aligned} E_1(q_1, \ldots, q_5) &= E_2(q_1, \ldots, q_5) \quad \text{if } q_5 \geq 0, \\ E_1(q_1, \ldots, q_5) &= E_3(q_1, \ldots, q_5) \quad \text{if } q_5 \leq 0. \end{aligned} \tag{5.18}$$

For $(q_1, \ldots, q_5) \in C_{13}$ we have

$$\begin{aligned} E_1(q_1, \ldots, q_5) &= E_3(q_1, \ldots, q_5) \quad \text{if } q_5 \geq 0, \\ E_1(q_1, \ldots, q_5) &= E_2(q_1, \ldots, q_5) \quad \text{if } q_5 \leq 0, \end{aligned} \tag{5.19}$$

with triple degeneracy only occurring at the origin, as seen in Figure 5.3. Other than the point of triple degeneracy, the energy eigenvalues $E_2(q)$ and $E_3(q)$ never coincide. Physically,

this is due to the Zeeman splitting that we built in to the local Hamiltonian when deriving (5.9).

Note that because $4r^6 \geq 27c^2$, then the energy eigenvalues (5.17) are sums of conjugate pairs and thus $E_1(q), E_2(q), E_3(q) \in \mathbb{R}$ for all $q \in M$. We also make the observation that the local Hamiltonian (5.9) is traceless and thus the roots add to zero

$$E_1(q) + E_2(q) + E_3(q) = 0. \tag{5.20}$$

Now that we know the energy eigenvalues of our local Hamiltonian, we can choose an open cover and local coordinates that take advantage of their symmetry. The energy eigenvalues are completely governed by the parameters c and r^2. These parameters are cylindrically symmetric about the q_5 axis. To make use of this we prescribe the open cover for M:

Let $\epsilon > 0$ and define

$$U_1 = M \backslash \Big(\{(q_1, q_2, 0, 0, q_5) \in \mathbb{R}^5\} \cup \{(0, 0, q_3, q_4, q_5) \in \mathbb{R}^5\} \Big), \tag{5.21a}$$

$$U_2 = \left\{ (q_1, q_2, q_3, q_4, q_5) \in \mathbb{R}^5 : 0 < \sqrt{2}|q_5| < \sqrt{q_1^2 + q_2^2} \text{ and } q_3^2 + q_4^2 < \epsilon \right\}, \tag{5.21b}$$

$$U_3 = \left\{ (q_1, q_2, q_3, q_4, q_5) \in \mathbb{R}^5 : 0 < \sqrt{2}|q_5| < \sqrt{q_3^2 + q_4^2} \text{ and } q_1^2 + q_2^2 < \epsilon \right\}, \tag{5.21c}$$

$$U_4 = \left\{ (q_1, q_2, q_3, q_4, q_5) \in \mathbb{R}^5 : \sqrt{2}|q_5| > \sqrt{q_1^2 + q_2^2} \text{ and } q_3^2 + q_4^2 < \epsilon \right\}, \tag{5.21d}$$

$$U_5 = \left\{ (q_1, q_2, q_3, q_4, q_5) \in \mathbb{R}^5 : \sqrt{2}|q_5| > \sqrt{q_3^2 + q_4^2} \text{ and } q_1^2 + q_2^2 < \epsilon \right\}. \tag{5.21e}$$

Now define the trivialisations

$$\Phi_1 : U_1 \to \mathbb{R}^5, \quad (q_1, q_2, q_3, q_4, q_5) \mapsto (\rho_1, \phi_1, \rho_2, \phi_2, z), \tag{5.22a}$$

$$\Phi_2 : U_2 \to \mathbb{R}^5, \quad (q_1, q_2, q_3, q_4, q_5) \mapsto (\rho_1, \phi_1, q_3, q_4, z), \tag{5.22b}$$

$$\Phi_3 : U_3 \to \mathbb{R}^5, \quad (q_1, q_2, q_3, q_4, q_5) \mapsto (q_1, q_2, \rho_2, \phi_2, z), \tag{5.22c}$$

$$\Phi_4 : U_4 \to \mathbb{R}^5, \quad (q_1, q_2, q_3, q_4, q_5) \mapsto (q_1, q_2, q_3, q_4, q_5), \tag{5.22d}$$

$$\Phi_5 : U_5 \to \mathbb{R}^5, \quad (q_1, q_2, q_3, q_4, q_5) \mapsto (q_1, q_2, q_3, q_4, q_5). \tag{5.22e}$$

We define the local coordinates ρ_1, ρ_2, and z as follows

$$\rho_1 = q_1^2 + q_2^2, \qquad \rho_2 = q_3^2 + q_4^2 \qquad z = q_5. \tag{5.23}$$

The angle $\phi_1 \in [0, 2\pi)$ is azimuthal in the $q_1 q_2$-plane measured anticlockwise from the positive q_1 axis. Similarly, $\phi_2 \in [0, 2\pi)$ is the azimuthal angle in the $q_3 q_4$-plane measured anticlockwise from the positive q_3 axis.

Figure 5.4 depicts the part of U_2 and U_4 with $q_3 = q_4 = 0$. The red cone surfaces, $z = \pm\rho_1/\sqrt{2}$, are two of the four regions where the local Hamiltonian is degenerate. For $q_3 = q_4 = 0$, U_4 describes the volume inside the cones while U_2 describes the volume outside the cones. As we will see in the section 5.4, these cones of degeneracy cause discontinuities in the Berry connection. Hence, a quantum state will accumulate Berry phase differently depending on whether it is parallel transported inside a cone (U_4) or outside a cone (U_2). The next step to take is to establish the NV vector bundle and derive the Berry connection from the local energy eigenvectors.

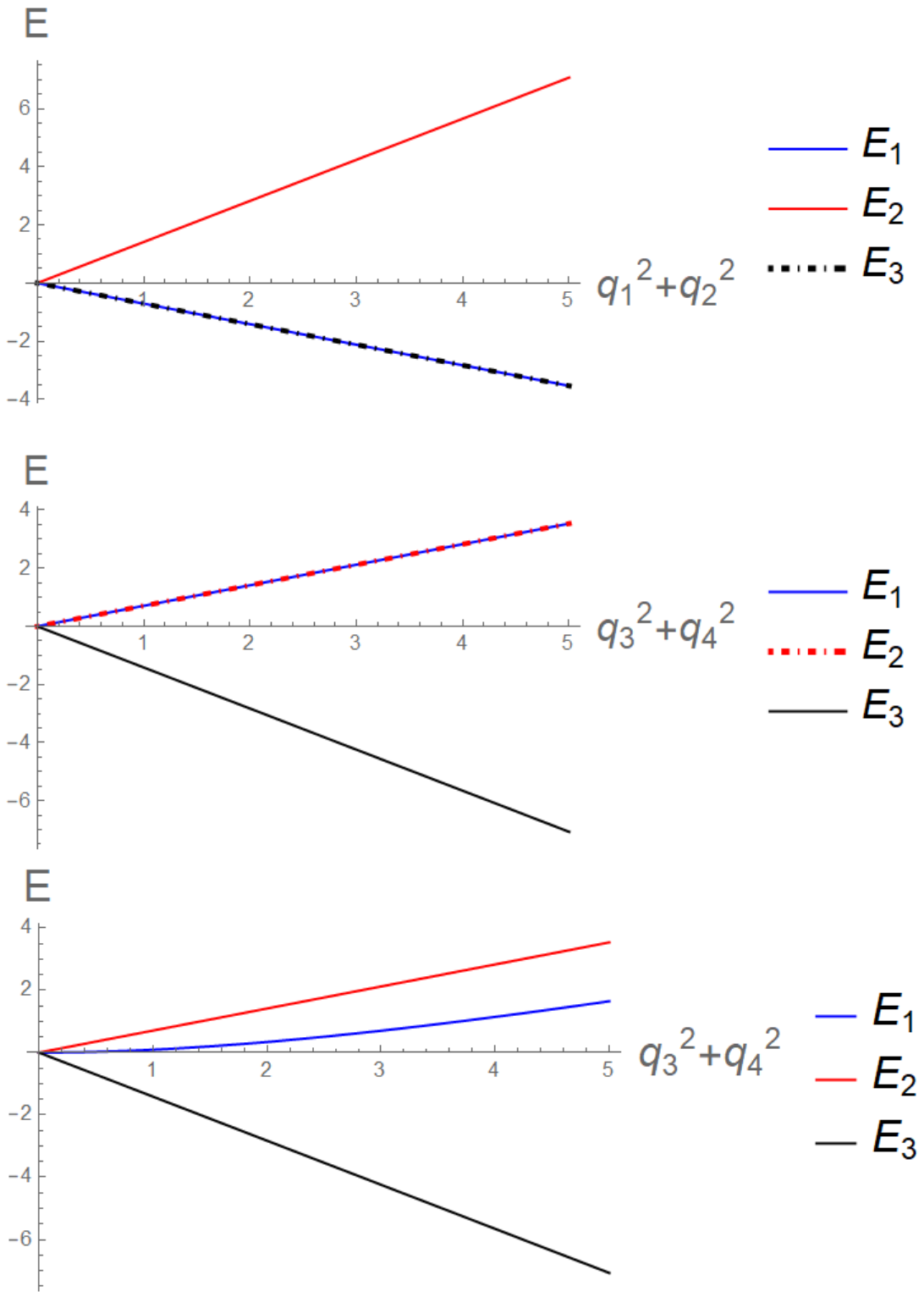

Figure 5.3: Top: Cross section of energy eigenvalues with $q_1, q_2 \in \mathbb{R}$, $2z^2 = q_1^2 + q_2^2$ and $q_3 = q_4 = 0$. Notice that $E_1 = E_3$ in this scenario.
Middle: Cross section of energy eigenvalues with $q_3, q_4 \in \mathbb{R}$, $2z^2 = q_3^2 + q_4^2$ and $q_1 = q_2 = 0$. Notice that $E_1 = E_2$ in this scenario.
Bottom: Cross section of energy eigenvalues with $q_3, q_4 \in \mathbb{R}$, $2z^2 = q_3^2 + q_4^2 + 20$ and $q_1 = q_2 = 0$. Notice that the only point of degeneracy is at the origin.

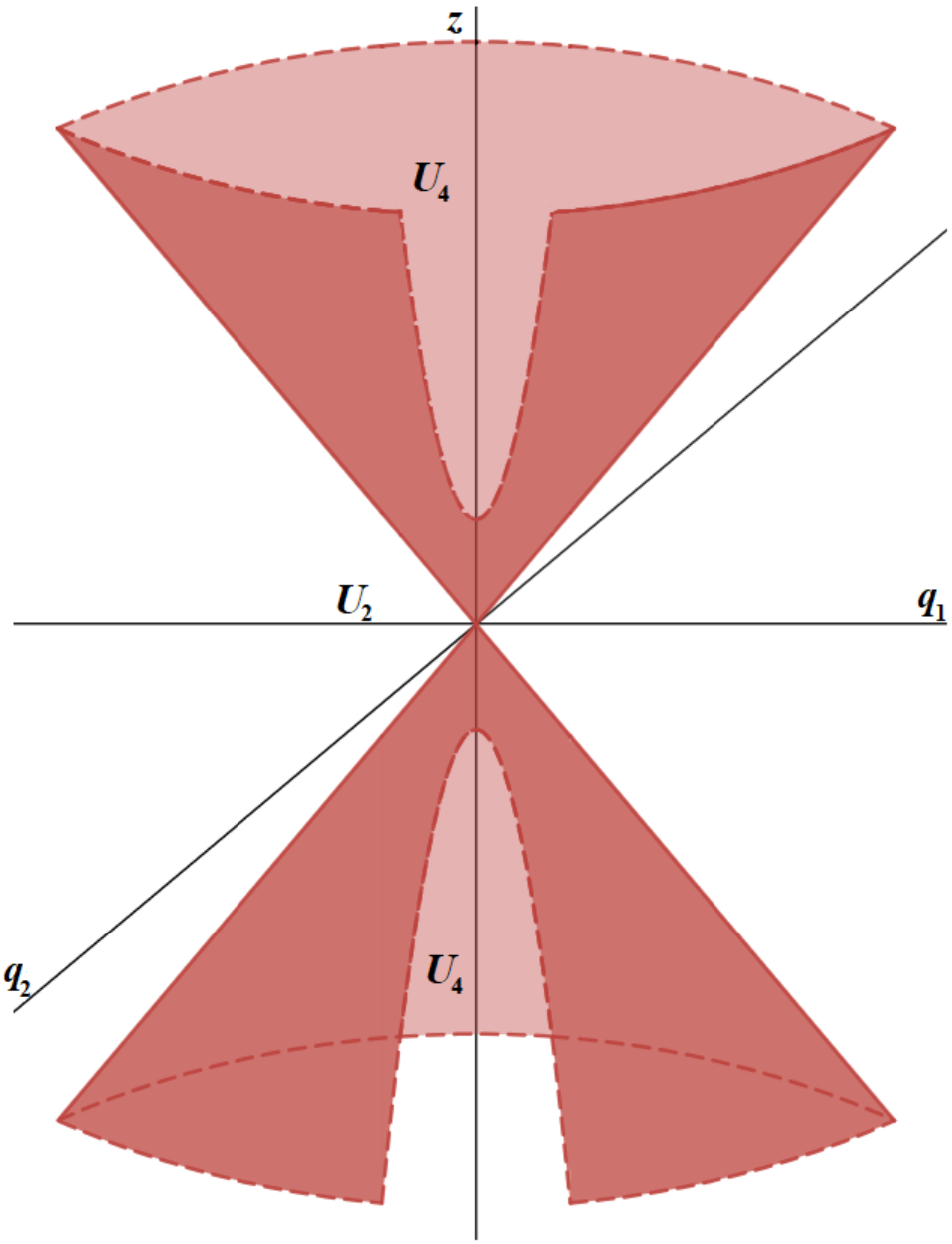

Figure 5.4: Diagram depicting the parts of U_4 and U_2 for $q_3 = q_4 = 0$. The red cone surfaces are regions where the local Hamiltonian is degenerate. Note that these cones extend infinitely and have been cut in this picture for the convenience of the reader.

