# Free Field Realisations in Logarithmic Conformal Field Theory

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For my family

*in tenebras lucent*

# Declaration

The work in this thesis is my own except where otherwise stated.

Michael Carrington Cromer

# Acknowledgements

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# Abstract

Invariances of conformal field theories (CFTs) would seem to suggest that correlation functions behave as power laws. However, logarithms also exhibit conformal invariance. When logarithms are permitted in two-dimensional CFTs, the corresponding state spaces characteristically involve reducible but indecomposable Virasoro modules with non-diagonalisable algebra action. Notably, such state spaces no longer naturally admit a grading into energy eigenspaces. Despite this non-diagonalisable energy operator, one finds significant physical motivation for the study of such representations, with many interesting statistical mechanical models exhibiting this behaviour: percolation, dilute polymers, self-avoiding walks, and more. A vast amount of effort has been made in the study of these two-dimensional logarithmic CFTs, both their internal structure and their fusion rules. However, it would be fair to say that logarithmic CFTs are still less well understood than their non-logarithmic counterparts. In recent years, the relevance of free-field oscillator algebras to the study of such representations has become more and more apparent. Many of the module structures in question might more appropriately be considered as Fock-type spaces.

In this thesis we develop free field realisations of logarithmic CFTs. We analyse some general features, examining staggered modules of the Virasoro algebra in particular, before providing a construction for staggered modules consisting of Fock spaces considered as Virasoro modules. We derive an explicit formula for a module invariant of staggered Fock modules, verifying that the given construction agrees with those seen to date in the literature. We then turn to more conjectural areas, examining how the non-diagonalisaiblity of the Virasoro representation can be reproduced by the inclusion of additional modes into the underlying oscillator algebras, and how the states created by these modes correspond to the vacuum evaluations of logarithmic fields. We take these as our motivating examples for a subsequent working definition of logarithmic vertex operator algebras, in the hope that not only do their state spaces correspond to the staggered structures developed to this point, but that they provide an additional avenue of approach in the construction and study of logarithmic conformal field theories.

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# Introduction

*He had seen everything, had experienced all emotions,*
*from exaltation to despair, had been granted a vision*
*into the great mystery, the secret places,*
*the primeval days before the Flood. He had journeyed*
*to the edge of the world and made his way back, exhausted,*
*but whole. He had carved his trials on stone tablets.*

*Gilgamesh*, c.2000BC

Conformal field theory (CFT) is an important area within quantum field theory, and the machinery of conformal symmetry is a powerful analytical tool, so much so that conformal field theories find important and widespread applications across many areas of theoretical physics, from string theory to statistical mechanics. Correlation functions of quantum fields which are conformally invariant are more heavily constrained than those which are not, making exact analytic solutions possible. This is what makes CFTs so amenable to study and calculation.

However, theoretical physics is not the only arena where CFT enjoys enormously active research applications. The constraints of conformal symmetry motivate rigorous formulations of the algebraic structure of quantum fields. Vertex algebras, the resulting objects, are of import in many areas outside of physics, for instance in the study of modular forms and in monstrous moonshine (e.g. [16]). From two-dimensional CFT we also have the Virasoro algebra, an infinite-dimensional Lie algebra and the generators of conformal symmetry. The representation theory of the Virasoro algebra is a deep and complex topic, these spaces being the state spaces of the corresponding field theories.

The scale invariance of conformal field theories seems to suggest that correlation functions must behave as power laws. However, logarithms can also obey conformal invariance [22]. When logarithmic behaviour is permitted in two-dimensional conformal field theory, one finds that the corresponding state spaces characteristically involve reducible but indecomposable Virasoro modules with non-diagonalisable action of the algebra's generators [41]. Notably, this means that the state spaces are no longer naturally graded into eigenspaces of the Virasoro zero mode (such a grading is typically related to the action of the Hamiltonian). Despite the suggestion of a non-diagonalisable energy operator, one finds significant physical motivation for the study of such representations. It transpires that many interesting statistical mechanical models exhibit this behaviour, including percolation, dilute polymers, self-avoiding walks, and more [7,10,34,40,45,46,46].

A vast amount of effort has been made in the study of these two-dimensional logarithmic CFTs, both their internal structure and their fusion rules [8,13,14,18,19,23,24,31]. However, it would be fair to say that logarithmic CFTs are still less well understood than their non-logarithmic counterparts. In recent years, the relevance of free-field oscillator algebras to the state spaces of logarithmic theories has become more and more apparent [37,39]. It appears that many of the module structures which are 'glued' together to produce the representations in question might more appropriately be considered as Fock-type spaces.

In this thesis we develop free field realisations of logarithmic CFTs. The first chapter is an overview of the Lie representation theory used throughout the rest of the work. It discusses in detail the structure of the Virasoro algebra $\mathfrak{Vir}$ and touches on its super-versions $\mathfrak{s}_N\mathfrak{Vir}$ ($N = 1$ and $N = 2$) as well the representation theory of all three algebras, particularly Verma modules. We also discuss the infinite-dimensional bosonic and fermionic oscillator algebras $\mathfrak{a}$ and $\mathfrak{b}$ and their Fock spaces; their infinite-dimensional $\mathbb{Z}$-graded highest weight representations. We make realisations of the Virasoro algebra as operators on these spaces and discuss the construction of $\mathfrak{Vir}$-intertwiners between them.

In the second chapter we briefly discuss the field-theoretic underpinnings of logarithmic conformal field theories, but quickly change focus to the representation theory, giving a definition for a *staggered module*, a type of indecomposable modules seen in logarithmic conformal field theory. These staggered modules involve a weakening of the requirement that $L_0$, the Virasoro zero mode, act diagonalisably. Instead of grading these modules into $L_0$ eigenspaces, we must relax this into a grading by *generalised* $L_0$ eigenvalues. We analyse some general features of such modules, turning to staggered modules of the Virasoro algebra in particular, before providing a construction for staggered modules consisting of Fock spaces considered as Virasoro modules. We compute module invariants for such staggered Fock modules, showing that the specified construction indeed agrees with module invariants calculated in the literature and therefore that it provides a means of realising logarithmic theories in terms of free fields.

The third chapter is an algebraic exploration of some features of staggered modules as developed in chapter two. We examine how the non-diagonalisability of the Virasoro representation can be reproduced by the inclusion of additional modes into the underlying oscillator algebras, and how the states created by these modes correspond to the vacuum evaluations of logarithmic fields. We take these as our motivating examples for a subsequent working definition of logarithmic vertex operator algebras, in the hope that not only do their state spaces correspond to the staggered structures developed to this point, but that they provide an additional avenue of approach in the construction and study of logarithmic conformal field theories.

*Chapter 1*

# The Virasoro Algebra

*"Would you tell me, please, which way I ought to go from here?"*
*"That depends a good deal on where you want to get to," said the Cat.*
*"I don't much care where-" said Alice.*
*"Then it doesn't matter which way you go," said the Cat.*

Lewis Carroll (C.L. Dodgson), *Alice's Adventures in Wonderland*

The Virasoro Algebra is an infinite-dimensional Lie algebra with a wide range of applications in theoretical and mathematical physics, where it appears chiefly as the symmetry algebra of 2-dimensional conformal field theories. While typically the conformal group is $SO(d+1,1)$, with a single spatial dimension there is a degeneracy in one of the usually-relevant constraints, and the algebra becomes infinite-dimensional [42]. The Virasoro algebra is nothing more than the central extension of the Witt algebra, which is the algebra of differential operators on the circle generated by objects of the form $z^n \partial_z$ for $n \in \mathbb{Z}$. This central extension is simply the addition of a new generator $C$ which is central; i.e. it commutes with the entire algebra. The commutation relations are also modified to include the new element $C$. This can be thought of as a kind of "quantum deformation" of the Witt algebra. In the case of the Virasoro algebra, this extension is universal (in the categorical sense) [25].

In what follows, we introduce the algebra itself and some of its important representation theory, which is vital for the main result of this work. For our immediate purposes, we only need to understand the algebra "as-is" in an abstract sense, so will spend little time discussing the underlying field theory, which will be relegated to notes interspersed through the text. This field content comes into play more heavily in later chapters, so is developed more fully as needed. However, we do assume a working knowledge of vertex (operator) algebras, the field structures used in conformal field theories. Introductory details for the reader unfamiliar with the subject can be found in Appendix A.

# 1.1 The Virasoro Algebra

In this section we define the Virasoro algebra itself, discuss some of its important representation theory, and introduce some fermionic extensions to Virasoro superalgebras.

## 1.1.1 Virasoro Basics

**1.1.1 Definition.** The *Virasoro Algebra*, denoted by $\mathfrak{Vir}$, is the infinite-dimensional algebra with generators

$$\{L_n, C \mid n \in \mathbf{Z}\} \tag{1.1}$$

and defining relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)C\delta_{m,-n} \tag{1.2}$$

and

$$[L_m, C] = 0 \quad \forall m \in \mathbb{Z}. \tag{1.3}$$

The Virasoro algebra forms what is called a *graded* algebra. In particular, it is $\mathbb{Z}$-*graded*, meaning that it has the following decomposition:

$$\mathfrak{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{Vir}_n \tag{1.4}$$

where

$$\mathfrak{Vir}_n = \mathbb{C}L_n,\ n \neq 0, \qquad \mathfrak{Vir}_0 = \mathbb{C}L_0 \oplus \mathbb{C}C, \tag{1.5}$$

satisfying

$$[\mathfrak{Vir}_m, \mathfrak{Vir}_n] \subseteq \mathfrak{Vir}_{m+n}. \tag{1.6}$$

This property is important when we begin discussing the representation theory of the Virasoro algebra, where we focus on *graded* representations: vector spaces which are themselves graded, and whose gradings are compatible with that of the Virasoro algebra. We will make this statement more precise in what follows.

**Remark:**

The Virasoro generators $L_m$ can be found within the field content of a (1+1)D conformal field theory as the modes in the Laurent expansion of the energy-momentum tensor $T$:

$$T(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}. \tag{1.7}$$

All algebraic relations between modes have a corresponding presentation in the form of an *operator product expansion* (see Appendix A for details on this and other field theory concepts). For instance, the commutation relations of the algebra may equivalently be written as

$$T(z)T(w) \sim \frac{\frac{1}{2}C}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{1.8}$$

We will typically work at the level of individual modes and algebraic relations through our analysis of representations of these algebras, though will frequently pause to touch on corresponding statements in the field theory. //

It will be convenient for what follows to split $\mathfrak{Vir}$ into subalgebras by index. We write

$$\begin{aligned} \mathfrak{Vir}_\pm &= \bigoplus_{n>0} \mathbb{C}L_{\pm n} \\ \mathfrak{Vir}_0 &= \mathbb{C}L_0 \oplus \mathbb{C}C \end{aligned} \tag{1.9}$$

and call $\mathfrak{Vir}_+$ the subalgebra of *positive* (resp. $\mathfrak{Vir}_-$ the *negative*) modes.

In addition to the Virasoro algebra itself, we heavily employ its *universal enveloping algebra*. Roughly speaking, this is the linear span of all formal monomials of algebra elements, under the quotient by the ideal generated by the Lie relations — i.e., all those of the kind $(ab - ba) - [a, b]$.

**1.1.2 Definition.** . The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined by

$$\mathcal{U}(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}/I \tag{1.10}$$

where $I$ is the ideal

$$I = \langle a \otimes b - b \otimes a - [a, b] \mid a, b \in \mathfrak{g} \rangle, \tag{1.11}$$

and the multiplication on $\mathcal{U}(\mathfrak{g})$ is the obvious one. The bracket of $\mathfrak{Vir}$ may be easily extended to all of $\mathcal{U}(\mathfrak{Vir})$ by repeated application of the identifications in $I$. We will typically suppress the tensor product notation when writing elements of $\mathcal{U}(\mathfrak{Vir})$.

$\mathcal{U}(\mathfrak{Vir})$ inherits the grading of $\mathfrak{Vir}$, in addition to the standard grading by tensor degree. This is done by defining the grade of a monomial element $L_{n_1} \cdots L_{n_k}$ to be the sum of the grades of its composite modes; $n_1 + \cdots + n_k$. Then we have the following decomposition of the enveloping algebra into graded subspaces $\mathcal{U}(\mathfrak{Vir})_n$:

$$\mathcal{U}(\mathfrak{Vir})_n = \bigoplus_{\sum n_i = n, m \in \mathbb{N}} \mathbb{C}C^m L_{n_1} \cdots L_{n_k}. \tag{1.12}$$

Note the presence of arbitrary powers of the central element $C$, a grade 0 object which therefore contributes nothing to the grade of a monomial (along with $L_0$).

Later, when we identify $C$ and $L_0$ with their vacuum eigenvalues $c$ and $h$, these arbitrary powers of degree 0 elements will drop away to be absorbed into the factor of $\mathbb{C}$. Thanks to the ability to reorder products by applying the bracket relations, we may define a standard ordering which describes a basis for $\mathcal{U}(\mathfrak{Vir})$. That such a choice is well-defined follows from the famous Poincaré-Birchoff-Witt theorem, and the resultant basis is colloquially known as a PBW basis. All such orderings are essentially equivalent, but our particular choice of ordering will make clear some important features of the enveloping algebra and will make future work in this regard much easier. We choose a basis of monomials of the form

$$L_{n_1} \cdots L_{n_k} \tag{1.13}$$

with weakly increasing index $n_1 \leq \cdots \leq n_k$. This shows that we have yet another decomposition

$$\mathcal{U}(\mathfrak{Vir}) \cong \mathcal{U}(\mathfrak{Vir}_-) \otimes \mathcal{U}(\mathfrak{Vir}_0) \otimes \mathcal{U}(\mathfrak{Vir}_+) \tag{1.14}$$

– note here that $\mathfrak{Vir}$ is first split into its positive, negative and zero mode subalgebras before taking the product of their three (different) universal enveloping algebras. Strictly speaking, without further quotients, there is no notion of multiplication between elements of this product, so the equivalence above should either be considered strictly one of vector spaces, or as one of an abuse of notation.

We will also take the opportunity to define here the anti-linear anti-involution $\dagger$ on $\mathfrak{Vir}$, whereby

$$L_n^\dagger = L_{-n}, \qquad C^\dagger = C. \tag{1.15}$$

This operation extends naturally to all of $\mathcal{U}(\mathfrak{Vir})$.

### 1.1.2 Virasoro Modules

**1.1.3 Definition.** A $\mathbb{Z}$-*graded* vector space is a vector space $V$ for which there exists a decomposition of $V$ into a direct sum of disjoint subspaces $V_i$:

$$V = \oplus_{i\in\mathbb{Z}} V_n, \tag{1.16}$$

and $V_i \cap V_j = \emptyset$ whenever $i \neq j$. If there is an inner product $\langle\cdot,\cdot\rangle$ on $V$, we also require $\langle V_i, V_j\rangle = 0$ for distinct graded subspaces.

If such a vector space carries a representation of the Virasoro algebra, we say that the two gradings are *compatible* if the Virasoro generators are *graded operators*, that is if

$$\mathfrak{Vir}_m \circ V_n \subseteq V_{m+n}. \tag{1.17}$$

In particular, we restrict our study to those graded representations which are also *highest* (*lowest*, depending on convention) *weight spaces*. One of the most important feature of such spaces is that the positive modes of $\mathfrak{Vir}$ (i.e. those $L_n$ with $n > 0$) are locally nilpotent, and for every $v \in V$ there is an upper bound on those $n$ for which $L_n v \neq 0$. This provides the proper framework under which to describe physical systems, wherein states are typically excited from a minimum-energy "vacuum" state by the action of the algebra.

We also require that such a representation carry a $\mathbb{Z}$-grading compatible with the action of the algebra (as one might also reasonably expect in a description of physical systems; indeed, $L_0$ forms in part[1] the energy operator for the system). These features alone heavily constrain the viable structures, but perhaps the most important requirement yet is that there exist a lower bound to the set of indices of non-empty graded subspaces $V_n$; that is:

$$\min\{n \in \mathbb{Z} | V_n \neq \emptyset\} = n_0 \in \mathbb{Z}. \tag{1.18}$$

and further that the subspace $V_{n_0}$ corresponding to this lower bound is a (complex) line ($\dim(V_{n_0}) = 1$). This corresponds to the physical notion of the single vacuum state of minimal energy.

In all, there are many ways in which highest weight spaces of the Virasoro algebra can be realised mathematically (in the sense of providing an explicit construction), each with more or less rigour and relevance depending upon the context in which they are employed. The definition used here is not the most sophisticated, but it has the benefit of being direct, and of being sufficient for our purposes.

**1.1.4 Definition.** A *highest weight space* $V$ (of the Virasoro algebra) is a vector space representation generated from a *highest weight vector* $v_0 \in V$ satisfying

$$L_n v_0 = 0 \qquad \forall n > 0 \tag{1.19}$$

and

$$L_0 v_0 = h v_0 \qquad C v_0 = c v_0 \tag{1.20}$$

for some constants $h, c \in \mathbb{C}$. The pair $(h, c)$ is called the *highest weight*.

To say that $V$ is generated from $v_0$ is to say that $V \cong \mathcal{U}(\mathfrak{Vir})\, v_0 \cong \mathcal{U}(\mathfrak{Vir}_-)\, v_0$, or that every $v \in V$ can be written as a linear combination of elements of the form

$$L_{-n_1} \cdots L_{-n_k} v_0 \tag{1.21}$$

for some weakly decreasing positive integers $n_1 \geq n_2 \geq \cdots \geq n_k > 0$. This property ensures that $V$ is a $\mathbb{Z}$-graded $\mathfrak{Vir}$ module and that each graded subspace $V_i$ is finite-dimensional. Indeed, since the $L_n$ are graded operators, the dimension

[1] It would form the energy operator itself but for the issue of chirality, a concept not treated here and not particularly relevant for what follows. In short, the underling space $\mathbb{C} \setminus \{0\}$, on which $\mathfrak{Vir}$ acts as infinitesimal generators of conformal transformations, admits solutions in both the complex variable $z$ and its conjugate $\overline{z}$ – giving two independent copies of the algebra, corresponding physically to states of left- and right-handed chirality. The "true" energy operator is $L_0 + \overline{L}_0$, the sum of zero modes of both chiralities. Since these two algebras are commuting, it is relatively "safe" to conceptualise the energy operator as $L_0$ alone.

of $V_i$ is bounded above by the partition number of $i$ — the number of ways to create linearly independent monomial objects of the type in (1.21) at grade $i$.[2]

An important example of a highest weight space is the formal one in which all monomially-generated objects of this type are linearly independent by construction, and the dimension of each graded subspace attains its upper bound. Such a module is known as a *Verma module.*

**1.1.5 Definition.** The Virasoro *Verma module* $\mathcal{V}_{h,c}$ of highest weight $(h, c)$ is induced from the trivial one-dimensional representation $\mathbb{C}v_{h,c}$ of $\mathfrak{Vir}_+ \oplus \mathfrak{Vir}_0$, where

$$L_n v_{h,c} = 0 \qquad \forall n > 0 \tag{1.22}$$

and

$$L_0 v_{h,c} = h v_{h,c} \qquad C v_{h,c} = c v_{h,c}. \tag{1.23}$$

Then $\mathcal{V}_{h,c}$ is defined by

$$\mathcal{V}_{h,c} := \mathcal{U}\left(\mathfrak{Vir}\right) \otimes_{\mathcal{U}(\mathfrak{Vir}_0 \oplus \mathfrak{Vir}_+)} \mathbb{C}v_{h,c} \tag{1.24}$$

The vanishing of the highest weight vector under the action of positive modes allows us to also write

$$\mathcal{V}_{h,c} \cong \mathcal{U}\left(\mathfrak{Vir}_- \oplus \mathfrak{Vir}_0\right) / \left\langle (L_0 - h), (C - c), L_n\ (n > 0)\right\rangle \tag{1.25}$$

as $\mathcal{U}\left(\mathfrak{Vir}\right)$-module. From this we see the reason behind choosing the PBW basis ordering of weakly increasing index.

A Verma module is clearly uniquely determined by its highest weight. Furthermore, any other highest weight module of the same weight is necessarily isomorphic to a quotient of this corresponding Verma module [28], simply because a Verma module has the "maximum possible" dimension for each graded subspace. Any study of the highest weight modules of the Virasoro algebra is therefore a study of its Verma modules. In particular, we are interested in conditions on reducibility and/or decomposability. Fortunately, the particular construction of $\mathcal{V}_{h,c}$ places us in a good position for determining its substructure. For instance, all Verma modules are indecomposable, for if we had $\mathcal{V}_{h,c} = X \oplus Y$ then either $X$ or $Y$ would contain $v_{h,c}$ and would therefore coincide with $\mathcal{V}_{h,c}$ itself by construction. At least for Verma modules of the Virasoro algebra, proper submodules of $\mathcal{V}_{h,c}$ are necessarily (direct sums of) highest weight modules, and the direct sum $\mathcal{J}_{h,c}$ of all proper submodules is also a graded submodule (and a proper submodule at that; it does not contain $v_{h,c}$, for instance) which is maximal with respect to inclusions. The quotient

$$\mathcal{M}_{h,c} = \mathcal{V}_{h,c}/\mathcal{J}_{h,c} \tag{1.26}$$

[2]Repeated application of the commutator shows that re-orderings of these monomials agree up to a linear combination of shorter ones at the same grade, so by induction don't contribute any further linearly independent basis elements.

is therefore both non-trivial and irreducible; what is more, by virtue of the above discussion on the "universality" of $\mathcal{V}_{h,c}$, is the *unique* non-trivial irreducible highest weight module of weight $(h, c)$.

Further structural analysis of Verma modules of the Virasoro algebra typically proceeds via the use of bilinear forms, as proper highest weight vectors (those not equal to the generating vector $v_{h,c}$ but which generate proper highest weight submodules) introduce degeneracies in the determinants of these forms. It can be shown [28] that there is a unique contravariant Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{V}_{h,c}}$ on each $\mathcal{V}_{h,c}$, commonly known as the *Shapovalov form*, such that

$$\langle v_{h,c}, v_{h,c} \rangle_{\mathcal{V}_{h,c}} = 1 \tag{1.27}$$

defined by

$$\langle U_1 v_{h,c}, U_2 v_{h,c} \rangle_{\mathcal{V}_{h,c}} := v_{h,c}^{\dagger} \left( U_1^{\dagger} U_2 v_{h,c} \right) \qquad \forall U_1, U_2 \in \mathcal{U}\left(\mathfrak{Vir}\right). \tag{1.28}$$

where $v_{h,c}^{\dagger} : \mathcal{V}_{h,c} \to \mathbb{C}$ is the linear functional satisfying

$$v_{h,c}^{\dagger}(v) = \begin{cases} 1 & v = v_{h,c} \\ 0 & v \neq v_{h,c} \end{cases} \tag{1.29}$$

This evaluation effectively takes only the piece of the product $U_1^{\dagger} U_2$ at grade 0, and even then picks up only the constant terms involving $h$ or $c$. This amounts to taking only those grade-0 terms which do not annihilate the highest weight vector — because of our ordering convention for basis monomials, in order to realise $U_1^{\dagger} U_2 v_{h,c}$ as an element of $\mathcal{V}_{h,c}$, we must commute generators through each other until they have weakly increasing indexes. The contravariance and Hermiticity of this form are manifest.

**1.1.6 Proposition.** *Graded subspaces* $(\mathcal{V}_{h,c})_m$ *and* $(\mathcal{V}_{h,c})_n$ *are orthogonal if* $m \neq n$.

*Proof.* Immediate, since for any $v_m = U_m v_{h,c} \in (\mathcal{V}_{h,c})_m$ and $v_n = U_n v_{h,c} \in (\mathcal{V}_{h,c})_n$ for some $U_m \in \mathcal{U}\left(\mathfrak{Vir}\right)_m$ and $U_n \in \mathcal{U}\left(\mathfrak{Vir}\right)_n$, we have $U_m^{\dagger} U_n \in \mathcal{U}\left(\mathfrak{Vir}\right)_{n-m}$. If $m \neq n$ then clearly $U_m^{\dagger} U_n v_{h,c}$ has no component at the 0th grade, and so $\langle v_m, v_n \rangle_{\mathcal{V}_{h,c}} = 0$. □

**1.1.7 Proposition.** *We have*

$$\operatorname{Ker} \langle \cdot, \cdot \rangle_{\mathcal{V}_{h,c}} \cong \mathcal{J}_{h,c}. \tag{1.30}$$

*Proof.* Firstly by contravariance the kernel is linear and closed under the $\mathcal{U}\left(\mathfrak{Vir}\right)$ action, and $v_{h,c} \notin \operatorname{Ker} \langle \cdot, \cdot \rangle_{\mathcal{V}_{h,c}}$, so forms a proper submodule and is therefore contained in the maximal such. For the other direction, suppose there existed some $v \in \mathcal{J}_{h,c}$ with $\langle v, w \rangle_{\mathcal{V}_{h,c}} \neq 0$ for at least one $w \in \mathcal{V}_{h,c}$. But then since $w$ can be written as $w = U v_{h,c}$ for some $U \in \mathcal{U}\left(\mathfrak{Vir}\right)$, by contravariance of the Shapovalov form this implies that $U^{\dagger} v$ is nonzero and proportional to $v_{h,c}$, and hence $v_{h,c} \in \mathcal{J}_{h,c}$, a contradiction. □

**1.1.7.1 Corollary.** $\langle\cdot,\cdot\rangle_{\mathcal{V}_{h,c}}$, *when restricted to the irreducible quotient* $\mathcal{M}_{h,c}$, *is non-degenerate.*

The Shapovalov form allows us to determine the reducibility of a Verma module. Its determinant $\det_{h,c}(n)$ at each graded subspace $(\mathcal{V}_{h,c})_n$ will vanish whenever there exists a vector in $\mathcal{J}_{h,c}$ at that grade. There is a famous result which gives a formula for this determinant.

| $r\backslash s$ | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | $\frac{1}{3}$ | 1 | 2 | $\frac{10}{3}$ | $\cdots$ |
| 2 | $\frac{5}{8}$ | $\frac{1}{8}$ | $\frac{-1}{24}$ | $\frac{1}{8}$ | $\frac{5}{8}$ | $\frac{35}{24}$ | |
| 3 | 2 | 1 | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | |
| 4 | $\frac{33}{8}$ | $\frac{21}{8}$ | $\frac{35}{24}$ | $\frac{5}{8}$ | $\frac{1}{8}$ | $\frac{-1}{24}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | | | | | $\vdots$ | |

Figure 1.1: The first few entries of the extended $(2,3)$ Kac table. The so-called "minimal models" of the non-extended table occupy the top-left corner. In this particular example there are two such entries, and both have $h=0$. Different shadings correspond to different submodule structures of the corresponding Verma modules: unshaded to indicate a braid-type structure; light shading to indicate a chain-type; and darker shading to indicate point-type (c.f. Figure 1.2 for a depiction of these structures).

**1.1.8 Theorem** (Kac determinant formula, [27], [11])**.** *We have*

$$\det_{h,c}(n) \propto \prod_{0\le r,s\le n} (h-h_{r,s})^{\pi(n-rs)} \tag{1.31}$$

*where* $\pi(k)$ *denotes the partition number of* $k$, *and* $h_{r,s}$ *is a quantity which also depends on* $c$*:*

$$h_{r,s}(c) = \frac{1}{48}\Big((13-c)(r^2+s^2)+\sqrt{(1-c)(25-c)}(r^2-s^2)-24rs-2(1-c)\Big). \tag{1.32}$$

For $h$ to take any one of these values $h_{r,s}(c)$ is a necessary and sufficient condition for the reducibility of the corresponding $\mathcal{V}_{h,c}$. There are several alternate parametrisations of these quantities. One such is when both $h$ and $c$ are cast in terms of a third parameter $t$;

$$\begin{aligned} c(t) &= 13-6(t+t^{-1}) \\ h_{r,s}(t) &= \frac{1}{4}\left((r^2-1)t-2(rs-1)+(s^2-1)t^{-1}\right), \end{aligned} \tag{1.33}$$

where for $t \in \mathbb{R} \setminus \{0\}$ we get $c \leq 1$ or $25 \leq c$, $1 \leq c \leq 25$ when $t$ is taken to be on $S^1 \subset \mathbb{C}$, and $c \in \mathbb{C}$ for generic $t \in \mathbb{C} \setminus \{0\}$. Many instances of physical interest correspond to $t \in \mathbb{Q} \setminus \{0\}$ and $c \leq 1$, so on many occasions we make use of a different but related parametrisation in terms of coprime positive integers $p$ and $q$, where

$$c_{p,q} := 1 - 6\frac{(q-p)^2}{pq} \tag{1.34}$$

(without loss of generality we take $p \leq q$, by convention), and

$$h^{p,q}_{r,s} := \frac{(rq-sp)^2 - (q-p)^2}{4pq} \tag{1.35}$$

but usually drop the superscript and just write $h_{r,s}$ when context makes the choice of $p, q$ clear. Given such a choice, it is then convenient to tabulate the values of $h_{r,s}$. Note the periodicity of (1.35) — typically, we restrict to $0 < r < p$ and $0 < s < q$. Such an array is known as the $p, q$ *Kac table*, or the *extended* $p, q$ Kac table if all positive $r, s$ are permitted. See Figure 1.1 for an example extended table.

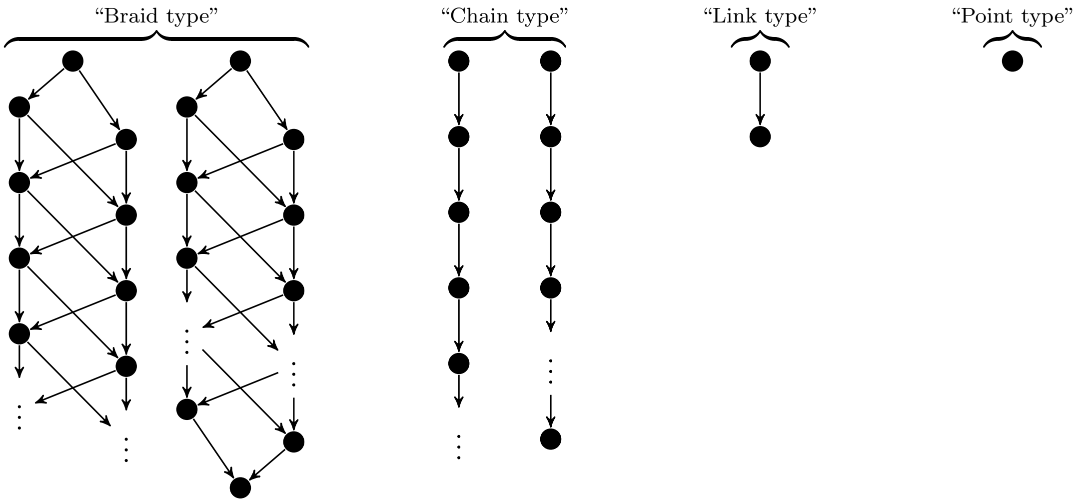


Figure 1.2: The six different structures seen in Verma modules. Vertices indicate singular vectors (as well as the vacuum vector at the top of the diagram) and arrows indicate the action of the Virasoro algebra; a vector at the head of an arrow lies inside the submodule generated by the one at its tail.

All this is in preparation of discussing the substructure of Verma, other highest weight, and more generally $\mathbb{Z}$-graded Virasoro modules. For Verma modules, the submodule structure is commonly depicted in what is known as an *embedding diagram*, showing how the generating vector $v_{h,c}$, sometimes called the *vacuum* vector, stands in relation to the *singular vectors* of that module.

**1.1.9 Definition.** A *singular vector* $v \in \mathcal{V}_{h,c}$ is an $L_0$ eigenvector such that

$$L_n v = 0 \qquad \forall n > 0 \tag{1.36}$$

and with $v \neq v_{h,c}$. Note we may immediately conclude from this definition that the Shapovalov vanishes on such vectors; they must belong to the maximal proper submodule.

In short, a singular vector is a highest weight vector which is also a proper *descendant* of the vacuum vector — i.e., it is attained from the vacuum[3] by the action of a nontrivial element of $\mathcal{U}(\mathfrak{Vir})$. A singular vector itself generates a highest weight submodule of its parent module. This submodule may contain its own singular vectors, which will generate further submodules. Every nontrivial proper submodule must be generated by at least one singular vector in this way. This is easily seen by noting that, since a proper submodule cannot contain the vacuum vector, there must be a nonzero lower bound on the grades of vector appearing in it. A *submodule diagram* is a convenient way to display this information: singular vectors are denoted by vertices, with directed edges between two vertices signifying that the submodule generated by the singular vector at the tail contains the one generated by the singular vector at the head. It is a well-known result that the subspace of singular vectors in a Verma module, intersected with any one graded subspace, is either $\emptyset$ or a line: up to scaling, there is at most one singular vector at any one grade [2].

Much work in the literature has gone into classifying the various types of submodule diagram found in the representation theory of the Virasoro algebra, e.g. [25]. There are six distinct types, all corresponding to various relative values of the parametrising variables. These are depicted in Figure 1.2.

Of particular interest in what follows will be the non-terminating braid and chain types. They correspond to rational $c \leq 1$. In fact, we will be concerned not with Verma modules themselves, but instead with what are known as *Fock spaces* (see Section 1.2) considered as Virasoro modules. Such spaces do contain highest weight vectors, but are not strictly speaking highest weight modules, as they are not entirely generated by their vacuum vectors. Their submodule structure is very similar — and in fact closely related — to that of Verma modules, but involves further subcategories of singular vector, allowing for arrows to point either "up" or "down" in the grading.

### 1.1.3 Virasoro Superalgebras

One may introduce a number $N$ of fermions into the fields of a conformal theory. Equivalently, one may extend $\mathfrak{Vir}$ by an infinite number of fermionic generators

[3] In the case of Verma modules, at least: in other Virasoro modules, such as Fock spaces (see Section 1.2 of this chapter), it is not true that all vectors are attainable from the vacuum in this way. This does not contradict our definition of singular vectors, which only requires them to be *unequal* to the vacuum vector, but the reader should be cognisant of the somewhat colloquial use of the term "singular vector".

grouped into $N$ many "families". We treat the simplest examples here, $N = 1$ and $N = 2$.

| $r\backslash s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $0$ | $\frac{3}{80}$ | $\frac{1}{10}$ | $\frac{7}{16}$ | $\frac{4}{5}$ | $\frac{23}{16}$ | $\frac{21}{10}$ | $\frac{243}{80}$ | $4$ | $\frac{419}{80}$ | $\frac{13}{2}$ | $\cdots$ |
| 2 | $\frac{7}{16}$ | $\frac{1}{10}$ | $\frac{3}{80}$ | $0$ | $\frac{19}{80}$ | $\frac{1}{2}$ | $\frac{83}{80}$ | $\frac{8}{5}$ | $\frac{39}{16}$ | $\frac{33}{10}$ | $\frac{71}{16}$ | $\cdots$ |
| 3 | $\frac{7}{6}$ | $\frac{169}{240}$ | $\frac{4}{15}$ | $\frac{5}{48}$ | $-\frac{1}{30}$ | $\frac{5}{48}$ | $\frac{4}{15}$ | $\frac{169}{240}$ | $\frac{7}{6}$ | $\frac{457}{240}$ | $\frac{8}{3}$ | $\cdots$ |
| 4 | $\frac{39}{16}$ | $\frac{8}{5}$ | $\frac{83}{80}$ | $\frac{1}{2}$ | $\frac{19}{80}$ | $0$ | $\frac{3}{80}$ | $\frac{1}{10}$ | $\frac{7}{16}$ | $\frac{4}{5}$ | $\frac{23}{16}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

(a) The $(p, q) = (3, 5)$ extended $N = 1$ Kac table.

| $r\backslash s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $0$ | $\frac{1}{16}$ | $\frac{1}{6}$ | $\frac{9}{16}$ | $1$ | $\frac{83}{48}$ | $\frac{5}{2}$ | $\frac{57}{16}$ | $\frac{14}{3}$ | $\frac{97}{16}$ | $\frac{15}{2}$ | $\frac{443}{48}$ | $11$ | $\cdots$ |
| 2 | $\frac{3}{8}$ | $\frac{1}{16}$ | $\frac{1}{24}$ | $\frac{1}{16}$ | $\frac{3}{8}$ | $\frac{35}{48}$ | $\frac{11}{8}$ | $\frac{33}{16}$ | $\frac{73}{24}$ | $\frac{65}{16}$ | $\frac{43}{8}$ | $\frac{323}{48}$ | $\frac{67}{8}$ | $\cdots$ |
| 3 | $1$ | $\frac{9}{16}$ | $\frac{1}{6}$ | $\frac{1}{16}$ | $0$ | $\frac{11}{48}$ | $\frac{1}{2}$ | $\frac{17}{16}$ | $\frac{5}{3}$ | $\frac{41}{16}$ | $\frac{7}{2}$ | $\frac{227}{48}$ | $6$ | $\cdots$ |
| 4 | $\frac{17}{8}$ | $\frac{21}{16}$ | $\frac{19}{24}$ | $\frac{5}{16}$ | $\frac{1}{8}$ | $-\frac{1}{48}$ | $\frac{1}{8}$ | $\frac{5}{16}$ | $\frac{19}{24}$ | $\frac{21}{16}$ | $\frac{17}{8}$ | $\frac{143}{48}$ | $\frac{33}{8}$ | $\cdots$ |
| 5 | $\frac{7}{2}$ | $\frac{41}{16}$ | $\frac{5}{3}$ | $\frac{17}{16}$ | $\frac{1}{2}$ | $\frac{11}{48}$ | $0$ | $\frac{1}{16}$ | $\frac{1}{6}$ | $\frac{9}{16}$ | $1$ | $\frac{83}{48}$ | $\frac{5}{2}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

(b) The $(p, q) = (4, 6)$ extended $N = 1$ Kac table.

Figure 1.3: The first few entries in the extended $N = 1$ Kac tables with $(p, q) = (3, 5)$ and $(4, 6)$. The entries in each table show $h_{r,s}$, with shading to show the structure of the corresponding reducible module. Unshaded entries correspond to bulk-type modules, light grey to edge-type, grey to corner type, and dark grey to centre type. These tables show both sectors (R and NS) combined as one; these disjoint cases fit together in a checkerboard pattern — recall from (1.45) that $r \equiv s \mod 2$ corresponds to the NS sector, $r \not\equiv s \mod 2$ to the R sector.

**1.1.10 Definition.** The infinite-dimensional Lie superalgebra $\mathfrak{s}_1\mathfrak{Vir}$, called the $N = 1$ Virasoro superalgebra, is the algebra with basis

$$\{L_n \mid n \in \mathbb{Z}\} \cup \{C\} \cup \{G_a \mid a \in \sigma + \mathbb{Z}\} \tag{1.37}$$

with either $\sigma = 0$ (the Ramond sector) or $\sigma = \frac{1}{2}$ (the Neveu-Schwarz sector),

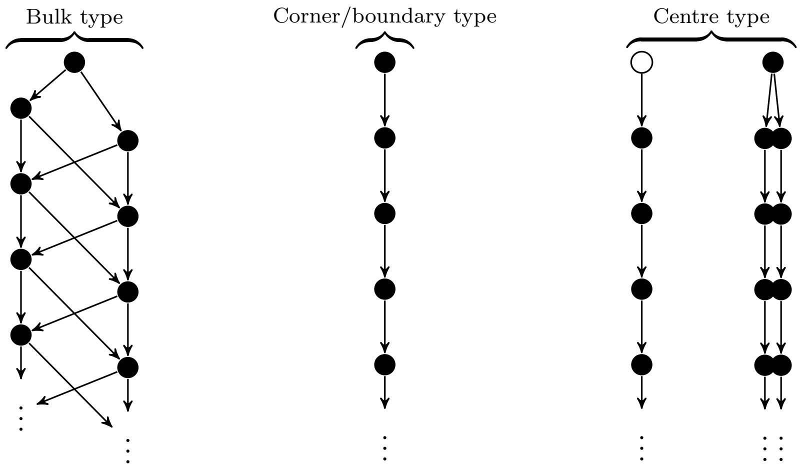


Figure 1.4: Submodule diagrams for Verma modules for the $N = 1$ Virasoro superalgebra. Each black vertex represents one singular vector of $\mathfrak{s}_1\mathfrak{Vir}$ in the Neveau-Schwarz sector, and two singular vectors in the Ramond sector. For corner type, a white vertex indicates a solitary singular vector, and doubled black vertices indicate singular vectors of multiplicity four. Like in Figure 1.2, arrows indicate the action of the Virasoro superalgebra.

subject to the relations

$$\begin{aligned} \{G_a, G_b\} &= 2L_{a+b} + \frac{1}{3}\left(a^2 - \frac{1}{4}\right) C\delta_{a,-b} \\ [L_m, G_a] &= \left(\frac{1}{2}m - a\right) G_{m+a} \\ [G_a, C] &= 0 \qquad \forall a, b, m \end{aligned} \tag{1.38}$$

in addition to those already defined for $\mathfrak{Vir}$ (the "$N = 0$ Virasoro superalgebra").

**1.1.11 Definition.** The infinite-dimensional Lie superalgebra $\mathfrak{s}_2\mathfrak{Vir}$, called the $N = 2$ Virasoro superalgebra, is the algebra with basis

$$\{L_n \mid n \in \mathbb{Z}\} \cup \{C\} \cup \{G_a^+ \mid a \in \sigma + \mathbb{Z}\} \cup \{G_b^- \mid b \in \sigma + \mathbb{Z}\} \cup \{J_m \mid m \in \mathbb{Z}\} \tag{1.39}$$

with either $\sigma = 0$ (the Ramond-Ramond sector) or $\sigma = \frac{1}{2}$ (the Neveu-Schwarz-Neveu-Schwarz sector)[4], subject to the relations

$$\begin{aligned} &[L_m, G_a^\pm] = \left(\tfrac{1}{2}m - n\right) G_{m+a}^\pm && [L_m, J_n] = -nJ_{m+n} \\ &\{G_a^+, G_b^-\} = \left(L_{a+b} + \tfrac{1}{2}(a-b)J_{a+b} + \tfrac{1}{6}(a^2 - \tfrac{1}{4})C\delta_{a,-b}\right) && \{G_a^\pm, G_b^\pm\} = 0 \\ &[G_m^\pm, J_n] = \mp G_{m+n}^\pm && [J_m, J_n] = \tfrac{1}{3}mC\delta_{m+n,0} \end{aligned} \tag{1.40}$$

and $C$ remains central, in addition to those relations already defined for $\mathfrak{Vir}$.

