Subregular W-Algebras

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Declaration

I hereby declare that:

(i) The thesis comprises only my original work towards the Doctor of Philosophy except where indicated in the preface.

(ii) Due acknowledgement has been made in the text to all other material used.

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Abstract

While rational W-algebras have enjoyed many years of attention, nonrational W-algebras are increasingly at the center of many important developments in mathematics and physics. Despite this, detailed knowledge of the structure and representation theory of such vertex operator algebras is lacking. Investigation into various examples has revealed that despite their complexity, nonrational W-algebras often exhibit rich features that make them suitable for logarithmic conformal field theory and other applications.

In this thesis, we first study the representation theory of a W-algebra known as the Bershadsky–Polyakov algebra. A classification of simple weight modules is achieved as well as the construction of some interesting nonsimple modules. These results, in conjunction with an 'inverse' to quantum hamiltonian reduction, are then used to determine the modular transformations and Grothendieck fusion rules of nonrational Bershadsky–Polyakov minimal models in terms of rational Zamolodchikov minimal models. Finally, we describe how features of the preceding analysis generalise to all subregular W-algebras of type A. There, an inverse quantum hamiltonian reduction is defined and is used to relate the representation theories of subregular and regular W-algebras despite the generic nonrationality of the former.

Overall the results of this thesis support a holistic approach to nonrational W-algebras with inverse quantum hamiltonian reductions playing a central role.

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Contents

Chapter 0

Introduction

0.1. Background

Logarithmic conformal field theories are increasingly playing a role in applications of conformal field theories to physics. These include percolation [39, 163], 4d-2d duality [29], quantum Hall transitions [164] and string theory [90, 126]. The vertex operator algebras associated to such conformal field theories are often nonrational. While rational vertex operator algebras, their representations/fusion rings and modular properties may be studied using general results like the Verlinde formula [94, 96, 156], such results for nonrational vertex operator algebras are not known.

There are small rank examples that are particularly well-understood such as admissible-level $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ [50, 52], admissible-level $\mathsf{L}_\mathsf{k}(\mathfrak{osp}(1|2))$ [47, 142, 168], the singlet and triplet algebras [8, 51, 85, 110], and the $\beta\gamma$ ghost vertex algebra [77, 137, 144]. In each case, a conjectural continuous version of the famed Verlinde formula appears to give sensible answers for Grothendieck and genuine fusion rules [51, 146]. What is lacking in general are results about the structure and representation theory of nonrational vertex operator algebras and a proof of such a 'logarithmic Verlinde formula'.

A particularly important class of logarithmic conformal field theories are fractional-level Wess-Zumino-Witten models. The associated vertex operator algebras are the fractional-level affine vertex operator algebras. In the case of admissible-level $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, it was quickly realised that if one works only with highest-weight representations, the usual Verlinde formula gives nonsensical answers [27, 34, 118].

It was later shown that fusion products in admissible-level $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ also contain representations that are not highest-weight [84]. In particular, a detailed analysis of the fusion rules for $\mathsf{L}_{-4/3}(\mathfrak{sl}_2)$ showed that the fusion of two highest-weight representations can give representations that are reducible-but-indecomposable and some whose energy eigenvalues are not bounded below.

To define a sensible 2d chiral conformal field theory with chiral symmetry algebra given by an admissible-level affine vertex operator algebra $\mathsf{L}_\mathsf{k}(\mathfrak{g})$, the appropriate 'larger' class of representations that needs to be considered appears to be the *relaxed highest-weight* representations of the underlying $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ and their spectral flows [69]. Such representations were classified for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ in [7], for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$ in [21] and for $\mathsf{L}_\mathsf{k}(\mathfrak{osp}(1|2))$ in [47, 168]. Additionally, if one has a classification of highest-weight modules for a certain quotient of the universal enveloping algebra $\mathsf{U}(\mathfrak{g})$, an algorithm for the classification of relaxed modules for general $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ is presented in [114]. For admissible levels, such a highest-weight classification was completed by Arakawa [18].

For admissible-level $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$, admissible-level $\mathsf{L}_\mathsf{k}(\mathfrak{osp}(1|2))$ and $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$, the characters and conjectural Grothendieck fusion rules of relaxed highest-weight modules are known [47,50,52,113, 115]. Genuine fusion rules are only known for certain levels and modules. Curiously the characters of relaxed modules for nondegenerate admissible-level $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ contain characters for modules of the Virasoro minimal model [52, 134], which is obtained from the simple vertex operator algebra $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ by quantum hamiltonian reduction.

Characters for relaxed highest-weight modules are also known for admissible-level $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_{n+1})$ [112]. Here, as in the $\mathfrak{sl}_2$ case, the character formulae contain characters of the *minimal* quantum hamiltonian reduction associated to $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_{n+1})$. Beyond this data in these specific cases, much is still unknown.

It is suspected that, in general, the characters of admissible-level affine vertex operator algebras $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ will depend on the characters of their quantum hamiltonian reductions $\mathsf{W}^\mathsf{k}(\mathfrak{g}, f)$ (so-called *W-algebras*) where $f \in \mathfrak{g}$ is nilpotent. Such reductions were first defined in terms of vertex algebras in the case of regular nilpotent elements [65] and more generally in [102, 106].

The relationship between characters of affine vertex operator algebras and of W-algebras, explored in more detail in [158], can be understood in terms of an 'inverse' to quantum hamiltonian reduction. The first example of such an inverse was described by Semikhatov [147] for the case of $\mathfrak{g} = \mathfrak{sl}_2$. It was later shown by Adamović that this inverse reduction can be deployed to understand some of the representation theory of the universal affine vertex operator algebra $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$ and, at certain levels, its simple quotient $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$ [2].

Crucially, the quantum hamiltonian reduction of $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_2)$ is the Virasoro vertex operator algebra Vir^k whose simple quotient is rational when k is admissible but not integral (the aforementioned Virasoro minimal models) in contrast to the nonrational $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$. Remarkably, the relaxed highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules admit concise descriptions in terms of Virasoro minimal model modules using inverse quantum hamiltonian reduction, explaining the relationship between characters.

In the $\mathfrak{g} = \mathfrak{sl}_3$ case, inverse quantum hamiltonian reductions can be described from all $\mathfrak{sl}_3$ W-algebras to $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_3)$, in addition to an inverse reduction relating the two nonaffine W-algebras: the Bershadsky–Polyakov algebra BP^k and the Zamolodchikov algebra [3, 4]. In the latter case, what is being inverted is an as-of-yet undefined partial quantum hamiltonian reduction that relates W-algebras corresponding to a fixed $\mathfrak{g}$.

As in the $\mathfrak{sl}_2$ case, the simple quotient of the Zamolodchikov algebra is rational at nondegenerate admissible level [17]. However, the inverse reduction relating the Zamolodchikov algebra and $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ naturally factors through the Bershadsky–Polyakov algebra which is expected to be nonrational at these levels.

The simple quotient BP_{k} of the Bershadsky–Polyakov algebra is known to be rational at admissible levels $\mathsf{k} = -3 + \mathsf{u}/2$ where $\mathsf{u} \in \mathbb{Z}_{\geqslant 3}$ is odd [15]. For all other admissible levels, it is expected that there is a class of 'relaxed highest-weight' modules for the Bershadsky–Polyakov algebra that are both necessary for logarithmic conformal field theory considerations and related to modules for the Zamolodchikov algebra using inverse quantum hamiltonian reduction. This also means that, at these levels, BP_{k} is a nonrational W-algebra. One of our aims here is to show that this is the case.

It therefore appears that it is necessary to investigate the structure of nonrational W-algebras to gain additional information about fractional-level affine vertex operator algebras. Moreover, W-algebras are interesting examples of vertex operator algebras in their own right as they appear frequently in both physics [87, 151, 170] and mathematics [75, 161]. While there are many examples of rational W-algebras where the conformal field theoretic information is known [15, 17, 22], nonrational W-algebras are still largely mysterious.

This project broadly aims to explore nonrational W-algebras by combining direct computations (where possible) with inverse quantum hamiltonian reductions. One important feature of this approach is that inverse quantum hamiltonian reduction is particularly effective at simplifying calculations important for conformal field theory applications. The examples of W-algebras explored in this thesis are all examples of *subregular W-algebras*.

Subregular W-algebras appear in the Schur indices of $4D$ superconformal field theories known as Argyres–Douglas theories [24, 29, 41] and the subregular nilpotent orbit also plays a crucial role in singularity theory [148]. As such, subregular W-algebras represent an important class of vertex operator algebras for which a greater understanding might indicate a path towards more general W-algebras, in addition to more interdisciplinary endeavours.

0.2. Outline

Chapter 1 starts with preliminary material on vertex operator algebras and their modules in Section 1.1. An important associative unital algebra associated to a vertex operator algebra is the Zhu algebra. The relationship between the representation theory of a vertex operator algebra and its Zhu algebra is reviewed in Section 1.1.4. An informal account of the Verlinde formula, relating the fusion product of vertex operator algebra modules to modular transformations of characters, is also given.

Section 1.2 is devoted to a very important family of vertex operator algebras known as universal affine vertex operator algebras $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ and their simple quotients $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$. The modular transformations and fusion rules of $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ when $\mathsf{k} \in \mathbb{Z}_{\geqslant 0}$ are reviewed in Section 1.2.2. Detailed analysis of such vertex operator algebras when k is admissible but not integral motivates studying W-algebras.

The construction of W-algebras by quantum hamiltonian reduction of affine vertex operator algebras is described in Section 1.3 following [**102**,**106**]. Noting that the W-algebras corresponding to $\mathfrak{g} = \mathfrak{sl}_2$ are well understood, we take up the task of analysing $\mathfrak{g} = \mathfrak{sl}_3$ W-algebras. The first of these is the prototypical W-algebra: the Zamolodchikov algebra [**170**]. The representation theory, modular transformations and fusion rules of its simple quotient $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$, where $\mathsf{k} = \frac{\mathsf{u}}{\mathsf{v}} - 3$ with $\mathsf{u}, \mathsf{v} \in \mathbb{Z}_{\geqslant 3}$ coprime, are well known and recalled in Section 1.3.3.

In Chapter 2 we construct the other $\mathfrak{sl}_3$ W-algebra, the universal Bershadsky–Polyakov vertex operator algebra BP^{k} and its simple quotient BP_{k}. The BP^{k}- and BP_{k}-modules of interest, untwisted and twisted relaxed highest-weight modules, are introduced in Section 2.2. We then explain how to identify these modules using the untwisted and twisted Zhu algebras of BP^{k}. This leads to a classification of untwisted highest-weight BP^{k}-modules (Theorem 2.2.6).

The twisted classification (Theorem 2.2.16) requires the identification [**15**] of the twisted Zhu algebra with a central extension of a Smith algebra [**149**] (Proposition 2.2.8). A classification of simple weight modules, with finite-dimensional weight spaces, of this extension is achieved in Theorem 2.2.15. This readily gives a classification of simple twisted relaxed highest-weight BP^{k}-modules by the twisted version of Zhu's theorem. For later use, we also introduce coherent families of modules for the twisted Zhu algebra of BP^{k}, following [**128**].

In Section 2.3 we convert the classification results for the universal Bershadsky–Polyakov algebras BP^{k} into the corresponding results for their simple quotients BP_{k} at nonintegral admissible level k. After reviewing the highest-weight theory of the simple affine vertex operator algebra $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$ [**18**, **103**], we prove a crucial result on the surjectivity of quantum hamiltonian reduction on simple highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$-modules in Section 2.3.2.

The classification of simple highest-weight BP_{k}-modules is easily deduced from this (Theorem 2.3.15). In Section 2.3.4, we show this lifts to a classification of simple relaxed highest-weight BP_{k}-modules using coherent families. The existence of reducible-but-indecomposable relaxed highest-weight BP_{k}-modules is similarly proved in Section 2.3.4. A simple consequence of these results is that BP_{k} is nonrational when k is admissible with $\mathsf{v} > 2$. We conclude this chapter by illustrating our classification results with some examples in Section 2.3.5.

The material presented in Chapter 2 appears in [**62**], and was obtained in collaboration with Kazuya Kawasetsu and David Ridout.

Chapter 3 is concerned with the relationship between the Bershadsky–Polyakov and Zamolodchikov algebras and its consequences. This takes the form of an embedding of BP^{k} into the Zamolodchikov algebra tensored with the 'half lattice' vertex algebra Π [4]. The idea, discussed in Section 3.1, behind such embeddings is that they are partial inverses to quantum hamiltonian reduction functors.

Importantly, for any nondegenerate admissible level, the embedding descends to an embedding of simple quotients $\mathsf{BP}_{\mathsf{k}} \hookrightarrow \mathsf{W}_3(\mathsf{u},\mathsf{v}) \otimes \Pi$. With the modular transformations and fusion rules of the rational minimal model $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ in hand, we use this fact to study the modular transformations and fusion rules for $\mathsf{BP}(\mathsf{u},\mathsf{v}) = \mathsf{BP}_{\mathsf{k}}$ whose simple relaxed highest-weight modules were classified in Chapter 2.

The results obtained thus far fit within the framework of the standard module formalism of [51,146] with spectral flows of relaxed $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules playing the role of the standard modules. Here it is convenient to modify the conformal structure of $\mathsf{BP}(\mathsf{u},\mathsf{v})$ so that the standard modules are untwisted. Section 3.2.3 then describes how to compute the characters of standard $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules in terms of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ characters using the inverse quantum hamiltonian reduction embedding. The modular S-matrix for standard one-point functions is computed in Section 3.2.5.

The standard module formalism also details how to extend this modularity to the simple highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules and is applied in Section 3.3. To minimise complications, we first consider $\mathsf{BP}(\mathsf{u},\mathsf{v})$ with $\mathsf{v} = 3$. These cases exemplify the general structure and, subject to the conjectured equalities (3.3.2) (the standard Verlinde formula for nonrational vertex operator algebras), the Grothendieck fusion rules of all simple weight modules are computed (Theorem 3.3.6). We conclude our $\mathsf{v} = 3$ studies by identifying simple currents of $\mathsf{BP}(\mathsf{u},3)$ and exploring the example $(\mathsf{u},\mathsf{v}) = (4,3)$ in Section 3.3.3.

The remainder of this chapter is devoted to attacking the general nonrational minimal model $\mathsf{BP}(\mathsf{u},\mathsf{v})$. Section 3.3.4 contains character formulae for all highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules and the modular S-matrix for the simplest class of these is obtained in Theorem 3.3.15. The standard Grothendieck fusion rules are then computed in Section 3.3.6 and simple currents are identified. Finally, these general results are illustrated with the example $(\mathsf{u},\mathsf{v}) = (3,4)$ in Section 3.3.7.

The material presented in Chapter 3 appears in [63], and was obtained in collaboration with David Ridout.

Chapter 4 focuses on generalising essential ingredients of the preceding analysis to subregular W-algebras of type-A $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$. In Section 4.1.1, regular W-algebras $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}})$ are introduced. For example, the aforementioned Zamolodchikov algebra is the $\mathfrak{sl}_3$ regular W-algebra. The representation theory of W_{n+1} minimal models, which are the simple vertex operator algebras $\mathsf{W}_{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}})$ when k is nondegenerate-admissible, is well known and described following [17].

The titular subregular W-algebras, of which the Bershadsky–Polyakov algebra is an example, are introduced in Section 4.1.2 along with the definition of an important 'spectral flow' automorphism and an identification of the $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$-modules of interest.

Unravelling the relationship between subregular W-algebras, regular W-algebras and the half-lattice vertex operator algebra requires free-field realisations of both the regular [123] and subregular [88] $\mathfrak{sl}_{n+1}$ W-algebras. Such free-field realisations are the focus of Section 4.2.

In Section 4.3, these free-field realisations are used to show that for generic k, there exists an 'inverse quantum hamiltonian reduction' embedding $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}}) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}}) \otimes \Pi$ with an explicitly known screening operator (Theorem 4.3.2). This embedding is made explicit in Section 4.3.2 where we decompose free-field strong generators of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$ [89] in terms of fields in $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}})$ and Π for all noncritical k. That the inverse quantum hamiltonian reduction embedding exists for all noncritical k is a consequence of these decompositions and the injectivity of certain free-field realisations.

With the inverse quantum hamiltonian reduction embedding in hand, Section 4.3.3 explores the consequences for the representation theory of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$. We show that taking tensor products of appropriate $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}})$- and Π-modules results in ($\mathbb{Z}$-graded) $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$-modules. Proposition 4.3.16 shows that relaxed highest-weight $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$-modules can be constructed as tensor products of irreducible $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}})$-modules with certain relaxed Π-modules, as encountered for the Bershadsky–Polyakov and Zamolodchikov algebras. When the embedding in Theorem 4.3.2 descends to an embedding $\mathsf{W}_{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}}) \hookrightarrow \mathsf{W}_{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}}) \otimes \Pi$ of simple quotients is determined in Section 4.3.4.

The material presented to this point in Chapter 4 appears in [61]. We conclude by using a similar free-field approach to prove the existence of an inverse quantum hamiltonian reduction embedding from the subregular W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$ to a hook-type W-algebra in Section 4.4, generalising a recent result for $\mathfrak{sl}_3$ [3].

Chapter 1

Vertex Operator Algebras

1.1. Vertex Operator Algebras and their Modules

The main objects of study in this thesis are vertex operator algebras. Here we recall the main definitions and several useful facts for our investigations. For more comprehensive accounts of the theory of vertex operator algebras, see the excellent textbooks [72, 100].

1.1.1. Vertex Algebras. Let $\mathsf{V} = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} \mathsf{V}_n$ be a $\frac{1}{2}\mathbb{Z}$-graded vector space over $\mathbb{C}$. A formal power series of the form, with $a \in \mathsf{V}_m$,

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \operatorname{End} \mathsf{V}[[z^{\pm}]], \tag{1.1.1}$$

whose coefficients $a_{(n)}$, the *modes* of $a(z)$, are homogeneous linear operators of degree $-n-1+m$ (i.e. $a_{(n)}(\mathsf{V}_p) \subset \mathsf{V}_{p-n-1+m}$), is a *field* if for any $v \in \mathsf{V}$, $a_{(n)}v = 0$ for $n \gg 0$. More general gradings of V are possible, but all vertex algebras encountered in this thesis will involve only $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded vector spaces.

Definition 1.1.1. *A* vertex algebra $(\mathsf{V}, \mathbb{1}, Y)$ *consists of the data*

- *A vector space* V*,*
- (vacuum vector) *a distinguished vector* $\mathbb{1} \in \mathsf{V}$*,*
- (vertex map) *a linear operator* $Y : \mathsf{V} \to$ *End* $\mathsf{V}[[z^{\pm}]]$ *whose image consists of fields,*

subject to the conditions:

- $Y(\mathbb{1}, z) = \mathrm{id}_{\mathsf{V}}$.
- *For all* $a \in \mathsf{V}$*,* $Y(a,z)\mathbb{1} \in \mathsf{V}[[z]]$ *and* $\lim_{z\to 0} Y(a,z)\mathbb{1} = a$*.*
- *All fields* $Y(a,z)$*,* $a \in \mathsf{V}$*, are* mutually local*. That is, for any* $a, b \in \mathsf{V}$*, there exists* $N \in \mathbb{Z}_{\geqslant 0}$ *such that the commutator of* $Y(a,z)$ *and* $Y(b,z)$ *satisfies*

$$(z-w)^N \big[Y(a,z), Y(b,z)\big] = 0. \tag{1.1.2}$$

We will often denote vertex algebras $(\mathsf{V}, \mathbb{1}, Y)$ by V, on the understanding that the remaining data is clear by context. Similarly, denote the image of the vertex map by $Y(a, z)$ by $a(z)$.

An important consequence of the locality condition in Definition 1.1.1 is the existence of *operator product expansions*: By [**100**, Thm. 2.3], $a(z)$ and $b(z)$ are mutually local if and only if, in the region $|z| > |w|$,

$$a(z)b(w) = \sum_{n\geqslant 0} \frac{(a_n b)(w)}{(z-w)^{n+1}} + {:}a(z)b(w){:}, \tag{1.1.3}$$

where the *normally-ordered product* ${:}a(z)b(w){:}$ is defined by

$$ {:}a(z)b(w){:} = \sum_{n\leq -1} a_{(n)} z^{-n-1} b(w) + b(w) \sum_{n\geq 0} a_{(n)} z^{-n-1}. \tag{1.1.4}$$

The equation (1.1.5) is called the *operator product expansion* of $a(z)$ and $b(z)$. The normally-ordered product is regular in $z - w$ and is often omitted. The operator product expansion of $a(z)$ and $b(z)$ will therefore be written as

$$a(z)b(w) \sim \sum_{n\geqslant 0} \frac{(a_n b)(z)}{(z-w)^{n+1}}. \tag{1.1.5}$$

In fact, it is often possible to specify a vertex algebra by declaring a set of 'generating' fields and their operator products expansions.

Theorem 1.1.2 ([**72**])**.** *Let V be a vector space and $\mathbb{1}$ a nonzero vector. Let S be a countable set and $\{a^\alpha\}_{\alpha\in S}$ a collection of vectors in V and a collection of fields*

$$a^\alpha(z) = \sum_{n\in\mathbb{Z}} a^\alpha_{(n)} z^{-n-1} \tag{1.1.6}$$

on V that satisfy:

- *$a^\alpha(z)\mathbb{1} = a^\alpha + z(\dots)$ for all α.*
- *All fields $a^\alpha(z)$ are mutually local.*
- *V is spanned by the vectors $a^{\alpha_1}_{(j_1)} \dots a^{\alpha_m}_{(j_m)} \mathbb{1}$ where $j_i < 0$.*

The assignment, for nonzero $a^{\alpha_1}_{(j_1)} \dots a^{\alpha_m}_{(j_m)} \mathbb{1}$,

$$Y\left(a^{\alpha_1}_{(j_1)} \dots a^{\alpha_m}_{(j_m)} \mathbb{1}, z\right) = \frac{1}{(-j_1-1)! \dots (-j_m-1)!} {:}\partial_z^{-j_1-1} a^{\alpha_1}(z) \dots \partial_z^{-j_m-1} a^{\alpha_m}(z){:} \tag{1.1.7}$$

defines a vertex algebra structure on V. This is the unique vertex algebra structure on V satisfying the conditions above, and such that the image of a^α under the vertex map $Y(-, z)$ is $a^\alpha(z)$.

The fields $\{a^\alpha(z)\}_{\alpha\in S}$ in Theorem 1.1.2 are said to *strongly generate* the vertex algebra V. If the monomials of the form $a^{\alpha_1}_{(j_1)}\dots a^{\alpha_m}_{(j_m)}\mathbb{1}$ with $j_i\in\mathbb{Z}$ (rather than $\mathbb{Z}_{\geqslant 0}$) span V, then the fields $\{a^\alpha\}_{\alpha\in S}$ *generate* V.

We will frequently define the vertex algebras of interest by giving a set of strong generators and their operator product expansions. Checking that the strong generating fields are mutually local is sufficient and often a straightforward computation.

Like any algebraic object, homomorphisms, subalgebras and ideals of vertex algebras are straightforward to define. For a vector space A, denote by $\mathsf{A}((z))$ the space of A–valued formal Laurent series in z.

Definition 1.1.3. • *A* vertex algebra homomorphism *is a linear map* $\rho:\mathsf{V}_1\to\mathsf{V}_2$ *that maps the vacuum vector of* V_1 *to that of* V_2 *and satisfies*

$$\rho\left(a(z)b\right)=\rho(a)(z)\rho(b), \tag{1.1.8}$$

for all $a,b\in\mathsf{V}_1$.

• *A* vertex subalgebra *is a subspace* $\mathsf{V}'\subset\mathsf{V}$ *that is invariant under taking derivatives of fields and satisfies* $\mathbb{1}\in\mathsf{V}'$ *and*

$$a(z)b\in\mathsf{V}'((z)), \tag{1.1.9}$$

for all $a,b\in\mathsf{V}'$.

• *A* vertex algebra ideal *is a subspace* $\mathsf{I}\subset\mathsf{V}$ *that is invariant under taking derivatives of fields and satisfies*

$$a(z)b\in\mathsf{I}((z)), \tag{1.1.10}$$

for all $a\in\mathsf{I}$ *and* $b\in\mathsf{V}$.

The quotient V/I of a vertex algebra V by an ideal I inherits the structure of a vertex algebra: by [72, Prop. 3.2.5], the condition $a(z)b\in\mathsf{I}((z))$ implies that $b(z)a\in\mathsf{I}((z))$. In other words, left ideals of vertex algebras are always two-sided ideals. The vertex algebra structure on V therefore descends to the quotient V/I.

Another way to build new vertex algebras out of old ones is as tensor products. Given two vertex algebras V_1 and V_2, the vector space $\mathsf{V}_1\otimes\mathsf{V}_2$ has the structure of a vertex algebra with vacuum vector $\mathbb{1}_1\otimes\mathbb{1}_2$ and vertex map defined by

$$(a_1\otimes a_2)(z)=a_1(z)\otimes a_2(z). \tag{1.1.11}$$

Definition 1.1.4. *A* vertex operator algebra *is a vertex algebra* V *with a distinguished element* $\omega \in \mathsf{V}_2$ *whose modes* $\{\omega_n\}$ *from the expansion*

$$\omega(z) = \sum_n \omega_{(n)} z^{-n-1} = \sum_n \omega_n z^{-n-2} \tag{1.1.12}$$

satisfy:

- $[\omega_{-1}, Y(a,z)] = \partial Y(a,z)$ *for all* $a \in \mathsf{V}$.
- $\omega_{-1}\mathbb{1} = 0$.
- $\omega_0|_{\mathsf{V}_n} = n\,\mathrm{id}_{\mathsf{V}_n}$
- *The relations of the Virasoro algebra*

$$[\omega_m, \omega_n] = (m-n)\omega_{m+n} + \frac{\mathsf{c}}{12}(m^3 - m)\delta_{m+n,0}, \tag{1.1.13}$$

  where c *is a central endomorphism of* V.

The field $\omega(z)$ (vector ω) is commonly called an *energy-momentum field* (vector) or a *conformal field* (vector). This is due to the intimate relationship between the Virasoro algebra and the conformal symmetry of 2d conformal field theories [32].

Additionally, the central endomorphism c is taken to be multiplication by a complex number (also denoted by c) called the *central charge* of V. A homogeneous vector $a \in \mathsf{V}_\mathsf{m}$ in a vertex operator algebra is said to have *conformal dimension* $\Delta_a = m$, and is alternatively expanded in modes according to

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} = \sum_n a_n z^{-n-\Delta_a}. \tag{1.1.14}$$

Note that if the conformal dimension of $a(z)$ is a half-integer, then the above sum containing the modes a_n must be taken over $n \in \mathbb{Z} + \frac{1}{2}$.

The definitions of homomorphisms, subalgebras, ideals, quotients and tensor products also apply to vertex operator algebras subject to additional conditions relating to the conformal vector. Chief among these is that homomorphisms of vertex operator algebras must preserve the conformal vector, and that the tensor product of vertex operator algebras V_1 and V_2 is naturally a vertex operator algebra with conformal vector $\omega_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \omega_2$.

The definitions presented here admit natural generalisations to superspaces $\mathsf{V} = \mathsf{V}_{\bar{0}} \oplus \mathsf{V}_{\bar{1}}$ (see [72, Rem. 3.2.1]). The necessary modifications to define a *vertex operator superalgebra* are parity conditions on elements/modes of V (such as $\mathbb{1}$ and ω being required to be even) and an additional sign in the definition of locality:

$$(z-w)^N a(z)b(w) = (-1)^{p(a)p(b)}(z-w)^N b(w)a(z). \tag{1.1.15}$$

That is, the modes of odd fields satisfy anticommutation relations instead of commutation relations.

1.1.2. Examples.

Example (Virasoro). *The 'simplest' vertex operator algebra is the one whose only generating field is an energy-momentum field $\omega(z)$. This vertex operator algebra is known as the* Virasoro vertex algebra $\mathsf{Vir}^{\mathsf{k}}$ *and its nonregular operator product expansion is*

$$\omega(z)\omega(w) \sim \frac{\mathsf{c}_{\mathsf{k}}^{\mathsf{Vir}}\mathbb{1}}{2(z-w)^4} + \frac{2\omega(w)}{(z-w)^2} + \frac{\partial\omega(w)}{z-w}, \quad \mathsf{c}_{\mathsf{k}}^{\mathsf{Vir}} = -\frac{6\mathsf{k}^2+11\mathsf{k}+4}{\mathsf{k}+2}, \tag{1.1.16}$$

where we parametrise the central charge $\mathsf{c}_{\mathsf{k}}^{\mathsf{Vir}}$ in terms of the level $\mathsf{k} \in \mathbb{C} \setminus \{-2\}$. *When the level is of the form $\mathsf{k} = \frac{\mathsf{u}}{\mathsf{v}} - 2$ for some coprime $\mathsf{u}, \mathsf{v}+1 \in \mathbb{Z}_{\geqslant 2}$, $\mathsf{Vir}^{\mathsf{k}}$ contains a nontrivial maximal ideal. The simple quotient vertex operator algebra $\mathsf{Vir}_{\mathsf{k}}$ at such levels is known as a Virasoro minimal model $\mathsf{M}(\mathsf{u},\mathsf{v})$.*

Example (Heisenberg). *Let $\mathfrak{h}$ be a finite-dimensional $\mathbb{C}$-vector space with basis $\{a^i\}$ and a nondegenerate symmetric bilinear form $(-|-)$. Consider the affinisation $\hat{\mathfrak{h}} = \mathfrak{h}[t,t^{-1}] \oplus \mathbb{C}K$ with K central and commutation relations, for $a, b \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$,*

$$[a_m, b_n] = m(a|b)\delta_{m+n,0}K, \tag{1.1.17}$$

writing $h_p = h \otimes t^p$ for $h \in \mathfrak{h}$ and $p \in \mathbb{Z}$. The fields

$$a(z) = \sum_n a_n z^{-n-1}, \quad a \in \mathfrak{h}, \tag{1.1.18}$$

are mutually local and strongly generate the Heisenberg vertex algebra H *whose operator product expansions are, for $a, b \in \mathfrak{h}$,*

$$a(z)b(w) \sim \frac{(a|b)K}{z-w}. \tag{1.1.19}$$

where the endomorphism K acts by multiplication by some $\mathsf{k} \in \mathbb{C}$. To make H a vertex operator algebra, let $\{b^i\}$ be the dual basis of $\mathfrak{h}$ relative to $(-|-)$ (identifying $\mathfrak{h}$ with $\mathfrak{h}^*$). The field

$$\omega(z) = \frac{1}{2\mathsf{k}} \sum_i :a^i(z)b^i(z): \tag{1.1.20}$$

is an energy-momentum field of central charge $c = \dim(\mathfrak{h})$. *The conformal dimension of the fields $a(z)$, $a \in \mathfrak{h}$, is* 1 *with respect to $\omega(z)$. The modes of the fields of* H *obey the commutation relations of $\hat{\mathfrak{h}}$ given in* (1.1.17).

Example (Bosonic ghosts). *Two very important vertex operator algebras are the ghost vertex operator algebras, related to the ghost fields encountered frequently in physics in processes such as Faddeev-Popov gauge fixing [77].*

The first of these is the $\beta\gamma$ or bosonic ghost vertex algebra B *introduced in physics in [137]. It has strong generators $\beta(z)$ and $\gamma(z)$ with operator product expansions*

$$\beta(z)\beta(w) \sim 0 \sim \gamma(z)\gamma(w), \quad \beta(z)\gamma(w) \sim \frac{-\mathbb{1}}{z-w}. \tag{1.1.21}$$

An energy-momentum field for B is given by $T^{\mathsf{B}}(z) = \frac{1}{2}(:\partial\beta(z)\gamma(z) - \partial\gamma(z)\beta(z):)$. The central charge associated to $T^{\mathsf{B}}(z)$ is -1, and both $\beta(z)$ and $\gamma(z)$ have conformal dimension $\frac{1}{2}$. The modes of $\beta(z)$ and $\gamma(z)$ obey the commutation relations

$$[\beta_n, \beta_n] = [\gamma_m, \gamma_n] = 0, \quad [\beta_m, \gamma_n] = -\delta_{m+n,0}\mathbb{1}. \tag{1.1.22}$$

where here, $\mathbb{1}$ is the identity endomorphism of B. To generalise the $\beta\gamma$ ghost vertex algebra, let A be a finite-dimensional $\mathbb{C}$-vector space and a nondegenerate symplectic form $(-|-)$. As for the Heisenberg vertex algebra, consider the affinisation $\widehat{A} = A[t, t^{-1}] \oplus \mathbb{C}K$ with K central and commutation relations, for $\delta, \zeta \in A$,

$$[\delta_{(m)}, \zeta_{(n)}] = (\delta|\zeta)\delta_{m+n,0}K, \tag{1.1.23}$$

writing $\phi_{(p)} = \phi \otimes t^p$ for $\phi \in A$ and $p \in \mathbb{Z}$. Define a field $\delta(z)$ for any $\delta \in A$ according to

$$\delta(z) = \sum_n \delta_{(n)} z^{-n-1}. \tag{1.1.24}$$

We will call the vertex algebra $\mathsf{B}(A)$ strongly generated by the fields $\delta(z)$ for $\delta \in A$, identifying K with the identity automorphism of $\mathsf{B}(A)$, the neutral ghost vertex algebra *(c.f. [102, Ex. 1.2]). This vertex algebra is also known as the Weyl vertex algebra or the symplectic bosons. The 'neutral' here refers to a charge assignment used in quantum hamiltonian reduction. The operator product expansion of these fields is given by, for $\delta, \zeta \in A$,*

$$\delta(z)\zeta(w) \sim \frac{(\delta|\zeta)\mathbb{1}}{z-w}. \tag{1.1.25}$$

The singular part of this operator product expansion enforces that the modes of the fields of $\mathsf{B}(A)$ satisfy the commutation relations (1.1.23) where again K is the identity automorphism of $\mathsf{B}(A)$.

Let $\{\delta_i\}$ be a basis of A. An energy-momentum field for $\mathsf{B}(A)$ is given by

$$T^{\mathsf{B}(A)}(z) = \frac{1}{2}\sum_i :\partial\delta_i(z)\delta^i(z):, \tag{1.1.26}$$

where $\{\delta^i\}$ is the dual basis of A to $\{\delta_i\}$ with respect to $(-|-)$. The conformal dimension of the field $\delta(z)$ is $\frac{1}{2}$ for any $\delta \in A$ and the central charge is $-\frac{1}{2}\mathrm{dim}(A)$. It is straightforward to show that if A is two dimensional, then $\mathsf{B}(A) \simeq \mathsf{B}$.

Example (Fermionic ghosts). *The second ghost vertex algebra is the bc or* fermionic ghost vertex superalgebra F *introduced in physics in [138]. It has strong generators $b(z)$ and $c(z)$ (both odd fields) with operator product expansions*

$$b(z)b(w) \sim 0 \sim c(z)c(w), \quad b(z)c(w) \sim \frac{\mathbb{1}}{z-w}. \tag{1.1.27}$$

An energy-momentum field for F is given by $T^{\mathsf{F}}(z) = {:}\partial b(z)c(z){:}$. The central charge associated to $T^{\mathsf{F}}(z)$ is -2, and $b(z)$ and $c(z)$ have conformal dimensions 0 and 1 respectively. The modes of the fields $b(z)$ and $c(z)$ obey the anticommutation relations

$$\{b_m, b_n\} = \{c_m, c_n\} = 0, \quad \{b_m, c_n\} = \delta_{m+n,0}\mathbb{1}. \tag{1.1.28}$$

where here, $\mathbb{1}$ is the identity endomorphism of F. To generalise the bc ghost vertex superalgebra, let A be a finite-dimensional $\mathbb{C}$-vector space with basis $\{\varphi_i\}$ and a nondegenerate symmetric form $(-|-)$. Suppose further that $A = A_+ \oplus A_-$ with $A_\pm$ isotropic. The vertex superalgebra $\mathsf{F}(A)$ strongly generated by odd fields $\varphi(z)$ for $\varphi \in A$ with operator product expansion, for $\varphi, \psi \in A$,

$$\varphi(z)\psi(w) \sim \frac{(\varphi|\psi)\mathbb{1}}{z-w} \tag{1.1.29}$$

is called the charged ghost vertex superalgebra *(c.f. [102, Ex. 1.3]). Being odd fields, the modes of the fields $\varphi(z)$ for $\varphi \in A$ satisfy the anticommutation relations, for $\varphi, \psi \in A$,*

$$\{\varphi_{(m)}, \psi_{(n)}\} = (\varphi|\psi)\delta_{m+n,0}\mathbb{1}. \tag{1.1.30}$$

The 'charged' here refers to the assignment of charges $+1$ for all fields $\varphi(z)$ with $\varphi \in A_+$ and -1 for all fields $\varphi^*(z)$ with $\varphi^* \in A_-$. The charged ghost vertex superalgebra can be decomposed in terms of subalgebras $\mathsf{F}(A)_m$ consisting of all fields of charge $m \in \mathbb{Z}$ according to

$$\mathsf{F}(A) = \bigoplus_{m\in\mathbb{Z}} \mathsf{F}(A)_m. \tag{1.1.31}$$

Let $\{\varphi_i^*\}$ be the basis of A_- dual to the basis $\{\varphi_i\}$ of A_+. For any $m \in \mathbb{C}^{\mathrm{dim} A_+}$, there is an energy-momentum field for $\mathsf{F}(A)$ given by

$$T^{\mathsf{F}(A)}(z) = -\sum_i m_i{:}\varphi_i^*(z)\partial\varphi_i(z){:} + \sum_i (1-m_i){:}\partial\varphi_i^*(z)\varphi_i(z){:}. \tag{1.1.32}$$

The conformal dimensions of the fields φ_i and φ_i^* are $(1-m_i)$ and m_i respectively, and the central charge is $-\sum_i(12m_i^2 - 12m_i + 2)$. Such general conformal structures on $\mathsf{F}(A)$ are encountered frequently in quantum hamiltonian reduction (see Section 1.3). As before, if A is two dimensional (i.e. $A_\pm \simeq \mathbb{C}$ as vector spaces) then $\mathsf{F}(A) \simeq \mathsf{F}$.

1.1.3. Vertex Algebra Modules. The notion of a vertex (operator) algebra module comes from allowing the modes of a field $a(z)$ of V to be endomorphisms on a vector space $\mathcal{M}$ in a way compatible with the original vertex algebra structure on V.

Definition 1.1.5. *Let V be a vertex algebra. A vector space $\mathcal{M}$ is a* vertex algebra module *for V if it is equipped with an operation $Y_M : \mathsf{V} \to \mathrm{End}\mathcal{M}[[z^\pm]]$ which assigns to each $a \in \mathsf{V}$ a field*

$$Y_{\mathcal{M}}(a,z) = \sum_{n\in\mathbb{Z}} a^{\mathcal{M}}_{(n)} z^{-n-1} \tag{1.1.33}$$

on $\mathcal{M}$ subject to the conditions:

- $Y_{\mathcal{M}}(\mathbb{1}, z) = \mathrm{id}_{\mathcal{M}}$.
- *For all $a, b \in \mathsf{V}$ and $c \in \mathcal{M}$, the three expressions*

$$Y_{\mathcal{M}}(a,z)Y_{\mathcal{M}}(b,w)c \in \mathcal{M}((z))((w)), \tag{1.1.34a}$$

$$Y_{\mathcal{M}}(b,w)Y_{\mathcal{M}}(a,z)c \in \mathcal{M}((w))((z)), \text{ and} \tag{1.1.34b}$$

$$Y_{\mathcal{M}}\left(Y(a,z-w)b,w\right)c \in \mathcal{M}((w))((z-w)) \tag{1.1.34c}$$

are expansions, in their respective domains ($|z| > |w|$, $|w| > |z|$ and $|w| > |z-w|$), of the same element of

$$\mathcal{M}[[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]. \tag{1.1.35}$$

If V is vertex operator algebra, a vertex algebra module $\mathcal{M}$ is a vertex operator algebra module *(or* V-module *for short) if $\mathcal{M}$ decomposes into generalised eigenspaces for $\omega_0^{\mathcal{M}} = \omega_{(1)}^{\mathcal{M}}$.*

Of course a vertex operator algebra is always a module over itself. This special module is referred to as the *vacuum module*.

Definitions of vertex operator algebra modules encountered in the literature often include restrictions on the grading $\mathcal{M}$ by $\omega_0^{\mathcal{M}}$. For example, as the operator ω_0 is the chiral part of the Hamiltonian of the associated 2d conformal field theory, it is sensible in many applications to require that the (generalised) eigenvalues of $\omega_0^{\mathcal{M}}$ are bounded below. We will refer to such a V-module as *positive-energy*. The (generalised) eigenspace of minimal $\omega_0^{\mathcal{M}}$-eigenvalue is called the *top space* of $\mathcal{M}$ and will be denoted by $\mathcal{M}^{\mathrm{top}}$.

There exists an 'algebra of modes' U, constructed as a certain topological completion of the universal enveloping algebra of a Lie algebra associated to a given vertex operator algebra V [72, Sec. 4.3]. Then a V-module is automatically a U-module as the former specifies the action of the elements of the latter. The converse is not true in general, but is true for *smooth* U-modules ([72, Thm. 5.1.6]).

Given a V-module $\mathcal{M}$ and a vertex algebra automorphism ω of V, one can 'twist' $\mathcal{M}$ by ω: Define $\omega^*(\mathcal{M})$ to be the image of $\mathcal{M}$ under an (arbitrarily chosen) isomorphism ω^* of vector spaces. The action of V on $\omega^*(\mathcal{M})$ is then defined by

$$a(z) \cdot \omega^*(v) = \omega^*(\omega^{-1}(a(z))v), \quad a(z) \in \mathsf{V},\ v \in \mathcal{M}. \tag{1.1.36}$$

In other words, $\omega(a(z))\cdot\omega^*(v) = \omega^*(a(z)v)$. In view of this, we shall drop the star that distinguishes the automorphism ω from the corresponding vector space isomorphism ω^*. The automorphism ω need not preserve the conformal vector as the formula (1.1.36) will define an action of V on $\omega(\mathcal{M})$ regardless. Of course if ω is a vertex operator algebra automorphism, the conformal vector is preserved by definition.

If V contains vectors with half-integer conformal dimension, the mode index of the corresponding fields can be taken to be half-integers or integers. That is, expanding such fields according to $a(z) = \sum_n a_n z^{-n-\Delta_a}$, we may take the sum over $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$. This choice corresponds to a choice of boundary conditions for fields of half-integer conformal dimension. The choice of mode index impacts what powers of z are present in the mode expansion of a field with half-integer conformal dimension.

V-modules as defined above correspond to the choice of half-integer mode indices for half-integer conformal dimension fields. These are referred to as *untwisted* V-modules. On the other hand, modifying the definition so that half-integer conformal dimension fields act with integer mode indices results in *twisted* V-modules.

The 'mode algebra' U^{tw} that acts on twisted V-modules is a similar topological completion of the algebra of modes where the mode index of all fields are integers. Of course if V is $\mathbb{Z}$-graded, then $\mathsf{U}^{\text{tw}} \simeq \mathsf{U}$. For more general V, this is not always the case.

1.1.4. Zhu Technology. A fundamental tool for classifying and constructing positive energy (untwisted) V-modules are functors induced between these modules and those of the corresponding (untwisted) Zhu algebra denoted by $\mathsf{Zhu}[\mathsf{V}]$. The idea behind this unital associative algebra was already well known to physicists (see [67] for example), but a mathematical account was first given by Zhu [171]. Here we follow the description of the Zhu algebra given in [145, App. B].

Suppose that V is a vertex operator algebra with mode algebra $\mathsf{U} = \mathsf{U}_< \otimes \mathsf{U}_0 \otimes \mathsf{U}_>$, where $\mathsf{U}_<$, U_0 and $\mathsf{U}_>$ denote the unital subalgebras of U generated by modes a_n with index $n < 0$, $n = 0$ and $n > 0$ respectively. Suppose further that U admits a PBW-type basis consisting of $\mathbb{1}$ and monomials of the form $a^{i_1}_{n_1} \dots a^{i_p}_{n_p}$ where $n_j \in \mathbb{Z}$ and $\{a^{i_j}(z)\}$ is a finite collection of fields in V not including $\mathbb{1}(z)$. All W-algebras encountered in this thesis satisfy this property [106, Rem. 4.2]. Let $\mathsf{U}'_>$ denote the ideal of $\mathsf{U}_>$ spanned by the basis elements $a^{i_1}_{n_1} \dots a^{i_p}_{n_p}$ with $n_j > 0$ and $p > 0$ (so that $\mathsf{U}_> = \mathbb{C}\mathbb{1} \oplus \mathsf{U}'_>$ as vector spaces).

Definition 1.1.6. *The* untwisted Zhu algebra *of* V *is the vector space*

$$\mathsf{Zhu}[\mathsf{V}] = \frac{\mathsf{U}_0}{\mathsf{U}_0 \cap (\mathsf{U}\mathsf{U}'_>)}, \tag{1.1.37}$$

equipped with the multiplication (defined for homogeneous a of conformal weight Δ_a and extended linearly)

$$\left[a_0\right]\left[b_0\right] = \left[a_0 b_0\right] = \sum_{n=0}^{\infty} \binom{\Delta_a}{n} \left[(a_{-\Delta_a+n} b)_0\right], \tag{1.1.38}$$

where $\left[u_0\right]$ *is the image in* $\mathsf{Zhu}[\mathsf{V}]$ *of* $u_0 \in \mathsf{U}_0$.

In [171], Zhu defined two functors between the categories of V- and $\mathsf{Zhu}[\mathsf{V}]$-modules. We shall refer to them as the Zhu functor and the Zhu induction functor. The first is quite easy to define.

Definition 1.1.7. *The* Zhu functor *assigns to any* V*-module* $\mathcal{M}$*, the* $\mathsf{Zhu}[\mathsf{V}]$*-module* $\mathsf{Zhu}[\mathcal{M}] = \mathcal{M}^{\mathsf{U}'_>}$*, the subspace of* $\mathcal{M}$ *whose elements are annihilated by* $\mathsf{U}'_>$.

The second amounts to inducing a $\mathsf{Zhu}[\mathsf{V}]$-module by treating it as a U_0-module equipped with a trivial $\mathsf{U}'_>$-action, and taking a quotient that imposes, among other things, the generalised commutation relations of V obtained from the operator product expansions. The details may be found in [121, 171].

Proposition 1.1.8 ([171])**.** *There exists a functor, which we call the* Zhu induction functor*, that assigns to any* $\mathsf{Zhu}[\mathsf{V}]$*-module* $\mathcal{N}$ *a* V*-module* $\mathsf{Ind}[\mathcal{N}]$ *such that* $\mathsf{Zhu}[\mathsf{Ind}[\mathcal{N}]] \simeq \mathcal{N}$.

The Zhu functor is thus a left inverse of the Zhu induction functor, at the level of isomorphism classes of modules. While it is not a right inverse in general, it is if we restrict to simple positive-energy V-modules: If $\mathcal{M}$ is a positive-energy V-module, then $\mathcal{M}^{\mathrm{top}}$ is naturally a $\mathsf{Zhu}[\mathsf{V}]$-module. In fact, it may be identified with $\mathsf{Zhu}[\mathcal{M}]$ if $\mathcal{M}$ is also simple, though this will not be true in general.

Theorem 1.1.9 ([171]). *$\mathsf{Zhu}[-]$ and $\mathsf{Ind}[-]$ induce a bijection between the sets of isomorphism classes of simple positive-energy V-modules and simple $\mathsf{Zhu}[\mathsf{V}]$-modules.*

To classify simple positive-energy V-modules, it is therefore sufficient to classify simple $\mathsf{Zhu}[\mathsf{V}]$-modules and apply $\mathsf{Ind}[-]$. Subject to identifying $\mathsf{Zhu}[\mathsf{V}]$, this is a dramatic simplification as the latter are modules over an associative algebra.

An important feature of the Zhu algebra $\mathsf{Zhu}[\mathsf{V}]$ is how it changes upon replacing V with any quotient V/I where I is an ideal of V. Denote by $\mathsf{Zhu}[\mathsf{I}]$ the image in $\mathsf{Zhu}[\mathsf{V}]$ of all elements in I. Then $\mathsf{Zhu}[\mathsf{I}]$ is a two-sided ideal in $\mathsf{Zhu}[\mathsf{V}]$ [76, Prop. 1.4.2].

Proposition 1.1.10 ([76]). *If I is an ideal of a vertex operator algebra V, then*

$$\mathsf{Zhu}[\mathsf{V}/\mathsf{I}] \simeq \mathsf{Zhu}[\mathsf{V}]/\mathsf{Zhu}[\mathsf{I}]. \tag{1.1.39}$$

Determining whether or not a simple positive-energy V-module $\mathcal{M}$ is a V/I-module can be reduced to checking whether $\mathsf{Zhu}[\mathcal{M}]$ is annihilated by $\mathsf{Zhu}[\mathsf{I}]$.

There is a parallel story for twisted V-modules. It was developed in different levels of generality by Kac and Wang [108] and by Dong, Li and Mason [58]. Much of the details are identical, except that the relevant mode algebra is the twisted mode algebra U^{tw} of V. This is discussed in detail in [36, App. A].