5.3 The NV bundle

We will now use our system's energy eigenvalues to compute our local energy eigenvectors. Let $v \in \mathbb{C}^3$ be an eigenvector of the matrix representation of our local Hamiltonian

$$\begin{aligned} &H(q_1,q_2,q_3,q_4,q_5)v = \lambda v, \\ &\begin{pmatrix} q_5v_1 + (q_1+iq_2)v_2 \\ (q_1-iq_2)v_1 + (q_3-iq_4)v_3 \\ (q_3+iq_4)v_2 - q_5v_3 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}. \end{aligned} \tag{5.24}$$

We find that our local eigenvectors of $H(q)$ take the form

$$|u_i(q)\rangle = \frac{1}{R_i(q)} \begin{pmatrix} (q_1+iq_2)(q_3-iq_4) \\ \big(E_i(q)-q_5\big)(q_3-iq_4) \\ E_i(q)\big(E_i(q)-q_5\big) - \big(q_1^2+q_2^2\big) \end{pmatrix}, \tag{5.25}$$

where $1/R_i(q)$ is a normalisation factor. Let us now determine $R_i(q)$

$$\begin{aligned} 1 = \langle u_i(q)|u_i(q)\rangle = \frac{1}{|R_i(q)|^2}\Big((q_1^2+q_2^2)(q_3^2+q_4^2) + \big(E_i(q)-q_5\big)^2(q_3^2+q_4^2) \\ + \Big(E_i(q)\big(E_i(q)-q_5\big) - \big(q_1^2+q_2^2\big)\Big)^2\Big), \\ \implies |R_i(q)|^2 = \Big((q_1^2+q_2^2)(q_3^2+q_4^2) + \big(E_i(q)-q_5\big)^2(q_3^2+q_4^2) \\ + \Big(E_i(q)\big(E_i(q)-q_5\big) - \big(q_1^2+q_2^2\big)\Big)^2\Big). \end{aligned}$$

We will use the charateristic polynomial (5.10) to remove E_i^3 term. Doing this will lead to

$$\begin{aligned} |R_i(q)|^2 = (q_1^2+q_2^2)\Big(q_1^2+q_2^2+q_3^2+q_4^2 - \big(E_i(q)(E_i(q)-q_5)+2q_5^2\big)\Big) \\ + \big(E_i(q)-q_5\big)\big(2E_i(q)-3q_5\big)\big(q_3^2+q_4^2\big) + 2E_i(q)(E_i(q)-q_5)q_5^2. \end{aligned} \tag{5.26}$$

This may still seem like a complicated expression, but under our local coordinates for U_1 this simplifies further

$$\begin{aligned} R_i^2 \circ \phi_1^{-1}(x) = \rho_1^2\Big(\rho_1^2+\rho_2^2 - \Big(E_i(x)\big(E_i(x)-z\big)+2z^2\Big)\Big) \\ + \Big(E_i(x)-z\Big)\Big(2E_i(x)-3z\Big)\rho_2^2 + 2E_i(x)\Big(E_i(x)-z\Big)z^2, \end{aligned} \tag{5.27}$$

where $x = (\rho_1, \phi_1, \rho_2, \phi_2, z)$.

As discussed in section 3.2, the transition maps are given by projections (3.23). That is, for all $p \in U_\alpha \cap U_\beta$, we have transition maps

$$t_{\alpha\beta}(p) = |u^i(p)\rangle_{\alpha\,\beta}\langle u_i(p)|. \tag{5.28}$$

Using the construction method discussed in section 2.2, we form our NV Berry bundle

$$\big(E, M, \pi, \mathcal{H}, U(3)\big), \tag{5.29}$$

with local basis sections

$$\iota_{\alpha\,i}(p) = \Big[\Big(p, \big|u_i(p)\big\rangle_\alpha\Big)\Big], \tag{5.30}$$

for all $p \in U_\alpha$.

Now, all that is left to calculate is the Berry connection. The local coordinates of U_1 provide a simple way of computing the diagonal components of our connection. First, we

note that the Berry connection is antihermitian. Thus, the diagonal elements are purely imaginary.

Let us take a look at our eigenvectors in the local coordinates of U_1:

$$|u_i(p)\rangle = \frac{1}{R(p)} \begin{pmatrix} \rho_1\rho_2 \exp\big(i(\phi_1 - \phi_2)\big) \\ \rho_2\big(E_i(p) - z\big)\exp\big(-i\phi_2\big) \\ E_i(p)\big(E_i(p) - z\big) - \rho_1^2 \end{pmatrix}. \tag{5.31}$$

Since $E_i(p)$ and $R(p)$ are real, while $A_\mu{}^i{}_i(p)$ is imaginary for all $p \in U_1$, then

$$A_\mu{}^i{}_i(p)dx^\mu = \left\langle u^i(p) \middle| \frac{\partial u_i}{\partial \phi_1}\middle|_p \right\rangle d\phi_1 + \left\langle u^i(p) \middle| \frac{\partial u_i}{\partial \phi_2}\middle|_p \right\rangle d\phi_2,$$

because the ϕ derivatives are the only terms that will produce a factor i. Thus we have,

$$A_\mu{}^j{}_j(p)dx^\mu = i\frac{\rho_1^2\rho_2^2}{R_j(p)^2}d\phi_1 - i\rho_2^2\frac{\rho_1^2 + \big(E_j(p) - z\big)^2}{R_j(p)^2}d\phi_2, \tag{5.32}$$

for $j = 1, 2, 3$. These three connection 1-forms provide the entire effective Berry connection: the terms needed to perform parallel transport.

The other six off diagonal terms of A are necessary for studying the Chern classes and curvature. We will now compute these terms. Note that orthogonality of $\big\{|u_i(p)\rangle\big\}$ for all $p \in M$ gives us the following relation for $i \neq j$

$$\begin{aligned} \left\langle u^i(p) \middle| \frac{\partial u_j}{\partial x^\mu} \right\rangle &= \langle u^i(p)| \left[-\frac{1}{R_j(p)}\frac{dR_j}{dx^\mu}|u_j(p)\rangle + \frac{1}{R_j(p)}\frac{d}{dx^\mu}\Big(R_j(p)|u_j(p)\rangle\Big)\right], \\ &= \frac{1}{R_j(p)}\langle u^i(p)|\frac{d}{dx^\mu}\Big(R_j(p)\,|u_j(p)\rangle\Big). \end{aligned}$$

Thus, for $(x^1, \ldots, x^5) = (\rho_1, \phi_1, \rho_2, \phi_2, z)$ and $i \neq j$ we have

$$\begin{aligned} A_1{}^i{}_j(p)dx^1 =& \frac{1}{R_i(p)R_j(p)}\Bigg(\rho_1\rho_2^2 + \rho_2^2\frac{\partial E_j(p)}{\partial \rho_1}\Big(E_i(p) - z\Big) \\ &+ \Big(E_i(p)\big(E_i(p) - z\big) - \rho_1^2\Big)\Big(\frac{\partial E_j}{\partial \rho_1}\big(2E_j(p) - z\big) - 2\rho_1\Big)\Bigg)d\rho_1. \end{aligned} \tag{5.33}$$

$$A_2{}^i{}_j(p)dx^2 = i\frac{\rho_1^2\rho_2^2}{R_i(p)R_j(p)}d\phi_1. \tag{5.34}$$

$$\begin{aligned} A_3{}^i{}_j(p)dx^3 = \frac{1}{R_i(p)R_j(p)}\Bigg(&\rho_1^2\rho_2 + \rho_2\big(E_i(p) - z\big)\Big(E_j(p) + \rho_2\frac{\partial E_j}{\partial \rho_2} - z\Big) \\ &+ \frac{\partial E_j(p)}{\partial \rho_2}\big(2E_j(p) - z\big)\Big(E_i(p)\big(E_i(p) - z\big) - \rho_1^2\Big)\Bigg)d\rho_2. \end{aligned} \tag{5.35}$$

$$A_4{}^i{}_j(p)dx^4 = -i\frac{\rho_2^2}{R_i(p)R_j(p)}\Big(\rho_1^2 + \Big(E_i(p) - z\Big)\Big(E_j(p) - z\Big)\Big)d\phi^2. \tag{5.36}$$

$$\begin{aligned} A_5{}^i{}_j(p)dx^5 =& \frac{1}{R_i(p)R_j(p)}\Bigg(\rho_2\Big(E_i(p) - z\Big)\Big(\frac{\partial E_j}{\partial z} - 1\Big) \\ &+ \Big(E_i(p)\big(E_i(p) - z\big) - \rho_1^2\Big)\Big(2E_j(p)\frac{\partial E_j}{\partial z} - E_j(p) - z\frac{\partial E_j}{\partial z}\Big)\Bigg)dz. \end{aligned} \tag{5.37}$$

Rather than recalculating the Berry connection for each set of local coordinates, we can use (2.89) and the following relevant equalities for all points in $U_1 \cap U_j$

$$\begin{aligned} d\rho_1 &= \frac{q_1 dq_1 + q_2 dq_2}{\sqrt{q_1^2+q_2^2}}, \qquad d\rho_2 = \frac{q_3 dq_3 + q_4 dq_4}{\sqrt{q_3^2+q_4^2}}, \qquad dz = dq_5, \\ d\phi_1 &= \frac{q_1 dq_2 - q_2 dq_1}{q_1^2+q_2^2}, \qquad d\phi_2 = \frac{q_3 dq_4 - q_4 dq_3}{q_3^2+q_4^2}. \end{aligned} \tag{5.38}$$

We now have all the information we need to begin studying the NV center's holonomy. We will first discuss how the diagonal terms of the local Berry connections can be simplified to functions of two variables. Then use this to study the general behaviour of Berry phase, for three simple adiabatic paths (5.42). We will then study the special case, $q_5 = 0$, before providing a method to induce any Berry phase. Lastly, we will have a short discussion on the non-vanishing Chern class.

5.4 Results

In this section we define the values θ_{21} and θ_{31} as the phase differences between the second and first eigenstates, and the third and first eigenstates respectively. That is to say, in $\mathcal{H}_p$'s projective space we have

$$a_1 e^{i\phi_1}|u_1(p)\rangle + a_2 e^{i\phi_2}|u_2(p)\rangle + a_3 e^{i\phi_3}|u_1(p)\rangle = a_1|u_1(p)\rangle + a_2 e^{i\theta_{21}}|u_2(p)\rangle + a_3 e^{i\theta_{31}}|u_1(p)\rangle, \tag{5.39}$$

for all $p \in M$ and $a_1, a_2, a_3 \geq 0$.