[4]We only consider $N = 2$ with matching fermion type.

**Remark:**

In terms of field theory, we have in the $N = 1$ case:

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \\ G(z) &= \sum_{n\in\sigma+\mathbb{Z}} G_n z^{-n-\frac{3}{2}} \end{aligned} \tag{1.41}$$

with additional relations

$$\begin{aligned} T(z)G(w) &\sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \\ G(z)G(w) &\sim \frac{\frac{2}{3}C}{(z-w)^3} + \frac{2T(w)}{z-w}, \end{aligned} \tag{1.42}$$

and in the $N = 2$ case:

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \\ G^{\pm}(z) &= \sum_{n\in\sigma+\mathbb{Z}} G^{\pm}_n z^{-n-\frac{3}{2}} \\ J(z) &= \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \end{aligned} \tag{1.43}$$

with additional relations

$$\begin{aligned} T(z)G^{\pm}(w) &\sim \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w} \\ G^{+}(z)G^{-}(w) &\sim \frac{\frac{1}{3}C}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w)+\frac{1}{2}\partial J(w)}{z-w} \\ T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} \\ G^{\pm}(z)J(w) &\sim \frac{\mp G^{\pm}(w)}{z-w} \\ J(z)J(w) &\sim \frac{\frac{1}{3}C}{(z-w)^2} \end{aligned} \tag{1.44}$$

One defines Verma-type modules of the Virasoro superalgebras in exactly the same way as for the $N = 0$ case. All of the terminology associated with graded representation spaces, such as the notion of highest weight vectors, transfers to the more general case entirely analogously. Some results no longer follow, however. These differences are chiefly seen in the Ramond sector, at least for modules of $\mathfrak{s}_1\mathfrak{Vir}$: the theory of Verma modules in the Neveu-Schwarz sector effectively follows through just as those of $\mathfrak{Vir}$ itself; all non-trivial submodules being generated by singular vectors, and the singular subspace at any one grade being at most one-dimensional.

By contrast, the Ramond sector for $N = 1$ differs on both counts. Nontrivial submodules which are not highest weight modules may exist, and there can be up to two linearly independent singular vectors of each parity at any one grade. This complicates the representation theory of $\mathfrak{s}_1\mathfrak{Vir}$ somewhat[5].

Singular vectors are defined in just the same way in these spaces as in the $N = 0$ case. As for the reducibility of these modules, there is a similar kind of parametrisation by positive integers and a corresponding organisation into Kac tables. The $N = 1$ analogues of the Kac determinant formula (see e.g. [29]) hold that in order for $\mathfrak{s}_1\mathfrak{Vir}$ Verma modules to be reducible we require positive integers[6] $p, q, r, s$ such that

$$h = h_{r,s} := \frac{1}{8pq}\left[(rq - sp)^2 - (q-p)^2\right] + \frac{1}{16}\delta_{\sigma,0}. \tag{1.45}$$

It is customary to demand $p \equiv q \mod 2$ and $\gcd\{p, \frac{1}{2}(q-p)\} = 1$. One can check that this implies $\gcd\{p, q\} \leq 2$. For the NS sector, we require that $r \equiv s \mod 2$ and for the R sector that $r \not\equiv s \mod 2$. For studying the so-called minimal models, belonging to the Kac table, one demands that $1 \leq r < p$ and $1 \leq s < q$ and that $p, q \geq 2$, but we make no such restriction here as we are instead interested in the entries of the *extended* Kac table, where $p$, $q$, $r$, and $s$ are allowed to be arbitrary positive integers (though still subject to the parity and coprimality constraints above). This can result in what might typically be considered "degenerate" cases where the standard Kac table is actually empty, but here we are interested in a broader context. Examples of extended Kac tables for the $N = 1$ case can be found in Figure 1.3.

There are four types of diagram for $N = 1$ spaces relevant to our purposes, each corresponding to the location of $r, s$ in the table relative to $p$ and $q$. We have:

- $p \mid r$ and $q \mid s$, called *corner* type;
- $p \mid r$ or $q \mid s$ (but not both), called *edge* type;
- $r \equiv \frac{1}{2}p \mod p$ and $s \equiv \frac{1}{2}q \mod q$, called *centre* type;
- all others, called *bulk* type.

[5] And these features only become more pronounced for $N > 1$. It would be fair to say that the representation theory of these higher-$N$ Virasoro superalgebras is only poorly understood

[6] We implicitly assume that $q \geq p$ in what follows, but one may easily verify that $p \leftrightarrow q$, while introducing factors of $(-1)$ in some equations, does not change the modules or representation theory in any way which would affect what follows (i.e., the number of factors is even whenever it would be relevant).

Note that, depending on $p, q$, some types may not appear in a particular extended table. Embedding diagrams for the $N = 1$ case, in the manner of Figure 1.2, can be found in Figure 1.4.

# 1.2 Fock Space as a Virasoro Module

"Fock Space" as a general term is used for the highest weight spaces generated by infinite-dimensional bosonic or femionic oscillator algebras (or combinations thereof). As-is, they have very little interesting structure and are always irreducible as modules of the oscillator algebras themselves. However, we are able to define a $\mathfrak{Vir}$-action on them, and in this sense they can have much more interesting structure.

## 1.2.1 Fock Space Basics

**1.2.1 Definition.** The *infinite-dimensional bosonic oscillator algebra*, $\mathfrak{a}$, is the Lie algebra with basis

$$\{a_n \mid n \in \mathbb{Z}\} \cup \{K\} \tag{1.46}$$

and relations

$$[a_m, a_n] = mK\delta_{m,-n} \tag{1.47}$$

with $K$ central.

**1.2.2 Definition.** The *infinite-dimensional fermionic oscillator algebra*, $\mathfrak{b}^\sigma$, is the Lie algebra with basis

$$\{b_n \mid n \in \sigma + \mathbb{Z}\} \cup \{\kappa\} \tag{1.48}$$

and relations

$$\{b_m, b_n\} = \kappa\delta_{m,-n} \tag{1.49}$$

with $\kappa$ central, and where either $\sigma = 0$ (the Ramond sector), or $\sigma = \frac{1}{2}$ (the Neveu-Schwarz sector).

// **Remark:**

The modes $a_n, b_n$ are the coefficients in the Laurent expansions of the free boson and free fermion fields;

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad b(z) = \sum_{n\in\sigma+\mathbb{Z}} b_n z^{-n-\frac{1}{2}}. \tag{1.50}$$

We will have occasion to use such objects when we discuss intertwining operators between Fock spaces. These fields have relations

$$\begin{aligned} a(z)a(w) &\sim \frac{1}{(z-w)^2} \\ b(z)b(w) &\sim \frac{1}{(z-w)}. \end{aligned} \tag{1.51}$$

Note that the symmetric and antisymmetric dependence on the variables $z, w$ in each case reflect respectively the bosonic and fermionic natures of the mode algebras $\mathfrak{a}$ and $\mathfrak{b}^\sigma$. //

These are $\mathbb{Z}$-graded algebras, just as $\mathfrak{Vir}$, and we therefore have corresponding notions of graded representations, highest weight spaces/vectors, etc. We will also keep a similar notion of anti-linear anti-involution † for $\mathfrak{a}$ and $\mathfrak{b}^\sigma$, whereby

$$K^\dagger = K, \qquad \kappa^\dagger = \kappa, \qquad x_n^\dagger = x_{-n},\, x_n \in \mathfrak{x} \tag{1.52}$$

for $\mathfrak{x} = \mathfrak{a}$ or $\mathfrak{b}^\sigma$. This of course extends in the obvious way to monomials of oscillator modes, and by linearity to all elements of $\mathcal{U}(\mathfrak{x})$.

Just as with Verma modules being the prototypical example of such a representation, we have a similar construction for $\mathfrak{a}$ and $\mathfrak{b}^\sigma$, called Fock space. These corresponding Fock spaces we will denote by $\mathcal{F}_\eta^{\mathfrak{a}}$ in the bosonic case and $\mathcal{F}_\updownarrow^{\mathfrak{b},\sigma}$ in the fermionic, where these are defined by:

**1.2.3 Definition.** *Fock space*, $\mathcal{F}^{\mathfrak{x}}$, for either $\mathfrak{x} = \mathfrak{a}$ or $\mathfrak{x} = \mathfrak{b}^\sigma$, is induced from the representation $V$ of $\mathfrak{x}_0 \oplus \mathfrak{x}_+$ generated from a vector $v$ in the following way:

$$x_n v = 0 \qquad x_n \in \mathfrak{x},\, n > 0 \tag{1.53}$$

and

$$a_0 v = \eta v, \qquad Kv = v \tag{1.54}$$

for $\eta \in \mathbb{C}$ if $\mathfrak{x} = \mathfrak{a}$, and

$$\kappa v = v \tag{1.55}$$

if $\mathfrak{x} = \mathfrak{b}^\sigma$[7]. If $\sigma = 0$, there is no $b_0$ eigenvalue for $v$ — the vectors $v$ and $b_0 v$ at grade 0 are linearly independent. Then we have

$$\mathcal{F}^{\mathfrak{x}} := \mathcal{U}(\mathfrak{x}) \otimes_{\mathcal{U}(\mathfrak{x}_0 \oplus \mathfrak{x}_+)} V. \tag{1.56}$$

The vacuum vector $1 \otimes v$ we typically write as $|\eta\rangle$ in the bosonic case, and either $|\uparrow\rangle^\sigma$ or $|\downarrow\rangle^\sigma$ in the fermionic (this arrow notation records the *parity* of the vacuum vector, a grading not present in the bosonic case, and one which we shall soon discuss). In most cases we will also record the choice of vacuum vector as a subscript, writing $\mathcal{F}_\eta^{\mathfrak{a}}$ and/or $\mathcal{F}_\updownarrow^{\mathfrak{b}}$ as necessary. The choice of sector for $\mathfrak{b}$ might also be unclear from context, and if this is the case, we shall include it as a second superscript (in the manner $\mathcal{F}_\updownarrow^{\mathfrak{b},\sigma}$) rather than clutter the notation by attempting to include it as a superscript to $\mathfrak{b}$. Sometimes we will wish to make statements about Fock spaces in general; so as to avoid tedious repetitions across near-identical cases, we will continue on occasion to use $\mathcal{F}^{\mathfrak{x}}$, without assuming a choice of either $\mathfrak{x} = \mathfrak{a}$ or $\mathfrak{b}^\sigma$, though noting any differences between these cases.

[7]There is no practical loss of generality in taking the vacuum eigenvalues of $K$ and $\kappa$ to be 1. If they were any other value (except 0), we could rescale the algebra to bring them back to unity. We do not consider the case where either is 0; one can easily see that in this case, the generator modes completely decouple from each other, becoming effectively commutative algebras. It is not possible to have the kind of representations of $\mathfrak{Vir}$ that we want on the resulting space.

There is no vacuum eigenvalue for $|\updownarrow\rangle^{\sigma}$, so it does not appear as a subscript to $\mathcal{F}$ as in the bosonic case. The choice of $\uparrow$ or $\downarrow$ as a label instead represents the *parity* of the vacuum vector, a choice nonexistent in the bosonic case. In addition to the $\sigma+\mathbb{Z}$-grading induced by that of $\mathfrak{b}^{\sigma}$ itself, there is a $\mathbb{Z}_2$-grading of $\mathcal{F}_{\updownarrow}^{\mathfrak{b},\sigma}$ which counts the *parity* of a vector; that is, whether the monomials of elements from $\mathfrak{b}^{\sigma}$ which comprise it are of even or of odd length. If even, the vector is defined to have the same parity as the vacuum, otherwise it has the opposite. For the purposes of examining the structure of a single $\mathcal{F}_{\updownarrow}^{\mathfrak{b},\sigma}$, this initial parity choice is entirely irrelevant, but it does become important when discussing structures involving more than one fermionic Fock space.

Note that while there are infinitely many choices for $\eta$ in the bosonic case, there are only two distinct choices for $\updownarrow$, and in fact only three distinct $\mathcal{F}_{\updownarrow}^{\mathfrak{b},\sigma}$ up to isomorphism if one takes into account the choices for $\sigma$. The choices $\uparrow$ and $\downarrow$ give distinct spaces if $\sigma=\frac{1}{2}$, but $\mathcal{F}_{\updownarrow}^{\mathfrak{b},0}$ can be thought of as containing both choices of vacuum vector simultaneously, since for example $b_0\,|\uparrow\rangle$ has the same parity and is at the same grade as $|\downarrow\rangle$.

As with $\mathcal{V}_{h,c}$, the dimension of the graded subspace of $\mathcal{F}_{\eta}^{\mathfrak{a}}$ at grade $n$ is equal to the number of weakly decreasing positive integer partitions of $n$. For the fermionic case, this dimension is equal to the number of strongly decreasing non-negative integer (for $\sigma=0$) or half-integer partitions (for $\sigma=\frac{1}{2}$) of $n$. Since all negatively-indexed modes of $\mathfrak{a}$ are mutually abelian (and those of $\mathfrak{b}$ are mutually anti-abelian), all choices of basis orderings for monomials are equivalent (up to a possible sign). However, for consistency's sake, we will take the convention of increasing index labels;

$$a_{-n_1}a_{-n_2}\cdots a_{-n_k}\,|\eta\rangle \tag{1.57}$$

and/or

$$b_{\sigma-n_1}b_{\sigma-n_2}\cdots b_{\sigma-n_k}\,|\updownarrow\rangle^{\sigma} \tag{1.58}$$

for positive integers $n_1\geq n_2\geq\cdots\geq n_k$.

As representation spaces of their respective oscillator algebras, both bosonic and fermionic Fock spaces are irreducible. Fock spaces have much more interesting structure when considered as $\mathfrak{Vir}$-modules.

We place a contravariant Hermitian form $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ on any one $\mathcal{F}^{\mathfrak{x}}$ by defining

$$\langle U_1\,|\cdot\rangle\,,U_2\,|\cdot\rangle\rangle_{\mathcal{F}}=\langle\cdot|\left(U_1^{\dagger}U_2\,|\cdot\rangle\right),\qquad\forall U_1,U_2\in\mathcal{U}\left(\mathfrak{x}\right) \tag{1.59}$$

just like the Shapovalov form of a Verma module (c.f. (1.28)), where we have written $|\cdot\rangle$ simply as a placeholder for either $|\eta\rangle$ or $|\updownarrow\rangle^{\sigma}$, and $\langle\cdot|$ is the $\mathbb{C}$-linear functional on $\mathcal{F}^{\mathfrak{x}}$ which evaluates to 1 on this vacuum vector and to 0 on every other element of the space. Colloquially we write this in the bra-ket notation

$$\langle\cdot|\,U_1^{\dagger}U_2\,|\cdot\rangle \tag{1.60}$$

rather than the more "bilinear form-like" $\langle U_1\,|\cdot\rangle\,,U_2\,|\cdot\rangle\rangle_{\mathcal{F}}$. Note that this form is positive-definite, and distinct monomials are orthogonal.

### 1.2.2 Bosonic Virasoro Action

We may define a $\mathfrak{Vir}$-action on $\mathcal{F}_\eta^{\mathfrak{a}}$ in the following way:

$$L_n := \frac{1}{2}\sum_{k\in\mathbb{Z}} :a_{n-k}a_k: -\lambda(n+1)a_n \tag{1.61}$$

where $:\cdot\cdot:$ denotes *normal ordering*;

$$:a_m a_n: := \begin{cases} a_m a_n & n \geq m \\ a_n a_m & n < m \end{cases} \tag{1.62}$$

and $\lambda \in \mathbb{C}$ is a constant. For future reference, we also define fermionic normal ordering:

$$:b_m b_n: := \begin{cases} b_m b_n & n \geq m \\ -b_n b_m & n < m \end{cases}, \tag{1.63}$$

which is chiefly the same but takes into account the fact that the generators of $\mathfrak{b}$ anti-commute.

Although these are infinite sums of products of elements of $\mathfrak{a}$, their evaluation on $\mathcal{F}_\eta^{\mathfrak{a}}$ is well-defined, because for each $v \in \mathcal{F}_\eta^{\mathfrak{a}}$, $a_k v = 0$ for all sufficiently large positive $k$, and the normal ordering of the operators in the sum takes care of the issue of placing positive-indexed modes to the right of negative ones. When $\mathcal{F}_\eta^{\mathfrak{a}}$ is considered as a $\mathfrak{Vir}$ module, we record the choice of $\lambda$ in an additional subscript, writing $\mathcal{F}_{\eta,\lambda}^{\mathfrak{a}}$. One also finds that

$$[L_m, a_n] = -na_{m+n} - n(n-1)\lambda\delta_{m,-n}. \tag{1.64}$$

These $L_m$ are a legitimate representation of $\mathfrak{Vir}$, but what is more, their grading respects the natural one defined on the Fock space by the $a_n$ themselves. That is, the gradings of $\mathfrak{Vir}$, $\mathfrak{a}$, and $\mathcal{F}_\eta^{\mathfrak{a}}$ are all mutually compatible. This means that both $L_0$ and $C$ are grade-0 operators, are diagonal, and we can find their eigenvalue data — the highest weight data, $(h, c)$ — just as for Verma modules. One finds the following correspondence:

$$h_{\eta,\lambda} = \frac{1}{2}\eta(\eta - 2\lambda) \qquad c_\lambda = 1 - 12\lambda^2. \tag{1.65}$$

Since these are quadratic in nature, there are generally two solutions for each, so nominally four modules with data $(\eta_{h,c}, \lambda_c)$ which have $\mathfrak{Vir}$-module data $(h_{\eta,\lambda}, c_\lambda)$. However, one may easily check that interchanging the two possible $\lambda$ interchanges the solutions for $\eta$, up to a factor of $(-1)$.

// **Remark:**

This construction corresponds to the following field realisation of $T(z)$ in terms of $a(z)$:

$$T(z) := \frac{1}{2} :a(z)a(z): +\lambda\partial a(z). \tag{1.66}$$

From the OPE

$$T(z)a(w) \sim \frac{-2\lambda}{(z-w)^3} + \frac{a(z)}{(z-w)^2} + \frac{\partial a(w)}{z-w} = \partial_w\left(\frac{a(w)}{z-w}\right) - \frac{2\lambda}{(z-w)^3} \tag{1.67}$$

we are also able to calculate the commutation relations $[L_m, a_n]$. //

### 1.2.3 (Ir)reducibility

Fock spaces are generically irreducible as $\mathfrak{Vir}$ modules. We can examine the circumstances under which we find nontrivial embedding structures by starting with the highest weight submodule generated exclusively from the Fock space vacuum, $\mathcal{U}(\mathfrak{Vir}) \cdot |\eta\rangle$. Due the universality and uniqueness properties of Verma modules, the space generated in this way is necessarily a quotient of the corresponding $\mathcal{V}_{h_{\eta,\lambda},c_\lambda}$. If the Verma module is irreducible, it coincides with its own irreducible quotient, in which the dimension of each graded subspace attains its maximum. This is the same as the number of integer partitions of each grade, and the same as the dimension of the graded subspaces of $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$. Thus $\mathcal{U}(\mathfrak{Vir}) \cdot |\eta\rangle$ fills all of $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$, and $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$ is irreducible.

However, it is not necessarily the case that all elements of $\mathcal{U}(\mathfrak{Vir}) \cdot |\eta\rangle$ must always be linearly independent. Note that if we extend $\dagger$ from $\mathcal{U}(\mathfrak{a})$ to infinite sums of bosonic oscillator modes, then provided $\lambda$ is real, then $L_n^\dagger = L_{-n}$, and $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ is contravariant Hermitian with respect to $\mathfrak{Vir}$, so by uniqueness coincides with $\langle\cdot,\cdot\rangle_{\mathcal{V}_{h,c}}$ on this submodule. Recall, however, that $\langle\cdot,\cdot\rangle_{\mathcal{V}_{h,c}}$ is degenerate when $\mathcal{V}_{h,c}$ has nontrivial submodule structure, but that $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ is positive-definite. In such an instance, then, there must exist elements $U \in \mathcal{U}(\mathfrak{Vir})$ such that

$$U^\dagger U\, |\eta\rangle = 0 \tag{1.68}$$

so $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$ is reducible as a $\mathfrak{Vir}$ module — either $U$ or one of its factors creates a singular vector, or there is an "unreachable" *subsingular* vector; $U$ or one of its factors evaluates to 0 on the vacuum vector.

**1.2.4 Definition.** Given a Fock space $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$, let $\mathcal{M}^{\mathfrak{a}}_{\eta,\lambda}$ denote the $\mathfrak{Vir}$ submodule generated by all singular vectors, that is:

$$\mathcal{M}^{\mathfrak{a}}_{\eta,\lambda} = \bigcup_{v \text{ singular}} \mathcal{U}(\mathfrak{Vir}) \cdot v \tag{1.69}$$

(note that the union is used in favour of the direct sum; this is because under certain circumstances, certain of the $\mathcal{U}(\mathfrak{Vir}) \cdot v$ are not disjoint — refer to Figure 1.5 for details).

Then a *subsingular vector* is defined to be a vector which, while not singular in $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$, becomes singular in the quotient $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}/\mathcal{M}^{\mathfrak{a}}_{\eta,\lambda}$.

Some Fock spaces contain what should rightly be called *sub-subsingular* vectors; those which require two iterations of the above procedure before they become singular. For our purposes, the distinction will be mostly irrelevant, so we will typically just refer to all vectors of this type as just "subsingular".

Distinguished vectors, singular and subsingular, appear in $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$ at the same grades as singular vectors in the corresponding $\mathcal{V}_{h,c}$. The submodule diagrams therefore appear similar, with the chief difference being that the directions of some arrows have reversed direction. For $\lambda$ real (and hence $c \leq 1$) the parametrisations (1.34) and (1.35) in terms of coprime integers $p, q$ give corresponding parametrisations for $\eta$ and $\lambda$:

$$\begin{aligned} \lambda_{p,q} &= \frac{1}{\sqrt{2pq}}\,(q-p) \\ \eta^{\pm}_{r,s} &= \frac{1}{\sqrt{2pq}}\,((q-p) \pm (rq - sp)) \end{aligned} \tag{1.70}$$

where again we have suppressed superscripts on $\eta$ recording the choice of $p, q$, as we typically work with the value of $c$ clear by context. In lieu of this, we opt to display the choice of solution to the quadratic in (1.65). It is also much more convenient and legible if we refer to $\mathcal{F}^{\mathfrak{a}}_{\eta^{\pm}_{r,s},\lambda_{p,q}}$ simply as $\mathcal{F}^{\pm}_{r,s}$.

We also have Kac tables for Fock spaces. The values of $h_{r,s}$ are the same as for the Kac tables of Verma modules, but now there exist two entries in each cell, one for each of $\eta^{\pm}_{r,s}$. Interestingly, this choice of $\pm$ for the vacuum eigenvalue is not irrelevant, since it is preserved by the intertwining maps we will construct below.

The diagram structure of a reducible Fock space as a Virasoro module depends on its position in its (extended) Kac table. There are three chief regions in each table, as follows:

- *Bulk*, where $r \nmid p$ and $s \nmid q$,
- *Edge*, where either $r \mid p$ xor $s \mid q$, and
- *Corner*, where both $r \mid p$ and $s \mid q$.

The particular submodule structures corresponding to each, and their dependence on the choice of $\eta^{\pm}_{r,s}$, are displayed in Figure 1.5.

### 1.2.4 Intertwining Maps

We introduce the notion of the *vertex*, or *screening*, operator, giving *screening currents* which act as maps from one Fock space to another. These maps may even be intertwiners of the corresponding $\mathfrak{Vir}$ representations — that is, they commute

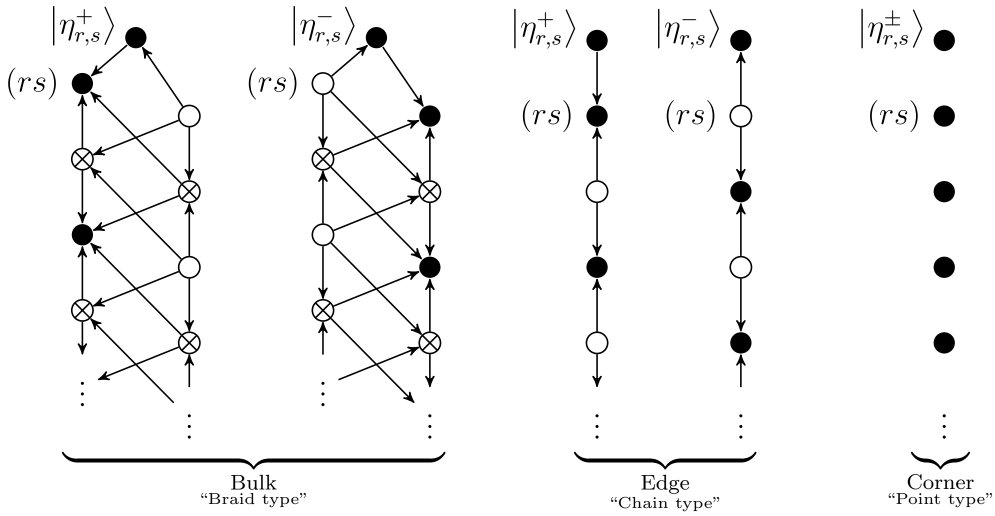


Figure 1.5: The various possible submodule structures of $\mathcal{F}^{\pm}_{r,s}$. Grading increases down the page, with arrows indicating nontrivial $\mathfrak{Vir}$ action. Filled nodes denote singular vectors, unfilled and partially filled denote sub- and sub-subsingular vectors respectively. Note how the choice of $\eta^{\pm}_{r,s}$ changes the submodule structure — in particular the type of the vector at grade $rs$, marked on the diagrams. It should be noted that in the bulk the first proper singular vector of $\mathcal{F}^{+}_{r,s}$ does not always occur at a higher grade than that of $\mathcal{F}^{-}_{r,s}$; this apparently being the case is simply due to our choice of diagram.

with the $\mathfrak{Vir}$-action — if the vertex operator's internal parameter meets certain conditions, based upon the spaces themselves.

Recall that the modes $a_n$ were the Laurent coefficients of the field

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}. \tag{1.71}$$

We now introduce a new operator, $\mathbf{q}$. This operator extends the algebra $\mathfrak{a}$ with the following relations:

$$[a_n, \mathbf{q}] = \delta_{n,0}, \qquad [\mathbf{q}, K] = 0. \tag{1.72}$$

The reader familiar with Hamiltonian mechanics may recognise this as a scaled version of the canonical commutation relation between a position operator ($\mathbf{q}$) and its conjugate momentum ($a_0$).

We treat $\mathbf{q}$ as an "integration constant", writing

$$\partial^{-1}a(z) := \mathbf{q} + a_0 \log(z) - \sum_{n\neq 0} \frac{1}{n} a_n z^{-n} \tag{1.73}$$

which allows us to define the *vertex operator of shift* $\mu$.

**1.2.5 Definition.** The *vertex operator of shift* $\mu$, $V_\mu(z)$, is defined to be

$$\begin{aligned} V_\mu(z) &:= \,:\exp\left(\mu\partial^{-1}a(z)\right): \\ &= e^{\mu q} z^{\mu\alpha_0} \exp\left(\mu \sum_{n>0} \frac{\alpha_{-n}}{n} z^n\right) \exp\left(-\mu \sum_{n>0} \frac{\alpha_n}{n} z^{-n}\right). \end{aligned} \tag{1.74}$$

And when acted upon a Fock space $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$ such that $\mu\eta \in \mathbb{Z}$, we may write[8]

$$V_\mu(z) = \sum_{n\in\mathbb{Z}} V_n z^{-n-h_\mu} \tag{1.75}$$

where $h_\mu = \frac{1}{2}\mu(\mu - 2\lambda)$ is the *conformal weight* of the vertex operator.

**Remark:**

From this definition we can calculate, for instance,

$$a(z)V_\mu(w) \sim \frac{\mu V_\mu(w)}{z-w} \tag{1.76}$$

and hence

$$\begin{aligned} T(z)V_\mu(w) &\sim \frac{\frac{1}{2}\mu\left(\mu - 2\lambda\right)V_\mu(w)}{(z-w)^2} + \frac{\mu a(w) V_\mu(w))}{z-w} \\ &= \frac{h_\mu V_\mu(w)}{(z-w)^2} + \frac{\partial V_\mu(w)}{z-w} \end{aligned} \tag{1.77}$$

Note that

$$a_0 e^{\mu\mathbf{q}} \left|\eta\right\rangle = \mu e^{\mu\mathbf{q}} \left|\eta\right\rangle + e^{\mu\mathbf{q}} a_0 \left|\eta\right\rangle = (\eta+\mu) e^{\mu\mathbf{q}} \left|\eta\right\rangle, \tag{1.78}$$

so we identify $e^{\mu\mathbf{q}}\left|\eta\right\rangle$ with $\left|\eta+\mu\right\rangle$, and this factor within the vertex operator achieves the mapping from one Fock space to another. The parameter $\mu$ is then the amount by which the vacuum eigenvalue is shifted. The other exponential terms in the definition of $V_\mu(z)$ have the effect of introducing factors consisting of combinations of bosonic oscillator modes, which produce a net creation or annihilation of states which commutes with the shift in vacuum eigenvalue.

In addition to satisfying the monodromy condition ($\mu\eta \in \mathbb{Z}$), if we also have $h_\mu = 1$, then we find

$$[L_m, V_n] = -nV_{m+n} \qquad \forall m,n \in \mathbb{Z}. \tag{1.79}$$

[8] This is because any non-integral power of $z$ appearing in $V_\mu(z)$ comes from the factor of $z^{\mu a_0}$; provided its evaluation $z^{\mu\eta}$ on $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$ has integral exponent, the series expansion is possible.

// **Remark:**

In field theory parlance, we say that in this case $V_\mu(z)$ is a (Viraosro) *primary field* of *weight one*. Since $h_\mu = 1$, we can in particular write

$$T(z)V_\mu(w) = \frac{V_\mu(w)}{(z-w)^2} + \frac{\partial V_\mu(w)}{z-w} = \partial_w\left(\frac{V_\mu(w)}{z-w}\right), \tag{1.80}$$

which is what allows us to calculate the commutation relation above. More details on the field theory of these algebras are to follow in later chapters, but for the time being our chief interest is with the commutation relations of the expansion coefficients as operators on Fock space. //

In particular, note that

$$[L_m, V_0] = 0 \qquad \forall m \in \mathbb{Z} \tag{1.81}$$

meaning, then, that:

**1.2.6 Proposition.** *If $V_\mu(z)$ is a vertex operator, and $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$ a Fock space such that*

$$\mu\eta \in \mathbb{Z} \tag{1.82}$$

*and*

$$h_\mu = \frac{1}{2}\mu(\mu - 2\lambda) = 1 \tag{1.83}$$

*then the operator $V_0 : \mathcal{F}^{\mathfrak{a}}_{\eta,\lambda} \to \mathcal{F}^{\mathfrak{a}}_{\eta+\mu,\lambda}$ defined by*

$$V_0 = \operatorname*{Res}_{z=0} V_\mu(z) \tag{1.84}$$

*is an intertwiner of Virasoro representations;* $[L_m, V_0] = 0$.

The intertwiner $V_0$ is a graded operator of weight 0 in the sense that it preserves the $L_0$ eigenvalue of any vector it acts upon. This is *not* to say that it preserves the grade of the element of $\mathcal{U}(\mathfrak{a})$ which creates it from $|\eta\rangle$. Instead it is the net creation or annihilation of bosonic modes together with a global shift in vacuum grading which combine to make $V_0$ a weight-0 operator.

Several vertex operators may be composed together to give more general intertwining maps, but the situation becomes significantly more complicated to treat. For the composition of $n$ many identical vertex operators, we have

$$\begin{aligned} V_\mu(z_1)\cdots V_\mu(z_n) = e^{n\mu q} \prod_{i<j}(z_i - z_j)^{\mu^2} \prod_i z_i^{\mu\alpha_0} \\ \times \exp\left(\mu \sum_{k>0} \frac{\alpha_{-k}}{k} p_k\right) \exp\left(-\mu \sum_{k>0} \frac{\alpha_k}{-k} p_{-k}\right) \end{aligned} \tag{1.85}$$

where $p_k = z_1^k + \cdots + z_n^k$ is the $k$th power sum. When the context is clear, we abbreviate (1.85) as $V_{n\times\mu}(z)$.

The "zero mode" $V_0$ is now harder to extract. In order to proceed, we follow the theory developed in [44] and outlined in [25] and make the projective change of variables $(z_1, \dots z_n) \mapsto (x, y_i, \dots y_n)$

$$x = z_1, \qquad y_i = \frac{z_i}{x}, \quad i = 2, 3, \dots, n \tag{1.86}$$

// **Remark:**

This change of variables is possible because the individual $z_i$ are never zero. They are not merely complex variables; they represent "insertion points" for the $n$ many distinct fields $V_\mu(z_i)$ on $\mathbb{C}^\times$, the punctured complex plane $(\mathbb{C}\backslash\{0\})$. From the physics side of the field theory, this space – equivalent to an infinite cylinder – corresponds to the world-sheet of a string. We also have $z_i \neq z_j$ whenever $i \neq j$ – otherwise the fields $V_\mu(z_i)$ would fail to be distinct. Hence we can conclude that the new variables $y_i$ satisfy $y_i \neq 0, 1, \infty, y_j$ for all $i = 1, 2, \dots n$ and $i \neq j$.

This rather geometric statement, together with the factors of $(z_i - z_j)$ in the expansion of $V_{n\times\mu}(z)$ (c.f. (1.85)) which become factors of $(1 - y_i)$ and $(y_i - y_j)$, reminiscent of the integration kernel of the generalised hypergeometric function, is suggestive of a particular well-known type of contour integral. Indeed, this approach is what allows us to find a nontrivial "residue" of the multi-variate $V_{n\times\mu}(z)$ and therefore to extract an intertwining operator from this composite of fields.

//

Under the change of variables, the monodromy condition becomes one only for the $x$ variable, because when extracting the "zero mode" we are able to split the integration contour into a loop around $x$ connected to a generalised Pochhammer contour around the $y_i$ [25].

Following this prescription, we obtain the requirements that

$$\lambda = \frac{1}{2}\mu - \frac{1}{\mu} \tag{1.87}$$

and

$$\eta = \lambda - \frac{1}{2}n\mu - \frac{1}{\mu}m \tag{1.88}$$

for some integer $m$. If these conditions are met, then there exists a contour $\Gamma$ such that

$$V_0 = \oint_\Gamma V_{n\times\mu}(z)\mathrm{d}z_1 \cdots \mathrm{d}z_n \tag{1.89}$$

is a *bona fide* non-trivial intertwiner of Virasoro representations $V_0 : \mathcal{F}_{\eta,\lambda} \to \mathcal{F}_{\eta+n\mu,\lambda}$.

The first of the above conditions is the familiar requirement that the vertex operators involved be Virasoro primaries of conformal weight 1; i.e. that

$h_\mu = 1$. The second, the monodromy condition, is perhaps more intuitive when re-expressed through an equivalent statement involving the vacuum conformal weights of the domain and image spaces:

$$\begin{aligned}\Delta h &= h_{\eta+n\mu} - h_\eta \\ &= n\left(\mu\eta + \frac{1}{2}(n-1)\mu^2 + 1\right).\end{aligned} \tag{1.90}$$

By inserting (1.88) into the above, we find it to be equivalent to the requirement that $\Delta h$ be an integral multiple of $n$. Using these restrictions, we see there may only exist at most a single pair of modules related by any one given triple $(n, \Delta h, \mu)$. Indeed, if $V_0 : \mathcal{F}_{\eta_1} \to \mathcal{F}_{\eta_2}$, is nontrivial, then

$$\eta_1 = \frac{1}{2}(1-n)\mu + \frac{\Delta h - n}{n\mu}, \qquad \eta_2 = \frac{1}{2}(1+n)\mu + \frac{\Delta h - n}{n\mu} \tag{1.91}$$

or, in other words,

$$\begin{aligned}h_{\eta_1} &= \frac{1}{2}\left[\frac{1}{4}(n^2-1)\mu^2 + (1-\Delta h) + \frac{\Delta h^2 - n^2}{n^2\mu^2}\right], \\ h_{\eta_2} &= \frac{1}{2}\left[\frac{1}{4}(n^2-1)\mu^2 + (1+\Delta h) + \frac{\Delta h^2 - n^2}{n^2\mu^2}\right].\end{aligned} \tag{1.92}$$

The central charge is also set by $c = 1 - 12(\frac{\mu}{2} - \frac{1}{\mu})^2$, uniquely determining the two modules. When the modules involved are reducible, within a fixed $(p, q)$ table, we find two possible solutions for $\mu$:

$$\mu^+ = \sqrt{\frac{2q}{p}}, \qquad \mu^- = -\sqrt{\frac{2p}{q}}. \tag{1.93}$$

Then

$$\eta^\pm_{r,s} + \mu^+ = \eta^\pm_{r\pm2,s} \qquad \eta^\pm_{r,s} + \mu^- = \eta^\pm_{r,s\pm2}, \tag{1.94}$$

so vertex operators preserve the choice $(\pm)$ of vacuum $\eta^\pm_{r,s}$, with $\mu^+$ changing the $r$ index and $\mu^-$ changing the $s$, both in steps of 2 with an appropriate sign. Note that the central charge $c$ is never changed.

### 1.2.5 Fock Superspaces

We create Fock space representations of the superspaces $\mathfrak{s}_1\mathfrak{Vir}$ and $\mathfrak{s}_2\mathfrak{Vir}$ by including the modes of $\mathfrak{b}$. The number of independent copies of $\mathfrak{b}$ required is equal to $N$, the fermion number of the superalgebra.

#### The $N = 1$ Fock Superspace

**1.2.7 Definition.** The $N = 1$ Fock superspace of vacuum eigenvalue $\eta$, vacuum parity $\updownarrow$, and sector choice $\sigma$, denoted $\mathcal{S}_1\mathcal{F}^\sigma_{\eta,\updownarrow}$, is defined to be

$$\mathcal{S}_1\mathcal{F}^\sigma_{\eta,\updownarrow} = \mathcal{F}^{\mathfrak{a}}_\eta \otimes_{\mathcal{U}(\mathfrak{b})} \mathcal{F}^{\mathfrak{b},\sigma}_{\updownarrow} \tag{1.95}$$

We typically do not notate vectors in this space nor the action of the algebras $\mathcal{U}(\mathfrak{a}), \mathcal{U}(\mathfrak{b}^\sigma)$ using full tensor product notation, opting for something much more compact. For instance, we write its vacuum vector as $|\eta, \updownarrow\rangle^\sigma$ instead of $|\eta\rangle \otimes |\updownarrow\rangle^\sigma$ and just denote the action of such operators as $a_n \otimes 1_{\mathcal{U}(\mathfrak{b}^\sigma)}$ and $1_{\mathcal{U}(\mathfrak{a})} \otimes b_n$ with simple juxtaposition, so the typical basis element of $\mathcal{S}_1\mathcal{F}^\sigma_{\eta,\updownarrow}$ appears as

$$a_{-n_1} \cdots a_{-n_i} b_{-m_1+\sigma} \cdots b_{-m_j} |\eta, \updownarrow\rangle^\sigma, \tag{1.96}$$

for some $n_1, \dots, n_i \in \mathbb{Z}$ and $m_1, \dots, m_j \in \sigma + \mathbb{Z}$.

This space is simple as a module of the combined oscillator algebras. It can however be made to carry an action of $\mathfrak{s}_1\mathfrak{Vir}$, the $N = 1$ Virasoro superalgebra, with a similar construction to that seen in the $N = 0$ case. We define

$$\begin{aligned} L_n &:= \frac{1}{2}\sum_{k\in\mathbb{Z}} :a_k a_{n-k}: -(n+1)\lambda a_n - \frac{1}{2}\sum_{k\in\sigma+\mathbb{Z}} (k+\frac{1}{2}) :b_{n-k}b_k: \\ G_n &:= \sum_{k\in\mathbb{Z}} a_k b_{n-k} - 2\lambda(n+\frac{1}{2})b_n \end{aligned} \tag{1.97}$$

for some constant $\lambda \in \mathbb{C}$, and keep track of this choice with an additional subscript $\mathcal{S}_1\mathcal{F}_{\eta,\updownarrow,\lambda}$. We can then calculate

$$\begin{aligned} [L_m, a_n] &= -na_{m+n} - n(n-1)\lambda\delta_{m,-n} \\ [G_m, a_n] &= -nb_{m+n} \\ [L_m, b_n] &= -\left(\frac{1}{2}m + n\right) b_{m+n} \\ \{G_m, b_n\} &= a_{m+n} + 2\lambda\left(n - \frac{1}{2}\right)\delta_{m,-n} \end{aligned} \tag{1.98}$$

**Remark:**

We have, at the level of fields,

$$T(z) = \frac{1}{2} :a(z)^2: +\lambda\partial a(z) - \frac{1}{2} :b(z)\partial b(z): \tag{1.99}$$

and

$$G(z) = a(z)b(z) + 2\lambda\partial b(z) \tag{1.100}$$

giving

$$\begin{aligned} T(z)a(w) &\sim \partial_w\left(\frac{a(w)}{z-w}\right) - \frac{2\lambda}{(z-w)^3} \\ G(z)a(w) &\sim \partial_w\left(\frac{b(w)}{z-w}\right) \\ T(z)b(w) &\sim \frac{\frac{1}{2}b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} \\ G(z)b(w) &\sim \frac{-2\lambda}{(z-w)^2} + \frac{a(w)}{z-w}. \end{aligned} \tag{1.101}$$

//

The values of $(h, c)$ for this representation can then be calculated as

$$h = \frac{1}{2}\eta(\eta - 2\lambda) + \frac{1}{16}\delta_{\sigma,0}, \qquad c = \frac{3}{2} - 12\lambda^2 \tag{1.102}$$

We find again that due to the quadratic relationship between $h$ and $\eta$, there are in general two inequivalent reducible Fock modules of the super-Virasoro algebra at each $(r, s)$, which nevertheless have the same values of $h$ and $c$. These solutions correspond to taking

$$\eta = \eta^{\pm}_{r,s} = \frac{1}{2\sqrt{pq}}\left[(q-p) \pm (rq - sp)\right]. \tag{1.103}$$

Note that there is no sector dependence in this equation, holding for either $\sigma = 0$ or $\sigma = \frac{1}{2}$. The two sectors are, however, distinct in that the difference $r - s$ is odd in the former ($\sigma = 0$) case and even in the latter ($\sigma = \frac{1}{2}$). Generically, interchanging the solutions $\eta^+ \leftrightarrow \eta^-$ has the effect of reversing all the arrows in the submodule diagram. Again, these diagrams are schematic descriptions of the modules involved, showing (sub)singular vectors which generate submodules, as vertices, with arrows between them indicating when one such generating vector can be obtained nontrivially from another by action of the super-Virasoro algebra. Singular vectors are annihilated by all positive modes of the algebra, subsingular are *not* singular but become so in the quotient by all vectors "downstream" of themselves, following the arrows in the diagram. In the Ramond sector, the multiplicity of singular and subsingular vectors is doubled at each level, as any vector $v \in \mathcal{S}_1\mathcal{F}_\eta$ is (sub)singular if and only if $b_0 v$ is also (sub)singular. The parity-reversing mapping afforded by $b_0$ can (nearly always) be achieved at the level of super-Virasoro generators by $G_0$ – but only for the vacuum vector(s) $|\eta, \updownarrow\rangle^0$. Indeed, note

$$G_0 \, |\eta, \updownarrow\rangle^0 = (\eta - \lambda)\, b_0 \, |\eta, \updownarrow\rangle^0 . \tag{1.104}$$

Both $G_0$ and $b_0$ are typically locally idempotent on the vacuum, except for when $\eta = \lambda$, in which case $G_0$ is the zero operator. This occurs exactly when $(rq - sp) = 0$ and provides the only type of Fock-type module of the $N = 1$ super-Virasoro algebra with significant structural differences with those of the $N = 0$ algebra.