Given a vertex operator algebra V with twisted mode algebra $\mathsf{U}^{\mathrm{tw}} = \mathsf{U}^{\mathrm{tw}}_{<} \otimes \mathsf{U}^{\mathrm{tw}}_{0} \otimes \mathsf{U}^{\mathrm{tw}}_{>}$, let $\mathsf{U}^{\mathrm{tw}}_{>}{}'$ be the ideal of $\mathsf{U}^{\mathrm{tw}}_{>}$ defined in the same way as $\mathsf{U}_{>}{}'$ in $\mathsf{U}_{>}$.

Definition 1.1.11.

- *The* twisted Zhu algebra *of V is the vector space*

$$\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{V}\big] = \frac{\mathsf{U}^{\mathrm{tw}}_{0}}{\mathsf{U}^{\mathrm{tw}}_{0} \cap (\mathsf{U}^{\mathrm{tw}}\mathsf{U}^{\mathrm{tw}}_{>}{}')}, \tag{1.1.40}$$

  equipped with the multiplication defined in (1.1.38), but where $\big[u_0\big]$ is now the image in $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{V}\big]$ of $u_0 \in \mathsf{U}^{\mathrm{tw}}_{0}$.
- *The* twisted Zhu functor *assigns to any twisted V-module $\mathcal{M}$ the $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{V}\big]$-module $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathcal{M}\big] = \mathcal{M}^{\mathsf{U}^{\mathrm{tw}}_{>}{}'}$ of elements of $\mathcal{M}$ that are annihilated by $\mathsf{U}^{\mathrm{tw}}_{>}{}'$.*

Using these definitions, the twisted versions of Zhu's theorems hold. This is because, as in the untwisted case, if $\mathcal{M}$ is a positive-energy twisted V-module, then $\mathcal{M}^{\mathrm{top}}$ is naturally a $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{V}\big]$-module.

Theorem 1.1.12 ([58]).

- *There exists a* twisted Zhu induction functor *that takes a* $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{V}]$*-module* $\mathcal{N}$ *to a* V*-module* $\mathsf{Ind}^{\mathrm{tw}}[\mathcal{N}]$ *satisfying* $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{Ind}^{\mathrm{tw}}[\mathcal{N}]] \simeq \mathcal{N}$.
- $\mathsf{Zhu}^{\mathrm{tw}}[-]$ *and* $\mathsf{Ind}^{\mathrm{tw}}[-]$ *induce a bijection between the sets of isomorphism classes of simple positive-energy twisted* V*-modules and simple* $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{V}]$*-modules.*

1.1.5. Modularity and Fusion. Given a 2d conformal field theory with corresponding vertex operator algebra V, the state space of the theory must be composed of V-modules. For a given collection of V-modules to be suitable for this purpose, the category $\mathcal{C}$ comprising the collection of V-modules must satisfy a number of necessary conditions including:

- $\mathcal{C}$ is closed under twisting modules by certain automorphisms of V such as conjugation.
- $\mathcal{C}$ is closed under the *fusion product* of V-modules.
- The partition function of $\mathcal{C}$ is modular invariant.

Additionally, there are a number of properties the category $\mathcal{C}$ is expected to have. For example, it is often the case that the category $\mathcal{C}$ has the structure of a modular tensor category. Such structure is known to be present in categories of modules for sufficiently nice vertex operator algebras such as the *strongly rational* ones [95] but it is not known if this true for generically.

The presence of tensor-categorical structure on $\mathcal{C}$ allows for diagrammatic and algorithmic approaches to many vertex-algebraic questions important for conformal field theory. See for example [116].

Given two V-modules $\mathcal{M}, \mathcal{N} \in \mathcal{C}$, the fusion product is written as

$$\mathcal{M} \times \mathcal{N} \simeq \bigoplus_{\mathcal{P} \in \mathcal{C}} \mathcal{N}_{\mathcal{M},\mathcal{N}}^{\ \ \mathcal{P}} \mathcal{P}, \tag{1.1.41}$$

where the *fusion coefficients* $\mathcal{N}_{\mathcal{M},\mathcal{N}}^{\ \ \mathcal{P}}$ are nonnegative integers. At the level of vertex operator algebra modules, the fusion product can be written as a quotient of the usual tensor product of two V-modules [82, 83]. Alternatively, the fusion product can be defined using a universal property with respect to intertwining maps in $\mathcal{C}$ [97].

Definition 1.1.13. *A 2d conformal field theory with vertex operator algebra* V *and category* $\mathcal{C}$ *of* V*-modules is* rational *if all objects in* $\mathcal{C}$ *are completely reducible and* $\mathcal{C}$ *has finitely many simple objects.*

In rational 2d conformal field theories, fusion products describe the primary fields that appear in the operator product expansion of two other primary fields. This assists in, for example, computing the correlation functions of the theory (see for example [70, Sec. 7.3.1]).

We also call the vertex operator algebra V *rational in* $\mathcal{C}$ if all objects in $\mathcal{C}$ are completely reducible and $\mathcal{C}$ has finitely many simple objects. Rational vertex operator algebras satisfy a number of nice properties.

One remarkable property of rational vertex operator algebras is expressed by the famous Verlinde formula [156] that relates the fusion coefficients in (1.1.41) to the modular properties of *characters* of V-modules in $\mathcal{C}$: Define the character of a V-module $\mathcal{M}$ to be the trace

$$\mathrm{ch}\left[\mathcal{M}\right](\mathsf{q}) = \mathrm{tr}_{\mathcal{M}}\, \mathsf{q}^{\omega_0 - \mathsf{c}/24}, \tag{1.1.42}$$

where $\omega(z)$ is the energy-momentum field in V and c is its central charge. Let V be rational in a category $\mathcal{C}$ and denote the simple objects in $\mathcal{C}$ by $\mathcal{M}_i$ for $i \in I$ in some finite indexing set.

Suppose that, in addition to being rational, $\dim\left(\mathsf{V}/C_2(\mathsf{V})\right) < \infty$, where $C_2(\mathsf{V})$ is the span of all elements of V that can be written as $a_{(-2)}b$ for some $a, b \in \mathsf{V}$. A vertex operator algebra satisfying this condition is *C_2-cofinite*. Many of the well known rational vertex operator algebras are also C_2-cofinite, but there are C_2-cofinite W-algebras that are not rational [8, 85].

It was shown by Zhu [171] that the $\mathbb{C}$-span of characters $\mathrm{ch}\left[\mathcal{M}_i\right](\mathsf{q})$ of such a rational C_2-cofinite vertex operator algebra admits an action of the modular group $\mathsf{SL}_2(\mathbb{Z})$. The action of the generating elements S and T of $\mathsf{SL}_2(\mathbb{Z})$ are written as, writing $\mathsf{q} = \mathsf{e}^{2\pi \mathsf{i}\tau}$ for τ in the upper half plane,

$$\mathsf{S}\left(\mathrm{ch}\left[\mathcal{M}_i\right](\mathsf{q})\right) = \mathrm{ch}\left[\mathcal{M}_i\right]\left(\mathsf{e}^{-2\pi\mathsf{i}/\tau}\right) = \sum_{j\in I} \mathsf{S}_{i,j}\, \mathrm{ch}\left[\mathcal{M}_j\right](\mathsf{q}) \tag{1.1.43}$$

and

$$\mathsf{T}\left(\mathrm{ch}\left[\mathcal{M}_i\right](\mathsf{q})\right) = \mathrm{ch}\left[\mathcal{M}_i\right]\left(\mathsf{e}^{2\pi\mathsf{i}(\tau+1)}\right) = \sum_{j\in I} \mathsf{T}_{i,j}\, \mathrm{ch}\left[\mathcal{M}_j\right](\mathsf{q}) \tag{1.1.44}$$

respectively. The matrix whose entries are $\mathsf{S}_{i,j}$ ($\mathsf{T}_{i,j}$) is known as the *S-(T-)matrix* of V. The Verlinde formula relates the entries of the S-matrix of V to the fusion coefficients of V.

Theorem 1.1.14 ([94, 96, 156])**.** *Let V be a simple $\mathbb{Z}_{\geqslant 0}$-graded vertex operator algebra having the following properties:*

- *V is C_2-cofinite.*
- *The only fields of conformal dimension zero are multiples of the vacuum, $V_0 = \mathbb{C}\mathbb{1}$.*
- *V is isomorphic, as a V-module, to its contragradient dual.*
- *V is rational in a V-module category $\mathcal{C}$*

Let $\{\mathcal{M}_i\}_{i\in I}$ be a complete set of representatives of the (isomorphism classes of) simple objects in $\mathcal{C}$ where I is a finite indexing set. Then the Verlinde formula holds:

$$\mathcal{N}_{\mathcal{M}_i,\mathcal{M}_j}^{\ \ \mathcal{M}_k} = \sum_{\ell\in I} \frac{\mathsf{S}_{i,\ell}\mathsf{S}_{j,\ell}\mathsf{S}^*_{k,\ell}}{\mathsf{S}_{vac,\ell}}, \tag{1.1.45}$$

where $*$ denotes complex conjugation and $\mathsf{S}_{vac,\ell}$ are the S-matrix elements of the S-action on the character of V.

The Verlinde formula therefore provides an easy means to compute the fusion rules of a rational C_2-cofinite conformal field theory whose modular properties are well understood. One drawback of this approach is that the q characters of a rational vertex operator algebras are often not linearly independent so determining the S-matrix is a formidable task. There are remedies for this, as we will see in Section 1.3.3.

There are many applications of 2d conformal field theory for which it is necessary to have a nonsemisimple V-module category. For example, percolation problems [39, 163], 4d-2d duality [29], quantum Hall transitions [164] and string theory [90, 125, 126]. In these cases, the Verlinde formula is no longer guaranteed to produce nonnegative integer fusion multiplicities.

A conjectural extension of the Verlinde formula from [51] will be discussed in Section 3.3 where it will be used to obtain nonnegative integer fusion coefficients for a nonrational vertex operator algebra.

1.2. Affine Vertex Operator Algebras

An important class of vertex operator algebras are those related to Wess-Zumino-Witten theories, which are 2d conformal field theories whose target space is a simple Lie group G [135, 165–167]. These *affine vertex operator algebras* are intimately related to the affine Lie algebra $\widehat{\mathfrak{g}}$ (where $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$) and are the amongst the best studied vertex operator algebras. Here, we recall their construction and the main features of their representation theory following [72, Ch. 2] and [76].

1.2.1. Definition and Modules.

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and consider the affine Kac-Moody algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K, \tag{1.2.1}$$

where K is central. The commutation relations of $\widehat{\mathfrak{g}}$ are, for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$

$$[a_m, b_n] = [a, b]_{m+n} + m\delta_{m+n,0}\langle a, b\rangle K, \tag{1.2.2}$$

where we write, for example, $a_m = a \otimes t^m$. Here we use the symmetric bilinear form $\langle -, - \rangle = \frac{1}{2\mathsf{h}^\vee}\kappa_{\mathfrak{g}}(-,-)$ where $\kappa_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$.

It is clear that the subset $\mathfrak{g}[t]$ spanned by all elements of the form a_m with $a \in \mathfrak{g}$ and $m \geqslant 0$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$. Let $\mathbb{C}_{\mathsf{k}}$ be the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts as zero and K acts as multiplication by the *level* $\mathsf{k} \in \mathbb{C}$.

To define a vertex operator algebra, we first need to specify a vector space. For this purpose, let

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \underset{U(\mathfrak{g}[t]\oplus\mathbb{C}K)}{\otimes} \mathbb{C}_{\mathsf{k}}, \tag{1.2.3}$$

where $U(\widehat{\mathfrak{g}})$ is the universal enveloping algebra of $\widehat{\mathfrak{g}}$ and likewise for $U(\mathfrak{g}[t] \oplus \mathbb{C}K)$. As a vector space, $\mathsf{V}^{\mathsf{k}}(\mathfrak{g}) \simeq U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$. To define a vertex algebra structure on $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$, the next step is to specify fields for all elements of $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$ such that the axioms in Definition 1.1.1 are satisfied. In this case, Theorem 1.1.2 makes it possible to specify fields for finitely many elements of $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$:

Let $\{J^a\}_{a=1,\dots,\dim\mathfrak{g}}$ be an ordered basis of $\mathfrak{g}$ and denote by $\mathbb{1}_{\mathsf{k}}$ is the image of the identity element of $U(\widehat{\mathfrak{g}}) \otimes \mathbb{C}_{\mathsf{k}}$ in $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$. In the language of vertex algebras, $\mathbb{1}_{\mathsf{k}}$ will be the vacuum vector $\mathbb{1}$. By the Poincaré-Birkhoff-Witt theorem, $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ admits a basis consisting of elements of the form

$$J^{a_1}_{j_1} \dots J^{a_m}_{j_m} \mathbb{1}_{\mathsf{k}}, \tag{1.2.4}$$

where $j_1 \leqslant \dots \leqslant j_m < 0$ and if $j_i = j_{i+1}$, $a_i \leqslant a_{i+1}$. Such elements define a natural $\mathbb{Z}$-grading of $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ by defining the grade of (1.2.4) to be $-\sum_i j_i$. Therefore by Theorem 1.1.2, all we require are mutually local fields $J^a(z)$, one for each element of the basis of $\mathfrak{g}$. Define

$$J^a(z) = \sum_n J^a_n z^{-n-1}. \tag{1.2.5}$$

The data consisting of vector space $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$, the vacuum vector $\mathbb{1} = \mathbb{1}_{\mathsf{k}}$ and generating fields $J^a(z)$ defines the structure of a vertex algebra on $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ [76, Thm. 2.4.1]. This vertex algebra is called the *universal affine vertex algebra* for $\mathfrak{g}$. The operator product expansions of $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ can be obtained from the commutation relations (1.2.2) and are

$$J^{a_i}(z)J^{a_j}(w) \sim \frac{\mathsf{k}\langle J^{a_i}, J^{a_j}\rangle \mathbb{1}}{(z-w)^2} + \frac{[J^{a_i}, J^{a_j}](w)}{z-w}. \tag{1.2.6}$$

Some important modules for $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ can be obtained by a similar induction procedure starting from highest-weight modules of $\mathfrak{g}$. First, note that the subset $\widehat{\mathfrak{g}}_0$ of $\widehat{\mathfrak{g}}$ (and of $\mathfrak{g}[t] \oplus \mathbb{C}K$) spanned by elements of the form a_0 is a Lie subalgebra isomorphic to $\mathfrak{g}$. In fact, the Zhu algebra of $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$

is the algebra generated by elements of $\widehat{\mathfrak{g}}_0$ and is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ [76, Thm. 3.1.1].

Denote by $\overline{\mathcal{K}}_{\overline{\lambda}}$ the Verma $\mathfrak{g}$-module with highest-weight $\overline{\lambda}$. Let $\overline{\mathcal{K}}_{\overline{\lambda},\mathsf{k}}$ be the $\mathfrak{g}[t]\oplus\mathbb{C}K$-module, isomorphic to $\overline{\mathcal{K}}_{\overline{\lambda}}$ as a vector space, where $\widehat{\mathfrak{g}}_0$ acts as $\mathfrak{g}$, $t\mathfrak{g}[t]$ acts as 0 and K acts as multiplication by $\mathsf{k}\in\mathbb{C}$. The vector space

$$\mathcal{K}_\lambda = U(\widehat{\mathfrak{g}}) \underset{U(\mathfrak{g}[t]\oplus\mathbb{C}K)}{\otimes} \overline{\mathcal{K}}_{\overline{\lambda},\mathsf{k}}, \tag{1.2.7}$$

where λ is the unique level-k $\widehat{\mathfrak{g}}$ weight with finite part $\overline{\lambda}$, has the structure of a $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra module for $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ [76]. The Verma modules $\mathcal{K}_\lambda$ for $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ are sometimes reducible and we denote their simple quotients by $\mathcal{L}_\lambda$. Both $\mathcal{K}_\lambda$ and $\mathcal{L}_\lambda$ are highest-weight $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$-modules in the sense that they are generated by vector that is annihilated by all positive modes and is an eigenvector for all zero modes h_0 where h is in the Cartan subalgebra of $\mathfrak{g}$.

An energy-momentum field for $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ when the level k is not equal to minus the dual Coxeter number $\mathsf{h}^\vee$ of $\mathfrak{g}$ (k is *noncritical*) is given by the *Sugawara construction* [150, 154]: Let $\{J_a\}_{a=1,\dots,\dim\mathfrak{g}}$ be the dual basis of $\mathfrak{g}$ to $\{J^a\}_{a=1,\dots,\dim\mathfrak{g}}$ with respect to the form $\langle -,-\rangle$. The field

$$T^{\text{Sug.}}(z) = \frac{1}{2(\mathsf{k}+\mathsf{h}^\vee)}\sum_{a=1}^{\dim\mathfrak{g}} {:}J_a(z)J^a(z){:} \tag{1.2.8}$$

is then an energy-momentum field for $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ with central charge

$$\mathsf{c}_{\mathsf{k}} = \frac{\mathsf{k}\dim\mathfrak{g}}{\mathsf{k}+\mathsf{h}^\vee}. \tag{1.2.9}$$

With respect to $T^{\text{Sug.}}(z)$, all the fields $J^a(z)$ have conformal dimension equal to 1.

1.2.2. Levels and Rationality. It was shown by Kac and Wakimoto that the vertex operator algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ is reducible (as a module over itself) if k is an *admissible level* [103]. That is, when k is of the form

$$\mathsf{k}+\mathsf{h}^\vee = \frac{\mathsf{u}}{\mathsf{v}} \quad \text{where } \mathsf{u},\mathsf{v}\in\mathbb{Z}_{\geqslant 1},\ (\mathsf{u},\mathsf{v})=1 \text{ and } \mathsf{u}\geqslant \begin{cases} \mathsf{h}^\vee & (\mathsf{r}^\vee,\mathsf{v})=1, \\ \mathsf{h} & (\mathsf{r}^\vee,\mathsf{v})=\mathsf{r}^\vee. \end{cases} \tag{1.2.10}$$

where $\mathsf{r}^\vee$ is the lacing number of $\mathfrak{g}$. Denote the simple quotient of $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ by $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$. When $\mathsf{k}\in\mathbb{Z}_{\geqslant 0}$, $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ is rational in the category $\mathcal{O}_{\mathsf{k}}$ consisting of highest-weight modules [76]. These are the vertex operator algebras corresponding to the aforementioned Wess-Zumino-Witten models.

Let $\mathsf{k}\in\mathbb{Z}_{\geqslant 0}$. The simple modules of $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ are the highest-weight modules $\mathcal{L}_\lambda$ with λ dominant integral and level k [76, Thm. 3.1.3]. Such modules are called 'integrable' as the action of the affine Lie algebra $\widehat{\mathfrak{g}}_{\mathsf{k}}$ on such modules can be integrated to define a representation of an infinite dimensional Lie group associated to $\widehat{\mathfrak{g}}_{\mathsf{k}}$. Denote the set of level k dominant integral $\widehat{\mathfrak{g}}$ weights by $\mathsf{P}^{\mathsf{k}}_{\geqslant}$. A particular consequence of this is that all $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules have finite-dimensional top spaces: The top space of the $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-module $\mathcal{L}_\lambda$ can be identified with the finite-dimensional highest-weight module $\overline{\mathcal{L}}_{\overline{\lambda}}$ of the Zhu algebra $\mathsf{Zhu}[\mathsf{L}_{\mathsf{k}}(\mathfrak{g})] \simeq U(\mathfrak{g})/\mathsf{Zhu}\left[\mathsf{I}^{\mathsf{k}}\right]$, where I^{k} is the maximal ideal in $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$.

The Weyl-Kac formula expresses the character of $\mathcal{L}_\lambda$ in terms of certain sums over the Weyl group W of $\mathfrak{g}$. What is important for the computation of fusion rules are the modular transformations of such characters. As $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ with $\mathsf{k} \in \mathbb{Z}_{\geqslant 0}$ is rational and C_2-cofinite [57, 76], the $\mathbb{C}$-span of the characters of $\mathcal{L}_\lambda$ where λ ranges over $\mathsf{P}^{\mathsf{k}}_{\geqslant}$ admits an action of $\mathsf{SL}_2(\mathbb{Z})$.

To describe the modular transformations of characters of $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ with $\mathsf{k} \in \mathbb{Z}_{\geqslant 0}$, let r be the rank of $\mathfrak{g}$, $|\Delta_+|$ the number of positive roots in $\mathfrak{g}$ and D the determinant of the matrix whose rows are the Dynkin labels of the simple coroots in $\mathfrak{g}$. As shown by Kac and Peterson [101], the S action on the characters of $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ is given by

$$\mathsf{S}\left(\mathrm{ch}\left[\mathcal{L}_\lambda\right](\mathsf{q})\right) \simeq \sum_{\lambda' \in \mathsf{P}^{\mathsf{k}}_{\geqslant}} \mathsf{S}^{\mathsf{k}}_{\lambda,\lambda'}\, \mathrm{ch}\left[\mathcal{L}_{\lambda'}\right](\mathsf{q}), \tag{1.2.11}$$

where the S-matrix elements are

$$\mathsf{S}^{\mathsf{k}}_{\lambda,\lambda'} = \frac{\mathsf{i}^{|\Delta_+|}}{\sqrt{D}(\mathsf{k}+\mathsf{h}^\vee)^{r/2}} \sum_{w \in \mathsf{W}} \det w\; \mathsf{e}^{-2\pi\mathsf{i}(\mathsf{k}+\mathsf{h}^\vee)\langle w(\overline{\lambda}+\overline{\rho}),\overline{\lambda}'+\overline{\rho}\rangle}. \tag{1.2.12}$$

The fusion rules of the rational vertex operator algebra $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ with $\mathsf{k} \in \mathbb{Z}_{\geqslant 0}$

$$\mathcal{L}_\lambda \times \mathcal{L}_{\lambda'} = \bigoplus_{\lambda'' \in \mathsf{P}^{\mathsf{k}}_{\geqslant}} \mathcal{N}^{\mathsf{k}\ \lambda''}_{\lambda,\lambda'} \mathcal{L}_{\lambda''} \tag{1.2.13}$$

can be obtained in a number of ways. Firstly, the Verlinde formula (1.1.45) expresses the the fusion coefficients $\mathcal{N}^{\mathsf{k}\ \lambda''}_{\lambda,\lambda'}$ in terms of a finite sum over ratios and products of the S-matrix elements (1.2.12). Secondly, the *Kac–Walton formula* [80, 99, 159, 160] relates the fusion coefficients to tensor product coefficients of simple $\mathfrak{g}$-modules:

$$\mathcal{N}^{\mathsf{k}\ \lambda''}_{\lambda,\lambda'} = \sum_{\substack{w \in \widehat{\mathsf{W}} \\ w\cdot\lambda'' \in \mathsf{P}^{\mathsf{k}}_{\geqslant}}} \det w\; \mathsf{N}_{\overline{\lambda},\overline{\lambda'}}^{\ \overline{w\cdot\lambda''}}. \tag{1.2.14}$$

Here, $\widehat{\mathsf{W}}$ is the affine Weyl group of $\widehat{\mathfrak{g}}$, $\overline{\lambda}$ is the projection of λ onto the weight space of $\mathfrak{g}$, and $\mathsf{N}_{\overline{\lambda},\overline{\lambda'}}^{\ \overline{\lambda''}}$ denotes the tensor product (Littlewood–Richardson) coefficients of the simple finite-dimensional $\mathfrak{g}$-modules $\overline{\mathcal{L}}_{\overline{\lambda}}$:

$$\overline{\mathcal{L}}_{\overline{\lambda}} \otimes \overline{\mathcal{L}}_{\overline{\lambda'}} \simeq \bigoplus_{\overline{\lambda''}} \mathsf{N}_{\overline{\lambda},\overline{\lambda'}}^{\ \overline{\lambda''}} \overline{\mathcal{L}}_{\overline{\lambda''}}. \tag{1.2.15}$$

A parallel-but-incomplete story exists for $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ when k is admissible but not a nonnegative integer. Indeed for such k, Kac and Wakimoto defined a class of $\mathfrak{g}$ weights called *admissible weights*, for which the characters of the associated simple highest-weight $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$-modules have desirable modular properties [104]. A famous result of Arakawa then states the following:

Theorem 1.2.1 ([18, Prop. 4.6])**.** *Let k be admissible. The $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$-module $\mathcal{L}_\lambda$ is a $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-module if and only if λ is an admissible weight.*

As the set of admissible weights for a given admissible k is finite [104] and all highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$-modules are completely reducible [18], this implies that $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$ is rational in the category $\mathcal{O}_{\mathsf{k}}$ of highest-weight modules. Kac and Wakimoto considered the modular properties of the corresponding $\mathbb{C}$-span of characters and obtained an S-matrix [104, Thm. 3.6].

The natural next step is to apply the Verlinde formula using this S-matrix and obtain fusion rules for nonintegral admissible level $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$. However, even in the relatively simple case of $\mathfrak{g} = \mathfrak{sl}_2$, the Verlinde formula was shown to give negative values for the fusion coefficients [34, 118, 129]. This disagreed with attempts at direct computations of fusion rules for these examples [11, 27, 57, 66, 81, 136] and is untenable from a 2d conformal field theory point of view.

A resolution was proposed (and checked for $\mathfrak{g} = \mathfrak{sl}_2$) in work by Creutzig and Ridout [50, 52]. There, it was shown to be necessary to work in a category containing modules that were not highest-weight. As this larger category of weight modules was known to be nonrational [7], an alternative approach to computing nonnegative integer fusion coefficients was necessary and is known as the *standard module formalism* [51].

Applying the standard module formalism to nonintegral admissible level $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ results in nonnegative fusion coefficients expressible in terms of fusion coefficients for Virasoro minimal modules, and an explanation of the previously obtained undesirable fusion coefficients as the result of ignoring the additional weight modules [50, 52]. The appearance of Virasoro minimal module fusion coefficients is in fact anticipated by the early observation that the characters of admissible level $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ are related to the characters of Virasoro minimal models [134]. This is particularly interesting as the latter is a *quantum hamiltonian reduction* of the former.

1.3. Quantum Hamiltonian Reduction

Originating in the work of Zamolodchikov and others [35, 117, 139, 170], and later generalised by various groups [54, 65, 102, 106], quantum hamiltonian reduction is a homological procedure that produces new vertex operator algebras from affine ones. The vertex operator algebras that result from quantum hamiltonian reduction are known as *W-algebras*. Here we follow the general construction of W-algebras described in [106, Sec. 1].

1.3.1. Construction. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, $\mathsf{k} \in \mathbb{C}$ and $x, f \in \mathfrak{g}$ satisfying

(a) The adjoint action $\operatorname{ad} x(-) = [x, -]$ of x on $\mathfrak{g}$ is diagonalisable with half-integer eigenvalues:

$$\mathfrak{g} = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_m, \tag{1.3.1}$$

where $\mathfrak{g}_m = \{g \in \mathfrak{g} \mid [x, g] = mg\}$.

(b) $f \in \mathfrak{g}_{-1}$.

(c) The adjoint action $\operatorname{ad} f$ of f restricted to $\mathfrak{g}_{1/2}$ defines a vector space isomorphism $\mathfrak{g}_{1/2} \simeq \mathfrak{g}_{-1/2}$.

Let $A_{\mathrm{ne}} = \mathfrak{g}_{1/2}$. A symplectic form $(-|-)_{\mathrm{ne}}$ on A_{ne} is given by,

$$(\varphi|\psi)_{\mathrm{ne}} = \langle f, [\varphi, \psi] \rangle, \quad \varphi, \psi \in A_{\mathrm{ne}}, \tag{1.3.2}$$

where $\langle -, - \rangle$ is the usual normalised Killing form. To see that this symplectic form is nondegenerate, the Killing form is invariant so $(\varphi|\psi)_{\mathrm{ne}} = \langle [f, \varphi], \psi \rangle$. As $\operatorname{ad} f : \mathfrak{g}_{1/2} \to \mathfrak{g}_{-1/2}$ is a vector space isomorphism, $(-|-)_{\mathrm{ne}}$ defines a nondegenerate pairing between $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_{-1/2}$. Recalling Section 1.1.2, we can therefore construct the neutral ghost vertex algebra $\mathsf{B}(A_{\mathrm{ne}})$.

Similarly, let $A_+ = \oplus_{m>0} \mathfrak{g}_m$, $A_- = (A_+)^*$ and $A_{\mathrm{ch}} = A_+ \oplus A_-$. Define a nondegenerate symmetric form $(-|-)_{\mathrm{ch}}$ on A_{ch} by

$$(A_+|A_+)_{\mathrm{ch}} = (A_-|A_-)_{\mathrm{ch}} = 0, \quad (\delta|\xi)_{\mathrm{ch}} = (\xi|\delta)_{\mathrm{ch}} = \xi(\delta), \quad \delta \in A_+, \xi \in A_-. \tag{1.3.3}$$

Again from Section 1.1.2, we can therefore construct the charged ghost vertex superalgebra $\mathsf{F}(A_{\mathrm{ch}})$.

Let $\mathsf{G} = \mathsf{F}(A_{\mathrm{ch}}) \otimes \mathsf{B}(A_{\mathrm{ne}})$ and $\mathsf{C} = \mathsf{V}^{\mathsf{k}}(\mathfrak{g}) \otimes \mathsf{G}$. The charge decomposition of $\mathsf{F}(A_{\mathrm{ch}})$ gives rise to a charge decomposition

$$\mathsf{C} = \bigoplus_{m \in \mathbb{Z}} \mathsf{C}_m \tag{1.3.4}$$

by setting the charge of all fields in $\mathsf{B}(A_{\mathrm{ne}})$ and $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ to be zero. To construct a differential on the graded vertex algebra C, fix a basis $\{J^a\}_{a \in S_j}$ for each $\mathfrak{g}_j$. Let $S = \sqcup_j S_j$ (so that $\{J^a\}_{a \in S}$ is a basis of $\mathfrak{g}$), $S_+ = \sqcup_{j>0} S_j$ (so that $\{J^a\}_{a \in S_+}$ is a basis of A_+) and $C^{a,b}_c \in \mathbb{C}$ be the structure constants of $\mathfrak{g}$ defined by

$$[J^a, J^b] = \sum_{c \in S} C^{a,b}_c J^c. \tag{1.3.5}$$

Denote the corresponding basis of A_{ne} by $\{\delta^a\}_{a \in S_{1/2}}$ and that of A_+ by $\{\varphi^a\}_{a \in S_+}$. Finally let $\{\psi^a\}_{a \in S_+}$ be the basis of A_- dual to $\{\varphi^a\}_{a \in S_+}$ with respect to $(-|-)_{\mathrm{ch}}$. That is,

$$(\varphi^a|\psi^b)_{\mathrm{ch}} = \psi^b(\varphi^a) = \delta_{a,b}. \tag{1.3.6}$$

In terms of these generating fields, the charge decomposition of C is defined by giving the fields $\{J^a(z)\}_{a\in S}$ and $\{\delta^a(z)\}_{a\in S_{1/2}}$ charge zero, $\{\varphi^a(z)\}_{a\in S_+}$ charge 1 and $\{\psi^a(z)\}_{a\in S_+}$ charge -1.

Define an odd field $d(z)\in\mathsf{C}$ of charge equal to -1 by

$$d(z)=\sum_{a\in S_+}J^a(z)\psi^a(z)-\frac{1}{2}\sum_{a,b,c\in S_+}C^{a,b}_c{:}\varphi^c(z)\psi^a(z)\psi^b(z){:} + \sum_{a\in S_+}\langle f,J^a\rangle\psi^a(z)+\sum_{a\in S_{1/2}}\psi^a(z)\delta^a(z), \tag{1.3.7}$$

where we have omitted tensor product symbols. By [102, Thm. 2.1], $d(z)d(w)\sim 0$. A simple consequence of this is that the zero mode $d=d_0:\mathsf{C}_m\to\mathsf{C}_{m-1}$ is a differential, i.e. $d^2=0$.

The homology of the chain complex (C,d) graded by charge, denoted by

$$\mathsf{W}^{\mathsf{k}}(\mathfrak{g},x,f)=H_{x,f}\left(\mathsf{V}^{\mathsf{k}}(\mathfrak{g})\right), \tag{1.3.8}$$

is called a *quantum hamiltonian reduction* of $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$, or a *W-algebra* for short. The vertex algebra structure on $\mathsf{W}^{\mathsf{k}}(\mathfrak{g},x,f)$ is inherited from that of C. As shown by Kac and Wakimoto [102, Thm. 4.1], the homology is concentrated on the zeroth degree component,

$$\mathsf{W}^{\mathsf{k}}(\mathfrak{g},x,f)=H^0_{x,f}\left(\mathsf{V}^{\mathsf{k}}(\mathfrak{g})\right). \tag{1.3.9}$$

Here we denote the components of homology with upper indices $H^j_{x,f}(-)$ (as in cohomology) rather than the usual lower indices. This notational choice is justified by the fact that we can multiply our charge assignments by -1 and obtain a new charge decomposition of C. Denote this graded vertex algebra by C'. Then $d(z)$ has charge 1, and (C',d) is a cochain complex. The cohomology of (C',d) coincides with the homology (C,d): By definition, the m'th component of the cohomology of (C',d) is equal to the $-m$'th component of the homology (C,d). The only component of the homology of (C,d) used in this thesis is the zeroth one so distinguishing between homology and cohomology is not necessary.

To make $\mathsf{W}^{\mathsf{k}}(\mathfrak{g},x,f)$ a vertex operator algebra, we require an energy-momentum field $L(z)\in\mathsf{W}^{\mathsf{k}}(\mathfrak{g},x,f)$. This can be achieved by defining a field $L(z)\in\mathsf{C}$ such that $d(z)$ is a primary field with conformal dimension 1. With this goal in mind, let $\mathsf{k}\neq-\mathsf{h}^\vee$, $\{\xi^a\}_{a\in S_{1/2}}$ be the dual basis to $\{\delta^a\}_{a\in S_{1/2}}$ with respect to $(-|-)_{\mathrm{ne}}$ and $\{m_a\}_{a\in S}\subset\mathbb{C}$ be the set defined by $[x,J^a]=m_aJ^a$. Define the field $L(z)$ by

$$L(z)=T^{\mathrm{Sug.}}(z)+\partial x(z)-\sum_{a\in S_+}m_a{:}\psi^a(z)\partial\varphi^a(z){:} + \sum_{a\in S_+}(1-m_a){:}\partial\psi^a(z)\varphi^a(z){:}+\frac{1}{2}\sum_{a\in S_{1/2}}{:}\partial\delta(z)\xi^a(z){:}. \tag{1.3.10}$$

Kac, Wakimoto and Roan showed that $L(z)$ is an energy-momentum field in C with central charge c given by [102, Thm. 2.2(a)]

$$\mathsf{c} = \frac{\mathsf{k}\dim\mathfrak{g}}{\mathsf{k}+\mathsf{h}^\vee} - 12\mathsf{k}\langle x,x\rangle - \sum_{a\in S_+}(12m_a^2 - 12m_a + 2) - \frac{1}{2}\dim\mathfrak{g}_{1/2}, \tag{1.3.11}$$

and showed that $d(z)$ was primary of conformal dimension 1 [102, Thm. 2.3]. The non-zero image of $L(z)$ in $\mathsf{W}^\mathsf{k}(\mathfrak{g},x,f)$, which we also denote by $L(z)$, gives $\mathsf{W}^\mathsf{k}(\mathfrak{g},x,f)$ the structure of a vertex operator algebra.

The most important examples of elements $x,f\in\mathfrak{g}$ satisfying the required conditions for quantum hamiltonian reduction are those arising from $\mathfrak{sl}_2$ triples $\{x,e,f\}\subset\mathfrak{g}$. In such cases, the nilpotent element $f\in\mathfrak{g}$ determines $x\in\mathfrak{g}$ up to conjugation by the Jacobson-Morozov theorem. For $x,f\in\mathfrak{g}$ belonging to $\mathfrak{sl}_2$ triples, we will therefore denote the corresponding W-algebra by $\mathsf{W}^\mathsf{k}(\mathfrak{g},f) = H_f\left(\mathsf{V}^\mathsf{k}(\mathfrak{g})\right)$.

Alternatively, any element $x\in\mathfrak{g}$ whose adjoint action grades $\mathfrak{g}$ with half-integer eigenvalues determines a nilpotent element $f\in\mathfrak{g}$ (up the action of the adjoint group of $\mathfrak{g}$) such that x and f belong to an $\mathfrak{sl}_2$ triple [106, Rem. 1.1].

The orbits of nilpotent elements in $\mathfrak{g}$ under the action of the adjoint group of $\mathfrak{g}$ are known as the *nilpotent orbits* of $\mathfrak{g}$. Therefore the W-algebra $\mathsf{W}^\mathsf{k}(\mathfrak{g},f)$ is determined, up to isomorphism, by the nilpotent orbit containing f.

All W-algebras encountered in this thesis are isomorphic to $\mathsf{W}^\mathsf{k}(\mathfrak{g},f)$ for some nilpotent $f\in\mathfrak{g}$. In light of this, we restrict attention to W-algebras of this form. Many of the subsequent results also apply to W-algebras not corresponding to $\mathfrak{sl}_2$ triples with appropriate modifications but this level of generality is not required here.

A similar homological construction as quantum hamiltonian reduction builds W-algebra modules out of $\mathsf{V}^\mathsf{k}(\mathfrak{g})$-modules. Given a $\mathsf{V}^\mathsf{k}(\mathfrak{g})$-module $\mathcal{M}$, form the chain complex $(\mathsf{C}(\mathcal{M}),d)$ where $\mathsf{C}(\mathcal{M}) = \mathcal{M}\otimes\mathsf{G}$. Here, all $\mathsf{V}^\mathsf{k}(\mathfrak{g})$ modes in $d=d_0$ act on $\mathcal{M}$ rather than on $\mathsf{V}^\mathsf{k}(\mathfrak{g})$. Giving $\mathcal{M}$ charge 0, the C-module $\mathsf{C}(\mathcal{M})$ is graded by charge. The homology $H_f(\mathcal{M})$ of this chain complex is a direct sum of $\mathsf{W}^\mathsf{k}(\mathfrak{g},f)$-modules, graded by charge with the module $\mathcal{M}$ given charge 0,

$$H_f(\mathcal{M}) = \bigoplus_{j\in\mathbb{Z}} H_f^j(\mathcal{M}). \tag{1.3.12}$$

In this way, quantum hamiltonian reduction defines a functor between suitable categories of $\mathsf{V}^\mathsf{k}(\mathfrak{g})$-modules and $\mathsf{W}^\mathsf{k}(\mathfrak{g},f)$-modules.

There is a natural notion of highest-weight and Verma $\mathsf{W}^\mathsf{k}(\mathfrak{g},f)$-modules. The general definition is presented in [102, Sec. 6]. We will outline these definitions in the specific case of $\mathfrak{g}=\mathfrak{sl}_3$ and $f=f_\theta$ in Section 2.2.1. As usual, highest-weight $\mathsf{W}^\mathsf{k}(\mathfrak{g},f)$-modules are required to be generated by a highest-weight vector with minimal conformal dimension with respect to the energy-momentum vector (1.3.10).

Theorem 1.3.1. *Let* $\mathsf{k} \neq -\mathsf{h}^\vee$ *and* λ *be a level-*k $\widehat{\mathfrak{g}}$ *weight.*

- *[106, Thm. 6.2]* $H^j_f(\mathcal{K}_\lambda) = 0$ *for all* $j \neq 0$.
- *[106, Thm. 6.3]* $H^0_f(\mathcal{K}_\lambda)$ *is a Verma* $\mathsf{W}^\mathsf{k}(\mathfrak{g}, f)$*-module with minimal conformal dimension*

$$\Delta = \frac{\langle \overline{\lambda}, \overline{\lambda} + 2\overline{\rho} \rangle}{2(\mathsf{k} + \mathsf{h}^\vee)} - \lambda(x). \tag{1.3.13}$$

- $H^0_f(-)$ *induces a surjection from the set of isomorphism classes of Verma* $\mathsf{V}^\mathsf{k}(\mathfrak{g})$*-modules to the the set of isomorphism classes of Verma* $\mathsf{W}^\mathsf{k}(\mathfrak{g}, f)$*-modules.*

Note that the above results do not describe the structure of the $\mathsf{W}^\mathsf{k}(\mathfrak{g}, f)$-module $H_f(\mathcal{L}_\lambda)$. In general this is a very difficult problem. A class of W-algebras for which the results exist in this direction are the W-algebras corresponding to the choice of nilpotent element $f = f_\theta$, the negative root vector corresponding to the highest root in $\mathfrak{g}$. The W-algebras of the form $\mathsf{W}^\mathsf{k}(\mathfrak{g}, f_\theta)$ are known as *minimal W-algebras*.

Theorem 1.3.2. *Let* $\mathsf{k} \neq -\mathsf{h}^\vee$ *and* λ *be a level-*k $\widehat{\mathfrak{g}}$ *weight with zeroth Dynkin label* λ_0.

- *[12, Thm. 6.7.4]* $H^0_{f_\theta}(\mathcal{L}_\lambda) = 0$ *if and only if* $\lambda_0 \in \mathbb{Z}_{\geqslant 0}$. *For* $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$, $H^0_{f_\theta}(\mathcal{L}_\lambda)$ *is a simple highest-weight* $\mathsf{W}^\mathsf{k}(\mathfrak{g}, f_\theta)$*-module.*
- $H^0_{f_\theta}(-)$ *induces a surjection from the set of isomorphism classes of simple highest-weight* $\mathsf{V}^\mathsf{k}(\mathfrak{g})$*-modules to the union of the set of isomorphism classes of simple highest-weight* $\mathsf{W}^\mathsf{k}(\mathfrak{g}, f_\theta)$*-modules and* $\{0\}$.
- *[12, Cor. 6.7.3] The restriction of* $H^0_{f_\theta}(-)$ *to the category* $\widehat{\mathscr{O}}_\mathsf{k}$ *of* $\mathsf{V}^\mathsf{k}(\mathfrak{g})$*-modules is exact.*

Recall that the isomorphism classes of W-algebras for a given $\mathfrak{g}$ are determined by nilpotent orbits in $\mathfrak{g}$. That is, $\mathsf{W}^\mathsf{k}(\mathfrak{g}, f_1) \simeq \mathsf{W}^\mathsf{k}(\mathfrak{g}, f_2)$ if and only if f_1 and f_2 belong to the same nilpotent orbit.

There is a well known partial ordering on the nilpotent orbits of a given $\mathfrak{g}$ known as the *closure* or *Chevalley ordering* [40]. The Chevalley ordering of nilpotent orbits in $\mathfrak{g}$ also defines a partial ordering on the isomorphism classes of W-algebras obtained from $\mathsf{V}^\mathsf{k}(\mathfrak{g})$.

It is suspected that, in addition to the usual quantum hamiltonian reduction, one can also perform a 'partial quantum hamiltonian reduction' between two W-algebras as long as they are related by this partial ordering.

Strong supporting evidence for this can be found in the twisted Zhu algebra: the associative algebra $\mathsf{Zhu}^{\mathrm{tw}}\left[\mathsf{W}^\mathsf{k}(\mathfrak{g}, f)\right]$ is isomorphic to the finite W-algebra corresponding to the same $\mathfrak{g}$ and f

[55]. Finite W-algebras are constructed by a hamiltonian reduction procedure from $U(\mathfrak{g})$, whose 'affinisation' is the usual quantum hamiltonian reduction of $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$.

Partial reductions of finite W-algebras corresponding to $\mathfrak{g} = \mathfrak{sl}_{n+1}$ were constructed by Morgan [133]. A partial reduction from the $f_1 \in \mathfrak{sl}_{n+1}$ finite W-algebra to the $f_2 \in \mathfrak{sl}_{n+1}$ finite W-algebra was shown to exist when the corresponding nilpotent orbits $\mathbb{O}_{f_i}$ satisfy $\mathbb{O}_{f_2} \geqslant \mathbb{O}_{f_1}$ in the Chevalley ordering.

While early work on partial quantum hamiltonian reductions exists in physics literature [124], the vertex algebraic content of such a construction is not yet known.

1.3.2. The Zamolodchikov Algebra. The first W-algebra was constructed by Zamolodchikov [170]. It is the quantum hamiltonian reduction of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ corresponding to the nilpotent orbit of $\mathfrak{sl}_3$ containing the *regular* nilpotent element $f_{\alpha_1} + f_{\alpha_2}$, where f_{α_i} is the negative root vector corresponding to the simple $\mathfrak{sl}_3$ root α_i.

Definition 1.3.3. *The* universal Zamolodchikov algebra $\mathsf{W}_3^{\mathsf{k}} = \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, f_{\alpha_1} + f_{\alpha_2})$ *is the vertex algebra strongly and freely generated by fields $T(z)$ and $W(z)$ with the following operator product expansions:*

$$T(z)T(w) \sim \frac{\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3}\mathbb{1}}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \qquad T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{(z-w)},$$
$$W(z)W(w) \sim \frac{2\Lambda(w)}{(z-w)^2} + \frac{\partial\Lambda(w)}{(z-w)} + A_{\mathsf{k}}\Bigg[\frac{\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3}\mathbb{1}}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{\frac{3}{10}\partial^2 T(w)}{(z-w)^2} + \frac{\frac{1}{15}\partial^3 T(w)}{(z-w)}\Bigg]. \tag{1.3.14}$$

Here, we set

$$\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3} = -\frac{2(3\mathsf{k}+5)(4\mathsf{k}+9)}{\mathsf{k}+3},$$
$$\Lambda(z) = :T(z)T(z): - \frac{3}{10}\partial^2 T(z), \tag{1.3.15}$$
$$A_{\mathsf{k}} = -\frac{(3\mathsf{k}+4)(5\mathsf{k}+12)}{2(\mathsf{k}+3)} = \frac{22 + 5\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3}}{16}.$$

Denote the simple quotient of $\mathsf{W}_3^{\mathsf{k}}$ by $\mathsf{W}_{3,\mathsf{k}}$. When the level k is of the form $\mathsf{k} = \frac{\mathsf{u}}{\mathsf{v}} - 3$ for some coprime $\mathsf{u}, \mathsf{v} \in \mathbb{Z}_{\geqslant 3}$, $\mathsf{W}_3^{\mathsf{k}}$ is not simple [131, 162]. In this case, the simple quotient is known as a W_3 minimal model and is denoted by $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$. These models are all rational and C_2-cofinite [16, 17].

The central charge is invariant under exchanging u and v:

$$\mathsf{c}_{\mathsf{u},\mathsf{v}}^{\mathsf{W}_3} = -\frac{2(3\mathsf{u}-4\mathsf{v})(4\mathsf{u}-3\mathsf{v})}{\mathsf{uv}} = 2 - \frac{24(\mathsf{u}-\mathsf{v})^2}{\mathsf{uv}}. \tag{1.3.16}$$

As the defining operator product expansions (1.3.14) only depend on k through $\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3}$, it follows that $\mathsf{W}_3(\mathsf{u},\mathsf{v}) = \mathsf{W}_3(\mathsf{v},\mathsf{u})$.

The particular normalisation for $W(z)$ in Definition 1.3.3 is not standard. Namely we have multiplied the standard definition of [170] by $\sqrt{A_{\mathsf{k}}}$ in order to cancel the poles that arise when $\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3} = -\frac{22}{5}$, i.e. $(\mathsf{u},\mathsf{v}) = (3,5)$ or $(5,3)$. In fact, W and Λ belong to the maximal ideal of $\mathsf{W}_3^{\mathsf{k}}$ at this central charge, hence are zero in the simple quotient $\mathsf{W}_3(3,5) = \mathsf{W}_3(5,3)$. It is not hard to see that the W_3 minimal model $\mathsf{W}_3(3,5)$ therefore coincides with the Virasoro minimal model $\mathsf{M}(2,5)$.

The representation theory of the rational W_3 minimal models is well understood. The classification of simple $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-modules was obtained in [60]. These modules are all highest-weight (that is, positive energy and generated by a vector in the top space) with one-dimensional top spaces.

Writing $T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n-2}$ and $W(z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-3}$, a highest-weight vector is therefore a simultaneous eigenvector of T_0 and W_0 that is annihilated by the T_n and W_n with $n > 0$. In light of Zhu's theorem (Theorem 1.1.9), it is sufficient to specify the eigenvalues of T_0 and W_0 on the highest-weight vectors of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-modules to describe the classification of simple $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-modules. Here, we adapt the parametrisation of the highest weights given in [38].