The first result we will discuss is that the effective NV Berry connection ($A^i{}_i$ for fixed $i = 1, 2, 3$) is dependent only on the ratios $r_1 = \rho_1/z$ and $r_2 = \rho_2/z$, assuming $z \neq 0$. To see this we note that under the local coordinates of U_1 (5.22a)

$$\begin{aligned} E_i \circ \phi_1^{-1}(x) = \epsilon z \Bigg(& \frac{\rho_1^2-\rho_2^2}{2z^2} + i\sqrt{\left(\left(\frac{\rho_1^2+\rho_2^2}{3z^2}+\frac{1}{3}\right)^3 - \frac{\rho_1^2-\rho_2^2}{2z^2}\right)^2}\Bigg)^{1/3} \\ & + \epsilon^{-1} z \left(\frac{\rho_1^2-\rho_2^2}{2z^2} - i\sqrt{\left(\frac{\rho_1^2+\rho_2^2}{3z^2}+\frac{1}{3}\right)^3 - \left(\frac{\rho_1^2-\rho_2^2}{2z^2}\right)^2}\right)^{1/3}, \\ = z\widehat{E_i}(r_1, r_2). & \end{aligned}$$

$$\begin{aligned} R_i \circ \phi_1^{-1}(x) = z^2 \Bigg[& \frac{\rho_1^2}{z^2}\left(\frac{\rho_1^2}{z^2} + \frac{\rho_2^2}{z^2} - \left(\widehat{E_i}(x)\left(\widehat{E_i}(x)-1\right)+2\right)\right) \\ & + \left(\widehat{E_i}(x)-1\right)\left(2\widehat{E_i}(x)-3\right)\frac{\rho_2^2}{z^2} + 2\widehat{E_i}(x)\left(\widehat{E_i}(x)-1\right)\Bigg]^{\frac{1}{2}}, \\ = z^2\widehat{R_i}(r_1, r_2). & \end{aligned}$$

We therefore have

$$A_\mu{}^i{}_i(p)dx^\mu = i\frac{r_1^2 r_2^2}{\widehat{R_i}(r_1,r_2)^2}d\phi_1 - ir_2^2\left(\frac{r_1^2}{\widehat{R_i}(r_1,r_2)^2} + \frac{\left(\widehat{E_i}(r_1,r_2)-1\right)^2}{\widehat{R_i}(r_1,r_2)^2}\right)d\phi_2. \tag{5.40}$$

What this means is that the accumulation of Berry phase in the NV system, is only dependent on the square of the relative field strengths of the magnetic field along the defect axis, and the incident microwaves. Mathematically, this means that when modelling Berry phase for $q_5 \neq 0$, the NV parameter space is effectively described by

$$M_{\text{effective}} = \{(r_1, \phi_1, r_2, \phi_2) \in \mathbb{R}^4 : r_1, r_2 > 0, \text{ and } r_1, r_2 \neq \sqrt{2}, \text{ and } \phi_1, \phi_2 \in [0, 2\pi)\}. \tag{5.41}$$

This is similar to what we saw in the spin-1/2 case. Experimentally, this means that large field strengths are not necessary to explore the NV center's Berry phase.

Since the Berry connection is only dependent on the ratios r_1 and r_2, we can use contour plots to show general behaviour of the Berry phases θ_{21} and θ_{31} for the adiabatic paths

$$\begin{aligned}\gamma_1(s) &= \Phi_1^{-1}(r_1, 2\pi s, r_2, 0, 1), \quad \gamma_2(s) = \Phi_1^{-1}(r_1, 0, r_2, 2\pi s, 1),\\ \gamma_3(s) &= \Phi_1^{-1}(r_1, 2\pi s, r_2, 2\pi s, 1).\end{aligned} \tag{5.42}$$

Physically, these paths correspond to the experimenter fixing the strength of the magnetic field B, the amplitudes and frequencies of incident microwaves, and then rotating the phases of the microwaves a full 2π.

Figure 5.5 depicts contour plots of the Berry phases θ_{21} and θ_{31} associated to adiabatic paths γ_1, γ_2 and γ_3. In these plots we have the ratios r_1 and r_2 vary from 0 to 100 to see the rough limiting behaviour. Notice that the θ_{21} and θ_{31} plots for γ_1 and γ_2 appear almost identical. For these plots the brightest yellow represent a phase of π while the darkest blue corresponds to -2π, and the red lines are the zero contours. This gives us three interesting remarks to make for the Berry phases associated to γ_1:

- In the limit $r_1, r_2 \to \infty$, the values of θ_{21} and θ_{31} will coincide.
- For $r_2 \gg 10r_1$, the Berry phases negligibly contribute to the system's dynamics.
- For $r_1 \gg 10r_2$, we will have $\theta_{21} \approx \theta_{31} \approx \pi$.
- The zero contour is asymptotically equivalent to $r_1 = \sqrt{2}r_2$.

The same is true for the path γ_2 but the ratios are swapped, $r_1 \leftrightarrow r_2$.

The last row of plots in Figure 5.5 corresponds to the Berry phases associated to the adiabatic process γ_3. Note that the white regions in these plots were excluded so that the larger ratio behaviour could have greater detail. Unlike the phases belonging to γ_1 and γ_2, the range of the Berry phases associated to γ_3 for large ratios is very small and centred on $-\pi$. Investigating this further, we found that in the limit $r_1, r_2 \to \infty$, we have $\theta_{21}, \theta_{31} \to -\pi$.

One last thing to note about the plots in Figure 5.5 is that the diagrams, θ_{21} for γ_1, and θ_{31} for γ_2 are tranposes of one another. Similarly, θ_{31} for γ_1 and θ_{21} for γ_2 are also transposes of each other. By the linearity of our horizontal lift, we also have that the plots of γ_3 are given by the sums of the γ_1 and γ_2 plots. This means that the γ_3 plots are also transposes of one another. Furthermore, we see this pattern continue in Figure 5.6 which depict the Berry phases associated to γ_1, γ_2 and γ_3 for ratios $0 \leq r_1, r_2 \leq 2$.

The most striking feature of the plots in Figure 5.6 are the points $(0, \sqrt{2})$ and $(\sqrt{2}, 0)$ which correspond to the cones of degeneracy (5.13). We can see that these cones cause discontinuities in each of the Berry phases belonging to the adiabatic paths γ_1, γ_2 and γ_3. Furthermore, the range of θ_{21} and θ_{31} in each of these plots completely covers $U(1)$. This means that we can restrict the field strengths in our experiments such that the ratios are less than or equal to 2 and still have access to the entire holonomy group of the NV center.

Consider the γ_1 plots in Figure 5.6. If we take a closer look at the discontinuity in θ_{21} at $(0, \sqrt{2})$ we notice that it will disappear after exponentiation:

$$\theta_{21} = \begin{cases} 2\pi, & r_1 = 0 \text{ and } r_2 < \sqrt{2} \\ -2\pi, & r_1 = 0 \text{ and } r_2 > \sqrt{2}. \end{cases} \tag{5.43}$$

Analysing the behaviour of θ_{21} near the point $(\sqrt{2}, 0)$ shows us that this is in fact a discontinuity that does not disappear after exponentiation. The plot in Figure 5.7 confirms the existence of this observable discontinuity. This figure depicts θ_{21} as a function of r_1 centered on $(\sqrt{2}, 0)$. We find that for phases taken in $(-\pi, \pi]$,

$$\theta_{21} = \begin{cases} -2\pi/3, & r_2 = 0 \text{ and } r_1 > \sqrt{2} \\ 2\pi/3, & r_2 = 0 \text{ and } r_1 < \sqrt{2}. \end{cases} \tag{5.44}$$

The contour plot for θ_{31} associated to γ_1 in Figure 5.6 shows distinct discontinuities at both cones $(\sqrt{2},0)$ and $(0,\sqrt{2})$. However, the discontinuity at $(0,\sqrt{2})$ jumps between 0 and -2π and thus disappears after exponentiation. The plot in Figure 5.8 depicts θ_{31} as a function of r_1 centered on $(\sqrt{2},0)$. We find that for phases taken in $(-\pi,\pi]$,

$$\theta_{31} = \begin{cases} 2\pi/3, & r_2 = 0 \text{ and } r_1 > \sqrt{2} \\ -2\pi/3, & r_2 = 0 \text{ and } r_1 < \sqrt{2} \end{cases} \tag{5.45}$$

Again, all of this holds for the path γ_2, except θ_{21} and θ_{31} will swap, and r_1 will swap with r_2. This means that the discontinuity at $(0,\sqrt{2})$ will not disappear for the Berry phases associated to γ_2. The most important note to make here is that discontinuities in the Berry phase can made continuous via exponentiation, and whether this happens is entirely path dependent.

The last row of diagrams in Figure 5.6 are the contour plots for θ_{21} and θ_{31} associated to γ_3. These plots have three interesting features; the two discontinuities and the zero contour. Firstly, because these plots are given by the sum of the plots belonging to γ_1 and γ_2 we have

$$\begin{aligned} \theta_{21} &= \begin{cases} -2\pi/3, & r_2 = 0 \text{ and } r_1 > \sqrt{2} \\ 2\pi/3, & r_2 = 0 \text{ and } r_1 < \sqrt{2} \\ 2\pi/3, & r_1 = 0 \text{ and } r_2 > \sqrt{2} \\ -2\pi/3, & r_1 = 0 \text{ and } r_2 < \sqrt{2} \end{cases}, \\ \theta_{31} &= \begin{cases} -2\pi/3, & r_1 = 0 \text{ and } r_2 > \sqrt{2} \\ 2\pi/3, & r_1 = 0 \text{ and } r_2 < \sqrt{2} \\ 2\pi/3, & r_2 = 0 \text{ and } r_1 > \sqrt{2} \\ -2\pi/3, & r_2 = 0 \text{ and } r_1 < \sqrt{2} \end{cases} \end{aligned} \tag{5.46}$$

where we have taken our phases to be in $(-\pi,\pi]$.

The zero contour for the γ_3 Berry phase plots in Figure 5.6, have been highlighted in red for the ease of the reader. They are both given as the first quadrant of a circle of radius $\sqrt{2}$ centred on $(0,0)$. This means that whenever we have

$$r_1^2 + r_2^2 = 2, \tag{5.47}$$

the path γ_3 will produce no Berry phase.

Let us now use our original effective Berry connection (5.32) to explore the special case $q_5 = 0$. Mathematically, this is equivalent to the limit $z \to 0^+$ and therefore $r_1, r_2 \to \infty$. This means we should expect our observations from Figure 5.5 to agree with any results we find under this regime. Physically, $q_5 = 0$ corresponds to setting the frequencies of the incident microwaves to $\omega_1 = \hbar D + \gamma_{\text{nv}} B$ and $\omega_2 = \hbar D - \gamma_{\text{nv}} B$. When this is true our energy eigenvalues simplify to

$$\begin{aligned} &E_1(q_1,\ldots,q_4,0) = 0, \qquad E_2(q_1,\ldots,q_4,0) = \sqrt{q_1^2+q_2^2+q_3^2+q_4^2}, \\ &E_3(q_1,\ldots,q_4,0) = -\sqrt{q_1^2+q_2^2+q_3^2+q_4^2}. \end{aligned} \tag{5.48}$$

Thus the effective local connection 1-form in U_1 is

$$\begin{aligned} {A_\mu}^1{}_1(p)dx^\mu &= i\frac{\rho_2^2}{\rho_1^2+\rho_2^2}d\phi_1 - i\frac{\rho_2^2}{\rho_1^2+\rho_2^2}d\phi_2, \\ {A_\mu}^2{}_2(p)dx^\mu = {A_\mu}^3{}_3(p)dx^\mu &= i\frac{\rho_1^2}{2\left(\rho_1^2+\rho_2^2\right)}d\phi_1 - i\frac{2\rho_1^2+\rho_2^2}{2\left(\rho_1^2+\rho_2^2\right)}d\phi_2. \end{aligned} \tag{5.49}$$