Indeed, corner, edge and bulk-type modules in the $N = 1$ case have identical substructure diagrams to their $N = 0$ namesakes (c.f. Figure 1.5), except that the multiplicity of singular vectors at each vertex in the Ramond sector is two, not one. The centre type modules generically appear similar to the bulk, except for the special case when $\eta = \lambda$ and there exists no mapping between the two vacua $|\eta, \uparrow\rangle^0$ and $|\eta, \downarrow\rangle^0$ within the action of the algebra $\mathfrak{s}_1\mathfrak{Vir}$. This particular case is exhibited in Figure 1.6.

Now, if we wish to construct intertwining maps between the $\mathfrak{s}_1\mathfrak{Vir}$ representations on such spaces, we must turn again to the vertex operators seen in 1.2.4.

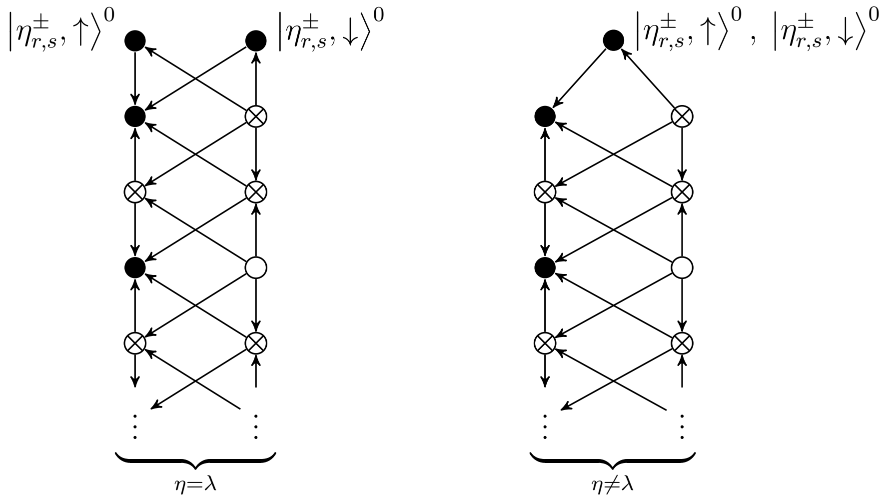


Figure 1.6: The structures of the $N = 1$ centre-type $\mathfrak{s}_1\mathfrak{Vir}$ modules in the Ramond sector. Typically (e.g. for $\eta \neq \lambda$) the same structure as those of the bulk type module, each vertex represents a set of (sub)singular vectors of multiplicity two, except for the uppermost pair in the left-hand diagram, which indicate just one singular vector each. We see how the collision of the parameters $\eta$ and $\lambda$, removing our ability to interchange between the two vacua of opposite parities using the action of $\mathfrak{s}_1\mathfrak{Vir}$, splits the spaces generated from these vectors into two disjoint submodules.

To be a legitimate intertwiner, any candidate map $\phi : \mathcal{S}_1\mathcal{F}_{\eta_1,\updownarrow_1} \to \mathcal{S}_1\mathcal{F}_{\eta_2,\updownarrow_2}$ must now commute not only with the $L_n$, but also with the $G_n$. We begin with the field defined in Definition 1.2.5;

$$\begin{aligned} V_\mu(z) &:= \, :\exp\left(\mu\partial^{-1}a(z)\right): \\ &= e^{\mu q} z^{\mu\alpha_0} \exp\left(\mu \sum_{n>0} \frac{\alpha_{-n}}{n} z^n\right) \exp\left(-\mu \sum_{n>0} \frac{\alpha_n}{n} z^{-n}\right). \end{aligned} \tag{1.105}$$

Since the "modifications" made to the $L_n$s in moving from the $N = 0$ to the $N = 1$ case is effectively the addition of series terms involving products of $b_n$s, which commute with the bosonic modes used in the construction of $V_\mu(z)$, we may infer as before that having $h_\mu = 1$ results in the relation $[L_m, V_n] = -nV_{m+n}$ remaining unchanged. However, $V_0$ no longer constitutes an intertwiner of Virasoro (super)algebra representations, as it does not commute with the $G_n$s. In fact, we can calculate that generally

$$[G_m, V_n] = \mu \sum_{k\in\mathbb{Z}} V_{n+k} b_{m-k}. \tag{1.106}$$

whenever $\mu$ is such that a series decomposition of $V_\mu(z)$ is possible.

It is clear that, in order to rectify this issue and produce genuine intertwining operator for $N = 1$ representations, it will be necessary to introduce fermionic components into $V_\mu(z)$ that take into account the behaviour of the $G_n$ modes.

While it is possible to proceed in an algebraic manner, it is much more expedient to attempt this construction at the level of fields, where the existence of intertwining operators is immediately apparent. It is sufficient to find an operator whose product expansion with the fields of the theory (in this case, $T(z)$ and $G(z)$) is a total derivative; the residue of this field about 0 is then an operator which commutes with the entire algebra.

Since one already has

$$T(z)V_\mu(z) \sim \frac{h_\mu V_\mu(w)}{(z-w)^2} + \frac{\partial V_\mu(w)}{z-w} \tag{1.107}$$

and

$$G(z)V_\mu(w) \sim \frac{\mu b(w)V_\mu(w)}{z-w}, \tag{1.108}$$

and one can also calculate

$$T(z)\left(b(w)V_\mu(w)\right) \sim \frac{\left(\frac{1}{2}+h_\mu\right)\left(b(w)V_\mu(w)\right)}{(z-w)^2} + \frac{\partial\left(b(w)V_\mu(w)\right)}{z-w} \tag{1.109}$$

and

$$G(z)\left(b(w)V_\mu(w)\right) \sim \frac{1}{\mu}\left[\frac{2h_\mu V_\mu(w)}{(z-w)^2} + \frac{\partial V_\mu(w)}{z-w}\right], \tag{1.110}$$

we see that there exists an even/odd couplet $(V_\mu^{(0)}(z), V_\mu^{(\frac{1}{2})}(z))$

$$\begin{aligned} V_\mu^{(0)}(z) &= \frac{1}{\sqrt{\mu}} V_\mu(z) \\ V_\mu^{(\frac{1}{2})}(z) &= \sqrt{\mu}\, b(z)V_\mu(z) \end{aligned} \tag{1.111}$$

which is preserved by $T(z)$ and interchanged by $G(z)$. If we tune $\mu$ such that $h_\mu = \frac{1}{2}$ (and happen to have $\eta$ such that series expansions exists for the fields in the couplet), then we find that

$$[L_m, V_n^{(\frac{1}{2})}] = -nV_{m+n} = [G_m, V_n^{(\frac{1}{2})}] \tag{1.112}$$

and note in particular that while only one fermion has been used in the construction of $V_\mu^{(\frac{1}{2})}(z)$, the field is overall bosonic (due to it having a net conformal weight of 1) so that it is appropriate to use the ordinary commutator bracket for $[G_m, V_n^{(\frac{1}{2})}]$ rather than the anticommutator. Since

$$V_\mu^{(\frac{1}{2})}(z) = e^{\mu q}\left(\sum_{m\in\sigma+\mathbb{Z}} b_m z^{-m-\frac{1}{2}}\right) z^{\mu a_0} \prod_{n>0} \exp\left(\frac{\mu}{n}a_{-n}z^n\right)\exp\left(-\frac{\mu}{n}a_n z^{-n}\right), \tag{1.113}$$

we see that a series expansion is possible provided that

$$\mu\eta - \frac{1}{2} \in \sigma + \mathbb{Z}. \tag{1.114}$$

One can produce more general intertwining maps by taking the composition of several fields $V_\mu^{\frac{1}{2}}(z)$. In this case, the requirement that $h_\mu = \frac{1}{2}$ remains the same, but rather than the condition for trivial monodromy being as in (1.114), we have something more complicated. For the product of $n$ many fields with the same $\mu$, which we abbreviate as $V_{n\times\mu}^{(\frac{1}{2})}(z)$, we have

$$\begin{aligned} V_{n\times\mu}^{(\frac{1}{2})}(z) &= V_\mu^{(\frac{1}{2})}(z_1)\cdots V_\mu^{(\frac{1}{2})}(z_n) \\ &= e^{n\mu\mathbf{q}}\left(\sum_{m_1,\dots m_n\in\sigma+\mathbb{Z}} b_{m_1}\cdots b_{m_n} z_1^{-m_1-\frac{1}{2}}\cdots z_n^{-m_n-\frac{1}{2}}\right) \\ &\qquad \times \prod_{i\neq j}(z_i-z_j)^{\frac{1}{2}\mu^2}\prod_{i=1}^{n} z_i^{\mu a_0}\prod_{k>0}\exp\left(\frac{\mu}{k}a_{-k}p_k\right)\exp\left(-\frac{\mu}{k}a_k p_{-k}\right) \end{aligned} \tag{1.115}$$

where again $p_k(z)$ is the $k$th power sum,

$$p_k = z_i^k + \cdots z_n^k. \tag{1.116}$$

As in the $N=0$ case, one proceeds by following the theory outlined in [25], making the projective change of variables $(z_1,\dots z_n)\mapsto(x,y_i,\dots y_n)$ and showing the existence a non-trivial contour as a connected sum of a loop in $x$ and a generalised Pochhammer contour in the $y_i$. While only the purely bosonic case is considered in [25], one can check with relative ease that the result is unchanged by the presence of fermions (or of any particular modes of any particular algebra, providing they do no more than introduce additional monomial powers of the variables, and do not change the basic form of the kernel of integration).

We find the monodromy condition

$$n\left(\frac{1}{2}(n-1)\mu^2 + \mu\eta - \frac{1}{2}\right) \in n\sigma + \mathbb{Z} \tag{1.117}$$

(and note that setting $n=1$ recovers the monodromy condition for a single field). We can consider this, as for (1.88), in the equivalent form of a condition on the quantity

$$\begin{aligned} \Delta h &= h_{\eta_L} - h_{\eta_R} \\ &= h_{\eta_R+n\mu} - h_{\eta_R} \\ &= n\left(\frac{1}{2}n\mu^2 + \mu\eta - \mu\lambda\right) \\ &= n\left(\frac{1}{2}(n-1)\mu^2 + \mu\eta + \frac{1}{2}\right), \end{aligned} \tag{1.118}$$

where the last equality follows because $h_\mu = \frac{1}{2}\mu(\mu - 2\lambda) = \frac{1}{2}$, obtaining $\Delta h - n \in n\sigma + \mathbb{Z}$, or just $\Delta h \in n\sigma + \mathbb{Z}$. This means we are also able to constrain

$$\eta_R = \frac{1}{2}(1-n)\mu + \frac{2\Delta h - n}{2n\mu}, \qquad \eta_L = \frac{1}{2}(1+n)\mu + \frac{2\Delta h - n}{2n\mu} \tag{1.119}$$

and therefore

$$\begin{aligned} h_R &= \frac{1}{2}\left[\frac{1}{4}(n^2-1)\mu^2 + \left(\frac{1}{2} - \Delta h\right) + \frac{4(\Delta h)^2 - n^2}{4n^2\mu^2}\right] \\ h_L &= \frac{1}{2}\left[\frac{1}{4}(n^2-1)\mu^2 + \left(\frac{1}{2} + \Delta h\right) + \frac{4(\Delta h)^2 - n^2}{4n^2\mu^2}\right], \end{aligned} \tag{1.120}$$

and we see that in order to determine $\mathcal{S}_1\mathcal{F}_{\eta_L}$ and $\mathcal{S}_1\mathcal{F}_{\eta_R}$ up to isomorphism, it is sufficient to specify $(\mu, \Delta h, n)$.

Of particular interest is when the two spaces involved are reducible, belonging to an extended Kac table. The chosen conformal weight, withh $h_\mu = \frac{1}{2}$, reveals that in the $(p, q)$ extended table there are two solutions for $\mu$:

$$\mu^+ = 2\sqrt{\frac{q}{p}}, \qquad \mu^- = -2\sqrt{\frac{p}{q}} \tag{1.121}$$

This means

$$\eta^\pm_{r,s} + n\mu^+ = \eta^\pm_{r\pm 4n,s}, \qquad \eta^\pm_{r,s} + n\mu^- = \eta^\pm_{r,s\pm 4n} \tag{1.122}$$

so that intertwiners map in steps of 4 around the extended Kac table, and hence preserve the sector (R or NS) as well as the choice of $\pm$ for $\eta^\pm$. It is interesting to note here the relative parity of the vacuum vectors of the domain and image modules under the mapping. Overall, the zero mode of the field $V^{\frac{1}{2}}_{n\times\mu}(z)$ is bosonic, being a graded operator of grading $0 \in \mathbb{Z}$. However, simple mode counting of the expansion in (1.115) shows that the parity of fermionic creation operators in the zero mode is the same as that of $n$. Thus, if $n$ is odd, then the intertwiner also changes the fermionic parity of any vector it acts upon, even though it must be an even operator. The only way to resolve this apparent contradiction is if the vacuum vectors of the two modules are of opposite parities — parity is, in some sense, a "local" property, defined only with respect to the relevant vacuum vector. Vacuum parity, usually a mostly irrelevant choice of convention, now becomes an important aspect when attempting to find Fock modules of the $N = 1$ Virasoro superalgebra which are related by intertwining maps.

### The $N = 2$ Fock Superspace

The extra complications arising from the addition of a single copy of $\mathfrak{b}$ in the $N = 1$ case are only exacerbated further when a second copy is introduced. In the $N = 2$ case, we make yet another highest weight construction, this time with four oscillator algebras (two copies of $\mathfrak{a}$ and two of $\mathfrak{b}$) from a vacuum $|\eta_1, \eta_2, \updownarrow_1, \updownarrow_2\rangle^\sigma$.

**1.2.8 Definition.** The $N = 2$ Fock superspace of vacuum eigenvalues $\eta_1$ and $\eta_2$, vacuum parities $\updownarrow_1$ and $\updownarrow_2$, and sector $\sigma$, denoted $\mathcal{S}_2\mathcal{F}^{\sigma}_{\eta_1,\eta_2,\updownarrow_1,\updownarrow_2}$, is defined to be

$$\mathcal{S}_2\mathcal{F}^{\sigma}_{\eta_1,\eta_2,\updownarrow_1,\updownarrow_2} = \mathcal{F}^{\mathfrak{a}}_{\eta_1} \otimes \mathcal{F}^{\mathfrak{a}}_{\eta_2} \otimes \mathcal{F}^{\mathfrak{b}}_{\updownarrow_1,\sigma} \otimes \mathcal{F}^{\mathfrak{b}}_{\updownarrow_2,\sigma} \tag{1.123}$$

This space, like its lower-$N$ cousins, is simple as a module of its combined oscillator algebras, and many of the same structural statements apply. However, when considered as a module of the $N = 2$ Virasoro superalgebra, the situation becomes extremely complicated. Very little is known about the Fock superspace representation theory of $\mathfrak{s}_2\mathfrak{Vir}$.

One begins the construction of the Virasoro superalgebra generators in a similar way as before. Since all oscillator algebras are mutually commuting, the $L_n$ modes (for instance) are simply the sum of two disjoint copies of what is seen in the $N = 1$ case;

$$L_n := \sum_{i=1}^{2} \left[ \frac{1}{2} \sum_{k \in \mathbb{Z}} : a^{(i)}_k a^{(i)}_{n-k} : \, -(n+1)\lambda_i a^{(i)}_n - \sum_{k \in \sigma + \mathbb{Z}} (k + \frac{1}{2}n) \, : b^{(i)}_k b^{(i)}_{n-k} : \right] \tag{1.124}$$

where $i$ labels the copies of the oscillator algebras, and $\lambda_i$ are complex numbers. One can derive, for instance, the central charge as

$$c = 3 - 12(\lambda_1^2 + \lambda_2^2). \tag{1.125}$$

This realisation utilises mutually commuting (real) bosonic $(a_1(z), a_2(z))$ and fermionic $(b_1(z), b_2(z))$ fields, but it can be convenient to make a change of "basis" to one *complex* field of each type:

$$\begin{aligned} a(z) &:= \frac{1}{\sqrt{2}} \left( a_1(z) + i a_2(z) \right), & \bar{a}(z) &:= \frac{1}{\sqrt{2}} \left( a_1(z) - i a_2(z) \right) \\ b(z) &:= \frac{1}{\sqrt{2}} \left( b_1(z) + i b_2(z) \right), & \bar{b}(z) &:= \frac{1}{\sqrt{2}} \left( b_1(z) - i b_2(z) \right) \end{aligned} \tag{1.126}$$

satisfying

$$\begin{aligned} & a(z)a(w) \sim \bar{a}(z)\bar{a}(w) \sim b(z)b(w) \sim \bar{b}(z)\bar{b}(w) \sim 0 \\ & a(z)\bar{a}(w) \sim \bar{a}(z)a(w) \sim \frac{1}{(z-w)^2} \\ & b(z)\bar{b}(w) \sim \bar{b}(z)b(w) \sim \frac{1}{z-w}. \end{aligned} \tag{1.127}$$

Together with the definition

$$\lambda := \frac{1}{\sqrt{2}} \left( \lambda_1 + i\lambda_2 \right), \qquad \bar{\lambda} := \frac{1}{\sqrt{2}} \left( \lambda_1 - i\lambda_2 \right), \tag{1.128}$$

we then have

$$\begin{aligned}
T(z) &= \,:a(z)\bar{a}(z): + \frac{1}{2}\left(\bar{\lambda}\partial a(z) + \lambda\partial\bar{a}(z)\right) - \frac{1}{2}\,:b(z)\partial\bar{b}(z) + \bar{b}(z)\partial b(z): \\
G(z) &= G^{+}(z) = \bar{a}(z)b(z) + \bar{\lambda}\partial b(z) \\
\bar{G}(z) &= G^{-}(z) = a(z)\bar{b}(z) + \lambda\partial\bar{b}(z) \\
J(z) &= \,:b(z)\bar{b}(z): - \left(\bar{\lambda}a(z) - \lambda\bar{a}(z)\right).
\end{aligned} \tag{1.129}$$

The reason for this choice of basis is to simplify some of the derivations at the field level — again, it is simpler to construct candidate intertwining maps using the properties of the operator product expansion.

In this basis, one makes choices of shifts $\mu_1, \mu_2$ to the vacuum eigenvalues $\eta_1, \eta_2$, defining

$$\mu := \frac{1}{\sqrt{2}}\left(\mu_1 + i\mu_2\right), \qquad \bar{\mu} := \frac{1}{\sqrt{2}}\left(\mu_1 - i\mu_2\right), \tag{1.130}$$

and beginning with the familiar vertex operator (now in two bosonic fields)

$$V_\mu(z) := \,:\exp\left(\bar{\mu}\partial^{-1}a(z) + \mu\partial^{-1}\bar{a}(z)\right):, \tag{1.131}$$

we see, for instance,

$$\begin{aligned}
T(z)V_\mu(w) &\sim \frac{(h_{\mu_1} + h_{\mu_2})V_\mu(w)}{(z-w)^2} + \frac{\partial V_\mu(w)}{z-w} \\
G(z)V_\mu(w) &\sim \frac{\bar{\mu}b(w)V_\mu(w)}{z-w} \\
\bar{G}(z)V_\mu(w) &\sim \frac{\mu\bar{b}(w)V_\mu(w)}{z-w} \\
J(z)V_\mu(w) &\sim \frac{-2\left(\mu\bar{\lambda} - \bar{\mu}\lambda\right)V_\mu(z)}{z-w},
\end{aligned} \tag{1.132}$$

which suggests by analogy with the $N=1$ case a quadruplet of fields

$$\left(V_\mu^{(0)}(z), V_\mu^{(\frac{1}{2})}(z), \bar{V}_\mu^{(\frac{1}{2})}(z), V_\mu^{(1)}(z)\right) \tag{1.133}$$

related by constant scalings to the four fields

$$\begin{aligned}
V_\mu^{(0)}(z) &\propto V_\mu(z) \\
V_\mu^{(\frac{1}{2})}(z) &\propto b(z)V_\mu(z) \\
\bar{V}_\mu^{(\frac{1}{2})}(z) &\propto \bar{b}(z)V_\mu(z) \\
V_\mu^{(1)}(z) &\propto \,:\left(\beta b(z)\bar{b}(z) + \bar{\alpha}a(z) + \alpha\bar{a}(z)\right)V_\mu(z):
\end{aligned} \tag{1.134}$$

for some constants $\beta, \alpha, \bar{\alpha}$. Indeed, one finds relations like

$$
\begin{aligned}
T(z)V_\mu^{(1)}(w) &\sim \partial\left(\frac{V_\mu^{(1)}(w)}{z-w}\right) \\
&\quad + k_1\left[\frac{(h_{\mu_1}+h_{\mu_2})}{(z-w)^2} + \frac{\left(\bar{\alpha}(\mu-2\lambda)+\alpha(\bar{\mu}-2\bar{\lambda})\right)V_\mu(z)}{(z-w)^3}\right] \\
G(z)V_\mu^{(1)}(w) &\sim \bar{\alpha}\frac{k_1}{k_{\frac{1}{2}}}\partial\left(\frac{V_\mu^{(\frac{1}{2})}(w)}{z-w}\right) \\
&\quad - k_1\left[\frac{(\beta+\bar{\alpha}\mu-\alpha\bar{\mu})\ :\bar{a}bV_\mu:(w)}{z-w} + \frac{(\bar{\mu}-2\bar{\lambda})b(w)V_\mu(w)}{(z-w)^2}\right] \\
\bar{G}(z)V_\mu^{(1)}(w) &\sim \alpha\frac{k_1}{\bar{k}_{\frac{1}{2}}}\partial\left(\frac{\bar{V}_\mu^{(\frac{1}{2})}(w)}{z-w}\right) \\
&\quad + k_1\left[\frac{(\beta+\bar{\alpha}\mu-\alpha\bar{\mu})\ :a\bar{b}V_\mu:(w)}{z-w} + \frac{(\mu-2\lambda)b(w)V_\mu(w)}{(z-w)^2}\right] \\
J(z)V_\mu^{(1)}(w) &\sim 0 - 2k_1\left[\frac{(\bar{\lambda}\mu-\lambda\bar{\mu})V_\mu^{(1)}(w)}{z-w} + \frac{(-\frac{1}{2}\beta+\bar{\lambda}\alpha-\lambda\bar{\alpha})V_\mu(z)}{(z-w)^2}\right]
\end{aligned}
\tag{1.135}
$$

where $k_1, k_{\frac{1}{2}}, \bar{k}_{\frac{1}{2}}$ are the constants of proportionality relating the fields in (1.134). This gives us a system of equations which constrains the candidate source of the intertwining operator; the field $V_\mu^{(1)}(z)$. This system, removing redundancies, is:

$$
\begin{aligned}
\mu - 2\lambda &= 0 \\
\beta + \bar{\alpha}\mu - \alpha\bar{\mu} &= 0.
\end{aligned}
\tag{1.136}
$$

The first of these two equations simply ensures that the field $V_\mu(z)$ is a weight-0 Virasoro primary, since then $h_\mu = \frac{1}{2}\mu(\mu-2\lambda) = 0$. Compare this to the $N = 0$ requirement that this field be primary of weight 1, and the $N = 1$ requirement that it be primary of weight $\frac{1}{2}$. This is in line with the general procedure of producing intertwining operators by creating primary fields of total weight 1 through products of free fields present in the theory. In each case, as $N$ increased, we found that progressively higher-order combinations of such fields were necessary, because we required additional degrees of freedom to fix the operator product expansions with the superpartner fields to $T(z)$.

The meaning of the second equation is less immediately obvious; it appears to contain at least one completely unconstrained degree of freedom, perhaps due to our ability to independently vary $\eta_1$ and $\eta_2$ through $\mu_1$ and $\mu_2$. We have suggestively named $\alpha, \bar{\alpha}$ to imply the solution $(\beta, \alpha, \bar{\alpha}) = (0, \mu, \bar{\mu})$, but in the absence of other constraints, this system is actually under-determined ($\alpha$ and $\bar{\alpha}$, unlike pairs $\mu$ and $\bar{\mu}$ or $\lambda$ and $\bar{\lambda}$, need not be genuine complex conjugates). If

$\beta = 0$ is in fact the case, however, then note by the placement of $\alpha, \bar{\alpha}$ in the definition of $V_\mu^{(1)}(z)$ that

$$V_\mu^{(1)}(z) \propto \partial \left( V_\mu^{(0)}(z) \right) . \tag{1.137}$$

As a total derivative, $V_\mu^{(1)}(z)$ has no residue, and therefore the intertwiner it provides is in fact the zero operator. Therefore the existence of nontrivial intertwiners obtained from $V_\mu^{(1)}(z)$ is contingent on taking $\beta \neq 0$. In this case, it is not clear if there is any *natural* choice for $\alpha$ and $\bar{\alpha}$.

## 1.3 Summary

We have examined the representation theory of the Virasoro algebra and its $N = 1$ and $N = 2$ superalgebra variants. While beginning with Verma modules, we have in fact focused our attention on another type of $\mathbb{Z}$-graded module for these algebras: various types of Fock space.

Fock spaces, induced Verma-style via the action of bosonic and fermionic oscillator algebras from a vacuum vector, nevertheless carry Virasoro representations wherein the Virasoro generators are realised as infinite quadratic sums of oscillator generators. We have seen that the structures of these Fock spaces is typically related, but not identical, to those of the Virasoro Verma modules of the same parameters.

The presence of the oscillator algebras affords us the opportunity of algebraic manipulations outside the action of the Virasoro elements themselves, and thus the ability to produce intertwining maps between Fock spaces using vertex operators – the normally ordered exponentials of oscillator modes (together with the important generator of vacuum momentum shifts, $\mathbf{q}$) and their field products of varying degree.

We have directed our attention at these intertwining maps, because their existence provides the basis of the work to follow, which focuses on the construction of indecomposable Virasoro modules where the zero mode $L_0$ acts non-diagonalisably.

*Chapter 2*

# Logarithmic Conformal Field Theory

> *"I don't know anyone that really likes algebra, and if you do, you need to look in the mirror and check yourself."*
>
> "Pitbull" (Armando Christian Perez), singer/songwriter

We now turn to the main topic of this work; logarithmic conformal field theory (LCFT, or logCFT). This is a kind of extension of standard conformal field theory in which the fields are permitted to have logarithmic-type singularities. In particular, we are interested in the new types of Virasoro representations which arise out of making this extension.

These modules have been called "staggered" in the literature [41] for at least two decades now, reflecting the fact that the Virasoro algebra action on these spaces is indecomposable and its zero mode, $L_0$, is non-diagonalisable. On the field theory side of things, this corresponds to the existence of multiplets of fields whose expansions with $T(z)$ have a particular "upper-diagonal" form.

## 2.1 Basics of logCFTs and Staggered Modules

When one studies two-dimensional conformal field theories one often works at the level of fields and their operator product expansion relations in some vertex operator algebra. These relations are heavily constrained by conformal symmetry. For instance, for a primary field $\phi$ of conformal weight $h$, the four-point correlator is determined up to a function of the cross-ratio $x$ of the field variables:

$$\begin{aligned}\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle &\sim \frac{1}{(z_1-z_3)^{2h}}\frac{1}{(z_2-z_4)^{2h}}F(x),\\ x &= \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}.\end{aligned} \tag{2.1}$$

When $\phi$ belongs to a Kac table, a differential equation may be found for $F$. Generically, these can be directly solved. However, for certain values of the "internal parameters" such as the central charge $c$ and the conformal weight $h$, this differential equation becomes (in a sense) degenerate: simply assuming polynomial behaviour of $F$ near its poles fails to give all solutions. The full set of solutions for $F$ are only found once one introduces *logarithmic* behaviour — hence the name *logarithmic* conformal field theory [22, 23]. Perhaps surprisingly, doing so does not affect conformal invariance.

In these instances, the primary fields $\phi$ appear in multiplets together with a number of non-primary *logarithmic partner fields* whose operator product expansions contain these logarithmic terms. The length of this multiplet is called the *rank* of the logarithmic theory. For instance, in a rank-2 theory with multiplet $(\phi, \Phi)$, we have

$$\begin{aligned} \langle \Phi(z)\Phi(w) \rangle &\sim \frac{A - 2B\log(z-w)}{(z-w)^{2h}} \\ \langle \Phi(z)\phi(w) \rangle &\sim \frac{B}{(z-w)^{2h}} \\ \langle \phi(z)\phi(w) \rangle &\sim 0 \end{aligned} \tag{2.2}$$

for some constants $A, B$ [22]. The vanishing of $\langle \phi\phi \rangle$ may be surprising, but is forced by consistency with conformal symmetry. It may be utilised (provided $B \neq 0$, an uncommon but not impossible scenario) to make a field redefinition $\Phi \mapsto \Phi' = \Phi + k\phi$ in order to tune $A$ to any desired value. It is not possible to alter $B$ at all in this way, and it is in fact an invariant of the multiplet, variously called *anomaly number*, *logarithmic coupling*, or *indecomposability parameters* by other authors. It can be thought of as measuring how "badly" the theory in question is non-semisimple, in the sense that the stress-energy field $T$ of the theory has non-diagonalisable upper triangular action on the multiplet (through the OPE). It should be noted that such couplings have not been studied in great depth for logarithmic CFTs with ranks greater than 2.

In non-logarithmic CFTs, typical objects of interest are representations of the Virasoro algebra which can be graded into finite-dimensional $L_0$ eigenspaces with eigenvalues bounded below. Verma modules and Fock spaces fall under this classification. In logarithmic theories we seek to weaken this requirement to modules $\mathcal{M}$ which can be graded into finite-dimensional *generalised* $L_0$ eigenspaces. We will use the term *staggered* for such modules.

**2.1.1 Definition.** For the purposes of this text, a *staggered module* of the Virasoro algebra is a Virasoro module $\mathcal{M}$ which admits a decomposition

$$\mathcal{M} = \bigoplus_{n=0}^{\infty} \mathcal{M}_n \tag{2.3}$$

such that each $\mathcal{M}_n$ is a finite-dimensional vector subspace, and $L_0 - h_n$ is nilpotent on $\mathcal{M}_n$ for some generalised eigenvalue $h_n$.

The maximum rank of generalised eigenvector[1] appearing in such an $\mathcal{M}$ is said to be the *rank* of the module as a whole. This notion of rank matches with that of the field theory: with a rank-$r$ multiplet of fields $(\Phi_0 = \phi, \Phi_1, \Phi_2, \ldots, \Phi_{r-1})$ one finds that

$$(L_0 - h\mathbf{1})\left|\Phi_k\right\rangle \propto \left|\Phi_{k-1}\right\rangle \tag{2.4}$$

where $\left|\Phi_i\right\rangle$ is the state corresponding to $\Phi_i(z)$, and $\left|\Phi_{-1}\right\rangle, \left|\Phi_{-2}\right\rangle$, etc. are defined to be the zero vector, so that a rank-$r$ multiplet of fields corresponds to a rank-$r$ generalised eigenspace and vice versa. The Virasoro module freely generated from this multiplet of states then produces a rank-$r$ module.

A staggered module automatically enjoys compatibility with the Virasoro algebra generators as graded operators. One can easily check that

$$(L_0 - h_v\mathbf{1})^n v = 0 \quad \Longrightarrow \quad (L_0 - (h_v - m)\mathbf{1})^n L_m v = 0 \tag{2.5}$$

so that $L_m$ sends the generalised eigenspace of weight $h$ into that of weight $h-m$. Note that this implies all $L_m$ must also be (weakly) upper triangular, in the sense that they cannot increase the rank of a vector. The set of rank-1 vectors within a staggered module therefore constitutes a proper rank-1 (i.e., without staggered structure) Virasoro submodule. Taking the quotient by this submodule leaves us with another staggered module with rank reduced by one. Iterating this process eventually leaves us with a final rank-1 quotient module. We can therefore characterise staggered modules by a decomposition into a chain of proper rank-1 subquotients, the length of the chain being equal to the rank of the staggered module. If each successive submodule admits an integer grading, for instance if they are all highest weight modules, then their respective gradings can differ from each other by only integers. Otherwise, $(L_0 - h)$ would always evaluate to zero at some rank, which would as a result break the decomposition into a direct sum of lower rank staggered modules.

**2.1.2 Proposition.** *The choice of rank-1 submodule is always well-defined and unique. Furthermore, due to the upper-diagonal action of the algebra, both the projection map onto the quotient space and the inclusion map of submodule into the containing staggered module are Virasoro homomorphisms.*

In light of this decomposition, we see that rank-2 staggered modules $\mathcal{M}$ in particular enjoy a presentation as the middle object in a non-split short exact sequence of non-staggered modules:

$$0 \longrightarrow V_L \xrightarrow{\iota} \mathcal{M} \xrightarrow{\pi} V_R \longrightarrow 0 \tag{2.6}$$

where the maps $\iota, \pi$ are $\mathfrak{Vir}$ homomorphisms and the spaces $V_L, V_R$ are graded Virasoro modules of rank 1 (non-staggered). These are called the *left* and *right*

[1]The *rank* of a generalised eigenvector $v$ with eigenvalue $h$ of some operator $M$ is the least positive integer $n$ such that

$$(M-h)^n\, v = 0$$

spaces respectively, and $\mathcal{M} \cong_{\mathrm{Vec}} V_L \oplus V_R$ (as vector spaces only). In the literature, the left and right spaces of a rank-2 staggered module are typically taken to be highest weight spaces, providing the historical definition of staggered module, though as noted we do not make that restriction here.

// **Remark:**

The definition of a staggered module as stated does not ensure that each generalised eigenspace is "structurally compatible" with every other or that each rank-1 submodule in the decomposition as a chain of subquotients is of the same type. We therefore introduce a further convention of nomenclature which fixes somewhat the object of study. If "$x$" is a property of modules (e.g. **Z**-graded, *Fock*, *Verma*, etc), then we will say that a staggered module is a *staggered "x" module* to mean that each rank-1 submodule in the described decomposition has property "$x$". Staggered Fock modules are the most important objects of study in the results to follow. //

One interesting question now is how the structure of a module with a given property $x$ manifests in a staggered $x$-module. For instance, one can ask whether the singular vectors of a reducible Fock module are still meaningful structural classification tools when one considers that Fock space in the context of the decomposition series of a staggered Fock module, and indeed more generally what determines the reducibility of such a module, how many distinct staggered structures can be constructed from any given decomposition series (or when such a construction is possible at all), and how to distinguish between inequivalent staggered structures.

This last point is an important consideration if we hope to assess candidate staggered modules as concrete constructions of the state spaces of known logarithmic theories. Luckily, in the case of rank 2 staggered modules, some considerable effort has been put into exactly this. We find that with some minimal structural assumptions, we can find an important internal parameter which remains invariant under module isomorphisms. To be precise, suppose that $V_L$ and $V_R$ from (2.6) are $\mathbb{N}$-graded non-staggered Virasoro modules with one-dimensional highest weight subspaces, each spanned by highest weight vectors $v_L$ and $v_R$ with weights $h_L$ and $h_R$ respectively (this is not the same as $V_L$ and $V_R$ being highest weight spaces themselves, as they need not be generated by these highest weight vectors). Let $x_L = \iota(v_L)$ and choose a representative $x_R \in \pi^{-1}(v_R)$. Since $L_n x_R \in \pi^{-1}(L_n v_R) = \pi^{-1}(\{0\})$ for $n > 0$, one can show that

$$w = (L_0 - h_R)x_R \in \iota(V_L) \tag{2.7}$$

is a Virasoro singular vector. Since the left and right spaces have integer gradings, $\Delta h = h_L - h_R \in \mathbb{Z}$. If $w$ is non-zero, then we can conclude in addition that $\Delta h \leq 0$. Depending on the particular choices of $V_L$ and $V_R$, we might also have $w = U x_L$ for some Virasoro creation operator $U \in \mathcal{U}(\mathfrak{Vir})$. This appears to be the case for all staggered modules seen "in the wild" (e.g. through the computation of fusion products of highest weight modules). If this is indeed the case, and $V_L$ is equipped with an adjoint-equivariant bilinear or sesquilinear form — such as the Shapovalov form — then we may compute the quantity

$$\beta := \left\langle x_L, U^\dagger x_R \right\rangle_{\iota(V_L)}, \tag{2.8}$$

possible since $x_L, U^\dagger x_R \in \iota(V_L)$, which, as a $\mathfrak{Vir}$-submodule of $\mathcal{M}$ with $\iota$ a homomorphism, may directly inherit the bilinear form of $V_L$. The adjoint $\dagger$ is the standard one, sending $L_n$ to $L_{-n}$ and $C$ to $C$, but the form can only be consistently applied on the restriction to $\iota(V_L)$.

One may easily check that the quantity $\beta$ defined in (2.8) does not depend on the choice of representative $x_R \in \pi^{-1}(v_R)$, only on the fixed normalisation of the vectors $x_L, x_R$ (or equivalently of $U_{\Delta h}$ and $x_R$, or any other pair of data in the inner product). This quantity is called the *beta invariant* of the staggered module, and has been of quite some importance in studying 2D logarithmic conformal field theories to date, as in addition to identifying the staggered module up to isomorphism, it (being related to the constant $B$ from the discussion of the field theory earlier in this section) also appears in correlation functions. It is therefore extremely useful to have a way of computing $\beta$ for candidate constructions of staggered modules corresponding to various logCFTs.

In fact, the historical importance of $\beta$ in the study of indecomposable Virasoro representations should not be understated, nor should its widespread application as a classification tool. The $\beta$ notation for this distinguishing parameter between different staggered $\mathfrak{Vir}$-modules appears as early as [17], and in numerous works since (see those already noted in the abstract and introduction; [7, 8, 10, 13, 14, 18, 19, 23, 24, 31, 34, 40, 45, 46] and many others). See in particular [31] for a comprehensive treatment of the links between $\beta$ and the isomorphism classes of staggered Virasoro modules, as well as a large amount of relevant context from the literature. In [31], it is shown that the isomorphism class of a staggered module comprising two highest weight Virasoro modules is uniquely determined[2] by its so-called *data*, the pair of vectors $(L_1 x_R, L_2 x_R)$ (in our notation), up to a notion of 'equivalence' of data which amounts to a different choice of vacuum representative $x_R \in \pi^{-1}(V_R)$ (called a *gauge* choice in by some, e.g. in [31] and [35]) which of course does not affect the module structure. Note that the subalgebra $\mathfrak{Vir}_+$ of annihilation operators is generated by $L_1$ and $L_2$ (in the sense that $[L_n, L_1] = (n-1)L_{n+1}$); the $U^\dagger$ appearing in (2.8), as an element of $\mathcal{U}(\mathfrak{Vir}_0 \oplus \mathfrak{Vir}_+)$, can be expanded as some net annihilator $U^\dagger = U_0(L_0 - h) + U_1 L_1 + U_2 L_2$ and thus the data $(L_1 x_R, L_2 x_R)$ appear directly in the computation of $\beta$.

Thus we are justified in our focus on $\beta$, for if there are two staggered modules with the same left and right spaces $V_L, V_R$, the same singular vector $U v_L \in V_L$,

[2] Of course, it goes without saying that the relevant modules $V_L$ and $V_R$ must *also* be specified; otherwise the question makes no sense.

and the same $\beta$ but with different data $(L_1 x_R, L_2 x_R)$ and $(L_1 y_R, L_2 y_R)$, then

$$\langle x_L, (U_1 L_1 + U_2 L_2)(x_R - y_R)\rangle = 0 \tag{2.9}$$

and so the difference $(x_R - y_R)$ vanishes under annihilation operators; it belongs to $\iota(V_L)$ and the spaces are therefore related by a gauge transform – the staggered modules are isomorphic.

### 2.1.1 Staggering Operators

Some features of the staggered structure do not depend on the constituent submodules or indeed on the presence of the Virasoro algebra at all. Suppose we were to consider rank-2 staggered $\mathfrak{g}$-modules $\mathcal{M}$ of an arbitrary Lie algebra $\mathfrak{g}$ constructed out of previously known $\mathfrak{g}$-representations $V_L, V_R$ in the manner of the short exact sequence in (2.6). Here the term rank refers not to the maximal rank of $L_0$ generalised eigenvectors, of course, but instead to the number of submodules in the decomposition chain (in the sense of the decomposition of staggered objects in terms of non-staggered ones described above) — an equivalent concept, at least for the staggered modules with generalised $L_0$ grading discussed in Section 2.1.