Simple highest-weight $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-modules are specified by pairs of $\widehat{\mathfrak{sl}}_3$ weights r,s where $\mathsf{r} = [r_0,r_1,r_2] \in \mathsf{P}_{\geqslant}^{\mathsf{u}-3}$ and $\mathsf{s} = [s_0,s_1,s_2] \in \mathsf{P}_{\geqslant}^{\mathsf{v}-3}$. The eigenvalues of T_0 and W_0 on the highest-weight vector of the simple $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-module corresponding to the weights r,s are given by

$$\begin{aligned}(1.3.17a)\quad \Delta(\mathsf{r},\mathsf{s}) = \frac{1}{3\mathsf{uv}}\Big(&\big(\mathsf{v}(r_1+1)-\mathsf{u}(s_1+1)\big)\big(\mathsf{v}(r_2+1)-\mathsf{u}(s_2+1)\big)\\ &+\big(\mathsf{v}(r_1+1)-\mathsf{u}(s_1+1)\big)^2+\big(\mathsf{v}(r_2+1)-\mathsf{u}(s_2+1)\big)^2-(\mathsf{u}-\mathsf{v})^2\Big),\end{aligned}$$

$$(1.3.17b)\quad w(\mathsf{r},\mathsf{s}) = \frac{\big(\mathsf{v}(r_0-r_1)-\mathsf{u}(s_0-s_1)\big)\big(\mathsf{v}(r_0-r_2)-\mathsf{u}(s_0-s_2)\big)\big(\mathsf{v}(r_1-r_2)-\mathsf{u}(s_1-s_2)\big)}{3(3\mathsf{uv})^{3/2}},$$

respectively. These eigenvalues are invariant under the free $\mathbb{Z}_3$-action

$$(1.3.18)\qquad \begin{bmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{bmatrix} \longmapsto \begin{bmatrix} r_2 & r_0 & r_1 \\ s_2 & s_0 & s_1 \end{bmatrix} \longmapsto \begin{bmatrix} r_1 & r_2 & r_0 \\ s_1 & s_2 & s_0 \end{bmatrix} \longmapsto \begin{bmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{bmatrix}.$$

Similarly, the conformal weight (1.3.17a) is invariant under the (nonfree) $\mathbb{Z}_2$-action

$$(1.3.19)\qquad \begin{bmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} r_0 & r_2 & r_1 \\ s_0 & s_2 & s_1 \end{bmatrix},$$

whilst (1.3.17b) changes sign. The $\mathbb{Z}_2$-action corresponds to the conjugation automorphism of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ which is given by $T(z) \leftrightarrow T(z)$ and $W(z) \leftrightarrow -W(z)$, . We therefore get an additional isomorphism corresponding to (1.3.19) if $w(\mathsf{r},\mathsf{s}) = 0$. But, (1.3.17b) shows that this happens if and only if two of the pairs (r_0,s_0), (r_1,s_1) and (r_2,s_2) coincide, in which case the conjugation isomorphism is already accounted for by one of the isomorphisms corresponding to the $\mathbb{Z}_3$-action (1.3.18).

The isomorphism classes of the simple $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-modules are therefore classified by elements of $(\mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant})/\mathbb{Z}_3$. Denote such modules by $\mathcal{W}(\mathsf{r},\mathsf{s})$ where we interpret (r,s) as an element of $(\mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant})/\mathbb{Z}_3$. In other words, let ∇ be $\mathbb{Z}_3$-action on the $\widehat{\mathfrak{sl}}_3$ weights defined by $\nabla(\mathsf{t}) = \nabla([t_0,t_1,t_2]) = [t_2,t_0,t_1]$. Then, $\mathcal{W}(\mathsf{r},\mathsf{s}) = \mathcal{W}(\nabla(\mathsf{r}),\nabla(\mathsf{s})) = \mathcal{W}(\nabla^2(\mathsf{r}),\nabla^2(\mathsf{s}))$.

1.3.3. Modularity and Fusion Rules for $\mathsf{W}_3(\mathsf{u},\mathsf{v})$. As $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ is rational and C_2-cofinite [16, 17], by Zhu's theorem the span of its characters admits an $\mathsf{SL}_2(\mathbb{Z})$-action. By the Verlinde formula (1.1.45), knowledge of the S-matrix for $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ also allows us to determine the fusion rules of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$. There is however a technical subtlety to determining the elements of the S-matrix from the characters of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$. Let $\mathcal{M}$ be a $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-module and consider its q character

$$\mathrm{ch}\big[\mathcal{M}\big](\tau) = \mathrm{tr}_{\mathcal{M}}\,\mathsf{q}^{T_0 - \mathsf{c}^{\mathsf{W}_3}_{\mathsf{u},\mathsf{v}}/24}. \tag{1.3.20}$$

Applying the $\mathbb{Z}_2$-action (1.3.19) to a given $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-module $\mathcal{W}(\mathsf{r},\mathsf{s})$ gives a new $\mathsf{W}_3(\mathsf{u},\mathsf{v})$-module whose highest-weight vector has conformal dimension $\Delta(\mathsf{r},\mathsf{s})$ and W_0 eigenvalue $-w(\mathsf{r},\mathsf{s})$. As $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ generically admits modules having $w(\mathsf{r},\mathsf{s}) \neq 0$, this means that the image of $\mathcal{W}(\mathsf{r},\mathsf{s})$ under the conjugation automorphism of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ will always have the same character as $\mathcal{W}(\mathsf{r},\mathsf{s})$ but is usually not isomorphic to it.

Put simply, the characters (1.3.20) are not linearly independent in general. Therefore definitively extracting S-matrix elements from an $\mathsf{SL}_2(\mathbb{Z})$-action on the span of such characters is not possible.

A remedy for the lack of linear independence of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ characters was provided in [23]. The proposal therein is to upgrade characters to *one-point functions* by inserting the zero mode of some $u \in \mathsf{W}_3(\mathsf{u},\mathsf{v})$:

$$\mathrm{ch}\big[\mathcal{W}(\mathsf{r},\mathsf{s})\big](\tau\,;u) = \mathrm{tr}_{\mathcal{W}(\mathsf{r},\mathsf{s})}\Big(u_0\mathsf{q}^{T_0 - \mathsf{c}^{\mathsf{W}_3}_{\mathsf{u},\mathsf{v}}/24}\Big). \tag{1.3.21}$$

These one-point functions are linearly independent for generic u since $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ is rational and C_2-cofinite [17]. The modular S-matrix elements obtained by transforming the one-point functions can then be written down unambiguously. Note as well that the characters (1.3.20) are precisely the one-point functions with $u = \mathbb{1}$.

As the highest-weight modules of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ are completely specified by the eigenvalues of T_0 and W_0 on the highest-weight vector, a suitable choice for u is W. However it might be the case that W is zero in $\mathsf{W}_3(\mathsf{u},\mathsf{v})$. The operator product expansions of $\mathsf{W}^{\mathsf{k}}_3$ in (1.3.14) show that W is a null vector in $\mathsf{W}_3^{\mathsf{k}}$ if and only if $\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3} = 0$ or $A_{\mathsf{k}} = 0$. However,

$$\mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3} = 0 \iff (\mathsf{u},\mathsf{v}) = (3,4),(4,3), \quad A_{\mathsf{k}} = 0 \iff (\mathsf{u},\mathsf{v}) = (3,5),(5,3). \tag{1.3.22}$$

The minimal model $\mathsf{W}_3(3,4) = \mathsf{W}_3(4,3)$ is the trivial vertex operator algebra, while $\mathsf{W}_3(3,5) = \mathsf{W}_3(5,3)$ is the Virasoro minimal model $\mathsf{M}(2,5)$. In either case, the q characters are linearly independent. So when W is zero in $\mathsf{W}_3(\mathsf{u},\mathsf{v})$, we can just take $u = \mathbb{1}$.

Here we specialise the results on modular properties of regular W-algebras (see Section 4.1.1) in [23,74,105] to the case of W_3. In addition, we deduce several identities satisfied by the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix elements that will be crucial in our later investigations into a closely related W-algebra called the Bershadsky–Polyakov algebra.

Denote by $\bar{\mathsf{t}} = [t_1, t_2]$ the projection of onto the weight space of $\mathfrak{sl}_3$ of an $\widehat{\mathfrak{sl}}_3$-weight $\mathsf{t} = [t_0, t_1, t_2]$. Let $\overline{\rho} = [1,1]$ denote the Weyl vector of $\mathfrak{sl}_3$, $\langle -,- \rangle = \frac{1}{6}\kappa_{\mathfrak{sl}_3}(-,-)$ its normalised Killing form and S_3 its Weyl group.

Theorem 1.3.4 ([105, Thm. 4.4], [23, Cor. 8.4])**.** *For coprime* $\mathsf{u},\mathsf{v} \in \mathbb{Z}_{\geqslant 3}$*, the S-transform of the* $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ *one-point function* (1.3.21) *is given by*

$$\mathrm{ch}\big[\mathcal{W}(\mathsf{r},\mathsf{s})\big]\left(-\frac{1}{\tau}\,;\frac{u}{\tau^{\Delta_u}}\right) = \sum_{(\mathsf{r}',\mathsf{s}')\in(\mathsf{P}_{\geqslant}^{\mathsf{u}-3}\times\mathsf{P}_{\geqslant}^{\mathsf{v}-3})/\mathbb{Z}_3} \mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')}\,\mathrm{ch}\big[\mathcal{W}(\mathsf{r}',\mathsf{s}')\big]\,(\tau\,;u)\,, \tag{1.3.23}$$

and the S-matrix entries are given, for $(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}') \in (\mathsf{P}_{\geqslant}^{\mathsf{u}-3}\times\mathsf{P}_{\geqslant}^{\mathsf{v}-3})/\mathbb{Z}_3$*, by*

$$\begin{aligned}\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')} &= \frac{1}{\sqrt{3}\mathsf{u}\mathsf{v}}\mathsf{e}^{2\pi\mathsf{i}\left(\langle\bar{\mathsf{r}}+\overline{\rho},\bar{\mathsf{s}}'+\overline{\rho}\rangle+\langle\bar{\mathsf{s}}+\overline{\rho},\bar{\mathsf{r}}'+\overline{\rho}\rangle\right)} \\ &\quad\cdot\sum_{w\in\mathsf{S}_3}\det w\,\mathsf{e}^{-2\pi\mathsf{i}\frac{\mathsf{v}}{\mathsf{u}}\langle w(\bar{\mathsf{r}}+\overline{\rho}),\bar{\mathsf{r}}'+\overline{\rho}\rangle}\sum_{w\in\mathsf{S}_3}\det w\,\mathsf{e}^{-2\pi\mathsf{i}\frac{\mathsf{u}}{\mathsf{v}}\langle w(\bar{\mathsf{s}}+\overline{\rho}),\bar{\mathsf{s}}'+\overline{\rho}\rangle}.\end{aligned} \tag{1.3.24}$$

As the modules $\mathcal{W}(\mathsf{r},\mathsf{s})$ are independent of the choice of representatives of the $\mathbb{Z}_3$-orbit (r,s), so too is the the S-matrix formula (1.3.24). To see this explicitly, observe that acting on r or s by the $\mathbb{Z}_3$-generator ∇ amounts to acting with an outer automorphism of $\widehat{\mathfrak{sl}}_3$. It is easy to check that on the projection onto the weight space of $\mathfrak{sl}_3$, ∇ acts as follows:

$$\nabla(\bar{\mathsf{t}}) = w_1 w_2(\bar{\mathsf{t}}) + \mathsf{k}(\mathsf{t})\omega_1. \tag{1.3.25}$$

Here, $\omega_1 = [1,0]$ is the first fundamental weight of $\mathfrak{sl}_3$ and $\mathsf{k}(\mathsf{t})$ is the level of t. Acting with ∇ on (1.3.24) on r and s gives

$$\begin{aligned}\mathsf{S}^{\mathsf{W}_3}_{(\nabla(\mathsf{r}),\mathsf{s}),(\mathsf{r}',\mathsf{s}')} &= \mathsf{e}^{-2\pi\mathsf{i}\mathsf{v}\langle\omega_1,\bar{\mathsf{r}}'+\overline{\rho}\rangle}\mathsf{e}^{-\mathsf{v}\langle\omega_1,\xi_{\bar{\mathsf{s}}'}\rangle}\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')}, \\ \mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\nabla(\mathsf{s})),(\mathsf{r}',\mathsf{s}')} &= \mathsf{e}^{+2\pi\mathsf{i}\mathsf{v}\langle\omega_1,\bar{\mathsf{r}}'+\overline{\rho}\rangle}\mathsf{e}^{+\mathsf{v}\langle\omega_1,\xi_{\bar{\mathsf{s}}'}\rangle}\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')},\end{aligned} \tag{1.3.26}$$

where $\xi_{\overline{\mathsf{s}'}} = -2\pi \mathrm{i}\frac{\mathsf{u}}{\mathsf{v}}(\overline{\mathsf{s}'} + \overline{\rho})$. Therefore, applying ∇ to both r and s leaves the S-matrix invariant. Applying ∇ to both r' and s' also leaves the S-matrix invariant because (1.3.24) is symmetric. The S-matrix may also be verified to be unitary, see for example [105, Prop. 4.4].

A similar calculation demonstrates that its square is the matrix whose $(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')$-entry is 0 unless $\overline{\mathsf{r}'} = [r_2, r_1]$ and $\overline{\mathsf{s}'} = [s_2, s_1]$, in which case it is 1. This is the matrix representing the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ conjugation (1.3.19).

Before moving on to the computing fusion rules for $\mathsf{W}_3(\mathsf{u},\mathsf{v})$, there are some properties of the S-matrix (1.3.24) that will prove useful for various computations encountered in this thesis. Some of these require extending (1.3.24) to allow arbitrary integral $\widehat{\mathfrak{sl}}_3$-weights r, r', s and s'. For example, it is straightforward to show that

$$\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},w\cdot\mathsf{s}),(\mathsf{r}',\mathsf{s}')} = \det w\, \mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')}, \qquad w \in \mathsf{S}_3, \tag{1.3.27}$$

where $w \cdot \mathsf{s} = w(\mathsf{s}+\rho) - \rho$ is the usual shifted action of the Weyl group. It follows that if $s_i = -1$, for $i = 1$ or 2, then it is fixed by the shifted action of the i-th simple Weyl reflection w_i and so

$$\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')} = 0. \tag{1.3.28}$$

Similarly, the well known decomposition of $\overline{w_0(\mathsf{s}+\rho)}$ as the Weyl reflection for the highest root θ followed by translation by $\mathsf{v}\theta$ leads to (1.3.27) also holding for $w = w_0$ (and therefore for any w in the affine Weyl group $\widehat{\mathsf{S}}_3$ of $\widehat{\mathfrak{sl}}_3$). Consequently, (1.3.28) continues to hold if $s_0 = -1$, hence $w_0 \cdot \mathsf{s} = \mathsf{s}$. It follows that the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix entry (1.3.24) vanishes when s lies on a shifted affine alcove boundary. Swapping the roles of s and v with r and u gives the analogous result for r.

Remarkably, ratios of the S-matrix elements (1.3.24) are related to characters of highest-weight $\mathfrak{sl}_3$-module. The same phenomenon is present in ratios of the S-matrix elements of $\mathsf{L}_\mathsf{k}(\mathfrak{g})$ for $\mathsf{k} \in \mathbb{Z}_{\geqslant 0}$ (see [70, Sec. 14.6.3]). Indeed the sums present in the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix are superficially of the same form as the $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$ S-matrix (1.2.12). This can be viewed as a consequence of the coset construction of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ that involves two copies of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$ at specific levels [19, 92].

For any $\mathfrak{sl}_3$-weight $\bar{\mathsf{t}} = [t_1, t_2]$, denote the character of the simple highest-weight $\mathfrak{sl}_3$-module $\overline{\mathcal{L}}_{\bar{\mathsf{t}}}$ by $\chi_{\bar{\mathsf{t}}}$. Let $0 = [\mathsf{v}-3, 0, 0]$.

Proposition 1.3.5. *Let* $\mathsf{u},\mathsf{v} \in \mathbb{Z}_{\geqslant 3}$ *be coprime and* $(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}') \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant}/\mathbb{Z}_3$. *Then,*

$$\frac{\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')}}{\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},0),(\mathsf{r}',\mathsf{s}')}} = \mathsf{e}^{2\pi\mathrm{i}\langle\bar{\mathsf{s}},\overline{\mathsf{r}'}+\overline{\rho}\rangle}\chi_{\bar{\mathsf{s}}}(\xi_{\overline{\mathsf{s}'}}). \tag{1.3.29}$$

PROOF. Substituting (1.3.24) into the left-hand-side of (1.3.29) and simplifying gives

$$\frac{\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')}}{\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},0),(\mathsf{r}',\mathsf{s}')}} = \mathsf{e}^{2\pi\mathsf{i}\langle\bar{\mathsf{s}},\bar{\mathsf{r}}'+\bar{\rho}\rangle}\frac{\sum_{w\in\mathsf{S}_3}\det w\,\mathsf{e}^{\langle w(\bar{\mathsf{s}}+\bar{\rho}),\xi_{\bar{\mathsf{s}}'}\rangle}}{\sum_{w\in\mathsf{S}_3}\det w\,\mathsf{e}^{\langle w(\bar{\rho}),\xi_{\bar{\mathsf{s}}'}\rangle}} = \mathsf{e}^{2\pi\mathsf{i}\langle\bar{\mathsf{s}},\bar{\mathsf{r}}'+\bar{\rho}\rangle}\chi_{\bar{\mathsf{s}}}(\xi_{\bar{\mathsf{s}}'}), \tag{1.3.30}$$

where the final equality is the Weyl character formula. ■

The roles of r and s in Proposition 1.3.5 can be reversed to obtain a similar result involving the character $\chi_{\bar{\mathsf{r}}}$ of $\overline{\mathcal{L}}_{\bar{\mathsf{r}}}$ instead.

Finally, a generalisation of Proposition 1.3.5 that will prove useful in Chapter 3 requires a choice of a dominant integral $\mathfrak{sl}_3$-weight $\bar{\mathsf{t}}$. We define

$$\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}\otimes\bar{\mathsf{t}}),(\mathsf{r}',\mathsf{s}')} \equiv \sum_{\bar{\mathsf{t}}'} \mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}+\mathsf{t}'),(\mathsf{r}',\mathsf{s}')}, \tag{1.3.31}$$

where the sum runs over the (finitely many) weights $\bar{\mathsf{t}}'$ of $\overline{\mathcal{L}}_{\bar{\mathsf{t}}}$, with multiplicity, and t' denotes the level-0 weight of $\widehat{\mathfrak{sl}}_3$ whose projection onto the weight space of $\mathfrak{sl}_3$ is $\bar{\mathsf{t}}'$. Note that we may define this sum for any dominant integral $\mathfrak{sl}_3$-weight $\bar{\mathsf{t}}$, even if $\mathsf{s}+\mathsf{t}' \notin \mathsf{P}^{\mathsf{v}-3}_{\geqslant}$, by directly substituting the right-hand side of (1.3.24) for the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix.

Proposition 1.3.6. *Let* $\mathsf{u},\mathsf{v} \in \mathbb{Z}_{\geqslant 3}$ *be coprime,* $(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}') \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant} \times \mathsf{P}^{\mathsf{v}-3}_{\geqslant}/\mathbb{Z}_3$ *and* $\bar{\mathsf{t}}$ *be a dominant integral* $\mathfrak{sl}_3$*-weight. Then,*

$$\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}\otimes\bar{\mathsf{t}}),(\mathsf{r}',\mathsf{s}')} = \mathsf{e}^{2\pi\mathsf{i}\langle\bar{\mathsf{s}},\bar{\mathsf{r}}'+\bar{\rho}\rangle}\chi_{\bar{\mathsf{t}}}(\xi_{\bar{\mathsf{s}}'})\,\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')}. \tag{1.3.32}$$

PROOF. Substituting (1.3.24) into the definition (1.3.31) gives

$$\begin{aligned}\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}\otimes\bar{\mathsf{t}}),(\mathsf{r}',\mathsf{s}')} = {} & \frac{1}{\sqrt{3}\mathsf{uv}}\mathsf{e}^{2\pi\mathsf{i}\left(\langle\bar{\mathsf{r}}+\bar{\rho},\bar{\mathsf{s}}'+\bar{\rho}\rangle+\langle\bar{\mathsf{s}}+\bar{\rho},\bar{\mathsf{r}}'+\bar{\rho}\rangle\right)}\sum_{w\in\mathsf{S}_3}\det w\,\mathsf{e}^{-2\pi\mathsf{i}\frac{\mathsf{v}}{\mathsf{u}}\langle w(\bar{\mathsf{r}}+\bar{\rho}),\bar{\mathsf{r}}'+\bar{\rho}\rangle} \\ & \cdot\sum_{\bar{\mathsf{t}}'}\mathsf{e}^{2\pi\mathsf{i}\langle\bar{\mathsf{t}}',\bar{\mathsf{r}}'+\bar{\rho}\rangle}\sum_{w\in\mathsf{S}_3}\det w\,\mathsf{e}^{-2\pi\mathsf{i}\frac{\mathsf{u}}{\mathsf{v}}\langle w(\bar{\mathsf{s}}+\bar{\rho}),\bar{\mathsf{s}}'+\bar{\rho}\rangle}\mathsf{e}^{-2\pi\mathsf{i}\frac{\mathsf{u}}{\mathsf{v}}\langle w(\bar{\mathsf{t}}'),\bar{\mathsf{s}}'+\bar{\rho}\rangle}.\end{aligned} \tag{1.3.33}$$

Since the weights of $\overline{\mathcal{L}}_{\bar{\mathsf{t}}}$ differ by elements of the root lattice $\overline{\mathsf{Q}}$ of $\mathfrak{sl}_3$, we may replace $\bar{\mathsf{t}}'$ by $\bar{\mathsf{t}}$ in the first exponential on the second line. Moreover, the weights of $\overline{\mathcal{L}}_{\bar{\mathsf{t}}}$ are permuted by S_3 so that

$$\begin{aligned}\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}\otimes\bar{\mathsf{t}}),(\mathsf{r}',\mathsf{s}')} &= \mathsf{e}^{2\pi\mathsf{i}\langle\bar{\mathsf{t}},\bar{\mathsf{r}}'+\bar{\rho}\rangle}\sum_{\bar{\mathsf{t}}'}\mathsf{e}^{-2\pi\mathsf{i}\frac{\mathsf{u}}{\mathsf{v}}\langle\bar{\mathsf{t}}',\bar{\mathsf{s}}'+\bar{\rho}\rangle}\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')} \\ &= \mathsf{e}^{2\pi\mathsf{i}\langle\bar{\mathsf{s}},\bar{\mathsf{r}}'+\bar{\rho}\rangle}\chi_{\bar{\mathsf{t}}}(\xi_{\bar{\mathsf{s}}'})\,\mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')}.\end{aligned} \tag{1.3.34}$$

■

As for Proposition 1.3.5, the roles of r and s in this proposition can be reversed to obtain a similar result involving the character $\chi_{\bar{\mathsf{r}}}$ of $\overline{\mathcal{L}}_{\bar{\mathsf{r}}}$ instead.

Recalling that W_3 minimal models are rational and C_2-cofinite, their fusion coefficients may be computed from the Verlinde formula (1.1.45). The superficial similarity between the W_3 S-matrix

(1.3.24) and that for the rational $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$ (1.2.12) suggests that the corresponding fusion coefficients are related.

As described in Section 1.2.2, for $\ell \in \mathbb{Z}_{\geqslant 0}$, the simple affine vertex operator algebra $\mathsf{L}_\ell(\mathfrak{sl}_3)$ of level ℓ is rational and C_2-cofinite [76]. Its simple modules are the integrable highest-weight $\widehat{\mathfrak{sl}}_3$-modules $\mathcal{L}_\mathsf{t}$ whose highest weights t lie in $\mathsf{P}^\ell_\geqslant$. Recall that the fusion products of $\mathsf{L}_\ell(\mathfrak{sl}_3)$ take the form

$$\mathcal{L}_\mathsf{t} \times \mathcal{L}_{\mathsf{t}'} \simeq \bigoplus_{\mathsf{t}'' \in \mathsf{P}^\ell_\geqslant} \mathcal{N}^{\ell\ \mathsf{t}''}_{\mathsf{t},\mathsf{t}'} \mathcal{L}_{\mathsf{t}''}, \tag{1.3.35}$$

where the fusion coefficients $\mathcal{N}^{\ell\ \mathsf{t}''}_{\mathsf{t},\mathsf{t}'}$ are known. Importantly, the $\mathsf{L}_\ell(\mathfrak{sl}_3)$ fusion coefficients satisfy

$$\mathcal{N}^{\ell\ \nabla(\mathsf{t}'')}_{\nabla(\mathsf{t}),\mathsf{t}'} = \mathcal{N}^{\ell\ \nabla(\mathsf{t}'')}_{\mathsf{t},\nabla(\mathsf{t}')} = \mathcal{N}^{\ell\ \mathsf{t}''}_{\mathsf{t},\mathsf{t}'}, \tag{1.3.36}$$

see [70, Eq. (16.9)] for example. Let $\overline{\mathsf{Q}}$ denote the root lattice of $\mathfrak{sl}_3$.

Theorem 1.3.7 ([74, Thm. 4.3])**.** *Let $\mathsf{u}, \mathsf{v} \in \mathbb{Z}_{\geqslant 3}$ be coprime. Then, the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ fusion coefficients are given by*

$$\mathcal{N}^{\mathsf{W}_3\ (\mathsf{r}'',\mathsf{s}'')}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')} = \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\mathsf{r},\mathsf{r}'} \mathcal{N}^{\mathsf{v}-3\ \mathsf{s}''}_{\mathsf{s},\mathsf{s}'}, \tag{1.3.37}$$

where we choose representatives of $(\mathsf{r},\mathsf{s}), (\mathsf{r}',\mathsf{s}'), (\mathsf{r}'',\mathsf{s}'') \in \mathsf{P}^{\mathsf{u}-3}_\geqslant \times \mathsf{P}^{\mathsf{v}-3}_\geqslant / \mathbb{Z}_3$ so that:

- *If $\mathsf{u} \in 3\mathbb{Z}$, then take $\bar{\mathsf{s}}, \overline{\mathsf{s}'}, \overline{\mathsf{s}''} \in \overline{\mathsf{Q}}$.*
- *If $\mathsf{v} \in 3\mathbb{Z}$, then take $\bar{\mathsf{r}}, \overline{\mathsf{r}'}, \overline{\mathsf{r}''} \in \overline{\mathsf{Q}}$.*
- *If $\mathsf{u}, \mathsf{v} \notin 3\mathbb{Z}$, then take either $\bar{\mathsf{r}}, \overline{\mathsf{r}'}, \overline{\mathsf{r}''} \in \overline{\mathsf{Q}}$ or $\bar{\mathsf{s}}, \overline{\mathsf{s}'}, \overline{\mathsf{s}''} \in \overline{\mathsf{Q}}$ (it does not matter which).*

For example, the fusion coefficients for $\mathsf{v} = 3$ take the form $\mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\lambda],[\lambda']} = \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\mathsf{r},\mathsf{r}'}$, with $\bar{\mathsf{r}}, \overline{\mathsf{r}'}, \overline{\mathsf{r}''} \in \overline{\mathsf{Q}}$, because in this case $\mathsf{s} = \mathsf{s}' = \mathsf{s}'' = [0,0,0]$. It follows that the $\mathsf{W}_3(\mathsf{u},3)$ fusion ring coincides with the subring of the $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$ fusion ring spanned by the $\mathcal{L}_\lambda$ with $\bar{\mathsf{r}} = [\lambda_1, \lambda_2] \in \overline{\mathsf{Q}}$. That this indeed constitutes a subring follows from (1.2.14).

Chapter 2

Bershadsky–Polyakov Algebras

2.1. Bershadsky–Polyakov Algebras From the Ground Up

Having seen the classification of modules and modularity of the W_3 minimal models $\mathsf{W}_3(\mathsf{u},\mathsf{v})$, we now move on to similar considerations for another W-algebra related to $\mathfrak{sl}_3$.

There are exactly three (up to isomorphism) W-algebras one can obtain by applying quantum hamiltonian reduction to $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$: $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ itself, the Zamolodchikov algebra $\mathsf{W}_3^{\mathsf{k}}$ and the *Bershadsky–Polyakov algebra* BP^{k} [35, 139]. As the structure, representation theory and rational minimal models for both $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ and $\mathsf{W}_3^{\mathsf{k}}$ were discussed in Chapter 1, we will focus on BP^{k} for the next two chapters.

2.1.1. The Minimal $\mathfrak{sl}_3$ W-Algebra.

To construct the W-algebra of interest using quantum hamiltonian reduction, we first fix a basis for $\mathfrak{sl}_3$. Let $M_{i,j}$ be the 3×3 matrix whose entries are all zeros except for the (i,j)'th entry which is 1. Then, we set

$$e_\theta = M_{1,3}, \quad \begin{aligned} e_{\alpha_1} &= M_{1,2}, & h_{\alpha_1} &= M_{1,1}-M_{2,2}, & f_{\alpha_1} &= M_{2,1}, \\ e_{\alpha_2} &= M_{2,3}, & h_{\alpha_2} &= M_{2,2}-M_{,33}, & f_{\alpha_2} &= M_{3,2}, \end{aligned} \quad f_\theta = M_{3,1}. \tag{2.1.1}$$

Here, $\theta = \alpha_1+\alpha_2$ is the highest root of $\mathfrak{sl}_3$ and we shall also set $h_\theta = h_{\alpha_1}+h_{\alpha_2} = M_{1,1}-M_{3,3}$. For the purpose of quantum hamiltonian reduction, choose the nilpotent element f_θ. The associated $\mathfrak{sl}_2$ triple in $\mathfrak{sl}_3$ is $\{h_\theta, e_\theta, f_\theta\}$, and the grading of $\mathfrak{sl}_3$ by eigenvalue of the adjoint action of $x=\frac{1}{2}h_\theta$ is

$$\mathfrak{sl}_3 = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} (\mathfrak{sl}_3)_i, \quad \text{where} \quad \begin{aligned} (\mathfrak{sl}_3)_1 &= \mathrm{span}_{\mathbb{C}}\{e_\theta\}, \\ (\mathfrak{sl}_3)_{\frac{1}{2}} &= \mathrm{span}_{\mathbb{C}}\{e_{\alpha_1}, e_{\alpha_2}\}, \\ (\mathfrak{sl}_3)_0 &= \mathrm{span}_{\mathbb{C}}\{h_{\alpha_1}, h_{\alpha_2}\}, \\ (\mathfrak{sl}_3)_{-\frac{1}{2}} &= \mathrm{span}_{\mathbb{C}}\{f_{\alpha_1}, f_{\alpha_2}\}, \\ (\mathfrak{sl}_3)_{-1} &= \mathrm{span}_{\mathbb{C}}\{f_\theta\}. \end{aligned} \tag{2.1.2}$$

As described in Section 1.3, to form the complex we need the universal affine vertex algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$, the neutral ghost vertex algebra $\mathsf{B}(A_{\mathrm{ne}})$ and the charged ghost vertex superalgebra $\mathsf{F}(A_{\mathrm{ch}})$. Let S be the set of roots of $\mathfrak{sl}_3$ (recalling that $f_\alpha = e_{-\alpha}$). Then $S_{1/2} = \{\alpha_1, \alpha_2\}$ and $S_+ = S_{1/2} \cup \{\theta\}$. Recall that the nonregular operator product expansions of the ghost vertex algebras F and B are

$$b(z)c(w) \sim \frac{\mathbb{1}}{z-w} \quad \text{and} \quad \beta(z)\gamma(w) \sim \frac{\mathbb{1}}{z-w}, \tag{2.1.3}$$

respectively. Here, in contrast to (1.1.21), we have redefined $\gamma(z) \mapsto -\gamma(z)$ to remove various minus signs in the forthcoming formulae. As A_{ne} is two dimensional, $\mathsf{B}(A_{\mathrm{ne}}) \simeq \mathsf{B}$. In fact, letting $\{\delta^{\alpha_1}, \delta^{\alpha_2}\}$ be the basis of A_{ne} corresponding to $\{e_{\alpha_1}, e_{\alpha_2}\}$,

$$\delta^{\alpha_i}(z)\delta^{\alpha_j}(w) \sim \frac{(1-\delta_{i,j})\mathbb{1}}{z-w}. \tag{2.1.4}$$

So we may write $\delta^{\alpha_1}(z) = \beta(z)$ and $\delta^{\alpha_2}(z) = \gamma(z)$. Similarly, the charged ghost vertex superalgebra $\mathsf{F}(A_{\mathrm{ch}}) = \mathsf{F}^{\alpha_1} \otimes \mathsf{F}^{\alpha_2} \otimes \mathsf{F}^{\theta}$ where F^{α} is the subalgebra of $\mathsf{F}(A_{\mathrm{ch}})$ generated by the fields $\{\varphi^\alpha, \psi^\alpha\}$ for $\alpha \in S_+$. In fact, $\mathsf{F}^\alpha \simeq \mathsf{F}$ for all $\alpha \in S_+$ by identifying φ^α and ψ^α with b and c respectively. In light of this, denote the generating fields by $\varphi^\alpha(z)$ and $\psi^\alpha(z)$ of F^α by $b^\alpha(z)$ and $c^\alpha(z)$ respectively.

Collect these ghost fields into a vertex operator superalgebra $\mathsf{G} = \mathsf{F}^{\alpha_1} \otimes \mathsf{F}^{\alpha_2} \otimes \mathsf{F}^{\theta} \otimes \mathsf{B}$ and let $\mathsf{C} = \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \otimes \mathsf{G}$. In what follows, we will frequently omit tensor product symbols and tensor products involving vacuum fields. Using (1.3.7), define the charge -1 field $d(z) \in \mathsf{C}$ by

$$\begin{aligned} d(z) = \big(e_\theta(z) + \mathbb{1}\big)c^\theta(z) + \big(e_{\alpha_1}(z) + \beta(z)\big)c^{\alpha_1}(z) \\ + \big(e_{\alpha_2}(z) + \gamma(z)\big)c^{\alpha_2}(z) + {:}b^\theta(z)c^{\alpha_2}(z)c^{\alpha_1}(z){:}. \end{aligned} \tag{2.1.5}$$

A straightforward computation verifies that $d(z)d(w) \sim 0$. We then form a differential complex by requiring that $d(z)$ is homogeneous of conformal weight 1 and equipping $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \otimes \mathsf{G}$ with the differential d_0. From (2.1.5), this requires that the conformal weights of $c^\theta(z)$ and $e_\theta(z)$ are 1 and 0 respectively. The latter has conformal weight 1 with respect to the Sugawara energy-momentum tensor $T^{\mathrm{Sug.}}$ defined in (1.2.8), so we instead use the conformal structure on $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ furnished by $T^{\mathrm{Sug.}} + (1/2)\partial h_\theta$. An appropriate conformal structure on C is therefore

$$L(z) = T^{\mathrm{Sug.}}(z) + \frac{1}{2}\partial h_\theta(z) + T^{\mathsf{F}^{\alpha_1}} + T^{\mathsf{F}^{\alpha_2}} + T^{\mathsf{F}^{\theta}} + T^{\mathsf{B}}, \tag{2.1.6}$$

where

$$T^{\mathsf{F}^{\alpha_i}} = \frac{1}{2}{:}\partial b^{\alpha_i}c^{\alpha_i} + \partial c^{\alpha_i}b^{\alpha_i}{:}, \quad T^{\mathsf{F}^{\theta}} = {:}\partial b^\theta c^\theta{:} \quad \text{and} \quad T^{\mathsf{B}} = \frac{1}{2}{:}\partial\gamma\beta - \partial\beta\gamma{:}. \tag{2.1.7}$$

This is just the specialisation of the energy-momentum tensor defined in (1.3.10) to the choice $\mathfrak{g} = \mathfrak{sl}_3$, $x = \frac{1}{2}h_\theta$ and $f = f_\theta$.

As described in Section 1.3.1, the W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, f_\theta)$ is defined as the zeroth homology $H^0(\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \otimes \mathsf{G}, d_0) = H^0(\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3))$ of this differential chain complex, where we write $H^0(-) = H^0_{f_\theta}(-)$. Theorem 4.1 in [106] describes strong generators for $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, f_\theta)$ which we denote by

$$\{L(z), J(z), G^+(z), G^-(z)\}, \tag{2.1.8}$$

where $L(z)$ is given by (2.1.6) and

$$\begin{aligned} J &= \frac{1}{3}(h_{\alpha_1} - h_{\alpha_2}) + {:}b^{\alpha_1}c^{\alpha_1}{:} - {:}b^{\alpha_2}c^{\alpha_2}{:} - {:}\beta\gamma{:}, \\ G^+ &= f_{\alpha_2} + h_{\alpha_2}\beta - {:}b^{\alpha_1}c^{\theta}{:} - {:}b^{\alpha_1}c^{\alpha_1}\beta{:} + 2{:}b^{\alpha_2}c^{\alpha_2}\beta{:} + {:}b^{\theta}c^{\theta}\beta{:} + {:}\beta\beta\gamma{:} + (\mathsf{k}+1)\partial\beta, \\ G^- &= f_{\alpha_1} - h_{\alpha_1}\gamma + {:}b^{\alpha_2}c^{\theta}{:} - 2{:}b^{\alpha_1}c^{\alpha_1}\gamma{:}, + {:}b^{\alpha_2}c^{\alpha_2}\gamma{:} - {:}b^{\theta}c^{\theta}\gamma{:} + {:}\gamma\gamma\beta{:} - (\mathsf{k}+1)\partial\gamma. \end{aligned} \tag{2.1.9}$$

suppressing the z-dependence of fields momentarily. As it happens, $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, f_\theta)$ is isomorphic to the (universal) *Bershadsky–Polyakov algebra* BP^{k} defined in [35, 139]. The strong generators of BP^{k} are likewise denoted by $\{L(z), J(z), G^{\pm}(z)\}$. This is no accident: BP^{k} was originally defined as a 'second' quantum hamiltonian reduction associated to $\mathfrak{sl}_3$, the first being the Zamolodchikov algebra $\mathsf{W}_3 = \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, f_{\alpha_1} + f_{\alpha_2})$ [170].

If instead of f_θ we had chosen the nilpotent element to be f_{α_1} or f_{α_2} and performed quantum hamiltonian reduction, the resulting W-algebra would be isomorphic to $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, f_\theta)$. This is because the nilpotent orbit in $\mathfrak{sl}_3$ containing f_θ also contains f_{α_1} and f_{α_2} [40].

As previously described, the homological construction of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, f_\theta)$ (and therefore BP^{k}) above extends naturally to $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-modules: Let $\mathcal{M}$ be a $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-module satisfying certain finiteness conditions and consider the differential complex formed by $\mathcal{M} \otimes \mathsf{G}$ and d_0 (where all $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ modes act on $\mathcal{M}$ rather than $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$). The zeroth homology of this complex, denoted by $H^0(\mathcal{M})$, is a BP^{k}-module [106]. We will use this fact extensively when studying the representation theory of BP^{k} and its simple quotients.

2.1.2. Operator Product Expansions. Content in our knowledge that the W-algebra we seek to understand is the Bershadsky–Polyakov algebra BP^{k}, we will now define BP^{k} once and for all in terms of strong generators and operator product expansions from [35, 139] and describe some of its properties.

Definition 2.1.1. *Given* $\mathsf{k} \in \mathbb{C}$, $\mathsf{k} \neq -3$, *the* level-k universal Bershadsky–Polyakov algebra BP^{k} *is the vertex operator algebra with vacuum* $\mathbb{1}$ *that is strongly and freely generated by fields* $J(z)$,

$G^+(z)$, $G^-(z)$ *and* $L(z)$ *satisfying the following operator product expansions:*

$$\begin{aligned}
L(z)L(w) &\sim -\frac{(2\mathsf{k}+3)(3\mathsf{k}+1)\mathbb{1}}{2(\mathsf{k}+3)(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)},\\
L(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)}, \qquad L(z)G^\pm(w) \sim \frac{\frac{3}{2}G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{(z-w)},\\
J(z)J(w) &\sim \frac{(2\mathsf{k}+3)\mathbb{1}}{3(z-w)^2}, \qquad J(z)G^\pm(w) \sim \frac{\pm G^\pm(w)}{(z-w)}, \qquad G^\pm(z)G^\pm(w) \sim 0,\\
G^+(z)G^-(w) &\sim \frac{(\mathsf{k}+1)(2\mathsf{k}+3)\mathbb{1}}{(z-w)^3} + \frac{3(\mathsf{k}+1)J(w)}{(z-w)^2}\\
&\quad + \frac{3{:}JJ{:}(w) + \frac{3}{2}(\mathsf{k}+1)\partial J(w) - (\mathsf{k}+3)L(w)}{z-w}.
\end{aligned} \tag{2.1.10}$$

The operator product expansions (2.1.10) are those satisfied by the fields in (2.1.6) and (2.1.9). From (2.1.10), we see that the conformal weights of the generating fields $J(z)$, $G^+(z)$, $G^-(z)$ and $L(z)$ are 1, $\frac{3}{2}$, $\frac{3}{2}$ and 2, respectively, whilst the central charge is

$$\mathsf{c}_{\mathsf{k}}^{\mathsf{BP}} = -\frac{(2\mathsf{k}+3)(3\mathsf{k}+1)}{\mathsf{k}+3}. \tag{2.1.11}$$

The mode algebra of BP^{k} can be obtained from its operator product expansions by expanding the homogeneous fields in the usual form

$$A(z) = \sum_{n\in\mathbb{Z}-\Delta_A+\varepsilon_A} A_n z^{-n-\Delta_A}, \tag{2.1.12}$$

where Δ_A is the conformal weight of $A(z)$ and $\varepsilon_A = \frac{1}{2}$, if $\Delta_A \in \mathbb{Z}+\frac{1}{2}$ and $A(z)$ is acting on a twisted BP^{k}-module (with respect to the automorphism $\mathsf{e}^{2\pi \mathsf{i} L_0}$), and $\varepsilon_A = 0$ otherwise. Imposing the constraints from the operator product expansions on the modes of the fields yields the following result.

Proposition 2.1.2. *The commutation relations of the modes of the generating fields of* BP^{k} *are*

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} - \frac{(2\mathsf{k}+3)(3\mathsf{k}+1)}{\mathsf{k}+3}\frac{m^3-m}{12}\delta_{m+n,0}\mathbb{1},\\
[L_m, J_n] &= -nJ_{m+n}, \qquad [L_m, G^\pm_s] = \left(\frac{m}{2}-s\right)G^\pm_{m+s},\\
[J_m, J_n] &= \frac{2\mathsf{k}+3}{3}m\delta_{m+n,0}\mathbb{1}, \qquad [J_m, G^\pm_s] = \pm G^\pm_{m+s}, \qquad [G^\pm_r, G^\pm_s] = 0,\\
[G^+_r, G^-_s] &= 3{:}JJ{:}_{r+s} - (\mathsf{k}+3)L_{r+s} + \frac{3}{2}(\mathsf{k}+1)(r-s)J_{r+s} + (\mathsf{k}+1)(2\mathsf{k}+3)\frac{r^2-\frac{1}{4}}{2}\delta_{r+s,0}\mathbb{1}.
\end{aligned} \tag{2.1.13}$$

We call the unital associative algebra generated by the modes, with $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z}+\frac{1}{2}$, of the fields of BP^{k} the *untwisted mode algebra* U. Likewise we call the unital associative algebra generated by the modes, with $m, n, r, s \in \mathbb{Z}$, of the fields of BP^{k} the *twisted mode algebra* U^{tw}.

The 'untwisted' and 'twisted' modifiers here reflect which BP^{k}-modules the modes that form the algebra act on.

Definition 2.1.3.

- *A* fractional level $\mathsf{k} \in \mathbb{C}$ *for the Bershadsky–Polyakov algebras is one that is not critical, meaning that* $\mathsf{k} \neq -3$, *and for which* BP^{k} *is not simple.*
- *The* level-k simple Bershadsky–Polyakov vertex operator algebra BP_{k} *is the unique simple quotient of* BP^{k}.

According to [93, Thms. 0.2.1 and 9.1.2], the fractional levels are precisely the k satisfying

$$\mathsf{k}+3 = \frac{\mathsf{u}}{\mathsf{v}}, \quad \text{where } \mathsf{u} \in \mathbb{Z}_{\geqslant 2},\ \mathsf{v} \in \mathbb{Z}_{\geqslant 1} \text{ and } \gcd\{\mathsf{u},\mathsf{v}\} = 1. \tag{2.1.14}$$

If k is fractional, then we shall refer to BP_{k} as a *Bershadsky–Polyakov minimal model* and use the special notation $\mathsf{BP}(\mathsf{u},\mathsf{v})$. The Bershadsky–Polyakov minimal models $\mathsf{BP}(\mathsf{u},2)$, where $\mathsf{u} \geqslant 3$ is odd, are rational [18] with only highest-weight modules and their direct sums. In addition to this chapter, the $\mathsf{v} > 2$ cases will be explored in Chapter 3 where we will describe the modular transformations and Grothendieck fusion rules of such Bershadsky–Polyakov minimal models.

We note that the central charge of the minimal model $\mathsf{BP}(\mathsf{u},\mathsf{v})$ takes the form

$$\mathsf{c}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}} = -\frac{(3\mathsf{u}-8\mathsf{v})(2\mathsf{u}-3\mathsf{v})}{\mathsf{u}\mathsf{v}} = 1 - \frac{6(\mathsf{u}-2\mathsf{v})^2}{\mathsf{u}\mathsf{v}}. \tag{2.1.15}$$

Whilst the central charge is invariant under exchanging $\frac{\mathsf{u}}{\mathsf{v}}$ with $\frac{4\mathsf{v}}{\mathsf{u}}$, the corresponding Bershadsky–Polyakov minimal models are not isomorphic.

2.1.3. Automorphisms. There are two types of automorphisms of BP^{k} that will prove useful for studying the representation theory of BP^{k}. These are the *conjugation* (vertex operator algebra) automorphism γ and the *spectral flow* (vertex algebra) automorphisms σ^ℓ, $\ell \in \mathbb{Z}$.

Proposition 2.1.4. *There exist conjugation and spectral flow automorphisms γ and σ^ℓ, $\ell \in \mathbb{Z}$, of* BP^{k}. *They are uniquely determined by the following actions on the generating fields:*

$$\begin{gathered}
\gamma(J(z)) = -J(z), \qquad \gamma(G^+(z)) = +G^-(z), \\
\gamma(G^-(z)) = -G^+(z), \qquad \gamma(L(z)) = L(z), \\
\sigma^\ell(J(z)) = J(z) - \frac{2\mathsf{k}+3}{3}\ell z^{-1}\mathbb{1}, \qquad \sigma^\ell(G^+(z)) = z^{-\ell}G^+(z), \\
\sigma^\ell(L(z)) = L(z) - \ell z^{-1}J(z) + \frac{2\mathsf{k}+3}{6}\ell^2 z^{-2}\mathbb{1}, \qquad \sigma^\ell(G^-(z)) = z^{+\ell}G^-(z).
\end{gathered} \tag{2.1.16}$$

That these actions define BP^{k} automorphisms amounts to checking that the operator product expansions (2.1.10) are preserved. It is also straightforward to check that the inverse of σ^{ℓ} is $\sigma^{-\ell}$.

The spectral flows σ^{ℓ} with $\ell \neq 0$ are not vertex operator algebra automorphisms because they do not preserve the conformal structure furnished by $L(z)$. This might sound alarming but is not an issue as our main use of these automorphism is constructing new BP^{k}-modules out of old ones by twisting as described in Section 1.1.3.

Note that conjugation has order 4, whilst spectral flow has infinite order. Together, they satisfy the dihedral group relation

$$\gamma\sigma^{\ell} = \sigma^{-\ell}\gamma, \tag{2.1.17}$$

though we do not have $\gamma^2 = \mathbb{1}$. Expanding the defining actions of the automorphisms from Proposition 2.1.4 in terms of modes gives a characterisation of σ and γ as automorphisms of the mode algebras of BP^{k}.

Proposition 2.1.5. *Conjugation and spectral flow act on the modes of the generating fields $J(z)$, $G^+(z)$, $G^-(z)$ and $L(z)$ of BP^{k} as follows:*

$$\begin{gathered}
\gamma(J_n) = -J_n, \quad \gamma(G^{\pm}_r) = \pm G^{\mp}_r, \quad \gamma(L_n) = L_n, \\
\sigma^{\ell}(J_n) = J_n - \frac{2\mathsf{k}+3}{3}\ell\delta_{n,0}\mathbb{1}, \quad \sigma^{\ell}(G^{\pm}_r) = G^{\pm}_{r\mp\ell}, \\
\sigma^{\ell}(L_n) = L_n - \ell J_n + \frac{2\mathsf{k}+3}{6}\ell^2\delta_{n,0}\mathbb{1}.
\end{gathered} \tag{2.1.18}$$

One particularly noteworthy feature of spectral flow is its relationship with the mode algebra U, U^{tw}. As σ^{ℓ}, $\ell \in \mathbb{Z}$, is a BP^{k} automorphism, the actions in (2.1.18) also define automorphisms of U and U^{tw} depending on the range of r. However if we begin with $r \in \mathbb{Z} + \frac{1}{2}$ and apply σ^{ℓ} with $\ell \in \mathbb{Z} + \frac{1}{2}$, the mode index of $G^{\pm}$ takes values in $\mathbb{Z}$ only. That is, the action of σ^{ℓ} with $\ell \in \mathbb{Z} + \frac{1}{2}$ exchanges the twisted and untwisted mode algebras.

Each BP^{k}-automorphism ω lifts to an autoequivalence of any category of BP^{k}-modules that is closed under twisting its objects by ω. The examples we have in mind are the category $\mathscr{W}_{\mathsf{k}}$ of weight modules, with finite-dimensional weight spaces (see Definition 2.2.1 below), and the analogous category $\mathscr{W}^{\mathrm{tw}}_{\mathsf{k}}$ of twisted modules.

The aforementioned extension of σ^{ℓ} allowing $\ell \in \mathbb{Z} + \frac{1}{2}$ defines equivalences between $\mathscr{W}_{\mathsf{k}}$ and $\mathscr{W}^{\mathrm{tw}}_{\mathsf{k}}$. We remark that one of the important consistency requirements for building a conformal field theory from a module category over a vertex operator algebra is that it is closed under twisting by automorphisms, especially conjugation.