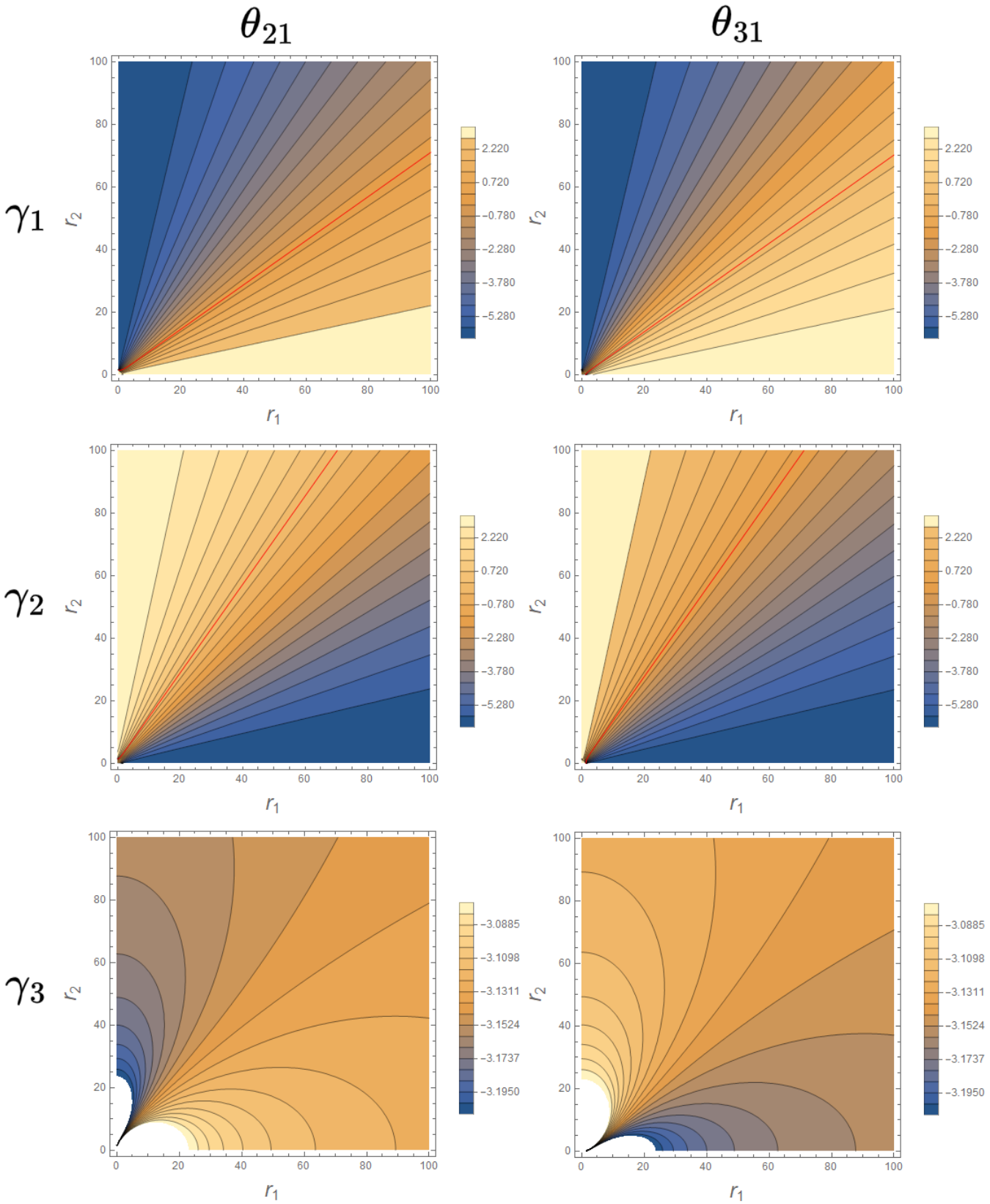

Figure 5.5: Contour plots for Berry phases θ_{21} and θ_{31} belonging to the paths γ_1, γ_2 and γ_3 defined in equation (5.42). In these plots we vary r_1 and r_2 from 0 to 100 to study the behaviour of Berry phase for large field strength ratios.

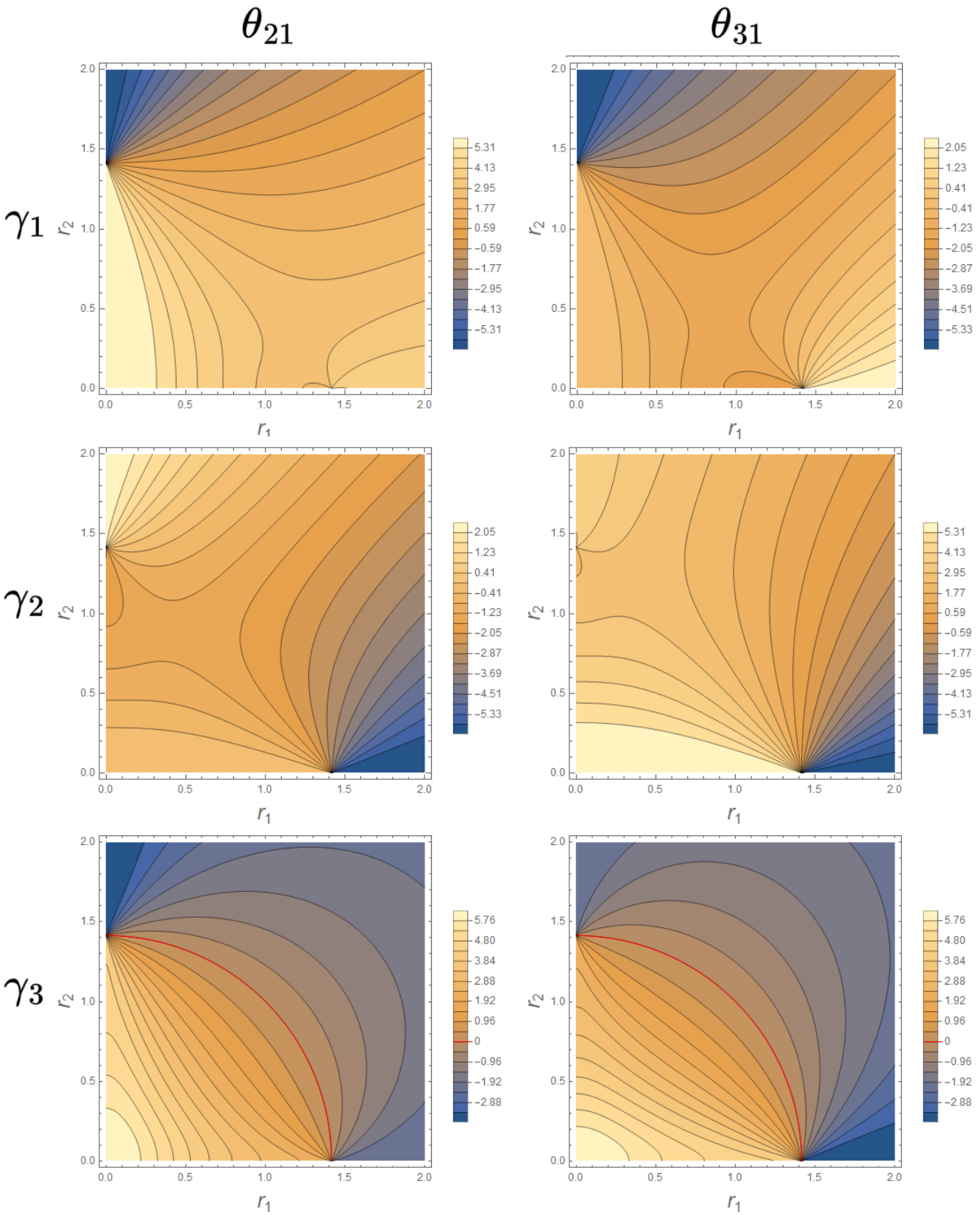

Figure 5.6: Contour plots for Berry phases θ_{21} and θ_{31} belonging to the paths γ_1, γ_2 and γ_3 defined in equation (5.42). In these plots we vary r_1 and r_2 from 0 to 2 to study the behaviour of Berry phase near the cones of degeneracy (5.13), at $(r_1, r_2) = (\sqrt{2}, 0), (0, \sqrt{2})$.

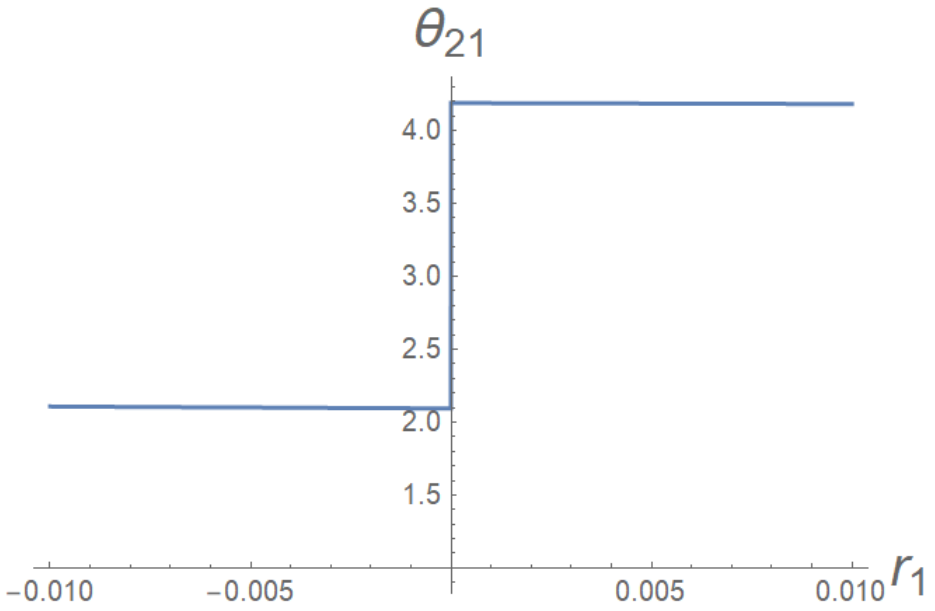

Figure 5.7: Cross section of the Berry phase θ_{21} centred on $r_1 = \sqrt{2}$ with $r_2 = 10^{-10}$. There is a clear discontinuity at the point $(r_1, r_2) = (\sqrt{2}, 0)$ that does not disappear under exponentiation. Furthermore, under exponentiation, this discontinuity goes from $2\pi/3$ to $-2\pi/3$. Note that making $r_2 = 0$ causes numeric problems as well as problems with the local coordinate system.

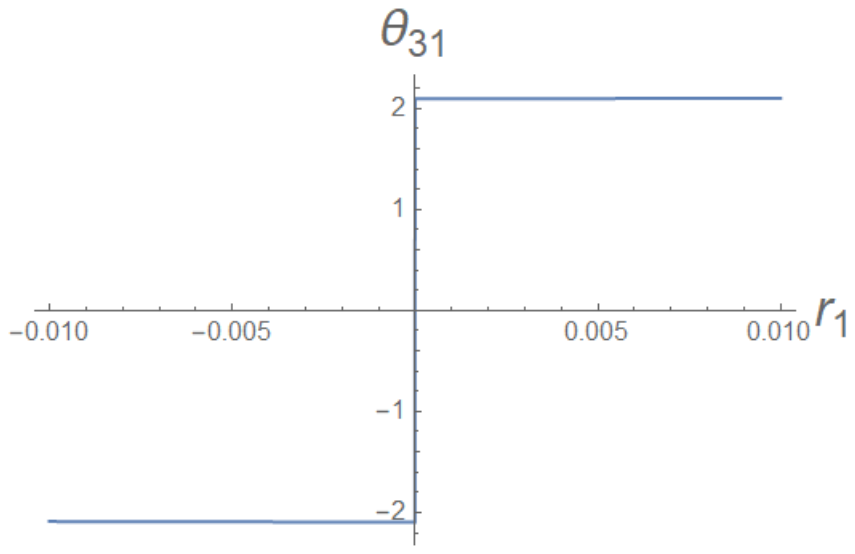

Figure 5.8: Cross section of the Berry phase θ_{31} centred on $r_1 = \sqrt{2}$ with $r_2 = 10^{-10}$. There is a clear discontinuity at the point $(r_1, r_2) = (\sqrt{2}, 0)$ from $-2\pi/3$ to $2\pi/3$.

This means that for $z = 0$, the Berry phase accumulated by the spin down state and the spin up state when transported along an adiabatic path γ, will be equivalent to

$$\begin{aligned}\theta_{31} = \theta_{21} &= -i \oint_\gamma \Big(A_\mu{}^2{}_2(p) - A_\mu{}^1{}_1(p) \Big) dx^\mu, \\ &= \frac{1}{2} \oint_\gamma \left(\frac{\rho_1^2 - 2\rho_2^2}{\rho_1^2 + \rho_2^2} d\phi_1 - \frac{2\rho_1^2 - \rho_2^2}{\rho_1^2 + \rho_2^2} d\phi_2 \right).\end{aligned} \tag{5.50}$$

In relation to the adiabatic paths (5.42) we find the associated Berry phases

$$\begin{aligned}\gamma_1 : &\quad \theta_{21} = \theta_{31} = \pi \frac{\rho_1^2 - 2\rho_2^2}{\rho_1^2 + \rho_2^2}, \\ \gamma_2 : &\quad \theta_{21} = \theta_{31} = \pi \frac{\rho_2^2 - 2\rho_1^2}{\rho_1^2 + \rho_2^2}, \\ \gamma_3 : &\quad \theta_{21} = \theta_{31} = -\pi.\end{aligned} \tag{5.51}$$

These results agree with our observations from Figure 5.5:

- The Berry phases associated to each path are equal.
- The Berry phases associated to γ_2 can be found by transforming those belonging to γ_1 under $\rho_1 \leftrightarrow \rho_2$.
- For γ_1 with $\rho_2 \gg 10\rho_1$, $\theta_{21} = \theta_{31} \approx -2\pi$.
- For γ_1 with $\rho_1 \gg 10\rho_2$, $\theta_{21} = \theta_{31} \approx \pi$.
- For γ_1, we have vanishing Berry phase whenever $\rho_1 = \sqrt{2}\rho_2$.
- For γ_3, the Berry phases are always $-\pi$.