Suppose that $\mathfrak{g}$ has generators $\{X^i | i \in I\}$ for some index set $I$ and structure constants $f^{ij}_k$, so that (using summation notation)

$$[X^i, X^j] = f^{ij}_k X^k. \tag{2.10}$$

Since the staggered module $\mathcal{M}$ has vector space structure $V_L \oplus V_R$, by writing the $\mathfrak{g}$-action on $\mathcal{M}$ in this basis as

$$\tilde{X}^i = \begin{bmatrix} X^i & V^i \\ 0 & X^i \end{bmatrix}, \tag{2.11}$$

where $X^i \mapsto \tilde{X}^i$ denotes the $2 \times 2$ block matrix form of the action of $\mathfrak{g}$ on the staggered module, we see that a nontrivial staggered structure corresponds exactly to a nontrivial set of off-diagonal operators

$$\{V^i : V_R \to V_L | i \in I\} \tag{2.12}$$

which provide the indecomposable part of the action. We will call these $V^i$ *staggering operators*, and simply requiring $X^i \mapsto \tilde{X}^i$ to be a Lie algebra representation forces the relationship

$$[X^i, V^j] + [V^i, X^j] = f^{ij}_k V^k, \tag{2.13}$$

with both sides considered as operators taking $V_R$ to $V_L$, and the $X^i$ evaluated according to their corresponding module actions on the left and right spaces in the two orderings from the bracket $[\cdot, \cdot]$. One can verify that antisymmetry and Jacobi identity for the $\tilde{X}^i$ follow easily from (2.13), so do not constitute additional constraints. Note that if $\mathfrak{g}$ is instead a more general graded Lie-type algebra (such as a superalgebra), then the commutator in (2.13) needs simply to be replaced with the graded commutator, and the staggering operators summarily assigned parity appropriate to their corresponding basis element.

Having the staggering operators respect any generalised grading of the staggered module itself is a requirement which must be imposed separately. In the case of the Virasoro algebra, and of other (super)algebras containing the Virasoro generators as a subalgebra, we expect

$$[L_m, V_n] + [V_m, L_n] = (m - n)V_{m+n} \tag{2.14}$$

with each $V_n$ being a graded operator of grade $n$. The central element $C$ should continue to act as $c\mathbf{1}$ in the staggered module, with no off-diagonal components (i.e. $V_C$ is the zero operator). One way in which we can satisfy this equation is to have

$$[L_m, V_n] = -nV_{m+n}; \tag{2.15}$$

or in other words, the staggering operators are the modes of a weight-1 Virasoro primary field. The zero mode $V_0$ of such a field is a Virasoro intertwiner $V_R \to V_L$ (that is, $[L_m, V_0] = 0$ for all modes $L_m$), so in this sense finding constructions for staggered modules is a by-product of, or at least related to, finding intertwining maps between Virasoro modules. In certain settings, we find that specifying *only* an intertwining map is sufficient information in order to be able to construct an entire family of staggering operators. Such a construction may be made when the left and right modules both carry an action of a second algebra, whose basis elements also behave — up to possible additions of central elements — as the modes of a weight-1 Virasoro primary field.

The structure of rank-2 staggered modules where the left and right spaces are Fock modules was examined in some detail in [9], where a general formula for $\beta$ was derived. We re-present that work here, and also seek to extend this analysis to more general situations. For instance, when the staggered module in question is a representation space for a larger algebra, such as one of the $N > 0$ Virasoro superalgebras, we must also supply staggering operators to be associated with the other, non-Virasoro generators. In what follows, we find that the core idea of Section 2.2 (summarised in (2.23)) may be appropriately extended in the context of the $N = 1$ and $N = 2$ cases, and that a single intertwining map between Fock spaces over these larger algebras also suffices to provide the full staggered structure.

## 2.2 Staggered Fock Spaces

Let us put now together the pieces developed up until this point and examine some particular constructions of (rank 2) staggered Fock modules. As we have seen, there are several equivalent ways of presenting such objects. Their characterisation as the middle object in a non-split short exact sequence can be made more explicit by specifying the set of staggering operators, which (apart from defining the action of the Virasoro algebra on the staggered space) provide the details of the $\mathfrak{Vir}$-module homomorphisms in the chain.

We examine staggered structures involving purely bosonic ($N = 0$) as well as $N = 1$ and $N = 2$ Fock (super)spaces. In each case we provide a valid family of staggering operators and demonstrate a way to systematically calculate $\beta$ for each of them.[3]

### 2.2.1 Bosonic ($N = 0$) Staggered Fock Modules

We now examine rank-2 bosonic staggered Fock modules, which as discussed we define to be Virasoro modules $\mathcal{M}$ which fit into the short exact sequence

$$0 \longrightarrow \mathcal{F}^{\mathfrak{a}}_{\eta_1,\lambda} \xrightarrow{\iota} \mathcal{M} \xrightarrow{\pi} \mathcal{F}^{\mathfrak{a}}_{\eta_2,\lambda} \longrightarrow 0 \tag{2.16}$$

We fix the nature of the indecomposable $\mathfrak{Vir}$ action by providing a family of staggering operators, as outlined in (2.13) in general and (2.14) for the Virasoro algebra in particular. As noted, the Laurent coefficients of a weight-1 primary field suffice for this when the generators of the algebra consist of the $L_n$ alone. Any such field whose modes map between the spaces in question therefore defines a staggered structure. Happily, we have examples of such fields.

**Remark:**

Firstly, we recall the following generic facts about Fock spaces $\mathcal{F}^{\mathfrak{a}}_{\eta,\lambda}$ from Chapter 1 They are generated from the algebra $\mathfrak{a}$ whose basis elements satisfy

$$[a_m, a_n] = m\delta_{m,-n}\mathbf{1}, \qquad [a_m, \mathbf{1}] = 0 \qquad (\forall m, n \in \mathbb{Z}), \tag{2.17}$$

and we write $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ for the corresponding field. The *Fock space* $\mathcal{F}^{\mathfrak{a}}_{\eta}$ with vacuum vector $|\eta\rangle$ (for $\eta \in \mathbb{C}$) is the $\mathfrak{a}$-module freely generated from the action of the algebra on $|\eta\rangle$ subject to the relations

$$a_0 |\eta\rangle = \eta |\eta\rangle, \qquad a_n |\eta\rangle = 0 \quad (\forall n > 0), \tag{2.18}$$

and this may be given a $\mathfrak{Vir}$ action by setting

$$L_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} :a_k a_{n-k}: - \lambda(n+1)a_n, \tag{2.19}$$

[3]It should be noted, however, that whether $\beta$ is meaningful for classifying $N = 2$ staggered modules or not is still unknown.

where $:\cdot\cdot:$ denotes normal ordering, for $\lambda \in \mathbb{C}$. The resulting $\mathfrak{Vir}$ representation has highest weight $h = \frac{1}{2}\eta(\eta - 2\lambda)$ and central charge $c = 1 - 12\lambda^2$, and

$$[L_m, a_n] = -na_{m+n} - \lambda n(n-1)\delta_{m,-n}\mathbf{1}. \tag{2.20}$$

For Fock spaces we explicitly constructed intertwining maps through the use of *screening operators*, the zero modes of vertex operators $V_\mu(z)$, where

$$\begin{aligned} V_\mu(z) &:= \,:\exp(\mu\partial^{-1}a(z)): \\ &= e^{\mu\mathbf{q}} z^{\mu a_0} \prod_{n>0} \exp(\frac{\mu}{n}a_{-n}z^n)\exp(-\frac{\mu}{n}a_n z^{-n}) \end{aligned} \tag{2.21}$$

with

$$\partial^{-1}a(z) := \mathbf{q} + a_0\log(z) - \sum_{n\neq 0}\frac{1}{n}a_n z^{-n} \tag{2.22}$$

where the constant of integration $\mathbf{q}$ is a new operator with "canonical" position-momentum commutation relations $[a_n, \mathbf{q}] = \delta_{n,0}$. $V_0$ is defined whenever $V_\mu(z)$ has trivial monodromy (which depends both on $\mu$ and on the eigenvalue of $a_0$ as evidenced by the presence of the factor $z^{\mu a_0}$), and this operator commutes with the Virasoro generators whenever $\mu$ is such that $V_\mu(z)$ is a Virasoro primary field of weight one. //

Now it is easy to verify that *any* intertwining map $\phi$ between the left and right spaces in this short exact sequence may be used to construct a set of staggering operators. Since $[L_m, \phi] = 0$ by definition, we can set

$$V_n := [a_n, \phi], \tag{2.23}$$

then by a simple application of the Jacobi identity we find that (2.14) is satisfied, and the $V_n$ thus defined form a valid set of staggering operators for Fock space representations of $\mathfrak{Vir}$. We find that the task of constructing staggered Fock modules has been reduced to finding pairs of Fock spaces related by Virasoro intertwiners.

Recalling our prior examination of such vertex operators, one can now check via the OPE of $a(z)$ and $V_\mu(w)$ that the modes $V_n$ in the resulting series expansion are in fact identically equal to those produced by (2.23), up to a global scale factor of $\mu$. That is, in fact,

$$V_\mu(z) = \frac{1}{\mu}\sum_{n\in\mathbb{Z}}[a_n, V_0]z^{-n-1}. \tag{2.24}$$

While this seems to suggest that the extra step of constructing $V_n$ as $[a_n, V_0]$ may be an overcomplication, note that single vertex operators were not the most general primary fields which produced intertwining operators. This procedure is in fact what allows us to maintain a nontrivial family of staggering operators with the correct commutation relations when considering more general intertwiners.

Indeed, recall that the constrained relationship between $\mu$, $\eta$, and $\lambda$ heavily restricts which Fock modules are non-trivially related by zero modes of vertex operators $V_\mu(z)$, and that typically we can do much better — that is, find more general intertwiners for a larger collection of Fock modules — by considering the "zero modes" of *compositions* of several vertex operators (again with similar monodromy constraints, etc.), but with multiple variables $z_i$ to work with, constructing a contour around which to integrate and showing that the resulting operator is non-trivial is a difficult task. Finding one such contour for *each* staggering operator $V_n$ would be even harder still, but thankfully once an intertwiner has been found, using (2.23) suffices instead.

### Admissible Fock Spaces

It is important to determine the possible scope of application of the content developed in Section 2.2. In particular, since reducible structures are of interest to us, we now identify the exact pairs of entries in extended $(p, q)$ Kac tables which correspond to a given choice[4] of $(n, \Delta h)$. Suppose that $\eta_2 = \eta^{\pm}_{r,s}$. Given a fixed $\Delta h$, we solve $\eta_1 = \eta_2 + n\mu$ for $(r, s)$. We find four separate relations, each corresponding to a different choice of $\eta_2 = \eta^{\pm}_{r,s}$ and $\mu = \mu^{\pm}$ (Figure 2.1).

| $s(r)$ | $\eta^+_{r,s}$ | $\eta^-_{r,s}$ |
|---|---|---|
| $\mu^+$ | $(r+n)\frac{q}{p} - \frac{\Delta h}{n}$ | $(r-n)\frac{q}{p} + \frac{\Delta h}{n}$ |
| $\mu^-$ | $(r+\frac{\Delta h}{n})\frac{q}{p} - n$ | $(r-\frac{\Delta h}{n})\frac{q}{p} + n$ |

Figure 2.1: $s$ given $r$ for $\mathcal{F}^{\mathfrak{a}}_{\eta_2,\lambda} = \mathcal{F}^{\pm}_{r,s}$ within a staggered Fock module.

The Kac table locations of the left and right modules are independent of the choice of $\eta^{\pm}$, so two distinct solutions can be retrieved by fixing either $\eta_R = \eta^+_{r,s}$ or $\eta_R = \eta^-_{r,s}$ and then using the two different $\mu^{\pm}$.

A linear relation with reduced rational slope $\frac{q}{p}$ and reduced rational intercept $\frac{b}{a}$ possesses integral solutions if and only if $a$ divides $p$. If it possesses one such solution $(r_0, s_0)$, then necessarily infinitely many integral solutions $(r_0+kp, s_0+kq)$ exist, $k \in \mathbb{Z}$. By examining the table in Figure 2.1, we see that integral solutions always exist when $n = 1$. For solutions to exist for arbitrary $n \in \mathbb{Z}^+$, we require

$$n | p\Delta h \quad (\mu^+), \qquad n | q\Delta h \quad (\mu^-) \tag{2.25}$$

since these are the conditions under which the denominator of the intercept divides the denominator of the gradient for the rational linear relations in Figure 2.1.

[4]Recall that we use $n$ for the multiplicity of vertex operator $V_\mu(z)$ used to create the intertwining operator, and $\Delta h$ for the difference $h_1 - h_2$ in $L_0$ vacuum eigenvalue between the two Fock spaces.

However, recall that in the case at hand — for intwertwiners between reducible Fock modules belonging to an extended Kac table — $\frac{\Delta h}{n}$ is an integer (1.88). We find the two distinct staggered modules have short exact sequences:

$$0 \longrightarrow \mathcal{F}^{(-)}_{(kp-n,kq+\frac{\Delta h}{n})} \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^{(-)}_{(kp+n,kq+\frac{\Delta h}{n})} \longrightarrow 0 \qquad (\mu^+) \tag{2.26}$$

and

$$0 \longrightarrow \mathcal{F}^{(-)}_{(kp+\frac{\Delta h}{n},kq-n)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^{(-)}_{(kp+\frac{\Delta h}{n},kq+n)} \longrightarrow 0 \qquad (\mu^-) \tag{2.27}$$

The left and right modules, and thus the staggered modules themselves, are insensitive of the choice of $k \in \mathbb{Z}$. Here, to reduce clutter, we have written $\mathcal{F}_{r,s}$ to indicate that $\eta = \eta_{r,s}$, while the superscript $(-)$ is used to indicate $\eta = \eta^-$. Choosing $\eta^+$ instead does not give distinct $\mathcal{M}$, but does relate entries at different $(r, s)$ in the extended table. The sequences in that case are:

$$0 \longrightarrow \mathcal{F}^{(+)}_{(kp+n,kq-\frac{\Delta h}{n})} \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^{(+)}_{(kp-n,kq-\frac{\Delta h}{n})} \longrightarrow 0 \qquad (\mu^+) \tag{2.28}$$

and

$$0 \longrightarrow \mathcal{F}^{(+)}_{(kp-\frac{\Delta h}{n},kq+n)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^{(+)}_{(kp-\frac{\Delta h}{n},kq-n)} \longrightarrow 0 \qquad (\mu^-) \tag{2.29}$$

Again, all choices of $k$ are identical. Staggered modules in these four sequences are isomorphic if and only if they share the same $\mu$.

These particular solutions occur in rectangular patterns symmetrically arranged around corner entries of the extended $(p, q)$ table with $(r, s) = (kp, kq)$. Left modules occur on the principal diagonal corners of this pattern, right modules on the off-diagonal. $\mu^+$ maps vertically and $\mu^-$ horizontally. Sequences involving $\eta^+$ have $\mathcal{F}^{\mathfrak{a}}_{\eta_1,\lambda}$ below or to the right of $\mathcal{F}^{\mathfrak{a}}_{\eta_2,\lambda}$ in the table, and those involving $\eta^-$ have $\mathcal{F}^{\mathfrak{a}}_{\eta_1,\lambda}$ above or to the left. Figure 2.2 gives pictorial examples of this.

Although not pertinent to the case at hand, it is interesting to see that a similar pattern appears when $\frac{\Delta h}{n} \notin \mathbb{Z}$. If we assume the conditions for the existence of a solution are satisfied, then locating viable $(r, s)$ in the table is tantamount to solving for $i, j \in \mathbb{Z}$ such that

$$iq - jp = \frac{p\Delta h}{n} \in \mathbb{Z} \quad (\mu^+), \qquad iq - jp = -\frac{q\Delta h}{n} \in \mathbb{Z} \quad (\mu^-), \tag{2.30}$$

possible since $p$ and $q$ are coprime. Then the sequences become (again identical for all $k$):

$$\left.\begin{array}{l} 0 \longrightarrow \mathcal{F}^{(+)}_{(kp+i+n,kq+j)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^{(+)}_{(kp+i-n,kq+j)} \longrightarrow 0 \\ 0 \longrightarrow \mathcal{F}^{(-)}_{(kp-i-n,kq-j)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^{(-)}_{(kp-i+n,kq-j)} \longrightarrow 0 \end{array}\right\} \mu^+ \tag{2.31}$$

$$\left.\begin{array}{l} 0 \longrightarrow \mathcal{F}^{(+)}_{(kp+i,kq+j+n)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^{(+)}_{(kp+i,kq+j-n)} \longrightarrow 0 \\ 0 \longrightarrow \mathcal{F}^{(-)}_{(kp-i,kq-j-n)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^{(-)}_{(kp-i,kq-j+n)} \longrightarrow 0 \end{array}\right\} \mu^-, \tag{2.32}$$

though we note that the particular $(i, j)$ need not be the same across the two cases $\mu^{\pm}$. These modules exhibit the same symmetric rectangular distribution around particular points in the extended $(p, q)$ table, but not around the "principal diagonal" corner entries $(kp, kq)$. Instead they are located symmetrically around the $(kp + i, kq)$ and $(kp, kq + j)$ entries (for $\mu^+$ and $\mu^-$ respectively). A cursory examination of the relations (2.30) shows that, necessarily, $p|i$ ($\mu^+$) and $q|j$ ($\mu^-$). Therefore these more general sequences appear symmetrically around *other* corner entries, at $(r, s) = (k_1 p, k_2 q)$ with $k_1 \neq k_2$. The other properties mentioned (direction of mappings in the table, relative locations of left and right modules, etc.) remain the same.

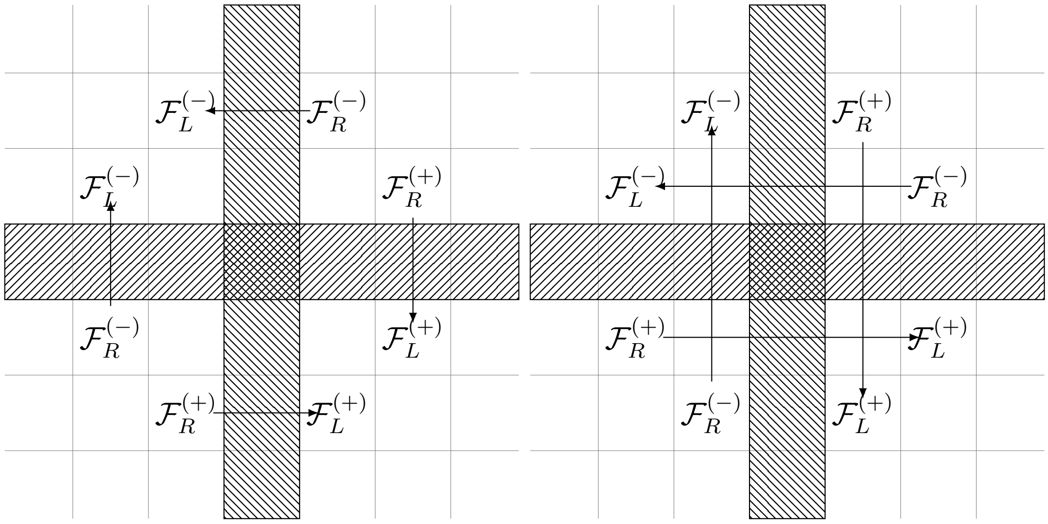


Figure 2.2: Example locations of left and right modules within an arbitrary extended Kac table. Shaded regions indicate edge entries; their intersections are corner entries $(r, s) = (k_1 p, k_2 q)$ — although we have principal diagonal corner entries ($k_1 = k_2$) in the particular examples shown here. We have chosen to show $(n, \Delta h) = (1, -2)$ (left-hand diagram) and $(n, \Delta h) = (2, -2)$ (right-hand diagram). Horizontal arrows correspond to $\mu^-$, vertical ones to $\mu^+$. When $(p, q)$ are large enough compared to $(n, \Delta h)$, $\mathcal{F}_L$ and $\mathcal{F}_R$ will belong to the bulk (implicitly shown here), although in particular instances $\mathcal{F}_L$ and $\mathcal{F}_R$ may themselves be edge or even corner entries.

It is worth noting that some authors (e.g. [35]) have considered "shifted" extended Kac tables, designed to contain entries at fractional $(r, s)$. Such modules were included in order to resolve some difficulties arising from the computation of fusion products. It is not clear (and remains to be studied) whether or not permitting fractional $(r, s)$ in this construction corresponds also to these staggered modules — since the condition (1.88) is not met, intertwiners do not arise for these modules, at least in the standard way of zero modes of screening fields.

**Calculation of $\beta$**

Recall that the parameter $\beta$ determines the isomorphism class of a rank-2 staggered modules. It is presented in abstract terms in (2.8), but here we derive an expression relevant to the current context. This has the benefit, once the calculation is carried to its full extent, of being an explicit formula for $\beta$.

$\beta$ is calculated via the inner product of two particular vectors in the staggered module. Given staggered module $\mathcal{F}_L \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}_R$ with vacua $|\eta_L\rangle, |\eta_R\rangle$ and intertwiner $V_0 : \mathcal{F}_R \to \mathcal{F}_L$, we had

$$\beta = \langle \eta_1 | U^\dagger_{\Delta h} |\eta_R\rangle \tag{2.33}$$

where $U_{\Delta h} \in \mathcal{U}(\mathfrak{Vir})$ is a creation operator such that

$$U_{\Delta h} |\eta_L\rangle = (L_0 - h_R) |\eta_R\rangle = V_0 |\eta_R\rangle \tag{2.34}$$

when evaluated in the full staggered module $\mathcal{M}$ (recall that $V_0$ is the staggering operator associated with $L_0$; we have $L_0 |\eta_R\rangle = h_2 |\eta_R\rangle + V_0 |\eta_R\rangle$ in $\mathcal{M}$). Since $V_0 |\eta_R\rangle$ is guaranteed to be singular in $\mathcal{F}_L$ by the nature of $V_0$ as an intertwiner, the only way (2.34) can fail to hold is if this vector occurs at too deep a conformal weight in $\mathcal{F}_L$ for there to be a path to it from the vacuum through the submodule diagram (c.f. Figure 1.5) — that is, there exists no such $U_{\Delta h}$. However, it is not hard to see that there will exist *some* (sub)singular vector of which it is a descendant[5], so this problem is an indication that $\mathcal{F}_L$ is in some sense the wrong choice of image module for $V_0$, and that the "correct" choice for the construction of a staggered module is some subspace or quotient of $\mathcal{F}_L$. If such a $U_{\Delta h}$ does not exist, then $\beta$ cannot be defined, so in order to progress we must assume that the necessary module restriction, if possible, has been made.

We make use of two different expressions for the vector in (2.34), in terms of the basis of generators of $\mathfrak{Vir}$ and of $\mathfrak{a}$. Let it have the following expansions:

$$\begin{aligned} U_{\Delta h} |\eta_L\rangle &= \sum_{|\tau| + \Delta h = 0} A_{(\tau)} L_{-(\tau)} |\eta_L\rangle \\ &= \sum_{|\upsilon| + \Delta h = 0} B_{(\upsilon)} a_{-(\upsilon)} |\eta_L\rangle \end{aligned} \tag{2.35}$$

for some (not necessarily nonzero) coefficients $A_{(\tau)}, B_{(\upsilon)}$, and where $\tau, \upsilon$ denote integer partitions which label the modes appearing in the monomials of creation operators which produce the vector. Notationally, $L_{-(\tau)}$ is to be understood as

$$L_{-(\tau)} = L_{-\tau_1} \cdots L_{-\tau_\ell} \tag{2.36}$$

for $\tau = (\tau_1, \ldots, \tau_\ell)$, *mutatis mutandis* for monomials in the generators of $\mathfrak{a}$ labelled by their respective partitions, following our previously chosen convention

[5] The only instance where this fails to hold is in the corner case, where there are no arrows in the module diagrams at all. $\beta$ does not exist (and cannot be defined) for indecomposable structures built from such modules, unless in fact $V_0 |\eta_R\rangle = |\eta_L\rangle$, in which case $\beta = 1$.

for ordering of basis monomials: indices non-decreasing read left to right. This is of course so that we may legitimately use partitions as labels.

Now we compute $U^\dagger_{\Delta h} |\eta_R\rangle$ by firstly calculating that — since $\tau_1, \dots \tau_\ell > 0$ — for each monomial, in the full staggered module, we have

$$\begin{aligned} L_{\tau_\ell} \cdots L_{\tau_1} |\eta_R\rangle &= (-\tau_1)(-\tau_1 - \tau_2) \cdots (-\tau_1 - \cdots - \tau_{\ell-1}) V_{\sum_i \tau_i} |\eta_R\rangle \\ &= \frac{\overline{C}_\tau}{\Delta h} [a_{\sum_i \tau_i}, V_0] |\eta_R\rangle \end{aligned} \tag{2.37}$$

where $\overline{C}_\tau$ is a combinatorial factor produced by iteratively commuting $L_n$ through the staggering operators;

$$\overline{C}_\tau = (-\tau_1 - \cdots - \tau_\ell)(-\tau_1 - \cdots - \tau_{\ell-1}) \cdots (-\tau_1), \tag{2.38}$$

a kind of "rising factorial" of the parts of the partition $\tau$, but with alternating sign. The staggering operator $V_{\sum_i \tau_i}$ appears because after the first annihilator mode $L_{\tau_1}$ acts upon $|\eta_R\rangle$, we have remaining only $V_{\tau_1} |\eta_R\rangle \in \iota(\mathcal{F}_L) \subset \mathcal{M}$, which as a rank-1 vector is thereafter acted on by the other modes without any staggered action. We only need to commute them through the staggering operator $V_{\tau_1}$ which, since

$$[L_m, V_n] = -n V_{m+n}, \tag{2.39}$$

is the reason for both the final index $\sum_i \tau_i$ and the combinatorial factor $\overline{C}_\tau$. The second equality in (2.37) follows from our definition of the staggering operators themselves, which are obtained from the intertwiner $V_0$ through taking the commutator with the appropriately-graded generator of $\mathfrak{a}$.

Now we may calculate

$$\begin{aligned} (U_{\Delta h})^\dagger |\eta_R\rangle &= \frac{1}{\Delta h} \left( \sum_{|\tau| + \Delta h = 0} \overline{C}_\tau \overline{A}_\tau \right) [a_{-\Delta h}, V_0] |\eta_R\rangle \\ &= \frac{1}{\Delta h} \left( \sum_{|\tau| + \Delta h = 0} \overline{C}_\tau \overline{A}_\tau \right) [a_{-\Delta h}, B_{(-\Delta h^1)} a_{\Delta h} + (\cdots)] |\eta_L\rangle \\ &= -B_{(-\Delta h^1)} \left( \sum_{|\tau| + \Delta h = 0} \overline{C}_\tau \overline{A}_\tau \right) |\eta_L\rangle \,, \end{aligned} \tag{2.40}$$

where the second equality follows because $[a_{-\Delta h}, V_0] |\eta_R\rangle = a_{-\Delta h} V_0 |\eta_R\rangle$ (recall that $\Delta h < 0$ for the staggered modules under consideration,[6] so that $a_{-\Delta h} |\eta_L\rangle = 0$), and the third from our assumed expansion of $V_0 |\eta_R\rangle = U_{\Delta h} |\eta_L\rangle$ in terms of oscillator modes.

// **Remark:**

[6] Because $\beta$ is undefined otherwise.

Since $U_{\Delta h}\,|\eta_L\rangle$ is by assumption the first proper singular vector of $\mathcal{F}_L$, it can be shown that the coefficient $B_{(-\Delta h^1)}$ is non-vanishing, a valid concern here.

Firstly let $\kappa : \mathcal{F}^{\mathfrak{a}}_\eta \to \mathbb{N}$ be the function

$$\kappa(v) = \max\left\{k \in \mathbb{N} \mid a_k v \neq 0\right\} \qquad \forall v \in \mathcal{F}^{\mathfrak{a}}_\eta. \tag{2.41}$$

Notice that $\kappa(v) = 0$ if and only if $v \propto |\eta\rangle$, with $\kappa(v) > 0$ otherwise.

Furthermore, if $v$ is singular with respect to the $\mathfrak{Vir}$ action, then it is trivial to prove that

$$a_{\kappa(v)} v \tag{2.42}$$

is also $\mathfrak{Vir}$-singular (and nonzero by construction, of course). With this in mind, if $v$ is the first proper singular vector in the grading, the only possibility is to have

$$a_{\kappa(v)} v \propto |\eta\rangle\,, \tag{2.43}$$

so $\kappa(v)$ coincides with the grade of $v$, and this forces $v$ to have a non-vanishing coefficient on the singleton term $a_{-\kappa(v)}$ in its expansion in terms of the generators of $\mathfrak{a}$. This applies to our current case, as noted. //

At this point we can already state that

$$\beta = -\langle\eta_L|\, B_{(-\Delta h^1)} \left(\sum_{|\tau|+\Delta h=0} \overline{C}_\tau \overline{A}_\tau\right) |\eta_L\rangle = -B_{(-\Delta h^1)} \left(\sum_{|\tau|+\Delta h=0} \overline{C}_\tau \overline{A}_\tau\right), \tag{2.44}$$

so can see that using our construction method, $\beta$ depends only on the nature of the first proper singular vector of the image space of the intertwiner. As it is, this expression contains both the $A_\tau$ and $B_\upsilon$, requiring expansions of this vector both in the generators of $\mathfrak{Vir}$ and of $\mathfrak{a}$. We can in fact do better, re-expressing $B_{(-\Delta h^1)}$ in terms of the $A_\tau$.

While computing expressions for the $A_\tau$ and $B_\upsilon$ in terms of each other is generically a computationally intensive task, this is made tractable by simplifications which arise in the particular case of $B_{(-\Delta h^1)}$ due to the fact that this coefficient multiplies the singleton monomial $a_{\Delta h}$. We proceed iteratively, one creation element of $\mathfrak{Vir}$ at a time, commuting successive $L$s through the accumulated $a$s to act on the vacuum. Terms containing products of more than one element of $\mathfrak{a}$, once created, may be discarded, because such terms may never contribute thereafter to the final coefficient on the singleton.

With this in mind, observe that

$$\begin{aligned} L_{-\tau_1}\cdots L_{-\tau_\ell}\,|\eta\rangle = (\tau_\ell)(\tau_\ell+\tau_{k-1})\cdots(\tau_\ell+\cdots+\tau_2)\left[\eta_L + (\tau_\ell - 1)\lambda\right] a_{-\sum_i \tau_i}\,|\eta\rangle \\ + (\text{products of multiple } a\text{s})\,|\eta\rangle\,, \end{aligned} \tag{2.45}$$

therefore

$$\begin{aligned} U_{\Delta h} \left| \eta_L \right\rangle = - \left( \sum_{|\tau| + \Delta h = 0} \frac{1}{\Delta h} \left[ \eta_L + (\tau_k - 1) \lambda \right] C_\tau A_\tau \right) a_{\Delta h} \left| \eta_L \right\rangle \\ + \left( \text{products of multiple } a\text{s} \right) \left| \eta \right\rangle , \end{aligned} \tag{2.46}$$

where $C_\tau$ is the combinatorial factor related to the partition $\tau$,

$$C_\tau = (\tau_\ell)(\tau_\ell + \tau_{k-1}) \cdots (\tau_\ell + \cdots + \tau_2)(\tau_\ell + \cdots + \tau_1), \tag{2.47}$$

which, like $\overline{C}_\tau$, is a kind of factorial of the parts of $\tau$. We have now identified $B_{(-\Delta h^1)}$, so obtain the main result for this section:

**2.2.1 Theorem.** *We have the following expression for the $\beta$ invariant of a staggered Fock module of the type considered above:*

$$\beta = \frac{1}{\Delta h} \left( \sum_{|\tau| + \Delta h = 0} \left[ \eta_L + (\tau_k - 1) \lambda \right] C_\tau A_\tau \right) \left( \sum_{|\tau| + \Delta h = 0} \overline{C}_\tau \overline{A}_\tau \right) \tag{2.48}$$

This expression, quadratic in the coefficients of $U_{\Delta h} \left| \eta_L \right\rangle$, almost has the form of an inner product of the creation operator $U_{\Delta h}$ with itself. These coefficients are themselves fixed by the requirement that $U_{\Delta h} \left| \eta_L \right\rangle$ be singular, and as such may be determined in terms of the parameters $\eta_L$ and $\lambda$ — in fact, these coefficients are bound simply by the commutation relations of $\mathfrak{Vir}$ to be rational linear combinations of polynomials of these parameters over the rationals.

As discussed, the staggered structure is completely determined by the choice of the tuple $(\Delta h, n, \mu)$. $\Delta h$ alone sets the degree of the singular vector (and hence the number and labels of the coefficients $A_\tau$), while $n$ and $\mu$ are actually relatable to the data $\eta_L, \eta_R, \lambda$ of the two modules in the short exact sequence. $\beta$ may therefore be computed ahead of time, once for each $\Delta h$, with $n$ and $\mu$ left as variables. We find, generically,

$$\beta = \frac{1}{\Delta h n \mu^2} \left( \sum_{|\tau| + \Delta h = 0} \left[ \frac{1}{2} (n + \tau_k) \mu^2 - \tau_k + \frac{\Delta h}{n} \right] C_\tau A_\tau \right) \left( \sum_{|\tau| + \Delta h = 0} \overline{C}_\tau \overline{A}_\tau \right), \tag{2.49}$$

the first few of these being

$$\begin{aligned}
\beta(\mu,n)|_{\Delta h=-1} &= \frac{n+1}{2n^2}\mu^{-2}(n\mu^2-2) \\
\beta(\mu,n)|_{\Delta h=-2} &= \frac{(n+1)(n+2)}{36n^4}\mu^{-2}\left(n^2(n^2-1)\mu^4+6n^2\mu^2+16-4n^2\right) \\
&\times\left(n(n-1)\mu^2+2n-4\right)(n\mu^2-2)(n\mu^2-4) \\
\beta(\mu,n)|_{\Delta h=-3} &= \frac{3}{2048n^{10}}\mu^{-10}(n(n+1)\mu^2+2n+6)(n(n+1)\mu^2-2n+6) \\
&\times(n(n+1)\mu^2-2n-6)^2(n(n-1)\mu^2+2n+6) \\
&\times(n(n-1)\mu^2+2n-6)^2(n(n-1)\mu^2-2n+6)(n^2\mu^2-6).
\end{aligned} \tag{2.50}$$

These expressions quickly become complicated[7]. For reference, we tabulate specific choices of both $\Delta h$ and $n$ in Figure 2.5, and choose to present these as functions of $\mu$ alone, since all of $\eta_L$, $\eta_R$, and $\lambda$ can be written in terms of this parameter, given values of $\Delta h$ and $n$.

These formulae hold for generic $\mu$, but are of most interest when $\mu=\mu^{\pm}$ of a reducible $(p,q)$ theory. We note that when we substitute particular $(p,q)$ into these equations, we reproduce the results of various authors (Figure 2.6). This suggests that modes of vertex operators are the "correct" staggering operators for free field realisations of staggered Virasoro modules, in as far as this construction agrees with staggered structures arising independently, and that Fock spaces provide effective means of computation within them. Particular values of interest from the literature are $(n,\Delta h)=(2,-2)$ and $(1,-2)$ because at $(p,q)=(2,3)$ the former should correspond to critical percolation with $\beta=\frac{-5}{8}$ and the latter to dilute polymers with $\beta=\frac{5}{6}$. Accounting for normalisation (a factor of $p^{-2}q^2$ for $(n,\Delta h)=(2,-2)$ and $p^2q^{-2}$ for $(1,-2)$), we find this to be the case.

Some authors derive or suggest rules for general formulae for $\beta$ in terms of $p$ and $q$ for particular types of staggered module (e.g. the $LM(2,q)$ modules of [35]). These are special cases of the formula (2.49). In some cases, these authors have commented on the surprisingly neat way in which $\beta(t)$ splits into linear factors, where $t$ parametrises the central charge $c$ as

$$c=13-6(t+\frac{1}{t}). \tag{2.51}$$

Clearly $t=\mu^2$. We can immediately deduce that only even powers of $\mu$ may appear, as not only is every explicit appearance of $\mu$ in (2.49) of the form $\mu^{2n}$, but also every implicit appearance via the $A_\tau$: they involve only integral powers of $h_L$ and $c$, with rational coefficients, themselves quadratic in $\mu^2$.

[7]Recall that $\beta$ is only determined up to a scale factor. In these examples we have chosen to scale so that the coefficient on the $L_{-1}^{-\Delta h}$, that is $A_{(1)^{-\Delta h}}$, is set to 1, though other authors (such as [7], [23], or [45], whose results are compared to (2.49) in Figure 2.6) have in the past taken other conventions. Such differences have been noted in calculations wherever appropriate.

The fact that $\beta$ factorises so neatly may be related to the structure of the $(p,q)$ extended Kac tables. Depending on the values of $p$ and $q$, the entries related via staggered module with some fixed $(n, \Delta h)$ may belong to the bulk, to the edges, or even to the corners. Of course, corner Fock modules have *no* submodule embedding structure, and are instead completely reducible (cf. Figure 1.5). This means that $\beta$ is undefined, since there is no operator $U_{\Delta h}$ which creates the first proper singular vector. Of course, since the formulae for $\beta$ are just polynomials, we can still evaluate them at these points.

Since $\frac{\Delta h}{n} \in \mathbb{Z}$, this "corner collision" occurs whenever $n$ is a multiple of $p$ and $\frac{\Delta h}{n}$ a multiple of $q$, or vice versa, according to $\mu = \mu^+$ or $\mu^-$ respectively. Therefore, $\beta(\mu)$ should be undefined at the collection of points

$$\prod_{p|n,\ q|\frac{\Delta h}{n},\ \gcd(p,q)=1} \left(\mu - \sqrt{\frac{2q}{p}}\right), \qquad \prod_{q|n,\ p|\frac{\Delta h}{n},\ \gcd(p,q)=1} \left(\mu + \sqrt{\frac{2p}{q}}\right)$$

from these two contributions. By noting symmetry, we see that this is actually a net factor of

$$\prod_{p|n,\ q|\frac{\Delta h}{n},\ \gcd(p,q)=1} \left(\mu^2 - \frac{2q}{p}\right)$$

When applied to the formulae seen in Figure 2.5, we see an interesting empirical relationship between these special values and the zeroes of $\beta$, in that every such degenerate point corresponds to a zero of $\beta$ treated as a function of $\mu^2$ — but not all zeroes of $\beta$ appear to be predictable in this way. As well as missing some of the type $(\mu^2 - a)$, those of the form $(\mu^2 + a)$ are entirely absent, which we note *cannot* come from this kind of consideration of the extended $(p,q)$ Kac tables with $p, q > 0$. This perhaps hints that the $\beta$ invariant "knows about" other staggered Fock modules arising from generic integers $(p,q)$. The regime where factors $(\mu^2 + a)$ could be produced corresponds to exactly one of these integers being negative, and hence $c > 1$. This regime is unexplored in this work, and the existence of staggered structures outside the commonly-studied extended Kac tables is the subject of future efforts.

### 2.2.2 Fermionic ($N > 0$) Staggered Fock Super-Modules

Following the example of the $N = 0$ case, we now turn to staggered Fock super-modules, which are modules $\mathcal{M}$ which fit into the short exact sequence

$$0 \longrightarrow \mathcal{S}_1\mathcal{F}^{\sigma}_{\eta_1,\updownarrow_1} \xrightarrow{\iota} \mathcal{M} \xrightarrow{\pi} \mathcal{S}_1\mathcal{F}^{\sigma}_{\eta_2,\updownarrow_2} \longrightarrow 0 \tag{2.52}$$

for the $N = 1$ case, and *mutatis mutandis* for $N = 2$. Initially we focus on $N = 1$.

### $N = 1$ Construction

We specify the construction by providing a family of staggering operators, as in (2.13). In the specific case of $\mathfrak{s}_1\mathfrak{Vir}$, this means we require a family of operators

$$\{V_n \mid n \in \mathbb{Z}\} \cup \{W_n \mid n \in \sigma + \mathbb{Z}\} \tag{2.53}$$

such that

$$\begin{aligned} [L_m, V_n] + [V_m, L_n] &= (m-n)V_{m+n} \\ [L_m, W_n] + [V_m, G_n] &= (\frac{1}{2}m - n)W_{m+n} \\ \{G_m, W_n\} + \{W_m, G_n\} &= 2V_{m+n}, \end{aligned} \tag{2.54}$$

where $V_n$ is the operator associated to the indecomposable action of $L_n$, and $W_n$ to that of $G_n$. It seems that having access to a single intertwining operator may not be sufficient, but in fact, we may proceed nearly completely analogously to the construction made for $N = 0$. Given such an intertwining map $V_0 : \mathcal{S}_1\mathcal{F}^{\sigma}_{\eta_2,\updownarrow_2} \to \mathcal{S}_1\mathcal{F}^{\sigma}_{\eta_1,\updownarrow_1}$, let us define

$$\begin{aligned} V_n &:= [a_n, V_0] \\ W_n &:= [b_n, V_0] \end{aligned} \tag{2.55}$$

where, as $V_0$ is to be associated to $L_0$ and is hence an even operator, these are genuine commutators (i.e., not anticommutators in the case of the $W_n$). Through the use of the (super)Jacobi identity, we can easily verify that these operators as defined satisfy (2.54), and hence comprise a valid family of staggering operators.

### Admissible $N = 1$ Fock Superspaces

Since the literature (e.g. [5, 6, 38]) focuses on rank-2 staggered modules where the left and right spaces are reducible, as with the purely bosonic case, we now attempt to locate positions in any given extended Kac table which can be related by an intertwiner of the required type. Given (1.122), solving $\eta_R = \eta^{\pm}_{r,s}$ for $r, s$ by imposing $\eta_L = \eta_R + n\mu$ for a fixed $\Delta h$ reveals four different equations relating $r$ and $s$, depending on the choice of $\eta^{\pm}$ and $\mu^{\pm}$ (Figure 2.3). When they take integral values, subject to the sector-dependent parity constraints, they correspond to entries in the extended Kac table which can be related by a non-trivial intertwiner. One can check that, in fact, while the function $s(r)$ takes a different form depending on the choice of $\eta^{\pm}$, the resulting module data $(c, h)$ does not, so while valid solutions for $(r, s)$ will differ in the two cases (i.e., will correspond to different locations in the extended Kac table), the resulting staggered structures will be identical as $\mathfrak{s}_1\mathfrak{Vir}$ modules.