2.2. Representation Theory of Bershadsky–Polyakov Algebras

Now we move on to the representation theory of BP^{k}. We are interested in a class of modules known as *relaxed highest-weight* modules for BP^{k}. Relaxed highest-weight modules have been shown to be essential to achieve consistent modular properties for many nonrational vertex operator algebras, for example admissible-level $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ [2, 7, 26, 50, 52, 84, 113, 141, 145]. The modular properties of BP_{k} at admissible levels, i.e. fractional levels with $\mathsf{u} \geqslant 3$, that are nonintegral will be explored in Chapter 3.

The first step in understanding the relaxed highest-weight BP^{k}-modules with finite-dimensional weight spaces is to classify the simple ones in both the untwisted and twisted sectors. The restriction on the dimension of the weight space is in order to have well defined characters. That reducible relaxed highest-weight modules can also be constructed (as we will see) is a strong indication that the simple Bershadsky–Polyakov vertex operator algebra BP_{k} is generically nonrational. When k is nonintegral admissible and BP^{k} is not simple, a related problem to be addressed is which simple relaxed highest-weight BP^{k}-modules are also BP_{k}-modules.

2.2.1. Weight Modules. In Section 2.1, we introduced the untwisted (twisted) mode algebra U (U^{tw}) of the universal Bershadsky–Polyakov vertex operator algebra BP^{k}. Any BP^{k}-module is obviously a U-module and likewise for the twisted versions. The converse is not true however. As these algebras are graded by conformal weight, we have the following generalised triangular decompositions, as in [106]:

$$\mathsf{U} = \mathsf{U}_{<} \otimes \mathsf{U}_0 \otimes \mathsf{U}_{>} \qquad \text{and} \qquad \mathsf{U}^{\mathrm{tw}} = \mathsf{U}^{\mathrm{tw}}_{<} \otimes \mathsf{U}^{\mathrm{tw}}_0 \otimes \mathsf{U}^{\mathrm{tw}}_{>}. \tag{2.2.1}$$

Here, $\mathsf{U}_{<}$, U_0 and $\mathsf{U}_{>}$ are the unital subalgebras generated by the modes A_n, for all homogeneous $A(z) \in \mathsf{BP}^{\mathsf{k}}$, with $n < 0$, $n = 0$ and $n > 0$, respectively (and similarly for their twisted versions). The following definitions are specialisations to BP^{k} of definitions proposed for general vertex operator algebras in [145].

Definition 2.2.1.

- *A vector v in a twisted or untwisted BP^{k}-module $\mathcal{M}$ is a* weight vector *of* weight (j, Δ) *if it is a simultaneous eigenvector of J_0 and L_0 with eigenvalues j and Δ called the* charge *and* conformal weight *of v, respectively. The nonzero simultaneous eigenspaces of J_0 and L_0 are called the* weight spaces *of $\mathcal{M}$. If $\mathcal{M}$ has a basis of weight vectors and each weight space is finite-dimensional, then $\mathcal{M}$ is a* weight module.

- *A vector in an untwisted* BP^{k}*-module is a* highest-weight vector *if it is a simultaneous eigenvector of* J_0 *and* L_0 *that is annihilated by the action of* $\mathsf{U}_>$*. An untwisted* BP^{k}*-module generated by a single highest-weight vector is called an* untwisted highest-weight module.
- *A vector in a twisted* BP^{k}*-module is a* highest-weight vector *if it is a simultaneous eigenvector of* J_0 *and* L_0 *that is annihilated by* G_0^+ *and the action of* $\mathsf{U}_>^{\mathrm{tw}}$*. A twisted* BP^{k}*-module generated by a single highest-weight vector is called a* twisted highest-weight module.
- *A vector in a twisted or untwisted* BP^{k}*-module is a* relaxed highest-weight vector *if it is a simultaneous eigenvector of* J_0 *and* L_0 *that is annihilated by the action of* $\mathsf{U}_>^{\mathrm{tw}}$ *or* $\mathsf{U}_>$*, respectively. A* BP^{k}*-module generated by a single relaxed highest-weight vector is called a* relaxed highest-weight module.

As $\mathsf{BP}(\mathsf{u},\mathsf{v})$ is a quotient of BP^{k} with $\mathsf{k}+3 = \frac{\mathsf{u}}{\mathsf{v}}$, these definitions also descend to $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules. A simple consequence of these definitions is that an untwisted relaxed highest-weight vector of BP^{k} is automatically a highest-weight vector. The same is not true for twisted relaxed highest-weight vector in general. The modules we are aiming to classify are therefore simple untwisted highest-weight modules and simple twisted relaxed highest-weight modules.

From the actions of the conjugation and spectral flow automorphisms in (2.1.18), we deduce the following useful facts.

Proposition 2.2.2.

- *If* $\mathcal{M}$ *is a twisted or untwisted* BP^{k}*-module and* $v \in \mathcal{M}$ *is a weight vector of weight* (j, Δ)*, then* $\gamma(v)$ *and* $\sigma^\ell(v)$ *are weight vectors in* $\gamma(\mathcal{M})$ *and* $\sigma^\ell(\mathcal{M})$ *of weights* $(-j, \Delta)$ *and* $(j+\frac{2\mathsf{k}+3}{3}\ell, \Delta+j\ell+\frac{2\mathsf{k}+3}{6}\ell^2)$*, respectively.*
- *Let* $\mathcal{M}$ *be an untwisted* BP^{k}*-module. Then,* $v \in \mathcal{M}$ *is a highest-weight vector of weight* (j, Δ) *if and only if* $\sigma^{1/2}(v)$ *is a highest-weight vector in the twisted module* $\sigma^{1/2}(\mathcal{M})$ *of weight* $(j + \frac{2\mathsf{k}+3}{6}, \Delta + \frac{1}{2}j + \frac{2\mathsf{k}+3}{24})$.
- $\mathcal{M}$ *is a simple untwisted highest-weight* BP^{k}*-module if and only if* $\sigma^{1/2}(\mathcal{M})$ *is a simple twisted highest-weight* BP^{k}*-module.*

In particular, to classify all simple highest-weight BP^{k}-modules, it is enough to only classify the untwisted ones. However many simple weight BP^{k}-modules that we will encounter in Chapter 3 are not highest-weight, nor even relaxed highest-weight. In particular, if $\mathcal{M}$ is a simple relaxed highest-weight BP^{k}-module, then $\sigma^\ell(\mathcal{M})$ is simple and weight, but is usually only relaxed highest-weight for a few choices of ℓ. We believe, however, that the simple objects of the categories $\mathscr{W}_{\mathsf{k}}$ and $\mathscr{W}_{\mathsf{k}}^{\mathrm{tw}}$ of untwisted and twisted weight BP^{k}-modules are all spectral flows of simple relaxed highest-weight BP^{k}-modules, which we will now classify.

2.2.2. Untwisted Zhu Algebra. One of the main tools that we will use in our classification of simple relaxed highest-weight BP^{k}-modules are the untwisted and twisted Zhu algebras from Section 1.1.4. This is because a classification of simple modules for the untwisted (twisted) Zhu algebra of BP^{k} gives a classification of simple untwisted (twisted) positive-energy BP^{k}-modules by Theorems 1.1.9 and 1.1.12. Additionally, simple positive-energy weight BP^{k}-modules coincide precisely with the simple relaxed highest-weight modules.

We begin with determining the untwisted Zhu algebra $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$. In many results in this section, it is possible to replace BP^{k} with $\mathsf{BP}(\mathsf{u},\mathsf{v})$ as $\mathsf{Zhu}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$ is a quotient of $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$ at the appropriate level by (1.1.39) and likewise for the twisted versions.

Proposition 2.2.3. $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$ *is a quotient of* $\mathbb{C}[J,L]$.

Proof. Since the fields $G^{\pm}(z)$ have half-integer conformal weights, they do not have zero modes when acting on untwisted modules. More generally, only the (homogeneous) fields of integer conformal weight have zero modes. Express the zero mode of such a field as a linear combination of monomials in the modes of the generating fields $J(z)$, $G^{\pm}(z)$ and $L(z)$. Next, use the commutation relations to order the modes so that the mode index weakly increases from left to right — it is easy to see that this is always possible despite the nonlinear nature of the commutation relations (2.1.13). Now remove any monomial which contains a positive mode. The image of the zero mode in $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$ is thus a polynomial in $\left[J_0\right]$ and $\left[L_0\right]$.

Since $\left[L_0\right]$ is central in $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$, the multiplication (1.1.38) of $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$ matches that of $\mathbb{C}[J,L]$. There is therefore a surjective homomorphism $\mathbb{C}[J,L] \to \mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$ determined by $J \mapsto \left[J_0\right]$ and $L \mapsto \left[L_0\right]$. ∎

It can be shown that $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right] \simeq \mathbb{C}[J,L]$ using the fact that the image of the field $(J^n L^m)(z)$ in $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$ is

$$\left[J_0^n \prod_{i=0}^{m} (L_0 + 2i)\right]. \tag{2.2.2}$$

It is however sufficient for our purposes to know that $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$ (and therefore $\mathsf{Zhu}[\mathsf{BP}_{\mathsf{k}}]$ for all k) is a quotient of $\mathbb{C}[J,L]$.

2.2.3. Simple Untwisted BP^{k}-Modules. Having identified $\mathsf{Zhu}\left[\mathsf{BP}^{\mathsf{k}}\right]$ (and therefore its quotient $\mathsf{Zhu}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$) as a quotient of the free abelian algebra $\mathbb{C}[J,L]$, we may identify its finite-dimensional simple modules as $\mathbb{C}[J,L]$-modules.

The Zhu induction of an arbitrary $\mathsf{Zhu}[\mathsf{BP}^{\mathsf{k}}]$-module is not guaranteed to be a weight BP^{k}-module. For the purposes of classifying simple relaxed highest-weight BP^{k}-modules, it is therefore necessary to restrict attention to a subclass of $\mathsf{Zhu}[\mathsf{BP}^{\mathsf{k}}]$-modules.

Definition 2.2.4. *A $\mathbb{C}[J,L]$-module is said to be* weight *if J and L act semisimply and their simultaneous eigenspaces are all finite-dimensional.*

The simple weight modules of $\mathbb{C}[J,L]$ are therefore one-dimensional. We shall denote them by $\mathbb{C}v_{j,\Delta}$, where j and Δ are the eigenvalues of J and L, respectively, on $v_{j,\Delta}$. As every simple $\mathsf{Zhu}[\mathsf{BP}^{\mathsf{k}}]$-module must also be simple as a $\mathbb{C}[J,L]$-module, we arrive at our first identification result.

Proposition 2.2.5. *Every simple weight $\mathsf{Zhu}[\mathsf{BP}^{\mathsf{k}}]$-module, and therefore every simple weight $\mathsf{Zhu}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-module, is isomorphic to some $\mathbb{C}v_{j,\Delta}$, where $j,\Delta \in \mathbb{C}$.*

Proposition 1.1.8 and Theorem 1.1.9 then guarantee that if $\mathbb{C}v_{j,\Delta}$ is a $\mathsf{Zhu}[\mathsf{BP}^{\mathsf{k}}]$-module, then there exists a simple untwisted BP^{k}-module $\mathcal{H}_{j,\Delta}$ which is uniquely determined (up to isomorphism) by the fact that its top space is isomorphic to $\mathbb{C}v_{j,\Delta}$ (as a $\mathbb{C}[J,L]$-module). As this top space is one-dimensional, $\mathcal{H}_{j,\Delta}$ is a highest-weight module.

Theorem 2.2.6. *Every simple untwisted relaxed highest-weight BP^{k}-module, and therefore every simple untwisted relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module, is isomorphic to some $\mathcal{H}_{j,\Delta}$, where $j,\Delta \in \mathbb{C}$.*

The fact that $\mathsf{Zhu}[\mathsf{BP}^{\mathsf{k}}] \simeq \mathbb{C}[J,L]$ means that all $\mathbb{C}v_{j,\Delta}$ give rise to simple untwisted relaxed highest-weight BP^{k}-modules $\mathcal{H}_{j,\Delta}$ by Zhu induction. Of course not all of these BP^{k}-modules will be $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules.

Note that there will be other simple weight BP^{k}- and $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules such as those obtained from the $\mathcal{H}_{j,\Delta}$ by applying spectral flow. Simple nonweight modules also exist in general [5], but they will not concern us here.

2.2.4. Twisted Zhu Algebra. We now move on to the twisted Zhu algebra of BP^{k}. In contrast to the untwisted case detailed in Section 2.2.2, the fields $G^{\pm}(z)$ do have zero modes when acting on twisted modules. We therefore expect that the representation theory of $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}^{\mathsf{k}}]$ will be more complicated than that of $\mathsf{Zhu}[\mathsf{BP}^{\mathsf{k}}]$.

Definition 2.2.7. *Let Z_{k} denote the (complex) unital associative algebra generated by J, G^+, G^- and L, subject to L being central and*

$$[J, G^{\pm}] = \pm G^{\pm}, \qquad [G^+, G^-] = f_{\mathsf{k}}(J, L), \tag{2.2.3}$$

where $f_{\mathsf{k}}(J, L) = 3J^2 - (\mathsf{k}+3)L - \frac{1}{8}(\mathsf{k}+1)(2\mathsf{k}+3)\mathbb{1}$.

Proposition 2.2.8. $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}^{\mathsf{k}}\big]$ *is a quotient of* Z_{k}.

Proof. Every homogeneous field of BP^{k} has a zero mode when acting on a twisted module. As in the proof of Proposition 2.2.3, it follows that the zero modes of the generating fields have images that generate $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}^{\mathsf{k}}\big]$. The fact that the generator $\big[L_0\big]$ is central is standard [58, 108], but is also easy to verify directly in this case.

We therefore start by using (1.1.38) to compute the products of the images of J_0 and $G_0^{\pm}$ in $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}^{\mathsf{k}}\big]$:

$$\begin{aligned} \big[J_0\big]\big[G_0^{\pm}\big] &= \sum_{n=0}^{\infty} \binom{1}{n} \big[(J_{n-1}G^{\pm})_0\big] \\ &= \big[(J_0G^{\pm})_0\big] + \big[(J_{-1}G^{\pm})_0\big] \\ &= \pm\big[G_0^{\pm}\big] + \big[{:}JG^{\pm}{:}_0\big], \end{aligned} \tag{2.2.4}$$

$$\begin{aligned} \big[G_0^{\pm}\big]\big[J_0\big] &= \sum_{n=0}^{\infty} \binom{3/2}{n} \big[(G^{\pm}_{n-3/2}J)_0\big] \\ &= \big[(G^{\pm}_{-3/2}J)_0\big] + \frac{3}{2}\big[(G^{\pm}_{-1/2}J)_0\big] \\ &= \big[(J_{-1}G^{\pm})_0\big] \pm \big[(\partial G^{\pm})_0\big] \pm \big[G_0^{\pm}\big] \\ &= \big[{:}JG^{\pm}{:}_0\big]. \end{aligned} \tag{2.2.5}$$

Here, we have noted that $G^{\pm}_{-3/2}J = G^{\pm}_{-3/2}J_{-1}\mathbb{1} = J_{-1}G^{\pm}_{-3/2}\mathbb{1} \mp G^{\pm}_{-5/2}\mathbb{1} = {:}JG^{\pm}{:} \mp \partial G^{\pm}$, that $G^{\pm}_{-1/2}J = \mp G^{\pm}$ (similarly) and that $(\partial G^{\pm})_0 = -\frac{3}{2}G_0^{\pm}$. With the surjection induced by $A \mapsto \big[A_0\big]$, $A = J, G^{\pm}, L$, this proves the first relation in (2.2.3). The same method works for the second relation. ■

It turns out that Z_{k} is in fact isomorphic to $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}^{\mathsf{k}}\big]$, though again we do not need this for what follows. One can establish this isomorphism by combining the fact that $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}^{\mathsf{k}}\big]$ is known [55] to be isomorphic to the finite W-algebra associated to $\mathfrak{sl}_3$ and the minimal nilpotent orbit. An explicit presentation of this finite W-algebra is given in [153].

The associative algebra Z_{k} is a central extension of a Smith algebra. This is well known, see [5, 15] for instance. Smith algebras were introduced and studied in [149] as examples of algebras generalising the universal enveloping algebra of $\mathfrak{sl}_2$. The representation theory of Z_k is therefore quite tractable. In particular, the classification of weight modules of Smith algebras is superficially very similar to that of $\mathfrak{sl}_2$.

2.2.5. Simple Twisted BP^k-Modules. As in the untwisted case, we wish to identify simple $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}^\mathsf{k}]$-modules as Z_k-modules. For this, we need a classification of the simple Z_k-modules. The similarity between Z_k and $\mathfrak{sl}_2$ allows us to follow the approach in [130] for $\mathfrak{sl}_2$ with appropriate modifications.

To begin, a triangular decomposition for Z_k is given by

$$\mathsf{Z}_\mathsf{k} = \mathbb{C}[G^-] \otimes \mathbb{C}[J, L] \otimes \mathbb{C}[G^+]. \tag{2.2.6}$$

The existence of this decomposition is an easy (central) extension of [149, Cor. 1.3], which guarantees a Poincaré–Birkhoff–Witt-style basis for Z_k. The 'Cartan subalgebra' of Z_k is then spanned by J and L.

Definition 2.2.9.

- *A vector in a Z_k-module is a* weight vector *of* weight (j, Δ) *if it is a simultaneous eigenvector of J and L with eigenvalues j and Δ, respectively. The nonzero simultaneous eigenspaces of J and L are called the* weight spaces. *If the Z_k-module has a basis of weight vectors and its weight spaces are all finite-dimensional, then it is a* weight module.
- *A vector in a Z_k-module is a* highest-weight vector *(*lowest-weight vector*) if it is a weight vector that is annihilated by G^+ (by G^-). A* highest-weight module *(*lowest-weight module*) is a Z_k-module that is generated by a single highest-weight vector (by a single lowest-weight vector).*
- *A weight Z_k-module is* dense *if its weights coincide with the set $[j] \times \{\Delta\}$, for some coset $[j] \in \mathbb{C}/\mathbb{Z}$ and some $\Delta \in \mathbb{C}$.*

These definitions are designed to be compatible with the definitions of weights and highest weights for BP^k-modules. As $\mathsf{Zhu}[\mathsf{BP}^\mathsf{k}]$ is abelian, the only weight $\mathsf{Zhu}[\mathsf{BP}^\mathsf{k}]$-modules whose weights coincide with $[j] \times \{\Delta\}$ for some coset $[j] \in \mathbb{C}/\mathbb{Z}$ and some $\Delta \in \mathbb{C}$ are infinite direct sums of simple modules of the form

$$\bigoplus_{n \in \mathbb{Z}} \left(\mathbb{C} v_{j+n,\Delta}\right)^{\oplus i_n} \tag{2.2.7}$$

for some $i_n \in \mathbb{Z}_{\geqslant 1}$. So simple dense modules are only possible in the twisted sector. This is only one of the many interesting features exhibited by the twisted sector of BP^k that has no analogue in the untwisted sector.

We note that Z_{k} possesses a "conjugation" automorphism $\overline{\gamma}$ (induced by the conjugation of BP^{k}) defined by

$$\overline{\gamma}(J) = -J, \quad \overline{\gamma}(G^+) = +G^-, \quad \overline{\gamma}(G^-) = -G^+, \quad \overline{\gamma}(L) = L. \tag{2.2.8}$$

Conjugating a highest-weight Z_{k}-module of highest weight (j, Δ) then results in a lowest-weight module of lowest weight $(-j, \Delta)$ and vice versa. The structures of highest- and lowest-weight Z_{k}-modules are therefore equivalent. We will focus primarily on highest-weight Z_{k}-modules and derive the analogous results for lowest-weight Z_{k}-modules using conjugation.

To construct highest-weight Z_{k}-modules, we realise them as quotients of Verma Z_{k}-modules. Let $\mathsf{Z}_{\mathsf{k}}^{\geqslant}$ denote the (unital) subalgebra of Z_{k} generated by J, L and G^+. Let $\mathbb{C}_{j,\Delta}$, with $j, \Delta \in \mathbb{C}$, be the one-dimensional $\mathsf{Z}_{\mathsf{k}}^{\geqslant}$-module, spanned by v, on which we have $Jv = jv$, $Lv = \Delta v$ and $G^+ v = 0$. The Verma Z_{k}-module $\overline{\mathcal{V}}_{j,\Delta}$ is then the induced module $\mathsf{Z}_{\mathsf{k}} \otimes_{\mathsf{Z}_{\mathsf{k}}^{\geqslant}} \mathbb{C}_{j,\Delta}$.

It is easy to check that $\overline{\mathcal{V}}_{j,\Delta}$ is a highest-weight module with highest-weight vector $v = \mathbb{1} \otimes v$ and one-dimensional weight spaces of weights $(j - n, \Delta)$, $n \in \mathbb{Z}_{\geqslant 0}$. Let $\overline{\mathcal{H}}_{j,\Delta}$ denote the unique simple quotient of $\overline{\mathcal{V}}_{j,\Delta}$.

For convenience, we define

$$\begin{aligned} h_{\mathsf{k}}^n(J, L) &= \sum_{m=0}^{n-1} f_{\mathsf{k}}(J - m\mathbb{1}, L) \\ &= n\left(n^2\mathbb{1} - \frac{3}{2}n(2J + \mathbb{1}) + \frac{1}{2}(6J^2 + 6J + \mathbb{1}) - (\mathsf{k} + 3)L - \frac{1}{8}(\mathsf{k} + 1)(2\mathsf{k} + 3)\mathbb{1}\right), \end{aligned} \tag{2.2.9}$$

where the f_{k} were defined in (2.2.3).

Proposition 2.2.10.

- *The Verma module $\overline{\mathcal{V}}_{j,\Delta}$ is simple, so $\overline{\mathcal{H}}_{j,\Delta} = \overline{\mathcal{V}}_{j,\Delta}$, unless $h_{\mathsf{k}}^n(j, \Delta) = 0$ for some $n \in \mathbb{Z}_{\geqslant 1}$.*
- *Verma Z_{k}-modules may have at most three composition factors. Exactly one of these is infinite-dimensional.*
- *If $h_{\mathsf{k}}^n(j, \Delta) = 0$ for some $n \in \mathbb{Z}_{\geqslant 1}$ and N is the minimal such n, then $\overline{\mathcal{H}}_{j,\Delta} \simeq \overline{\mathcal{V}}_{j,\Delta} / \overline{\mathcal{V}}_{j-N,\Delta}$ and $\dim \overline{\mathcal{H}}_{j,\Delta} = N$.*

Proof. The first statement follows easily by noting that every proper nonzero submodule of $\overline{\mathcal{V}}_{j,\Delta}$ is generated by a singular vector of the form $(G^-)^n v$, $n \in \mathbb{Z}_{\geqslant 1}$. The condition to be a singular vector is

$$0 = G^+(G^-)^n v = \sum_{m=0}^{n-1} (G^-)^{n-1-m} [G^+, G^-] (G^-)^m v = \sum_{m=0}^{n-1} (G^-)^{n-1-m} f_{\mathsf{k}}(J, L)(G^-)^m v \tag{2.2.10}$$

$$= \sum_{m=0}^{n-1} (G^-)^{n-1} f_k(J - m\mathbb{1}, L)v = (G^-)^{n-1} \sum_{m=0}^{n-1} f_k(j - m\mathbb{1}, \Delta)v = h_k^n(j, \Delta)(G^-)^{n-1}v.$$

Since h_k^n is a cubic polynomial in n, there can be at most three roots in $\mathbb{Z}_{\geqslant 1}$, hence at most three highest-weight vectors. The remaining statements are now clear. ∎

Unlike $\mathfrak{sl}_2$, there exist nonsemisimple finite-dimensional Z_k-modules. Examples include highest-weight modules obtained by quotienting a Verma module with three composition factors by its unique simple submodule.

This proposition completes the classification of finite-dimensional Z_k-modules and highest-weight Z_k-modules. To obtain the analogous classification of lowest-weight Z_k-modules, we apply the conjugation automorphism $\overline{\gamma}$. The conjugate of a simple Verma module $\overline{\mathcal{V}}_{j,\Delta}$ is the lowest-weight Verma module of lowest weight $(-j, \Delta)$.

However, if the $\overline{\mathcal{V}}_{j,\Delta}$ is not simple and N is the smallest positive integer such that $h_k^N(j, \Delta) = 0$, then the conjugate of $\overline{\mathcal{H}}_{j,\Delta}$ is isomorphic to $\overline{\mathcal{H}}_{N-j-1,\Delta}$. This is in contrast to $\mathfrak{sl}_2$ where simple finite-dimensional modules are self-conjugate.

It remains to construct simple weight Z_k-modules that are neither highest- nor lowest-weight. Such modules are necessarily dense. As for $\mathfrak{sl}_2$, the classification of simple dense Z_k-modules is greatly simplified by identifying the centraliser C_k of the Cartan subalgebra $\mathbb{C}[J, L]$ in Z_k.

Lemma 2.2.11. *The centraliser C_k is the polynomial algebra $\mathbb{C}[J, L, G^+G^-]$.*

Proof. Note first that G^+G^- obviously commutes with J, by (2.2.3). Consider a Poincaré–Birkhoff–Witt basis of Z_k given by elements of the form $J^aL^b(G^+)^c(G^-)^d$, for $a, b, c, d \in \mathbb{Z}_{\geqslant 0}$. It is easy to check that such a basis element belongs to C_k if and only if $c = d$. To show that J, L and G^+G^- generate C_k, it therefore suffices to show that $(G^+)^c(G^-)^c$ may be written as a polynomial in J, L and G^+G^-, for each $c \in \mathbb{Z}_{\geqslant 0}$.

Proceeding by induction, this is clear for $c = 0$. So take $c \geqslant 1$ and assume that $(G^+)^{c-1}(G^-)^{c-1}$ is a polynomial in J, L and G^+G^-. Then, the commutation rules (2.2.3) give

$$\begin{aligned}(G^+)^c(G^-)^c &= (G^+G^-)(G^+)^{c-1}(G^-)^{c-1} + G^+[(G^+)^{c-1}, G^-](G^-)^{c-1} \\ &= (G^+G^-)(G^+)^{c-1}(G^-)^{c-1} + \sum_{n=1}^{c-1} (G^+)^n f_k(J, L)(G^+)^{c-1-n}(G^-)^{c-1}.\end{aligned} \tag{2.2.11}$$

The first term on the right-hand side is a polynomial in J, L and G^+G^-, by the inductive hypothesis. For the remaining terms, note that as L is central and $G^+J = (J - \mathbb{1})G^+$, we have $(G^+)^nJ = (J -$

$n\mathbb{1})(G^+)^n$ and hence

$$\sum_{n=1}^{c-1}(G^+)^n f_{\mathsf{k}}(J,L)(G^+)^{c-1-n}(G^-)^{c-1} = \sum_{n=1}^{c-1} f_{\mathsf{k}}(J-n\mathbb{1},L)(G^+)^{c-1}(G^-)^{c-1}, \tag{2.2.12}$$

which is likewise a polynomial in J, L and G^+G^-. ■

Just as for the centraliser of $\mathfrak{sl}_2$ (see [130, Lem. 3.4.2]), the weight spaces of a simple weight Z_{k}-module are simple C_{k}-modules. It is easy to see that C_{k} is abelian, so we have the following result.

Proposition 2.2.12. *The weight spaces of simple weight Z_{k}-modules are one-dimensional.*

Specifying these weight spaces therefore requires knowledge of the eigenvalues of J, L and G^+G^- on a given simple weight Z_{k}-module. The latter will vary with the weight (j,Δ) in general, so it is convenient to note that we may replace G^+G^- by a central element of Z_{k} whose eigenvalue is therefore constant. Such 'Casimir elements' for Smith algebras are known [149, Prop. 1.5].

Lemma 2.2.13. *The element*

$$\Omega = G^+G^- + G^-G^+ + 2J^3 + J - 2J\Big((\mathsf{k}+3)L + \frac{1}{8}(\mathsf{k}+1)(2\mathsf{k}+3)\mathbb{1}\Big) \tag{2.2.13}$$

is central in Z_{k} and we have $\overline{\gamma}(\Omega) = -\Omega$ and $\mathsf{C}_{\mathsf{k}} = \mathbb{C}[J,L,\Omega]$.

Proof. We start by noting that

$$[G^+G^-,G^+] = -G^+ f_{\mathsf{k}}(J,L) = -G^+\Big(3J^2 - (\mathsf{k}+3)L - \tfrac{1}{8}(\mathsf{k}+1)(2\mathsf{k}+3)\mathbb{1}\Big). \tag{2.2.14}$$

Since $[J^n,G^+] = G^+((J+\mathbb{1})^n - J^n)$, we can cancel the terms appearing on the right-hand side (starting with $3J^2$) by adding counterterms to G^+G^-. In this way, we arrive at an element $\widetilde{\Omega} \in \mathsf{Z}_{\mathsf{k}}$ that commutes with J, G^+ and L:

$$\widetilde{\Omega} = G^+G^- + J^3 - \frac{3}{2}J^2 + \frac{1}{2}J - J\Big((\mathsf{k}+3)L + \frac{1}{8}(\mathsf{k}+1)(2\mathsf{k}+3)\mathbb{1}\Big). \tag{2.2.15}$$

By using $G^+G^- = G^-G^+ + f_{\mathsf{k}}(J,L)$, we obtain a second expression for $\widetilde{\Omega}$. Adding the two expressions, we see that

$$\Omega = 2\widetilde{\Omega} + (\mathsf{k}+3)L + \frac{1}{8}(\mathsf{k}+1)(2\mathsf{k}+3)\mathbb{1} \tag{2.2.16}$$

also commutes with J, G^+ and L. But, the explicit form (2.2.13) shows that it also commutes with G^- as the conjugation automorphism (2.2.8) sends Ω to $-\Omega$. ■

By (2.2.13), the eigenvalue of Ω on a highest-weight vector (+) or lowest-weight vector (−) of weight (j, Δ) is given by

$$\omega^{\pm}_{j,\Delta} = (2j \pm 1)\Big(j(j \pm 1) - (\mathsf{k}+3)\Delta - \frac{1}{8}(\mathsf{k}+1)(2\mathsf{k}+3)\Big). \tag{2.2.17}$$

These eigenvalues satisfy the following relations:

$$\omega^{-}_{-j,\Delta} = -\omega^{+}_{j,\Delta} = \omega^{+}_{-j-1,\Delta}. \tag{2.2.18}$$

We note that the first equality is consistent with conjugation.

Dense Z_k-modules can now be constructed by inducing C_k-modules: Let $\mathbb{C}_{j,\Delta,\omega}$ be a one-dimensional C_k-module, spanned by v, on which we have $Jv = jv$, $Lv = \Delta v$ and $\Omega v = \omega v$, for some $j, \Delta, \omega \in \mathbb{C}$. Define the Z_k-module $\overline{\mathcal{R}}_{j,\Delta,\omega} = \mathsf{Z}_\mathsf{k} \otimes_{\mathsf{C}_\mathsf{k}} \mathbb{C}_{j,\Delta,\omega}$.

A basis of $\overline{\mathcal{R}}_{j,\Delta,\omega}$ is given by $v = \mathbb{1} \otimes v$ and the $(G^{\pm})^n v$ with $n \in \mathbb{Z}_{\geqslant 1}$. This implies that the weights of $\overline{\mathcal{R}}_{j,\Delta,\omega}$ coincide with $[j] \times \{\Delta\}$ and so $\overline{\mathcal{R}}_{j,\Delta,\omega}$ is a dense Z_k-module generated by v.

Proposition 2.2.14.

- *For each $n \in \mathbb{Z}_{\geqslant 0}$, $(G^-)^{n+1}v$ is a highest-weight vector of $\overline{\mathcal{R}}_{j,\Delta,\omega}$ if and only if $\omega = \omega^{+}_{j-n-1,\Delta}$.*
- *For each $n \in \mathbb{Z}_{\geqslant 0}$, $(G^+)^{n+1}v$ is a lowest-weight vector of $\overline{\mathcal{R}}_{j,\Delta,\omega}$ if and only if $\omega = \omega^{-}_{j+n+1,\Delta}$.*
- *The dense Z_k-module $\overline{\mathcal{R}}_{j,\Delta,\omega}$ is simple if and only if $\omega \neq \omega^{+}_{i,\Delta}$ (equivalently $\omega \neq \omega^{-}_{i,\Delta}$) for any $i \in [j]$.*
- *$\overline{\mathcal{R}}_{j,\Delta,\omega}$ has at most four composition factors. If it is not simple, then one composition factor is infinite-dimensional highest-weight and another is infinite-dimensional lowest-weight; any other composition factors are finite-dimensional.*

Proof. The existence criteria for highest- and lowest-weight vectors is straightforward calculation using (2.2.18). The simplicity of $\overline{\mathcal{R}}_{j,\Delta,\omega}$ is equivalent to the absence of highest- and lowest-weight vectors. However, $\omega \neq \omega^{-}_{j-n,\Delta}$ for all $n \in \mathbb{Z}_{\geqslant 0}$ implies that $\omega \neq \omega^{+}_{j-n-1,\Delta}$ for all $n \in \mathbb{Z}_{\geqslant 0}$, by (2.2.18). Combining with $\omega \neq \omega^{+}_{j+n,\Delta}$ for all $n \in \mathbb{Z}_{\geqslant 0}$, we get the desired condition. The statements about composition factors now follow from the fact that $\omega - \omega^{\pm}_{i,\Delta}$ is a cubic polynomial in i, so it can have at most three roots $i \in [j]$. ∎

It follows from this proposition that we have isomorphisms $\overline{\mathcal{R}}_{j,\Delta,\omega} \simeq \overline{\mathcal{R}}_{j+m,\Delta,\omega}$, $m \in \mathbb{Z}$, when these modules are simple. We shall therefore denote these simple dense Z_k-modules by $\overline{\mathcal{R}}_{[j],\Delta,\omega}$, where $[j] \in \mathbb{C}/\mathbb{Z}$.

We have now seen how to construct various simple Z_k-modules by inducing modules over subalgebras of Z_k and taking a simple quotient when necessary. The modules we have constructed can be divided into four classes: finite-dimensional, infinite-dimensional highest-weight, infinite-dimensional lowest-weight and dense. A natural question is whether there are any more simple weight Z_{k}-modules. The classification of weight $\mathfrak{sl}_2$-modules consists of the same four classes, so the following result is perhaps unsurprising.

Theorem 2.2.15. *Every simple weight Z_{k}-module is isomorphic to one of the modules in the following list of pairwise-nonisomorphic modules:*

- *The finite-dimensional highest-weight (and lowest-weight) modules $\overline{\mathcal{H}}_{j,\Delta}$ with $j,\Delta \in \mathbb{C}$ such that $h^n_{\mathsf{k}}(j,\Delta) = 0$ for some $n \in \mathbb{Z}_{\geqslant 1}$.*
- *The infinite-dimensional highest-weight modules $\overline{\mathcal{H}}_{j,\Delta} = \overline{\mathcal{V}}_{j,\Delta}$ with $j,\Delta \in \mathbb{C}$ such that $h^n_{\mathsf{k}}(j,\Delta) \neq 0$ for all $n \in \mathbb{Z}_{\geqslant 1}$.*
- *The infinite-dimensional lowest-weight modules $\overline{\gamma}(\overline{\mathcal{H}}_{j,\Delta}) = \overline{\gamma}(\overline{\mathcal{V}}_{j,\Delta})$ with $j,\Delta \in \mathbb{C}$ such that $h^n_{\mathsf{k}}(j,\Delta) \neq 0$ for all $n \in \mathbb{Z}_{\geqslant 1}$.*
- *The infinite-dimensional dense modules $\overline{\mathcal{R}}_{[j],\Delta,\omega}$ with $[j] \in \mathbb{C}/\mathbb{Z}$ and $\Delta, \omega \in \mathbb{C}$ such that $\omega \neq \omega^+_{i,\Delta}$ for any $i \in [j]$.*

Proof. The classification for simple weight modules having a highest- and/or lowest-weight, i.e. the first three cases, follows from Proposition 2.2.10 in the same way as the analogous result for $\mathfrak{sl}_2$.

If the simple weight module has no highest- or lowest-weight, choose an arbitrary weight space. This is a simple C_{k}-module, hence it is one-dimensional and spanned by v say. As there are no highest- or lowest-weight vectors, G^+ and G^- act freely on v and so the simple weight module is dense and so isomorphic to one of the $\overline{\mathcal{R}}_{[j],\Delta,\omega}$ in the list. ∎

As in the untwisted case, the fact that $\mathsf{Zhu}^{\mathrm{tw}}\left[\mathsf{BP}^{\mathsf{k}}\right]$ is a quotient of Z_{k} means that every simple $\mathsf{Zhu}^{\mathrm{tw}}\left[\mathsf{BP}^{\mathsf{k}}\right]$-module is also simple as a Z_{k}-module. Theorem 1.1.12 then guarantees that every simple weight $\mathsf{Zhu}^{\mathrm{tw}}\left[\mathsf{BP}^{\mathsf{k}}\right]$-module $\overline{\mathcal{M}}$ corresponds to a simple twisted relaxed highest-weight BP^{k}-module $\mathcal{M} = \mathsf{Ind}^{\mathrm{tw}}\left[\overline{\mathcal{M}}\right]$ which is uniquely determined (up to isomorphism) by the fact that its top space is isomorphic to $\overline{\mathcal{M}}$ (as a Z_{k}-module).

Theorem 2.2.16. *Every simple twisted relaxed highest-weight BP^{k}-module, and hence every simple twisted relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module, is isomorphic to one of the modules in the following list of pairwise-nonisomorphic modules:*

- *The highest-weight modules $\mathcal{H}^{\mathrm{tw}}_{j,\Delta}$ with $j,\Delta \in \mathbb{C}$ such that $h^n_{\mathsf{k}}(j,\Delta) = 0$ for some $n \in \mathbb{Z}_{\geqslant 1}$.*
- *The highest-weight modules $\mathcal{H}^{\mathrm{tw}}_{j,\Delta} = \mathcal{V}^{\mathrm{tw}}_{j,\Delta}$ with $j,\Delta \in \mathbb{C}$ such that $h^n_{\mathsf{k}}(j,\Delta) \neq 0$ for all $n \in \mathbb{Z}_{\geqslant 1}$.*

- *The conjugate highest-weight modules* $\gamma(\mathcal{H}^{\mathrm{tw}}_{j,\Delta}) = \gamma(\mathcal{V}^{\mathrm{tw}}_{j,\Delta})$ *with* $j, \Delta \in \mathbb{C}$ *such that* $h^n_{\mathsf{k}}(j, \Delta) \neq 0$ *for all* $n \in \mathbb{Z}_{\geqslant 1}$.
- *The relaxed highest-weight modules* $\mathcal{R}^{\mathrm{tw}}_{[j],\Delta,\omega}$ *with* $[j] \in \mathbb{C}/\mathbb{Z}$ *and* $\Delta, \omega \in \mathbb{C}$ *such that* $\omega \neq \omega^+_{i,\Delta}$ *for all* $i \in [j]$.

The fact that $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}^{\mathsf{k}}]$ is isomorphic to Z_{k} means that all modules listed in Theorem 2.2.16 appear as simple twisted relaxed highest-weight BP^{k}-modules. We also remark that just like in the untwisted case, spectral flow will allow us to construct simple twisted weight BP^{k}-modules that are not relaxed highest-weight in general.

2.2.6. Coherent Families and Reducible BP^{k}-Modules. A crucial observation of Mathieu [128] concerning simple dense $\mathfrak{g}$-modules, for $\mathfrak{g}$ a simple Lie algebra, is that they may be naturally arranged into coherent families. Here, we extend this observation to dense Z_{k}-modules in preparation for showing that it also extends to $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-modules.

Definition 2.2.17. *A* coherent family of Z_{k}-modules *is a weight module* $\overline{\mathcal{C}}$ *for which:*

- L *and* Ω *act as multiples,* Δ *and* ω *respectively, of the identity on* $\overline{\mathcal{C}}$.
- *There exists* $d \in \mathbb{Z}_{\geqslant 0}$ *such that for all* $j \in \mathbb{C}$*, the dimension of the weight space* $\overline{\mathcal{C}}(j, \Delta)$ *of weight* (j, Δ) *is* d.
- *For each* $U \in \mathsf{C}_{\mathsf{k}}$*, the function taking* $j \in \mathbb{C}$ *to* $\mathrm{tr}_{\overline{\mathcal{C}}(j,\Delta)} U$ *is polynomial in* j.

Coherent families are by definition highly decomposable. Indeed, that the weight space $\overline{\mathcal{C}}(j, \Delta)$ has dimension d for all $j \in \mathbb{C}$ implies that a coherent family of Z_{k}-modules can be decomposed into submodules according to

$$\overline{\mathcal{C}} = \bigoplus_{[j]\in\mathbb{C}/\mathbb{Z}} \overline{\mathcal{C}}_{[j]}. \tag{2.2.19}$$

If all of the $\overline{\mathcal{C}}_{[j]}$ are semisimple as Z_{k}-modules, then $\overline{\mathcal{C}}$ is said to be *semisimple*. If any of the $\overline{\mathcal{C}}_{[j]}$ are simple as Z_{k}-modules, then $\overline{\mathcal{C}}$ is said to be *irreducible*. These are the same definitions introduced by Mathieu for simple Lie algebras.

It follows immediately from Proposition 2.2.12 that the common dimension d of the weight spaces $\overline{\mathcal{C}}(j, \Delta)$ of an irreducible coherent family of Z_{k}-modules is 1.

We would like to form a coherent family of Z_{k}-modules by summing over some collection of dense modules $\overline{\mathcal{R}}_{[j],\Delta,\omega}$, $[j] \in \mathbb{C}/\mathbb{Z}$, whilst holding Δ and ω fixed. Recall from Proposition 2.2.14 that $\overline{\mathcal{R}}_{j,\Delta,\omega}$ is simple if and only if j is not a root of a certain cubic polynomial. When j is such a root, there are a number of choices for how to define $\overline{\mathcal{R}}_{[j],\Delta,\omega}$:

- The first is to define $\overline{\mathcal{R}}_{[j],\Delta,\omega}$ to be $\overline{\mathcal{R}}^{\mathrm{ss}}_{j,\Delta,\omega}$, where the *semisimplification* $\mathcal{M}^{\mathrm{ss}}$ of a (finite-length) module $\mathcal{M}$ is the direct sum of its composition factors. This is well defined as $\overline{\mathcal{R}}^{\mathrm{ss}}_{j,\Delta,\omega} \simeq \overline{\mathcal{R}}^{\mathrm{ss}}_{j+1,\Delta,\omega}$.
- An alternative is to define $\overline{\mathcal{R}}_{[j],\Delta,\omega}$ to be $\overline{\mathcal{R}}^{+}_{[j],\Delta,\omega} = \overline{\mathcal{R}}_{j^+,\Delta,\omega}$, where we choose $j^+ \in [j]$ to have smaller real part than those of the solutions $i \in [j]$ of $\omega = \omega^+_{i,\Delta}$. This ensures that $\overline{\mathcal{R}}^{+}_{[j],\Delta,\omega}$ has no highest-weight vectors.
- We may instead define $\overline{\mathcal{R}}_{[j],\Delta,\omega}$ to be $\overline{\mathcal{R}}^{-}_{[j],\Delta,\omega} = \overline{\mathcal{R}}_{j^-,\Delta,\omega}$, where we choose $j^- \in [j]$ to have larger real part than those of the solutions $i \in [j]$ of $\omega = \omega^-_{i,\Delta}$. This ensures that $\overline{\mathcal{R}}^{-}_{[j],\Delta,\omega}$ has no lowest-weight vectors.

For each of the three choices above, we can define an irreducible coherent family of Z_{k}-modules by taking the direct sum of the $\overline{\mathcal{R}}_{[j],\Delta,\omega}$ over $[j] \in \mathbb{C}/\mathbb{Z}$. That is,

$$\overline{\mathcal{C}}^{\#}_{\Delta,\omega} = \bigoplus_{[j]\in\mathbb{C}/\mathbb{Z}} \overline{\mathcal{R}}^{\#}_{j,\Delta,\omega}, \tag{2.2.20}$$

where $\# \in \{\mathrm{ss}, +, -\}$, is an irreducible coherent family. All coherent families are not created equal however. It is easy to see that $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ is semisimple, whilst $\overline{\mathcal{C}}^{+}_{\Delta,\omega}$ and $\overline{\mathcal{C}}^{-}_{\Delta,\omega}$ are nonsemisimple with G^+ and G^- acting injectively, respectively. Note that $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ is the unique irreducible semisimple coherent family of Z_{k}-modules on which L acts as multiplication by Δ and Ω acts as multiplication by ω, up to isomorphism. Coherent families can be twisted by the BP^{k} automorphisms of Proposition 2.1.4, and we find that

$$\overline{\gamma}(\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}) \simeq \overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,-\omega}, \quad \overline{\gamma}(\overline{\mathcal{C}}^{\pm}_{\Delta,\omega}) \simeq \overline{\mathcal{C}}^{\mp}_{\Delta,-\omega}. \tag{2.2.21}$$

For classifying simple $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules, the semisimple coherent families $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ are most suitable. We shall also see $\overline{\mathcal{C}}^{+}_{\Delta,\omega}$ and $\overline{\mathcal{C}}^{-}_{\Delta,\omega}$ in Section 2.3.4 when considering the existence of nonsemisimple $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules.

Proposition 2.2.18.

- *Every simple weight Z_{k}-module embeds into a unique irreducible semisimple coherent family.*
- *Every irreducible semisimple coherent family of Z_{k}-modules has an infinite-dimensional highest-weight submodule.*

Proof. By Theorem 2.2.15, a simple dense Z_{k}-module $\mathcal{M}$ is isomorphic to some $\overline{\mathcal{R}}_{[j],\Delta,\omega}$, where $[j] \in \mathbb{C}/\mathbb{Z}$ and $\Delta, \omega \in \mathbb{C}$ satisfy $\omega \neq \omega^+_{i,\Delta}$ for any $i \in [j]$. As $\overline{\mathcal{R}}^{\mathrm{ss}}_{[j],\Delta,\omega} = \overline{\mathcal{R}}_{[j],\Delta,\omega}$, we have an embedding $\mathcal{M} \hookrightarrow \overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$. The target is obviously unique, up to isomorphism, since no other irreducible semisimple coherent family has the correct L- and Ω-eigenvalues.

A simple highest-weight Z_{k}-module $\mathcal{M}$ is isomorphic to $\overline{\mathcal{H}}_{j,\Delta}$, for some $j, \Delta \in \mathbb{C}$. Take $\omega = \omega^+_{j,\Delta}$, so that $\overline{\mathcal{R}}_{j,\Delta,\omega}$ is not simple and there is a highest-weight vector of weight (j,Δ) in $\overline{\mathcal{R}}^{\mathrm{ss}}_{j,\Delta,\omega}$, by Proposition 2.2.14. This vector generates a copy of $\overline{\mathcal{H}}_{j,\Delta}$, so we again have an embedding $\mathcal{M} \hookrightarrow \overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ with unique target.

Finally, if $\mathcal{M}$ is a simple lowest-weight Z_k-module, then we have an embedding $\overline{\gamma}(\mathcal{M}) \hookrightarrow \overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ for some unique $\Delta, \omega \in \mathbb{C}$. By (2.2.21), we have $\mathcal{M} \hookrightarrow \overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,-\omega}$. This covers all possibilities, by Theorem 2.2.15, so the first statement is established.

For the second, a given irreducible semisimple coherent family $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ is uniquely specified by choosing $\Delta, \omega \in \mathbb{C}$. As $\omega - \omega^+_{i,\Delta}$ is a cubic polynomial in i, there is at least one solution in $\mathbb{C}$, $i = j$ say. Then, $\overline{\mathcal{R}}_{j,\Delta,\omega}$ is not simple and has an infinite-dimensional highest-weight submodule, by Proposition 2.2.14, hence so does $\overline{\mathcal{R}}^{\mathrm{ss}}_{j,\Delta,\omega} \subset \overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$. ■

2.3. BP Minimal Models

Recall from Section 1.1.4 that if I is an ideal of a vertex operator algebra V, then $\mathsf{Zhu}[\mathsf{V}/\mathsf{I}] \simeq \mathsf{Zhu}[\mathsf{V}]/\mathsf{Zhu}[\mathsf{I}]$. If J^k denotes the maximal ideal of BP^k, then classifying the relaxed highest-weight modules of $\mathsf{BP}_\mathsf{k} = \mathsf{BP}^\mathsf{k}/\mathsf{J}^\mathsf{k}$ is then just a matter of classifying those of BP^k and then testing which have Zhu-images annihilated by $\mathsf{Zhu}\left[\mathsf{J}^\mathsf{k}\right]$. Unfortunately, it is hard to compute $\mathsf{Zhu}\left[\mathsf{J}^\mathsf{k}\right]$ in general even for admissible k.

An alternative route to the desired classification is provided by Arakawa's celebrated classification [18] of the highest-weight modules of all simple admissible-level affine vertex operator algebras $\mathsf{L}_\mathsf{k}(\mathfrak{g})$, specialised to $\mathfrak{g} = \mathfrak{sl}_3$ and his results [12] on minimal quantum hamiltonian reduction.