We will now use the original Berry connection (5.32) to study the Berry phase discontinuity that occurs due to the cone C_{13}. In general for a closed adiabatic path $\gamma : [0, 1] \to U_1$

$$\begin{aligned}\theta_{21} = &\oint_\gamma \frac{\rho_1^2 \rho_2^2 \Big(R_1(p)^2 - R_2(p)^2 \Big)}{R_1(p)^2 R_2(p)^2} d\phi_1 \\ &- \oint_\gamma \rho_2^2 \left(\frac{\rho_1^2 \Big(R_1(p)^2 - R_2(p)^2 \Big)}{R_1(p)^2 R_2(p)^2} + \frac{\big(E_2(p) - z \big)^2 R_1(p)^2 - \big(E_1(p) - z \big)^2 R_2(p)^2}{R_1(p)^2 R_2(p)^2} \right) d\phi_2, \\ \theta_{31} = &\oint_\gamma \frac{\rho_1^2 \rho_2^2 \Big(R_1(p)^2 - R_3(p)^2 \Big)}{R_1(p)^2 R_3(p)^2} d\phi_1 \\ &- \oint_\gamma \rho_2^2 \left(\frac{\rho_1^2 \Big(R_1(p)^2 - R_3(p)^2 \Big)}{R_1(p)^2 R_3(p)^2} + \frac{\big(E_3(p) - z \big)^2 R_1(p)^2 - \big(E_1(p) - z \big)^2 R_3(p)^2}{R_1(p)^2 R_3(p)^2} \right) d\phi_2.\end{aligned} \tag{5.52}$$

The cone C_{13} borders U_3 and U_5, so we can use (5.38) to transform (5.52) into the local

coordinates of U_5:

$$\begin{aligned}\theta_{21} &= \oint_\gamma \frac{\left(q_3^2+q_4^2\right)\left(R_1(p)^2-R_2(p)^2\right)}{R_1(p)^2R_2(p)^2}\left(q_1dq_2-q_2dq_1\right) - \oint_\gamma\left(\frac{\rho_1^2\left(R_1(p)^2-R_2(p)^2\right)}{R_1(p)^2R_2(p)^2}\right.\\ &\qquad\left.+\frac{\left(E_2(p)-z\right)^2R_1(p)^2-\left(E_1(p)-z\right)^2R_2(p)^2}{R_1(p)^2R_2(p)^2}\right)\left(q_3dq_4-q_4dq_3\right),\\ \theta_{31} &= \oint_\gamma \frac{\left(q_3^2+q_4^2\right)\left(R_1(p)^2-R_3(p)^2\right)}{R_1(p)^2R_3(p)^2}\left(q_1dq_2-q_2dq_1\right) - \oint_\gamma\left(\frac{\rho_1^2\left(R_1(p)^2-R_3(p)^2\right)}{R_1(p)^2R_3(p)^2}\right.\\ &\qquad\left.+\frac{\left(E_3(p)-z\right)^2R_1(p)^2-\left(E_1(p)-z\right)^2R_3(p)^2}{R_1(p)^2R_3(p)^2}\right)\left(q_3dq_4-q_4dq_3\right).\end{aligned} \tag{5.53}$$

Consider the adiabatic path in U_5

$$\gamma_{r,\epsilon}(s) = \Phi_5\left(0,0,r\cos(2\pi s), r\sin(2\pi s), r/\sqrt{2}+\epsilon\right), \tag{5.54}$$

with $r,\epsilon>0$. This path will induce the Berry phases

$$\begin{aligned}\theta_{21,\epsilon} &= -2\pi r^2\int_0^1\left(\frac{\left(E_2\left(\gamma_{r,\epsilon}(s)\right)-z(\epsilon)\right)^2R_1\left(\gamma_{r,\epsilon}(s)\right)^2-\left(E_1\left(\gamma_{r,\epsilon}(s)\right)-z(\epsilon)\right)^2R_2\left(\gamma_{r,\epsilon}(s)\right)^2}{R_1\left(\gamma_{r,\epsilon}(s)\right)^2R_2\left(\gamma_{r,\epsilon}(s)\right)^2}\right)ds,\\ \theta_{31,\epsilon} &= -2\pi r^2\int_0^1\left(\frac{\left(E_3\left(\gamma_{r,\epsilon}(s)\right)-z(\epsilon)\right)^2R_1\left(\gamma_{r,\epsilon}(s)\right)^2-\left(E_1\left(\gamma(s)\right)-z(\epsilon)\right)^2R_3\left(\gamma_{r,\epsilon}(s)\right)^2}{R_1\left(\gamma_{r,\epsilon}(s)\right)^2R_3\left(\gamma_{r,\epsilon}(s)\right)^2}\right)ds,\end{aligned}$$

with $z(\epsilon)=r/\sqrt{2}+\epsilon$. Because R_i and E_i are only dependent on $q_1^2+q_2^2$, $q_3^2+q_4^2$ and q_5, they will remain constant along the entire path. Thus we get

$$\begin{aligned}\theta_{21} &= -2\pi r^2\frac{\left(E_2\left(\gamma(1)\right)-z(\epsilon)\right)^2R_1\left(\gamma(1)\right)^2-\left(E_1\left(\gamma(1)\right)-z(\epsilon)\right)^2R_2\left(\gamma(1)\right)^2}{R_1\left(\gamma(1)\right)^2R_2\left(\gamma(1)\right)^2},\\ \theta_{31} &= -2\pi r^2\frac{\left(E_3\left(\gamma(1)\right)-z(\epsilon)\right)^2R_1\left(\gamma(1)\right)^2-\left(E_1\left(\gamma(1)\right)-z(\epsilon)\right)^2R_3\left(\gamma(1)\right)^2}{R_1\left(\gamma(1)\right)^2R_3\left(\gamma(1)\right)^2}.\end{aligned} \tag{5.55}$$

Now consider the adiabatic path in U_3

$$\gamma'_{r,\epsilon}(s) = \Phi_3(0,0,r,2\pi s, r/\sqrt{2}-\epsilon), \tag{5.56}$$

with and $0<\epsilon<r/\sqrt{2}$. The Berry phases produced by γ' are

$$\begin{aligned}\theta'_{21,\epsilon} &= -2\pi r^2\frac{\left(E_2\left(\gamma'_{r,\epsilon}(1)\right)-z'(\epsilon)\right)^2R_1\left(\gamma'_{r,\epsilon}(1)\right)^2-\left(E_1\left(\gamma'_{r,\epsilon}(1)\right)-z'(\epsilon)\right)^2R_2\left(\gamma'_{r,\epsilon}(1)\right)^2}{R_1\left(\gamma'_{r,\epsilon}(1)\right)^2R_2\left(\gamma'_{r,\epsilon}(1)\right)^2},\\ \theta'_{31,\epsilon} &= -2\pi r^2\frac{\left(E_3\left(\gamma'_{r,\epsilon}(1)\right)-z'(\epsilon)\right)^2R_1\left(\gamma'_{r,\epsilon}(1)\right)^2-\left(E_1\left(\gamma'_{r,\epsilon}(1)\right)-z'(\epsilon)\right)^2R_3\left(\gamma'_{r,\epsilon}(1)\right)^2}{R_1\left(\gamma'_{r,\epsilon}(1)\right)^2R_3\left(\gamma'_{r,\epsilon}(1)\right)^2},\end{aligned} \tag{5.57}$$

with $z'(\epsilon)=r/\sqrt{2}-\epsilon$.

We are interested in the limit of the phases (5.55) and (5.57) as $\epsilon\to 0^+$. The left column of Figure 5.9 depicts the Berry phases θ_{21} and θ_{31} associated with the paths $\gamma_{\sqrt{2}/2,\epsilon}$ (in blue) and $\gamma'_{\sqrt{2}/2,\epsilon}$ (in orange) as functions of ϵ. These plots characterise a discontinuity in

the NV center's holonomy. Let $\Gamma(\gamma)$ denote the linear map (2.97), associated with parallel transporting quantum states about the path $\gamma_{r,\epsilon}$. What we find is

$$\begin{aligned}\lim_{\epsilon\to 0^+}\Gamma(\gamma_{r,\epsilon}) &= \begin{pmatrix}1 & 0 & 0\\ 0 & \exp\left(-i\frac{2\pi}{3}\right) & 0\\ 0 & 0 & \exp\left(i\frac{2\pi}{3}\right)\end{pmatrix},\\ \lim_{\epsilon\to 0^+}\Gamma(\gamma'_{r,\epsilon}) &= \begin{pmatrix}1 & 0 & 0\\ 0 & \exp\left(i\frac{2\pi}{3}\right) & 0\\ 0 & 0 & \exp\left(-i\frac{2\pi}{3}\right)\end{pmatrix}.\end{aligned} \tag{5.58}$$

In the right column of Figure 5.9 we have plots of the Berry phase θ_{21} and θ_{31} associated to the path $\gamma_{\sqrt{2}/2,\epsilon}$ and z has been allowed to vary from -0.8 to 0.8. The discontinuities in the plots occur precisely when $\gamma_{r,\epsilon}$ intersects the cones at $2z^2 = r^2$.

Now note that the energy eigenvalues are fixed on $\gamma_{r,\epsilon}$ for any value of r and ϵ. This means that we can plot how the energy changes as $\gamma_{r,\epsilon}$ is moved past the cones of degeneracy. This is depicted in Figure 5.10 for $r = \sqrt{2}/2$. As we can see, the energy becomes degenerate when $\gamma_{r,\epsilon}$ touches one of the cones at $z = \pm 0.5$.

With an equivalent experimental setup to that given in section 5.1, one could attempt the following experiment: Assign the first microwave an amplitude significantly larger than the other ($b_1 \gg b_2$), then set the frequency ω_1 and magnetic field along the defect axis such that $\hbar D + \gamma_{nv}B - \omega_1 < \sqrt{2}|b_1|$. Set up the quantum state one wishes to study, rotate the phase of both microwaves a full 2π, and record any Berry phase. Increment the first microwave's frequency down such that $\hbar D + \gamma_{nv}B - \omega_1 < \sqrt{2}|b_1|$. Again, set up the same quantum state, rotate the phase of both microwaves a full 2π, and record any Berry phase. Continue this process, with $\hbar D + \gamma_{nv}B - \omega_1$ approaching $\sqrt{2}|b_1|$ from below. Now repeat the experiment starting with $\hbar D + \gamma_{nv}B - \omega_1 > \sqrt{2}|b_1|$ and increase ω_1 in increments so that $\hbar D + \gamma_{nv}B - \omega_1$ approaches $\sqrt{2}|b_1|$ from above. One should expect to find a discontinuity in the Berry phase at $\hbar D + \gamma_{nv}B - \omega_1 = \sqrt{2}|b_1|$ of around $2\pi/3$.

It should be noted that to measure a Berry phase we measure the populations of the eigenstates $|u_1\rangle$, $|u_2\rangle$ and $|u_3\rangle$ over an ensemble of nitrogen vacancy centers. In our experimental set up we can measure these populations by firing a green laser at our samples. This causes each spin state to decay into the zero spin state via a different processes. Each process emits a particular wavelength of light which can be used to measure populations. This method is also used to initialise any state. A green laser is fired at the sample, forcing it into the spin 0 state, and then the microwaves are used to manipulate the system into any state needed [52].

The next result we will discuss is the holonomy group of the NV center. Consider γ_1 from (5.42). A contour plot of the integrand of θ_{21} is depicted in Figure 5.11. The red curve represents the zero contour for the integrand of θ_{21}, while the blue curve is the zero contour for the integrand of θ_{31}. Note that these curves never intersect. Now notice that the effective Berry connection (5.32) lacks a $d\rho_1$, $d\rho_2$ and a dz term. This means that no Berry phase will accumulate for any adiabatic path that only varies in ρ_1, ρ_2 and z. Furthermore, when $z = 0$ we have Berry phases of the form (5.50). We found that when $\rho_1 = \sqrt{2}\rho_2$, the $d\phi_1$ coefficient vanishes. This means that that path

$$\gamma(s) = \Phi_1(\sqrt{2}\rho_2, \alpha s, \rho_2, 0), \qquad \alpha \in [0, 2\pi] \text{ and } \rho_2 \in \mathbb{R}_{>0} \tag{5.59}$$

will make no contribution to the Berry phases θ_{21} and θ_{31}.

Now consider the adiabatic path. Choose any $p \in M$ and smoothly vary ρ_1, ρ_2 and z until we have reached a point on the zero contour for the integrand of θ_{31}. Then rotate ϕ_1 as many times as one needs to reach the desired value of θ_{21}. If ϕ_1 is not the value it started as, then smoothly vary ρ_1 to $\sqrt{2}\rho_2$, and z to 0. Rotate ϕ_1 back to its initial value, and finish by smoothly varying ρ_1, ρ_2 and z back to the starting point, p.