Recall that a reduced rational linear relation $y(x) = \frac{a}{b}x + \frac{c}{d}$ has integral solutions if and only if the denominator of the intercept divides the denominator

| $s(r)$ | $\eta^+_{r,s}$ | $\eta^-_{r,s}$ |
|---|---|---|
| $\mu^+$ | $\frac{q}{p}r + (2n-1)\frac{q}{p} - (\frac{\Delta h}{n} + \frac{1}{2})$ | $\frac{q}{p}r - (2n-1)\frac{q}{p} + (\frac{\Delta h}{n} + \frac{1}{2})$ |
| $\mu^-$ | $\frac{q}{p}r - (2n-1) + (\frac{\Delta h}{n} + \frac{1}{2})\frac{q}{p}$ | $\frac{q}{p}r + (2n-1) - (\frac{\Delta h}{n} + \frac{1}{2})\frac{q}{p}$ |

Figure 2.3: $s$ as a function of $r$ where $(r, s)$ denotes an entry in a (fixed but arbitrary) extended $(p, q)$ Kac table

of the slope; that is if and only if $d|b$. If it possesses one integral solution $(x_0, y_0)$, it necessarily possesses infinitely many of the form $(x_0 + kb, y_0 + ka)$, $k \in \mathbb{Z}$. By examining the relationships in Figure 2.3, we find conditions on $n$ and $\Delta h$ for the possible existence of integral solutions (and hence the potential to relate two entries in the table through intertwiners). We do so by first noting that $\frac{q}{p}$ is either already reduced, or that a greatest common divisor of 2 may be removed, corresponding to $p \equiv q \equiv 1$ or $0 \mod 2$ respectively. In either case, we write $\bar{p}, \bar{q}$ for the reduced forms of these numbers. Then we note that it is impossible to satisfy the aforementioned divisibility constraint $(d|b)$ unless

$$\mu = \mu^+ : \quad 2n|\bar{p}(2\Delta h + n), \qquad \mu = \mu^- : \quad 2n|\bar{q}(2\Delta h + n), \tag{2.56}$$

or in other words, either $\bar{p}\left(\frac{\Delta h}{n} + \frac{1}{2}\right)$ or $\bar{q}\left(\frac{\Delta h}{n} + \frac{1}{2}\right)$ is an integer. Like with the $N = 0$ case, we choose to first make the simplifying assumption that, in fact, $\left(\frac{\Delta h}{n} + \frac{1}{2}\right) \in \mathbb{Z}$ itself. Then we have short exact sequences

$$0 \longrightarrow \mathcal{S}_1\mathcal{F}^{(-)}_{(kp-2n-1,kq+\frac{\Delta h}{n}+\frac{1}{2})} \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}_1\mathcal{F}^{(-)}_{(kp+2n-1,kq+\frac{\Delta h}{n}+\frac{1}{2})} \longrightarrow 0 \qquad (\mu^+) \tag{2.57}$$

and

$$0 \longrightarrow \mathcal{S}_1\mathcal{F}^{(-)}_{(k\bar{p}+\frac{\Delta h}{n}+\frac{1}{2},k\bar{q}-2n-1)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}_1\mathcal{F}^{(-)}_{(k\bar{p}+\frac{\Delta h}{n}+\frac{1}{2},k\bar{q}+2n-1)} \longrightarrow 0 \qquad (\mu^-) \tag{2.58}$$

with the isomorphism class of the staggered module independent of $k \in \mathbb{N}$, with the proviso that the indexed modules actually exist (the corresponding locations in the extended table must have non-negative index labels, for instance) and that an intertwiner actually exists between them. As before, we have made an abbreviation in writing $\mathcal{S}_1\mathcal{F}_{r,s}$ to indicate that $\eta = \eta_{r,s}$, while the superscript $(-)$ is used to indicate $\eta = \eta^-$. Given the multiple different possible relationships between the vacuum parities of the two modules, according to the fermionic parity of the particular intertwining map involved, we have also chosen to suppress these subscript labels.

As with the purely bosonic case, staggered modules $\mathcal{M}$ constructed using the positive root choice of the $a_0$ eigenvalue, $\eta^+$, give staggered modules isomorphic to those already generated, though at different $(r, s)$ in the extended table. Their sequences are:

$$0 \longrightarrow \mathcal{S}_1\mathcal{F}^{(+)}_{(k\bar{p}+2n+1,k\bar{q}-\frac{\Delta h}{n}-\frac{1}{2})} \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}_1\mathcal{F}^{(+)}_{(k\bar{p}-2n+1,k\bar{q}-\frac{\Delta h}{n}-\frac{1}{2})} \longrightarrow 0 \qquad (\mu^+) \tag{2.59}$$

and

$$0 \longrightarrow \mathcal{S}_1\mathcal{F}^{(+)}_{(k\bar{p}-\frac{\Delta h}{n}-\frac{1}{2},k\bar{q}+2n+1)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}_1\mathcal{F}^{(+)}_{(k\bar{p}-\frac{\Delta h}{n}-\frac{1}{2},k\bar{q}-2n+1)} \longrightarrow 0 \qquad (\mu^-) \tag{2.60}$$

Again, all valid choices of $k$ give isomorphic $\mathcal{M}$.

These staggered modules are arranged in patterns around corner entries in the related extended Kac table in much the same way as the $N = 0$ case, except incrementing the value of $n$ steps these modules through the table in multiples of 4, not of 2 (consider the relative factors of $\sqrt{2}$ leading to the differences between (1.94) and (1.122)).

Now we relax the condition that $\left(\frac{\Delta h}{n} + \frac{1}{2}\right)$ itself be an integer for the more general construction. Solving the linear relations in Figure 2.3 is then tantamount to solving the Diophantine equations

$$\begin{aligned} \mu = \mu^+ : &\quad i\bar{q} - j\bar{p} = \bar{p}(\frac{\Delta h}{n} + \frac{1}{2}) \in \mathbb{Z}, \\ \mu = \mu^- : &\quad i\bar{q} - j\bar{p} = -\bar{q}(\frac{\Delta h}{n} + \frac{1}{2}) \in \mathbb{Z}, \end{aligned} \tag{2.61}$$

for some integers $i, j$ (note that these are not necessarily the same integers across the two cases $\mu = \mu^{\pm}$). Then we find the sequences

$$\left.\begin{array}{l} 0 \longrightarrow \mathcal{S}_1\mathcal{F}^{(+)}_{(k\bar{p}+i+2n+1,k\bar{q}+j)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}_1\mathcal{F}^{(+)}_{(k\bar{p}+i-2n+1,kq+j)} \longrightarrow 0 \\ 0 \longrightarrow \mathcal{S}_1\mathcal{F}^{(-)}_{(kp-i-2n-1,kq-j)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}_1\mathcal{F}^{(-)}_{(kp-i+2n-1,kq-j)} \longrightarrow 0 \end{array}\right\} \mu^+ \tag{2.62}$$

$$\left.\begin{array}{l} 0 \longrightarrow \mathcal{S}_1\mathcal{F}^{(+)}_{(kp+i,kq+j+2n+1)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}_1\mathcal{F}^{(+)}_{(kp+i,kq+j-2n+1)} \longrightarrow 0 \\ 0 \longrightarrow \mathcal{S}_1\mathcal{F}^{(-)}_{(kp-i,kq-j-2n-1)} \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}_1\mathcal{F}^{(-)}_{(kp-i,kq-j+2n-1)} \longrightarrow 0 \end{array}\right\} \mu^- \tag{2.63}$$

### Calculation of $\beta$

In order to find the $\beta$ invariant for $\mathcal{M}$, we must compute an appropriate inner product in addition to identifying an element $U_{\Delta h}$ of $\mathcal{U}\left(\mathfrak{s}_1\mathfrak{Vir}\right)$ for which $U_{\Delta h}\left|\eta + n\mu\right\rangle = V_0\left|\eta\right\rangle$ for the left and right modules seen in (2.52) — in which case, we may write

$$\beta = \left\langle \eta + n\mu \right| U^{\dagger}_{\Delta h} \left|\eta\right\rangle , \tag{2.64}$$

where we have also suppressed parity for the vacua, $\updownarrow$, because of the parity ambiguity of the operator $U_{\Delta h}$: if this operator is even, then the parities of the two vectors in the inner product must agree; $\beta$ otherwise evaluates to zero. Likewise if this operator is odd, they must disagree. Having a meaningful definition for $\beta$ requires us to choose the parities of the two vectors in the inner product on a case-by-case basis so as to give a non-zero result. In the NS sector, there is nothing more to do than to make an initial choice and follow it through as demanded by context (the fixed parity of the operator $V_0$, for instance), but in the R sector, vacuum vectors of both parities appear in the same module, so there are always two possible ways to define $\beta$. However, beginning from $|\eta,\uparrow\rangle \in \mathcal{S}_1\mathcal{F}_{\eta_2}$, we see that

$$|\eta,\downarrow\rangle = \sqrt{2}b_0 \,|\eta,\uparrow\rangle\,, \tag{2.65}$$

so that interchanging both $\langle\eta+n\mu|$ and $|\eta\rangle$ with their parity-flipped counterparts introduces a factor of $2b_0^2 = 1$, so the two methods of computing $\beta$ thankfully give identical results.

We shall compute $\beta$, as in the $N = 0$ case, by comparing the distinguished vector $U_{\Delta h}\,|\eta+n\mu\rangle$ via its two expansions: one in terms of the basis of generators of $\mathfrak{s}_1\mathfrak{Vir}$; the other in the oscillator basis. Firstly, suppose that the element $U_{\Delta h}$ of the universal enveloping algebra exists such that the desired property holds:

$$U_{\Delta h}\,|\eta+n\mu\rangle = V_0\,|\eta\rangle\,. \tag{2.66}$$

As with the $N = 0$ case, $V_0\,|\eta\rangle$ is guaranteed to be singular in $\mathcal{S}_1\mathcal{F}_{\eta+n\mu}$, so the only way this can fail to occur is if this vector occurs at too deep a conformal weight in $\mathcal{S}_1\mathcal{F}_{\eta+n\mu}$ for there to be a path to it from the vacuum through the embedding diagram. Again, there will then exist *some* (sub)singular vector of which it is a descendant[8], so this problem is an indication that $\mathcal{S}_1\mathcal{F}_{\eta+n\mu}$ is the wrong choice of image module, and that the "correct" choice for the construction of a staggered module of this type is some subspace or submodule restriction of $\mathcal{S}_1\mathcal{F}_{\eta+n\mu}$.

Let the vector (2.66) have the following expansions:

$$\begin{aligned} U_{\Delta h}\,|\eta+n\mu\rangle &= \sum_{|\xi|+|\nu|+\Delta h=0} A_{(\xi,\nu)} L_{-(\xi)} G_{-(\nu)}\,|\eta+n\mu\rangle \\ &= \sum_{|\rho|+|\tau|+\Delta h=0} B_{(\rho,\tau)} a_{-(\rho)} b_{-(\tau)}\,|\eta+N\mu\rangle \end{aligned} \tag{2.67}$$

for some (not necessarily non-zero) coefficients $A_{(\xi,\nu)}, B_{(\rho,\tau)}$, and where $\xi,\nu,\rho,\tau$ denote integral or half-integral (sector-dependent) non-negative, non-increasing partitions[9] which label the modes appearing in the monomials of creation operators which produce the vector. Notationally, $L_{-(\xi)}$ is to be understood as

$$L_{-(\xi)} = L_{-\xi_1}\cdots L_{-\xi_\ell} \tag{2.68}$$

[8]The only instance where this fails to hold is in the corner case, where there are no arrows in the embedding diagrams at all, so staggered modules are trivial, in the sense that either $\Delta h = 0$ (and so $\beta$ is not defined) or the staggered module decomposes as a direct sum whose staggered submodule(s) have $\Delta h = 0$.

[9]Of the partitions labelling fermionic modes, only those which are non-repeating (i.e., *strictly* decreasing) contribute meaningfully.

for $\xi = (\xi_1, \ldots, \xi_\ell)$, *mutatis mutandis* for monomials of other generators $G, a, b$ labelled by their respective partitions. Recall the chosen ordering for the basis of monomials of generators: $L$s appearing in the left of the monomial, $G$s on the right, completed by demanding that indices be non-decreasing within each of these two. This is of course so that we may legitimately use partitions as labels. Now we compute $U^\dagger_{\Delta h} |\eta\rangle$ by firstly calculating that, in the full staggered module,

$$G_{\nu_m} \cdots G_{\nu_1} L_{\xi_\ell} \cdots L_{\xi_1} |\eta\rangle = \overline{C}_{\xi,\nu} \times \begin{cases} \frac{1}{\Delta h} V_{-\Delta h} |\eta\rangle & \nu \text{ even} \\ W_{-\Delta h} |\eta\rangle & \nu \text{ odd} \end{cases} \tag{2.69}$$

where the parity of a partition is simply whether it is of odd or even length, and where $\overline{C}_{\xi,\nu}$ is a combinatorial factor relating to the partitions themselves;

$$\begin{aligned} \overline{C}_{\xi,\nu} &= (-\xi_1)(-\xi_1 - \xi_2) \cdots (-\xi_1 - \cdots - \xi_\ell) \\ &\quad \times (-\xi_1 - \cdots - \xi_\ell - \nu_1 - \nu_2) \\ &\quad \cdots \\ &\quad \times (-\xi_1 - \cdots - \xi_\ell - \nu_1 - \nu_2 - \cdots - \nu_{(2\lfloor m/2 \rfloor - 1)} - \nu_{2\lfloor m/2 \rfloor}) \\ &= (-1)^{\ell + \lfloor m/2 \rfloor} \prod_{i=1}^{\ell} \left( \sum_{j=1}^{i} \xi_j \right) \prod_{i=1}^{\lfloor m/2 \rfloor} \left( \sum_{k=1}^{\ell} \xi_k + \sum_{j=1}^{2i} \nu_j \right) \end{aligned} \tag{2.70}$$

where we treat empty products as factors of unity (relevant when $\ell = 0$ or $m = 0, 1$). Then we see that

$$U^\dagger_{\Delta h} |\eta\rangle = \sum_{|\xi| + |\nu| + \Delta h = 0} \left( \overline{C}_{\xi,\nu} \overline{A}_{(\xi,\nu)} \times \begin{cases} \frac{1}{\Delta h} [a_{-\Delta h}, V_0] |\eta\rangle & \nu \text{ even} \\ [b_{-\Delta h}, V_0] |\eta\rangle & \nu \text{ odd.} \end{cases} \right) \tag{2.71}$$

Note that this analysis does not yet depend on the choice of sector. Now, the left entries of each of the commutators in (2.71) annihilate the vacuum $|\eta\rangle$, and $V_0 |\eta\rangle$ has a known expansion in terms of oscillators since it equals $U_{\Delta h} |\eta + N\mu\rangle$ by assumption. Since this vector is homogeneous of degree $\Delta h$, only the singleton monomials $a_{\Delta h}$ and $b_{\Delta h}$ survive this commutation (possibly including factors of $b_0$ in the R sector), though in the NS sector contributions from both bosonic and fermionic singleton monomials are impossible since there is an index mismatch.

We conclude, in the first case (c.f. (2.67)),

$$\frac{1}{\Delta h} [a_{-\Delta h}, V_0] |\eta\rangle = -\left( B_{((-\Delta h),\emptyset)} + B_{((-\Delta h),(0))} b_0 \right) |\eta + n\mu\rangle \tag{2.72}$$

and in the second

$$[b_{-\Delta h}, V_0] |\eta\rangle = \left( B_{(\emptyset,(-\Delta h))} - B_{(\emptyset,(-\Delta h,0))} b_0 \right) |\eta + n\mu\rangle \,. \tag{2.73}$$

These relations display the (mutually exclusive) multiple cases simultaneously — at most two of the four coefficients appearing could contribute to any one particular $\beta$: obviously in the R sector we may distinguish singular vectors by parity,[10] so that either the two first or the two second terms in each of (2.72) and (2.73) can appear, and in the NS sector, not only is the mode $b_0$ non-existent (so its coefficients obviously cannot contribute), but we have either $\Delta h \in \mathbb{Z}$ or $\Delta h \in \frac{1}{2} + \mathbb{Z}$, so either $B_{((-\Delta h),\emptyset)}$ or $B_{(\emptyset,(-\Delta h))}$ is zero. One may carefully check the circumstances which determine the parity of $U_{\Delta h}$ to further constrain these coefficients in the R sector. There are two cases:

- $n$ **even.** In this case, $U_{\Delta h}$ creates an even number of fermions. Only even-length partitions of fermions appear in the oscillator basis expansion, so $[a_{-\Delta h}, V_0]$ is the relevant operator. Since the basis monomial $a_{\Delta h} b_0$ is odd, its coefficient $B_{((-\Delta h),(0))}$ must be zero, and therefore only $B_{((-\Delta h),\emptyset)}$ appears. Since $V_0$ and therefore $[a_{\Delta h}, V_0]$ is even overall, the vacuum parities of the two modules agree.

- $n$ **odd.** Conversely, only odd-length partitions of fermions appear in $U_{\Delta h}$, so $[b_{-\Delta h}, V_0]$ is the relevant operator. Since the basis monomial $b_{\Delta h} b_0$ is even, its coefficient $B_{(\emptyset,(-\Delta h,0))}$ must be zero, and therefore only $B_{(\emptyset,(-\Delta h))}$ appears. $V_0$ is even overall, so $[b_{-\Delta h}, V_0]$ is odd, although due to its construction must create an even number of fermionic modes. The vacuum parities therefore differ.

Hence we are able to ignore coefficients on terms involving $b_0$ entirely and can write

$$\begin{aligned}\beta &= \langle \eta + n\mu | \, U^\dagger_{\Delta h} \, |\eta\rangle \\ &= \left( \sum_{|\xi|+|\nu|+\Delta h = 0} \overline{C}_{\xi,\nu} \overline{A}_{(\xi,\nu)} \right) \times \begin{cases} -B_{((-\Delta h),\emptyset)} & n \text{ even} \\ B_{(\emptyset,(-\Delta h))} & n \text{ odd.} \end{cases}\end{aligned} \tag{2.74}$$

Repeating the calculations which led to (2.69) with monomials of creation instead of annihilation operators, we are able to find explicit values for $B_{(-\Delta h),\emptyset}$ and $B_{\emptyset,(-\Delta h)}$ in terms of the coefficients $A$ appearing in (2.67) together with some combinatorial factors. Due to the commutation relations between the basis elements of $\mathfrak{s}_1\mathfrak{Vir}$ and the $a$s and $b$s, the full expansion of a general monomial of $L$s and $G$s in terms of this oscillator basis can become very complicated. Luckily in order to compute the desired coefficients manually, acting each mode in the monomial one after another, we only need to retain first-order contributions, as objects of order $O(a^2)$, $O(ab)$, $O(b^2)$ or greater, once produced, cannot contribute to the top-level monomials $a_{\Delta h}$ and $b_{\Delta h}$. We find that

$$\begin{aligned}L_{-(\xi)} G_{-(\nu)} |\eta\rangle &= L_{\xi_1} \cdots L_{-\xi_\ell} G_{-\nu_1} \cdots G_{-\nu_m} |\eta\rangle \\ &= C_{-\xi,-\nu} \times \begin{cases} \frac{-1}{\Delta h} a_{\Delta h} |\eta\rangle & n \text{ even} \\ b_{\Delta h} |\eta\rangle & n \text{ odd} \end{cases} + (\text{higher terms})\end{aligned} \tag{2.75}$$

[10] Consider that for $U \in \mathcal{U}(\mathfrak{s}_1\mathfrak{Vir})$ of fixed parity, $U|\eta\rangle$ is singular if and only if $Ub_0|\eta\rangle$ is also singular.

where again $C_{-\xi,-\nu}$ is a combinatorial factor similar to that appearing in (2.70);

$$\begin{aligned} C_{-\xi,-\nu} = &\left(\eta + 2\lambda(x - \frac{1}{2})\right) \prod_{i=1}^{\lfloor m/2 \rfloor} \left( \sum_{j=1}^{i} \nu_{m+1-j} \right) \\ &\times \prod_{i=1}^{\ell} \left( \sum_{k=1}^{m} \nu_k - \frac{1}{4}\left(1 - (-1)^m\right) \xi_{\ell+1-i} + \sum_{j=1}^{i} \xi_{\ell+1-j} \right) \end{aligned} \tag{2.76}$$

where $x = \nu_m$ only if $\nu$ is not the empty partition. Otherwise, $x = \frac{1}{2}\xi_\ell$. Note that, unlike in (2.70), the parts of the partitions appear in reverse order here, and without the alternating factor of $-1$. Another chief difference is the presence of the term $\frac{1}{4}\left(1 - (-1)^m\right) \xi_{\ell+1-i}$, which is a compact way of notating the different forms of commutators $[L_m, a_n] = -na_{m+n}$ and $[L_m, b_n] = -\left(\frac{1}{2}m + n\right) b_{m+n}$. The appearance of one or the other is determined by the parity of the length $m$ of the partition $\nu$.

Combining this all together, we find

**2.2.2 Theorem.** *For the staggered $N = 1$ super-Fock modules of $\mathfrak{s}_1\mathfrak{Vir}$ of the type considered above, we have the following expression for $\beta$:*

$$\beta = (\Delta h)^{-\frac{1}{2}(1+(-1)^n)} \left( \sum_{|\xi|+|\nu|+\Delta h=0} \overline{C}_{\xi,\nu} \overline{A}_{(\xi,\nu)} \right) \left( \sum_{|\xi|+|\nu|+\Delta h=0} C_{-\xi,-\nu} A_{(\xi,\nu)} \right). \tag{2.77}$$

This formula is entirely general for all staggered modules constructed using this method. It looks something like a modified inner product of the vector $U_{\Delta h} |\eta + n\mu\rangle$. Interestingly, it does not explicitly depend in upon $V_0$ or any of the other staggering operators in any way (the dependence upon these operators is implicit throughout the derivation of this expression via the assumed forms of the singular vector and through setting $V_n = [a_n, V_0]$).

The coefficients $\overline{C}_{\xi,\nu}$ and $C_{-\xi,-\nu}$, being purely combinatorial, do not depend upon the particulars of the staggered module. Their values for low-lying $\Delta h$ have been tabulated in Figure 2.4. The coefficients $A_{(\xi,\nu)}$ do depend upon the module, but only in that the vector they define must be singular in the left module of (2.52), independently of the staggered structure. The singularity of this vector is sufficient to constrain these coefficients up to a global scale, so in principle general formulae for $\beta$ may be computed without actually specifying values for $n$ or $\mu$ ($\Delta h$ fixes the grading of the singular vector itself, so it is necessary to choose a value for it before we are able to calculate $\beta$ in closed form).

Under the assumptions that a vector at a particular grade is singular, we are able to calculate the coefficients $A_{(\xi,\nu)}$ and thus find a closed form for $\beta$ for those stagered modules with $|\Delta h|$ equal to this grade, again up to a global scale factor.

| $\xi,\nu$ | $\emptyset,\left(\tfrac{1}{2}\right)$ | $\emptyset,(1)$ | $\emptyset,(1,0)$ | $\emptyset,\left(\tfrac{1}{2}^2\right)$ | $(1),\emptyset$ | $(1),(0)$ | $\emptyset,\left(\tfrac{3}{2}\right)$ |
|---|---|---|---|---|---|---|---|
| $\overline{C}_{\xi,\nu}$ | $1$ | $1$ | $-1$ | $-1$ | $-1$ | $-1$ | $1$ |
| $C_{-\xi,-\nu}$ | $\eta$ | $\eta+\lambda$ | $\eta-\lambda$ | $\eta$ | $\tfrac{1}{2}(\eta-\lambda)$ | $\eta$ | $\eta+2\lambda$ |
| | $\emptyset,\left(\tfrac{1}{2}^3\right)$ | $(1),\left(\tfrac{1}{2}\right)$ | $\emptyset,(2)$ | $\emptyset,(2,0)$ | $\emptyset,\left(\tfrac{3}{2},\tfrac{1}{2}\right)$ | $\emptyset,(1^2)$ | $\emptyset,(1^2,0)$ |
| | $-1$ | $-1$ | $1$ | $-2$ | $-2$ | $-2$ | $-2$ |
| | $\eta$ | $\eta$ | $\eta+3\lambda$ | $2(\eta-\lambda)$ | $2\eta$ | $2(\eta+\lambda)$ | $2(\eta-\lambda)$ |
| | $\emptyset,\left(\tfrac{1}{2}^4\right)$ | $(1),(1)$ | $(1),(1,0)$ | $(1),\left(\tfrac{1}{2}^2\right)$ | $(2),\emptyset$ | $(2),(0)$ | |
| | $2$ | $-1$ | $2$ | $2$ | $-2$ | $-2$ | |
| | $2\eta$ | $\tfrac{3}{2}(\eta+\lambda)$ | $2(\eta-\lambda)$ | $2\eta$ | $2(\eta+3\lambda)$ | $(\eta-\lambda)$ | |

Figure 2.4: Example calculations of all $\overline{C}_{\xi,\nu}$ and $C_{-\xi,-\nu}$ with $|\Delta h| \leq 2$. In each instance $\eta$ indicates the $a_0$ eigenvalue of the left-hand space in (2.52).

The first few $\beta$ are as follows, for the NS sector:

$$\begin{aligned}
\beta_{\left(\Delta h=-\frac{1}{2}\right)} &= \left(-\frac{1}{2}\right)^{-\frac{1}{2}(1+(-1)^n)} \eta = \frac{\left(-\sqrt{2}\right)^{1+(-1)^n}(1+n)}{2n\mu}\left(n\mu^2-1\right) \\
\beta_{(\Delta h=-1)} &= 0 \\
\beta_{\left(\Delta h=-\frac{3}{2}\right)} &= \left(-\frac{3}{2}\right)^{-\frac{1}{2}(1+(-1)^n)} \frac{1}{2^{19}n^9\mu^9}\big[(n+3)(n^2-1)n^3\mu^6-(3n^2-32n-3)(n+1)n^2\mu^4 \\
&\qquad -4(n+9)(n+3)(n-1)n\mu^2+12(n^2-9)(n+1)\big] \\
&\quad \times\left((n^2-1)n^2\mu^2+32n^2\mu^2-4(n^2-9)\right)\left((n+1)n\mu^2+2(n-3)\right)^2 \\
&\quad \times\left((n-1)n\mu^2-2(n+3)\right)^2 \\
\beta_{(\Delta h=-2)} &= 0
\end{aligned} \tag{2.78}$$

and for the R sector:

$$\begin{aligned}
\beta_{(\Delta h=-1)} &= \mathrm{n/a}^{11} \\
\beta_{(\Delta h=-2)} &= 0
\end{aligned} \tag{2.79}$$

In these expressions for $\beta$, the scale factor has been chosen so that the monomial with the lowest appearing conformal weights (alternatively, the monomial corresponding to the longest partition) has a coefficient of 1. For instance, at $\Delta h = -\frac{3}{2}$ in the NS sector, this means we have the coefficient of $G^3_{-\frac{1}{2}}$ set to 1. Curiously we find that taking integral values of $\Delta h$ appears to make $\beta$ vanish in the NS sector

[11] There are too many relations at depth 1 for there to exist a singular vector.

(and see some similar, but less compelling, vanishing $\beta$ in the R sector). This is not due to any vanishing of the singular vector, of course, but rather to the vanishing of the combination of the factors $C_{-\xi,-\nu}$ and the expansion coefficients $A_{-\xi,-\nu}$.

While much of the above is entirely analogous to the $N = 0$ case, this vanishing of $\beta$ at certain $\Delta h$ is new, and implies deeper structural differences in the $N = 1$ case beyond these superficial similarities. Indeed, when we turn to the literature in order to compare our predicted values of $\beta$ — e.g., [5,6] — we find the staggered modules therein have left and right spaces that are *not* a multiple of four entries apart in the extended Kac tables, as we found to be required of staggered modules constructed in this manner in (1.122).

This obviously shows that the use of intertwining maps as in (1.115) is not as complete a construction of staggered $N = 1$ Fock modules as in the $N = 0$ case. There may exist other intertwiners which fill these gaps, but recall that the dependence of our construction on intertwining maps is a simplification based on the fact that in order to find solutions to (2.14) it was *sufficient* to take the $V_n$ to be the modes of a weight-1 Virasoro primary field. That this is not *necessarily* the case opens up yet further possibilities. From the singular vector data of the left and right spaces involved in such constructions, it may be possible to bootstrap the action of the staggering operators, but this would not necessarily reveal the details of any underlying analytic structure. Without a clear systematic procedure, examining the details of additional $N = 1$ staggered constructions will have to be the subject of future work.

### $N = 2$ Construction

Fock-type modules of the $N = 2$ Virasoro superalgebra $\mathfrak{s}_2\mathfrak{Vir}$ are less well understood than those of lower $N$, so our analysis of staggered $N = 2$ Fock supermodules is correspondingly less detailed, insofar as no analogous classification structures like the Kac table are known for this setting. This is not to say that no work has been done in the area (consider for instance [12, 43]); rather that to establish a setting in which the details of staggered $N = 2$ Fock modules could be rigorously discussed would go beyond the scope of the current work.

The more basic algebraic details of a staggered structure may be discussed, however, as they do not rely on an understanding of the submodule structures at all. We extend the now-familiar method of construction for a family of staggering operators to meet the needs of the larger algebra: given a $\mathfrak{s}_2\mathfrak{Vir}$ intertwining map $V_0 : \mathcal{S}_2\mathcal{F}_R \mapsto \mathcal{S}_2\mathcal{F}_L$ between two Fock supermodules, we define:

$$\begin{aligned} V_n &:= [\overline{A}a_n + A\overline{a}_n, V_0] \\ W_n &:= [\overline{A}b_n, V_0] \\ \overline{W}_n &:= [A\overline{b}_n, V_0] \\ X_n &:= 0 \end{aligned} \tag{2.80}$$

where $V_n$ is the staggering operator associated with $L_n$, $W_n$ and $\overline{W}_n$ with $G_n$ and $\overline{G}_n$ respectively, and $X_n$ with $J_n$. $A$ and $\overline{A}$ are free parameters. One may check with appropriate applications of the super-Jacobi identity that these definitions satisfy the relations demanded by self-consistency of the representation:

$$\begin{aligned}
[L_m, V_n] + [V_m, L_n] &= (m-n)V_{m+n} \\
[L_m, W_n] + [V_m, G_n] &= \left(\frac{1}{2}m - n\right) W_{m+n} \\
[L_m, \overline{W}_n] + [V_m, \overline{G}_n] &= \left(\frac{1}{2}m - n\right) \overline{W}_{m+n} \\
[L_m, X_n] + [V_m, J_n] &= -nX_{m+n} \\
\{G_m, W_n\} + \{W_m, G_n\} &= 0 \\
\{\overline{G}_m, \overline{W}_n\} + \{\overline{W}_m, \overline{G}_n\} &= 0 \\
\{G_m, \overline{W}_n\} + \{W_m, \overline{G}_n\} &= V_{m+n} + \frac{1}{2}(m-n)X_{m+n} \\
[G_m, X_n] + [W_m, J_n] &= -W_{m+n} \\
[\overline{G}_m, X_n] + [\overline{W}_m, J_n] &= \overline{W}_{m+n} \\
[J_m, X_n] + [X_m, J_n] &= 0
\end{aligned} \tag{2.81}$$

**Remark:**

It may come as some surprise that each $X_n$ is chosen to be identically zero. However, when a more careful analysis is made, the conclusion seems hard to avoid. From (2.81) with our construction as described, the $X_n$ must satisfy

$$\begin{aligned}
[L_m, X_n] &= -nX_{m+n} \\
[G_m, X_n] &= 0 \\
[\overline{G}_m, X_n] &= 0 \\
[J_m, X_n] &= [J_n, X_m]
\end{aligned} \tag{2.82}$$

Each of the staggering operators is (at the level of fields) a coefficient in a Laurent expansion in some primary field whose weight is determined by the corresponding field of algebra generators: the $V_n$ must partner with the $L_n$, which are the modes of $T(z)$, a weight-2 field; the $W_n$ and $\overline{W}_n$ to the modes of $G(z)$ and $\overline{G}(z)$, weight-$\frac{3}{2}$ fields; and the $X_n$ to $J(z)$, a weight-1 field. $V_0$ itself, as an intertwining map, is already a coefficient of a primary field of weight 1. Each $V_n$ (or rather, the field that they comprise) must therefore be given its net weight of 2 through multiplication by another field of weight 1 — a linear combination of the bosons suffice[12]. For the $W_n$ and $\overline{W}_n$, the deficit in weight is $\frac{1}{2}$, so we must use

[12] One can attempt to introduce terms quadratic in the fermion fields, these also being of weight $\frac{1}{2} + \frac{1}{2} = 1$, but will quickly arrive at a contradiction when attempting to enforce the relations in (2.81).

the fermion fields. For the $X_n$, then, we are forced to use the modes of a weight-0 field, since there is no additional conformal weight to add. However, the only weight-0 field in our theory is the identity field $\mathbf{1}(z) = 1$, whose coefficients are constants and which therefore commute with $V_0$. The "Jacobi-like" construction consequently demands that each $X_n$ vanish. //

One sees that there are infinitely many staggered constructions of this type corresponding to the full range of choices for $A$ and $\overline{A}$. The fact that staggered structures are only distinct up to a global normalisation choice suggests that only the ratio $\frac{A}{\overline{A}}$ is a free parameter; the moduli space in this case is the Riemann sphere.

### Admissibility and $\beta$ for $N = 2$

We do not delve into the complicated world of reducibility criteria for $N = 2$ Fock superspaces in this work, so admissibility of reducible modules as the left and right spaces in a short exact sequence construction of a staggered module is not our goal. It is currently unknown if $\beta$ is even relevant for $N = 2$ staggered modules. However, several key features remain the same, even in this more complicated setting.

We saw that intertwining maps for $N = 2$ Fock supermodules had one additional "vacuum shift" parameter $\mu$ compared to their lower-$N$ counterparts: the presence of two copies of $\mathfrak{a}$ demands two different $a_0$ eigenvalues, one corresponding to each, and generic intertwining maps can potentially alter both independently. These were mirrored by further parameters setting the relative strengths of multiplying factors of each weight-1 combination of the free fields themselves. We saw also that staggered structures have similar internal parameters, in some sense setting the relative strengths of the two bosons in determining the indecomposable part of the $\mathfrak{s}_2\mathfrak{Vir}$ action on the staggered module.

In this setting there is no reason to expect that the actual calculation of $\beta$ will be any different than what has gone before: after selecting left and right spaces which may be related by an intertwining map, the relevant singular vector is fixed; so are the combinatorial factors involved in its creation from the left vacuum vector.

| $n\backslash\Delta h$ | $-1$ | $-2$ | $-3$ |
|---|---|---|---|
| 1 | $\mu^{-2}(\mu^2-2)$ | $-2\mu^{-2}(\mu^4-4)(\mu^2-4)$ | $12\mu^{-10}(\mu^4-16)(\mu^4-4)(\mu^2-6)$ |
| 2 | $\frac{3}{4}\mu^{-2}(\mu^2-1)$ | $\frac{1}{2}\mu^2(\mu^4-4)(\mu^2-1)$ | $\frac{45}{2^{14}}\mu^{-10}(9\mu^4-25)(\mu^4-1)(\mu^2-3)$ $\times(\mu^2-1)^2(3\mu^2+1)(\mu^2+5)$ |
| 3 | $\frac{1}{3}\mu^{-2}(3\mu^2-2)$ | $\frac{20}{729}\mu^{-2}(36\mu^4+27\mu^2-10)(3\mu^2+1)(3\mu^2-2)(3\mu^2-4)$ | $\mu^{-2}(\mu^4-4)(\mu^4-1)(3\mu^2-2)$ |
| 4 | $\frac{5}{16}\mu^{-2}(2\mu^2-1)$ | $5\mu^{-2}(5\mu^4+2\mu^2-1)(3\mu^2+1)(2\mu^2-1)(\mu^2-1)$ | $\frac{3}{4194304}\mu^{-10}(10\mu^2+7)(10\mu^2-1)(10\mu^2-7)^2$ $\times(8\mu^2-3)(6\mu^2+7)(6\mu^2+1)^2(6\mu^2-1)$ |
| 5 | $\frac{3}{25}\mu^{-2}(5\mu^2-2)$ | $\frac{14}{625}\mu^{-2}(100\mu^4+25\mu^2-14)(10\mu^2+3)(5\mu^2-2)(5\mu^2-4)$ | $\frac{6}{9765625}\mu^{-10}(25\mu^2-6)(15\mu^2+8)(15\mu^2-2)$ $\times(15\mu^2-8)^2(5\mu^2+4)(5\mu^2+1)^2(5\mu^2-1)$ |
| 6 | $\frac{7}{36}\mu^{-2}(3\mu^2-1)$ | $\frac{28}{729}\mu^{-2}(315\mu^4+54\mu^2-32)(15\mu^2+4)(3\mu^2-1)(3\mu^2-2)$ | $\frac{1}{4096}\mu^{-10}(7\mu^2+3)(7\mu^2-1)(7\mu^2-3)^2$ $\times(6\mu^2-1)(5\mu^2+3)(5\mu^2+1)^2(5\mu^2-1)$ |

Figure 2.5: Various $\beta(\mu)$ at low-lying $(n,\Delta h)$. Note that these evaluations only correspond to legitimate constructions (due to the monodromy condition of (1.88)) when $n|\Delta h$, although it is still of course possible to set $n$ and $\Delta h$ independently in these polynomials. We note with interest that these particular examples factorise neatly, whereas those with $n\nmid\Delta h$ do not.

| **Dense logCFTs** | $(p,q)$ | $c$ | $\beta_{1,3}$ | $\beta_{1,4}$ | $\beta_{1,5}$ | $\beta_{1,6}$ | $\beta_{1,7}$ | $\beta_{1,8}$ | $\beta_{1,9}$ | **Ref** |
|---|---|---|---|---|---|---|---|---|---|---|
| - | $(1,4)$ | $-\frac{25}{2}$ | | | | | | | $-3$ $(1,-1)$ | [45] |
| - | $(1,3)$ | $-7$ | | | | | $-2$ $(1,-1)$ | $8$ $(2,-2)$ | | [45] |
| Dense Polymer | $(1,2)$ | $-2$ | | | $-1$ $(1,-1)$ | | $-\frac{9}{2}$ $(1,-2)$ | | $-\frac{75}{4}$ $(1,-3)$ | [7,23,45] |
| Percolation | $(2,3)$ | $0$ | | $-\frac{1}{2}$ $(1,-1)$ | $-\frac{5}{8}$ $(2,-2)$ | | $-\frac{35}{3}$ $(1,-3)$ | $-\frac{13475}{216}$ $(2,-6)$ | | [7,23,45] |
| Ising | $(3,4)$ | $\frac{1}{2}$ | | | $-\frac{35}{24}$ $(1,-2)$ | $-\frac{13475}{243}$ $(2,-4)$ | $-\frac{49049}{17496}$ $(3,-6)$ | | $-\frac{40415375}{944784}$ $(1,-5)$ | [45] |
| Tricritical Ising | $(4,5)$ | $\frac{7}{10}$ | | | | $-\frac{693}{100}$ $(1,-3)$ | $-\frac{6114399291}{1078465600}$ $(2,-6)$ | $-\frac{3^{14}7^{4}11^{2}13^{2}17^{1}23^{1}}{2^{12}5^{2}19^{1}11423^{2}}(?)$ | $-\frac{3^{13}7^{2}13^{3}17^{2}19^{1}23^{1}}{2^{6}5^{4}11^{2}18433^{2}}(?)$ | [45] |
| 3-State Potts | $(5,6)$ | $\frac{4}{5}$ | | | | | $-\frac{676039}{59895}$ $(1,-4)$ | $-\frac{2^{12}7^{2}11^{3}17^{2}19^{1}23^{2}29^{1}}{3^{2}5^{3}41^{2}17234^{2}}(?)$ | $-\frac{3^{9}7^{6}11^{3}13^{2}17^{3}19^{1}23^{2}29^{1}}{5^{5}9887^{2}281623^{2}}(?)$ | [45] |

| **Dilute logCFTs** | $(p,q)$ | $c$ | $\beta_{3,1}$ | $\beta_{4,1}$ | $\beta_{5,1}$ | $\beta_{6,1}$ | $\beta_{7,1}$ | $\beta_{8,1}$ | $\beta_{9,1}$ | **Ref** |
|---|---|---|---|---|---|---|---|---|---|---|
| Dilute Polymer | $(2,3)$ | $0$ | $\frac{5}{6}$ $(1,-2)$ | | $\frac{67375}{676}$ $(1,-5)$ | | $\frac{2^{1}5^{5}7^{2}11^{2}13^{2}17}{3^{1}23^{2}271^{2}}(?)$ | | $\frac{5^{6}7^{5}11^{3}13^{2}1^{2}19^{2}23^{1}}{2^{4}3^{1}23607803^{2}}(?)$ | [45] |
| $O(n\to 1)$ | $(3,4)$ | $\frac{1}{2}$ | | $\frac{175}{12}$ $(1,-3)$ | $\frac{49049}{15552}$ $(2,-6)$ | | $\frac{5^{2}7^{1}11^{2}13^{1}17^{1}19^{1}}{2^{1}3^{2}113^{2}}(?)$ | $\frac{5^{9}7^{5}11^{5}13^{4}17^{2}19^{2}23^{1}}{2^{12}29^{2}47^{2}787^{2}1201^{2}}(?)$ | | [45] |

Figure 2.6: Known values of $\beta$ for various staggered modules. The $A_{(-\Delta h^1)} = 1$ normalisation convention is used, not $A_{(1-\Delta h)} = 1$, to match with those in the referred works. Unless marked with a (?), entries in the table have been verified to match with those predicted by (2.49) — i.e., these staggered modules do indeed have Fock space realisations of the type constructed here. The relevant $(n, \Delta h)$ are given in small font beneath each value. Blank spaces indicate trivial or absent staggering. $\beta_{r,s}$ indicates that the right module occurs at $(r, s)$ in the appropriate extended table. Dense logCFTs take $\mu = \mu^-$, dilute ones $\mu = \mu^+$, and both have $\eta = \eta^-$. There is then only one way to assign the left module, at least for the small values of $(r, s)$ considered here.