What results is a classification of the highest-weight modules for Bershadsky–Polyakov minimal models. This can then be combined with the coherent families of Section 2.2.6 to obtain a full classification of simple (twisted and untwisted) relaxed highest-weight modules. Additionally, several classes of nonsimple relaxed highest-weight modules for Bershadsky–Polyakov minimal models can be defined in a natural way (when $\mathsf{v} \geqslant 3$).

2.3.1. Admissible-Level $\mathfrak{sl}_3$ Minimal Models. Recall from (2.1.14) the fractional levels of BP^k and their parametrisation in terms of u and v. In addition to BP^k, the affine vertex operator algebra $\mathsf{V}^\mathsf{k}(\mathfrak{sl}_3)$ is also not simple when k is a fractional level. For such k, the simple quotient will be denoted by $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3) = \mathsf{A}_2(\mathsf{u},\mathsf{v})$. The vertex operator algebras of the form $\mathsf{A}_2(\mathsf{u},1)$ where $\mathsf{u} \in \mathbb{Z}_{\geqslant 0}$ are the familiar rational $\mathfrak{sl}_3$ minimal models described in Section 1.2.2.

Definition 2.3.1. *An* admissible level k *for the affine vertex operator algebras associated to* $\mathfrak{sl}_3$*, and the Bershadsky–Polyakov algebras, is a fractional level for which* $\mathsf{u} \geqslant 3$.

Every highest-weight module for the affine Kac–Moody algebra $\widehat{\mathfrak{sl}}_3$ is a $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-module [76]. Let $\mathcal{L}_\lambda$ denote the simple highest-weight $\widehat{\mathfrak{sl}}_3$-module of highest weight $\lambda = \lambda_0\omega_0 + \lambda_1\omega_1 + \lambda_2\omega_2$, where the λ_i are the Dynkin labels and the ω_i are the fundamental weights. Recall that a weight is said to be level-ℓ if $\lambda_0 + \lambda_1 + \lambda_2 = \ell$. Let $\mathsf{P}^{\ell}_{\geqslant}$ denote the set of dominant integral level-ℓ weights of $\widehat{\mathfrak{sl}}_3$. This set is empty unless $\ell \in \mathbb{Z}_{\geqslant 0}$ as the Dynkin labels of dominant integral weights are nonnegative integers. Let w_i, $i = 0, 1, 2$, denote the Weyl reflection corresponding to the simple root α_i of $\widehat{\mathfrak{sl}}_3$.

The following definition specialises that of [103] to $\widehat{\mathfrak{sl}}_3$ (see also [70, App. 18.B]).

Definition 2.3.2. *Let* k *be an admissible level. A level-*k admissible weight λ *of* $\widehat{\mathfrak{sl}}_3$ *is one of the form*

$$\lambda = w \cdot \left(\lambda^I - \frac{\mathsf{u}}{\mathsf{v}}\lambda^{F,w}\right), \tag{2.3.1}$$

where $w \in \{\mathbb{1}, w_1\}$ *is a Weyl transformation of* $\mathfrak{sl}_3$, $\cdot$ *is the shifted Weyl group action,* $\lambda^I \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$, $\lambda^{F,w} \in \mathsf{P}^{\mathsf{v}-1}_{\geqslant}$ *and* $\lambda_1^{F,w_1} \geqslant 1$. *A weight of the form* (2.3.1) *will be called a* $w = \mathbb{1}$ *or* $w = w_1$ *admissible weight according as to which w is used.*

Importantly, the set of $w = \mathbb{1}$ admissible weights is disjoint to that of $w = w_1$ admissible weights. One can define w admissible weights for other elements of the Weyl group by imposing appropriate restrictions on $\lambda^{F,w}$. The set of such w admissible weights will always be equal to that of $\mathbb{1}$ or w_1 however [104, Prop. 2.1].

In [18], Arakawa classified the highest-weight modules of all simple admissible-level affine vertex operator algebras $\mathsf{L}_{\mathsf{k}}(\mathfrak{g})$, where the definition of admissible levels depends on the simple Lie algebra $\mathfrak{g}$. Specialised, the classification to $\mathfrak{g} = \mathfrak{sl}_3$ gives the following.

Theorem 2.3.3 ([18])**.** *For* $\mathsf{k} = \frac{\mathsf{u}}{\mathsf{v}} - 3$ *admissible, the simple level-*k *highest-weight module* $\mathcal{L}_\lambda$ *is an* $\mathsf{A}_2(\mathsf{u}, \mathsf{v})$*-module if and only if* λ *is admissible.*

Recall that the universal Bershadsky–Polyakov algebra BP^{k} is the minimal quantum hamiltonian reduction of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$; $H^0(\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)) = \mathsf{BP}^{\mathsf{k}}$. This fact allows us to construct BP^{k}-modules by applying the minimal quantum hamiltonian reduction functor $H^0(-)$ to $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-modules. Before moving on to the more difficult case of $\mathsf{BP}(\mathsf{u}, \mathsf{v})$, we note that the action of the minimal quantum hamiltonian reduction functor on highest-weight $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-modules satisfies a number of desirable properties. The following are specialisations to $\mathfrak{g} = \mathfrak{sl}_3$ of general results regarding Verma modules described in Section 1.3.1.

Theorem 2.3.4.

- *[106, Thm. 6.3] If $\mathcal{K}_\lambda$ denotes the Verma module of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ with highest weight λ, then $H^0(\mathcal{K}_\lambda)$ is isomorphic to the Verma module $\mathcal{V}_{j,\Delta}$ of BP^{k} with*

$$j = \frac{\lambda_1 - \lambda_2}{3} \quad \text{and} \quad \Delta = \frac{(\lambda_1 - \lambda_2)^2 - 3(\lambda_1 + \lambda_2)\big(2(\mathsf{k}+1) - \lambda_1 - \lambda_2\big)}{12(\mathsf{k}+3)}. \tag{2.3.2}$$

- *[12, Thm. 6.7.4] $H^0(\mathcal{L}_\lambda) = 0$ if and only if $\lambda_0 \in \mathbb{Z}_{\geqslant 0}$. For $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$, we have instead $H^0(\mathcal{L}_\lambda) \simeq \mathcal{H}_{j,\Delta}$, where j and Δ are given by (2.3.2).*
- *[12, Cor. 6.7.3] The restriction of $H^0(-)$ to the category $\widehat{\mathcal{O}}_{\mathsf{k}}$ of level-k $\widehat{\mathfrak{sl}}_3$-modules is exact.*
- *$H^0(-)$ induces a surjection from the set of isomorphism classes of simple highest-weight $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-modules to the union of $\{0\}$ and the set of isomorphism classes of simple highest-weight BP^{k}-modules. Moreover, there are at most two inequivalent $\mathcal{L}_\lambda$ mapping onto the same $\mathcal{H}_{j,\Delta}$.*

Proof. We only prove the last assertion. It follows from the second assertion above and by inverting (2.3.2) to obtain two solutions (λ_1, λ_2) for each (j, Δ). We have to ensure that at least one solution gives $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$. But, a simple calculation gives

$$\lambda_0 = \mathsf{k} - \lambda_1 - \lambda_2 = -1 \pm \sqrt{4(\mathsf{k}+3)\Delta + (\mathsf{k}+1)^2 - 3j^2}, \tag{2.3.3}$$

so the zeroth Dynkin labels of the two solutions sum to -2. ■

The second point in Theorem 2.3.4 specifies the action of quantum hamiltonian reduction on all highest-weight modules $\mathcal{L}_\lambda$ of $\mathsf{A}_2(\mathsf{u}, \mathsf{v})$ by Theorem 2.3.3. What it does not tell us is whether $H^0(\mathcal{L}_\lambda)$ with λ admissible satisfying $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$ is a $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module. Even nicer would be if all simple highest-weight $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module can be described in this way. Showing that this is indeed the case is our next job.

Definition 2.3.5. *For k admissible, we shall call a level-k weight λ of $\widehat{\mathfrak{sl}}_3$* surviving *if it is admissible and $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$. Theorem 2.3.4 then ensures that $H^0(\mathcal{L}_\lambda)$ is nonzero (and is moreover a simple BP^{k}-module).*

Lemma 2.3.6.

- *Every $w = w_1$ admissible weight is surviving.*
- *A $w = \mathbb{1}$ admissible weight λ is surviving if and only if $\lambda_0^{F,\mathbb{1}} \geqslant 1$.*
- *$w_0\cdot$ gives a (j, Δ)-preserving bijection between the $w = \mathbb{1}$ surviving weights and the $w = w_1$ admissible weights.*
- *If λ and μ are distinct $w = \mathbb{1}$ surviving weights, then $H^0(\mathcal{L}_\lambda)$ and $H^0(\mathcal{L}_\mu)$ are not isomorphic.*

Proof. The zeroth Dynkin label of a level-k admissible $\widehat{\mathfrak{sl}}_3$-weight λ has one of the following two forms:

$$\lambda_0 = \lambda_0^I - \frac{\mathsf{u}}{\mathsf{v}}\lambda_0^{F,\mathbb{1}} \quad (w = \mathbb{1}) \qquad \text{or} \qquad \lambda_0 = \lambda_0^I + \lambda_1^I - \frac{\mathsf{u}}{\mathsf{v}}\left(\lambda_0^{F,w_1} + \lambda_1^{F,w_1}\right) + 1 \quad (w = w_1). \tag{2.3.4}$$

Consider first a $w = \mathbb{1}$ admissible weight λ. Since $\lambda^{F,\mathbb{1}} \in \mathsf{P}_{\geqslant}^{\mathsf{v}-1}$, we clearly have $\lambda_0 \in \mathbb{Z}$ if and only if $\lambda_0^{F,\mathbb{1}} = 0$. On the other hand, a $w = w_1$ admissible weight λ necessarily has $0 < \lambda_0^{F,w_1} + \lambda_1^{F,w_1} < \mathsf{v}$, since $\lambda^{F,w_1} \in \mathsf{P}_{\geqslant}^{\mathsf{v}-1}$ and $\lambda_1^{F,w_1} \geqslant 1$. It follows that the Dynkin label λ_0 can never be an integer in this case. This proves the first two statements.

For the third, let μ be a level-k weight. Explicit calculation shows that the Dynkin labels of $w_0 \cdot w_1 \cdot \mu$ are

$$\left[\mu_2 - \frac{\mathsf{u}}{\mathsf{v}},\ \mu_0,\ \mu_1 + \frac{\mathsf{u}}{\mathsf{v}}\right]. \tag{2.3.5}$$

Let $\lambda = w_1 \cdot \left(\lambda^I - \frac{\mathsf{u}}{\mathsf{v}}\lambda^{F,w_1}\right)$ be a $w = w_1$ admissible weight. Then, $w_0 \cdot \lambda$ has the form $\mu = \mu^I - \frac{\mathsf{u}}{\mathsf{v}}\mu^{F,\mathbb{1}}$ with

$$\mu^I = \left[\lambda_2^I,\ \lambda_0^I,\ \lambda_1^I\right] \quad \text{and} \quad \mu^{F,\mathbb{1}} = \left[\lambda_2^{F,w_1} + 1,\ \lambda_0^{F,w_1},\ \lambda_1^{F,w_1} - 1\right]. \tag{2.3.6}$$

It is easy to see that $\mu^I \in \mathsf{P}_{\geqslant}^{\mathsf{u}-3}$ and $\mu^{F,\mathbb{1}} \in \mathsf{P}_{\geqslant}^{\mathsf{v}-1}$, so μ is a $w = \mathbb{1}$ admissible weight. Moreover, $\mu_0^{F,\mathbb{1}} \geqslant 1$ implies that μ is surviving. Since $w_0 \cdot (-)$ is self-inverse, we have the desired bijection between $w = \mathbb{1}$ surviving weights and $w = w_1$ admissible weights. To show that it is (j, Δ)-preserving, we show that the functions $j(\lambda)$ and $\Delta(\lambda)$ defined by (2.3.2) are invariant under $\lambda \mapsto w_0 \cdot \lambda$. This is clear from $(w_0 \cdot \lambda)_1 = \mathsf{k} + 1 - \lambda_2$ and $(w_0 \cdot \lambda)_2 = \mathsf{k} + 1 - \lambda_1$.

Finally, let λ and μ be surviving weights and suppose that $H^0(\mathcal{L}_\lambda) \simeq H^0(\mathcal{L}_\mu)$, so that $j(\lambda) = j(\mu)$ and $\Delta(\lambda) = \Delta(\mu)$. We have just seen that λ and $w_0 \cdot \lambda$ always give the same j and Δ. But, if λ is a $w = \mathbb{1}$ surviving weight, then $\mu = w_0 \cdot \lambda$ is a $w = w_1$ surviving weight. Since the intersection of the sets of $w = \mathbb{1}$ and $w = w_1$ admissible weights is empty [104, Prop. 2.1], we have $\lambda \neq \mu$. As there are at most two weights corresponding to a given choice of j and Δ (Theorem 2.3.4), this shows that there are never two distinct $w = \mathbb{1}$ surviving weights giving the same j and Δ. ∎

In light of the second point of Lemma 2.3.6, 'surviving' shall be understood to mean '$w = \mathbb{1}$ surviving' unless otherwise indicated. Likewise from now on, we will drop the label w from $\lambda^{F,w}$, understanding that we mean $w = \mathbb{1}$ unless otherwise indicated. Denote the set of surviving level-k weights by Σ_{k}.

Let I^{k} denote the maximal ideal of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$, so that $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3) = \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)/\mathsf{I}^{\mathsf{k}}$. If k is an admissible level, then by Theorem 2.3.3 we have that $\mathsf{I}^{\mathsf{k}} \cdot \mathcal{L}_\lambda = 0$ if and only if λ is an admissible weight. If, in addition, $\mathsf{v} \geqslant 2$, then

$$H^0(\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)) = H^0(\mathcal{L}_{\mathsf{k}\omega_0}) \simeq \mathcal{H}_{0,0} = \mathsf{BP}_\mathsf{k}, \tag{2.3.7}$$

by Theorem 2.3.4. Moreover, the exactness of $H^0(-)$ means that the maximal ideal J^k of BP^k is then isomorphic to $H^0(\mathsf{I}^\mathsf{k})$. It follows that $H^0(\mathcal{L}_\lambda)$ is a BP_k-module if and only if

$$H^0(\mathsf{I}^\mathsf{k}) \cdot H^0(\mathcal{L}_\lambda) = 0. \tag{2.3.8}$$

Proposition 2.3.7. *Let* k *be admissible with* $\mathsf{v} \geqslant 2$*. If* $\mathcal{L}_\lambda$ *is an* $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$*-module, then* $H^0(\mathcal{L}_\lambda)$ *is a* BP_k*-module.*

Proof. The quantum hamiltonian reduction functor $H^0(-)$ acts on modules by tensoring with the ghost vertex operator superalgebra G and taking the zeroth homology with respect to the differential d_0, where $d(z)$ is given by (2.1.5). Denote the homology class of a (degree-0) cocycle a by $[a]$ (not to be confused with the notation for Zhu algebra images in Section 1.1.4).

Given (degree-0) cocycles a and v in the differential complexes $\mathsf{I}^\mathsf{k} \otimes \mathsf{G}$ and $\mathcal{L}_\lambda \otimes \mathsf{G}$, respectively, the action of $[a] \in H^0(\mathsf{I}^\mathsf{k})$ on $[v] \in H^0(\mathcal{L}_\lambda)$ is given by

$$[a] \cdot [v] \equiv [a](z)[v] = [a(z)v] \in H^0(\mathsf{I}^\mathsf{k} \cdot \mathcal{L}_\lambda). \tag{2.3.9}$$

For λ admissible, we therefore obtain $H^0(\mathsf{I}^\mathsf{k}) \cdot H^0(\mathcal{L}_\lambda) \subseteq H^0(\mathsf{I}^\mathsf{k} \cdot \mathcal{L}_\lambda) = 0$. That is, $H^0(\mathcal{L}_\lambda)$ is a BP_k-module. ■

Proposition 2.3.7 motivates restricting attention to fractional levels of the form

$$\mathsf{k} = -3 + \frac{\mathsf{u}}{\mathsf{v}} \quad \text{with} \quad \mathsf{u} \geqslant 3 \quad \text{and} \quad \mathsf{v} \geqslant 2. \tag{2.3.10}$$

The restriction on u means that k is an admissible level for $\mathfrak{sl}_3$, whilst the restriction on v guarantees that the minimal quantum hamiltonian reduction of $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3) = \mathsf{A}_2(\mathsf{u},\mathsf{v})$ is $\mathsf{BP}_\mathsf{k} = \mathsf{BP}(\mathsf{u},\mathsf{v})$. The main issue with using quantum hamiltonian reduction for $\mathsf{v} = 1$ admissible levels is that $H^0(\mathsf{A}_2(\mathsf{u},1)) = 0$ for $\mathsf{u} \geqslant 3$. A classification of simple modules for Bershadsky–Polyakov minimal models at admissible levels with $\mathsf{v} = 1$ is known [6].

To obtain a classification of simple highest-weight BP_k-modules from Arakawa's classification of simple highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$-modules (Theorem 2.3.3), a converse of Proposition 2.3.7 is needed; we need to know that all simple highest-weight BP_k-modules are isomorphic to $H^0(\mathcal{L}_\lambda)$ for some surviving weight.

Theorem 2.3.8. *Let* k *be admissible with* $\mathsf{v} \geqslant 2$*. Then, every simple highest-weight* $\mathsf{BP_k}$*-module is isomorphic to the minimal quantum hamiltonian reduction of some simple highest-weight* $\mathsf{L_k}(\mathfrak{sl}_3)$*-module.*

Note that if $\lambda_0 \in \mathbb{Z}_{\geqslant 0}$, then $H^0(\mathcal{L}_\lambda) = 0$ is a $\mathsf{BP_k}$-module, irrespective of whether or not it is an $\mathsf{L_k}(\mathfrak{sl}_3)$-module. It is therefore enough to show that if $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$ and $\mathcal{L}_\lambda$ is not a $\mathsf{L_k}(\mathfrak{sl}_3)$-module, then $H^0(\mathcal{L}_\lambda)$ is not a $\mathsf{BP_k}$-module. Equivalently, we must show that $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$ and $\mathsf{I^k} \cdot \mathcal{L}_\lambda \neq 0$ implies that $H^0(\mathsf{I^k}) \cdot H^0(\mathcal{L}_\lambda) \neq 0$.

2.3.2. Surjectivity of Reduction. This section is devoted to the rather technical proof of Theorem 2.3.8. If the reader is content with accepting Theorem 2.3.8 as true, skipping to Section 2.3.3 is advised.

We adopt the notation of Section 2.1.1 and assume throughout that $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$ so that $H^0(\mathcal{L}_\lambda) \neq 0$ (and that the level k is admissible with $\mathsf{v} \geqslant 2$). With these assumptions, the aim is to prove the following assertion:

$$\mathsf{I^k} \cdot \mathcal{L}_\lambda \neq 0 \quad \Rightarrow \quad H^0(\mathsf{I^k}) \cdot H^0(\mathcal{L}_\lambda) \neq 0. \tag{2.3.11}$$

We will prove (2.3.11) (and therefore Theorem 2.3.8) by exhibiting elements $\chi \in \mathsf{I^k}$ and $v \in \mathcal{L}_\lambda$ for which $\chi \otimes |0\rangle$ and $v \otimes |0\rangle$ are (degree-0) closed with respect to d_0, such that the element $\chi_n v \otimes |0\rangle$ is not exact, for some $n \in \mathbb{Z}$. This then gives a nonzero element of $H^0(\mathsf{I^k}) \cdot H^0(\mathcal{L}_\lambda)$ as

$$[\chi \otimes |0\rangle] \cdot [v \otimes |0\rangle] \equiv [\chi \otimes |0\rangle](z)[v \otimes |0\rangle] = [\chi(z) v \otimes |0\rangle] \neq 0. \tag{2.3.12}$$

Proving (2.3.11) requires several finer details of minimal ($f = f_\theta$) quantum hamiltonian reduction for $\mathsf{V^k}(\mathfrak{sl}_3)$ that we will now explore.

As we deformed the energy-momentum tensor of $\mathsf{V^k}(\mathfrak{sl}_3)$ to $T^{\text{Sug.}} + (1/2)\partial h_\theta$ in Section 2.1.1, we now have two distinct mode conventions for affine fields. For an affine generator a with conformal weight $\widetilde{\Delta}$ with respect to the deformed energy-momentum tensor, we expand $a(z)$ in modes according to

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} = \sum_{n \in \mathbb{Z} - \widetilde{\Delta}} a_{(n)} z^{-n-\widetilde{\Delta}}. \tag{2.3.13}$$

The expansions of the ghost fields will always be taken with respect to their conformal weight under $\sum_{\alpha \in \Delta} T^{\mathsf{F}^\alpha} + T^{\mathsf{B}}$. We start with a well known fundamental result for the highest-weight vector v of $\mathcal{L}_\lambda$, recalling that we are assuming throughout that $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$ and that k is of the form (2.3.10).

Let $|0\rangle$ denote the vacuum vector of G. Here again we denote the homology class of a cocycle a by $[a]$.

Lemma 2.3.9. *For all $n \in \mathbb{Z}_{\geqslant 0}$, $((e_\theta)_{-1})^n v \otimes |0\rangle$ is closed and inexact. In particular, $[v \otimes |0\rangle] \neq 0$.*

Proof. A direct calculation using (2.1.5) shows that $d(v \otimes |0\rangle) = 0$ and, more generally, that

$$d\Big(((e_\theta)_{-1})^n v \otimes |0\rangle\Big) = 0 \tag{2.3.14}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. On the other hand, we have the commutation relation

$$[d, b_0^\theta] = (e_\theta)_{(0)} + \mathbb{1} = (e_\theta)_{-1} + \mathbb{1}, \tag{2.3.15}$$

where $\mathbb{1}$ is the vacuum vector in $\mathsf{C} = \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \otimes \mathsf{G}$. Hence

$$d\Big(b_0^\theta \big((e_\theta)_{-1}\big)^n v \otimes |0\rangle\Big) = \big((e_\theta)_{-1}\big)^{n+1} v \otimes |0\rangle + \big((e_\theta)_{-1}\big)^n v \otimes |0\rangle. \tag{2.3.16}$$

Therefore in homology,

$$\Big[\big((e_\theta)_{-1}\big)^n v \otimes |0\rangle\Big] = (-1)^n \Big[v \otimes |0\rangle\Big]. \tag{2.3.17}$$

The image of the closed subspace $\operatorname{span}\Big\{\big((e_\theta)_{-1}\big)^n v \otimes |0\rangle : n \in \mathbb{Z}_{\geqslant 0}\Big\} \subset \mathsf{C}$ in homology is therefore spanned by $\big[v \otimes |0\rangle\big]$. If λ_0 were a non-negative integer, then $\big((e_\theta)_{-1}\big)^{\lambda_0+1} v = 0$ would force this spanning homology class to be 0. However, we are assuming that $\lambda_0 \notin \mathbb{Z}_{\geqslant 0}$ and, in this case, [12, Lemma 4.6.1, Prop. 4.7.1] proves the contrary instead. ∎

We next consider the structure of the maximal ideal I^{k} of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$.

Lemma 2.3.10. *I^{k} is generated by a single singular vector χ whose $\mathfrak{sl}_3$-weight and conformal weight (with respect to $T^{Sug.}$) are $(\mathsf{u}-2)\theta$ and $(\mathsf{u}-2)\mathsf{v}$, respectively. Moreover, $\chi \otimes |0\rangle$ is closed.*

Proof. This follows easily from [103, Cor. 1], which says that the maximal submodule of a Verma module whose highest weight is admissible is generated by singular vectors of known weight. The Verma module whose quotient is $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ has highest weight $\mathsf{k}\omega_0$ which is an 'admissible weight' because k is.

The only generating singular vector that is nonzero in the quotient $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ of this Verma module has weight $w \cdot (\mathsf{k}\omega_0)$, where w is the Weyl reflection corresponding to the root $-\theta + \mathsf{v}\delta$. Here, δ denotes the standard imaginary root of $\widehat{\mathfrak{sl}}_3$. Denote this singular vector by χ. The $\mathfrak{sl}_3$- and conformal weights of χ are now easily computed.

The fact that $\chi \otimes |0\rangle$ is closed follows from χ being a highest-weight vector. ∎

Suppose now that $\chi(z)v = 0$. Because χ generates I^{k}, it follows that $\mathsf{I}^{\mathsf{k}} \cdot v = 0$. Since v generates $\mathcal{L}_\lambda$, as a $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$-module, and I^{k} is a two-sided ideal of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$, we get $\mathsf{I}^{\mathsf{k}} \cdot \mathcal{L}_\lambda = 0$.

The hypothesis of (2.3.11), that $\mathcal{L}_\lambda$ is not an $\mathsf{L}_k(\mathfrak{sl}_3)$-module, therefore requires that $\chi_n v \neq 0$ for some $n \in \mathbb{Z}$. As χ has $\mathfrak{sl}_3$-weight $(\mathsf{u}-2)\theta$, such an n must be of the form $-(\mathsf{u}-2)-i$ for some $i \in \mathbb{Z}_{\geqslant 0}$. Let $N \in \mathbb{Z}_{\geqslant 0}$ be minimal satisfying $\chi_{-(\mathsf{u}-2)-N} v \neq 0$.

As $\mathcal{L}_\lambda$ is simple, the submodule generated by $\chi_{-(\mathsf{u}-2)-N} v$ must contain a multiple of v. That is, there exists a Poincaré–Birkhoff–Witt monomial $U \in \mathsf{U}(\widehat{\mathfrak{sl}}_3)$ such that

$$U \chi_{-(\mathsf{u}-2)-N} v = v \tag{2.3.18}$$

(rescaling χ if necessary). We choose an ordering for the various factors of U so that

$$(f_\alpha)_{n\leqslant 0} < (h_\alpha)_{n<0} < (e_\alpha)_{n<0} < (f_\alpha)_{n>0} < (h_\alpha)_{n>0} < (e_\alpha)_{n\geqslant 0}. \tag{2.3.19}$$

This means, for example, that the $(f_\alpha)_n$ with $n \leqslant 0$ are ordered to the left in U while the $(e_\alpha)_n$ with $n \geqslant 0$ are ordered to the right in U. As χ is a singular vector, $(e_\alpha)_0 \chi = 0$ and $(e_\alpha)_n v = 0$ for all $n \geqslant 0$. Hence

$$(e_\alpha)_n \chi_{-(\mathsf{u}-2)-N} v = \big((e_\alpha)_0 \chi\big)_{-(\mathsf{u}-2)-N+n} v = 0. \tag{2.3.20}$$

We may therefore assume that U contains no $(e_\alpha)_n$-modes with $n \geqslant 0$. Similarly,

$$(h_\alpha)_n \chi_{-(\mathsf{u}-2)-N} v = (\mathsf{u}-2)\theta(h_\alpha) \chi_{-(\mathsf{u}-2)-(N-n)} v = 0, \tag{2.3.21}$$

for $n > 0$, by the minimality of N. Thus, we may assume that U contains no $(h_\alpha)_n$-modes with $n > 0$ either. Finally, v is not in the image of any $(f_\alpha)_n$, with $n \leqslant 0$, $(h_\alpha)_n$, with $n < 0$, or $(e_\alpha)_n$, with $n < 0$. All these modes may therefore also be excluded from U.

Given a partition $\xi = [\xi_1 \geqslant \xi_2 \geqslant \cdots]$, let $\ell(\xi)$ denote its length and $|\xi|$ denote its weight. We write $(f_\alpha)_\xi = (f_\alpha)_{\xi_1} (f_\alpha)_{\xi_2} \cdots$. By the above discussion, there exists partitions ξ, π and ρ such that

$$U \chi_{-(\mathsf{u}-2)-N} v = (f_\theta)_\xi (f_{\alpha_2})_\pi (f_{\alpha_1})_\rho \chi_{-(\mathsf{u}-2)-N} v = v \tag{2.3.22}$$

with

$$\ell(\pi) = \ell(\rho), \quad \ell(\xi) + \ell(\pi) = \mathsf{u}-2 \quad \text{and} \quad |\xi| + |\pi| + |\rho| = \mathsf{u}-2+N. \tag{2.3.23}$$

The following useful result imposes bounds on the size on the parts of ξ, π and ρ.

Lemma 2.3.11. *Let $F(z)$, $F \in \mathfrak{sl}_3$, be an affine field and let U_0 be a monomial in the negative root vectors $(f_\alpha)_0$ of $\widehat{\mathfrak{sl}}_3$. Then, the modes of the field $(U_0\chi)(w)$ satisfy*

$$[F_m, (U_0\chi)_n] = (F_0 U_0 \chi)_{m+n}, \quad \textit{for all } m, n \in \mathbb{Z}. \tag{2.3.24}$$

Proof. Observe that $U_0\chi$ is annihilated by the F_m with $m > 0$. Consequently, (2.3.24) follows easily from the operator product expansion

$$F(z)(U_0\chi)(w) \sim \frac{(F_0U_0\chi)(w)}{z-w}. \tag{2.3.25}$$ ∎

Lemma 2.3.12. *If any of the parts of ξ, π or ρ are greater than* 1, *then*

$$(f_\theta)_\xi(f_{\alpha_2})_\pi(f_{\alpha_1})_\rho\chi_{-(\mathsf{u}-2)-N}v = 0. \tag{2.3.26}$$

Proof. Suppose without loss of generality that ξ has a part $\xi_i > 1$. Then, we can form a new partition ξ' from ξ by subtracting 1 from ξ_i and reordering parts if necessary. Note that $\ell(\xi') = \ell(\xi)$ and $|\xi'| = |\xi| - 1$. Then, Lemma 2.3.11 and N being minimal give

$$\begin{aligned}
0 &= (f_\theta)_{\xi'}(f_{\alpha_2})_\pi(f_{\alpha_1})_\rho\chi_{-(\mathsf{u}-2)-(N-1)}v \\
&= \Big(\big((f_\theta)_0\big)^{\ell(\xi')}\big((f_{\alpha_2})_0\big)^{\ell(\pi)}\big((f_{\alpha_1})_0\big)^{\ell(\rho)}\chi\Big)_{-(\mathsf{u}-2)+|\xi'|+|\pi|+|\rho|-N+1}v \\
&= \Big(\big((f_\theta)_0\big)^{\ell(\xi)}\big((f_{\alpha_2})_0\big)^{\ell(\pi)}\big((f_{\alpha_1})_0\big)^{\ell(\rho)}\chi\Big)_0 v \\
&= (f_\theta)_\xi(f_{\alpha_2})_\pi(f_{\alpha_1})_\rho\chi_{-(\mathsf{u}-2)-N}v,
\end{aligned} \tag{2.3.27}$$

where the final equality is obtained by applying Lemma 2.3.11 to the middle term in (2.3.22). ∎

As (2.3.22) is nonzero by assumption, all parts of ξ, π and ρ must be 1. Imposing the constraints (2.3.23) on such partitions then gives

$$(f_\theta)_\xi(f_{\alpha_2})_\pi(f_{\alpha_1})_\rho\chi_{-(\mathsf{u}-2)-N}v = (f_\theta)_1^{\mathsf{u}-2-N}(f_{\alpha_2})_1^N(f_{\alpha_1})_1^N\chi_{-(\mathsf{u}-2)-N}v = v. \tag{2.3.28}$$

By rescaling χ again, if necessary, we arrive at following key result.

Proposition 2.3.13. *If N is the minimal integer such that $\chi_{-(\mathsf{u}-2)-N}v \neq 0$, then*

$$(f_{\alpha_2})_1^N(f_{\alpha_1})_1^N\chi_{-(\mathsf{u}-2)-N}v = (e_\theta)_{-1}^{\mathsf{u}-2-N}v. \tag{2.3.29}$$

By Lemma 2.3.9, the right-hand side of (2.3.29) is inexact when tensored with $|0\rangle$. To show that $\chi_{-(\mathsf{u}-2)-N}v$ is inexact, it suffices to replace the action of $(f_{\alpha_i})_1$ with elements that commute with the differential $d = d_0$, i.e. closed elements. From Section 2.1.1, in particular (2.1.9), we know four elements of C that are closed with respect to d. The two denoted by G^+ and G^- decompose as $f_{\alpha_2} + \dots$ and $f_{\alpha_1} + \dots$ respectively.

Lemma 2.3.14. *For all $i, j \in \mathbb{Z}_{\geqslant 0}$, we have*

$$(G^+_{(1/2)})^i(G^-_{(1/2)})^j\big(\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle\big) = (f_{\alpha_2})_1^i(f_{\alpha_1})_1^j\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle. \tag{2.3.30}$$

Proof. We begin with the G^- case. Taking the mode expansion of the decomposition of G^- in fields in $\mathsf{C} = \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \otimes \mathsf{G}$ from (2.1.9) gives

$$G^-_{(1/2)} = (f_{\alpha_1})_{(1/2)} - \sum_{m\in\mathbb{Z}} (h_{\alpha_1})_{(m)}\gamma_{-m+1/2} + \dots \tag{2.3.31}$$
$$= (f_{\alpha_1})_1 - \sum_{m\in\mathbb{Z}} (h_{\alpha_1})_m\gamma_{-m+1/2} + \dots,$$

where the ellipses stands for terms containing only G modes, all of which annihilate the ghost vacuum $|0\rangle \in \mathsf{G}$. Proceeding by induction,

$$G^-_{(1/2)}\Big((f_{\alpha_1})_1^j\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle\Big) = (f_{\alpha_1})_1^{j+1}\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle \tag{2.3.32}$$
$$- \sum_{m=1}^{\infty}(h_{\alpha_1})_m(f_{\alpha_1})_1^j\chi_{-(\mathsf{u}-2)-N}v \otimes \gamma_{-m+1/2}|0\rangle,$$

for any $j \in \mathbb{Z}_{\geqslant 0}$. For all $m \geqslant 1$, we have that $(h_{\alpha_2})_m v = 0$ as v is a highest-weight vector. Therefore the summands in the last term of (2.3.32) can be written as

$$(h_{\alpha_1})_m(f_{\alpha_1})_1^j\chi_{-(\mathsf{u}-2)-N}v = [(h_{\alpha_1})_m, (f_{\alpha_1})_1^j]\chi_{-(\mathsf{u}-2)-N}v + (f_{\alpha_1})_1^j[(h_{\alpha_1})_m, \chi_{-(\mathsf{u}-2)-N}]v. \tag{2.3.33}$$

These summands are all actually zero: the first commutator is a sum of terms obtained from $\big((f_{\alpha_1})_1\big)^j$ by replacing one of the $(f_{\alpha_1})_1$ by $-2(f_{\alpha_1})_{m+1}$ which are all zero by Lemma 2.3.12. The second term is proportional to $\chi_{-(\mathsf{u}-2)-(N-m)}v$ which is zero by minimality of N. Therefore

$$G^-_{(1/2)}\Big((f_{\alpha_1})_1^j\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle\Big) = (f_{\alpha_1})_1(f_{\alpha_1})_1^j\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle, \tag{2.3.34}$$

Iterating (2.3.34) gives the $G^- \leftrightarrow f_{\alpha_1}$ part of (2.3.30). The $G^+ \leftrightarrow f_{\alpha_2}$ part is proved in a similar way. ■

To summarise, $\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle$ is closed and

$$\big(G^+_{(1/2)}\big)^N\big(G^-_{(1/2)}\big)^N\big(\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle\big) \tag{2.3.35}$$

is inexact. Moreover, $[d, G^\pm_{(1/2)}]$ as $G^\pm$ is closed. Suppose that $\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle$ is exact. Then so too is

$$\big(G^+_{(1/2)}\big)^N\big(G^-_{(1/2)}\big)^N\big(\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle\big) = (f_{\alpha_2})_1^N(f_{\alpha_1})_1^N\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle = (e_\theta)_{-1}^{\mathsf{u}-2-N}v \tag{2.3.36}$$

But this contradicts Lemma 2.3.9. Therefore, $\chi_{-(\mathsf{u}-2)-N}v\otimes|0\rangle$ is closed and inexact. The homology class $\big[\chi_{-(\mathsf{u}-2)-N}v \otimes |0\rangle\big]$ is a nonzero element of $H^0(\mathsf{I}^{\mathsf{k}}) \cdot H^0(\mathcal{L}_\lambda)$ and Theorem 2.3.8 is proved.

2.3.3. Simple Highest-Weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-Modules. By combining the results of Section 2.3.1 with Proposition 2.2.2, we arrive at the following classification result.

Theorem 2.3.15. *Let* k *be admissible with* $\mathsf{v} \geqslant 2$*. Then the simple highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules are, up to isomorphism,*

- *(Untwisted)* $\mathcal{H}_{j,\Delta}$*, where* j *and* Δ *are determined from the Dynkin labels of a surviving weight* $\lambda \in \Sigma_{\mathsf{k}}$ *by* (2.3.2).
- *(Twisted)* $\mathcal{H}^{\mathrm{tw}}_{j,\Delta}$*, where* j *and* Δ *are determined from the Dynkin labels of a surviving weight* $\lambda \in \Sigma_{\mathsf{k}}$ *by*

$$\begin{aligned} j &= \frac{\lambda_1 - \lambda_2}{3} + \frac{2\mathsf{k}+3}{6}, \\ \Delta &= \frac{(\lambda_1-\lambda_2)^2 - 3(\lambda_1+\lambda_2)\big(2(\mathsf{k}+1)-\lambda_1-\lambda_2\big)}{12(\mathsf{k}+3)} + \frac{\lambda_1-\lambda_2}{6} + \frac{2\mathsf{k}+3}{24}. \end{aligned} \tag{2.3.37}$$

Moreover, the $\mathcal{H}_{j,\Delta}$ *and* $\mathcal{H}^{\mathrm{tw}}_{j,\Delta}$ *determined by the surviving weights are all mutually nonisomorphic.*

In light of this classification, we let $\mathcal{H}_\lambda = \mathcal{H}_{j,\Delta}$ and $\mathcal{H}^{\mathrm{tw}}_\lambda = \mathcal{H}^{\mathrm{tw}}_{j,\Delta}$, where j and Δ are given in terms of $\lambda \in \Sigma_{\mathsf{k}}$ by (2.3.2) and (2.3.37), respectively. Note that this implies that

$$\mathcal{H}^{\mathrm{tw}}_\lambda \simeq \sigma^{1/2}(\mathcal{H}_\lambda), \tag{2.3.38}$$

by Proposition 2.2.2. With this new notation, the vacuum module is $\mathcal{H}_{0,0} = \mathcal{H}_{\mathsf{k}\omega_0}$.

Nothing *a priori* guarantees that all highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules are simple. Indeed reducible $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules are expected to make an appearance eventually as this work is motivated by constructing logarithmic conformal field theories. These hopes (at least for highest-weight modules) are quickly dashed by the following strengthening of Theorem 2.3.15, following [17, Thm. 10.10].

Theorem 2.3.16. *Let* k *be admissible with* $\mathsf{v} \geqslant 2$*. Then, every highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-module, untwisted or twisted, is simple.*

Proof. We prove this for untwisted modules as the twisted case follows immediately from (2.3.38) and the invertibility of spectral flow. Since the simple quotient of any highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module $\mathcal{H}$ is isomorphic to some $\mathcal{H}_\lambda$ with $\lambda \in \Sigma_{\mathsf{k}}$, by Theorem 2.3.15, it is enough to show that $\mathcal{H}$ cannot have a composition factor isomorphic to $\mathcal{H}_\mu$ for some $\mu \in \Sigma_{\mathsf{k}}$ distinct from λ. Indeed, it is enough to show that the Verma module $\mathcal{V}_\lambda = \mathcal{V}_{j,\Delta}$ of BP^{k} does not have such a composition factor.

Recall that $\mathcal{K}_\lambda$ denotes the Verma module of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ of highest weight λ and let $\big[\mathcal{K}_\lambda : \mathcal{L}_\nu\big]$ denote the multiplicity with which $\mathcal{L}_\nu$ appears as a composition factor of $\mathcal{K}_\lambda$. By Theorem 2.3.4, quantum hamiltonian reduction takes $\mathcal{K}_\lambda$ to $\mathcal{V}_\lambda$ and only $\mathcal{L}_\mu$ and $\mathcal{L}_{w_0\cdot\mu}$ are sent to $\mathcal{H}_\mu$. As reduction is exact, we must have $\left[\mathcal{V}_\lambda : \mathcal{H}_\mu\right] = \left[\mathcal{K}_\lambda : \mathcal{L}_\mu\right] + \left[\mathcal{K}_\lambda : \mathcal{L}_{w_0\cdot\mu}\right]$ (noting that μ and $w_0 \cdot \mu$ are distinct since $\mu \in \Sigma_{\mathsf{k}}$).

It follows that if $\mathcal{V}_\lambda$ has $\mathcal{H}_\mu$, $\mu \neq \lambda$, as a composition factor, then $\mathcal{K}_\lambda$ has either $\mathcal{L}_\mu$ or $\mathcal{L}_{w_0\cdot\mu}$ as a composition factor. But, λ, μ and $w_0 \cdot \mu$ are all admissible $\widehat{\mathfrak{sl}}_3$-weights (corresponding to $w = \mathbb{1}$, $\mathbb{1}$ and w_1, respectively, see Lemma 2.3.6), hence they are dominant. This is therefore impossible by the linkage principle for Verma $\widehat{\mathfrak{sl}}_3$-modules. ∎

Because the Bernšteĭn–Gel'fand–Gel'fand category $\mathscr{O}_{\mathsf{u},\mathsf{v}}$ of $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules (i.e. the category of highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules) admits contragredient duals, it follows from Theorem 2.3.16 that every extension between nonisomorphic highest-weight modules $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ splits. It is likewise easy to see that a nonsplit self-extension of $\mathcal{H}_\lambda$ requires a nonsemisimple action of J_0 or L_0, and such an extension is necessarily not in $\mathscr{O}_{\mathsf{u},\mathsf{v}}$. Therefore, $\mathscr{O}_{\mathsf{u},\mathsf{v}}$ is semisimple and has finitely many isomorphism classes of simple objects by Theorem 2.3.15. In other words, $\mathsf{BP}(\mathsf{u},\mathsf{v})$ is *rational in category* $\mathscr{O}_{\mathsf{u},\mathsf{v}}$.

The twisted modules appearing in Theorem 2.3.15 have top spaces that may or may not be finite-dimensional. By Proposition 2.2.10, the dimensionality of the top space of $\mathcal{H}^{\mathrm{tw}}_\lambda = \mathcal{H}^{\mathrm{tw}}_{j,\Delta}$ is determined by a remarkable relationship between the polynomial h^n_{k} from Proposition 2.2.10 and the eigenvalue formulae (2.3.37).

Proposition 2.3.17. *The top space of the simple twisted highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-module* $\mathcal{H}^{\mathrm{tw}}_\lambda$ *is finite-dimensional if and only if* $\lambda^F_1 = 0$*. When* $\lambda^F_1 = 0$*, the dimension of this top space is* $\lambda^I_1 + 1$.

Proof. By Proposition 2.2.10, $(\mathcal{H}^{\mathrm{tw}}_{j,\Delta})^{\mathrm{top}}$ is finite-dimensional if and only if $h^n_{\mathsf{k}}(j,\Delta) = 0$ for some $n \in \mathbb{Z}_{\geqslant 1}$ and, if it is finite-dimensional, then the dimension is the smallest such n. Substituting (2.3.37) into the definition (2.2.9) of h^n_{k} and simplifying, we find that

$$h^n_{\mathsf{k}}(j,\Delta) = n(n-\lambda_1-1)\Big(n+\lambda_2+1-\frac{\mathsf{u}}{\mathsf{v}}\Big). \tag{2.3.39}$$

The only roots in $\mathbb{Z}_{\geqslant 1}$ of h^n_{k} are therefore $n = \lambda_1+1$ and $n = \frac{\mathsf{u}}{\mathsf{v}} - \lambda_2 - 1$. Recall that surviving weights can be expanded as $\lambda = \lambda^I - \frac{\mathsf{u}}{\mathsf{v}}\lambda^F$ where $\lambda^I \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$ and $\lambda^F \in \mathsf{P}^{\mathsf{v}-1}_{\geqslant}$. If $n = \lambda_1 + 1$, we must therefore have $\lambda^F_1 = 0$ and therefore $n = \lambda^I_1 + 1 \in \mathbb{Z}_{\geqslant 1}$. On the other hand, if $n = -(\lambda^I_2 + 1) + \frac{\mathsf{u}}{\mathsf{v}}(\lambda^F_2 + 1)$ is an integer then $\lambda^F_2 = \mathsf{v} - 1$ which contradicts the fact that λ is a surviving weight, i.e. $\lambda^F_0 \geqslant 1$. ∎

Corollary 2.3.18. *Given* k *admissible with* $\mathsf{v} \geqslant 2$*, there are (up to isomorphism):*

- $\frac{1}{4}(\mathsf{u}-1)(\mathsf{u}-2)\mathsf{v}(\mathsf{v}-1)$ *simple untwisted highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules;*
- $\frac{1}{2}(\mathsf{u}-1)(\mathsf{u}-2)(\mathsf{v}-1)$ *simple twisted highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules that have finite-dimensional top spaces;*

- $\frac{1}{4}(\mathsf{u}-1)(\mathsf{u}-2)(\mathsf{v}-1)(\mathsf{v}-2)$ *simple twisted highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules that have infinite-dimensional top spaces;*

It is easy to see that there are no simple twisted highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules with infinite-dimensional top spaces when $\mathsf{v}=2$. This is in accord with the fact that the $\mathsf{BP}(\mathsf{u},2)$ with $\mathsf{u}\geqslant 3$ are rational and C_2-cofinite [**15**] as the top spaces of modules for such a vertex operator algebra are necessarily finite-dimensional [**56**].

Recall that the conjugation automorphism γ of $\mathsf{BP}(\mathsf{u},\mathsf{v})$, given in (2.1.18), negates J_0 and preserves L_0. At the level of their eigenvalues, this is effected in (2.3.2) by exchanging the Dynkin labels λ_1 and λ_2 of λ. The result of this exchange is clearly still a surviving weight, by Lemma 2.3.6.

Proposition 2.3.19. *For each $\lambda\in\Sigma_{\mathsf{k}}$, we have:*

- $\gamma\big(\mathcal{H}_{[\lambda_0,\lambda_1,\lambda_2]}\big)\simeq\mathcal{H}_{[\lambda_0,\lambda_2,\lambda_1]}$.
- *If $\lambda_1^F=0$, then $\gamma\big(\mathcal{H}_\lambda^{\mathrm{tw}}\big)\simeq\mathcal{H}_\mu^{\mathrm{tw}}$, where $\mu=[\lambda_2-\frac{\mathsf{u}}{\mathsf{v}},\lambda_1,\lambda_0+\frac{\mathsf{u}}{\mathsf{v}}]$, hence $\mu^I=[\lambda_2^I,\lambda_1^I,\lambda_0^I]$ and $\mu^F=[\lambda_2^F+1,0,\lambda_0^F-1]$. Otherwise, $\gamma\big(\mathcal{H}^{\mathrm{tw}}_{[\lambda_0,\lambda_1,\lambda_2]}\big)$ is not highest-weight (though it is relaxed highest-weight).*

Proof. The result of conjugating a simple untwisted highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module is clear from the above remarks, because the top spaces are one-dimensional.

For the twisted case, it is clear that the conjugate of $\mathcal{H}_\lambda^{\mathrm{tw}}$ is highest-weight only if its top space is finite-dimensional (otherwise the top space of the conjugate module will be an infinite-dimensional lowest-weight Z_{k}-module). By Proposition 2.3.17, the top space is finite-dimensional if and only if $\lambda_1^F=0$. Assuming this, recall that the charge j and conformal weight Δ of the highest-weight vector of $\mathcal{H}_\lambda^{\mathrm{tw}}$ are related to λ by (2.3.37). The highest-weight vector of $\gamma\big(\mathcal{H}_\lambda^{\mathrm{tw}}\big)$ has charge λ_1-j and conformal weight Δ as the dimension of the top space of $\mathcal{H}_\lambda^{\mathrm{tw}}$ is λ_1+1.