This adiabatic path can be used to make θ_{21} any value, while also keeping θ_{31} fixed. One could also have transported to the zero contour for the integrand of θ_{21} instead. This would

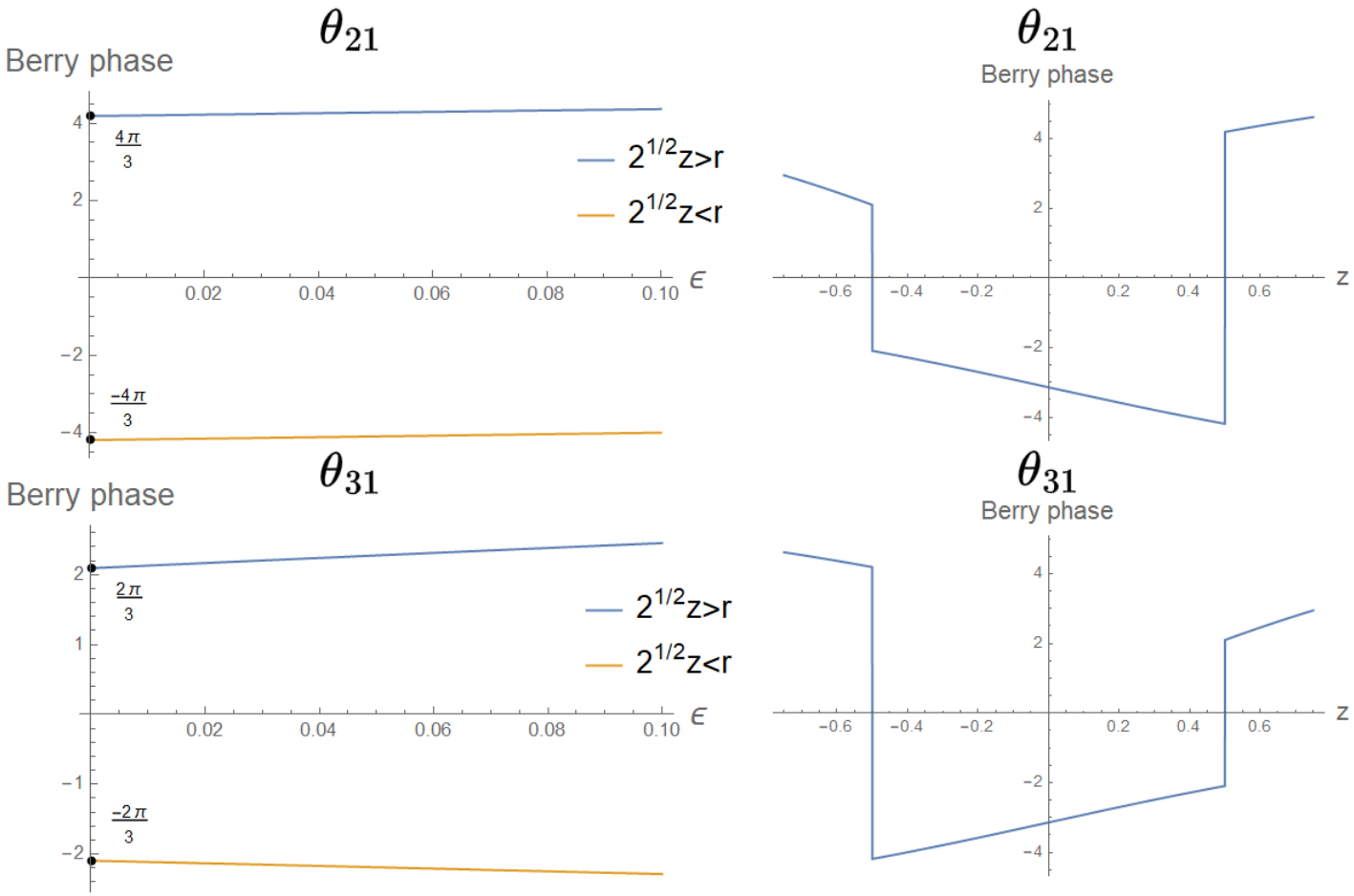

Figure 5.9: Left column: Plots of Berry phases θ_{21} and θ_{31} from (5.55) (in blue) and θ'_{21} and θ'_{31} from (5.57) (in orange) as functions of ϵ, with $r = \sqrt{2}/2$.
Right column: Plots of Berry phases θ_{21} and θ_{31} from (5.55) with $r = \sqrt{2}/2$ as a function of z. Notice that the discontinuities occur at $z = \pm 0.5$, which corresponds to γ intersecting one of the two cones.

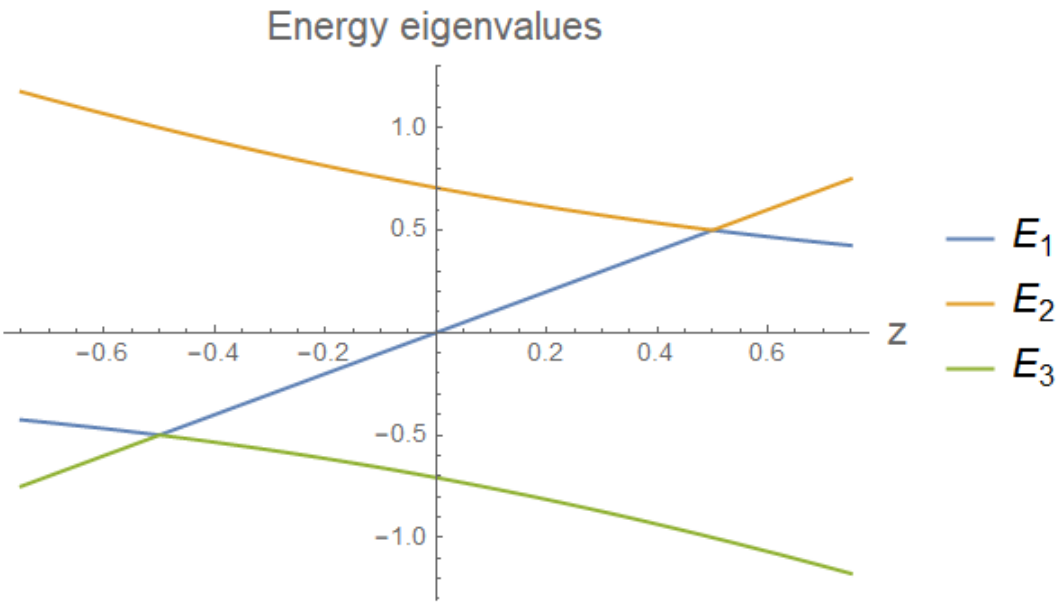

Figure 5.10: Plots of energy eigenvalues as functions of z, for every point on γ with $r = \sqrt{2}/2$. Note that the energy degeneracy only occurs for $z = \pm 0.5$, which corresponds to γ intersecting one of the two cones.

then allow us to tune θ_{31} to any value while keeping θ_{21} fixed. Thus, a combination of the two paths can be used to completely cover $U(1) \otimes U(1)$ for any starting point $p \in M$. This shows us that the holonomy group for the NV-center is

$$U(1) \otimes U(1). \tag{5.60}$$

Physically, if our experiment starts with a non-degenerate Hamiltonian $H(p)$, then for any quantum state $|\psi(p)\rangle = c^i |u_i(p)\rangle$, we can find a sequence of changes in our experimental parameters that will induce any phase difference in $|\psi(p)\rangle$ we want. Furthermore, this sequence can be done one parameter at a time, for example we could first vary the magnetic field strength along the defect axis, then vary one of the microwave frequencies, and after that change the amplitude of the other. As long as each variation is smooth and slow enough to be adiabatic, then our theory will apply. However, we must also account for dynamic phase. This could be added directly into the model via the connection recovery method in section 2.9, or after the Berry phase calculation is made.

In terms of dealing with dynamic phase in an experiment, we can transport a quantum state about some path adiabatically, measure the overall phase difference, then repeat the experiment but transport the state faster about the path. Plotting the phase data against the time it took to make the measurement, we will find a positive linear plot (as long as each path was adiabatic). This is because dynamic phase is proportional to time. Furthermore, because the path was unchanged in each experiment, the y-axis intercept of this plot is our Berry phase.

The last result we will discuss is the Chern class of the NV center. Seeing as there are 5 variables in our parameter space, we may have a non-trivial second Chern class. Recall that because the curvature 2-form associated to the Berry connection is always traceless, then the second Chern class has the simplified form (3.82)

$$c_2(\mathcal{F}) = \frac{-1}{8\pi^2}\mathrm{tr}(\mathcal{F} \wedge \mathcal{F}). \tag{5.61}$$

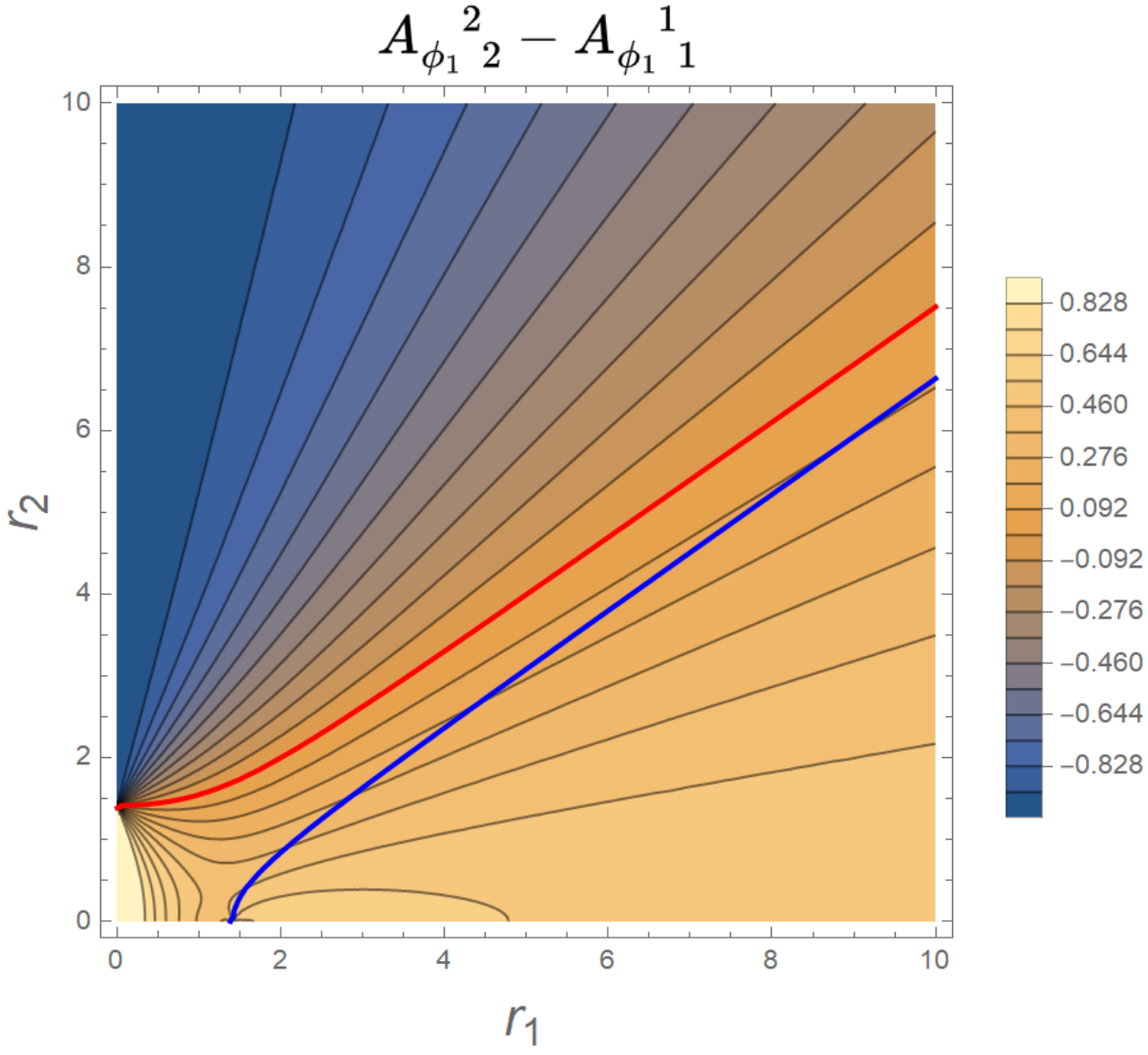

Figure 5.11: Contour plot of the integrand of θ_{21} for a general smooth path. The red curve is the zero contour, while the blue curve is the zero contour for the integrand of θ_{31}. Note that the red and blue curve never intersect