*Chapter 3*

# Towards Logarithmic Vertex Algebras

*We are the music makers, and we are the dreamers of dreams*

Arthur O'Shaughnessy, *Ode*
(*via Willy Wonka*)

In this chapter we examine staggered modules as state spaces of what might be termed "logarithmic" vertex algebras. These are vertex algebras modified so that the field content is permissive of fields containing logarithmic terms. It is hoped that through examining such examples we will be able to determine an appropriate definition for logarithmic vertex algebras themselves.

This treatment of logarithmic fields in the theory of vertex algebras differs from others (notably [3, 4]) in that other authors have, to date, typically considered specific types of twisted modules of standard (non-logarithmic) vertex algebras in which the twisted fields involve the logarithm of the formal variable. Our approach is to instead suggest enlargements of the state spaces of vertex algebras to accommodate new states whose corresponding fields are logarithmic and whose state space structure is indecomposable.

The relationship to staggered modules becomes apparent when we examine how staggered structures naturally arise in the context of induced modules — not of oscillator algebras $\mathfrak{a}$ and $\mathfrak{b}$ themselves, but of slight enlargements of them. We will likewise see that these enlargements correspond to the inclusion of vacuum evaluations of logarithmic fields.

## 3.1 Induced Staggered Modules

Instead of fixing two Virasoro highest weight spaces and fitting them into a short exact sequence via a family of staggering operators, as was done in Chapter 2, we may alternatively produce staggered modules by an extension of the allowed operators acting on a single Fock space. In fact, the viable staggered constructions of this kind include some which cannot be produced through our "Jacobi-like" construction method of (2.23), and we begin with an example of such an instance.

### 3.1.1 $\mu = 0$ Induced Staggered Fock Modules

The above construction of staggered Fock modules presumed the existence of an intertwining map extracted as the zero-mode of a screening field. For the modes of such fields $V_\mu(z)$ to be well-defined and to satisfy the correct commutation relations (2.13), we relied on being able to tune the parameter $\mu$ to a particular (non-zero) value as determined by the $a_0$ vacuum eigenvalue $\eta$ (1.88). If $\mu = 0$, then the field $V_\mu(z)$ becomes trivial, reducing to the identity map, and this approach breaks down. Clearly screening fields are the incorrect objects to consider in what amounts to staggering a Fock space with itself.

Observe that the condition (2.13) was entirely general, and the use of screening fields merely sufficient to meet these conditions when applied to the particular case of the Virasoro algebra, i.e.

$$[L_m, V_n] + [V_m, L_n] = (m-n)V_{m+n}, \tag{3.1}$$

since any family of staggering operators $\{V_n \mid n \in \mathbb{Z}\}$ which consisted of the Laurent coefficients of a weight-1 primary field would do the trick.

Now consider that the bosonic field $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ is a weight-1 field (when $\lambda = 0$) whose coefficients are operators from any given bosonic Fock space to itself. Rather than simply choosing its coefficients to be the staggering operators by setting $V_n := a_n$, however, we find we are forced by the $\delta$ term in the commutation relation

$$[L_m, a_n] = -na_{m+n} - n(n-1)\lambda\delta_{m,-n} \tag{3.2}$$

to take

$$V_n := a_n - \lambda\delta_{n,0}. \tag{3.3}$$

Using this definition, we are able to fit any Fock space $\mathcal{F}_\eta$ into a short exact sequence with itself as in (2.16). In fact, comparing to the corresponding conditions (2.54) and (2.81) for $N > 0$, we find that setting

$$\begin{aligned} V_n &= a_n - \lambda\delta_{n,0} \\ W_n &= b_n \end{aligned} \tag{3.4}$$

for $N = 1$, and

$$\begin{aligned} V_n &= \overline{A}a_n + A\overline{a}_n - \frac{1}{2}\left(\overline{A}\lambda + A\overline{\lambda}\right)\delta_{n,0} \\ W_n &= \overline{A}b_n \\ \overline{W}_n &= A\overline{b}_n \\ X_n &= \left(\overline{A}\lambda - A\overline{\lambda}\right)\delta_{n,0} \end{aligned} \tag{3.5}$$

for $N = 2$ gives similar "$\mu = 0$" staggered structures, with two copies of the same Fock (super)space as the left and right pieces of the short exact sequence of (2.16).

// **Remark:**

As it is, this type of staggered space happens to have some interesting features. Of the staggered Fock modules seen so far, it stands as a relatively unique example, in that its family of staggering operators arises from the "lifting" of a staggered structure consisting of oscillator ($\mathfrak{a}$ and $\mathfrak{b}$) modules only (not of $\mathfrak{Vir}$ modules) to one *also* of $\mathfrak{Vir}$ modules. This "lifting" happens automatically when the modes of $\mathfrak{Vir}$ (and of its super-versions) are expressed in the usual way as infinite sums of monomials of the (now-indecomposable) oscillator modes.

Following (2.13), a staggered structure involving only $\mathfrak{a}$ would require a family of staggering operators $\{x_n \mid n \in \mathbb{Z}\}$ such that

$$[a_m, x_n] + [x_m, a_n] = 0 \tag{3.6}$$

where the right hand side vanishes as $k\mathbf{1} \in \mathfrak{a}$ must be diagonal. While allowing the $x_n$ to be any collection of scalar values would indeed be permitted, we are guided by the need to respect the grading to select only one nonzero $x_n$; the one associated to $a_0$:

$$x_n = \delta_{n,0}. \tag{3.7}$$

When this indecomposable representation of the $a_n$ is used to create a representation of $\mathfrak{Vir}$, since

$$L_n := \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_{n-k} a_k : -(n+1)\lambda a_n, \tag{3.8}$$

and in the obvious basis we can think of each $a_n$ as a $2 \times 2$ block matrix

$$\begin{bmatrix} a_n & \delta_{n,0} \\ 0 & a_n \end{bmatrix}, \tag{3.9}$$

this indeed gives an indecomposable $\mathfrak{Vir}$ representation with

$$V_n = a_n - \lambda \delta_{n,0}. \tag{3.10}$$

This prescription also reproduces the staggering operators of (3.4) and (3.5): by letting

$$x_n = \delta_{n,0}, \qquad y_n = 0 \tag{3.11}$$

for the family of staggering operators associated to $a_n$ and $b_n$ respectively, we get (3.4); by letting

$$\begin{aligned} x_n = \overline{A}\delta_{n,0}, &\qquad \overline{x}_n = A\delta_{n,0} \\ y_n = 0, &\qquad \overline{y}_n = 0 \end{aligned} \tag{3.12}$$

for the operators associated to $a_n$, $\overline{a}_n$, $b_n$, and $\overline{b}_n$, we get (3.5). Here the same remarks apply as those in Section 2.2.2: whereas for $N < 2$ there was only one non-trivial parameter, which was in the end insensitive to overall scaling, allowing us to fix a specific non-trivial construction (e.g. $V_n = [a_n, V_0]$ for the Jacobi-like construction, or $\delta_{n,0}$ in the present case), for $N = 2$ there are two such parameters so we must keep the ability to vary both independently, for instance here via the coefficients $A, \overline{A}$. These choices decouple from each other for the oscillators, essentially meaning we have two completely independent staggered oscillator structures, but the corresponding staggered structures for the $N = 2$ Virasoro superalgebra enjoy no such property since the commutation relations of the modes ultimately mix the two parameters. //

While it is interesting that this type of staggered Virasoro module arises from a staggered module of the underlying oscillator algebra(s), it is not the key point of this example — we find that it has another, perhaps more fundamental[1] interpretation and construction.

Consider now the extended bosonic oscillator algebra $\mathfrak{a} \oplus \mathbb{C}\mathbf{q}$, which we shall abbreviate as $\widehat{\mathfrak{a}}$. Recall that $\mathbf{q}$ was the operator appearing as the constant of integration in the antiderivative of the bosonic field

$$\partial^{-1}a(z) = \mathbf{q} + \log(z)a_0 - \sum_{n\neq 0} \frac{1}{n} a_n z^{-n} \tag{3.13}$$

with commutation relations

$$[a_n, \mathbf{q}] = \delta_{n,0} = x_n \tag{3.14}$$

We find in addition that

$$[L_n, \mathbf{q}] = a_n - \lambda\delta_{n,0} = V_n, \tag{3.15}$$

(refer to (3.10) for the definition of $V_n$ in this setting) replicating in both cases the staggering operators of the staggered $\mathfrak{a}$ and $\mathfrak{Vir}$ Fock modules. Since $\mathbf{q}$ commutes with all the bosonic creation operators, by including a vector $\mathbf{q}\,|\eta\rangle$ in addition to the original vacuum vector $|\eta\rangle$ we find that the standard procedure of freely generating $\mathcal{F}$ from a vacuum now produces two independent copies of $\mathcal{F}$ instead, related by a factor of $\mathbf{q}$. Thanks to the commutation relation (3.15), the resulting space

$$\mathcal{F} \oplus \mathbf{q}\mathcal{F} \tag{3.16}$$

is isomorphic to the full staggered module. In fact, we not only calculate $[L_n, \mathbf{q}] = a_n - \lambda\delta_{n,0}$ but also $[G_n, \mathbf{q}] = b_n$, and introducing a second[2] mode $\overline{\mathbf{q}}$ in the $N = 2$

[1] In the sense that it underlies both, and makes contact with the field-theoretic content of the theory.

[2] Thinking of course of two modes $\mathbf{q}_i$ for the two bosonic fields $\partial^{-1}a^{(i)}(z)$, $i = 1, 2$ satisfying

$$[a_n^{(i)}, \mathbf{q}_j] = \delta_{n,0}\delta_{i,j}$$

case such that

$$\begin{aligned}[a_n, \overline{\mathbf{q}}] &= [\overline{a}_n, \mathbf{q}] = \delta_{n,0} \\ [a_n, \mathbf{q}] &= [\overline{a}_n, \overline{\mathbf{q}}] = 0\end{aligned} \tag{3.17}$$

which results in

$$\begin{aligned}[L_n, A\mathbf{q} + \overline{A}\overline{\mathbf{q}}] &= \overline{A}a_n + A\overline{a}_n - \frac{1}{2}\left(\overline{A}\lambda + A\overline{\lambda}\right) \\ [G_n, A\mathbf{q} + \overline{A}\overline{\mathbf{q}}] &= \overline{A}b_n \\ [\overline{G}_n, A\mathbf{q} + \overline{A}\overline{\mathbf{q}}] &= A\overline{b}_n \\ [J_n, A\mathbf{q} + \overline{A}\overline{\mathbf{q}}] &= \left(\overline{A}\lambda - A\overline{\lambda}\right)\delta_{n,0}\end{aligned} \tag{3.18}$$

so that in fact the operator(s) $\mathbf{q}$ are enough to generate all of the $\mu = 0$ staggered structures considered above. This means that

$$\mathcal{F} \oplus \mathbf{q}\mathcal{F} \tag{3.19}$$

is the staggered module for $N = 0$ and 1, and for $N = 2$ it is

$$\mathcal{F} \oplus \left(A\mathbf{q} + \overline{A}\overline{\mathbf{q}}\right)\mathcal{F} \tag{3.20}$$

for appropriate constants $A, \overline{A}$.

We may go further with this construction, which obviously generalises to higher powers of $\mathbf{q}$. We can consider such objects as

$$\bigoplus_{k=0}^{n-1} \mathbf{q}^k \mathcal{F}, \tag{3.21}$$

for instance, which provide easy access to theories of arbitrary logarithmic rank, the above being an example of a rank-$n$ staggered module.

### $\mu = 0$ Modules of Arbitrary Finite Rank

Consider the space $\mathcal{M}_\eta$ induced from the vacuum $|\eta\rangle$ by the action of the algebra $\widehat{\mathfrak{a}}$, just as the Fock space $\mathcal{F}_\eta$ is induced from this vector by the action of $\mathfrak{a}$ alone. This space can be thought of as

$$\mathcal{M}_\eta \cong \mathbb{C}[\mathbf{q}] \otimes \mathcal{F}_\eta, \tag{3.22}$$

i.e. the polynomial extension of the Fock space by powers of the operator $\mathbf{q}$. This space is isomorphic as a vector space to an infinite sum of disjoint copies of $\mathcal{F}_\eta$, but the extra mode $\mathbf{q}$ complicates its structure as a $\widehat{\mathfrak{a}}$-module. For instance,

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and resulting in

$$\mathbf{q} = \frac{1}{\sqrt{2}}\left(\mathbf{q}_1 + i\mathbf{q}_2\right), \qquad \overline{\mathbf{q}} = \frac{1}{\sqrt{2}}\left(\mathbf{q}_1 - i\mathbf{q}_2\right)$$

while we can produce an arbitrary staggered $\mathfrak{a}$ module of finite rank from this space by taking the quotient by $\mathbf{q}^n$, the resulting space cannot be an $\widehat{\mathfrak{a}}$-module (nor can any finitely-generated space on which $\mathbf{q}$ acts non-semisimply, or indeed any arbitrary space on which $\mathbf{q}$ is even locally nilpotent, if $a_0$ has eigenvectors), because consider that if

$$\mathbf{q}^n v \neq 0, \qquad \mathbf{q}^{n+1} v = 0 \tag{3.23}$$

for some nonzero $a_0$ eigenvector $v$ and positive integer $n$, then

$$0 = a_0 \left(\mathbf{q}^{n+1} v\right) = \mathbf{q}^{n+1} a_0 v + \left[a_0, \mathbf{q}^{n+1}\right] v = (n+1)\mathbf{q}^n v, \tag{3.24}$$

a clear contradiction. Therefore induced modules of $\widehat{\mathfrak{a}}$ itself must be of infinite rank. This is acceptable for the construction of finite rank staggered modules of the type discussed above, of course, since we only require the action of the algebra $\mathfrak{a}$ to be well-defined in order to construct a representation of $\mathfrak{Vir}$ on this space.

Let $\mathcal{M}_\eta^{(r)}$ be the rank-$r$ staggered Fock module generated by taking the obvious quotient;

$$\mathcal{M}_\eta^{(r)} = \frac{\mathbb{C}\left[\mathbf{q}\right]}{\langle \mathbf{q}^r \rangle} \otimes \mathcal{F}_\eta. \tag{3.25}$$

We also have an obvious basis for the highest weight subspace consisting of

$$\begin{aligned}
\left|\eta, 0\right\rangle &= \left|\eta\right\rangle, \\
\left|\eta, 1\right\rangle &= \mathbf{q}\left|\eta\right\rangle, \\
\left|\eta, 2\right\rangle &= \mathbf{q}^{(2)}\left|\eta\right\rangle, \\
&\cdots \\
\left|\eta, r-1\right\rangle &= \mathbf{q}^{(r-1)}\left|\eta\right\rangle
\end{aligned} \tag{3.26}$$

where we have adopted the bracketed exponent notation, relatively common in dealing with modes in state spaces of vertex algebras, indicating normalisation by a factorial:

$$x^{(k)} := \frac{1}{k!} x^k. \tag{3.27}$$

Then

$$L_n \left|\eta, k\right\rangle = 0, \qquad L_{-n}\left|\eta, k\right\rangle = q^{(k)} L_{-n}\left|\eta\right\rangle + a_{-n}\left|\eta, k-1\right\rangle, \qquad n > 0 \tag{3.28}$$

and

$$L_0 \left|\eta, k\right\rangle = h_\eta \left|\eta, k\right\rangle + (\eta - \lambda)\left|\eta, k-1\right\rangle + \frac{1}{2}\left|\eta, k-2\right\rangle. \tag{3.29}$$

### $\mathcal{M}_\eta$ as an $\widehat{\mathfrak{a}}$-Module

$\mathcal{M}_\eta$ as an $\widehat{\mathfrak{a}}$ module has some interesting features, which we examine in a little detail below. Note firstly that the bilinear form of $\mathcal{F}_\eta$ does not extend to $\mathcal{M}_\eta$; we must define $\mathbf{q}^\dagger$. One thing of note is that the operator $\mathbf{q}$ conserves the generalised eigenvalue of any vector it acts upon; it is therefore a weight-0 operator along with $a_0$. In re-defining the adjoint to take this into account, we must allow for the possibility that

$$a_0^\dagger = A_0 a_0 + A_q \mathbf{q}, \qquad q^\dagger = Q_0 a_0 + Q_q \mathbf{q} \tag{3.30}$$

for some constants $A_0, A_q, Q_0, Q_q \in \mathbb{C}$. Demanding that this be an involution compatible with the algebra structure, i.e., $[x,y]^\dagger = [y^\dagger, x^\dagger]$ imposes

$$a_0^\dagger = re^{i\theta} a_0 + c_q \mathbf{q}, \qquad \mathbf{q}^\dagger = c_0 a_0 - re^{-i\theta}\mathbf{q} \tag{3.31}$$

for $r, c_0, c_q \in \mathbb{R}$, angle $\theta$, and with $r = \sqrt{1 - c_0 c_q}$, which restricts $c_0 c_q \leq 1$. We would also want any proposed adjoint operation to be compatible with any conceptual inner product. Mirroring that of (1.28), we seek to define something like

$$\langle U_1 |\eta\rangle, U_2 |\eta\rangle\rangle = \langle\eta| \left(U_1^\dagger U_2 |\eta\rangle\right) \tag{3.32}$$

for $U_1, U_2 \in \mathcal{U}(\widehat{\mathfrak{a}})$ and with the linear functional $\langle\eta|(v) = \delta_{v,|\eta\rangle}$, but one quickly runs into difficulties, not least of which is that this does not generically result in a symmetric form:

$$\langle \mathbf{q}|\eta\rangle, |\eta\rangle\rangle = c_0 \eta, \qquad \langle |\eta\rangle, \mathbf{q}|\eta\rangle\rangle = 0. \tag{3.33}$$

One can conclude instantly (using this and similar symmetry constraints) that, for generic $\eta$, we must take

$$a_0^\dagger = a_0, \qquad \mathbf{q}^\dagger = -\mathbf{q} \tag{3.34}$$

which results in a space whose generators $q^k|\eta\rangle$, $k = 0, 1, 2, \ldots$ are all null (of vanishing norm) except $|\eta\rangle$, the first.

In the vacuum module, where $\eta = 0$, we have a second option: to take

$$a_0^\dagger = \mathbf{q}, \qquad \mathbf{q}^\dagger = a_0, \tag{3.35}$$

in which case the generators $q^k|\eta\rangle$, $k = 0, 1, 2, \ldots$ are of non-vanishing norm and are pairwise orthogonal.

This adjoint is interesting in that it provides an alternate representation of $\mathfrak{Vir}$ on $\mathcal{M}_0$. Let us define

$$\begin{aligned}\Lambda_n := L_{-n}^\dagger &= \begin{cases} \frac{1}{2}\sum_{k\neq 0,n} :a_{n-k}a_k: + \mathbf{q}a_n - (n+1)\lambda a_n & n \neq 0 \\ \frac{1}{2}\sum_{k\neq 0} :a_{-k}a_k: + \frac{1}{2}\mathbf{q}^2 - \lambda\mathbf{q} & n = 0 \end{cases} \\ &= \begin{cases} L_n + (\mathbf{q} - a_0)\, a_n & n \neq 0 \\ L_0 + \frac{1}{2}\left(\mathbf{q}^2 - a_0^2\right) - \lambda\left(\mathbf{q} - a_0\right) & n = 0 \end{cases}\end{aligned} \tag{3.36}$$

by virtue of defining † to be compatible with the bracket of $\widehat{\mathfrak{a}}$, this adjoint operation extends to $\mathfrak{Vir}$ as well, and one finds that

$$[\Lambda_m, \Lambda_n] = (m-n)\Lambda_{m+n} + \frac{1}{12}(m^3 - m)C\delta_{m,n}. \tag{3.37}$$

We therefore have a representation of $\mathfrak{Vir}$ in which the generators have the opposite "direction" in the staggering; operators which otherwise behave as the Virasoro generators but which increase the rank of vectors on which they act. Since now $[\Lambda_n, \mathbf{q}] = 0$, we have (using the notation of the previous section),

$$\Lambda_n |0,k\rangle = 0, \qquad \Lambda_{-n} |0,k\rangle = \mathbf{q}^{(k)} L_{-n} |0\rangle + (k+1)a_{-n} |0,k+1\rangle, \qquad n > 0 \tag{3.38}$$

and

$$\begin{aligned} L_0 |0,k\rangle &= -(k+1)\lambda |0,k+1\rangle + \frac{1}{2}(k+1)(k+2) |0,k+2\rangle \\ &= h_0 |0,k\rangle + (k+1)(0-\lambda) |0,k+1\rangle + \frac{1}{2}(k+1)(k+2) |0,k+2\rangle, \end{aligned} \tag{3.39}$$

exactly the type of behaviour seen in (3.28) and (3.29), albeit with additional factors of $(k+1)$ and $(k+2)$ inserted to maintain the factorial normalisation previously established. This representation of $\mathfrak{Vir}$ on $\mathcal{M}_0$ is truly dual to the standard one, not only through the adjoint but through the fact that $\mathfrak{Vir}$ generators now typically increase the rank of a vector rather than decrease it. Instead of the vacuum $|0\rangle = |0,0\rangle$ being in a sense a "terminal" object, it is now "initial", and $\mathcal{M}_0$ with this representation does not technically satisfy our definition of a staggered module (perhaps contrary to intuition) — $(\Lambda_0 - h)$, for instance, is not locally nilpotent for any $h$ and $\Lambda_0$ has no eigenvectors at all, generalised or otherwise. Since there are not yet any obvious major implications of such a representation on $\mathcal{M}_0$, it is not clear that the definition of staggered module should be expanded to accommodate this type of structure.

Of course, the $\Lambda_n$ do have utility in computing inner products on $\mathcal{M}_0$ with the adjoint so defined. It modifies the status of certain vectors — their singularity, subsingularity, etc. The relevant expressions for computing these quantities involve commutators of the form $[\Lambda_m, L_n]$ for $m > 0$ and $n < 0$. We find in this case that

$$\begin{aligned} [\Lambda_m, L_n] &= [L_m, L_n] + (\mathbf{q} - a_0)[a_m, L_n] + [\mathbf{q} - a_0 L_n]a_m \\ &= [L_m, L_n] + (\mathbf{q} - a_0)(ma_m + m(m-1)\lambda\delta_{m,-n}) - a_n a_m \\ &= [L_m, L_n] + (m\mathbf{q} - ma_0 - a_n)a_m + m(m-1)(\mathbf{q} - a_0)\lambda\delta_{m,-n}. \end{aligned} \tag{3.40}$$

As can be seen, this results in a modification to the "standard" bilinear form of the Verma module by an additional term wherein every position previously occupied by a commutator is replaced in turn with the term following $[L_m, L_n]$ in the last line of the above. It should be noted that while $a_0$ acts as the zero operator on the rank-0 subspace, it can no longer be discarded from general inner products, since it acts nontrivially on vectors of higher rank.

These alternate types of adjoint and Virasoro representation may or may not prove useful in the greater context of the fiend and representation theory — whether it can meaningfully be applied in Ward identities, for instance. Inspecting this would be beyond the scope of the current work.

Regardless of the adjoint we impose upon the vacuum $\widehat{\mathfrak{a}}$-module $\mathcal{M}_0$, its status as an induced module makes a strong connection with the state spaces of logarithmic fields. Since

$$\partial^{-1}a(z) = \mathbf{q} + \log(z)a_0 - \sum_{n\neq 0} \frac{1}{n} a_n z^{-n}, \tag{3.41}$$

we have in the fashion of the state-field correspondence that

$$\begin{aligned}\lim_{|z|\to 0} \partial^{-1}a(z) \left|0\right\rangle &= \lim_{|z|\to 0} \left(\mathbf{q} - \sum_{n>0} \frac{1}{n} a_{-n} z^n\right) \left|0\right\rangle \\ &= \mathbf{q} \left|0\right\rangle .\end{aligned} \tag{3.42}$$

This relationship suggests, just as $\mathcal{F}_0$ is the state space of the Heisenberg vertex operator algebra and is generated through the state-field correspondence by the field $a(z)$ via its derivatives and normally ordered products thereof, that $\mathcal{M}_0$ is generated through the state-field correspondence by the *logarithmic* field $\partial^{-1}a(z)$ via its derivatives and normally ordered products, and that $\mathcal{M}_0$ is the state space of some corresponding *logarithmic* vertex operator algebra.

While there are analytical issues to resolve in defining the normally ordered product of logarithmic fields, algebraically there is little issue: since $\mathbf{q}$ never sends any vector to zero in $\mathcal{M}_0$, we conclude that it is a creation operator, and correspondingly define

$$:\mathbf{q}x: \; = \mathbf{q}x \tag{3.43}$$

for all $x$ in $\widehat{\mathfrak{a}}$. Since $\mathbf{q}$ commutes with all $a_n$ except $a_0$, there is little difference between this and any other definition extending the domain of $:\cdot:$ to all of $\widehat{\mathfrak{a}}$, and hence little danger of encountering any troubles with it in the future. Indeed, we find with this working definition that

$$\lim_{|z|\to 0} :\underbrace{\partial^{-1}a(z)\partial^{-1}a(z)\cdots\partial^{-1}a(z)}_{n}: \left|0\right\rangle = \mathbf{q}^n \left|0\right\rangle \tag{3.44}$$

while considering a normally-ordered product of many terms to be a nested application;

$$:x_1 x_2 \cdots x_n: \; = \; :x_1 \left( :x_2 \left( :\cdots :x_{n-1} x_n: : \right) : \right) : \, . \tag{3.45}$$

This does indeed appear to be the correct interpretation, since other vectors of $\mathcal{M}_0$ are also generated in this way. For instance,

$$\mathbf{q}a_{-1} \left|0\right\rangle = \lim_{|z|\to 0} :a(z)\partial^{-1}a(z): , \tag{3.46}$$

*et cetera.*

Using such a definition, working purely with modes and not attempting to perform any contour integrals, this of course results in

$$\begin{aligned}
\partial^{-1}a(z)\partial^{-1}a(w) - :\partial^{-1}a(z)\partial^{-1}a(w): &= \log(z)\,[a_0, \mathbf{q}] + \left[\sum_{m>0}\frac{1}{m}a_m z^{-m}, \sum_{n<0}\frac{1}{n}a_n w^{-n}\right] \\
&= \log(z) - \sum_{k>0}\frac{1}{k}\left(\frac{w}{z}\right)^k \\
&= \log(z) + \log\left(1 - \frac{w}{z}\right) \\
&= \log(z-w),
\end{aligned} \tag{3.47}$$

as expected, which converges since $|z| > |w|$. This utilises a well-known expansion of the logarithm;

$$\log(1-x) = -\sum_{k>0}\frac{1}{k}x^k \tag{3.48}$$

for $|x| < 1$ [21]. We shall have occasion to use this expansion many more times in the sequel.

### 3.1.2 Symplectic Fermions

We now consider another type of staggered module of the Virasoro algebra, also generated from modes of an oscillator algebra, and also able to be induced as the vacuum module of a "logarithmically" extended version of the same oscillator algebra. This is the symplectic fermion representation, so-called because the generating set of the theory comprises two weight-one fields (or alternatively one two-component complex fermion) whose products may be given in terms of a symplectic form.

These fields, typically denoted

$$\chi^\alpha(z), \qquad \alpha \in \{+,-\} \tag{3.49}$$

obey the relations

$$\chi^\alpha(z)\chi^\beta(w) \sim \frac{\epsilon^{\alpha\beta}}{(z-w)^2} \tag{3.50}$$

where $\epsilon^{\alpha\beta}$ is a totally antisymmetric $2\times 2$ tensor; we can specify an ordered basis with $\epsilon^{+-} = -\epsilon^{-+} = 1$ and $\epsilon^{++} = \epsilon^{--} = 0$.

These fields have expansions

$$\chi^\alpha(z) = \sum_{n\in\mathbb{Z}} \chi^\alpha_n z^{-n-1}, \tag{3.51}$$

so the fermionic behaviour truly comes from the symplectic form, since the mode expansions appear bosonic in nature. Nevertheless, we do indeed find

$$\{\chi^\alpha_m, \chi^\beta_n\} = m\epsilon^{\alpha\beta}\delta_{m,-n}. \tag{3.52}$$

An energy-momentum tensor can be defined by setting

$$T(z) := \frac{1}{2}\epsilon_{\alpha\beta} :\chi^\alpha(z)\chi^\beta(z): \tag{3.53}$$

where $\epsilon_{\alpha\beta}$ is the inverse of $\epsilon^{\alpha\beta}$;

$$\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta^\beta_\alpha. \tag{3.54}$$

Thus

$$L_n = \frac{1}{2}\epsilon_{\alpha\beta}\sum_{k\in\mathbb{Z}} :\chi^\alpha_{n-k}\chi^\beta_k: \tag{3.55}$$

and we construct the state space by inducing from the vacuum $|0\rangle$ satisfying

$$\chi^\alpha_n |0\rangle = 0, \qquad n \geq 0. \tag{3.56}$$

Up to vacuum parity, this is the *unique* irreducible highest weight representation of this algebra [20], and it has $(c, h) = (-2, 0)$. There is an indecomposable extension [30] of this representation, freely generated from a four-dimensional space of vacuum states spanned by

$$\{|0\rangle, |\theta^\alpha\rangle, |\omega\rangle\} \tag{3.57}$$

with action of the oscillator zero modes given by

$$\begin{aligned} \chi^\alpha_0 |\omega\rangle &= -|\theta^\alpha\rangle \\ \chi^\alpha_0 \left|\theta^\beta\right\rangle &= \epsilon^{\alpha\beta}|0\rangle \\ \chi^\alpha_0 |0\rangle &= 0 \end{aligned} \tag{3.58}$$

which allows us to compute e.g.

$$L_0|\omega\rangle = |0\rangle, \qquad L_0|\theta^\alpha\rangle = 0, \qquad L_0|0\rangle = 0 \tag{3.59}$$

which shows that this is a logarithmic-type theory. It is technically a rank-2 staggered module, but with two additional fermionic ground states $|\theta^\alpha\rangle$ "between" the generalised eigenvector pair $|0\rangle$ and $|\omega\rangle$ which are in a sense invisible to the $\mathfrak{Vir}$ zero mode. Not so for the other $L_n$:

$$\begin{aligned} L_n|\omega\rangle = 0, \qquad L_{-n}|\omega\rangle &= P_{-n}(\chi^\pm)|\omega\rangle + \frac{1}{2}\epsilon_{\beta\gamma}\left(:\chi^\beta_{-n}\chi^\gamma_0: + :\chi^\beta_0\chi^\gamma_{-n}:\right)|\omega\rangle \\ &= P_{-n}(\chi^\pm)|\omega\rangle - \chi^-_{-n}\left|\theta^+\right\rangle + \chi^+_{-n}\left|\theta^-\right\rangle \\ L_n|\theta^\alpha\rangle = 0 \qquad L_{-n}|\theta^\alpha\rangle &= Q_{-n}(\chi^\pm)|\theta^\alpha\rangle + \frac{1}{2}\epsilon_{\beta\gamma}\left(:\chi^\beta_{-n}\chi^\gamma_0: + :\chi^\beta_0\chi^\gamma_{-n}:\right)|\theta^\alpha\rangle \\ &= P_{-n}(\chi^\pm) + \chi^\alpha_{-n}|0\rangle \end{aligned} \tag{3.60}$$

for all $n > 0$, and where $P_{-n}$ is a polynomial of $\chi^+$ and $\chi^-$ modes specific to $L_{-n}$ (in reality it is all the pairs of creation operators appearing in the expansion of $L_{-n}$ except for the pair containing the zero mode).

### Symplectic Fermions as a (Logarithmic) $bc$ System

The $bc$ system consists of a pair of anticommuting fields whose conformal dimensions $h_b$ and $h_c$ add to 1. We parametrise these possible systems by setting $h_b = \lambda$ and $h_c = 1 - \lambda$ for some $\lambda \geq \frac{1}{2}$. We make a comparison between the symplectic fermions discussed above and the $bc$ system at $\lambda = 1$.

We have

$$b(z)c(w) \sim \frac{1}{z-w} \tag{3.61}$$

for all $\lambda$, which of course implies

$$\{b_m, c_n\} = \delta_{m,-n} \tag{3.62}$$

Now obviously the $\lambda = 1$ $bc$ system has the wrong weights to be directly comparable to the symplectic fermions. This is evidenced by the missing factor of $m$ in the above anticommutation relation. We rectify this by identifying (say) $\chi^+(z)$ with $b(z)$ but $\chi^-(z)$ with $\partial c(z)$. Then one can compute directly from the field product

$$b(z)\partial c(w) \sim \partial_w \frac{1}{(z-w)^{h_b+h_c}} = \frac{1}{(z-w)^2} \sim \chi^+(z)\chi^-(w) \tag{3.63}$$

that the modes of these fields obey the same relations, so generate the same highest weight space on the vacuum. One must nevertheless consider such comparisons with caution: $\partial c$, being a derivative, lacks a mode at at $z^{-1}$ if $c$ is purely meromorphic, and the constant piece $c_0$ also drops away. In the prescription at hand, the "missing" mode on $z^{-1}$ is to be identified with $\chi_0^-$, which actually acts as the zero operator on the vacuum module anyway, so in this instance this hand-waving poses no real problem. The same concerns apply to $c_0$, which has no corresponding mode of $\chi^-(z)$ since it does not appear in $\partial c(z)$. If we truly wish to identify the symplectic fermions and the $bc$ system, $c$ must have logarithmic singularities, and this identification must involve a strict subalgebra of the modes of $c(z)$.

Indeed, typically $b(z)$ and $c(z)$ are written

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-\lambda}, \qquad c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n-1+\lambda} \tag{3.64}$$

but note that $\log(z)$ is also of weight 0 (its first derivative is $z^{-1}$, of weight $-1$), meaning we should redefine

$$c(z) = \log(z)c_{\log} + \sum_{n\in\mathbb{Z}} c_n z^{-n}, \tag{3.65}$$

writing

$$c(z) = \mathbf{q}^- + \log(z)\chi_0^- - \sum_{n\neq 0} \frac{1}{n}\chi_n^- z^{-n}; \tag{3.66}$$

that is,

$$c_{\log} = \chi_0^-, \quad c_0 = \mathbf{q}^-, \quad c_n = \frac{1}{n}\chi_n^-, \ (n \neq 0). \tag{3.67}$$

The already-known relations of the modes $\{c_n\}$ enforce $\{b_n, \mathbf{q}^-\} = \delta_{m,0}$. It is the new mode $c_{\log}$ for which we must find relations. However, this is fully determined by (3.61):

$$\log(w)\sum_{m\in\mathbb{Z}}\{b_m, c_{\log}\} + \sum_{m,n\in\mathbb{Z}}\{b_m, c_n\}z^{-m-1}w^{-n} = \sum_{k\in\mathbb{Z}} z^{-k-1}w^k, \tag{3.68}$$

implying $\{b_m, c_{\log}\} = 0$, consistent with our identification of $\chi_m^+ = b_m$ and each $\chi_n^-$ with the modes of $\partial c$.

The behaviour of the mode $\mathbf{q}^-$ hints at a possible induced structure as in Section 3.1.1. Indeed, if we allow the new[3] mode to act freely on the vacuum $|0\rangle$, the new state $\mathbf{q}^- |0\rangle$ satisfies

$$\begin{aligned} \chi_0^+\mathbf{q}^- |0\rangle &= |0\rangle \\ \chi_0^-\mathbf{q}^- |0\rangle &= 0, \end{aligned} \tag{3.69}$$

and since this also ensures the modes of $\mathfrak{Vir}$ act in the right way on this vector, we can confidently identify $\mathbf{q}^- |0\rangle$ with $|\theta^-\rangle$. Is it possible to integrate $b(z)$ to introduce a corresponding mode $\mathbf{q}^+$ which generates $|\theta^+\rangle$? Indeed, let us write

$$\partial^{-1}b(z) = \mathbf{q}^+ + \log(z)b_0 - \sum_{n\neq 0}\frac{1}{n}b_n z^{-n}. \tag{3.70}$$

Then we expect

$$c(z)\partial^{-1}b(w) \sim \partial_w^{-1}\frac{1}{z-w} \sim -\log(z-w), \tag{3.71}$$

and just by examining the orders of the variables attached to each mode, we must expect

$$\{\mathbf{q}^+, c_n\} = 0, \qquad n \in \mathbb{Z}\backslash\{0\}. \tag{3.72}$$

Indeed, treating $\mathbf{q}^+$ as a creation operator (so that it is placed to the right of any normal ordering), we find (writing $\partial^{-1}b_+$ and $c_-$ for the annihilation and

[3] Recall that, while $\mathbf{q}^- = c_0$ is not *new*, it was not previously identified with any $\chi_n^-$ since it does not appear in $\partial c$.

creation operators of the two fields)

$$\begin{aligned}
c(z)\partial^{-1}b(w) - \, :c(z)\partial^{-1}b(w): \, &= \{c_+(z), \partial^{-1}b_-(w)\} \\
&= \log(z)\{c_{\log}, \mathbf{q}^+\} + \sum_{m>0}\{c_m, \mathbf{q}^+\}z^{-n} \\
&\quad - \log(z)\sum_{n<0}\frac{1}{n}\{c_{\log}, b_n\}w^{-n} - \sum_{m>0}\sum_{n<0}\frac{1}{n}\{c_m, b_n\}z^{-m}w^{-n} \\
&= \log(z)\{c_{\log}, \mathbf{q}^+\} + \sum_{m>0}\{c_m, \mathbf{q}^+\}z^{-n} + \sum_{k>0}\frac{1}{n}\left(\frac{w}{z}\right)^k \\
&= \log(z)\{c_{\log}, \mathbf{q}^+\} + \sum_{m>0}\{c_m, \mathbf{q}^+\}z^{-n} - \log\left(1 - \frac{w}{z}\right)
\end{aligned} \tag{3.73}$$

where we have again used the series evaluation (3.48) as the ordering of the product implies $|w| < |z|$. In order for the above to be consistent, we see we must take

$$\{c_n, \mathbf{q}^+\} = -\delta_{n,0}, \qquad \{c_{\log}, \mathbf{q}^+\} = 0 \tag{3.74}$$

so that, in all,

$$\{\chi_n^\alpha, \mathbf{q}^\beta\} = \epsilon^{\alpha\beta}\delta_{n,0} \tag{3.75}$$

and the four distinct states $|0\rangle$, $\mathbf{q}^+ |0\rangle$, $\mathbf{q}^- |0\rangle$ and $\frac{1}{2}\epsilon_{\alpha\beta}\mathbf{q}^\alpha\mathbf{q}^\beta |0\rangle$ satisfy

$$\begin{aligned}
\frac{1}{2}\chi_0^\gamma\epsilon_{\alpha\beta}\mathbf{q}^\alpha\mathbf{q}^\beta |0\rangle &= -\mathbf{q}^\gamma |0\rangle \\
\chi_0^\alpha\mathbf{q}^\beta |0\rangle &= \epsilon^{\alpha\beta} |0\rangle
\end{aligned} \tag{3.76}$$

and, for example,

$$L_0\epsilon_{\alpha\beta}\mathbf{q}^\alpha\mathbf{q}^\beta |0\rangle = |0\rangle \,. \tag{3.77}$$

So we clearly can identify

$$\begin{aligned}
\omega &= \frac{1}{2}\epsilon_{\alpha\beta}\mathbf{q}^\alpha\mathbf{q}^\beta |0\rangle \\
\theta^\alpha &= \mathbf{q}^\alpha |0\rangle
\end{aligned} \tag{3.78}$$

and can also give state-field correspondences

$$\begin{aligned}
\frac{1}{2}\epsilon_{\alpha\beta}\mathbf{q}^\alpha\mathbf{q}^\beta |0\rangle &= \lim_{z\to 0}\frac{1}{2}\epsilon_{\alpha\beta}\left(:\partial^{-1}\chi^\alpha\partial^{-1}\chi^\beta:\right)(z) |0\rangle \\
\mathbf{q}^\alpha |0\rangle &= \lim_{z\to 0}\partial^{-1}\chi^\alpha(z) |0\rangle
\end{aligned} \tag{3.79}$$

wherein

$$:\mathbf{q}^\alpha\mathbf{q}^\beta: \, = \mathbf{q}^\alpha\mathbf{q}^\beta, \qquad \{\mathbf{q}^\alpha, \mathbf{q}^\beta\} = 0. \tag{3.80}$$

Note that these particular relations between the $\mathbf{q}^\pm$ are consistent with what has been introduced so far, but not explicitly implied by it, e.g. notably with the particular definition of the normally ordered product of two fields used in (3.73).

Explicit relations such as these will be found to come readily from more careful considerations of the operator product expansion for logarithmic fields. Since $\partial^{-1}b$ and $c$ are fermionic, $\frac{1}{2}\epsilon_{\alpha\beta}\mathbf{q}^\alpha\mathbf{q}^\beta$ is the highest degree of $\mathbf{q}$ operators that may be applied to $|0\rangle$; the chain-like structure of powers of $\mathbf{q}$ seen in $\mathcal{M}_\eta$ naturally terminates in this instance.

### 3.1.3 $\mu \neq 0$ Induced Staggered Fock Modules

In Section 3.1.1 we saw how staggered Fock modules could be induced from the vacuum vector in the same way as ordinary Fock spaces. In Section 3.1.2 we saw how this same idea could be used to produce other types of indecomposable structures involving the Virasoro algebra. This general construction relied on the introduction of new modes $\mathbf{q}$ into the underlying oscillator algebras such that the commutation relations with the $\mathfrak{Vir}$ generators reproduced the staggering operators necessary to the staggered structure. In the case of the symplectic fermions this was slightly obscured by the fact that there were two such "staggered structures" (as we have defined them), and consequently two such operators. Their fermionic nature ensured that the resulting space was generated by only finitely many vacuum vectors. An analogous bosonic structure would be that of the two complex bosons $a(z)$ and $\overline{a}(z)$; their field integrals would introduce commuting creation modes $\mathbf{q}$ and $\overline{\mathbf{q}}$ which would generate a non-negative integer lattice of ground states $\mathbf{q}^m\overline{\mathbf{q}}^n\,|0\rangle$, $m, n \geq 0$.