We therefore need to find $\mu\in\Sigma_{\mathsf{k}}$ corresponding to the charge λ_1-j and conformal weight Δ under (2.3.37). Solving for μ in term of λ, we find two solutions:

$$\mu=[\lambda_2-\mathsf{k}-3,\lambda_1,\lambda_0+\mathsf{k}+3]\ \text{ and }\ \mu=[\mathsf{k}+1-\lambda_2,-\lambda_0-2,\mathsf{k}+1-\lambda_1]\,. \tag{2.3.40}$$

It is easy to check that the first solution is a $w=\mathbb{1}$ surviving weight by writing it in the form

$$\mu_0=\lambda_2^I-\frac{\mathsf{u}}{\mathsf{v}}(\lambda_2^F+1),\quad \mu_1=\lambda_1^I\quad\text{and}\quad \mu_2=\lambda_0^I-\frac{\mathsf{u}}{\mathsf{v}}(\lambda_0^F-1). \tag{2.3.41}$$

Indeed, $\lambda_0^F\geqslant 1$ implies that $\mu^I=[\lambda_2^I,\lambda_1^I,\lambda_0^I]\in\mathsf{P}_{\geqslant}^{\mathsf{u}-3}$, $\mu^F=[\lambda_2^F+1,0,\lambda_0^F-1]\in\mathsf{P}_{\geqslant}^{\mathsf{v}-1}$ and $\mu_0^F\geqslant 1$, hence that $\mu\in\Sigma_{\mathsf{k}}$. The second solution is a $w=w_1$ surviving weight obtained from the

$w = \mathbb{1}$ one by applying the shifted action of w_0 and therefore defines the same $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module by Lemma 2.3.6. ∎

To determine when the spectral flow of a simple highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module is another such module, it suffices to consider the untwisted case by Proposition 2.2.2. The twisted sector still plays a crucial role here as $\sigma(\mathcal{H}_\lambda)$ will be highest-weight if and only if $\mathcal{H}_\lambda^{\mathrm{tw}} = \sigma^{1/2}(\mathcal{H}_\lambda)$ has a finite-dimensional top space. By Proposition 2.3.17, this is the case if and only if $\lambda_1^F = 0$

Proposition 2.3.20. *If $\lambda \in \Sigma_{\mathsf{k}}$ satisfies $\lambda_1^F = 0$, then $\sigma(\mathcal{H}_\lambda) \simeq \mathcal{H}_\mu$, where $\mu = [\lambda_2 - \frac{\mathsf{u}}{\mathsf{v}}, \lambda_0 + \frac{\mathsf{u}}{\mathsf{v}}, \lambda_1] \in \Sigma_{\mathsf{k}}$, hence $\mu^I = [\lambda_2^I, \lambda_0^I, \lambda_1^I]$ and $\mu^F = [\lambda_2^F + 1, \lambda_0^F - 1, 0]$. If $\lambda_1^F \neq 0$, then $\sigma(\mathcal{H}_\lambda)$ is not highest-weight (nor relaxed highest-weight).*

Proof. Let $\lambda \in \Sigma_{\mathsf{k}}$ with $\lambda_1^F = 0$ and denote by v the highest-weight vector of $\mathcal{H}_\lambda$. The highest-weight vector of $\sigma(\mathcal{H}_\lambda)$ is easily checked to be $(G^-_{\overline{1}/2})^{\lambda_1^I}\sigma(v)$. The charge and conformal weight of $(G^-_{\overline{1}/2})^{\lambda_1^I}\sigma(v)$ is computed using Proposition 2.2.2. The weight μ is the unique $w = \mathbb{1}$ surviving weight that gives these eigenvalues under (2.3.2), as in the proof of Proposition 2.3.19. ∎

Combining this with the dihedral relation (2.1.17) and Proposition 2.3.19, we obtain the following characterisation of the spectral flow orbit of a simple untwisted highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module $\mathcal{H}_\lambda$. We recall from Proposition 2.2.2 that a twisted member $\sigma^{\ell+1/2}(\mathcal{H}_\lambda)$, $\ell \in \mathbb{Z}$, of this orbit is highest-weight if and only if its untwisted predecessor $\sigma^\ell(\mathcal{H}_\lambda)$ is.

Theorem 2.3.21. *Take $\lambda \in \Sigma_{\mathsf{k}}$ and define $\mu, \nu, \bar{\mu}, \bar{\nu} \in \Sigma_{\mathsf{k}}$ by*

$$\begin{aligned} \mu^I &= [\lambda_2^I, \lambda_0^I, \lambda_1^I], & \mu^F &= [\lambda_2^F + 1, \lambda_0^F - 1, 0], & & & \nu^I &= [\lambda_1^I, \lambda_2^I, \lambda_0^I], & \nu^F &= [1, \mathsf{v} - 2, 0], \\ \bar{\mu}^I &= [\lambda_1^I, \lambda_2^I, \lambda_0^I], & \bar{\mu}^F &= [\lambda_1^F + 1, 0, \lambda_0^F - 1] & &\text{and} & \bar{\nu}^I &= [\lambda_2^I, \lambda_0^I, \lambda_1^I], & \bar{\nu}^F &= [1, 0, \mathsf{v} - 2]. \end{aligned} \tag{2.3.42}$$

- *$\sigma(\mathcal{H}_\lambda)$ is highest-weight if and only if $\lambda_1^F = 0$. In this case, $\sigma(\mathcal{H}_\lambda) \simeq \mathcal{H}_\mu$.*
- *$\sigma^{-1}(\mathcal{H}_\lambda)$ is highest-weight if and only if $\lambda_2^F = 0$. In this case, $\sigma^{-1}(\mathcal{H}_\lambda) \simeq \mathcal{H}_{\bar{\mu}}$.*
- *$\sigma^2(\mathcal{H}_\lambda)$ is highest-weight if and only if $\lambda^F = [1, 0, \mathsf{v} - 2]$. In this case, $\sigma^2(\mathcal{H}_\lambda) \simeq \mathcal{H}_\nu$.*
- *$\sigma^{-2}(\mathcal{H}_\lambda)$ is highest-weight if and only if $\lambda^F = [1, \mathsf{v} - 2, 0]$. In this case, $\sigma^{-2}(\mathcal{H}_\lambda) \simeq \mathcal{H}_{\bar{\nu}}$.*
- *For $|\ell| \in \mathbb{Z}_{\geqslant 3}$, $\sigma^\ell(\mathcal{H}_\lambda)$ is highest-weight if and only if $\mathsf{v} = 2$. In this case, $\sigma^{\pm 3}(\mathcal{H}_\lambda) \simeq \mathcal{H}_\lambda$.*

Note that when $\mathsf{v} = 2$, every $\lambda \in \Sigma_{\mathsf{k}}$ has $\lambda^F = [1, 0, 0]$. The spectral flow orbits thus take the form

$$\cdots \xmapsto{\sigma^{1/2}} \mathcal{H}_\lambda \xmapsto{\sigma^{1/2}} \mathcal{H}_\lambda^{\mathrm{tw}} \xmapsto{\sigma^{1/2}} \mathcal{H}_\mu \xmapsto{\sigma^{1/2}} \mathcal{H}_\mu^{\mathrm{tw}} \xmapsto{\sigma^{1/2}} \mathcal{H}_\nu \xmapsto{\sigma^{1/2}} \mathcal{H}_\nu^{\mathrm{tw}} \xmapsto{\sigma^{1/2}} \mathcal{H}_\lambda \xmapsto{\sigma^{1/2}} \cdots, \tag{2.3.43}$$

where μ and ν are as in (2.3.42) (with $\mu^F = \nu^F = [1, 0, 0]$).

It follows from Theorem 2.3.21 that, for k admissible with $\mathsf{v} \geqslant 3$, the spectral flow orbit of a simple highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module always contains exactly one simple twisted highest-weight module with an infinite-dimensional top space and exactly one simple twisted conjugate highest-weight module with an infinite-dimensional top space.

Definition 2.3.22. *Let* k *be admissible with* $\mathsf{v} \geqslant 3$. *We say that* $\lambda \in \Sigma_{\mathsf{k}}$ *is* type-n *whenever the spectral flow orbit* $\{\sigma^{\ell}(\mathcal{H}_\lambda) : \ell \in \frac{1}{2}\mathbb{Z}\}$ *contains precisely n (untwisted) highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules. In this case, we shall also refer to the spectral flow orbit of* $\mathcal{H}_\lambda$*, as well as any twisted or untwisted module isomorphic to one in the orbit, as being of type-n.*

When $\mathsf{v} = 3$, all the twisted and untwisted highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules are type-3. On the other hand, for $\mathsf{v} > 3$, there are $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules of every type. The vacuum module $\mathcal{H}_{\mathsf{k}\omega_0}$ is always an untwisted type-3 module.

Corollary 2.3.23. *Let* k *be admissible with* $\mathsf{v} \geqslant 3$. *Then, every type-n module is isomorphic to a unique* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-module of the form* $\sigma^{\ell}(\mathcal{H}_\lambda)$*, for some* $\ell \in \frac{1}{2}\mathbb{Z}$*, where* $\lambda \in \Sigma_{\mathsf{k}}$ *satisfies one of the following conditions:*

$n = 1$	$n = 2$	$n = 3$
$\lambda_1^F, \lambda_2^F \neq 0$	$\lambda_1^F = 0$ *and* $\lambda_2^F \neq 0, \mathsf{v}-2$	$\lambda_1^F = 0$ *and* $\lambda_2^F = \mathsf{v}-2$

We visualise the type-n spectral flow orbits in Figure 1. The representatives chosen in Corollary 2.3.23 are the leftmost for each type in this figure.

2.3.4. Relaxed Highest-Weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-Modules. Having completed the classification of simple highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules, it remains to classify relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules. As every simple untwisted relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module is highest-weight, the classification of simple untwisted relaxed highest-weight modules was completed in Theorem 2.3.15.

Recall from Section 2.2.5 that there exist simple twisted BP^{k}-modules whose top spaces are not highest-weight Z_{k}-modules. The $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules whose top spaces are simple lowest-weight Z_{k}-modules are conjugates of the simple twisted highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules classified in Theorem 2.3.15. By Theorem 2.2.16 what remains is to determine when the simple BP^{k}-module $\mathcal{R}^{\mathrm{tw}}_{[j],\Delta,\omega}$ is a $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module.

A simple twisted relaxed highest-weight BP^{k}-module $\mathcal{M}$ is a BP_{k}-module if and only if its top space $\mathcal{M}^{\mathrm{top}} = \mathsf{Zhu}^{\mathrm{tw}}[\mathcal{M}]$ is annihilated by $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{J}^{\mathsf{k}}]$, where J^{k} denotes the maximal ideal of BP^{k}.

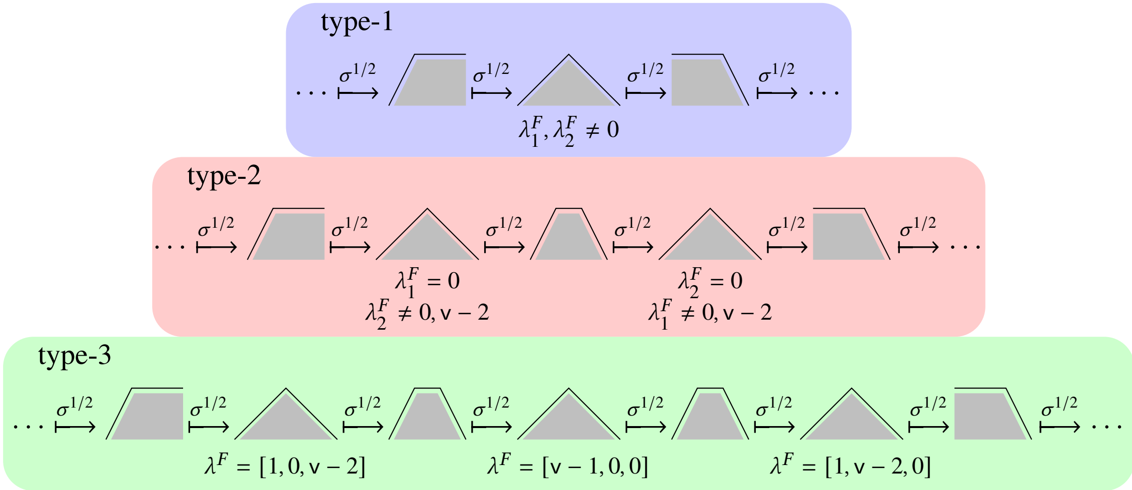

FIGURE 1. A picture of the weights of the three types of spectral flow orbits through a simple highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module for k admissible with $\mathsf{v} \geqslant 3$. The J_0-eigenvalue increases from left to right, whilst the L_0-eigenvalue increases from top to bottom. The conditions stated for the Dynkin labels of λ^F constrain the surviving weight $\lambda \in \Sigma_\mathsf{k}$ of the corresponding untwisted module. The $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules encompassed by the ellipses in these spectral flow orbits are not relaxed highest-weight modules

A consequence of Theorem 2.3.15 is that $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{J}^\mathsf{k}\big]$ annihilates $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathcal{H}_\lambda^{\mathrm{tw}}\big] \simeq \overline{\mathcal{H}}_{j,\Delta}$, with j and Δ determined by λ as in (2.3.37), if and only if $\lambda \in \Sigma_\mathsf{k}$. We extend this to the simple relaxed highest-weight modules $\mathcal{R}^{\mathrm{tw}}_{[j],\Delta,\omega}$ of Theorem 2.2.16 using an argument similar to that of [114, Prop. 4.2]. As there, the crucial objects in this analysis are coherent families of Z_k-modules from Section 2.2.6.

Proposition 2.3.24. *The irreducible semisimple coherent family* $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ *of* $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}^\mathsf{k}\big]$*-modules is a* $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$*-module if and only if one of its infinite-dimensional submodules is.*

PROOF. It is clear that $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ being a $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$-module implies that every one of its submodules is too, in particular the infinite-dimensional ones.

Following closely the general methodology developed in [114], consider the subalgebra $\mathsf{A}_\mathsf{k} = \mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{J}^\mathsf{k}\big] \cap \mathsf{C}_\mathsf{k}$, where we recall that $\mathsf{C}_\mathsf{k} = \mathbb{C}[J, L, \Omega]$ (Lemma 2.2.13). A given simple weight $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}^\mathsf{k}\big]$-module $\mathcal{M}$ is a $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$-module if and only if A_k annihilates some nonzero element of $\mathcal{M}$. This fact is proved in exactly the same way as the affine version in [114, Lem. 4.1].

For each $a \in \mathsf{A}_\mathsf{k} \subset \mathbb{C}[J, L, \Omega]$, there is a polynomial p_a in three variables such that a acts on the weight space $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}(j, \Delta, \omega)$ as multiplication by $p_a(j, \Delta, \omega)$. After choosing a coherent family $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$, Δ and ω are fixed and p_a can be treated as a single-variable polynomial in j.

Suppose one of the infinite-dimensional submodules of $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ is a $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$-module. That is, it is annihilated by $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{J}^\mathsf{k}\big]$ and thus by A_k. Therefore, for every $a \in \mathsf{A}_\mathsf{k}$, we have

$p_a(j,\Delta,\omega) = 0$ for infinitely many distinct values of j. As $p_a(-,\Delta,\omega)$ is a single-variable polynomial in j, it must be the zero polynomial. So all $a \in \mathsf{A}_\mathsf{k}$ act as zero on $\overline{\mathcal{C}}^{ss}_{\Delta,\omega}$ and therefore $\overline{\mathcal{C}}^{ss}_{\Delta,\omega}$ is a $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$-module. ∎

The top space of every (simple) $\mathcal{R}^{\mathrm{tw}}_{[j],\Delta,\omega}$ embeds into some irreducible semisimple coherent family and every such family has an infinite-dimensional *highest-weight* submodule $\overline{\mathcal{H}}_{j',\Delta}$, by Proposition 2.2.18. From Theorem 2.3.15, we have classified all the simple highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules in terms of surviving weights. By Proposition 2.3.24, the irreducible semisimple coherent families that are $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$-modules are precisely those containing simple *highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules whose top spaces are infinite-dimensional:

Let Γ_k denote the set of ($w = \mathbb{1}$) admissible $\widehat{\mathfrak{sl}}_3$-weights λ of level k with $\lambda_0^F \neq 0$, so that $\lambda \in \Sigma_\mathsf{k}$ (Lemma 2.3.6), and $\lambda_1^F \neq 0$, i.e. that $\mathcal{H}^{\mathrm{tw}}_\lambda$ has an infinite-dimensional top space (Proposition 2.3.17). Then, Γ_k parametrises the isomorphism classes of the simple highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules with infinite-dimensional top spaces.

For each $\lambda \in \Gamma_\mathsf{k}$, compute j and Δ using (2.3.37), then substitute into (2.2.17) to compute ω:

$$\omega = \omega^+_{j,\Delta} = -\frac{2}{27}(\lambda_1 - \lambda_2 + \mathsf{k} + 3)(2\lambda_1 + \lambda_2 - \mathsf{k})(\lambda_1 + 2\lambda_2 - 2\mathsf{k} - 3). \tag{2.3.44}$$

This gives the eigenvalues of J, L and Ω on the highest-weight vector of $(\mathcal{H}^{\mathrm{tw}}_\lambda)^{\mathrm{top}}$. The $\mathcal{R}^{\mathrm{tw}}_{[j'],\Delta,\omega}$ are, for all $[j'] \in \mathbb{C}/\mathbb{Z}$ satisfying $\omega^+_{i,\Delta} \neq \omega$ for every $i \in [j']$, simple relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules (by Theorem 2.2.16 and Proposition 2.3.24) and all such modules are obtained, up to isomorphism, in this way.

Theorem 2.3.25. *Let* k *be admissible with* $\mathsf{v} \geqslant 3$ *and let* j *be such that* $\mathcal{R}^{\mathrm{tw}}_{[j],\lambda} = \mathcal{R}^{\mathrm{tw}}_{[j],\Delta,\omega}$ *(where* Δ *and* ω *are determined by* λ*) is simple. Then,* $\mathcal{R}^{\mathrm{tw}}_{[j],\lambda}$ *is a (twisted)* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-module if and only if* $\lambda \in \Gamma_\mathsf{k}$.

Recall from Corollary 2.3.18 (and [15]) that there are no highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules with infinite-dimensional top spaces when $\mathsf{v} = 2$.

Corollary 2.3.26. *Let* k *be admissible with* $\mathsf{v} = 2$*. Then, every simple (twisted) relaxed highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-module is highest-weight.*

For the remainder of this chapter, we will restrict attention to admissible levels of the form

$$\mathsf{k} = -3 + \frac{\mathsf{u}}{\mathsf{v}} \text{ with } \mathsf{u},\mathsf{v} \geqslant 3 \tag{2.3.45}$$

to ensure that $\mathsf{BP}(\mathsf{u},\mathsf{v})$ admits modules whose top spaces are dense Z_k-modules. The levels of (2.3.45) are also known as *nondegenerate admissible levels*.

Suppose that $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ is a $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$-module. By Proposition 2.2.14, the direct summands $\overline{\mathcal{R}}_{[j],\Delta,\omega}$ are not simple for at least one, and at most three, $[j] \in \mathbb{C}/\mathbb{Z}$ and each nonsimple summand has precisely one infinite-dimensional highest-weight submodule.

Lemma 2.3.27. *If* k *is nondegenerate admissible, then each irreducible semisimple coherent family* $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ *of* $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$*-modules has precisely three infinite-dimensional highest-weight submodules. The map* $\Gamma_\mathsf{k} \to \mathbb{C}^2$ *given by* $\lambda \mapsto (\Delta,\omega)$ *is thus* 3*-to-*1*. Moreover, the highest weights* $\lambda = \lambda^I - \frac{\mathsf{u}}{\mathsf{v}}\lambda^F$ *of these three submodules are related by the following* $\mathbb{Z}_3$*-action generated by the action* $\nabla(\lambda) = [\lambda_2 - \frac{\mathsf{u}}{\mathsf{v}}, \lambda_0, \lambda_1 + \frac{\mathsf{u}}{\mathsf{v}}]$*.*

The orbit of a given $\lambda \in \Gamma_\mathsf{k}$ *(as well as its fractional and integral parts) under* ∇ *is given by*

$$
\begin{gathered}
\cdots \longmapsto [\lambda_0,\lambda_1,\lambda_2] \longmapsto [\lambda_2 - \tfrac{\mathsf{u}}{\mathsf{v}}, \lambda_0, \lambda_1 + \tfrac{\mathsf{u}}{\mathsf{v}}] \longmapsto [\lambda_1, \lambda_2 \tfrac{\mathsf{u}}{\mathsf{v}}, \lambda_0 + \tfrac{\mathsf{u}}{\mathsf{v}}] \longmapsto \cdots, \\
\cdots \longmapsto [\lambda_0^I,\lambda_1^I,\lambda_2^I] \longmapsto [\lambda_2^I,\lambda_0^I,\lambda_1^I] \longmapsto [\lambda_1^I,\lambda_2^I,\lambda_0^I] \longmapsto \cdots, \\
\cdots \longmapsto [\lambda_0^F,\lambda_1^F,\lambda_2^F] \longmapsto [\lambda_2^F+1,\lambda_0^F,\lambda_1^F-1] \longmapsto [\lambda_1^F,\lambda_2^F+1,\lambda_0^F-1] \longmapsto \cdots.
\end{gathered}
\tag{2.3.46}
$$

Proof. That the weights obtained from $\lambda \in \Gamma_\mathsf{k}$ remain in Γ_k under the $\mathbb{Z}_3$-action is clear. Therefore, the three highest-weight BP^k-modules corresponding to the distinct $\mathfrak{sl}_3$ weights $\nabla^i(\lambda)$, $i \in \{0,1,2\}$ are $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules with infinite-dimensional top spaces if any is.

That Δ and ω are invariant under ∇ can be checked by direct computation. The three highest-weight modules therefore arise as submodules of the same irreducible semisimple coherent family, and are nonisomorphic as their highest weights can only coincide if both u and v are divisible by 3. ■

By combining (2.1.18), Lemma 2.2.13 and Proposition 2.3.24, we have the following set of results.

Theorem 2.3.28. *Let* k *be nondegenerate admissible. Then:*

- *There are* $\frac{1}{3}|\Gamma_\mathsf{k}| = \frac{1}{12}(\mathsf{u}-1)(\mathsf{u}-2)(\mathsf{v}-1)(\mathsf{v}-2)$ *irreducible semisimple coherent families of* $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$*-modules* $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$*, up to isomorphism.*
- *The families of twisted relaxed highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules* $\mathcal{R}^{\mathrm{tw}}_{[j],\lambda} = \mathcal{R}^{\mathrm{tw}}_{[j],\Delta,\omega}$ *are in* 1*-to-*1 *correspondence with* $\Gamma_\mathsf{k}/\mathbb{Z}_3$*, where* $\mathbb{Z}_3$ *acts freely as in* (2.3.46)*.*
- *For each* $\lambda \in \Gamma_\mathsf{k}$*, the twisted relaxed highest-weight module* $\mathcal{R}^{\mathrm{tw}}_{[j],\lambda}$ *is a simple* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-module for all cosets* $[j] \in \mathbb{C}/\mathbb{Z}$ *except three, namely the three distinct cosets that contain a root i of the polynomial* $\omega^+_{i,\Delta} - \omega$*.*

- *The conjugate of the simple twisted relaxed highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-module* $\mathcal{R}^{\text{tw}}_{[j],\Delta,\omega}$ *is also a simple twisted relaxed highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-module:* $\gamma(\mathcal{R}^{\text{tw}}_{[j],\Delta,\omega}) \simeq \mathcal{R}^{\text{tw}}_{[-j],\Delta,-\omega}$.

This completes the classification of simple relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules. Note that if (Δ,ω) corresponds to a coherent family of $\mathsf{Zhu}^{\text{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-modules, then closure under conjugation requires that so must $(\Delta,-\omega)$. In fact, it is easy to check that under the Γ_{k}-preserving $\mathbb{Z}_2$-action

$$\begin{aligned}
[\lambda_0,\lambda_1,\lambda_2] &\longleftrightarrow [\lambda_2-\frac{\mathsf{u}}{\mathsf{v}},\lambda_1,\lambda_0+\frac{\mathsf{u}}{\mathsf{v}}],\\
[\lambda^I_0,\lambda^I_1,\lambda^I_2] &\longleftrightarrow [\lambda^I_2,\lambda^I_1,\lambda^I_0],\\
[\lambda^F_0,\lambda^F_1,\lambda^F_2] &\longleftrightarrow [\lambda^F_2+1,\lambda^I_1,\lambda^I_0-1],
\end{aligned} \tag{2.3.47}$$

the associated weight (Δ,ω) gets mapped to $(\Delta,-\omega)$. With (2.3.46), this defines an action of the $\mathfrak{sl}_3$ Weyl group $\mathsf{W}=\mathsf{S}_3$ on Γ_{k}. The orbits clearly have length 6 unless $\omega=0$, in which case Lemma 2.3.27 forces them to have length 3.

The spectral flow images $\sigma^{\ell}(\mathcal{R}^{\text{tw}}_{[j],\lambda})$, $\ell\neq 0$, of these simple twisted relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules are likewise simple $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules, but they are not relaxed highest-weight because their conformal weights are not bounded below. However such modules will play a crucial role in the modular transformations and fusion rules of $\mathsf{BP}(\mathsf{u},\mathsf{v})$ in Chapter 3.

The irreducible semisimple Z_{k}-modules $\overline{\mathcal{C}}^{\text{ss}}_{\Delta,\omega}$ are not the only coherent families we know. Indeed in Section 2.2.6, we also introduced two other coherent families $\overline{\mathcal{C}}^{\pm}_{\Delta,\omega}$ that contain reducible-but-indecomposable submodules. Just as $\overline{\mathcal{C}}^{\text{ss}}_{\Delta,\omega}$ played a crucial role in classifying simple relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules, we will now see how the coherent families $\overline{\mathcal{C}}^{\pm}_{\Delta,\omega}$ allow us to construct nonsemisimple $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules. Here as before, assume k is nondegenerate admissible.

Recall that the simple direct summands of $\overline{\mathcal{C}}^{\pm}_{\Delta,\omega}$ are $\overline{\mathcal{R}}_{[j],\Delta,\omega}$, for all but (up to) three $[j]\in\mathbb{C}/\mathbb{Z}$, and that its nonsimple direct summands are denoted by $\overline{\mathcal{R}}^{\pm}_{[j],\Delta,\omega}$.

Proposition 2.3.29. *Let* $\lambda\in\Gamma_{\mathsf{k}}$ *and let* j*,* Δ *and* ω *be defined by* (2.3.37) *and* (2.3.44)*. Then, the nonsimple* Z_{k}*-module* $\overline{\mathcal{R}}^{\pm}_{[j],\Delta,\omega}$ *has exactly two composition factors,* $\overline{\mathcal{H}}_{j,\Delta}$ *and* $\overline{\gamma}(\overline{\mathcal{H}}_{-j-1,\Delta})$*, both of which are* $\mathsf{Zhu}^{\text{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$*-modules. Moreover, we have the following nonsplit short exact sequences:*

$$\begin{gathered}
0\longrightarrow \overline{\gamma}(\overline{\mathcal{H}}_{-j-1,\Delta})\longrightarrow \overline{\mathcal{R}}^{+}_{[j],\Delta,\omega}\longrightarrow \overline{\mathcal{H}}_{j,\Delta}\longrightarrow 0,\\
0\longrightarrow \overline{\mathcal{H}}_{j,\Delta}\longrightarrow \overline{\mathcal{R}}^{-}_{[j],\Delta,\omega}\longrightarrow \overline{\gamma}(\overline{\mathcal{H}}_{-j-1,\Delta})\longrightarrow 0.
\end{gathered} \tag{2.3.48}$$

Proof. We only consider $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$ as the argument for $\overline{\mathcal{R}}^-_{[j],\Delta,\omega}$ is identical. First, note that $\overline{\mathcal{H}}_{j,\Delta}$ is an infinite-dimensional $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-module, by Theorem 2.3.15. The irreducible semisimple coherent family $\overline{\mathcal{C}}^{\mathrm{ss}}_{\Delta,\omega}$ is therefore a $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-module too, by Proposition 2.3.24, hence so is the lowest-weight module $\overline{\gamma}(\overline{\mathcal{H}}_{-j-1,\Delta}) \subset \overline{\mathcal{R}}^{\mathrm{ss}}_{[j],\Delta,\omega}$. As $\overline{\mathcal{R}}^{\mathrm{ss}}_{[j],\Delta,\omega}$ is the semisimplification of $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$, they have the same composition factors. To demonstrate that there are no more factors beyond the two already found, it suffices to show that $\overline{\mathcal{H}}_{-j-1,\Delta}$ is infinite-dimensional. Since the conjugate of $\overline{\mathcal{H}}_{-j-1,\Delta}$ is a $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-module, $\overline{\mathcal{H}}_{-j-1,\Delta}$ must correspond to some $\mu \in \Sigma_{\mathsf{k}}$, by Theorem 2.3.15.

Proceeding as in the proof of Lemma 2.3.6, we find that the unique solution is $\mu = [\lambda_0, \lambda_2 - \frac{\mathsf{u}}{\mathsf{v}}, \lambda_1 + \frac{\mathsf{u}}{\mathsf{v}}]$, hence $\mu^I = [\lambda_0^I, \lambda_2^I, \lambda_1^I]$ and $\mu^F = [\lambda_0^F, \lambda_2^F + 1, \lambda_1^F - 1]$. Because $\mu_1^F = \lambda_2^F + 1 \neq 0$, it follows that $\mu \in \Gamma_{\mathsf{k}}$ and so $\overline{\mathcal{H}}_{-j-1,\Delta}$ is infinite-dimensional, as desired. This establishes the first exact sequence in (2.3.48). It is nonsplit because G^+ acts injectively on $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$ by construction. ■

It remains to show that the nonsimple $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}^{\mathsf{k}}]$-modules $\overline{\mathcal{R}}^\pm_{[j],\Delta,\omega}$ appearing in the short exact sequences (2.3.48) are actually $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-modules.

Proposition 2.3.30. *Let $\lambda \in \Gamma_{\mathsf{k}}$ and let j, Δ and ω be defined by (2.3.37) and (2.3.44). Then, the nonsimple Z_{k}-module $\overline{\mathcal{R}}^\pm_{[j],\Delta,\omega}$ is a $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-module.*

Proof. We use a simplified version of the argument in [114, Thm. 5.3]. We shall also only detail the argument for $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$. The highest-weight module in the short exact sequence (2.3.48) containing $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$ is a $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-module. In other words, $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{J}^{\mathsf{k}}] \cdot \overline{\mathcal{H}}_{j,\Delta} = 0$ where J^{k} is the maximal ideal of BP^{k}. The same short exact sequence implies that $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{J}^{\mathsf{k}}] \cdot \overline{\mathcal{R}}^+_{[j],\Delta,\omega} \subseteq \overline{\gamma}(\overline{\mathcal{H}}_{-j-1,\Delta})$, where $\overline{\gamma}(\overline{\mathcal{H}}_{-j-1,\Delta})$ is also a $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-module.

That Z_{k} is noetherian is an easy generalisation of [149, Cor. 1.3]. The ideal

$$\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{J}^{\mathsf{k}}] \subset \mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}^{\mathsf{k}}] \simeq \mathsf{Z}_{\mathsf{k}} \tag{2.3.49}$$

is therefore generated by a finite number of elements $a_1, \dots, a_n$ which we may, without loss of generality, choose to be eigenvectors of J. Let j_i denote the J-eigenvalue of a_i, $i = 1, \dots, n$.

Choose $j' \in [j]$ such that $j' \leqslant j - \max\{j_1, \dots, j_n\}$. Then, a_i takes the J-eigenspace of $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$ of eigenvalue j' into the J-eigenspace of $\overline{\gamma}(\overline{\mathcal{H}}_{-j-1,\Delta})$ of eigenvalue $j' + a_i \leqslant j$. But, the eigenvalues of J acting on $\overline{\gamma}(\overline{\mathcal{H}}_{-j-1,\Delta})$ are bounded below by $j+1$, hence a_i annihilates the J-eigenspace of $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$ of eigenvalue j', for each i. It follows that $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{J}^{\mathsf{k}}]$ annihilates this eigenspace. As this eigenspace generates $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$, this shows that $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{J}^{\mathsf{k}}]$ (being an ideal) annihilates $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$ and therefore $\overline{\mathcal{R}}^+_{[j],\Delta,\omega}$ is a $\mathsf{Zhu}^{\mathrm{tw}}[\mathsf{BP}(\mathsf{u},\mathsf{v})]$-module. ■

Using the (twisted) Zhu induction functor discussed in Section 1.1.4, we can induce twisted relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules from each of the $\mathsf{Zhu}^{\mathrm{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$-modules $\overline{\mathcal{R}}^{\pm}_{[j],\Delta,\omega}$.

Consider the submodule of this induced module obtained by summing all the submodules whose intersection with the top space $\overline{\mathcal{R}}^{\pm}_{[j],\Delta,\omega}$ is zero. Quotienting by this submodule results in a twisted $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module which we shall denote by $\mathcal{R}^{\mathrm{tw},\pm}_{[j],\Delta,\omega}$ (or $\mathcal{R}^{\mathrm{tw},\pm}_{\lambda}$ for short as j, Δ and ω are determined by some $\lambda \in \Gamma_{\mathsf{k}}$), that has $\overline{\mathcal{R}}^{\pm}_{[j],\Delta,\omega}$ as its top space and has the property that its nonzero submodules intersect this top space nontrivially. The $\mathcal{R}^{\mathrm{tw},\pm}_{\lambda}$ are clearly nonsemisimple $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules, because their top spaces are.

Theorem 2.3.31. *When* k *is nondegenerate admissible, the vertex operator algebra* $\mathsf{BP}(\mathsf{u},\mathsf{v})$ *admits nonsemisimple modules. In physical language, the corresponding minimal model conformal field theory is* logarithmic.

As all the modules in the short exact sequences (2.3.48) admit Zhu inductions giving twisted $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules, it is reasonable to expect that there are analogous short exact sequences of $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules. Ensuring that this is the case is why we quotiented the Zhu induction of $\overline{\mathcal{R}}^{\pm}_{[j],\Delta,\omega}$ by the sum of all submodules that intersect trivially with the top space.

For this, it is convenient to introduce new modules $\mathcal{W}^{\mathrm{tw},\pm}_{\lambda} = \mathcal{W}^{\mathrm{tw},\pm}_{[j],\Delta,\omega}$ that are obtained by treating $\overline{\mathcal{R}}^{\pm}_{[j],\Delta,\omega}$ as a module over the twisted mode algebra $\mathsf{U}^{\mathrm{tw}}_{0}$ of (2.2.1), letting $\mathsf{U}^{\mathrm{tw}}_{>}$ act as 0, and then inducing to a U^{tw}-module. It follows that $\mathcal{W}^{\mathrm{tw},\pm}_{\lambda}$ is a “relaxed Verma” BP^{k}-module whose top space is $\overline{\mathcal{R}}^{\pm}_{[j],\Delta,\omega}$.

As such, we may consider the sum $\mathcal{N}^{\mathrm{tw},\pm}_{\lambda}$ of all the submodules of $\mathcal{W}^{\mathrm{tw},\pm}_{\lambda}$ whose intersection with the top space $\overline{\mathcal{R}}^{\pm}_{[j],\Delta,\omega}$ is zero. Because this top space is nonsemisimple, $\mathcal{N}^{\mathrm{tw},\pm}_{\lambda}$ is a proper submodule of the maximal submodule $\mathcal{M}^{\mathrm{tw},\pm}_{\lambda}$ of $\mathcal{W}^{\mathrm{tw},\pm}_{[j],\lambda}$. Additionally,

$$\mathcal{R}^{\mathrm{tw},\pm}_{\lambda} \simeq \frac{\mathcal{W}^{\mathrm{tw},\pm}_{\lambda}}{\mathcal{N}^{\mathrm{tw},\pm}_{\lambda}}. \tag{2.3.50}$$

We now proceed in an analogous fashion to [113, Sec. 4].

Theorem 2.3.32. *Let* k *be nondegenerate admissible and let* $\lambda \in \Gamma_{\mathsf{k}}$*. We then have the following nonsplit short exact sequences of* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules:*

$$\begin{gathered} 0 \longrightarrow \gamma\big(\mathcal{H}^{\mathrm{tw}}_{\mu}\big) \longrightarrow \mathcal{R}^{\mathrm{tw},+}_{\lambda} \longrightarrow \mathcal{H}^{\mathrm{tw}}_{\lambda} \longrightarrow 0, \\ 0 \longrightarrow \mathcal{H}^{\mathrm{tw}}_{\lambda} \longrightarrow \mathcal{R}^{\mathrm{tw},-}_{\lambda} \longrightarrow \gamma\big(\mathcal{H}^{\mathrm{tw}}_{\mu}\big) \longrightarrow 0, \end{gathered} \tag{2.3.51}$$

where $\mu = [\lambda_0, \lambda_2 - \frac{\mathsf{u}}{\mathsf{v}}, \lambda_1 + \frac{\mathsf{u}}{\mathsf{v}}] \in \Gamma_{\mathsf{k}}$.

Proof. As is now familiar, we shall also only detail the argument for $\overline{\mathcal{R}}_\lambda^+$. Let j, Δ and ω be defined by (2.3.37) and (2.3.44). As Zhu induction is exact, the short exact sequence (2.3.48) gives

$$\mathcal{V}_{j,\Delta}^{\mathrm{tw}} \simeq \frac{\mathcal{W}_{[j],\Delta,\omega}^{\mathrm{tw},+}}{\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big)}. \tag{2.3.52}$$

Hence, $\mathcal{H}_\lambda^{\mathrm{tw}}$ is also a quotient of $\mathcal{W}_\lambda^{\mathrm{tw},+} = \mathcal{W}_{[j],\Delta,\omega}^{\mathrm{tw},+}$ and (2.3.50) gives

$$\frac{\mathcal{R}_\lambda^{\mathrm{tw},+}}{\mathcal{M}_\lambda^{\mathrm{tw},+}/\mathcal{N}_\lambda^{\mathrm{tw},+}} \simeq \frac{\mathcal{W}_\lambda^{\mathrm{tw},+}}{\mathcal{M}_\lambda^{\mathrm{tw},+}} \simeq \mathcal{H}_\lambda^{\mathrm{tw}}, \tag{2.3.53}$$

since relaxed highest-weight modules have unique irreducible quotients. Thus, $\mathcal{H}_\lambda^{\mathrm{tw}}$ is a quotient of $\mathcal{R}_\lambda^{\mathrm{tw},+}$. This is the rightmost part of the short exact sequence in (2.3.51). For the leftmost part, note that the unique maximal submodule of $\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big)$ is $\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big) \cap \mathcal{N}_\lambda^{\mathrm{tw},+}$, because the only submodule of $\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big)$ intersecting its top space nontrivially is $\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big)$ itself. Therefore

$$\gamma\big(\mathcal{H}_\mu^{\mathrm{tw}}\big) = \frac{\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big)}{\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big) \cap \mathcal{N}_\lambda^{\mathrm{tw},+}} \simeq \frac{\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big) + \mathcal{N}_\lambda^{\mathrm{tw},+}}{\mathcal{N}_\lambda^{\mathrm{tw},+}}. \tag{2.3.54}$$

This is clearly a submodule of $\mathcal{W}_\lambda^{\mathrm{tw},+}/\mathcal{N}_\lambda^{\mathrm{tw},+} \simeq \mathcal{R}_\lambda^{\mathrm{tw},+}$ and therefore $\gamma\big(\mathcal{H}_\mu^{\mathrm{tw}}\big)$ embeds into $\mathcal{R}_\lambda^{\mathrm{tw},+}$. All that remains is to show that the sequence of $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules constructed from the embedding $\gamma\big(\mathcal{H}_\mu^{\mathrm{tw}}\big) \hookrightarrow \mathcal{R}_\lambda^{\mathrm{tw},+}$ and the surjection $\mathcal{R}_\lambda^{\mathrm{tw},+} \twoheadrightarrow \mathcal{H}_\lambda^{\mathrm{tw}}$ is exact. Using (2.3.50) and (2.3.54),

$$\frac{\mathcal{R}_\lambda^{\mathrm{tw},+}}{\gamma\big(\mathcal{H}_\mu^{\mathrm{tw}}\big)} \simeq \frac{\mathcal{W}_\lambda^{\mathrm{tw},+}}{\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big) + \mathcal{N}_\lambda^{\mathrm{tw},+}} \simeq \frac{\mathcal{V}_{j,\Delta}^{\mathrm{tw}}}{\big(\gamma\big(\mathcal{V}_{-j-1,\Delta}^{\mathrm{tw}}\big) + \mathcal{N}_\lambda^{\mathrm{tw},+}\big)/\gamma\big(\mathcal{H}_\mu^{\mathrm{tw}}\big)}. \tag{2.3.55}$$

The $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module $\mathcal{R}_\lambda^{\mathrm{tw},+}/\gamma\big(\mathcal{H}_\mu^{\mathrm{tw}}\big)$ is therefore a twisted highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module. By Theorem 2.3.16, it is also simple and the quotient on the right-hand side of (2.3.55) is isomorphic to $\mathcal{H}_\lambda^{\mathrm{tw}}$. ∎

To summarise the results of this chapter for $\mathsf{BP}(\mathsf{u},\mathsf{v})$ with $\mathsf{v} \geqslant 3$, we have proven that such Bershadsky–Polyakov minimal models admit infinitely many simple modules (in addition to classifying the relaxed ones) as well as reducible-but-indecomposable modules. This nonrationality is a significant obstacle to computing modular transformations and fusion rules, which is essential data for constructing logarithmic conformal field theories. The next chapter details how to navigate this obstacle.

2.3.5. Examples. We conclude this chapter by illustrating the classification for certain examples of $\mathsf{BP}(\mathsf{u},\mathsf{v})$.

Example ($\mathsf{BP}(3,2)$). *For $\mathsf{k} = -\frac{3}{2}$, the central charge of the minimal model is $\mathsf{c} = 0$. Since $\lambda^I \in \mathsf{P}_\geqslant^0 = \{[0,0,0]\}$ and $\lambda^F \in \mathsf{P}_\geqslant^1$ is constrained by $\lambda_0^F \geqslant 0$, the only surviving weight is*

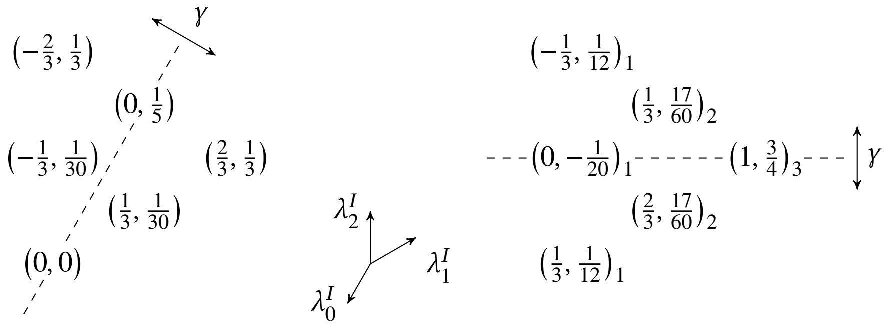

FIGURE 2. The charges and conformal weights (j,Δ) of the untwisted (left) and twisted (right) simple highest-weight $\mathsf{BP}(5,2)$-modules, arranged by the Dynkin labels of the integral parts λ^I of the corresponding surviving weights λ. The subscript on the twisted labels gives the dimension of the top space. Conjugation γ is indicated by reflection about the dashed line and spectral flow σ by 120° anticlockwise rotation about each triangle's centre.

$\lambda = [0,0,0] - \frac{3}{2}[1,0,0] = [\mathsf{k},0,0]$. *There is therefore a unique simple untwisted highest-weight module* $\mathcal{H}_{-3\omega_0/2} = \mathcal{H}_{0,0}$ *and a unique simple twisted highest-weight module* $\mathcal{H}^{\mathrm{tw}}_{-3\omega_0/2} = \mathcal{H}^{\mathrm{tw}}_{0,0}$ *(up to isomorphism). This is clearly the trivial minimal model.*

EXAMPLE ($\mathsf{BP}(5,2)$). *For* $\mathsf{k} = -\frac{1}{2}$, *the central charge is* $\mathsf{c} = \frac{2}{5}$ *and we have* $\lambda^I \in \mathsf{P}^2_{\geqslant}$ *and* $\lambda^F = [1,0,0]$. *Therefore there are* $\left|\mathsf{P}^2_{\geqslant}\right| = 6$ *simple untwisted highest-weight modules and* 6 *simple twisted highest-weight modules. As* λ^F *always has* $\lambda^F_1 = 0$, *all twisted highest-weight modules have finite-dimensional top spaces. We illustrate these modules and the charges and conformal weights of their highest-weight vectorsin Figure 2, arranging them according to* λ^I. *This example is one of the Bershadsky–Polyakov minimal models considered in [15].*

EXAMPLE ($\mathsf{BP}(4,3)$). *Moving outside cases from [15], consider the minimal model with* $\mathsf{k} = -\frac{5}{3}$ *and* $\mathsf{c} = -1$. *This minimal model was studied in [5]. Here* $\lambda^I \in \mathsf{P}^1_{\geqslant}$, *so* $\lambda^I = [1,0,0]$, $[0,1,0]$ *or* $[0,0,1]$. *Similarly,* $\lambda^F \in \mathsf{P}^2_{\geqslant}$ *and hence* $\lambda^F = [2,0,0]$, $[1,1,0]$ *or* $[1,0,1]$ *(recall that* λ *must be a surviving weight). Hence there are* $\left|\mathsf{P}^1_{\geqslant}\right|\left|\mathsf{P}^1_{\geqslant}\right| = 9$ *simple untwisted highest-weight modules and* 9 *simple twisted highest-weight modules. Of the twisted highest-weight modules,* 6 *have finite-dimensional top spaces whilst the top spaces of the other* 3 *are infinite-dimensional. All highest-weight modules are type-*3.

As in the previous example, we arrange the highest-weight data in Figure 3. By Theorem 2.3.28, there is one family of generically simple relaxed highest-weight $\mathsf{BP}(4,3)$*-modules* $\mathcal{R}^{\mathrm{tw}}_{[j],-1/8,0}$, $j \neq -\frac{1}{6},-\frac{1}{2},-\frac{5}{6} \pmod 1$. *The corresponding semisimple coherent family has* 3 *distinct highest-weight submodules, each isomorphic to one of the* 3 *twisted highest-weight modules with infinite dimensional top spaces. This family must be closed under conjugation and so* $\omega = 0$, *as can be checked explicitly.*

$\left(\frac{1}{9},-\frac{1}{18}\right)$ $\left(\frac{7}{9},\frac{5}{18}\right)$ $\left(\frac{4}{9},\frac{1}{9}\right)$ $\left(-\frac{7}{9},\frac{5}{18}\right)$ $\left(-\frac{1}{9},-\frac{1}{18}\right)$ $\left(-\frac{4}{9},\frac{1}{9}\right)$ $\left(-\frac{1}{3},\frac{1}{2}\right)$ $\left(\frac{1}{3},\frac{1}{2}\right)$ $(0,0)$

$\left(\frac{1}{18},-\frac{1}{72}\right)_1$ $\left(\frac{13}{18},\frac{47}{72}\right)_2$ $\left(\frac{7}{18},\frac{23}{72}\right)_1$ $\left(-\frac{5}{6},-\frac{1}{8}\right)_\infty$ $\left(-\frac{1}{6},-\frac{1}{8}\right)_\infty$ $\left(-\frac{1}{2},-\frac{1}{8}\right)_\infty$ $\left(-\frac{7}{18},\frac{23}{72}\right)_1$ $\left(\frac{5}{18},\frac{47}{72}\right)_2$ $\left(-\frac{1}{18},-\frac{1}{72}\right)_1$

FIGURE 3. The charges and conformal weights (j,Δ) of the untwisted (left) and twisted (right) simple highest-weight $\mathsf{BP}(4,3)$-modules, arranged by the Dynkin labels of the integral (small-scale) and fractional (large-scale) parts λ^F of the corresponding surviving weights λ. The subscript on the twisted labels gives the dimension of the top space.

By Theorem 2.3.32, we know of six nonsemisimple twisted relaxed highest-weight $\mathsf{BP}(4,3)$-*modules characterised by the following nonsplit short exact sequences:*

$$\begin{gathered}
0 \longrightarrow \gamma\big(\mathcal{H}^{\mathrm{tw}}_{-5/6,-1/8}\big) \longrightarrow \mathcal{R}^{\mathrm{tw},+}_{[-1/6],-1/8,0} \longrightarrow \mathcal{H}^{\mathrm{tw}}_{-1/6,-1/8} \longrightarrow 0,\\
0 \longrightarrow \gamma\big(\mathcal{H}^{\mathrm{tw}}_{-1/2,-1/8}\big) \longrightarrow \mathcal{R}^{\mathrm{tw},+}_{[-1/2],-1/8,0} \longrightarrow \mathcal{H}^{\mathrm{tw}}_{-1/2,-1/8} \longrightarrow 0,\\
0 \longrightarrow \gamma\big(\mathcal{H}^{\mathrm{tw}}_{-1/6,-1/8}\big) \longrightarrow \mathcal{R}^{\mathrm{tw},+}_{[-5/6],-1/8,0} \longrightarrow \mathcal{H}^{\mathrm{tw}}_{-5/6,-1/8} \longrightarrow 0,\\
0 \longrightarrow \mathcal{H}^{\mathrm{tw}}_{-1/6,-1/8} \longrightarrow \mathcal{R}^{\mathrm{tw},-}_{[-1/6],-1/8,0} \longrightarrow \gamma\big(\mathcal{H}^{\mathrm{tw}}_{-5/6,-1/8}\big) \longrightarrow 0,\\
0 \longrightarrow \mathcal{H}^{\mathrm{tw}}_{-1/2,-1/8} \longrightarrow \mathcal{R}^{\mathrm{tw},-}_{[-1/2],-1/8,0} \longrightarrow \gamma\big(\mathcal{H}^{\mathrm{tw}}_{-1/2,-1/8}\big) \longrightarrow 0,\\
0 \longrightarrow \mathcal{H}^{\mathrm{tw}}_{-5/6,-1/8} \longrightarrow \mathcal{R}^{\mathrm{tw},-}_{[-5/6],-1/8,0} \longrightarrow \gamma\big(\mathcal{H}^{\mathrm{tw}}_{-1/6,-1/8}\big) \longrightarrow 0.
\end{gathered} \tag{2.3.56}$$

An interesting feature of this minimal model is its relation to the $\beta\gamma$ ghost vertex algebra B*: It was shown in [5, Sec. 5.2] that the Bershadsky–Polyakov minimal model vertex operator algebra* $\mathsf{BP}(4,3)$ *embeds into* B *with* $\mathsf{c}=-1$. *Recall that* B *is strongly generated by β and γ, both of conformal weight $\frac{1}{2}$, subject to the operator product expansions*

$$\beta(z)\beta(w) \sim 0 \sim \gamma(z)\gamma(w) \quad \text{and} \quad \beta(z)\gamma(w) \sim \frac{-\mathbb{1}}{z-w}. \tag{2.3.57}$$

An embedding $\mathsf{BP}(4,3) \hookrightarrow \mathsf{B}$ *is then given by*

$$J \longmapsto \frac{1}{3}{:}\beta\gamma{:}, \quad G^+ \longmapsto \frac{1}{3\sqrt{3}}{:}\beta\beta\beta{:}, \quad G^- \longmapsto -\frac{1}{3\sqrt{3}}{:}\beta\beta\beta{:}, \quad L \longmapsto \frac{1}{2}({:}\partial\beta\gamma{:} - {:}\partial\gamma\beta{:}). \tag{2.3.58}$$

This suggests, and it is easy to check [5, Prop. 5.9], that $\mathsf{BP}(4,3)$ *is (isomorphic to) the* $\mathbb{Z}_3$*-orbifold of* B *corresponding to the automorphism* $\mathsf{e}^{2\pi \mathrm{i} J_0}$*. We will see in Section 3.3.7 that this special relationship is reflected in the fusion rules of* $\mathsf{BP}(4,3)$.