In local coordinates we have

$$\begin{aligned}
\mathcal{F}_{\mu\,\nu}(p)\mathcal{F}_{\sigma\,\eta}(p) &= \sum_{\ell=1}^{3}\left\langle \frac{\partial u_\ell}{\partial x^\mu}\middle|_p \middle| \frac{\partial u_\ell}{\partial x^\nu}\middle|_p \right\rangle \left\langle \frac{\partial u_\ell}{\partial x^\sigma}\middle|_p \middle| \frac{\partial u_\ell}{\partial x^\eta}\middle|_p \right\rangle |u_\ell(p)\rangle\langle u_\ell(p)| \\
&+ 2\sum_{\ell,k=1}^{3}\left\langle \frac{\partial u_k}{\partial x^\mu}\middle|_p \middle| \frac{\partial u_k}{\partial x^\nu}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_\ell}{\partial x^\sigma}\middle|_p \right\rangle \left\langle u_\ell(p)\middle| \frac{\partial u_\ell}{\partial x^\eta}\middle|_p \right\rangle |u_k(p)\rangle\langle u_\ell(p)| \\
&- 2\sum_{\ell,k=1}^{3}\left\langle \frac{\partial u_k}{\partial x^\mu}\middle|_p \middle| \frac{\partial u_k}{\partial x^\nu}\middle|_p \right\rangle \left\langle u_\ell(p)\middle| \frac{\partial u_k}{\partial x^\sigma}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_k}{\partial x^\eta}\middle|_p \right\rangle |u_k(p)\rangle\langle u_\ell(p)| \\
&+ \sum_{j,k,\ell=1}^{3}\left\langle u_j(p)\middle| \frac{\partial u_k}{\partial x^\mu}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_k}{\partial x^\nu}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_\ell}{\partial x^\sigma}\middle|_p \right\rangle \left\langle u_\ell(p)\middle| \frac{\partial u_\ell}{\partial x^\eta}\middle|_p \right\rangle |u_j(p)\rangle\langle u_\ell(p)| \\
&+ \sum_{j,k,\ell=1}^{3}\left\langle u_j(p)\middle| \frac{\partial u_k}{\partial x^\mu}\middle|_p \right\rangle \left\langle u_j(p)\middle| \frac{\partial u_j}{\partial x^\nu}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_\ell}{\partial x^\sigma}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_k}{\partial x^\eta}\middle|_p \right\rangle |u_j(p)\rangle\langle u_\ell(p)| \\
&- \sum_{j,k,\ell=1}^{3}\left\langle u_j(p)\middle| \frac{\partial u_k}{\partial x^\mu}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_k}{\partial x^\nu}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_\ell}{\partial x^\sigma}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_k}{\partial x^\eta}\middle|_p \right\rangle |u_j(p)\rangle\langle u_\ell(p)| \\
&- \sum_{j,k,\ell=1}^{3}\left\langle u_j(p)\middle| \frac{\partial u_k}{\partial x^\mu}\middle|_p \right\rangle \left\langle u_j(p)\middle| \frac{\partial u_j}{\partial x^\nu}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_\ell}{\partial x^\sigma}\middle|_p \right\rangle \left\langle u_\ell(p)\middle| \frac{\partial u_\ell}{\partial x^\eta}\middle|_p \right\rangle |u_j(p)\rangle\langle u_\ell(p)|.
\end{aligned}$$

This may seem daunting, but in the trace the last four terms cancel. We can show this by relabelling the ν index with the η index in the first and last two terms. This causes an odd permutation of $(\mu\nu\sigma\eta)$ and thus reordering the differentials back into order will leave an overall negative sign. The reason why the ν and η indices are exchanged is because these are the only places that each term differs from the rest. We will show how the last two cancel because it is not as obvious as the first two: In the trace, the last term consists of sum over j and k of the functions

$$\begin{aligned}
&\left\langle u_j(p)\middle| \frac{\partial u_k}{\partial x^\mu}\middle|_p \right\rangle \left\langle u_j(p)\middle| \frac{\partial u_j}{\partial x^\nu}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_j}{\partial x^\sigma}\middle|_p \right\rangle \left\langle u_j(p)\middle| \frac{\partial u_j}{\partial x^\eta}\middle|_p \right\rangle dx^\mu\wedge dx^\nu\wedge dx^\sigma\wedge dx^\eta, \\
&= \left\langle u_j(p)\middle| \frac{\partial u_k}{\partial x^\mu}\middle|_p \right\rangle \left\langle u_j(p)\middle| \frac{\partial u_j}{\partial x^\eta}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_j}{\partial x^\sigma}\middle|_p \right\rangle \left\langle u_j(p)\middle| \frac{\partial u_j}{\partial x^\nu}\middle|_p \right\rangle dx^\mu\wedge dx^\eta\wedge dx^\sigma\wedge dx^\nu, \\
&= -\left\langle u_j(p)\middle| \frac{\partial u_k}{\partial x^\mu}\middle|_p \right\rangle \left\langle u_j(p)\middle| \frac{\partial u_j}{\partial x^\nu}\middle|_p \right\rangle \left\langle u_k(p)\middle| \frac{\partial u_j}{\partial x^\sigma}\middle|_p \right\rangle \left\langle u_j(p)\middle| \frac{\partial u_j}{\partial x^\eta}\middle|_p \right\rangle dx^\mu\wedge dx^\nu\wedge dx^\sigma\wedge dx^\eta.
\end{aligned}$$

This implies that the last two terms cancel in the trace.

Thus the second Chern class of the NV center takes the local form

$$\begin{aligned}
c_2(\mathcal{F}) &= -\frac{1}{8\pi^2}\sum_{j=1}^{3}\left\langle \frac{\partial u_j}{\partial x^\mu}\middle| \frac{\partial u_j}{\partial x^\nu}\right\rangle \left\langle \frac{\partial u_j}{\partial x^\sigma}\middle| \frac{\partial u_j}{\partial x^\eta}\right\rangle dx^\mu\wedge dx^\nu\wedge dx^\sigma\wedge dx^\eta, \\
&= -\frac{1}{8\pi^2}\sum_{i,j,k=1}^{3} A_\mu{}^j{}_k A_\nu{}^k{}_j A_\sigma{}^j{}_i A_\eta{}^i{}_j dx^\mu\wedge dx^\nu\wedge dx^\sigma\wedge dx^\eta.
\end{aligned} \tag{5.62}$$

The Chern class does not vanish everywhere as illustrated by the contour plot in Figure 5.12. This plot depicts the coefficient of the $d\rho_1\wedge d\phi_1\wedge d\rho_2\wedge d\phi_2$ term for $z=1$, as functions of ρ_1 and ρ_2. Notice the singularities at $(\rho_1,\rho_2)=(0,\sqrt{2}),(\sqrt{2},0)$. In the Berry phase plots Figure 5.6, these points were finite discontinuities. However, the Chern class is divergent at these points. The reason for this is because the curvature is dependent on the entire Berry connection and not just the diagonal terms, like the Berry phase is. It is precisely the off diagonal terms where we encounter derivatives in the energy eigenvalues (5.33).

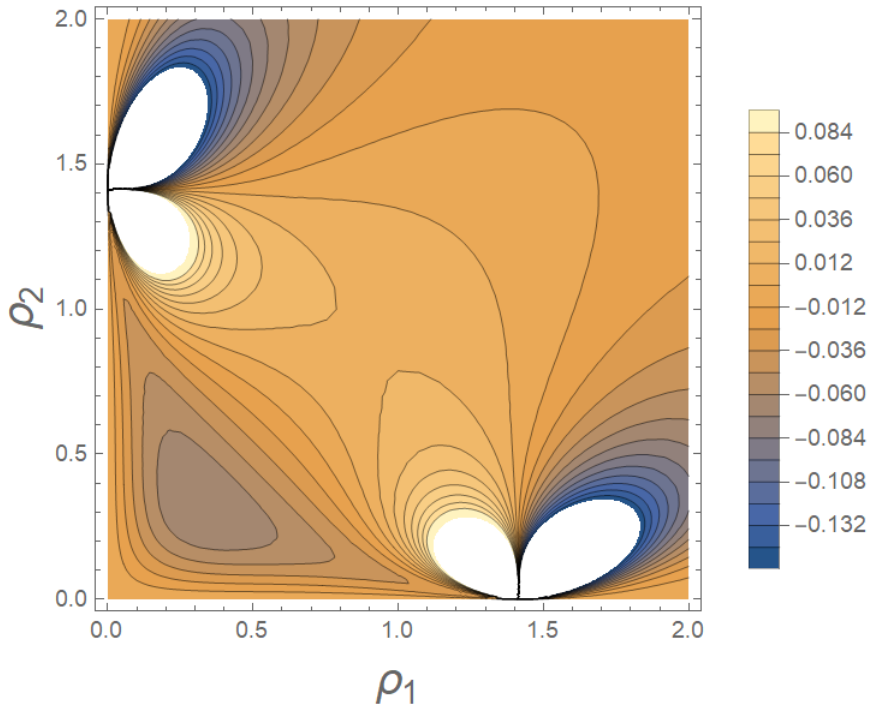

Figure 5.12: Contour plot the Chern class coefficient $\Big(c_2(\mathcal{F})\Big)_{1234}$. This contour plot shows that the Chern class for the NV center is non-vanishing. Again the white regions have been excluded due to the steepness. Note that the points $(\rho_1, \rho_2) = (0, \sqrt{2}), (\sqrt{2}, 0)$, are now poles.

Chapter 6

Conclusion

We began in Chapter 2 by discussing some of the underlying concepts in differential geometry in relation to fibre bundles. Importantly, this included a method to construct fibre bundles and an introduction to Ehresmann connections. This connection provided a way to parallel transport geometric objects about a vector bundle, by imposing a system of linear first order ODEs. In order to ensure the existence and uniqueness of parallel transport, we ivoked the Picard–Lindelöf theorem [9, Sections 1.1 and 1.2]. Localising this led to our first result: a development of local horizontal lift for vector bundles. This provided local IVPs whose solutions describe parallnel transport. We ended this chapter by developing a method to recover a connection from information on how geometric objects parallel transport.

In Chapter 3 we developed our geometric model for Berry phases. This started with a discussion on geometric phase as well as the adiabatic theorem, and provided an example of geometric phase by solving the Schrödinger equation. From there we showed how to construct a complex n-dimensional vector bundle from a family of n-dimensional local Hamiltonians $\{H^\alpha\}$. We defined the base manifold, or parameter space, M, to be largest subset of $\mathbb{R}^m$, such that H^α is non-degenerate for all $p \in M$. The fibre was taken to be the Hilbert space that our Hamiltonians acted on. The structure group was the set of $n \times n$ unitary matrices and the transition maps were taken to be the projections

$$t_{\alpha,\beta}(p) = \left|u^i(p)\right\rangle_\alpha \,{}_\beta\langle u_i(p)| \tag{6.1}$$

where $\left|u^i(p)\right\rangle_\alpha$ is the i^{th} eigenstate of $H^\alpha(p)$.

Next, we introduced the idea of variable basis sections by using local energy eigenstates as the local basis vectors in our fibre. This reduced complexity in coefficients and made the use of the adiabatic theorem natural. But, it required an extension in the derivation of local horizontal lifts in Chapter 2. We found that a horizontal lift remains unchanged when using variable local fibre coordinates. However, the IVPs provided by a horizontal lift do include a new term. From there we used the connection reconstruction method from Section 2.9 to determine the local Berry connections based on our findings in the geometric phase (see example 3.6). We ended Chapter 3 by discussing the globally defined Berry curvature and the Chern class of a Berry bundle. Because the Berry curvature is traceless, we showed that a parameter space must have at least four real dimensions in order to have a non-vanishing Chern class. Experimentally, this means that in order to observe a non-zero Chern number, one must be able to vary at least four independent experimental variables.

Chapter 4 was an example on how to use the model we built in Chapter 3. We studied a spin-1/2 system, whose Berry bundle consisted of a real 3-dimensional base manifold and a complex 2-dimensional fibre. Physically, the system was described by a spin-1/2 particle in a uniform magnetic field (B_x, B_y, B_z). We found that its parameter space was $\mathbb{R}^3\backslash\{0\}$, meaning that the only point of degeneracy corresponded to turning the magnetic field off. We also discovered that because the Berry connection was independent of the field strength, we could effectively describe the system's holonomy by only considering paths on S^2. Fur-

thermore, our model agreed with experimentation in that the Berry phase accumulated by transporting states about closed adiabatic paths is equal to the area it encloses [16, 22, 45]. This meant that the holonomy group of a spin-1/2 system was shown to be $U(1)$.

In Chapter 5 we studied a spin-1 nitrogen vacancy (NV) center. The reason we wanted to study the NV center is because it has a wide set of applications from quantum computation [47, 50], to communication [19, 51]. We started with an experimental set-up [52], and used it to determine the local Hamiltonian describing our system. We then transformed into a rotating frame and used the rotating wave approximation to simplify the local Hamiltonian. From this simplified Hamiltonian we constructed an NV Berry bundle, which has a real 5-dimensional parameter space and a complex 3-dimensional fibre. The parameter space was given by $\mathbb{R}^5 \backslash (C_{12} \cup C_{13})$ where C_{12} and C_{13} are 2-dimensional cones.