We would like to think of these structures in the following general way:

**3.1.1 Definition.** Let $\mathcal{M}$ be a rank-2 staggered module with family of staggering operators $\{V_n \mid n \in \mathbb{Z}\}$. Furthermore let there be an operator $Q$ such that

$$[L_n, Q] = V_n. \tag{3.81}$$

Then we call $Q$ a *co-staggering* operator for $\mathcal{M}$.

**3.1.2 Proposition.** *Let $\mathcal{M}$, $\{V_n\}_{n\in\mathbb{Z}}$, and $Q$ be as in Definition 3.1.1. Define $\mathcal{M}_{(1)} \subset \mathcal{M}$ to be the submodule consisting of all rank-1 generalised eigenvectors of $L_0$ (i.e., the span of all genuine $L_0$ eigenvectors), and*

$$\mathcal{M}_{(2)} = \mathcal{M}/\mathcal{M}_{(1)} \tag{3.82}$$

*to be the (non-staggered) quotient by this maximal non-staggered submodule. We then have*

$$\mathcal{U}\left(\mathfrak{Vir}\right) \cdot \left(Q\mathcal{M}_{(2)}\right) \tag{3.83}$$

*as a staggered submodule of $\mathcal{M}$, where $Q\mathcal{M}_{(2)}$ is a module which consists of all objects of the type $Qv$ for $v \in \mathcal{M}_{(2)}$; that is, with linear identification $Qv+Qw := Q(v+w)$ for vectors $v$ and $w$ and $kQv = Q(kv)$ for scalars $k$. The action of the Virasoro algebra is then defined be*

$$L_n(Qv) := Q(L_n v) + [L_n, Q]v \tag{3.84}$$

*which one can see results in an object which must interpreted as an element of $\mathcal{M}_{(1)} \oplus Q\mathcal{M}_{(2)}$. One can also check that the image space given in* (3.83) *behaves as it should and is a genuine Virasoro module. Since $Q\mathcal{M}_{(2)}$ is isomorphic to $\mathcal{M}_{(2)}$, we can realise this whole apparatus as a (sub)module of $\mathcal{M}$. Note that the submodule generated in this way therefore equals $\mathcal{M}$ itself only if the combined images of all the staggering operators exhausts $\mathcal{M}_{(1)}$.*

*Proof.* Reasonably simple. Since the staggering operators associated with $Q$ are maps $\mathcal{M}_{(2)} \to \mathcal{M}_{(1)}$ and by definition the commutation relations of the co-staggering operator reproduce the indecomposable action of $\mathfrak{Vir}$ on $\mathcal{M}$, we have it that the resulting module must firstly be staggered and secondly be strictly contained in $\mathcal{M}$. In addition, if the combined mappings of the staggering operators are onto, the underlying space is isomorphic as a vector space to $\mathcal{M}_{(1)} \oplus \mathcal{M}_{(2)} \cong_{\text{Vec}} \mathcal{M}$. $\square$

**Remark:**

One can easily see how these concepts generalise further to staggered modules of higher rank, as we saw implicitly in Section 3.1.1, for instance, by simply considering induced structures like that in (3.1.2) which begin from a higher power of $Q$. In this case, however, one must be careful about the exact commutation relations to be enforced between the generators of $\mathfrak{Vir}$ and the co-staggering operator, since higher-rank staggered structures can demand more than one family of staggering operator. In fact, we require $\frac{1}{2}r(r-1)$ many families in a generic rank-$r$ module — the number of strictly upper-diagonal entries when the Virasoro generators are treated as $r \times r$ upper-diagonal block matrices acting on the total space.

This seems a circuitous route by which to make more formal contact with the structures seen so far. Certainly the extra abstraction seems unnecessary when dealing with spaces induced simply by the inclusion of the new mode $\mathbf{q}$, not least because we have already examined such staggered modules $\mathcal{M}_\eta \cong \mathcal{F}_\eta[\mathbf{q}]$ with a less convoluted polynomial-type extension.

However, recall that the type of staggered Fock (super)modules constructed in Chapter 2 is completely at odds with the induced staggered modules so far. All of the co-staggering operators to this point have been of grade 0, and so the generating sets of vacuum vectors have been in the same highest-weight generalised eigenspace of $L_0$. The different-ranked pieces in the decomposition chain of the full staggered module (e.g., $\mathcal{M}_{(1)}$ and $\mathcal{M}_{(2)}$ of Proposition 3.1.2) have therefore been able to be identified with each other, with the co-staggering operator playing little more role than that of a formal variable keeping the ranked pieces separate.

Contrast this with more general indecomposable structures, such as the ones seen in Chapter 2, which cannot simply be realised as $\mathcal{F}_{\eta,\lambda}[Q]$ for some co-staggering operator $Q$, since the staggering operators map between genuinely distinct spaces. For example, $\mathcal{F}_{\eta,\lambda}$ is contained nowhere in $\mathcal{U}(\mathfrak{Vir}) \cdot Q\mathcal{F}_{\eta,\lambda}$; it does not correspond to the rank-1 subspace of the rank-2 staggered module induced by $Q$, because the staggering operators $[L_n, Q]$ do not map to $\mathcal{F}_{\eta,\lambda}$ but rather to $\mathcal{F}_{\eta+n\mu,\lambda}$ for some vacuum shift $n\mu$.

So is it actually possible to create a co-staggering operator $Q$ for the types of staggered Fock modules defined in Chapter 2? One can note certain similarities, comparing the staggering operators of these modules

$$V_n := [a_n, V_0] \tag{3.85}$$

to the staggering operators of the $\mu = 0$ induced modules seen above;

$$a_n - \lambda\delta_{n,0}, \tag{3.86}$$

which correspond to the co-staggering operator $\mathbf{q}$. If we substitute the co-staggering operator into this prescription, we find that setting

$$Q := [\mathbf{q}, V_0] \tag{3.87}$$

gives, by an application of the Jacobi identity,

$$[L_n, Q] = V_n \tag{3.88}$$

and even

$$[G_n, Q] = W_n \tag{3.89}$$

in the case of $N = 1$ and

$$\begin{aligned} [G_n, Q] &= W_n \\ [\overline{G}_n, Q] &= \overline{W}_n \\ [J_n, Q] &= X_n \end{aligned} \tag{3.90}$$

where $Q := [A\mathbf{q} + \overline{A}\overline{\mathbf{q}}, V_0]$ for $N = 2$ (compare (2.80) and (3.18) for the source of these claims).

So the surprising answer to the question posed above is: yes! We apparently *can* produce such a co-staggering operator $Q$ — with suspicious ease. It seems remarkable that such a simple operator $\mathbf{q}$ should hold so much power. It points to how deeply staggered modules are tied to logarithmic fields; how $\mathfrak{Vir}$ and its superalgebra cousins, being realisable in terms of an oscillator algebra, are altered when this oscillator algebra is extended to allow for indecomposable structures. Not only this, but the indecomposable structures are in a sense *natural* in that they arise only from allowing the new modes to act freely upon a vacuum, as one would allow any creation mode to act in a state space.

Unfortunately, of course, there is a complication, which brings the whole thing crashing down:[4]

$$[\mathbf{q}, V_0] = 0 \tag{3.91}$$

[4]It *is* logarithmic conformal field theory, after all!

if we just calculate at the level of oscillator modes. This is because $\mathbf{q}$ only commutes nontrivially with $a_0$, which "disappears" from $V_{n\times\mu}(z)$ when evaluated on a particular module, as is done when extracting the zero mode $V_0$. There are multiple potential ways to address this issue, each with their own advantages and disadvantages.

### Integrals of Vertex Operators

Given an intertwiner $V_0$ of vacuum shift $n\mu$ and the subsequent family of staggering operators $V_n = [a_n, V_0]$ created from it, regardless of the multiplicity $n$ of screening fields used to generate it, we can form a weight-1 primary field in the following way:

$$V_{[n\mu]}(z) := \sum_{n\in\mathbb{Z}} V_n z^{-n-1}, \tag{3.92}$$

i.e., this means

$$T(z)V_{[n\mu]}(w) \sim \partial_w \left( \frac{V_{[n\mu]}(w)}{z-w} \right). \tag{3.93}$$

Then if we write

$$\widehat{V}_{[n\mu]}(z) = \partial^{-1}V_{[n\mu]}(z) := Q + \log(z)V_0 - \sum_{n\neq 0} \frac{1}{n} V_n z^{-n} \tag{3.94}$$

we expect

$$T(z)\widehat{V}_{[n\mu]}(w) \sim \frac{V_{n\mu}(w)}{z-w} \tag{3.95}$$

which implies of course that

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}}\sum_{n\neq 0} \frac{-1}{n}[L_m, V_n]z^{-m-2}w^{-n} + \sum_{m\in\mathbb{Z}}[L_m, Q]z^{-m-2} = \left(\sum_{i\in\mathbb{Z}} V_i w^{-i-1}\right)\left(\sum_{k\in\mathbb{Z}} z^{-k-1}w^k\right) \\
\implies \quad &\sum_{m\in\mathbb{Z}}\sum_{n\neq 0} V_{m+n}z^{-m-2}w^{-n} + \sum_{m\in\mathbb{Z}}[L_m, Q]z^{-m-2} = \sum_{i,k\in\mathbb{Z}} V_i z^{-k-1}w^{k-i-1} \\
\implies \quad &\sum_{m\in\mathbb{Z}}[L_{m-1}, Q]z^{-m-1} = \sum_{m\in\mathbb{Z}} V_{m-1}z^{-m-1}.
\end{aligned} \tag{3.96}$$

giving

$$[L_m, Q] = V_m. \tag{3.97}$$

This circumvents the issue of determining the internal structure of $Q$ and whether or not it makes sense to try and think of it in terms of a decomposition of oscillator modes. By ignoring the problem, so to speak, or rather by turning it into one of field theory, we also pave the way to building higher-rank staggered modules of this type. We can consider some $Q$-induced module which involves higher powers of $Q$. The main issue to resolve is how to commute

$$[Q, V_n], \tag{3.98}$$

which, at least for the case of a single screening field ($n = 1$), can be computed in certain cases from the general expression

$$V_\mu(z)V_\nu(w) = (z-w)^{\mu\nu} \; :V_\mu(z)V_\nu(w):, \tag{3.99}$$

giving, if $\mu^2 = M \in \mathbb{N}$ (this only occurs in a discrete series of modules where $c = 13 - 3M - \frac{12}{M}$ for $M = 1, 2, \ldots$),

$$V_{\sqrt{M}}(z)V_{\sqrt{M}}(w) \sim 0, \quad \Longrightarrow \widehat{V}_{[n\sqrt{M}]}(z)V_{[n\sqrt{M}]}(w) \sim 0 \tag{3.100}$$

implying $[Q, V_n] = 0$.

These are exactly the vertex fields for which repeated composition makes sense; those for which the vacuum evaluation of $z^{\mu a_0}$ remains singly-valued regardless of how many multiples of $\mu$ are added to $a_0$. The total space can be generated from any vacuum vector $|\eta\rangle$ with $\eta$ of the form $\frac{m}{\mu} = \frac{m}{\sqrt{M}}$ for some $m \in \mathbb{Z}$. We do not need to be concerned in the case of higher-ranked modules about generating higher-order families of staggering operators, since the effect of acting a mode $L_n$ on any one vector $Q^k v$ is fully determined by repeated applications of the commutation relations.

However, one detail to work out is exactly what is meant by something like

$$V_i V_j \left|\eta\right\rangle. \tag{3.101}$$

Since $V_j$ takes the module generated by $|\eta\rangle$ into that generated by $|\eta + n\mu\rangle$, what exactly is meant by the index $i$ on $V_i$ in the above? An application of the composite vertex field $V_{n\times\mu}(z_1, \cdots, z_n)$ to $|\eta + n\mu\rangle$ has a different vacuum evaluation of $(z_1 \cdots z_n)^{\mu a_0}$ compared to when acting on $|\eta\rangle$, hence a different index shift on its variables, hence a different intertwining operator $V_0$ generated from the contour integration — hence, crucially, a different but related family of staggering operators, index-shifted from the first.

While true, it should be noted that it is not the composite field $V_{n\times\mu}(z_1, \cdots, z_n)$ that we are working with, but the "fake" screening field $V_{[n\mu]}(z)$ of a single variable which we created as a proxy weight-1 primary in order to introduce its integral $\widehat{V}_{[n\mu]}(z)$ and thus the constant of integration, the co-staggering operator $Q$:

$$V_{[n\mu]}(z) = \sum_{k\in\mathbb{Z}} [a_k, V_0] z^{-k-1}. \tag{3.102}$$

As such, it contains no appearances of $a_0$ and hence has no sensitivity on its own to the vacuum evaluation. The only thing which needs to be taken care of is those multiples of $a_0$ introduced under commutation with the $\mathfrak{Vir}$ generators (since, for example, $[L_n, a_{-n}] = na_0 - n(n+1)\lambda$, and $V_0$ consists of an infinite sum of normally ordered homogeneous polynomials in the generators of $\mathfrak{a}$).

There is a way to compute this directly. Note that the method of achieving the different vacuum $|\eta\rangle \mapsto |\eta + n\mu\rangle$ is entirely through the action of $\mathbf{q}$, or rather its exponential $e^{n\mu\mathbf{q}}$. This means we need not view the representation of $\mathfrak{a}$, nor the representation of $\mathfrak{Vir}$, as ever actually changing — all of the difference in vacuum evaluation coming purely from twist-like terms which appear when commuting the operators through this exponential. There is, in this sense, only one highest weight representation of $\mathfrak{a}$ (assuming $K \neq 0$), and hence only one $\mathfrak{Vir}$ representation on Fock space for each value of the central charge: the vacuum representation.

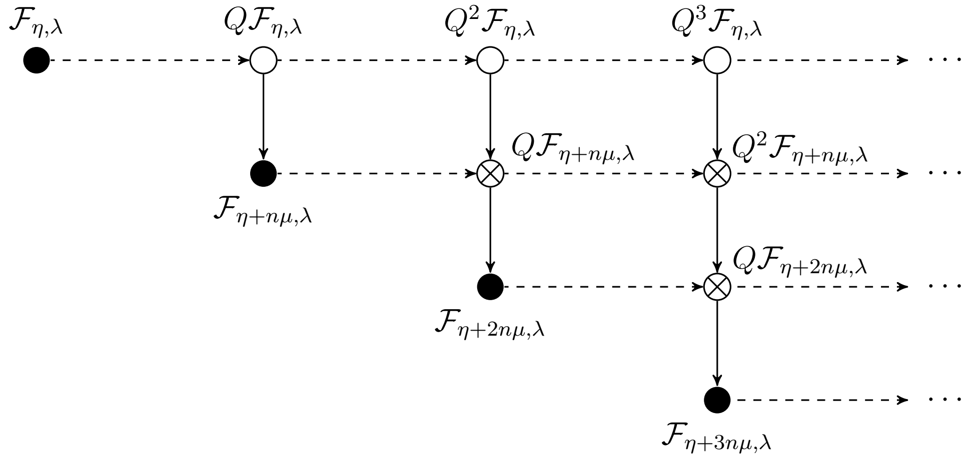


Figure 3.1: A diagrammatic representation of the indecomposable $\mathfrak{Vir}$ module induced by the polynomial extension of a Fock module by the co-staggering operator $Q$. Nodes represent subsets which can be identified with some Fock module (as vector spaces). Arrows indicate maps under certain module actions. Dotted arrows indicate the action of $Q$. Solid vertical arrows show the action of the Virasoro algebra. Each $r$th vertical chain, when taken in isolation, forms a rank-$r$ staggered module. Filled nodes denote the "bottom" subspaces consisting of rank-1 vectors, unfilled nodes denote the "top" subspaces of rank-$r$ vectors, and partially filled nodes denote subspaces of intermediate rank. The module constructed in Proposition 3.1.2 can therefore be identified with the length-2 chain in this diagram.

Let $\widetilde{a}_n$ and $\widetilde{L}_n$ denote this "true vacuum" representation; the one on $|0\rangle$ with

$$\widetilde{a}_n\,|0\rangle = 0, \quad n \geq 0 \qquad \Longrightarrow \qquad \widetilde{L}_n\,|0\rangle = 0, \quad n \geq -1. \tag{3.103}$$

We can simulate arbitrary Fock space representations generated from other vacua $|\eta\rangle$, $\eta \in \mathbb{C}$ by an additive shift:

$$\widetilde{a}_n \mapsto a_n^{(\eta)} := \widetilde{a}_n + \eta\delta_{n,0}. \tag{3.104}$$

One can easily calculate that this results in a corresponding $\mathfrak{Vir}$ representation

$$L_n^{(\eta)} = \widetilde{L}_n + \eta\widetilde{a}_n, \quad n \neq 0, \qquad L_0^{(\eta)} = \widetilde{L}_0 + \eta\widetilde{a}_0 + h_\eta \tag{3.105}$$

and that these "new" representation satisfy (for instance)

$$a_0^{(\eta)}\,|0\rangle = \eta\,|0\rangle\,, \qquad L_0^{(\eta)}\,|0\rangle = h_\eta\,|0\rangle\,, \tag{3.106}$$

as well as all the other relations expected of these algebras acting on the vacuum $|\eta\rangle$. Finally observe that

$$[\widetilde{L}_n, e^{\eta \mathbf{q}}] = e^{\eta \mathbf{q}} (\eta \widetilde{a}_n), \quad n \neq 0 \tag{3.107}$$

and

$$[\widetilde{L}_0, e^{\eta \mathbf{q}}] = e^{\eta \mathbf{q}} (h_\eta + \eta \widetilde{a}_0), \tag{3.108}$$

which correctly[5] reproduces the action of the $L_n^{(\eta)}$ on $|0\rangle$. This means that instead of considering the action of the Virasoro algebra on the domain and image space of an intertwining map as that of two distinct representations, we instead consider this as the action of one fixed representation, the vacuum representation $\widetilde{L}_n$, with an appropriate exponential of $\mathbf{q}$ simply another creation operator mediating and modifying the action. When this is the case and we have an intertwiner $\widetilde{V}_0$ for the vacuum representation, we can compute

$$\begin{aligned} [L_m^{(\eta)}, \widetilde{V}_n] &= [\widetilde{L}_m, \widetilde{V}_n] + \eta[\widetilde{a}_m, \widetilde{V}_n] \\ &= -n\widetilde{V}_{m+n} + \eta\mu\widetilde{V}_{m+n} \\ &= -(n - \eta\mu)\widetilde{V}_{m+n} \end{aligned} \tag{3.109}$$

This aligns with our understanding of how the screening operators affect the conformal weight of the vacuum in a nonlinear fashion. The conformal grade of the very same combination of modes $\widetilde{V}_n$ depends on the $a_0$ eigenvalue of the vector it acts upon. It is clear that in a module shifted to have vacuum eigenvalue $\eta$, we need to take

$$V_n = \widetilde{V}_{n+\eta\mu}, \tag{3.110}$$

and this holds true not just for the vacuum representation $\widetilde{L}_n$ but for any pair of representations whose $a_0$ eigenvalues differ by $\eta$. Thus we interpret

$$V_{n_1} V_{n_2} \cdots V_{n_k} |\eta\rangle = \widetilde{V}_{n_1+\eta\mu+(k-1)\mu^2} \widetilde{V}_{n_2+\eta\mu+(k-2)\mu^2} \cdots \widetilde{V}_{n_1+\eta\mu} |0\rangle \tag{3.111}$$

Does this make it impossible to induce higher-ranked modules with $Q$ in a grading-consistent way, given that $[L_n, Q] = V_n$ was fixed? Not necessarily; while $Q$ itself never changes, the effective representation of $\mathfrak{Vir}$ does. This action, combined with taking $[Q, V_n] = 0$ for all $n$, induces a very large space compared to the fixed-rank staggered modules of Chapter 2. Indeed, the induced module $\mathcal{F}_{\eta,\lambda}[Q]$ consists of an infinite chain of rank-$r$ staggered modules for $r = 1, 2, \ldots$ (c.f. Figure 3.1), all with $\mathcal{F}_{\eta,\lambda}$ as the top-rank space.

---

[5]The $\widetilde{a}_0$ term appearing in $\widetilde{L}_0$'s relation is what allows

$$[\widetilde{L}_0, e^{\eta \mathbf{q}} e^{\chi \mathbf{q}}] = e^{\eta \mathbf{q}} e^{\chi \mathbf{q}} (h_{\eta+\chi} + \chi \widetilde{a}_0),$$

providing an extra term so the quadratic $h_\eta$ and $h_\chi$ add correctly to give $h_{\eta+\chi}$

The Virasoro generators have the following block matrix action on each rank-$r$ staggered subspace of this $Q$-induced module:

$$\begin{bmatrix} L_n & \widetilde{V}_{n+\eta\mu+(r-2)\mu^2} & & & \\ & L_n & \widetilde{V}_{n+\eta\mu+(r-3)\mu^2} & & \\ & & \ddots & \ddots & \\ & & & L_n & \widetilde{V}_{n+\eta\mu} \\ & & & & L_n \end{bmatrix} \tag{3.112}$$

Like with Sections 3.1.1 and Section 3.1.2 we also note a kind of state-field correspondence on the true vacuum for the field $\widehat{V}_\mu(z)$. Although a grade-0 operator, $\widetilde{V}_0$ is a net annihilator of oscillator modes; it satisfies

$$\widetilde{V}_0 \left|0\right\rangle = 0. \tag{3.113}$$

Thus

$$\begin{aligned} \lim_{z\to 0} \widehat{V}_\mu(z) \left|0\right\rangle &= \lim_{z\to 0} \left( Q + \log(z)[a_0, \widetilde{V}_0] - \sum_{n\neq 0} \frac{1}{n}[a_n \widetilde{V}_0] z^{-n} \right) \left|0\right\rangle \\ &= \lim_{z\to 0} \left( Q\left|0\right\rangle + \sum_{n>0} \frac{1}{n} z^n \widetilde{V}_0 a_{-n} \left|0\right\rangle \right) \\ &= Q\left|0\right\rangle \end{aligned} \tag{3.114}$$

as one expects.

### Extension by a Logarithmic Variable

There is another way of interpreting the assignment

$$Q := [q, V_0] \tag{3.115}$$

and which has strong similarities with that of Section 3.1.3. Let us begin with the operator product expansion

$$a(z)V_\mu(w) \sim \frac{\mu V_\mu(w)}{z-w}, \tag{3.116}$$

giving

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} [a_m, V_n] z^{-m-1} w^{-n-1} = \left( \sum_{i\in\mathbb{Z}} V_i w^{-i-1} \right) \left( \sum_{k\in\mathbb{Z}} z^{-k-1} w^k \right) \tag{3.117}$$

which implies[6]

$$[a_m, V_n] = \mu V_{m+n}, \tag{3.118}$$

but also the operator product expansion can be used to evaluate the more interesting

$$\partial^{-1}a(z)V_\mu(w) \sim \log(z-w). \tag{3.119}$$

We can use the logarithmic operator product expansion (see Section 3.2) to evaluate this, and we find, along with $[a_m, V_n] = V_{m+n}$, that we are forced to take

$$[\mathbf{q}, V_n] = -\mu \log(w) V_n \tag{3.120}$$

since this is demanded by (3.119).

In a sense this is an absurd requirement; that the commutator of one mode with another depend on the variable of their parent field(s).[7] In another sense it is just what is necessary for the commutator to be non-trivial, since:

$$[\mathbf{q}, z^{\mu a_0}] = -\mu \log(w) z^{\mu a_0}, \tag{3.121}$$

matches the only term appearing in $V_\mu(w)$ which commutes non-trivially with $\mathbf{q}$. Somehow demanding consistency between logarithmic modules and logarithmic-type operator products has broken the boundary between modes and fields.

However, this in itself is not so much of a problem if we re-examine what features of the vertex operator are important from the point of view of producing *intertwiners*. These vertex fields operators a sad double life: they are defined through the normally ordered exponential of the logarithmic field $\partial^{-1}a(z)$, but are not well-defined as fields themselves unless they are acted upon a Fock module *a posteriori*, at which point the inherent variability from the factor of $z^{\mu a_0}$ they contain is "frozen out". It is *these* fields, not the vertex operators themselves, which are contour-integrated to extract intertwining operators; formal polynomial combinations of creation and annihilation operators of $\mathfrak{a}$ together with the vacuum shift operator $e^{\mu\mathbf{q}}$. It is these terms which have the desired commutation relations with the Virasoro algebra. In fact;

$$[L_n, z^{\mu a_0}] = 0 \quad \forall n, \tag{3.122}$$

so the only thing which is achieved by the vacuum evaluation of this term and subsequent contour integration is the removal of the indeterminate $z$ and an index shift on the $V_n$ to give the correct conformal grade to each. Looking again at (3.120), we seem to require

$$Q := -\mu \log(w) V_0, \tag{3.123}$$

[6]This looks like it "proves" our making use of this prescription in (2.23) to form the staggering operators of Chapter 2, but note that here we are only dealing with a single vertex operator ($n = 1$), not a general composition of them.

[7]Not the last time we will see such a thing occurring for logarithmic fields.

so the above facts suggest identifying the staggered an extension of the base Fock module by formal variables $z$ and $\zeta = \log(z)$ which satisfy

$$\partial_z \zeta = \frac{1}{z}, \qquad \partial_\zeta z = z. \tag{3.124}$$

This would require a modification of the $\mathfrak{Vir}$ action so that

$$[L_n, -\mu\zeta V_0] = V_n. \tag{3.125}$$

This can be done by setting

$$L_n \mapsto L_n - \frac{1}{\mu}\left(a_n - \lambda\delta_{n,0}\right)\partial_\zeta - \frac{1}{2\mu^2}\delta_{n,0}\partial_\zeta^2. \tag{3.126}$$

One finds that this prescription satisfies the commutation relations of the Virasoro algebra, and has the desired commutation properties. Provided the composition of the modes $V_n$ is well defined (c.f. the considerations made in Section 3.1.3), the resulting module induced from the action of this $Q$ bears some strong similarities to that seen in Figure 3.1. Vectors of rank $r$ are multiplied by powers $\zeta^r$ of the formal variable, but since each $\zeta$ is accompanied by a factor of $V_0$, in fact each rank $r$ subspace $Q^r\mathcal{F}_\eta$ can be identified with its image $V_0^r\mathcal{F}_\eta$ in $\mathcal{F}_{\eta+rn\mu}$ from the outset (every nontrivial $V_0$ is injective).

As was noted in Section 3.1.3, the identity of an intertwiner $V_0$ changes depending on the $a_0$ eigenvalue of the module, but can only ever change to another of $V_\mu(z)$'s modes. Using the notation of that section to denote true vacuum representations (i.e. on $|0\rangle$),

$$V_n = \widetilde{V}_{n+\eta\mu}. \tag{3.127}$$

which depends on the number of other $V_n$ to the right through the parameter $\eta$, as in (3.111).

It is still possible to realise this $Q$ as the constant piece of a field with logarithmic terms. If (recall (3.92)):

$$V_{[n\mu]}(z) := [a(z), V_0], \tag{3.128}$$

then we have

$$\partial_z^{-1}V_{[n\mu]}(z) := [\partial_z^{-1}a(z), V_0] \tag{3.129}$$

giving the staggering operators and co-staggering operator as previously defined. It is, however, unclear as to the benefits of choosing this construction over that of Section 3.1.3.

## 3.2 Logarithmic Vertex Algebras

By now we have already laid the principles of what we would like to call a logarithmic vertex algebra. We have been manipulating logarithmic field expressions in operator product expansions to compute commutation relations (and more) without much regard for the deeper validity of what was happening. In this section we would like to address this, introducing some definitive concepts which will hopefully pave the way for future work in making vertex algebras of logarithmic more rigorous.

Such logarithmic extensions have been considered by other authors. As noted in the introduction to this chapter, one might contrast the approach outlined here with those of e.g. [3,4] where twisted modules of standard (non-logarithmic) vertex algebras in which the twisted fields involve the logarithm of the formal variable are given. Our approach is to instead suggest enlargements of the state spaces of vertex algebras to accommodate new states whose corresponding fields are logarithmic and whose state space structure is indecomposable. The extension to logarithmic fields has been seen before, in [36], [1], and others, but we point out that the approach here appears unique in its motivation by the appearance of staggered modules as induced modules and the naturalness of their construction once the underlying algebras are enlarged to include modes which generate the staggering operators through the Lie bracket; the fact that these modes apparently correspond to the vacuum evaluation of logarithmic fields appears almost accidentally.

One of the core principles behind the operator product expansion is that the (time-ordered)[8] product $A(z)B(w)$ of two fields $A(z)$ and $B(w)$, up to a possible addition of terms regular in the variables $z$ and $w$, is a function of $z$ and $w$ which converges to a finite sum of singular terms in the domain $|z| > |w|$. These are not just any singular terms, in fact the result consists only of poles in $(z - w)$ of finite order;

$$A(z)B(w) \sim \sum_{k=1}^{N} \frac{C_k(w)}{(z-w)^k}. \tag{3.130}$$

for some $N \in \mathbb{N} \setminus \{0\}$ and some coefficient fields $C_k(w)$, which can have at worst poles at 0 or $\infty$.

This is an *analytic* approach to field theory; one can extract the coefficient fields, the "structure constants" of the vertex algebra, by performing contour integrals around $w$ — which must be delicately deformed into two oppositely-oriented loops around the origin; this accounts for the necessary ordering differences of the fields $A$ and $B$ along the contour about $w$ as it passes through the two domains $|z| > |w|$ and $|z| < |w|$. It seems that any attempt to introduce logarithmic singularities is doomed to fall at the very first hurdle, since such an integral is impossible.

Though still essentially analytic in nature, there is an alternative approach to presenting the information contained in (3.130). The time-ordered difference of this expression is, on the left hand side, the commutator of two fields, while on the right hand side it is the difference of two convergent expressions with disjoint

[8]This refers to the time ordering of events, where time is associated to the magnitude of the complex variable

domains of convergence. More details on how this is correctly interpreted may be found in Appendix A, but suffice it to say that we write the difference of these two competing convergent expansions as a single formal power series without regard for any convergent behaviour. This is in order to interpret it as an object without singularities away from the special[9] points 0 and $\infty$.

As long as one is careful about the specific space such an identity is being evaluated in, all of this is perfectly legal. The entire algebraic machinery of the operator product expansion is consistent with treating, for instance, derivatives of field products as one would intuitively expect;

$$\partial_z^m \partial_w^n \left(A(z)B(w)\right) = (\partial^m A)(z)(\partial^n B)(w) \sim \partial_z^m \partial_w^n \sum_{k=1}^{N} C_k(w)(z-w)^{-k}. \quad (3.131)$$

When attempting to include logarithmic behaviour into this machinery, it is hard to decide which properties may be relaxed and which are too important to alter. It is also difficult to decide just how and where to introduce this logarithmic behaviour, since one can see how potentially many different starting points may be used.

At the very least, it seems that having access to the usual operations of calculus is desirable, so we will seek to make this logarithmic behaviour consistent with the derivative operations mentioned above. Since the logarithmic singularity is the first antiderivative of the first order pole (or rather, since we would like to be thinking in terms of derivatives, the first order pole is the derivative of the logarithmic singularity), we would like to introduce a family of objects $\partial_w^{-k}\delta(z-w)$ which may be used to express logarithmic field commutators in the manner of (A.11). Using (A.8), we are able to show that if such a formal series as $\partial_w^{-1}\delta(z-w)$ exists, then it satisfies

$$\partial_w \left((z-w)\partial_w^{-1}\delta(z-w)\right) = -\partial_w^{-1}\delta(z-w) \quad (3.132)$$

so that multiplication by powers of $(z-w)$, instead of annihilating $\partial_w^{-1}\delta(z-w)$, generates all its antiderivatives:

$$\partial_w^{-k-1}\delta(z-w) = \frac{(-1)^k}{k!}(z-w)^k \partial_w^{-1}\delta(z-w). \quad (3.133)$$

Note that we are not requesting the existence of a new allowed operation $\partial_w^{-1}$, only the existence of antiderivatives of $\delta(z-w)$ *a priori*. Naïvely, the antiderivative of $\delta(z-w)$ with respect to $w$ is

$$\log(w) + \sum_{n\neq 0} \frac{1}{n}\left(\frac{w}{z}\right)^n + C(z), \quad (3.134)$$

[9]In typical CFT usage, such points represent the infinite past and infinite future, so actually sit outside the valid domain, any any singularities there can be “safely” ignored.

and since $\partial_z f(z-w) = -\partial_w f(z-w)$ for any function of the difference $(z-w)$, we must have either $\partial_z C(z) = 0$ or $\partial_z C(z) - = \frac{1}{z}$, since these are the only terms of $\delta(z-w)$ which do not contain a power of $w$ (and *mutando mutandis* for constant terms not containing powers of $z$). In fact, enforcing that $\partial_z \partial_w^{-1} \delta(z-w) \propto \delta(z-w)$ gives

$$C(z) = -\log(z) + C \tag{3.135}$$

for some constant of integration $C$, so we should expect

$$\partial_w^{-1} \delta(z-w) = C + \log\left(\frac{w}{z}\right) + \sum_{n \neq 0} \frac{1}{n} \left(\frac{w}{z}\right)^n. \tag{3.136}$$

The presence of the constant $C$ is a curious "gauge-type" choice which makes little to no contact with the field content. It is typical of the kind of indeterminacies that arise in treating logarithmic singularities — the indeterminacy in the choice of sheet for the complex logarithm, for instance. This degeneracy of absolute angle appears when we consider the time-ordered expansions of two logarithmic type singularities — for not only should the existence of this antiderivative apply on the right hand side of (A.11), but on the left too. This means that we should have, in particular,

$$\partial_w^{-1} \left( \iota_{z,w} \frac{1}{z-w} - \iota_{w,z} \frac{1}{z-w} \right) = \iota_{w,z} \left( \log(z-w) + C_1(z) \right) - \iota_{z,w} \left( \log(z-w) + C_2(z) \right) \tag{3.137}$$

where considerations similar to the above fix $C_i(z) = C_i$ for constants $C_i \in \mathbb{C}$, $i = 1, 2$. Here we make use of a very convenient expansion of the logarithm [21]. We have

$$\log(1-x) = -\sum_{n>0} \frac{1}{n} x^n \tag{3.138}$$

whenever $|x| < 1$. Therefore

$$\iota_{z,w} \log(z-w) = \iota_{z,w} \left( \log(z) - \log\left(1 - \frac{w}{z}\right) \right) = \log(z) - \sum_{n>0} \frac{1}{n} \left(\frac{w}{z}\right)^n \tag{3.139}$$

with a similar expansion for $\iota_{w,z}$, except with an additional additive term[10] of $\log(-1)$ from $\log(z-w) = \log(-w) + \log\left(1 - \frac{z}{w}\right)$ and $\log(-w) = \log(w) + \log(-1)$. Therefore

$$\partial_w^{-1} \delta(z-w) = C + \log\left(\frac{w}{z}\right) + \sum_{n \neq 0} \frac{1}{n} \left(\frac{w}{z}\right)^n, \tag{3.140}$$

where $C = \log(-1) + C_1 - C_2$ is the indeterminate constant of integration. One can compute further that

$$\partial_w^{-n} \log(z-w) = \frac{(-1)^{n+1}}{n!} (z-w)^n \left( \log(z-w) - H_n \right), \tag{3.141}$$

[10] We will use this term to capture the multiple possible values of the complex logarithm. $\log(z)$ and $\log(w)$ we can assume to be on the principal branch, or otherwise simply *a particular* fixed sheet in common.

where

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{3.142}$$

is the $n$th harmonic number, but note that the additional polynomial part $\frac{-H_n}{n!} \sum_{k=1}^{n} \frac{1}{k}(z-w)^n$ in (3.141) drops away when we take the time-ordered difference of the two expansions (its expansions are the same in both domains), consistent with our knowledge from (3.133) that higher order antiderivatives of $\delta(z-w)$ can be generated via multiplication by the variable $(z-w)$.

// **Remark:**

The issue of choosing a branch of log, or rather of having logarithmic correlation functions be indeterminate up to a constant, could be avoided if the field theory were made achiral; that is, if fields were generically nontrivial functions of both $z$ and $\overline{z}$, and in such a way that it is $\log|z-w|$ which appears in operator product expansions. An issue with this is that the modulus function, while continuous everywhere, is complex differentiable *nowhere*, so applications of derivatives do not make sense on the right hand side of operator product expansions. Whether or not this is a problem for internal consistency of calculation is a somewhat subtle point, since the application of derivatives to both sides of an operator product expansion in the first place is, more precisely, just a shorthand for how singular terms of the derivative fields behave compared to those of the original fields. In reality being able to make this shorthand comes from properties of contour integrals of finite-order poles, at least if we are taking the analytic route to defining operator product expansions. One could argue that now such an approach is generically invalid, but it is not clear what interpretation should replace it.

In either case, we have

$$\log|z-w| = \log(z-w) - i\Theta(z-w) \tag{3.143}$$

where $\Theta$ is the argument function. This is where both the problem with the differentiability of $\log|z-w|$ ($\Theta$ is not complex-differentiable) and the indeterminacy of $\partial_w^{-1}\delta(z-w)$ ($\Theta(x) = \Theta_0(x) + 2n\pi$ for some integer $n$ and any other choice of branch $\Theta_0$) comes from. We can conceivably pull this additive piece of $\Theta$ from the integration constant $C$ in order to change between the two interpretations of $\log(z-w)$ and $\log|z-w|$ as necessary. //

In order to actually compute logarithmic operator product expansions *a priori* (rather than relying on already knowing the expansions of some derivative fields and then taking antiderivatives) we will use an algebraic approach, which corresponds to the analytic one in non-logarithmic applications. We have in standard field theory that

$$A(z)B(w) = \sum_{k=1}^{N} \frac{C_k(w)}{(z-w)^k} + \,:A(z)B(w): \tag{3.144}$$

so we will opt to take as our *definition* of expansion, even for fields with logarithmic behaviour,

$$A(z)B(w) \sim A(z)B(w) - \,:A(z)B(w):\,, \tag{3.145}$$

so provided we have a good understanding of what a normally ordered product of logarithmic modes look like, we are in principle able to compute expansions of logarithmic fields.

We should therefore expect the generic logarithmic operator product expansion to be of the following form:

$$\begin{aligned} A(z)B(w) \sim \sum_{n=1}^{N} \frac{C_{n,0}(w)}{(z-w)^n} &+ \log(z-w) \sum_{n=-N_1}^{\infty} C_{n,1}(w)(z-w)^n \\ &+ \log(z-w)^2 \sum_{n=-N_2}^{\infty} C_{n,2}(w)(z-w)^n + \cdots \\ &\cdots + \log(z-w)^L \sum_{n=-N_L}^{\infty} C_{n,L}(w)(z-w)^n \end{aligned} \tag{3.146}$$

for some maximum power $L \in \mathbb{N}$ of complex logarithm. While we could plausibly allow for unbounded degree of log appearing in this expansion, we expect to operate with vertex algebraic structures produced from fields whose logarithmic singularities are the result of a finite number of antiderivatives applied to fields with regular singularities. This has held in every "example" of a logarithmic field theory we have seen so far. What is more, we have found that such fields can correspond to reasonable states in some vector space — since the logarithmic terms come from such antiderivatives, they correspond to coefficients which were previously associated to singular terms, so which annihilate the vacuum state.

For a formal Laurent series $A(z)$ then, let us make the following partitioning of its modes:

$$A(z) = A_L(z) + A_+(z) + A_-(z) \tag{3.147}$$

where $A_+(z)$ contains only *negative* powers of the variable,[11] $A_-(z)$ contains only *non-negative* powers, and $A_L(z)$ contains only terms with a non-zero power of $\log(z)$ — so all the logarithmic behaviour of $A(z)$ is pushed into $A_L(z)$. We expect, for such fields to have well-defined vacuum evaluations, that both $A_L(z)$ and $A_+(z)$ must annihilate the vacuum vector. Therefore we define

$$:A(z)B(w):\; := A(z)B(w) - [A_L(z) + A_+(z), B_-(w)] \tag{3.148}$$

[11] One must excuse this backwards-seeming notion, because of course the + then refers to the sign of the indices labelling the *modes*, which are (in this case) positive. Both this and the opposite labelling choice have equal potential for causing confusion.

**3.2.1 Example.** By way of example, recall the field $\partial^{-1}a(z) = \mathbf{q} + a_0 \log(z) - \sum_{n\neq 0} \frac{1}{n} a_n z^{-n}$ first used to construct the screening operators in (1.2.5). We have, of course

$$\begin{aligned} \partial^{-1}a_L(z) &= a_0 \log(z) \\ \partial^{-1}a_-(z) &= \mathbf{q} + \sum_{n>0} \frac{1}{n} a_{-n} z^n \\ \partial^{-1}a_+(z) &= -\sum_{n>0} \frac{1}{n} a_n z^{-n} \end{aligned} \tag{3.149}$$

and therefore we may calculate, as the only non-zero contributing terms,

$$\begin{aligned} [\partial^{-1}a_L(z) + \partial^{-1}a_+(z), \partial^{-1}a_-(w)] &= [a_0, \mathbf{q}] \log(z) - \sum_{m>0}\sum_{n>0} \frac{1}{mn} [a_m, a_{-n}] z^{-m} w^n \\ &= \log(z) - \sum_{n>0} \frac{1}{n} \left(\frac{w}{z}\right)^n \\ &= \log(z) + \log\left(1 - \frac{w}{z}\right) \end{aligned} \tag{3.150}$$

where the last equality follows by the assumption of time-ordering ($|z| > |w|$) and the series identity (3.48), giving us

$$\partial^{-1}a(z)\partial^{-1}a(w) \sim [\partial^{-1}a_L(z) + \partial^{-1}a_+(z), \partial^{-1}a_-(w)] = \log(z - w) \tag{3.151}$$

as we know we must expect from other arguments above. Where has the $a_0^2 \log(z) \log(w)$ term, quadratic in the logarithm, disappeared to? Of course, it has the same form in either the time-ordered or the normally ordered product, so it disappears in their difference.