Chapter 3

Inverting Quantum Hamiltonian Reduction

In the last chapter, we introduced Bershadsky–Polyakov minimal models denoted by $\mathsf{BP}(\mathsf{u},\mathsf{v})$ and classified their simple relaxed modules. Additionally, we proved the existence of reducible-but-indecomposable $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules.

The characters, modular transformations and fusion rules of $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules are of crucial importance in constructing and analysing logarithmic conformal field theories with $\mathsf{BP}(\mathsf{u},\mathsf{v})$ symmetry.

To compute this 'physics data', in this chapter we will leverage the fact that BP^{k} is a quantum hamiltonian reduction of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$, whose other quantum hamiltonian reduction (the Zamolodchikov algebra $\mathsf{W}_3^{\mathsf{k}}$ of Section 1.3.2) is well-understood. The idea is that the partial quantum hamiltonian reductions of Section 1.3 can be inverted, and that such inverses can be used to relate the representation theories of the W-algebras involved.

3.1. The Idea

An 'inverse' to quantum hamiltonian reduction was first introduced by Semikhatov [147] for the case of $\mathfrak{g} = \mathfrak{sl}_2$ (see Section 3.1.2). It takes the form of an embedding $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_2, f) \otimes \Pi$ where Π is a lattice vertex algebra to be defined shortly. It was later shown by Adamović that this inverse reduction can be deployed to understand some of the representation theory of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$ and, at certain levels, $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ [2].

Particularly, the inverse quantum hamiltonian reduction for $\mathfrak{g} = \mathfrak{sl}_2$ neatly explains the appearance of Virasoro minimal model characters in the characters of nondegenerate admissible-level relaxed $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$-modules [52, 113]; the Virasoro vertex operator algebra $\mathsf{Vir}^{\mathsf{k}}$ is a quantum hamiltonian reduction of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$.

An important consequence of this relation is that the modular S-transforms and Grothendieck fusion rules of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ at these levels are naturally expressed in terms of their Virasoro minimal model analogues.

It is expected that inverse quantum hamiltonian reductions can be constructed for more general pairs of W-algebras, and will provide nontrivial relationships between the representation theories of the W-algebras involved.

Recall from Section 1.3.1 that the (conjectured) condition for the existence of a partial quantum hamiltonian reduction from $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_1)$ to $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_2)$ is that $f_1 < f_2$ under the ordering on the corresponding nilpotent orbits of $\mathfrak{g}$.

The philosophy we adopt here is that despite not having a complete understanding of how partial quantum hamiltonian reduction should be defined for W-algebras, it can be inverted. That is, for all pairs of nilpotent elements $f_1, f_2 \in \mathfrak{g}$ such that $f_1 < f_2$, we believe that there exists an embedding

$$\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_1) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_2) \otimes \mathsf{V} \tag{3.1.1}$$

where V is a vertex operator algebra representing the degrees of freedom 'lost' in performing partial quantum hamiltonian reduction. Taking the tensor product of suitable $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_2)$- and V-modules gives $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_1)$-modules by restriction.

The inverse reduction (3.1.1) should also descend to an embedding of simple quotients at some levels k, and in some sense relate the conformal field theory data of $\mathsf{W}_{\mathsf{k}}(\mathfrak{g}, f_1)$ and $\mathsf{W}_{\mathsf{k}}(\mathfrak{g}, f_2)$ irrespective of rationality.

3.1.1. The Half-Lattice Vertex Algebra. To construct a given inverse quantum hamiltonian reduction embedding (3.1.1), first a suitable vertex operator algebra V needs to be determined. For the cases studied here, we require a "half-lattice" vertex operator algebra Π [33]. This vertex operator algebra appears in all known examples of inverse quantum hamiltonian reduction. The half-lattice Π is part of the vertex-algebraic content of the 'original' inverse reduction identified by Semikhatov[147]. Here we follow the construction of Π presented in [4, Sec. 3].

Consider the abelian Lie algebra $\mathfrak{h} = \text{span}_{\mathbb{C}}\{c, d\}$, equipped with the symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle c, c\rangle = \langle d, d\rangle = 0 \quad \text{and} \quad \langle c, d\rangle = 2. \tag{3.1.2}$$

The group algebra $\mathbb{C}[\mathbb{Z}c] = \text{span}_{\mathbb{C}}\{e^{nc} | \, n \in \mathbb{Z}\}$ has the structure of an $\mathfrak{h}$-module according to the formula

$$h(e^{nc}) = n\langle h, c\rangle e^{nc}. \tag{3.1.3}$$

Denote by H the Heisenberg vertex algebra defined by $\mathfrak{h}$ and $\langle\cdot,\cdot\rangle$.

Definition 3.1.1. *The* half lattice vertex algebra Π *is the lattice vertex algebra* $\mathsf{H}\otimes\mathbb{C}[\mathbb{Z}c]$ *where the action of* $h\in\mathfrak{h}$ *on* $\mathbb{C}[\mathbb{Z}c]$ *is identified with the action of the zero mode* h_0 *of* $h(z)\in H$.

A set of (strong) generating fields for Π is then $\{c(z), d(z), \mathrm{e}^{mc}(z) : m\in\mathbb{Z}\}$. The operator product expansions of these fields are easily determined:

$$\begin{aligned} c(z)c(w)&\sim 0, & c(z)d(w)&\sim\frac{2\,\mathbb{1}}{(z-w)^2}, & d(z)d(w)&\sim 0,\\ c(z)\mathrm{e}^{mc}(w)&\sim 0, & d(z)\mathrm{e}^{mc}(w)&\sim\frac{2m\,\mathrm{e}^{mc}(w)}{z-w}, & \mathrm{e}^{mc}(z)\mathrm{e}^{nc}(w)&\sim 0.\end{aligned}\tag{3.1.4}$$

The half lattice vertex algebra admits a two-parameter family of energy-momentum fields given by

$$t(z)=\frac{1}{2}{:}c(z)d(z){:}+\alpha\partial c(z)+\beta\partial d(z),\quad \alpha,\beta\in\mathbb{C}.\tag{3.1.5}$$

The corresponding central charge is $2-48\alpha\beta$. The choice of conformal structure on Π will depend on the inverse quantum hamiltonian reduction being considered.

3.1.2. Example. As mentioned previously, the first inverse quantum hamiltonian reduction was described by Semikhatov [147] and its vertex- and representation-theoretic content was analysed by Adamović [2]: Let $\mathfrak{g}=\mathfrak{sl}_2$. The universal affine vertex algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$ has strong generators $h(z)$, $e(z)$ and $f(z)$ with operator product expansions (see (1.2.6))

$$\begin{aligned} h(z)e(w)&\sim\frac{2e(w)}{z-w}, \quad h(z)f(w)\sim\frac{-2f(w)}{z-w}, \quad e(z)e(w)\sim f(z)f(w)\sim 0,\\ h(z)h(w)&\sim\frac{2\mathsf{k}\mathbb{1}}{(z-w)^2}, \quad e(z)f(w)\sim\frac{\mathsf{k}\mathbb{1}}{(z-w)^2}+\frac{h(w)}{z-w}.\end{aligned}\tag{3.1.6}$$

For $\mathsf{k}\neq-2$, the Sugawara construction (1.2.8) defines an energy-momentum field $T^{\text{Sug.}}(z)$ (i.e. makes $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$ a vertex *operator* algebra) given by

$$T^{\text{Sug.}}(z)=\frac{1}{2(\mathsf{k}+2)}\left(\frac{1}{2}{:}h(z)h(z){:}+{:}e(z)f(z){:}+{:}f(z)e(z){:}\right).\tag{3.1.7}$$

The only nonaffine W-algebra obtained from $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$ is the W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_2,f)$ specified by the identity embedding $\mathfrak{sl}_2\hookrightarrow\mathfrak{sl}_2$. In fact, this W-algebra is isomorphic to the Virasoro vertex operator algebra $\mathsf{Vir}^{\mathsf{k}}$ [28]. Denote the energy-momentum field of $\mathsf{Vir}^{\mathsf{k}}$ by $T(z)$ and recall the $T(z)T(w)$ operator product expansion

$$T(z)T(w)\sim\frac{\mathsf{c}^{\mathsf{Vir}}_{\mathsf{k}}\mathbb{1}}{2(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w},\quad \mathsf{c}^{\mathsf{Vir}}_{\mathsf{k}}=-\frac{6\mathsf{k}^2+11\mathsf{k}+4}{\mathsf{k}+2}.\tag{3.1.8}$$

Taking $\mathsf{V} = \Pi$ in (3.1.1) and giving Π the conformal structure defined by

$$t(z) = \frac{1}{2}{:}c(z)d(z){:} + \frac{\mathsf{k}}{4}\partial c(z) - \frac{1}{2}\partial d(z), \tag{3.1.9}$$

an inverse quantum hamiltonian reduction embedding is straightforward to construct.

Theorem 3.1.2. *[2,147] For $\mathsf{k} \neq -2$, there exists a vertex operator algebra embedding $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{Vir}^{\mathsf{k}} \otimes \Pi$ given by*

$$\begin{gathered} h(z) \longmapsto 2b(z), \quad e(z) \longmapsto \mathrm{e}^{c}(z), \\ f(z) \longmapsto {:}((\mathsf{k}+2)T(z) - (\mathsf{k}+1)\partial a(z) - a(z)a(z))\, \mathrm{e}^{-c}(z){:}, \end{gathered} \tag{3.1.10}$$

where $a(z) = -\frac{\mathsf{k}}{4}c(z) + \frac{1}{2}d(z)$ and $b(z) = \frac{\mathsf{k}}{4}c(z) + \frac{1}{2}d(z)$. This embedding descends to an embedding of simple quotients $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{Vir}_{\mathsf{k}} \otimes \Pi$ if and only if $\mathsf{k}+1 \notin \mathbb{Z}_{\geqslant 1}$.

An important set of levels are the *nondegenerate admissible* levels for $\mathfrak{sl}_2$. These are all k satisfying $\mathsf{k} = \frac{\mathsf{u}}{\mathsf{v}} - 2$ for some coprime $\mathsf{u}, \mathsf{v} \in \mathbb{Z}_{\geqslant 2}$. At such k, $\mathsf{Vir}_{\mathsf{k}}$ is rational and is also known as a Virasoro minimal model [78, 92, 98].

The inverse quantum hamiltonian reduction of Theorem 3.1.2 therefore relates the nonrational $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ to the rational $\mathsf{Vir}_{\mathsf{k}}$, where k is a nondegenerate admissible level. No such embedding exists for the admissible levels with $\mathsf{v} = 1$, at which $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ is rational.

3.2. From W_3 Minimal Models to BP Minimal Models

Despite the minimal models $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ with $\mathsf{v} \geqslant 3$ being nonrational, modular transformations and fusion rules can be unravelled using a general proposed framework known as the *standard module formalism* [51, 146]. This is the same method used successfully for the aforementioned nondegenerate admissible level $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ in the category of weight modules [50, 52].

A key requirement for applying the standard module formalism is the existence of continuous families of modules with linearly independent characters. For $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ with $\mathsf{v} \geqslant 3$, the modules $\mathcal{R}^{\mathrm{tw}}_{[j],\lambda}$ of Theorem 2.3.25 are good candidates for standard modules. What is therefore needed are linearly independent 'characters' of $\mathcal{R}^{\mathrm{tw}}_{[j],\lambda}$, which we will show can be taken to be certain one-point functions related to one-point functions of $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-modules. This relationship between $\mathsf{BP}(\mathsf{u}, \mathsf{v})$- and $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-modules is a straightforward consequence of the existence of an inverse quantum hamiltonian reduction between $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ and $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$ in the sense of Section 3.1.

The fact that $\mathcal{R}^{\mathrm{tw}}_{[j],\lambda}$ is a twisted $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module complicates modularity and fusion computations. Additionally, applying spectral flow σ^{ℓ} with $\ell \neq 0$ to $\mathcal{R}^{\mathrm{tw}}_{[j],\lambda}$ always results in a $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module that is not positive-energy with respect to the conformal structure furnished by $L(z)$. It will therefore be prudent to consider an alternative conformal structure on $\mathsf{BP}(\mathsf{u},\mathsf{v})$. Once this is done, the standard module formalism can be applied and certain fusion rules for $\mathsf{BP}(\mathsf{u},\mathsf{v})$ follow.

Crucial to the proceeding analysis is the presence of the known modular transformations and fusion rules for $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ from Section 1.3.3. That is, despite $\mathsf{BP}(\mathsf{u},\mathsf{v})$ with $\mathsf{v} \geqslant 3$ being nonrational, its relation to the rational $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ via inverse quantum hamiltonian reduction greatly assists in determining its modular transformations and fusion rules.

3.2.1. Ordering $\mathfrak{sl}_3$ W-Algebras. The ordering of $\mathfrak{sl}_3$ W-algebras is particularly simple. This is because there are only three distinct nilpotent orbits in $\mathfrak{sl}_3$. The $\mathfrak{sl}_3$ W-algebras corresponding to these nilpotent orbits are the affine vertex algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$, the Bershadsky–Polyakov algebra BP^{k} and the Zamolodchikov algebra $\mathsf{W}_3^{\mathsf{k}}$ of Section 1.3.2. The partial ordering of Section 1.3.1 for $\mathfrak{sl}_3$ is the ordering $\mathsf{W}_3^{\mathsf{k}} > \mathsf{BP}^{\mathsf{k}} > \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$.

The inverse quantum hamiltonian reduction embeddings for $\mathfrak{sl}_3$ are all known. The embedding corresponding to $\mathsf{W}_3^{\mathsf{k}} > \mathsf{BP}^{\mathsf{k}}$ was described in [4], while that corresponding to $\mathsf{BP}^{\mathsf{k}} > \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ was described in [3]. In both cases, when the embedding descends to an embedding of simple quotients is known.

3.2.2. Inverse Quantum Hamiltonian Reduction from W_3 to BP. The inverse quantum hamiltonian reduction relevant for our purposes is the embedding of the Bershadsky–Polyakov minimal model vertex operator algebra $\mathsf{BP}(\mathsf{u},\mathsf{v})$ in the tensor product of the half-lattice vertex operator algebra Π and the minimal model $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ from [4]. We review their main results, adapted to our choice of conformal structure. We also twist their embedding by the conjugation automorphism (2.1.18) in order to prioritise highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules over their conjugates.

Theorem 3.2.1 ([4, Thms. 3.6 and 6.2])**.** *For k nondegenerate-admissible (i.e. $\mathsf{k} = -3+\frac{\mathsf{u}}{\mathsf{v}}$ for some coprime $\mathsf{u},\mathsf{v} \in \mathbb{Z}_{\geqslant 3}$), there exists a vertex operator algebra embedding $\mathsf{BP}(\mathsf{u},\mathsf{v}) \hookrightarrow \mathsf{W}_3(\mathsf{u},\mathsf{v}) \otimes \Pi$ given by*

$$J(z) \longmapsto b(z), \qquad L(z) \longmapsto T(z)+t(z), \qquad G^-(z) \longmapsto \mathsf{e}^{-c}(z),$$

$$\begin{aligned} G^+(z) \longmapsto {}&:\Big(\frac{3(\mathsf{u}-\mathsf{v})}{\mathsf{v}}\partial a(z)a(z) - a(z)^3 - \frac{(\mathsf{u}-\mathsf{v})^2}{\mathsf{v}^2}\partial^2 a(z) \\ &+\frac{\mathsf{u}}{\mathsf{v}}T(z)a(z) - \frac{\mathsf{u}(\mathsf{u}-\mathsf{v})}{2\mathsf{v}^2}\partial T(z) - \sqrt{\frac{\mathsf{u}^3}{3\mathsf{v}^3}}W(z)\Big)\mathsf{e}^{c}(z):, \end{aligned} \tag{3.2.1}$$

where the fields $t(z), a(z), b(z) \in \Pi$ are given by

$$\begin{aligned} t(z) &= \frac{1}{2}{:}c(z)d(z){:} - \frac{3\kappa}{2}\partial c(z) + \frac{3}{4}\partial d(z), \\ a(z) &= -\kappa c(z) + \frac{1}{2}d(z), \quad b(z) = \kappa c(z) + \frac{1}{2}d(z), \end{aligned} \tag{3.2.2}$$

with $\kappa = \frac{2\mathsf{k}+3}{6}$. Moreover, such an embedding does not exist when $\mathsf{u} \geqslant 2$ and $\mathsf{v} = 1$ or 2.

The energy-momentum field $t(z) \in \Pi$ has central charge $\mathsf{c}_{\mathsf{k}}^{\Pi}$ related to κ according to

$$\mathsf{c}_{\mathsf{k}}^{\Pi} = 2 + 54\kappa, \tag{3.2.3}$$

which satisfies

$$\mathsf{c}_{\mathsf{k}}^{\mathsf{BP}} = \mathsf{c}_{\mathsf{k}}^{\Pi} + \mathsf{c}_{\mathsf{k}}^{\mathsf{W}_3}. \tag{3.2.4}$$

With respect to $t(z)$, both $a(z)$ and $b(z)$ have conformal weight 1, though a is not quasiprimary, whilst that of $\mathsf{e}^{mc}(z)$ is $-\frac{3m}{2}$. The central charge and conformal dimensions are our motivation for choosing $t(z)$ as the energy-momentum field in Π.

Like the $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules of Chapter 2, we are interested in positive-energy (indecomposable) weight modules of Π, meaning those on which the h_0, with $h \in \mathfrak{h}$, act semisimply and t_0 has eigenvalues that are bounded below. Such modules may be induced [33] from the $\mathbb{Z}c$-modules generated by elements $\mathsf{e}^h \in \mathbb{C}[\mathfrak{h}]$ on which $h' \in \mathfrak{h}$ acts as $h' \cdot \mathsf{e}^h = \langle h', h\rangle\, \mathsf{e}^h$. The following is adapted from [4] to accommodate our choice of conformal structure.

Proposition 3.2.2 ([4, Prop. 3.4])**.** *The (twisted) weight Π-module generated from e^{rb+jc} is a positive-energy module if and only if $r = \frac{3}{2}$. In this case, the twisted Π-module is simple and the minimal t_0-eigenvalue is $\frac{9}{4}\kappa$.*

The eigenvalue of b_0 on $\mathsf{e}^{3b/2+jc}$ is $j + 3\kappa$. We therefore define $\Pi_{[j]}$, $[j] \in \mathbb{C}/\mathbb{Z}$, to be the simple positive-energy weight Π-module generated by $\mathsf{e}^{3b/2+(j-3\kappa)c}$ so that the b_0-eigenvalues of $\Pi_{[j]}$ coincide with $[j]$. The notation reflects the fact that the isomorphism class of this module only depends on $[j]$ rather than j itself.

Recall from Section 1.3.3 that the minimal model $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ is rational. Its modules are all highest-weight and are denoted by $\mathcal{W}(\mathsf{r},\mathsf{s})$ where $\mathsf{r} = [r_0, r_1, r_2] \in \mathsf{P}_{\geqslant}^{\mathsf{u}-3}$ and $\mathsf{s} = [s_0, s_1, s_2] \in \mathsf{P}_{\geqslant}^{\mathsf{v}-3}$. This parametrisation is connected to that of the relaxed $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules $\mathcal{R}_{[j],[\lambda]}^{\mathsf{tw}}$ in the following way:

Given $\lambda \in \Gamma_{\mathsf{u},\mathsf{v}} = \Gamma_{\mathsf{k}}$, let the Dynkin labels of $\lambda^I \in \mathsf{P}_{\geqslant}^{\mathsf{u}-3}$ be $\mathsf{r} = [r_0, r_1, r_2]$. Let ω_i, $i = 0, 1, 2$, denote the fundamental weights of $\widehat{\mathfrak{sl}}_3$ and let the Dynkin labels of $\widetilde{\lambda}^F = \lambda^F - \omega_0 - \omega_1 \in \mathsf{P}_{\geqslant}^{\mathsf{v}-3}$ be $\mathsf{s} = [s_0, s_1, s_2]$. In other words, let

$$r_0 = \lambda_0^I, \quad r_1 = \lambda_1^I, \quad r_2 = \lambda_2^I \qquad \text{and} \qquad s_0 = \lambda_0^F - 1, \quad s_1 = \lambda_1^F - 1, \quad s_2 = \lambda_2^F. \tag{3.2.5}$$

Then, the $\mathbb{Z}_3$-action (2.3.46) becomes the cycle

$$\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\right] \overset{\nabla}{\longmapsto} \left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ s_2 & s_0 & s_1 \end{smallmatrix}\right] \overset{\nabla}{\longmapsto} \left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ s_1 & s_2 & s_0 \end{smallmatrix}\right] \overset{\nabla}{\longmapsto} \left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\right]. \tag{3.2.6}$$

This is precisely the $\mathbb{Z}_3$ symmetry of the (r, s) parametrisation of $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-modules described in Section 1.3.2. We shall therefore frequently parametrise weights $\lambda \in \Gamma_{\mathsf{u},\mathsf{v}}$ by r and s, or by the labels r_i and s_i, $i = 0, 1, 2$:

$$\lambda = \Gamma(\mathsf{r}, \mathsf{s}) = \Gamma\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\right] = \sum_{i=0}^{2} r_i \omega_i - \tfrac{\mathsf{u}}{\mathsf{v}}\Big(\omega_0 + \omega_1 + \sum_{i=0}^{2} s_i \omega_i\Big). \tag{3.2.7}$$

Extending this parametrisation to $\Sigma_{\mathsf{u},\mathsf{v}} = \Sigma_{\mathsf{k}}$ means extending the allowed range of s_0, s_1 and s_2 to include $\mathsf{v} - 2$, -1 and $\mathsf{v} - 2$, respectively (but still subject to $s_0 + s_1 + s_2 = \mathsf{v} - 3$). Of course only the $\lambda = \Gamma(\mathsf{r}, \mathsf{s})$ with $\mathsf{r} = [r_0, r_1, r_2] \in \mathsf{P}_{\geqslant}^{\mathsf{u}-3}$ and $\mathsf{s} = [s_0, s_1, s_2] \in \mathsf{P}_{\geqslant}^{\mathsf{v}-3}$ are relevant for labelling $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-modules.

We will also parametrise $\mathsf{W}_3(\mathsf{u}, \mathsf{v})$-modules in terms of $[\lambda] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$ and use the notation $\mathcal{W}_{[\lambda]} = \mathcal{W}(\mathsf{r}, \mathsf{s})$ for $\lambda = \Gamma(\mathsf{r}, \mathsf{s})$ when it is helpful.

By restriction, $\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$ is a $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module with several desirable properties.

Theorem 3.2.3 ([4, Thms. 5.12 and 6.3])**.** *Let k be nondegenerate-admissible. Then, for each $[\lambda] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$ and $[j] \in \mathbb{C}/\mathbb{Z}$:*

- *$\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$ is an indecomposable top-dense $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module on which G_0^- acts injectively.*
- *Every nonzero $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-submodule of $\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$ has nonzero intersection with its top space.*
- *If $[j]$ is not in the ∇-orbit of $[j^{\mathrm{tw}}(\lambda)]$, then $\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$ is a simple $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module.*

In light of the classification results in Chapter 2, the $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-module $\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$ (at least when it is simple) must be a module we have seen already. Our notational choices suggest that $\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$, when simple, is related to $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$.

For the nonsimple cases, recall that each family of simple top-dense relaxed highest-weight $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules, corresponding to a fixed $[\lambda] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$ and parametrised by $[j] \in \mathbb{C}/\mathbb{Z}$, has three 'gaps' corresponding to the $[j^{\mathrm{tw}}(\nabla^i(\lambda))]$, $i \in \mathbb{Z}_3$. Theorem 2.3.32 shows that these gaps in fact also correspond to top-dense, nonsimple $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules. Each of these 'gap modules' may be taken to be indecomposable, with two possible choices related through conjugation.

As we will be concerned with the modular properties of the characters of these twisted $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules, it does not matter which choice we make for the gap modules. Since G_0^- acts injectively on $\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$, we shall choose the indecomposable gap modules such that G_0^- acts injectively on them (the '-' modules in (2.3.51)). They will be denoted using the same notation $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$ as their simple cousins, where $[j] = [j^{\mathrm{tw}}(\nabla^i(\lambda))]$, $i \in \mathbb{Z}_3$.

Proposition 3.2.4. *Let* k *be nondegenerate-admissible,* $[\lambda] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$ *and* $[j] \in \mathbb{C}/\mathbb{Z}$*. Then,*

$$\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]} \simeq \mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}. \tag{3.2.8}$$

Proof. Note that the $\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$ are completely specified by their top spaces (Theorem 3.2.3), as are the $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$. It therefore suffices to show that the top spaces of each coincide as modules over the twisted Zhu algebra of $\mathsf{BP}(\mathsf{u},\mathsf{v})$. The classification of such modules was completed in Theorem 2.2.15. It therefore suffices to determine the J_0-, L_0- and Ω-eigenspaces in the top space of $\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$.

The check for J_0 is immediate, while that for $L_0 = T_0 + t_0$ follows from the equality

$$\Delta_\lambda + \frac{9\kappa}{4} = \Delta^{\mathrm{tw}}(\lambda), \tag{3.2.9}$$

where $\Delta_\lambda = \Delta(\mathsf{r},\mathsf{s})$ with $\lambda = \Gamma(\mathsf{r},\mathsf{s})$. The action of Ω on the top space of $\mathcal{W}_{[\lambda]} \otimes \Pi_{[j]}$ is obtained from (3.2.1) and (2.2.13). That it agrees with (2.3.44) can be checked directly. ∎

3.2.3. Characters for Standard $\mathsf{BP}(\mathsf{u},\mathsf{v})$-Modules. For a $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module $\mathcal{M}$, we define its character to be

$$\operatorname{ch}\big[\mathcal{M}\big]\left(\theta \,|\, \zeta \,|\, \tau\right) = \mathsf{y}^{\kappa} \operatorname{tr}_{\mathcal{M}} \left(\mathsf{z}^{J_0} \mathsf{q}^{L_0 - \mathsf{c}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}}/24}\right), \tag{3.2.10}$$

where $\mathsf{y} = \mathsf{e}^{2\pi \mathrm{i}\theta}$, $\mathsf{z} = \mathsf{e}^{2\pi \mathrm{i}\zeta}$ and $\mathsf{q} = \mathsf{e}^{2\pi \mathrm{i}\tau}$. The additional factor involving κ is for convenience in deriving modular transformations. These characters do not always distinguish inequivalent simple modules as they do not keep track of the eigenvalue of Ω. This will be fixed in Section 3.2.4 by upgrading to one-point functions.

Our working hypothesis, for $\mathsf{k} = \frac{\mathsf{u}}{\mathsf{v}} - 3$ nondegenerate-admissible, is that the standard modules of $\mathsf{BP}(\mathsf{u},\mathsf{v})$ are spectral flows of the top-dense $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$ (with $[j] \in \mathbb{R}/\mathbb{Z}$ and $[\lambda] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$). Having standard modules in the twisted module category $\mathscr{W}^{\mathrm{tw}}_{\mathsf{u},\mathsf{v}}$, while the vacuum module belongs to the untwisted module category $\mathscr{W}_{\mathsf{u},\mathsf{v}}$, is inconvenient for Verlinde considerations. Hence we shall modify the conformal structure of the vertex operator algebra $\mathsf{BP}(\mathsf{u},\mathsf{v})$, under which certain spectral flows $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$ are both untwisted and relaxed.

The minimal model $\mathsf{BP}(\mathsf{u},\mathsf{v})$ admits a one-parameter family of conformal structures given by

$$\widetilde{L}(z) = L(z) + \alpha \partial J(z), \quad \alpha \in \mathbb{C}, \tag{3.2.11}$$

with corresponding central charges $\widetilde{\mathsf{c}}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}} = \mathsf{c}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}} - 24\alpha^2\kappa$. As modules are then graded by the eigenvalue of J_0 and $\widetilde{L}_0 = L_0 - \alpha J_0$, the appropriate choice of characters for this new conformal structure is

$$\widetilde{\mathrm{ch}}\big[\mathcal{M}\big]\,(\theta \,|\, \zeta \,|\, \tau) = \mathsf{y}^{\kappa}\,\mathrm{tr}_{\mathcal{M}}\left(\mathsf{z}^{J_0}\mathsf{q}^{\widetilde{L}_0 - \widetilde{\mathsf{c}}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}}/24}\right) = \mathrm{ch}\big[\mathcal{M}\big]\left(\theta + \alpha^2\tau \,\middle|\, \zeta - \alpha\tau \,\middle|\, \tau\right). \tag{3.2.12}$$

In general, a $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module that is positive-energy with respect to $L(z)$ is not positive-energy with respect to $\widetilde{L}(z)$. This is certainly true of the relaxed modules $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$.

Proposition 3.2.5. *Let k be nondegenerate-admissible and assume that $\alpha \in \frac{1}{2}\mathbb{Z}$. Then,*

$$\widetilde{\mathcal{R}}_{[j],[\lambda]} = \sigma^{\alpha}\big(\mathcal{R}^{\mathrm{tw}}_{[j-2\alpha\kappa],[\lambda]}\big) \tag{3.2.13}$$

is a relaxed highest-weight module with respect to $\widetilde{L}(z)$.

Proof. It follows from (2.1.18) and (3.2.11) that

$$\sigma^{\ell}\big(\widetilde{L}_0\big) = L_0 - (\ell+\alpha)J_0 + \ell(\ell+2\alpha)\kappa\mathbb{1} = \widetilde{L}_0 - \ell J_0 + \ell(\ell+2\alpha)\kappa\mathbb{1}. \tag{3.2.14}$$

If v_j denotes a relaxed highest-weight vector of $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$ of J_0-eigenvalue j, then

$$\begin{aligned} \widetilde{L}_0\sigma^{\ell}(v_j) &= \sigma^{\ell}\Big(\big(L_0 + (\ell-\alpha)J_0 + \ell(\ell-2\alpha)\kappa\mathbb{1}\big)v_j\Big) \\ &= \Big(\Delta^{\mathrm{tw}}(\lambda) + (\ell-\alpha)j + \ell(\ell-2\alpha)\kappa\Big)\sigma^{\ell}(v_j), \end{aligned} \tag{3.2.15}$$

hence the $\widetilde{L}_0$-eigenvalue is j-independent if and only if $\ell = \alpha$. ■

Note that the shift in j on the right-hand side of (3.2.13) ensures that the J_0-eigenvalues of $\widetilde{\mathcal{R}}_{[j],[\lambda]}$ coincide with the coset $[j] \in \mathbb{C}/\mathbb{Z}$. To ensure that the relaxed modules in (3.2.13) are untwisted, we shall choose the conformal structure on $\mathsf{BP}(\mathsf{u},\mathsf{v})$ given by $\widetilde{L}(z)$ with $\alpha = \frac{1}{2}$. The corresponding modules $\widetilde{\mathcal{R}}_{[j],[\lambda]}$ are untwisted and relaxed as desired. We therefore take our *standard* modules to be those of the form

$$\sigma^{\ell}\big(\widetilde{\mathcal{R}}_{[j],[\lambda]}\big), \tag{3.2.16}$$

with $\ell \in \mathbb{Z}$, $[j] \in \mathbb{R}/\mathbb{Z}$ and $[\lambda] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$. As spectral flow features heavily in what follows, we will occasionally denote the spectral flow of $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module $\mathcal{M}$ by $\sigma^{\ell}(\mathcal{M}) = \mathcal{M}^{\ell}$.

With the standard modules now identified, the first step is to compute their characters. Our approach is to use Proposition 3.2.4 to compute the characters of $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$. The characters of the standard modules can then be obtained using the following lemma.

Lemma 3.2.6. *Given any* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-module* $\mathcal{M}$ *that possesses a character and* $\ell \in \frac{1}{2}\mathbb{Z}$*, we have*

$$\begin{aligned} &\operatorname{ch}\big[\mathcal{M}^\ell\big]\,(\theta\,|\,\zeta\,|\,\tau) = \operatorname{ch}\big[\mathcal{M}\big]\,\Big(\theta + 2\ell\zeta + \ell^2\tau\,\Big|\,\zeta + \ell\tau\,\Big|\,\tau\Big) \\ \text{and}\quad &\widetilde{\operatorname{ch}}\big[\mathcal{M}^\ell\big]\,(\theta\,|\,\zeta\,|\,\tau) = \widetilde{\operatorname{ch}}\big[\mathcal{M}\big]\,(\theta + 2\ell\zeta + \ell(\ell+1)\tau\,|\,\zeta + \ell\tau\,|\,\tau)\,. \end{aligned} \tag{3.2.17}$$

Proof. The first character identity follows easily from (2.1.18):

$$\begin{aligned} \operatorname{ch}\big[\mathcal{M}^\ell\big]\,(\theta\,|\,\zeta\,|\,\tau) &= \operatorname{tr}_{\sigma^\ell(\mathcal{M})}\Big(\mathsf{y}^\kappa \mathsf{z}^{J_0} \mathsf{q}^{L_0 - \mathsf{c}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}}/24}\Big) \\ &= \operatorname{tr}_{\mathcal{M}}\Big(\mathsf{y}^\kappa \mathsf{z}^{\sigma^{-\ell}(J_0)} \mathsf{q}^{\sigma^{-\ell}(L_0) - \mathsf{c}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}}/24}\Big) \\ &= \operatorname{tr}_{\mathcal{M}}\Big(\mathsf{y}^\kappa \mathsf{z}^{J_0 + 2\kappa\ell\mathbb{1}} \mathsf{q}^{L_0 + \ell J_0 + \kappa\ell^2\mathbb{1} - \mathsf{c}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}}/24}\Big) \\ &= \operatorname{ch}\big[\mathcal{M}\big]\,\Big(\theta + 2\ell\zeta + \ell^2\tau\,\Big|\,\zeta + \ell\tau\,\Big|\,\tau\Big)\,. \end{aligned} \tag{3.2.18}$$

The second follows in the same way, but using that $\widetilde{L}_0 = L_0 - \frac{1}{2}J_0$. ■

By Proposition 3.2.4, the character of $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$ is the product of the characters of $\mathcal{W}_{[\lambda]}$ and $\Pi_{[j]}$, which are defined as

$$\operatorname{ch}\big[\mathcal{W}_{[\lambda]}\big]\,(\tau) = \operatorname{tr}_{\mathcal{W}_{[\lambda]}} \mathsf{q}^{T_0 - \mathsf{c}^{\mathsf{W}_3}_{\mathsf{u},\mathsf{v}}/24} \quad \text{and} \quad \operatorname{ch}\big[\Pi_{[j]}\big]\,(\zeta\,|\,\tau) = \operatorname{tr}_{\Pi_{[j]}}\Big(\mathsf{z}^{b_0} \mathsf{q}^{t_0 - \mathsf{c}^{\Pi}_{\mathsf{u},\mathsf{v}}/24}\Big). \tag{3.2.19}$$

Being modules over a lattice vertex operator algebra, the $\Pi_{[j]}$ have easily computed characters.

Proposition 3.2.7. *For all* $[j] \in \mathbb{C}/\mathbb{Z}$*, we have*

$$\operatorname{ch}\big[\Pi_{[j]}\big]\,(\zeta\,|\,\tau) = \frac{\mathsf{z}^j}{\eta(\tau)^2} \sum_{m\in\mathbb{Z}} \mathsf{z}^m, \tag{3.2.20}$$

where $\eta(\tau) = \mathsf{q}^{1/24}\prod_{n=1}^{\infty}(1-\mathsf{q}^n)$ *is the Dedekind eta function.*

The characters of the $\mathcal{W}_{[\lambda]}$ may be found in many places, for example [37, 74]. Explicit expressions are not needed for our purposes however. This is because the modular transformations of the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ characters are what is needed for the Verlinde formula, not the characters themselves. By Proposition 3.2.4, we have that

$$\begin{aligned} \operatorname{ch}\big[\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}\big]\,(\theta\,|\,\zeta\,|\,\tau) &= \mathsf{y}^\kappa \operatorname{ch}\big[\mathcal{W}_{[\lambda]}\big]\,(\tau) \operatorname{ch}\big[\Pi_{[j]}\big]\,(\zeta\,|\,\tau) \\ &= \frac{\mathsf{y}^\kappa \mathsf{z}^j \operatorname{ch}\big[\mathcal{W}_{[\lambda]}\big]\,(\tau)}{\eta(\tau)^2} \sum_{m\in\mathbb{Z}} \mathsf{z}^m. \end{aligned} \tag{3.2.21}$$

Proposition 3.2.8. *Let* k *be nondegenerate-admissible. Then, for all* $\ell \in \frac{1}{2}\mathbb{Z}$, $[j] \in \mathbb{C}/\mathbb{Z}$ *and* $[\lambda] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$, *the standard characters have the form*

$$\widetilde{\operatorname{ch}}\big[\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}\big]\,(\theta\,|\,\zeta\,|\,\tau) = \mathrm{e}^{2\pi\mathrm{i}\kappa\left(\theta-\ell(\ell+1)\tau\right)}\,\frac{\operatorname{ch}\big[\mathcal{W}_{[\lambda]}\big]\,(\tau)}{\eta(\tau)^2}\sum_{m\in\mathbb{Z}} \mathrm{e}^{2\pi\mathrm{i}m(j+2\kappa\ell)}\,\delta(\zeta+\ell\tau-m). \tag{3.2.22}$$

Proof. The untilded character of $\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}$ is related to that of $\mathcal{R}^{\mathrm{tw}}_{[j-\kappa],[\lambda]}$ in (3.2.21) by (3.2.13) and Lemma 3.2.6:

$$\begin{aligned}\operatorname{ch}\big[\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}\big]\,(\theta\,|\,\zeta\,|\,\tau) &= \operatorname{ch}\big[\sigma^{\ell+1/2}(\mathcal{R}^{\mathrm{tw}}_{[j-\kappa],[\lambda]})\big]\,(\theta\,|\,\zeta\,|\,\tau)\\ &= \frac{\mathsf{y}^{\kappa}\mathsf{z}^{j+2\ell\kappa}\mathsf{q}^{(\ell+1/2)j+(\ell^2-1/4)\kappa}\operatorname{ch}\big[\mathcal{W}_{[\lambda]}\big]\,(\tau)}{\eta(\tau)^2}\sum_{m\in\mathbb{Z}}\mathsf{z}^m\mathsf{q}^{(\ell+1/2)m}.\end{aligned} \tag{3.2.23}$$

By (3.2.12),

$$\widetilde{\operatorname{ch}}\big[\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}\big]\,(\theta\,|\,\zeta\,|\,\tau) = \frac{\mathsf{y}^{\kappa}\mathsf{z}^{j+2\ell\kappa}\mathsf{q}^{\ell j+\ell(\ell-1)\kappa}\operatorname{ch}\big[\mathcal{W}_{[\lambda]}\big]\,(\tau)}{\eta(\tau)^2}\sum_{m\in\mathbb{Z}}\mathsf{z}^m\mathsf{q}^{\ell m}. \tag{3.2.24}$$

The delta function in the character is obtained using the identity $\sum_{m\in\mathbb{Z}}\mathrm{e}^{2\pi\mathrm{i}mx} = \sum_{m\in\mathbb{Z}}\delta(x-m)$. ∎

3.2.4. One-Point Functions for Standard $\mathsf{BP}(\mathsf{u},\mathsf{v})$-Modules. As mentioned previously, an important feature of standard modules is that their characters are linearly independent. The standard characters of (3.2.22) do not satisfy this property. This can be seen directly from (3.2.22) (or rather (3.2.19)), as the characters for $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ do not keep track of the eigenvalue of W_0. The conjugation automorphism (1.3.19) of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ negates W_0-eigenvalues and preserves T_0-eigenvalues. Therefore if $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ admits a highest-weight vector with a nonzero W_0-eigenvalue, its character will be the same as its (distinct) conjugate. By Proposition 3.2.4, the characters (3.2.22) of $\mathsf{BP}(\mathsf{u},\mathsf{v})$ for such u and v will have the same issue.

A remedy for the lack of linear independence of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ characters was provided in [23] and detailed in Section 1.3.3. The proposal therein is to upgrade $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ characters to *one-point functions* by inserting the zero mode of some $u \in \mathsf{W}_3(\mathsf{u},\mathsf{v})$. As explained previously, it is always possible to choose $u \in \mathsf{W}_3(\mathsf{u},\mathsf{v})$ that gives linearly independent one-point functions and allows for the computation of an S-matrix and fusion rules.

We can similarly upgrade the definition of $\mathsf{BP}(\mathsf{u},\mathsf{v})$-characters to one-point functions as follows:

$$\begin{aligned}\operatorname{ch}\big[\mathcal{M}\big]\,(\theta\,|\,\zeta\,|\,\tau\,;u) &= \mathsf{y}^{\kappa}\operatorname{tr}_{\mathcal{M}}\left(u_0\mathsf{z}^{J_0}\mathsf{q}^{L_0-\mathsf{c}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}}/24}\right),\\ \widetilde{\operatorname{ch}}\big[\mathcal{M}\big]\,(\theta\,|\,\zeta\,|\,\tau\,;u) &= \mathsf{y}^{\kappa}\operatorname{tr}_{\mathcal{M}}\left(u_0\mathsf{z}^{J_0}\mathsf{q}^{\widetilde{L}_0-\widetilde{\mathsf{c}}^{\mathsf{BP}}_{\mathsf{u},\mathsf{v}}/24}\right),\end{aligned} \qquad u \in \mathsf{BP}(\mathsf{u},\mathsf{v}). \tag{3.2.25}$$

Here we would also like to choose u such that the one-point functions of standard modules are linearly independent. The one-point functions of interest here are those of standard modules, which by Proposition 3.2.4 are always $\mathsf{W}_3(\mathsf{u},\mathsf{v}) \otimes \Pi$-modules.

It therefore suffices to take u to be an element of $\mathsf{W}_3(\mathsf{u},\mathsf{v}) \otimes \Pi$ and know that the one-point functions (3.2.25), with $\mathcal{M}$ standard, are well defined. In particular, choose $u = \mathbb{1} \otimes \mathbb{1}$ when $(\mathsf{u},\mathsf{v}) \in \{(3,4),(4,3),(3,5),(5,3)\}$ and $u = W \otimes \mathbb{1}$ otherwise. As the Π part of u is always $\mathbb{1}$, we may treat u as an element of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$.

Proposition 3.2.9. *Let k be nondegenerate-admissible. Then, for all $\ell \in \frac{1}{2}\mathbb{Z}$, $[j] \in \mathbb{C}/\mathbb{Z}$ and $[\lambda] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$, we have*

(3.2.26)
$$\widetilde{\mathrm{ch}}\big[\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}\big]\,(\theta\,|\,\zeta\,|\,\tau\,;u) = \mathrm{e}^{2\pi\mathrm{i}\kappa\left(\theta-\ell(\ell+1)\tau\right)}\,\frac{\mathrm{ch}\big[\mathcal{W}_{[\lambda]}\big]\,(\tau\,;u)}{\eta(\tau)^2}\sum_{m\in\mathbb{Z}}\mathrm{e}^{2\pi\mathrm{i}m(j+2\kappa\ell)}\delta(\zeta+\ell\tau-m).$$

Moreover, if we take $u = \mathbb{1}$ when $(\mathsf{u},\mathsf{v}) \in \{(3,4),(4,3),(3,5),(5,3)\}$ and $u = W$ otherwise, then these standard one-point functions are linearly independent.

3.2.5. Modular Transformations of Standard One-Point Functions. Recall from Theorem 1.3.4 that the S-transform of the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ one-point functions takes the following simple form:

$$\mathrm{ch}\big[\mathcal{W}_{[\lambda]}\big]\left(-\frac{1}{\tau}\,;\,\frac{u}{\tau^{\Delta_u}}\right) = \sum_{[\lambda']\in\Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3}\mathsf{S}^{\mathsf{W}_3}_{[\lambda],[\lambda']}\,\mathrm{ch}\big[\mathcal{W}_{[\lambda']}\big]\,(\tau\,;u)\,, \tag{3.2.27}$$

where we have defined $\mathsf{S}^{\mathsf{W}_3}_{[\lambda],[\lambda']} = \mathsf{S}^{\mathsf{W}_3}_{(\mathsf{r},\mathsf{s}),(\mathsf{r}',\mathsf{s}')}$ for $\lambda = \Gamma(\mathsf{r},\mathsf{s})$ and $\lambda' = \Gamma(\mathsf{r}',\mathsf{s}')$. The explicit form of the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix $\mathsf{S}^{\mathsf{W}_3}_{[\lambda],[\lambda']}$ is given in (1.3.24).

Define the following transformations on the parameter space $(\theta\,|\,\zeta\,|\,\tau\,;u)$:

$$\begin{aligned}\mathsf{S}\colon\ (\theta\,|\,\zeta\,|\,\tau\,;\,u) &\longmapsto \left(\theta-\frac{\zeta^2}{\tau}-\frac{\zeta}{\tau}+\zeta\,\middle|\,\frac{\zeta}{\tau}\,\middle|\,-\frac{1}{\tau}\,;\,\frac{u}{\tau^{\Delta_u}}\right),\\ \mathsf{T}\colon\ (\theta\,|\,\zeta\,|\,\tau\,;\,u) &\longmapsto (\theta\,|\,\zeta\,|\,\tau+1\,;\,u)\,.\end{aligned} \tag{3.2.28}$$

That this defines an $\mathsf{SL}_2(\mathbb{Z})$-action is a straightforward computation:

$$\mathsf{S}^2 = (\mathsf{ST})^3 = \mathsf{C}\colon\ (\theta\,|\,\zeta\,|\,\tau\,;\,u) \longmapsto \left(\theta+2\zeta\,\middle|\,-\zeta\,\middle|\,\tau\,;\,(-1)^{\Delta_u}u\right). \tag{3.2.29}$$

Obviously, C squares to the identity as required.