We found that the coefficients of the Berry connections were functions of the ratios in the magnetic field strengths. This allowed us to effectively reduce our parameter space to a 4-dimensional manifold. Physically, this means that in an experiment, a wide range of field strengths are not required.

The cones of degeneracy were found to cause observable discontinuities in the Berry phases. Furthermore, these discontinuities are path dependent. In the special case when the diagonal terms of our local Hamiltonian are set to zero, the Berry phases between our second and first eigenstates (θ_{21}), and our third and first eigenstates (θ_{31}), coincided. This was used to develop a method for constructing an adiabatic closed path to induce any value of θ_{21} and θ_{31}. Thus, the holonomy group of the NV center is $U(1) \otimes U(1)$. We concluded this chapter by determining the second Chern class of our NV Berry bundle, and showed it does not vanish everywhere in the parameter space. This means that the NV system could have a non-zero Chern number.

This thesis has shown that, to an experimenter it is important to understand the geometry and topology of a system's parameter space. This is especially true for experiments in the fields of quantum computation and communication, where it is critical to make precise and accurate measurements [41, 42]. Berry phase can change an observation, but as seen in section 5.4, it is also possible to control. An experimenter may be interested in using this model to search for topologically protected paths, in order to minimise the variation of Berry phase in their experiment. It can also be used to find paths that are affected or unaffected by discontinuities in the Berry connection.

Future work:

For further study into the NV Berry bundle we suggest explicitly calculating its Chern number using (5.62). Chern numbers are of great interest because they are observable [30, 32, 56]. To do this one must assign an orientation to the parameter space and study its fundamental homology cycles. Then, assign a pairing of Chern classes and the cycles, or in other words, integrate the Chern classes over the cycles. This is a complicated task, so we recommend that the topology of the NV center's parameter space is studied in greater detail. Keep in mind that the Berry connection can be used to reduce the parameter space to an effective base manifold (S^2 in the case of the spin-1/2 bundle), possibly simplifying the system.

In Chapters 4 and 5 we found that we could cancel a non-zero field strength, which corresponded to a radial coordinate in the spin-1/2 parameter space (4.22), and the q_5 axis in the NV parameter space (5.40). This begs the question of whether this is a standard property of Berry connections (3.67). One could study this for a general Berry connection. We suspect that this is a property of the normalisation of energy eigenstates. Furthermore, when the Berry connection does simplify in this way, what are the properties of the effective parameter spaces?

Our model for Berry phase was designed so that it could be adapted to non-adiabatic paths. This can be done by using a different connection that couples the IVPs describing parallel transport. It would be interesting to study a quantum system using a non-adiabatic connection. One could even incorporate time into the model such that for slowly evolving paths, the connection behaves like the Berry connection (3.67), but for quickly evolving

paths, non-adiabatic features become more prevalent. In order to remain physically relevant, a non-adiabatic connection should be recovered from experimental results or the Schrödinger equation [6, 33, 46], using the method described in Section 2.9.

The paper [8] studies the NV Hamiltonian (5.9) for the special case $q_5 = 0$. One could try to reproduce the results from [8] by extending our model from a vector bundle to a bundle gerbe [34]. This will upgrade the Berry connection from a 1-form to a 2-form and the Chern class will be generalised to the Dixmier-Douady class [14]. Furthermore, making use of a bundle gerbe will allow for study of higher Berry phases [37].

Our model for Berry phases is only relevant for finite dimensional Hilbert spaces. One could attempt to extend this model by allowing for infinite dimensional fibres. This could open the model to much more complicated systems such as L^2 spaces. There does seem to be some interest in this topic [11, 40, 44], but not so much in recent times.

Appendix A

Principal and associated bundles

Note that the material covered in this appendix will not be used in this thesis. These definitions and examples are provided to bridge the gap between our model and similar Berry phase models that make use of principal bundles [35, Section 10.6] and [20, 24].

A *principal bundle* is defined as a fibre bundle whose fibre, F, is the same as its structure group, G. We usually denote a principal bundle by $P(M, G)$ or say "a G bundle over M". A very useful property of principal bundles is that we can define both a left and right action of G on the fibre: Let $\omega_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha)$ be a local trivialisation. The right action of $a \in G$ on $b = \omega_\alpha(p, g) \in \pi^{-1}(U_\alpha)$ is given by $R_a(b) = ba = \omega_\alpha(p, ga)$, for $a \in G$.

Right multiplication is independent of the local trivialisation because the right and left actions commute. For all $p \in U_i \cap U_j$, $b \in \pi^{-1}(p)$ and $a \in G$, if

$$b = \omega_\beta(p, g_\beta) = \omega_\alpha(p, t_{\alpha\beta} g_\beta),$$

then

$$ba = \phi_\beta(p, g_\beta a) = \phi_\alpha(p, (t_{\alpha\beta} g_\beta)a) = \phi_\alpha(p, t_{\alpha\beta}(g_\beta a)). \tag{A.1}$$

Furthermore, because the right action is given by group multiplication, it is both transitive and free: for any $b_1, b_2 \in \pi^{-1}(p)$ there exists $a \in G$ such that $b_1 = R_a(b_2)$ and if $R_a(b_1) = b_1$ then $a = id_G$.

The transitive and free nature of the right action comes with the very useful consequence that if $\pi(b) = p$ then we can construct the entire fibre above p by the right action: $G_p = \{R_a(b) : a \in G\}$. This gives us a very easy and straight forward way to define trivialisations. So easy in fact, that mathematicians named the method *canonical trivialisation*. The idea of *canonical trivialisation* is to take a useful local section $\sigma_\alpha : U_\alpha \to P$ for each open set in our chosen cover $\{U_\alpha\}$ of M, and then for every $p \in U_\alpha$ fix $\sigma_\alpha(p)$ so that it is locally centred on the identity in G. By useful local section we mean a section that we wish to study, like a wave function $|\psi\rangle$ in a Berry phase model.

The proper definition of *canonical trivialisation* states that for a given local section $\sigma_\alpha(p)$ over U_α, a local trivialisation is given by

$$\omega_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha) : (p, g_b) \mapsto b = \sigma_\alpha(p)g. \tag{A.2}$$

Hence, for our chosen local section σ_α, we have for all $p \in M$

$$\sigma_\alpha(p) = \omega_\alpha(p, id).$$

For canonical trivialisations $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ and $\sigma_\beta : U_\beta \to \pi^{-1}(U_\beta)$ with $p \in U_\alpha \cap U_\beta$ our trivialisations transform in the following way: for all $g \in G$

$$\sigma_\alpha(p)g = \omega_\alpha(p, g) = \omega_\beta(p, t_{\beta\alpha}(p)g) = \omega_\beta(p, e)t_{\beta\alpha}(p)g = \sigma_\beta(p)t_{\beta\alpha}(p)g. \tag{A.3}$$

Note that this is different to how a section transforms from domain to domain (2.7).

Example A.0.1. Local tangent vectors of principal bundles.

Consider a principal bundle (P, π, M, G) with family of local sections $\{\sigma_\alpha : U_\alpha \to P\}$ where $U_\alpha \subseteq M$ are the domains of our base manifold. We will use these sections to form our canonical trivialisations

$$\phi_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha) : (p, g) = \sigma_\alpha(p)g. \tag{A.4}$$

Consider some smooth path $\hat{\gamma} : [0, 1] \to P$ with $\pi \circ \hat{\gamma} = \gamma$, and assume that $\hat{\gamma}(0) \in \pi^{-1}(U_\alpha)$. To find the vector tangent to $\hat{\gamma}(t)$ at $t = 0$ we will take the derivative of $f \circ \phi_\alpha \circ \hat{\gamma}(t)$ at $t = 0$ for some $f \in C^\infty(U_\alpha \times G)$. Because $\pi^{-1}(U_\alpha)$ is open and $\hat{\gamma}(0) \in \pi^{-1}(U_\alpha)$ then there exists an open neighbourhood $\hat{\gamma}(0) \in V \subset \pi^{-1}(U_\alpha)$ and $0 < \epsilon \leq 1$ such that $\hat{\gamma}(t) \in V \subset \pi^{-1}(U_\alpha)$ for all $t \in [0, \epsilon)$. Thus we have

$$\hat{\gamma}(t) = \sigma_\alpha\big(\gamma(t)\big)g(t) \tag{A.5}$$

for all $t \in [0, \epsilon)$ and some $g : [0, \epsilon) \to G$. This means that

$$f \circ \phi_\alpha \circ \hat{\gamma}(t) = f\Big(\phi_\alpha\Big(\sigma_\alpha\big(\gamma(t)\big)g(t)\Big)\Big) = f\big(\gamma(t), g(t)\big), \tag{A.6}$$

for for all $t \in [0, \epsilon)$. Hence,

$$\frac{d}{dt}\Big(f \circ \phi_\alpha \circ \hat{\gamma}(t)\Big)\Big|_{t=0} = \frac{d\gamma^\mu}{dt}\Big|_{t=0} \frac{\partial f}{\partial x^\mu}\Big|_{\hat{\gamma}(0)} + \frac{dg}{dt}\Big|_{t=0} \frac{\partial f}{\partial g}\Big|_{\hat{\gamma}(0)}. \tag{A.7}$$

Equation (A.0.1) shows us that even though principal bundles have a very different trivialisation method, via canonical trivialisation, tangent vectors behave the same way as they do for vector bundles. The local tangent vector, X, to the path $\hat{\gamma}(t)$ at $t = 0$ is given by

$$X = \frac{d\gamma^\mu}{dt}\Big|_{t=0} \frac{\partial}{\partial x^\mu}\Big|_{\hat{\gamma}(0)} + \frac{dg}{dt}\Big|_{t=0} \frac{\partial}{\partial g}\Big|_{\hat{\gamma}(0)}. \tag{A.8}$$

If we have a principal bundle but we would rather work with a different type of fibre bundle, we can use an *associated bundle*.

Given a principal bundle $P(M, G)$ we may construct an *associated fibre bundle*. Let G act on some manifold F on the left, $\rho : G \times F \to F$, and define an action of G on $P \times F$ such that $(u, f) \mapsto (ug, \rho\big(g^{-1}\big)f)$. Then we define our *associated bundle* as

$$E = (P \times F)/G \tag{A.9}$$

in which $(u, f) \sim (ug, \rho\big(g^{-1}\big)f)$. Locally, $P \times F$ is diffeomorphic to $M \times G \times F$ thus $(P \times F)/G$ is locally diffeomorphic to $M \times F$ which is exactly what we want in a fibre bundle.

We define the projection $\pi_E : E \to M$ in the following way

$$\pi_E\Big(\big[(u, f)\big]\Big) = \pi(u) \tag{A.10}$$

where $\pi : P \to M$ is the projection on P. The projection π_E is well defined because

$$\pi_E\Big(\big[(ug, \rho\big(g^{-1}\big)f)\big]\Big) = \pi(ug) = \pi(u) = \pi_E\Big(\big[(u, f)\big]\Big),$$

for all $g \in G$. Lastly, we need to define the transition maps. If $t_{\alpha\beta}$ is a transition map on P then $\rho \circ t_{\alpha\beta}$ defines a transition map on E.

Associated bundles are particularly useful for forming vector bundles from a principal bundle and vice versa as example A.0.2 demonstrates.

Example A.0.2. Let F be a $k-$dimensional vector space and ρ a $k-$dimensional representation of G. The associated vector bundle $P \times_\rho V$ is thus defined by identifying the points $(u, v) \sim (ug, \rho(g)^{-1}v)$.
For example, for $P(M, GL(k, \mathbb{R}))$ the associated fibre bundle is a vector bundle over M with fibre $\mathbb{R}^k$. Let us explore the fibre bundle structure of the associated bundle $E = P \times_\rho V$.
The projection: $\pi_E(u, v) = \pi(u)$ and since $(u, v) \sim (ug, \rho(g)^{-1}v)$ then we have $\pi(ug) = \pi_E(ug, \rho(g)^{-1}v) = \pi_E(u, v) = \pi(u)$.
Local trivialisations: $\omega_\alpha : U_\alpha \times V \to \pi_E^{-1}(U_\alpha)$.
Transition functions: $\rho(t_{\alpha\beta}) : V \to V$ where $t_{\alpha\beta} \in G$.

Conversely a vector bundle, (E, π, M, V, G), induces a principal bundle $P(E) = P(M, G)$, where we use the same transition functions.

Example A.0.2 can be adapted to our geometric model introduced in chapter 3 to convert it to a principal bundle in order to match it to similar models like that in [20].

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