It is reasonable to have qualms about this blasé appeal to algebraic manipulations in the definition of the normally ordered product of logarithmic fields, especially when so much of the non-logarithmic theory depends on (equivalent) analytic constructions. In particular, is it possible to take $z \mapsto w$ to get a suitable definition for $:AB:(w)$, when a typical implementation of such a thing from the field theory (at least in the setting of theoretical physics) makes use of contour integration techniques not strictly applicable here?

Well, $\log(z)$ and its powers are perfectly regular at $z = w$ for nonzero $w$, and is amenable to a series expansion, so there is not even any real analytic concern in taking the Taylor series

$$A(z) = A(w) + (z - w)\frac{\mathrm{d}A(w)}{\mathrm{d}w} + \frac{1}{2}(z - w)^2 \frac{\mathrm{d}^2 A(w)}{\mathrm{d}w^2} + \cdots \tag{3.152}$$

and taking $z$ to $w$.

**3.2.2 Example.** Consider a slightly more complicated example involving the second antiderivative of the bosonic field. We write

$$\hat{a}(z) = \partial_z^{-2} a(z) = \mathbf{q}_1 + \mathbf{q}_0 z + a_0 z\left(\log(z) - 1\right) - a_1 \log(z) + \sum_{n\neq 0,1} \frac{1}{n(n-1)} a_n z^{-n+1} \tag{3.153}$$

where the integration constant $\mathbf{q}_1$ is a new mode with as-yet unknown commutation relations, and $\mathbf{q}_0$ is a relabelling of the old integration constant $\mathbf{q}$ in order to distinguish the two. We have

$$\begin{aligned} \hat{a}_+(z) &= -a_0 z + \sum_{n>1} \frac{1}{n(n-1)} a_n z^{-n+1} \\ \hat{a}_L(z) &= -a_1 \log(z) + a_0 z \log(z) \\ \hat{a}_-(z) &= \mathbf{q}_1 + \mathbf{q}_0 z + \sum_{n>0} \frac{1}{n(n+1)} a_{-n} z^{n+1} \end{aligned} \tag{3.154}$$

so that we get

$$\begin{aligned} [\hat{a}_+(z), \hat{a}_-(w)] &= -[a_0, \mathbf{q}_1] z + \sum_{n>1} \frac{1}{n(n-1)} [a_n, \mathbf{q}_1] z^{-n+1} \\ &\quad - zw + zw \sum_{n>1} \frac{1}{n(n-1)(n+1)} \left(\frac{w}{z}\right)^n \\ &= -[a_0, \mathbf{q}_1] z + \sum_{n>1} \frac{1}{n(n-1)} [a_n, \mathbf{q}_1] z^{-n+1} \\ &\quad + \frac{1}{2}(z-w)^2 \sum_{n>0} \frac{1}{n} \left(\frac{w}{z}\right)^n - \frac{3}{2} zw + \frac{3}{4} w^2 \end{aligned} \tag{3.155}$$

and

$$[\hat{a}_L(z), \hat{a}_-(w)] = \log(z) \left( [a_0, \mathbf{q}_1] z - [a_1, \mathbf{q}_1] + zw - \frac{1}{2} w^2 \right) \tag{3.156}$$

and at this stage it seems untenable that there should be any appropriate algebra relations for $\mathbf{q}_1$ which will allow these expressions to yield what we naively expect for the field expansion of $\hat{a}$ with itself, namely:

$$[\hat{a}_+(z) + \hat{a}_L(z), \hat{a}_-(w)] = -\frac{1}{2}(z-w)^2 \left( \log(z-w) - \frac{3}{2} \right). \tag{3.157}$$

Indeed, while we have a few terms of an expanded $(z-w)^2$ with factors of the right series to make use of the identity (3.48), any choice at all for these relations leaves us with insufficient powers of the variables to collect a factor of $(z-w)^2$ in its entirety.

Unless, that is, if we make an unconventional choice and set

$$[a_n, \mathbf{q}_1] = -\frac{1}{n+1} z^{n+1}, \tag{3.158}$$

at least for all $n > 0$, so that commutators involving $\mathbf{q}_1$ somehow "remember" the variable associated to the field of the other algebra element. We have seen similar crossover memory between the field and algebra before, in Section 3.1.3. In this case, we find

$$\begin{aligned}[\hat{a}_+(z), \hat{a}_-(w)] = &\left(1 - \sum_{n>1} \frac{1}{n(n-1)(n+1)}\right) z^2 \\ &\frac{1}{2}(z-w)^2 \sum_{n>0} \frac{1}{n}\left(\frac{w}{z}\right)^n - \frac{3}{2}zw + \frac{3}{4}w^2 \\ = &\frac{1}{2}(z-w)^2 \left(\sum_{n>0} \frac{1}{n}\left(\frac{w}{z}\right)^n + \frac{3}{2}\right)\end{aligned} \tag{3.159}$$

because the infinite sum $\sum_{n>1} \frac{1}{n(n-1)(n+1)}$ evaluates to $\frac{1}{4}$, and

$$[\hat{a}_L(z), \hat{a}_-(w)] = -\frac{1}{2}(z-w)^2 \log(z) \tag{3.160}$$

giving, in all,

$$[\hat{a}_+(z) + \hat{a}_L(z), \hat{a}_-(w)] = -\frac{1}{2}(z-w)^2 \left(\log(z-w) - \frac{3}{2}\right) \tag{3.161}$$

after a single use of (3.48) and a regrouping of

$$\log(z-w) = \log(z) + \log\left(1 - \frac{w}{z}\right). \tag{3.162}$$

This agrees with our naïve expectations for the value of this logarithmic OPE.

Despite the fact that using (3.158) yields the "correct" expression for the field expansion of $\hat{a}(z)\hat{a}(w)$, the presence of the variable $z$ is, perhaps, more concerning than whatever benefit is gained from it. Certainly it holds strong implications for the kinds of spaces which might serve as the base vector spaces of states within any proposed definition of a logarithmic vertex algebra. The commutator with $\mathbf{q}_1$ is required to "remember" the variable associated to a field. To compute commutators involving $\mathbf{q}_1$ strictly within the space of states, where the field coefficients act as linear operators in the absence of formal variables, we must come up with a satisfactory answer to the question: which variable should be used?

Extensions of state spaces by a formal variable seem like a legitimate option, since this preserves their nature as vector spaces and by linearity does not interfere with the field map. If this were a separate special variable, say $\zeta$, not appearing in any field, it would in effect be a new grading consisting of $\mathbb{Z}$ many copies of the base state space. This does not remove the awkwardness of having to treat commutators as requiring contextual information external to the linear operators themselves, but there seems to be little choice in the matter if logarithmic field expansions are to behave as "expected". An obvious means of pushing our understanding further in this aspect would be to compute the same expansions with higher-order antiderivatives (for integration constant modes $\mathbf{q}_2, \mathbf{q}_3, \ldots$), hopefully obtaining some kind of

All of the above leads us to tentatively define a logarithmic vertex algebra.

### 3.2.1 Definition

This subsection presents a collection of conjecture and intuition relating to the proper formulation of logarithmic vertex algebras and some of their structure. Sadly, to develop what follows in any great detail would represent a body of work many times larger than suitable for the remaining space constraints of this document.

It is hoped that these concepts can provide a basis for later work. The need to find a useful definition of a logarithmic vertex (operator) algebra is driven by the apparent fact that they are the field structures associated to indecomposable representations of the Virasoro algebra. With such a theoretical basis from which to work, these so-called staggered structures are likely to be treatable in a much more consistent and comprehensive manner than has been the case in the past.

**3.2.3 Definition** (Logarithmic Vertex Algebra)**.** A *logarithmic vertex algebra* $(V, |0\rangle, T, Y)$ is a tuple consisting of the following objects:

- (*State space*) A vector space $V$
- (*Vacuum vector*) A vector $|0\rangle \in V$
- (*Translation operator*) A linear map $T : V \to V$
- (*(Logarithmic) vertex operators*) A linear map

  $$Y(\cdot, z) : V \to \operatorname{End} V[[z^{\pm 1}]][\log(z)] \tag{3.163}$$

  taking each $A \in V$ to a logarithmic field

  $$Y(A, z) = \sum_{\ell=0}^{L_A} \sum_{n \in \mathbb{Z}} A_{n,\ell} z^{-n-h_A} \log(z)^{\ell} \tag{3.164}$$

  where the constants $L_A, h_A \in \mathbb{Z}$ are allowed to depend on the particular vector $A$.

and subject to the following axioms:

- (*Vacuum axiom*) $Y(|0\rangle, z) = \mathrm{Id}_V$, and for any $A \in V$ we have

  $$Y(A, z)\,|0\rangle \in V[[z]], \tag{3.165}$$

and

$$Y(A,z)\left|0\right\rangle\big|_{z=0} = A. \tag{3.166}$$

An immediate consequence of this is $A_{-h_A}\left|0\right\rangle = A$, and furthermore that $A_{n,0}\left|0\right\rangle = 0$ for all $n > -h_A$ and $A_{n,\ell}\left|0\right\rangle = 0$ for all $n$ and $\ell > 0$.

- (*Translation axiom*) For any $A \in V$,

$$[T, Y(A,z)] = \partial_z Y(A,z) \tag{3.167}$$

  and $T\left|0\right\rangle = 0$.

- (~~*Locality axiom*~~ *Logarithmic OPE*) Instead of the standard locality axiom, we require that for every pair of states $A, B \in V$ there exist non-negative integers $N$ and $L$ such that

$$\begin{aligned}(z-w)^N Y(A,z)Y(B,w) \sim 0 \\ &+ \log(z-w)\sum_{n=-N_1}^{\infty} C_{n,1}(w)(z-w)^{n+N} + \cdots \\ &\cdots + \log(z-w)^L \sum_{n=-N_L}^{\infty} C_{n,L}(w)(z-w)^{n+N}.\end{aligned} \tag{3.168}$$

  where each $C_{n,\ell}(w \in \mathrm{End}V[[z^{\pm}]]$.

This last axiom differs stylistically from the locality axiom of non-logarithmic vertex algebras (c.f. Definition A.2.1) in that it does not give some kind of statement relating to the commutation of two fields, and does not easily translate to some concept of the structure constants of an algebra. But a logarithmic vertex algebra, being an extension of a non-logarithmic one, must have some measure of this familiar structure — the restriction of its fields to the set of those with pairwise non-logarithmic expansion products must be a vertex algebra. In addition, the fields with logarithmic behaviour seen in any kind of relevant contexts have been those which were nothing more than antiderivatives of such fields. Fields with arbitrarily bad logarithmic behaviour are, in some some sense, quite unmotivated, but must exist within the algebraic structure of a logarithmic vertex operator if we are to retain a multiplicative type operation. we look at a particular subset of logarithmic fields whose "bad behaviour" in this regard is not so extreme.

**3.2.4 Definition** (Weak Locality)**.** We will say that two fields $A(z)$ and $B(w)$ are *weakly local* if there exists non-negative integers $N$, $m$, and $n$ such that

$$(z-w)^N \partial_z^m A(z)\partial_w^n B(w) = (z-w)^N \partial_w^n B(w)\partial_z^m A(z) \tag{3.169}$$

holds. That is, the fields themselves are not local, but there exist sufficiently high derivatives which are.

This definition captures fields whose expansions contain at most a single power of $\log(z)$. It is fields whose expansions contain higher powers of the logarithm which make it particularly difficult to state a logarithmic analogue of the locality property in terms recognisable as such. Antiderivatives of non-logarithmic fields are mutually weakly local with every other, and it is this mutuality, and their closeness to regularity, which motivates the following.

**3.2.5 Conjecture** (Logarithmic Dong's Lemma)**.** Let $A_1(z_1), A_2(z_2), \ldots, A_a(z_a)$ and $B_1(w_1), B_2(w_2), \ldots B_b(w_b)$ be mutually weakly local fields. Then we conjecture that the field product

$$(:A_1A_2\cdots A_a:)(z)\,(:B_1B_2\cdots B_b:)(w) \tag{3.170}$$

satisfies the *Logarithmic OPE* axiom of Definition 3.2.3.

**Remark:**

The above conjecture, if true, gives a way to study (certain) logarithmic vertex algebras from their generating fields, just as we are able to do so with non-logarithmic ones. Weakly local fields should be "all we need": it has been apparent thus far that interesting staggered structures appear from when an antiderivative operation is allowed on base fields relevant in the equivalent non-staggered structures. However, it is also readily apparent that introducing logarithmic field behaviour has far deeper repercussions than the very simple examples relevant to staggered Virasoro modules. This can be seen, for instance, in how even the very simple step of considering the *second* antiderivative of a bosonic field demands the introduction of structure dramatically different from anything seen in the non-logarithmic case (consider, e.g., (3.158).

**3.2.6 Example** (Logarithmic Free Boson)**.** Subject to Conjecture 3.2.5, we then have a logarithmic vertex algebra generated by normally ordered products of the field

$$\begin{aligned}\partial^{-N-1}a(z) = \sum_{n=0}^{N}\frac{1}{n!}\mathbf{q}_{N-n}z^n + \frac{1}{N!}\sum_{n=0}^{N}(-1)^{N-n}\binom{N}{n}a_{N-n}z^n\left[\log(z)-H_n\right]\\ + (-1)^{N+1}\sum_{n\neq 0,1,2,\ldots N}\frac{1}{n(n-1)\cdots(n-N)}a_n z^{-n+N}\end{aligned} \tag{3.171}$$

and its derivatives, where $H_n$ is the $n$th harmonic number, that is;

$$H_n = \sum_{k=1}^{n}\frac{1}{k} \tag{3.172}$$

with $H_0 = 0$ by definition.

One reliable tactic in the study of vertex algebras is to make contact with either the field content or the underlying Lie algebraic structure as convenient — for instance, working with infinite collections of difficult combinatorial relations at the level of modes can often be avoided entirely simply a single equivalent statement at the level of the fields. If (3.158) is any indication, it may unfortunately prove impossible to make use of such conveniences in studying structures on general logarithmic fields.

The axioms and structural concepts of well-understood areas of mathematics often have many equivalent formulations. Vertex algebras are no exception, having been subjected to intense scrutiny from all sides for several decades: there are combinatorial formulations; analytic ones; ones which rely on fields as Laurent series of operators; ones which rely on formal differences of convergent expansions; ones which utilise only the linear operators; ones which never do. There are various methods of proof of important statements, each using techniques peculiar to the assumption of good behaviour in one of the above aspects; all equivalent.

Of course, when making some extension of a previously well-known area of mathematics, equivalences between its different formulations almost always break down, and it is usually very difficult to intuit which of its previously equivalent statements, if any, is appropriate to take as the "true" formulation, the one which is used to guide the development of the extended theory. Again, vertex algebras are no exception in this regard, and the inclusion of logarithms into the field content — apparently necessary for the natural appearance of staggered structures — will certainly require a careful, detailed approach.

There is sufficient self-consistency apparent in the logarithmic content explored up until now to inspire confidence in its development. It seems to provide a fruitful ground for investigation: a combination of reassuringly reasonable familiarity and surprising (exciting!) new features. It is hoped that this can be the subject of future work.

*Chapter 4*

# Conclusion

*Nature does not hurry, yet everything is accomplished*

Laozi, *Tao Te Ching*

We have seen how staggered structures seen in rank-2 logarithmic conformal field theories can be realised using a free field construction. We verified, though comparisons of the module invariant $\beta$, that the staggered Fock spaces developed here agree with the staggered modules in the literature. We did so by defining the *staggering operators* of a staggered module, which capture the indecomposable "gluing" of two Virasoro modules, and then giving a method for constructing these operators whenever the two Virasoro modules were related by an intertwining map.

The resulting formulae for $\beta$ in these staggered modules and supermodules displayed a common theme, like a modified inner product of the first singular vector of the left hand space in the short exact sequence construction of the staggered module. There was some discussion of the factorisation of $\beta$ as a function of the vacuum momentum shift $\mu$, and an apparent empirical relationship was noted between the zeroes of $\beta(\mu)$ and certain degeneracies of the staggered construction; where the left and right spaces in the short exact sequence collided with the corner entries in the relevant extended Kac table.

After repeating this construction for the $N = 1$ and $N = 2$ Viasoro superalgebras, to some degree of success (but noting that certain constructions seen in the literature were "missing"), we then turned to an examination of induced modules which exhibit staggered structure. These induced structures relied on the introduction of new modes into the oscillator algebras, which arose as the integration constants of antiderivatives of the free fields involved. Taking antiderivatives in this way introduced logarithmic behaviour into the fields of the theory, making contact between staggered modules and the types of state spaces demanded by the existence of logarithmic fields, and what could possibly lead to a self-consistent systematic understanding of logarithmic vertex algebras themselves. We make some conjectural efforts in this direction, although a proper examination of what threatens to be such a monstrous effort remains outside the scope of this work.

There are many avenues for future developments from this work. An obvious topic of interest is, of course, this continued development of logarithmic vertex algebras in a rigorous setting. Since the construction of staggering operators was defined in detail only for rank-2 staggered modules, but nevertheless has an obvious generalisation, another possible course of future work is to examine higher-rank staggered modules in more detail in this manner. Yet another is to fill out the study of staggered supermodules at $N \geq 2$, though this relies on further foundational work on the structure of the Fock superspaces that are involved as $\mathfrak{s}_N\mathfrak{Vir}$-modules. Finally, all of the developmental work here — staggering operators and their algebraic constraints, costaggering operators and their relationship to both the staggering operators and the staggered module they induce, logarithmic fields and their vacuum evaluations and logarithmic product expansions — all could be translated to apply to algebraic structures in the absence of both the Virasoro algebra and the oscillator algebras with very little extra effort. We could use this machinery to describe indecomposable staggered-type representations for algebras associated to arbitrary vertex algebras.

*Appendix A*

# Fields and Vertex Algebras

*And Jonathan said unto David, Come, and let us go out into the field. And they went out both of them into the field.*

*1 Samuel 20:11, King James Edition*

We provide a brief overview of the theory of vertex algebras and vertex operator algebras, important objects in (and an attempt at providing a mathematically rigorous foundation for) the study of quantum field theories. All of this content and more can be found treated in great depth in many dedicated works, e.g. [15], [26], or [32].

## A.1 Fields

There are some subtleties to be taken of care of in regards to the expansion of infinite series in different domains of convergence. For what follows, we will use the following notations:

$$\begin{aligned}
\mathbb{C}[z] &= \left\{ \sum_{n=0}^{N} a_n z^n \mid N \in \mathbb{N},\, a_n \in \mathbb{C} \right\} \\
\mathbb{C}(z) &= \left\{ \sum_{n=-N_1}^{N_2} a_n z^n \mid N_1, N_2 \in \mathbb{N},\, a_n \in \mathbb{C} \right\} \\
\mathbb{C}[[z]] &= \left\{ \sum_{n=0}^{\infty} a_n z^n \mid a_n \in \mathbb{C} \right\} \\
\mathbb{C}((z)) &= \left\{ \sum_{n=-N}^{\infty} a_n z^n \mid N \in \mathbb{N},\, a_n \in \mathbb{C} \right\} \\
\mathbb{C}[[z^{\pm}]] &= \left\{ \sum_{n=-\infty}^{\infty} a_n z^n \mid a_n \in \mathbb{C} \right\}
\end{aligned} \tag{A.1}$$

where in each case $z$ is a formal variable (we do not require convergence of any kind of these elements as functions on $\mathbb{C}$). In order, they are the (formal) *Taylor polynomials*, *Laurent polynomials*, *Taylor series*, *Laurent half-series*, and general *Laurent series*. Each of these sets except the Laurent series are closed under multiplication, forming commutative algebras. Attempting to take the product of two Laurent series in this way would result in infinite sums for the coefficient of each power of the variable. While the sequences are formal and need not converge, the individual coefficients must be definite elements of the scalar field. This concept of course need not be restricted to having coefficients in $\mathbb{C}$.

**A.1.1 Definition** ($R$-valued formal series)**.** Note that all of the concepts and notations of (A.1) may be extended to *$R$-valued formal series* for some $\mathbb{C}$-vector space $R$. This is done by simply replacing $\mathbb{C}$ with $R$ in the name of the set and requiring that all coefficients be from $R$. That is,

$$\begin{aligned}
R[z] &= \left\{ \sum_{n=0}^{N} a_n z^n \mid N \in \mathbb{N},\, a_n \in R \right\} \\
R(z) &= \left\{ \sum_{n=-N_1}^{N_2} a_n z^n \mid N_1, N_2 \in \mathbb{N},\, a_n \in R \right\} \\
R[[z]] &= \left\{ \sum_{n=0}^{\infty} a_n z^n \mid a_n \in R \right\} \\
R((z)) &= \left\{ \sum_{n=-N}^{\infty} a_n z^n \mid N \in \mathbb{N},\, a_n \in R \right\} \\
R[[z^{\pm}]] &= \left\{ \sum_{n=-\infty}^{\infty} a_n z^n \mid a_n \in R \right\}.
\end{aligned} \tag{A.2}$$

The notation for the Laurent series is a kind of special case of the notation we will employ for multivairate polynomials and series. For instance,

$$R[[z,w]] := \left\{ \sum_{m,n=-\infty}^{\infty} a_{m,n} z^m w^n \mid a_{m,n} \in R \right\}, \tag{A.3}$$

and similarly for the other sets defined in (A.1). One can now attempt to examine the possibility of multiplication maps like $R((z)) \times R((w)) \to R((z,w))$, though care must be taken when interpreting such products as elements of larger spaces, as different inclusion maps may be possible. This difference is in fact what permits the existence of non-trivial vertex algebras.

**A.1.2 Definition** (Fields)**.** Let $A(z) \in \operatorname{End} V[[z^{\pm}]]$, i.e. a formal Laurent series whose coefficients are linear operators on some $\mathbb{C}$-vector space $V$. Then $A(z)$ is said to be a *field* if

$$A(z)v \in V((z)), \qquad \forall v \in V. \tag{A.4}$$

Fields make it tenable to take compositions of operator-valued Laurent series acting on some vector space. We would like to develop a reasonable analogue of commutativity for operator-valued series. Note that the composition $A(z)B(z)$ does not necessarily have a well-defined $V$ action for any two arbitrary series $A(z)$, $B(w)$ on $V$.

We look at the product $A(z)B(w)$ in more detail. When acted upon a vector $v \in V$ the result may be interpreted as an object both in $V((z))((w))$ and in the larger space $\mathbb{V}[[z^{\pm}, w^{\pm}]]$. The other ordering $B(w)A(z)$, acting on the same vector, results in an element both of $V((w))((z))$ and of $\mathbb{V}[[z^{\pm}, w^{\pm}]]$. If $A(z)$ and $B(w)$ commute, then the two must be equal and the result must belong to the intersection of $V((z))((w))$ and $V((w))((z))$ in $V[[z^{\pm}, w^{\pm}]]$, which is $V[[z, w]][z^{-1}, w^{-1}]$. One finds, however, that the algebraic structures satisfying this property are in a sense trivial. A slightly weaker condition is that these series are two different expansions[1] of the same rational function from $V((z, w))[(z - w)^{-1}]$. A discussion on why this particular space of rational functions might be thought of as a mathematically astute choice can be found in [15]. This space of functions is a *physically* meaningful choice for the simple fact that if $A(z)$ and $B(w)$ are thought of as operators (standing in for interactions) creating elements of a space of physical states, then their composition must a) be some definite *third* operator, and b) can by physical symmetries only depend upon the separation $(z - w)$ of the two variables.

If we assume that this is the case, one can then study the different evaluations of elements of this space through the two different inclusion maps

$$\begin{tikzcd}[column sep=small] & V((z))((w)) \arrow[dr] & \\ V((z,w))[(z-w)^{-1}] \arrow[ur] \arrow[dr] & & V[[z^{\pm}, w^{\pm}]] \\ & V((w))((z)) \arrow[ur] & \end{tikzcd} \tag{A.5}$$

Within the context of quantum field theory, we conceptualise the two orderings $A(z)B(w)$ and $B(w)A(z)$ as two different orderings of events — the *time-ordered* difference. For our purposes here, these two orderings correspond to the cases $|z| > |w|$ and $|w| > |z|$ respectively, though only for the purposes of determining the images of the inclusion maps (i.e. $z$ and $w$ are formal variables only and convergence is not a concern, but these two "domains" instead inform the validity of making expansions in terms of $\frac{w}{z}$ and/or $\frac{z}{w}$). The most important example of

[1]Such expansions and comparisons should more precisely be made over $\mathbb{C}$, by taking the evaluation of these series with a linear functional on $V$. This amounts to a condition on the matrix elements of the field products with respect to some basis of $V$.

this is the first-order pole, whose time-ordered difference looks like

$$\begin{aligned}\iota_{z,w}\frac{1}{z-w} - \iota_{w,z}\frac{1}{z-w} &= \iota_{z,w}\frac{1}{z}\frac{1}{1-\frac{w}{z}} + \iota_{w,z}\frac{1}{w}\frac{1}{1-\frac{z}{w}} \\ &= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{w}{z}\right)^k + \frac{1}{w}\sum_{k=0}^{\infty}\left(\frac{z}{w}\right)^k \\ &= \sum_{k\in\mathbb{Z}} z^{-k-1}w^k \\ &= \delta(z-w),\end{aligned} \tag{A.6}$$

where $\iota_{z,w}$ denotes the expansion of the expression following it in the domain $|z| > |w|$, and vice versa for $\iota_{w,z}$. This expression earns the label $\delta(z-w)$ and the title of "delta function", because

$$f(z)\delta(z-w) = \left(\sum_{n,k\in\mathbb{Z}} f_n z^{n-k+1}w^k\right) = \left(\sum_{m,j\in\mathbb{Z}} f_m z^{-j+1}w^{j+m}\right) = f(w)\delta(z-w) \tag{A.7}$$

for $f(z)\sum_{n\in\mathbb{Z}} f_n z^n$. In particular,

$$(z-w)\delta(z-w) = 0. \tag{A.8}$$

Higher-order poles $(z-w)^{-k}$ are also just suitably normalised derivatives of the first order one. From (A.8) we can prove by induction that

$$(z-w)^{n+1}\partial_w^n\delta(z-w) = 0 \tag{A.9}$$

and hence that

$$\iota_{z,w}\frac{1}{(z-w)^{n+1}} - \iota_{w,z}\frac{1}{(z-w)^{n+1}} = \partial_w^{(n)}\delta(z-w) \tag{A.10}$$

(where $\partial_w^{(n)} = \frac{1}{n!}\partial_w^n$).

We may use this fact to write down commutators of fields, which as the difference of two orderings of a field composition is the difference of two expansions of the same element of $\mathrm{End}V((z,w))[(z-w)^{-1}]$. This gives the general form

$$[A(z), B(w)] = \sum_{k=1}^{N} C_k(w)\partial_w^{(k-1)}\delta(z-w), \tag{A.11}$$

thus motivating the definition:

**A.1.3 Definition** (Locality). Fields $A(z)$ and $B(w)$ are termed *local* if there exists a $N \in \mathbb{N}$ such that

$$(z-w)^N[A(z), B(w)] = 0, \tag{A.12}$$

since multiplication by a sufficiently high power of $(z-w)$ removes all of the singularities from the composition of the fields, leaving us with an element of $\mathrm{End}V((z,w))$, which has the same convergent expansion in $\mathrm{End}V[[z^{\pm},w^{\pm}]]$ regardless of the ordering of fields, causing the commutator to vanish.

This is a kind of notational shorthand to indicate that $A(z)B(w)$ and $B(w)A(z)$ have convergent expansions, in the domains $|z|>|w|$ and $|w|>|z|$ respectively, which agree up to a linear combination of poles in $(z-w)$ of finite order.[2] The expansion of the field product $A(z)B(w)$ can therefore be written as a singular piece plus a piece which stays finite as $z\to w$. This can be taken as the *definition* of the normally-ordered product $:A(z)B(w):$, the regular part of the product. We see from (A.11) that this regular piece must satisfy

$$:A(z)B(w): \;=\; :B(w)A(z): \tag{A.13}$$

and, in addition, that

$$:AB:(z)=\lim_{w\to z}\,:A(z)B(w): \tag{A.14}$$

exists.

## A.2 Vertex Algebras

The expansions field products $A(z)B(w)$ as the sum of singular terms $\sum_{k=1}^{N}C_k(w)(z-w)^{-k}$ and a regular, normally-ordered piece $:A(z)B(w):$ indicates an algebraic structure on the fields over a particular vector space, with the fields $C_k(w)$ to be thought of as structure constants. This algebraic structure is known as a *vertex algebra*, and for historical reasons relating to its development from the study of quantum field theories, its definition contains data in addition to the set of fields.

**A.2.1 Definition.** A *vertex algebra* $(V,|0\rangle,T,Y)$ is a tuple consisting of the following objects:

- (*State space*) A vector space $V$
- (*Vacuum vector*) A vector $|0\rangle\in V$
- (*Translation operator*) A linear map $T:V\to V$
- (*Vertex operators*) A linear map

$$Y(\cdot,z):V\to \mathrm{End}V[[z^{\pm 1}]] \tag{A.15}$$

[2]Equivalently that their commutator, as a formal Laurent series in two variables with coefficients in $\mathrm{End}V$, may be expressed as a linear combination of $\delta(z-w)$ and its derivatives, as in (A.11)

taking each $A \in V$ to a field on $V$

$$Y(A,z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-h_A} \tag{A.16}$$

where the constant $h_A \in \mathbb{Z}$ is allowed to depend on the particular vector $A$.

and subject to the following axioms:

- (*Vacuum axiom*) $Y(|0\rangle, z) = \mathrm{Id}_V$, and for any $A \in V$ we have
$$Y(A,z)\,|0\rangle \in V[[z]], \tag{A.17}$$
and
$$Y(A,z)\,|0\rangle|_{z=0} = A. \tag{A.18}$$
An immediate consequence of this is $A_{-h_A}\,|0\rangle = A$, and furthermore that $A_n\,|0\rangle = 0$ for all $n > -h_A$.

- (*Translation axiom*) For any $A \in V$,
$$[T, Y(A,z)] = \partial_z Y(A,z) \tag{A.19}$$
and $T\,|0\rangle = 0$.

- (*Locality axiom*) For every pair of vectors $A, B \in V$, the fields $Y(A,z)$ and $Y(B,w)$ are local.

**A.2.2 Example** (Heisenberg vertex algebra)**.** A standard example is the *heisenberg vertex algebra*. Its state space is the bosonic Fock space $\mathcal{F}_0$ generated from the vacuum vector $|0\rangle$. We can construct a translation operator by setting

$$T = \sum_{k=1}^{\infty} a_{-k} a_{k-1}. \tag{A.20}$$

It is easy to check that this is a well-defined linear operator on the state space, as only finitely many terms of the sum can give non-zero result when acted upon any particular $v \in \mathcal{F}_0$. It is no coincidence that this operator is equal to the mode $L_{-1}$ of the $\lambda = 0$ representation of $\mathfrak{Vir}$ on $\mathcal{F}_0$.

Finally, the linear map $Y$ taking states to vertex operators. We define it on a single basis vector of $\mathcal{F}_0$ and extend linearly. To a basis vector

$$a_{-n_1} a_{-n_2} \cdots a_{-n_k}\,|0\rangle \tag{A.21}$$

(for not necessarily distinct positive integers $n_1, n_2, \ldots, n_k$) we associate the normally ordered product

$$:\partial^{(n_1-1)} a(z) \partial^{n_2-1} a(z) \cdots \partial^{(n_k-1)} a(z):\ . \tag{A.22}$$

Of course, it remains to prove that these assignments satisfy the various axioms of a vertex algebra, not least that these infinitely many fields are actually mutually local. For this in particular we turn to *Dong's lemma*.

**A.2.3 Lemma** (Dong's lemma)**.** *If $A(z)$, $B(w)$, and $C(x)$ are three mutually local fields, then the fields $:AB:(z)$ and $C(w)$ are local.*

*Proof.* A repeated application of the definition of locality. Refer to (e.g.) [28,33] for a formal proof, but the outline is as follows.

Since the three fields are mutually local, we may find integers $n$ large enough so that the products $(z-w)^n A(z)B(w)$, $(z-x)^n A(z)C(x)$ and $(w-x)^n B(w)C(x)$ are commutative. By writing (e.g.) $(z-x) = (z-w) - (w-x)$ we may also make an expansion of powers one of these differences in terms of the other two; by taking $n$ initially large enough we may do so in such a way that all terms in the expansion contain enough factors to commute $C$ through both $A$ and $B$, and $A$ and $B$ through each other, where necessary. □

**A.2.4 Example** (Heisenberg vertex algebra continued)**.** One can check, now, that

$$\partial_z^{(m)} a(z) \partial_w^{(n)} a(w) \sim (-1)^m \binom{m+n}{m} \frac{1+m+n}{(z-w)^{2+m+n}} \tag{A.23}$$

for all $m, n \geq 0$, so all derivatives of the field $Y(a_{-1}, z)$ (corresponding to the fields $Y(a_{-k}, z)$ for $k = 1, 2, 3, \ldots$) are mutually local, hence all their normally ordered products are mutually local, and *locality* holds in the heisenberg vertex algebra.

It is now easy to check that the other axioms of a vertex algebra hold. For instance, for *translation* it suffices to check that $[T, Y(a_{-k}, z)] = \partial_z Y(a_{-k}, z)$ for $k > 0$. We will not do this here, but will point out another important feature of the heisenberg vertex algebra, and that is its grading.

Recall that the state space $\mathcal{F}_0$ of this vertex algebra, a bosonic Fock space, carries a natural $\mathbb{Z}$-grading which is compatible with the action of the algebra. The field associated to a monomial of algebra generators preserves this sense of grading; in the expansion of the field as a Laurent series we find that the total degree of each term — the sum of the coefficient's grade as an operator on $\mathcal{F}_0$ and the power of the variable it is attached to — is constant. What is more, this total grading is preserved by the OPE, in that the product expansion of a pair of fields will have degree equal to the sum of their individual degrees.

**A.2.5 Definition** ($\mathbb{Z}$ Graded vertex algebra)**.** A vertex algebra $(V, |0\rangle, T, Y)$ is said to be $\mathbb{Z}$-*graded* if $V$ is $\mathbb{Z}$-graded, $T$ is a graded operator of degree 1, and for each $v \in V_m$ (the $m$th graded piece of $V$) we have $Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n+m}$ with each $v_n$ an operator of grade $n$.

**A.2.6 Definition.** A $\mathbb{Z}$-graded vertex algebra $(V, |0\rangle, T, Y)$ is said to be *conformal* if it possesses a distinguished vector $\omega \in V$ of grade 2 such that

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{A.24}$$

where the coefficients $L_n$ satisfy the Virasoro commutation relation for some central charge $c_\omega$. If in addition the vector space $V$ consists only of finite-dimensional graded subspaces with some minimal grade $h$ for which all $V_n$ with $n < h$ are trivial (in other words, if $V$ is a highest weight space), the we say that $(V, |0\rangle, T, Y)$ is a *vertex operator algebra*.

**A.2.7 Example** (Heisenberg vertex algebra continued II)**.** Consider the vector

$$\omega = \left(\frac{1}{2}a_{-1} + \lambda a_{-2}\right)|0\rangle \in \mathcal{F}_0. \tag{A.25}$$

We have, in the heisenberg vertex algebra,

$$Y(\omega, z) = \frac{1}{2} :a(z)^2: +\lambda\partial a(z) \tag{A.26}$$

whose modes $L_n$ satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)(1 - 12\lambda^2)\delta_{m,-n}, \tag{A.27}$$

meaning that the heisenberg vertex algebra is indeed a vertex *operator* algebra.

We go no further here with our discussion of vertex algebras, the purpose only being to provide some introduction and background on the sporadic appearances of field theory within the body of the main text. Chapter 3 does contain somewhat more than a basic amount of field theory, but in this case we direct the interested reader to any of the excellent stand-alone works on the topic already mentioned at the start of this appendix; e.g. [15], [26], or [32].

# Nomenclature

$(p, q)$ integer parameters fixing the central charge of various reducible $\mathfrak{Vir}$ modules, page 13

$(r, s)$ integer parameters fixing the vacuum $L_0$ eigenvalue of reducible $\mathfrak{Vir}$ modules given a fixed central charge, page 13

$\beta$ an important module invariant for rank-2 staggered modules, page 45

$\mathfrak{a}$ bosonic oscillator algebra, page 19

$\chi^\alpha(z)$ symplectic fermion fields, page 80

$\Delta h$ difference in $L_0$ vacuum eigenvalue for two Fock spaces related by an intertwiner, page 29

$\eta^{\pm}_{r,s}$ $a_0$ vacuum eigenvalue(s) for $r, s$ entry of Kac table ($p, q$ assumed given), page 24

$\eta$ $a_0$ vacuum eigenvalue for generic bosonic Fock module, page 20

$\mathfrak{b}$ fermionic oscillator algebra, page 19

$\mathcal{F}^{\mathfrak{a}}_{\eta}$ bosonic Fock space, page 20

$\mathcal{F}^{\mathfrak{b},\sigma}_{\updownarrow}$ fermionic Fock space, page 20

$\mathcal{F}^{\pm}_{r,s}$ shorthand notation for $\mathcal{F}^{\mathfrak{a}}_{\eta^{\pm}_{r,s},\lambda_{p,q}}$, page 24

$|\eta\rangle$ bosonic Fock vacuum, page 20

$|\updownarrow\rangle^{\sigma}$ fermionic Fock vacuum, page 20

$\Lambda_n$ extended adjoint representation of $\mathfrak{Vir}$ on $\mathcal{M}_0$, page 77

$\lambda_{p,q}$ the $\lambda$ of Fock spaces from the $p, q$ Kac table, page 24

$\lambda$ "deformation-like" parameter which sets a $\mathfrak{Vir}$ representation on a $\mathcal{F}^{\mathfrak{a}}_{\eta}$, page 22

$\langle\cdot,\cdot\rangle_{\mathcal{V}_{h,c}}$ Shapovalov form on Verma module $\mathcal{V}_{h,c}$, page 11

$\mathcal{V}_{h,c}$ Virasoro Verma module of highest weight $(h, c)$, page 10

$\mu$ vacuum shift parameter of a vertex operator, page 26

$\mu^{\pm}$ allowed $\mu$ given some fixed extended Kac table, page 29

$:\cdot:$ normal ordering of operators, page 22

$\mathfrak{q}$ "canonical position operator", extends $\mathfrak{a}$, page 25

$\mathcal{S}_1\mathcal{F}^{\sigma}_{\eta,\updownarrow}$ Fock superspace with 1 fermion, page 29

$\sigma$ 0 (Ramond) or $\frac{1}{2}$ (Neveu-Schwarz) fermionic sector, page 15

$\mathcal{M}$ generic staggered module, page 42

$\mathcal{M}_\eta$ module induced from $\widehat{\mathfrak{a}}$ acting on $|\eta\rangle$, page 75

$\mathfrak{s}_N\mathfrak{Vir}$ $N$-fermion Virasoro superalgebra, page 15

$\mathcal{U}(\cdot)$ universal enveloping algebra of a Lie algebra, page 7

$\mathfrak{Vir}$ Virasoro algebra, page 6

$\mathfrak{Vir}_n$ $n$th graded subspace, page 6

$\widehat{\mathfrak{a}}$ the algebra $\mathfrak{a}$ extended by $\mathbf{q}$, page 74

$\widetilde{a}_n, \widetilde{L}_n$ true vacuum ($|0\rangle$) representations of $\mathfrak{a}$ and $\mathfrak{Vir}$, page 90

$a(z)$ free boson field, page 19

$a_n^{(\eta)}, L_n^{(\eta)}$ representations of $\mathfrak{a}$, $\mathfrak{Vir}$ on $\mathcal{F}_\eta$ as compared to those on the true vacuum module $\mathcal{F}_0$, page 90

$a_n$ bosonic osciallator mode, page 19

$A_\tau, B_{(v)}$ expansions coefficients of first singular vector in a given $\mathcal{M}$, page 53

$b(z)$ free fermion field, page 19

$b_n$ fermionic osciallator mode, page 19

$C$ Virasoro central element, page 6

$C_\tau, \overline{C}_\tau$ combinatorial factors related to integer partition $\tau$, page 53

$c_{p,q}$ $C$ eigenvalue for reducible Verma/Fock module, page 13

$G(z), G^\pm(z)$ field(s) of fermionic super-Virasoro generators $G_n, G_n^\pm$, page 17

$G_n, G_n^\pm$ fermionic super-Virasoro generator(s), page 15

$h_\mu$ conformal weight of vertex operator $V_\mu(z)$ (given particular $\lambda$), page 26

$H_n$ $n$th harmonic number; sum of the first $n$ integer reciprocals, page 97

$h_{r,s}$ $L_0$ vacuum eigenvalue for reducible Verma/Fock module, page 13

$J(z)$ field of bosonic super Viraosro generators $J_n$, page 17

$J_n$ bosonic super-Virasoro generator, page 16

$L_n$ Virasoro generator, page 6

$L_{-(\tau)}$ monomial of generators of $\mathfrak{Vir}$ as specified by integer partition $\tau$, page 53

$N$ (usually) the number of fermionic fields included in a given superalgebra, page 15

$p_k$ $k$th power sum of (given) fixed number of variables, page 28

$Q$ co-staggering operator, page 85

$T(z)$ energy-momentum tensor; field of Virasoro generators, page 6

$U_{\Delta n}$ particular element of $\mathcal{U}(\mathfrak{Vir})$ relating elements of a given $\mathcal{M}$, page 52

$U_n$ generic element of $\mathcal{U}(\mathfrak{Vir})$ of weight $n$, page 52

$V_\mu^{(0)}(z), V_\mu^{(\frac{1}{2})}(z)$ even/odd vertex operator couplet in $N=1$ Fock superspace, page 33

$V_0$ Fock space intertwiner arising as the zero mode of some $V_\mu(z)$ or composition thereof, page 27

$V_\mu(z)$ vertex operator of vacuum shift $\mu$, page 26

$V_n$ bosonic staggering operators (equivalently, modes of a sreening field), page 48

$V_{[n\mu]}(z), \widehat{V}_{[n\mu]}(z)$ Virasoro primary fields constructed from staggering operator data, page 88

$V_{n\times\mu}(z)$ $n$-fold composition of $V_\mu(z)$, page 27

$W_n$ fermionic staggering operators, page 58

$x^{(n)}$ bracketed index notation for normalisation of powers by a factor of $n!$, page 76

$X_n$ weight-1 bosonic staggering operators, page 67

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