Theorem 3.2.10. *Let k be nondegenerate-admissible. Then, for each $\ell \in \mathbb{Z}$, $[j] \in \mathbb{R}/\mathbb{Z}$ and $[\lambda] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$, the S-transform of the one-point function of $\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}$ is given by*

$$\mathsf{S}\Big\{\widetilde{\mathrm{ch}}\big[\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}\big]\Big\} = \frac{|\tau|}{-\mathrm{i}\tau}\sum_{\ell'\in\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}}\sum_{[\lambda']\in\Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3}\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell,[j],[\lambda]}\,\widetilde{\mathrm{ch}}\big[\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}\big]\,\mathrm{d}[j'], \tag{3.2.30}$$

where the entries of the 'S-matrix' (integral kernel) are

(3.2.31)
$$\mathsf{S}_{\ell,[j],[\lambda]}^{\ell',[j'],[\lambda']} = \mathsf{S}_{[\lambda],[\lambda']}^{\mathsf{W}_3} \mathsf{e}^{-2\pi \mathsf{i}\left(2\kappa\ell\ell'+\ell(j'-\kappa)+(j-\kappa)\ell'\right)}.$$

Proof. Our strategy is to evaluate and simplify both sides of (3.2.30). Starting with the left-hand side, we have

(3.2.32)
$$\begin{aligned}
\mathsf{S}\Big\{\widetilde{\mathrm{ch}}\big[\widetilde{\mathcal{R}}_{[j],[\lambda]}^{\ell}\big]\,(\theta \,|\, \zeta \,|\, \tau\,;u)\Big\} &= \widetilde{\mathrm{ch}}\big[\widetilde{\mathcal{R}}_{[j],[\lambda]}^{\ell}\big]\left(\theta - \frac{\zeta^2}{\tau} - \frac{\zeta}{\tau} + \zeta \,\middle|\, \frac{\zeta}{\tau} \,\middle|\, -\frac{1}{\tau}\,;\frac{u}{\tau^{\Delta_u}}\right) \\
&= \exp\left[2\pi \mathsf{i}\kappa\left(\theta - \frac{\zeta^2}{\tau} - \frac{\zeta}{\tau} + \zeta + \frac{\ell(\ell+1)}{\tau}\right)\right] \\
&\qquad \cdot \frac{\mathrm{ch}\big[\mathcal{W}_{[\lambda]}\big]\left(-\frac{1}{\tau}\,;\frac{u}{\tau^{\Delta_u}}\right)}{-\mathsf{i}\tau\eta(\tau)^2} \sum_{m\in\mathbb{Z}} \mathsf{e}^{2\pi \mathsf{i} m(j+2\kappa\ell)}\delta\left(\frac{\zeta}{\tau} - \frac{\ell}{\tau} - m\right) \\
&= \frac{|\tau|}{-\mathsf{i}\tau\eta(\tau)^2} \sum_{[\lambda']} \mathsf{S}_{[\lambda],[\lambda']}^{\mathsf{W}_3}\,\mathrm{ch}\big[\mathcal{W}_{[\lambda']}\big]\,(\tau\,;u) \\
&\qquad \cdot \exp\left[2\pi \mathsf{i}\kappa\left(\theta - \frac{\zeta^2}{\tau} - \frac{\zeta}{\tau} + \zeta + \frac{\ell(\ell+1)}{\tau}\right)\right] \sum_{m\in\mathbb{Z}} \mathsf{e}^{2\pi \mathsf{i}(j+2\kappa\ell)m}\delta(\zeta - \ell - m\tau) \\
&= \frac{|\tau|}{-\mathsf{i}\tau\eta(\tau)^2} \sum_{[\lambda']} \mathsf{S}_{[\lambda],[\lambda']}^{\mathsf{W}_3}\,\mathrm{ch}\big[\mathcal{W}_{[\lambda']}\big]\,\mathsf{e}^{2\pi \mathsf{i}\kappa(\theta+\ell)} \sum_{m\in\mathbb{Z}} \mathsf{e}^{-2\pi \mathsf{i}\left((j-\kappa)m+\kappa m(m+1)\tau\right)}\delta(\zeta + m\tau - \ell),
\end{aligned}$$

using (3.2.26), (3.2.27) and the well known S-transform of Dedekind's eta function. Here, and below, the $[\lambda']$-sums run over $\Gamma_{\mathsf{u,v}}/\mathbb{Z}_3$.

Inserting the S-matrix elements (3.2.31) into the right-hand side of (3.2.30),

(3.2.33)
$$\begin{aligned}
&\frac{|\tau|}{-\mathsf{i}\tau} \sum_{\ell'\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \sum_{[\lambda']} \mathsf{S}_{\ell,[j],[\lambda]}^{\ell',[j'],[\lambda']}\,\widetilde{\mathrm{ch}}\big[\widetilde{\mathcal{R}}_{[j'],[\lambda']}^{\ell'}\big]\,\mathrm{d}[j'] \\
&\quad = \frac{|\tau|}{-\mathsf{i}\tau\eta(\tau)^2} \sum_{[\lambda']} \mathsf{S}_{[\lambda],[\lambda']}^{\mathsf{W}_3}\,\mathrm{ch}\big[\mathcal{W}_{[\lambda']}\big] \\
&\qquad \cdot \sum_{\ell'\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \mathsf{e}^{-2\pi \mathsf{i}\left(2\kappa\ell\ell'+\ell(j'-\kappa)+(j-\kappa)\ell'\right)}\mathsf{e}^{2\pi \mathsf{i}\kappa\left(\theta-\ell'(\ell'+1)\tau\right)} \sum_{m\in\mathbb{Z}} \mathsf{e}^{2\pi \mathsf{i}(j'+2\kappa\ell')m}\delta(\zeta+\ell'\tau-m)\,\mathrm{d}[j'] \\
&\quad = \frac{|\tau|}{-\mathsf{i}\tau\eta(\tau)^2} \sum_{[\lambda']} \mathsf{S}_{[\lambda],[\lambda']}^{\mathsf{W}_3}\,\mathrm{ch}\big[\mathcal{W}_{[\lambda']}\big] \sum_{\ell'\in\mathbb{Z}} \mathsf{e}^{-2\pi i\left((j-\kappa)\ell'-\kappa\ell\right)}\mathsf{e}^{2\pi i\kappa\left(\theta-\ell'(\ell'+1)\tau\right)}\delta(\zeta+\ell'\tau-\ell) \\
&\quad = \frac{|\tau|}{-\mathsf{i}\tau\eta(\tau)^2} \sum_{[\lambda']} \mathsf{S}_{[\lambda],[\lambda']}^{\mathsf{W}_3}\,\mathrm{ch}\big[\mathcal{W}_{[\lambda']}\big]\,\mathsf{e}^{2\pi \mathsf{i}\kappa(\theta+\ell)} \sum_{m\in\mathbb{Z}} \mathsf{e}^{-2\pi \mathsf{i}\left((j-\kappa)m+\kappa m(m+1)\tau\right)}\delta(\zeta+m\tau-\ell). \qquad \blacksquare
\end{aligned}$$

An explicit formula for the (diagonal) T-matrix of the standard one-point functions is very easy to derive. Only the S-matrix of one-point functions is needed here however.

The 'matrix element' $\mathsf{S}_{\ell,[j],[\lambda]}^{\ell',[j'],[\lambda']}$ is symmetric because the corresponding $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix element $\mathsf{S}_{[\lambda],[\lambda']}^{\mathsf{W}_3}$ is. It is also easy to check that the $\mathsf{BP}(\mathsf{u},\mathsf{v})$ S-matrix is unitary and its square represents conjugation, properties which again follow from those of the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix.

3.3. Fusion Rules for BP Minimal Models

Crucial to the standard module formalism is that all simple objects in the category of weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules can be resolved in terms of the nonsimple standard modules to which we give the special notation

$$\sigma^{\ell+1/2}(\mathcal{R}_{\lambda}^{\text{tw}}) = \sigma^{\ell+1/2}(\mathcal{R}_{[j^{\text{tw}}(\lambda)],[\lambda]}^{\text{tw}}) = \sigma^{\ell}(\widetilde{\mathcal{R}}_{[j^{\text{tw}}(\lambda)+\kappa],[\lambda]}) = \widetilde{\mathcal{R}}_{\lambda}^{\ell}, \quad \ell \in \mathbb{Z},\ \lambda \in \Gamma_{\mathsf{u},\mathsf{v}}. \tag{3.3.1}$$

Note that this notation breaks the ∇-orbit symmetry for the nonsimple 'gap' modules: $\mathcal{R}_{\lambda}^{\text{tw}} \simeq \mathcal{R}_{\mu}^{\text{tw}}$ if and only if $\lambda = \mu$ in $\Gamma_{\mathsf{u},\mathsf{v}}$.

In this section, we shall derive the desired resolutions and determine the consequent modularity of a subset of the remaining simple modules (the type-3 modules). This includes the modularity of the vacuum module, recalling that the the S-matrix elements of the vacuum module play a prominent role in the usual Verlinde formula (1.1.45). We then move on to the computation of fusion coefficients and rules.

For logarithmic vertex operator algebras such as $\mathsf{BP}(\mathsf{u},\mathsf{v})$ with $\mathsf{v} \geqslant 3$, the Verlinde formula is no longer guaranteed to produce nonnegative integer fusion coefficients. Fortunately the standard module formalism provides a conjectural extension that has been successfully tested in a wide range of examples. This is known as the *standard Verlinde formula* [**51**, **146**]. The ingredients of the standard Verlinde formula are the modular transformations of the standard modules (3.2.31) and that of the vacuum module.

As all modular transformations are obtained by analysis of $\mathsf{BP}(\mathsf{u},\mathsf{v})$ one-point functions, the quantities computed by the standard Verlinde formula do not distinguish between a $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module and its semisimplification. This is of course not a problem for rational vertex operator algebras where all modules are completely reducible. So rather than computing the fusion coefficients of $\mathsf{BP}(\mathsf{u},\mathsf{v})$, the standard Verlinde formula computes the *Grothendieck* fusion coefficients. These are the structure constants of the Grothendieck group of the category of standard modules, equipped with (the image of) the fusion product. It is often possible, as we will see, to 'upgrade' Grothendieck fusion rules to fusion rules using conformal grading considerations to rule out non-split extensions.

To consistently equip the Grothendieck group with the fusion product, one needs to know that fusing with a standard module defines an exact functor. This appears to be very difficult to establish, so we shall have to conjecture that it does hold. In fact, we believe that a slightly stronger statement is true: the category of weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules is rigid. Assuming this, the standard Verlinde conjecture is as follows.

CONJECTURE. *Let* k *be admissible-nondegenerate. Then, for* $\ell,\ell' \in \mathbb{Z}$*,* $[j],[j'] \in \mathbb{R}/\mathbb{Z}$ *and* $[\lambda],[\lambda'] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$*, the Grothendieck fusion rules of the standard* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules are given by*

(3.3.2a)
$$\big[\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}\big] \boxtimes \big[\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}\big] = \sum_{\ell''\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \sum_{[\lambda'']\in\Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3} \begin{pmatrix} & \ell'',[j''],[\lambda''] & \\ \ell,[j],[\lambda] & & \ell',[j'],[\lambda'] \end{pmatrix} \big[\widetilde{\mathcal{R}}^{\ell''}_{[j''],[\lambda'']}\big]\, \mathrm{d}[j''].$$

The Grothendieck fusion coefficients are given by

(3.3.2b)
$$\begin{pmatrix} & \ell'',[j''],[\lambda''] & \\ \ell,[j],[\lambda] & & \ell',[j'],[\lambda'] \end{pmatrix} = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \sum_{[\mu]\in\Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3} \frac{\mathsf{S}^{m,[k],[\mu]}_{\ell,[j],[\lambda]}\, \mathsf{S}^{m,[k],[\mu]}_{\ell',[j'],[\lambda']} \Big(\mathsf{S}^{m,[k],[\mu]}_{\ell'',[j''],[\lambda'']}\Big)^*}{\mathsf{S}^{m,[k],[\mu]}_{\mathrm{vac.}}}\, \mathrm{d}[k],$$

where $\mathsf{S}^{m,[k],[\mu]}_{\mathrm{vac.}}$ *is the S-matrix element corresponding to the vacuum module* $\mathcal{H}_{\mathsf{k}\omega_0}$*, and the asterisk indicates complex conjugation.*

The standard Verlinde formula (3.3.2b) is superficially very similar to the Verlinde formula (1.1.45). The key difference is the need to integrate (rather than sum) over part of the parametrisation of standard modules. The results obtained in the remainder of this section will implicitly assume that this conjecture holds.

The Bershadsky–Polyakov minimal models with with $\mathsf{v} = 3$ have the desirable feature that every highest-weight module is type-3 (Corollary 2.3.23). This means that the resolutions of these modules all have the same form up to spectral flow. We will therefore begin with the analysis of $\mathsf{BP}(\mathsf{u},3)$, purely to present the analysis with a minimum of complications. The more technically demanding case of $\mathsf{v} > 3$ will be discussed in Section 3.3.4.

For convenience, we will frequently use the following notation for $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules:

(3.3.3)
$$\begin{gathered} \mathcal{H}_\lambda = \mathcal{H}(\mathsf{r},\mathsf{s}) = \mathcal{H}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\big], \quad \mathcal{H}^{\mathrm{tw}}_\lambda = \mathcal{H}^{\mathrm{tw}}(\mathsf{r},\mathsf{s}) = \mathcal{H}^{\mathrm{tw}}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\big], \\ \mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]} = \mathcal{R}^{\mathrm{tw}}_{[j]}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\big], \quad \mathcal{R}^{\mathrm{tw}}_\lambda = \mathcal{R}^{\mathrm{tw}}(\mathsf{r},\mathsf{s}) = \mathcal{R}^{\mathrm{tw}}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\big], \\ \widetilde{\mathcal{R}}_{[j],[\lambda]} = \widetilde{\mathcal{R}}_{[j]}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\big], \quad \widetilde{\mathcal{R}}_\lambda = \widetilde{\mathcal{R}}(\mathsf{r},\mathsf{s}) = \widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\big], \end{gathered}$$

when $\lambda = \Gamma(\mathsf{r},\mathsf{s}) = \Gamma\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\big]$. The ranges $\mathsf{r} \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$ and $\mathsf{s} \in \mathsf{P}^{\mathsf{v}-3}_{\geqslant}$ cover all $\lambda \in \Gamma_{\mathsf{u},\mathsf{v}}$, while allowing s_0, s_1 and s_2 to take the values $\mathsf{v}-2$, -1 and $\mathsf{v}-2$ respectively (but still subject to $s_0+s_1+s_2 = \mathsf{v}-3$) gives the remaining elements of $\Sigma_{\mathsf{u},\mathsf{v}}$.

By recasting Theorem 2.3.32 in terms of r and s and applying Proposition 2.3.19, we obtain a short exact sequence that will prove very useful in resolving highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules:

Proposition 3.3.1. *Let* k *be nondegenerate-admissible and choose* $\Gamma(\mathsf{r},\mathsf{s}) \in \Sigma_{\mathsf{u},\mathsf{v}}$ *leftmost in its orbit, as pictured in Figure 1. Then, we have the following nonsplit short exact sequence:*

$$0 \longrightarrow \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+1 & s_2-1 \end{smallmatrix}\right]^1 \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+1 & s_2-1 \end{smallmatrix}\right] \longrightarrow \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1 & s_2 \end{smallmatrix}\right] \longrightarrow 0. \tag{3.3.4}$$

Here, $\Gamma\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+1 & s_2-1 \end{smallmatrix}\right] \in \Gamma_{\mathsf{u},\mathsf{v}}$ *is the rightmost in its orbit. It is type-n under the following conditions:*

$n=1$	$n=2$	$n=3$
$s_2 \neq 1$	$s_1 \neq \mathsf{v}-4$ *and* $s_2 = 1$	$\mathsf{s}_1 = [0, \mathsf{v}-4, 1]$

3.3.1. One-Point Functions for Highest-Weight $\mathsf{BP}(\mathsf{u},3)$**-Modules.** For $\mathsf{v}=3$, the allowed weights s such that $\Gamma(\mathsf{r},\mathsf{s}) \in \Sigma_{\mathsf{u},\mathsf{v}}$ are $[0,-1,1]$, $[1,-1,0]$ and $[0,0,0]$. All highest-weight $\mathsf{BP}(\mathsf{u},3)$-modules are type-3, and those leftmost in their spectral flow orbits have $\mathsf{s} = [0,-1,1]$. The short exact sequence of Proposition 3.3.1 is thus

$$0 \longrightarrow \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\right]^1 \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\right] \longrightarrow \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & -1 & 1 \end{smallmatrix}\right] \longrightarrow 0. \tag{3.3.5}$$

The highest weight of $\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ is in $\Gamma_{\mathsf{u},3}$, hence it is the rightmost in its orbit. As the spectral flow orbit is type-3, it is obtained from the leftmost by spectrally flowing twice. That is, by Theorem 2.3.21,

$$\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\right] \simeq \mathcal{H}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & -1 & 1 \end{smallmatrix}\right]^2. \tag{3.3.6}$$

As spectral flow is exact, we can therefore splice the exact sequence (3.3.5) with that obtained by applying σ^3 to the corresponding exact sequence with quotient $\mathcal{H}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & -1 & 1 \end{smallmatrix}\right]$. Iterating this results in infinite length resolutions of type-3 $\mathsf{BP}(\mathsf{u},3)$-modules in terms of nonsimple standard modules:

Proposition 3.3.2. *Let* k *be admissible with* $\mathsf{v}=3$*. Then, every simple highest-weight* $\mathsf{BP}(\mathsf{u},3)$*-module is resolved by the nonsimple standard modules as follows:*

$$\cdots \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\right]^9 \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & 0 & 0 \end{smallmatrix}\right]^6 \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & 0 & 0 \end{smallmatrix}\right]^3 \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\right] \to \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & -1 & 1 \end{smallmatrix}\right] \to 0, \tag{3.3.7a}$$

$$\cdots \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & 0 & 0 \end{smallmatrix}\right]^{10} \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & 0 & 0 \end{smallmatrix}\right]^7 \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\right]^4 \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & 0 & 0 \end{smallmatrix}\right]^1 \to \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\right] \to 0, \tag{3.3.7b}$$

$$\cdots \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & 0 & 0 \end{smallmatrix}\right]^{11} \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\right]^8 \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & 0 & 0 \end{smallmatrix}\right]^5 \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & 0 & 0 \end{smallmatrix}\right]^2 \to \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\right] \to 0. \tag{3.3.7c}$$

Proof. The first resolution is the result of the aforementioned splicing the short exact sequence (3.3.5). The other two resolutions are obtained from the first by applying a suitable amount of spectral flow. ■

The one-point functions of a highest-weight $\mathsf{BP}(\mathsf{u},3)$-module can be easily read off from its corresponding resolution. For reasons that will become clear shortly, we centre our analysis on the highest-weight $\mathsf{BP}(\mathsf{u},3)$-modules with $\mathsf{s} = [1,-1,0]$. We will also suppress the arguments of one-point functions for readability.

Corollary 3.3.3. *Let* k *be admissible with* $\mathsf{v} = 3$*. Then, for all* $\mathsf{r} \in \mathsf{P}_{\geqslant}^{\mathsf{u}-3}$ *and* $\ell \in \frac{1}{2}\mathbb{Z}$*, we have*

$$\widetilde{\mathrm{ch}}\Big[\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell}\Big] = \sum_{n=0}^{\infty}(-1)^n\Big(\widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}\Big[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & 0 & 0 \end{smallmatrix}\Big]^{\ell+9n+1}\Big] - \widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\Big]^{\ell+9n+4}\Big] + \widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}\Big[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & 0 & 0 \end{smallmatrix}\Big]^{\ell+9n+7}\Big]\Big). \tag{3.3.8}$$

As the r-labels of the three summands appearing on the right-hand side of (3.3.8) are related by the $\mathbb{Z}_3$-action, we can rewrite (3.3.8) in the following alternative form:

$$\widetilde{\mathrm{ch}}\Big[\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell}\Big] = \sum_{n=0}^{\infty}(-1)^n\Big(\widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}(\nabla^{-1}(\mathsf{r}),0)^{\ell+9n+1}\Big] - \widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}(\mathsf{r},0)^{\ell+9n+4}\Big] + \widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}(\nabla(\mathsf{r}),0)^{\ell+9n+7}\Big]\Big). \tag{3.3.9}$$

where 0 is shorthand for the s-triple $[0,0,0]$. The S-transforms of the highest-weight one-point functions thus follow from the standard ones, computed in Theorem 3.2.10. For this, it is convenient to introduce notation for the J_0-eigenvalue of a highest weight with $\mathsf{s} = [1,-1,0]$:

$$j(\mathsf{r}) \equiv j\big(\Gamma(\mathsf{r},[1,-1,0])\big) = \tfrac{1}{3}(r_1 - r_2). \tag{3.3.10}$$

Theorem 3.3.4. *Let* k *be admissible with* $\mathsf{v} = 3$*, and* $\mathsf{r} \in \mathsf{P}_{\geqslant}^{\mathsf{u}-3}$*. Then for all* $\ell \in \mathbb{Z}$*, the S-transform of the one-point function of* $\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell}$ *is given by*

$$\mathsf{S}\Big\{\widetilde{\mathrm{ch}}\Big[\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell}\Big]\Big\} = \frac{|\tau|}{-\mathrm{i}\tau}\sum_{\ell'\in\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}}\sum_{[\lambda']\in\Gamma_{\mathsf{u},3}/\mathbb{Z}_3}\mathsf{S}_{\ell,\mathsf{r}}^{\ell',[j'],[\lambda']}\,\widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}_{[j'],[\lambda']}^{\ell'}\Big]\,\mathrm{d}[j'], \tag{3.3.11}$$

where the entries of the 'highest-weight S-matrix' are given by

$$\mathsf{S}_{\ell,\mathsf{r}}^{\ell',[j'],[\lambda']} = \mathsf{S}_{[\Gamma(\mathsf{r},0)],[\lambda']}^{\mathsf{W}_3}\,\frac{\mathrm{e}^{-2\pi\mathrm{i}\left(2\kappa(\ell-1/2)\ell'+(\ell-1/2)(j'-\kappa)+j(\mathsf{r})\ell'\right)}}{2\cos\big(3\pi(j'-\kappa)\big)}. \tag{3.3.12}$$

Proof. By Corollary 3.3.3, the S-matrix entry $\mathsf{S}_{\ell,\mathsf{r}}^{\ell',[j'],[\lambda']}$ from the S-transform of the one-point function of $\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell}$ may be written as an infinite linear combination of standard S-matrix entries (3.2.31). Recall that $\widetilde{\mathcal{R}}_{\mu} = \widetilde{\mathcal{R}}_{[j(\mu)+2\kappa],[\mu]}$ and note that the $\mu \in \Gamma_{\mathsf{u},\mathsf{v}}$ appearing in the standard one-point functions on the right-hand side of (3.3.9) all belong to the same class $[\Gamma(\mathsf{r},0)]$ in $\Gamma_{\mathsf{u},3}/\mathbb{Z}_3$. Comparing each $j(\mu)$ with $j(\mathsf{r})$ and noting that $\mathsf{u} = 9(\kappa + \frac{1}{2})$ (since $\mathsf{v} = 3$) gives

$$\mathsf{S}_{\ell,\mathsf{r}}^{\ell',[j'],[\lambda']} = \sum_{n=0}^{\infty}(-1)^n\Big(\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+9n+1,[j(\mathsf{r})-2\kappa],[\Gamma(\mathsf{r},0)]} - \mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+9n+4,[j(\mathsf{r})+\kappa-\frac{1}{2}],[\Gamma(\mathsf{r},0)]} + \mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+9n+7,[j(\mathsf{r})+4\kappa],[\Gamma(\mathsf{r},0)]}\Big). \tag{3.3.13}$$

Here we have also used the linear independence of standard one-point functions (Proposition 3.2.9). From (3.2.31), we obtain

$$\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+9n+1,[j(\mathsf{r})-2\kappa],[\Gamma(\mathsf{r},0)]} = \mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r},0)],[\lambda']}\mathsf{e}^{-2\pi\mathsf{i}\left(2\kappa(\ell+9n+1)\ell'+(\ell+9n+1)(j'-\kappa)+(j(\mathsf{r})-3\kappa)\ell'\right)}, \tag{3.3.14a}$$

$$\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+9n+4,[j(\mathsf{r})+\kappa-\frac{1}{2}],[\Gamma(\mathsf{r},0)]} = \mathsf{e}^{-6\pi\mathsf{i}(j'-\kappa)}\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+9n+1,[j(\mathsf{r})-2\kappa],[\Gamma(\mathsf{r},0)]}, \tag{3.3.14b}$$

$$\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+9n+7,[j(\mathsf{r})+4\kappa],[\Gamma(\mathsf{r},0)]} = \mathsf{e}^{-12\pi\mathsf{i}(j'-\kappa)}\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+9n+1,[j(\mathsf{r})-2\kappa],[\Gamma(\mathsf{r},0)]}. \tag{3.3.14c}$$

Substituting into (3.3.13) gives

(3.3.15)

$$\begin{aligned}\mathsf{S}_{\ell,\mathsf{r}}^{\ell',[j'],[\lambda']} &= \Big(1-\mathsf{e}^{-6\pi\mathsf{i}(j'-\kappa)}+\mathsf{e}^{-12\pi\mathsf{i}(j'-\kappa)}\Big)\\ &\qquad\cdot\mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r},0)],[\lambda']}\sum_{n=0}^{\infty}(-1)^n\mathsf{e}^{-2\pi\mathsf{i}\left(2\kappa(\ell+9n+1)\ell'+(\ell+9n+1)(j'-\kappa)+(j(\mathsf{r})-3\kappa)\ell'\right)}\\ &= \frac{1+\mathsf{e}^{-18\pi\mathsf{i}(j'-\kappa)}}{1+\mathsf{e}^{-6\pi\mathsf{i}(j'-\kappa)}}\mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r},0)],[\lambda']}\mathsf{e}^{-2\pi\mathsf{i}\left(2\kappa(\ell+1)\ell'+(\ell+1)(j'-\kappa)+(j(\mathsf{r})-3\kappa)\ell'\right)}\frac{1}{1+\mathsf{e}^{-18\pi\mathsf{i}(j'-\kappa)}}\\ &= \mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r},0)],[\lambda']}\frac{\mathsf{e}^{-2\pi\mathsf{i}\left(2\kappa(\ell-1/2)\ell'+(\ell-1/2)(j'-\kappa)+j(\mathsf{r})\ell'\right)}}{2\cos\big(3\pi(j'-\kappa)\big)}.\end{aligned}$$ ∎

A particularly important type-3 weight is $\mathsf{k}\omega_0 = \Gamma\left[\begin{smallmatrix}\mathsf{u}-3 & 0 & 0\\ 1 & -1 & 0\end{smallmatrix}\right]$, whose corresponding highest-weight $\mathsf{BP}(\mathsf{u},3)$-module is the vacuum module. This is due to its distinguished role in the standard Verlinde formula (3.3.2b). Recall the special notation $\mathsf{S}_{\text{vac.}}^{\ell',[j'],[\lambda']} = \mathsf{S}_{0,[\mathsf{u}-3,0,0]}^{\ell',[j'],[\lambda']}$.

Corollary 3.3.5. *Let* k *be admissible with* $\mathsf{v} = 3$. *Then,*

$$\mathsf{S}_{\text{vac.}}^{\ell',[j'],[\lambda']} = \mathsf{S}^{\mathsf{W}_3}_{\text{vac.},[\lambda']}\frac{\mathsf{e}^{2\pi\mathsf{i}\kappa\ell'}\mathsf{e}^{\pi\mathsf{i}(j'-\kappa)}}{2\cos\big(3\pi(j'-\kappa)\big)}, \qquad \mathsf{S}^{\mathsf{W}_3}_{\text{vac.},[\lambda']} = \mathsf{S}^{\mathsf{W}_3}_{[\Gamma([\mathsf{u}-3,0,0],0)],[\lambda']}. \tag{3.3.16}$$

Note that $\mathcal{W}([\mathsf{u}-3,0,0],0)$ is the vacuum module of $\mathsf{W}_3(\mathsf{u},3)$ because (1.3.17) gives

$$\Delta\left[\begin{smallmatrix}\mathsf{u}-3 & 0 & 0\\ 0 & 0 & 0\end{smallmatrix}\right] = w\left[\begin{smallmatrix}\mathsf{u}-3 & 0 & 0\\ 0 & 0 & 0\end{smallmatrix}\right] = 0. \tag{3.3.17}$$

Interestingly, the vacuum S-matrix element (3.3.16) diverges when $\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}$ is nonsimple. This same phenomenon was encountered in the analysis of the nonrational admissible-level $\mathfrak{sl}_2$ minimal models reported in [52]. To see this divergence, recall that $\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}$ is nonsimple when $[j'] = [j^{\mathrm{tw}}((\nabla^i(\lambda')) + \kappa]$ for some $i \in \mathbb{Z}_3$. As $\nabla^i(\lambda') \in \Gamma_{\mathsf{u},3}$, it can be written as $\Gamma(\mathsf{r},0)$ for some $\mathsf{r} \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$. However,

$$j^{\mathrm{tw}}\left(\Gamma(\mathsf{r},0)\right) + \kappa = \tfrac{1}{3}\left(r_1 - r_2\right) - \tfrac{1}{2}, \tag{3.3.18}$$

so the denominator of (3.3.16) becomes $\cos\bigl(\pi(r_1 - r_2) - \frac{3\pi}{2}\bigr) = 0$.

3.3.2. Grothendieck Fusion Rules for $\mathsf{BP}(\mathsf{u},3)$. With the 'S-matrix elements' (3.2.31) and (3.3.16) in hand, we are now in a position to compute the Grothendieck fusion rules of the $\mathsf{BP}(\mathsf{u},3)$-modules using the standard Verlinde formula (3.3.2b). The first and most important fusion rule is that between the standard modules.

The S-matrix for an $\ell \neq 0$ standard module can be expressed in terms of that of an $\ell = 0$ standard module by

$$\mathsf{S}^{m,[k],[\mu]}_{\ell,[j],[\lambda]} = \mathsf{e}^{-2\pi i\ell(2\kappa\ell'+j'-\kappa)}\,\mathsf{S}^{m,[k],[\mu]}_{0,[j],[\lambda]}. \tag{3.3.19}$$

By the standard Verlinde formula, this means that

$$\begin{pmatrix} & \ell'', [j''], [\lambda''] & \\ \ell, [j], [\lambda] & & \ell', [j'], [\lambda'] \end{pmatrix} = \begin{pmatrix} & \ell'' - \ell' - \ell, [j''], [\lambda''] & \\ 0, [j], [\lambda] & & 0, [j'], [\lambda'] \end{pmatrix}, \tag{3.3.20}$$

for all $\ell, \ell' \in \frac{1}{2}\mathbb{Z}$. In other words, the Grothendieck fusion rule for $\bigl[\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}\bigr] \boxtimes \bigl[\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}\bigr]$ may be obtained by applying $\sigma^{\ell+\ell'}$ to the rule for $\bigl[\widetilde{\mathcal{R}}_{[j],[\lambda]}\bigr] \boxtimes \bigl[\widetilde{\mathcal{R}}_{[j'],[\lambda']}\bigr]$.

Theorem 3.3.6. *Let* k *be admissible with* $\mathsf{v} = 3$. *Then for all* $\ell, \ell' \in \frac{1}{2}\mathbb{Z}$, $[j], [j'] \in \mathbb{R}/\mathbb{Z}$ *and* $[\lambda], [\lambda'] \in \Gamma_{\mathsf{u},3}/\mathbb{Z}_3$, *the Grothendieck fusion rules of the standard* $\mathsf{BP}(\mathsf{u},3)$*-modules are*

$$\bigl[\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}\bigr] \boxtimes \bigl[\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}\bigr] = \sum_{[\lambda''] \in \Gamma_{\mathsf{u},3}/\mathbb{Z}_3} \mathcal{N}^{\mathsf{W}_3\,[\lambda'']}_{[\lambda],[\lambda']} \Bigl(\bigl[\widetilde{\mathcal{R}}^{\ell+\ell'+2}_{[j+j'-4\kappa],[\lambda'']}\bigr] + \bigl[\widetilde{\mathcal{R}}^{\ell+\ell'-1}_{[j+j'+2\kappa],[\lambda'']}\bigr]\Bigr). \tag{3.3.21}$$

Proof. The desired Grothendieck fusion coefficients are given by the standard Verlinde formula (3.3.2b) with $\ell = \ell' = 0$. Using (3.2.31) and (3.3.16),

$$\begin{pmatrix} & \ell'', [j''], [\lambda''] & \\ 0, [j], [\lambda] & & 0, [j'], [\lambda'] \end{pmatrix} = \sum_{[\mu]} \frac{\mathsf{S}^{\mathsf{W}_3}_{[\lambda],[\mu]}\mathsf{S}^{\mathsf{W}_3}_{[\lambda'],[\mu]}\left(\mathsf{S}^{\mathsf{W}_3}_{[\lambda''],[\mu]}\right)^*}{\mathsf{S}^{\mathsf{W}_3}_{\mathrm{vac.},[\mu]}} \sum_{m \in \mathbb{Z}} \mathsf{e}^{-2\pi i(j+j'-j''-2\kappa\ell'')m} \cdot \int_{\mathbb{R}/\mathbb{Z}} \mathsf{e}^{2\pi i(\ell''-1/2)(k-\kappa)}\, 2\cos\bigl(3\pi(k-\kappa)\bigr)\, \mathrm{d}[k] \tag{3.3.22}$$

$$= \mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\lambda],[\lambda']} \delta\big([j''] - [j + j' - 2\kappa\ell'']\big)\big(\delta_{\ell'',2} + \delta_{\ell'',-1}\big)$$
$$= \mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\lambda],[\lambda']} \Big(\delta\big([j''] - [j + j' - 4\kappa]\big)\delta_{\ell'',2} + \delta\big([j''] - [j + j' + 2\kappa]\big)\delta_{\ell'',-1}\Big).$$

Substituting this result into (3.3.2a) and applying $\sigma^{\ell+\ell'}$ to both sides recovers (3.3.21). ∎

The asymmetry in the shifts of the spectral flow indices and J_0-eigenvalues in (3.3.21) is a reflection of our choice of conformal structure: the fields the conformal weights of G^+ and G^- have different conformal weight with respect to $\widetilde{L}(z)$ (1 and 2 respectively).

As the Grothendieck fusion rules do not depend on the choice of conformal grading, it is easy to translate (3.3.21) to a more symmetric setting. Under the conformal structure defined by $L(z)$, G^+ and G^- both have conformal weight $\frac{3}{2}$. The positive-energy relaxed modules under this conformal structure are the twisted $\mathsf{BP}(\mathsf{u},3)$-modules $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$ whose Grothendieck fusion rules are much more symmetric:

(3.3.23)
$$\big[\sigma^{\ell}(\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]})\big] \boxtimes \big[\sigma^{\ell'}(\mathcal{R}^{\mathrm{tw}}_{[j'],[\lambda']})\big] = \big[\widetilde{\mathcal{R}}^{\ell-1/2}_{[j+\kappa],[\lambda]}\big] \boxtimes \big[\widetilde{\mathcal{R}}^{\ell'-1/2}_{[j'+\kappa],[\lambda']}\big]$$
$$= \sum_{[\lambda'']\in\Gamma_{\mathsf{u},3}/\mathbb{Z}_3} \mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\lambda],[\lambda']} \Big(\big[\widetilde{\mathcal{R}}^{\ell+\ell'+1}_{[j+j'-2\kappa],[\lambda'']}\big] + \big[\widetilde{\mathcal{R}}^{\ell+\ell'-2}_{[j+j'+4\kappa],[\lambda'']}\big]\Big)$$
$$= \sum_{[\lambda'']\in\Gamma_{\mathsf{u},3}/\mathbb{Z}_3} \mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\lambda],[\lambda']} \Big(\big[\sigma^{\ell+\ell'+3/2}(\mathcal{R}^{\mathrm{tw}}_{[j+j'-3\kappa],[\lambda'']})\big] + \big[\sigma^{\ell+\ell'-3/2}(\mathcal{R}^{\mathrm{tw}}_{[j+j'+3\kappa],[\lambda'']})\big]\Big).$$

The remaining Grothendieck fusion rules of $\mathsf{BP}(\mathsf{u},3)$-module are straightforward to compute. As every highest-weight $\mathsf{BP}(\mathsf{u},3)$-module is type-3, any such module can be written as a spectral flow of one whose highest weight corresponds to $\mathsf{s} = [1,-1,0]$. Moreover, as the standard one-point functions are linearly independent, (3.3.8) lifts to the following identity in the Grothendieck group:

(3.3.24)
$$\Big[\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]\Big] = \sum_{n=0}^{\infty} (-1)^n \Big(\Big[\widetilde{\mathcal{R}}\Big[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & 0 & 0 \end{smallmatrix}\Big]^{9n+1}\Big] - \Big[\widetilde{\mathcal{R}}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & 0 \end{smallmatrix}\Big]^{9n+4}\Big] + \Big[\widetilde{\mathcal{R}}\Big[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & 0 & 0 \end{smallmatrix}\Big]^{9n+7}\Big]\Big).$$

Corollary 3.3.7. *Let* k *be admissible with* $\mathsf{v} = 3$. *Then for all* $\ell, \ell' \in \frac{1}{2}\mathbb{Z}$, $[j'] \in \mathbb{R}/\mathbb{Z}$, $\mathsf{r} \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$, *and* $[\lambda'] \in \Gamma_{\mathsf{u},3}/\mathbb{Z}_3$, *we have the following Grothendieck fusion rules:*

(3.3.25)
$$\Big[\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell}\Big] \boxtimes \big[\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}\big] = \sum_{[\lambda'']\in\Gamma_{\mathsf{u},3}/\mathbb{Z}_3} \mathcal{N}^{\mathsf{W}_3\quad [\lambda'']}_{[\Gamma(\mathsf{r},0)],[\lambda']} \big[\widetilde{\mathcal{R}}^{\ell+\ell'}_{[j(\mathsf{r})+j'],[\lambda'']}\big].$$

Proof. As in the proof of Theorem 3.3.4, we rewrite the standard modules in the form required by the standard-by-standard rules:

(3.3.26)
$$\Big[\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]\Big] = \sum_{n=0}^{\infty} (-1)^n \Big(\big[\widetilde{\mathcal{R}}^{9n+1}_{[j(\mathsf{r})-2\kappa],[\Gamma(\mathsf{r},0)]}\big] - \big[\widetilde{\mathcal{R}}^{9n+4}_{[j(\mathsf{r})+\kappa-1/2],[\Gamma(\mathsf{r},0)]}\big] + \big[\widetilde{\mathcal{R}}^{9n+7}_{[j(\mathsf{r})+4\kappa],[\Gamma(\mathsf{r},0)]}\big]\Big).$$

By fusing both sides with $\big[\widetilde{\mathcal{R}}_{[j'],[\lambda']}\big]$ and applying the standard-by-standard rules (3.3.21), almost every term cancels and we arrive at the $\ell = \ell' = 0$ version of (3.3.26). Applying $\sigma^{\ell+\ell'}$ to both sides recovers (3.3.26) in full generality. ∎

The highest-weight-by-standard Grothendieck fusion rules can also be obtained by directly applying the standard Verlinde formula (3.3.2b), in light of Theorems 3.2.10 and 3.3.4. This approach is particularly quick as the cosine in the denominator of (3.3.12) does not depend on the weight $\Gamma(\mathsf{r}, 0)$.

The only remaining Grothendieck fusion rules are the highest-weight-by-highest-weight rules. For this, recall from Theorem 1.3.7 that $\mathsf{W}_3(\mathsf{u}, 3)$ fusion coefficients may be expressed in terms of fusion coefficients for the rational $\mathfrak{sl}_3$ minimal model $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$:

$$\mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\lambda],[\lambda']} = \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\mathsf{r},\mathsf{r}'}. \tag{3.3.27}$$

As made clear in Theorem 1.3.7, such decompositions require choosing representatives of the involved $\lambda \in [\lambda]$ so that $\bar{\mathsf{r}} = [r_1, r_2] \in \overline{\mathsf{Q}}$, the root lattice of $\mathfrak{sl}_3$. Since

$$\overline{\nabla(\mathsf{r})} - \bar{\mathsf{r}} = \mathsf{u}\omega_1 \bmod \overline{\mathsf{Q}}, \tag{3.3.28}$$

$\mathsf{u} \notin 3\mathbb{Z}$ implies that such representatives always exist and are unique.

Corollary 3.3.8. *Let* k *be admissible with* $\mathsf{v} = 3$*. Then for all* $\ell, \ell' \in \frac{1}{2}\mathbb{Z}$ *and all* $\mathsf{r}, \mathsf{r}' \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$*, we have the following Grothendieck fusion rules:*

$$\Big[\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell}\Big] \boxtimes \Big[\mathcal{H}\Big[\begin{smallmatrix} r_0' & r_1' & r_2' \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell'}\Big] = \sum_{\mathsf{r}'' \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}} \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\mathsf{r},\mathsf{r}'} \Big[\mathcal{H}\Big[\begin{smallmatrix} r_0'' & r_1'' & r_2'' \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell+\ell'}\Big]. \tag{3.3.29}$$

Proof. Substituting the primed version of (3.3.26) and applying term by term the highest-weight-by-standard Grothendieck fusion rule (3.3.25) results in

$$\begin{aligned} &\Big[\mathcal{H}\Big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell}\Big] \boxtimes \Big[\mathcal{H}\Big[\begin{smallmatrix} r_0' & r_1' & r_2' \\ 1 & -1 & 0 \end{smallmatrix}\Big]^{\ell'}\Big] \\ &\quad = \sum_{[\Gamma(\mathsf{r}'',0)]} \mathcal{N}^{\mathsf{W}_3\quad [\Gamma(\mathsf{r}'',0)]}_{[\Gamma(\mathsf{r},0)],[\Gamma(\mathsf{r}',0)]} \sum_{n=0}^{\infty} (-1)^n \Big(\big[\widetilde{\mathcal{R}}^{9n+1}_{[j(\mathsf{r})+j(\mathsf{r}')-2\kappa],[\Gamma(\mathsf{r}'',0)]}\big] \\ &\qquad\qquad - \big[\widetilde{\mathcal{R}}^{9n+4}_{[j(\mathsf{r})+j(\mathsf{r}')+\kappa-1/2],[\Gamma(\mathsf{r}'',0)]}\big] \\ &\qquad\qquad + \big[\widetilde{\mathcal{R}}^{9n+7}_{[j(\mathsf{r})+j(\mathsf{r}')+4\kappa],[\Gamma(\mathsf{r}'',0)]}\big]\Big). \end{aligned} \tag{3.3.30}$$

What remains to show is that for each $[\Gamma(\mathsf{r}'', 0)] \in \Gamma_{\mathsf{u},3}/\mathbb{Z}_3$, the sum over n is $\Big[\mathcal{H}\Big[\begin{smallmatrix} r_0'' & r_1'' & r_2'' \\ 1 & -1 & 0 \end{smallmatrix}\Big]\Big]$ for some unique $\mathsf{r}'' \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$. There are three candidates for r'' as the $\mathbb{Z}_3$-orbit is fixed. The desired r'' must also satisfy $[j(\mathsf{r}'')] = [j(\mathsf{r}) + j(\mathsf{r}')]$ (by the double primed version of (3.3.26)).

To show that this constraint picks exactly one representative of the $\mathbb{Z}_3$-orbit, recall from (3.3.10) that $j(\mathsf{r}) \in \frac{1}{3}\mathbb{Z}$. On the other hand, an easy calculation gives $j\big(\nabla(\mathsf{r})\big) - j(\mathsf{r}) \in \mathbb{Z} + \frac{\mathsf{u}}{3}$. Since $\mathsf{u} \notin 3\mathbb{Z}$, it follows that the three elements of the $\mathbb{Z}_3$-orbit have distinct charges modulo 1. There thus exists a unique r'' that corresponds to a weight in the required $\mathbb{Z}_3$-orbit and satisfies $[j(\mathsf{r}'')] = [j(\mathsf{r})+j(\mathsf{r}')]$.

It only remains to replace the $\mathsf{W}_3(\mathsf{u},3)$ fusion coefficients in (3.3.30) by $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$ ones. We may choose the representative r'' to satisfy $\overline{\mathsf{r}''} \in \overline{\mathsf{Q}}$, but we cannot assume that r or r' satisfy the analogous constraints. Thus, (1.3.37) gives

$$\mathcal{N}^{\mathsf{W}_3 \quad [\Gamma(\mathsf{r}'',0)]}_{[\Gamma(\mathsf{r},0)],[\Gamma(\mathsf{r}',0)]} = \mathcal{N}^{\mathsf{u}-3 \ \mathsf{r}''}_{\nabla^m(\mathsf{r}),\nabla^n(\mathsf{r}')} = \mathcal{N}^{\mathsf{u}-3 \ \nabla^{-m-n}(\mathsf{r}'')}_{\mathsf{r},\mathsf{r}'}, \tag{3.3.31}$$

for some $m,n \in \mathbb{Z}_3$. This fusion coefficient is zero unless $\bar{\mathsf{r}} + \overline{\mathsf{r}'} - \overline{\nabla^{-m-n}(\mathsf{r}'')} \in \overline{\mathsf{Q}}$, by the Kac–Walton formula (1.2.14). It can therefore be nonzero for at most one $-m-n \in \mathbb{Z}_3$, by (3.3.28). We may therefore replace the sum in (3.3.30) by one over all $\mathsf{r}'' \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$, dropping the constraint $[j(\mathsf{r}'')] = [j(\mathsf{r})+j(\mathsf{r}')]$, because the $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$ fusion coefficient is zero when this constraint is not satisfied. ∎

Theorem 3.3.6, Corollary 3.3.8 and Corollary 3.3.7 taken together specify all Grothendieck fusion rules of $\mathsf{BP}(\mathsf{u},3)$ minimal models.

An interesting consequence of what we have found is the Grothendieck fusion of $\mathcal{H}\left[\begin{smallmatrix} 0 & \mathsf{u}-3 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ with another highest-weight module. Let $\mathsf{0} = [\mathsf{u}-3,0,0]$. The fusion rule (3.3.29) and (1.3.36) now give

$$\begin{aligned}
\Big[\mathcal{H}\left[\begin{smallmatrix} 0 & \mathsf{u}-3 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]\Big] \boxtimes \Big[\mathcal{H}\left[\begin{smallmatrix} r_0' & r_1' & r_2' \\ 1 & -1 & 0 \end{smallmatrix}\right]\Big] &= \sum_{\mathsf{r}'' \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}} \mathcal{N}^{\mathsf{u}-3 \ \mathsf{r}''}_{\nabla(\mathsf{0}),\mathsf{r}'} \Big[\mathcal{H}\left[\begin{smallmatrix} r_0'' & r_1'' & r_2'' \\ 1 & -1 & 0 \end{smallmatrix}\right]\Big] \\
&= \sum_{\mathsf{r}'' \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}} \mathcal{N}^{\mathsf{u}-3 \ \nabla^{-1}(\mathsf{r}'')}_{\mathsf{0},\mathsf{r}'} \Big[\mathcal{H}\left[\begin{smallmatrix} r_0'' & r_1'' & r_2'' \\ 1 & -1 & 0 \end{smallmatrix}\right]\Big] \\
&= \sum_{\mathsf{r}'' \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}} \delta_{\mathsf{r}'',\nabla(\mathsf{r}')} \Big[\mathcal{H}\left[\begin{smallmatrix} r_0'' & r_1'' & r_2'' \\ 1 & -1 & 0 \end{smallmatrix}\right]\Big] = \Big[\mathcal{H}\left[\begin{smallmatrix} r_2' & r_0' & r_1' \\ 1 & -1 & 0 \end{smallmatrix}\right]\Big].
\end{aligned} \tag{3.3.32}$$

This, and another nearly identical calculation for $\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & \mathsf{u}-3 \\ 1 & -1 & 0 \end{smallmatrix}\right]$, proves the following proposition. These computations prove the existence of *simple currents* in the fusion ring of $\mathsf{BP}(\mathsf{u},\mathsf{v})$. That is, modules that act invertibly on the fusion ring under the fusion product.

Proposition 3.3.9. *Let* k *be admissible with* $\mathsf{v}=3$. *Then,* $\mathcal{H}\left[\begin{smallmatrix} 0 & \mathsf{u}-3 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ *and* $\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & \mathsf{u}-3 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ *are simple currents of order* 3, *inverse to one another. Their highest weights (with respect to* J_0 *and* L_0*) are*

$$(j,\Delta) = \left(+\tfrac{\mathsf{u}-3}{3}, \tfrac{\mathsf{u}-3}{2}\right) \qquad \text{and} \qquad (j,\Delta) = \left(-\tfrac{\mathsf{u}-3}{3}, \tfrac{\mathsf{u}-3}{2}\right), \tag{3.3.33}$$

respectively.

Additionally, observe that Corollary 3.3.8 shows the existence of a set of $\mathsf{BP}(\mathsf{u},3)$-modules labelled by weights $\mathsf{r}'' \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$ having Grothendieck fusion rules with coefficients given by the fusion coefficients of $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$. One might reasonably suspect that if the Grothendieck fusion rules (3.3.29) can be upgraded to genuine fusion rules, these highest-weight $\mathsf{BP}(\mathsf{u},3)$-modules can be identified with $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$-modules in a fusion-preserving way. The following proposition says that this is the case.

Proposition 3.3.10. *Let* k *be admissible with* $\mathsf{v}=3$*. Then, the fusion subring of* $\mathsf{BP}(\mathsf{u},3)$*-modules generated by the* $\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\right] \in \Sigma_{\mathsf{u},3}$*, is isomorphic to the fusion ring of the affine vertex operator algebra* $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$.

Proof. By (3.3.29), the Grothendieck fusion subring generated by the $\left[\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\right]\right]$ is clearly isomorphic to the fusion ring of $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$. An obvious isomorphism consists of identifying the simple highest-weight $\mathsf{BP}(\mathsf{u},3)$-module $\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ with the simple highest-weight $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$-module $\mathcal{L}_{\mathsf{r}}$ whose highest weight is $\mathsf{r}=[r_0,r_1,r_2]$.

To show that this gives an isomorphism of fusion rings, we only need to show that the $\mathsf{BP}(\mathsf{u},3)$-modules of the form $\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ generate a semisimple fusion subring. That is, that (3.3.29) can be lifted to a genuine fusion rule. For this, it is useful to consider the weight $\mathsf{r}=[\mathsf{u}-4,1,0]$. The $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$ fusion coefficients on the right-hand side of (3.3.29) (when one of the weights on the left-hand side is $[\mathsf{u}-4,1,0]$) that appear can be computed using the Kac–Walton formula (1.2.14):

$$\mathcal{L}_{[\mathsf{u}-4,1,0]} \times \mathcal{L}_{[r_0',r_1',r_2']} \simeq \mathcal{L}_{[r_0'-1,r_1'+1,r_2']} \oplus \mathcal{L}_{[r_0'+1,r_1',r_2'-1]} \oplus \mathcal{L}_{[r_0',r_1'-1,r_2'+1]}. \tag{3.3.34}$$

Here, the modules appearing on the right-hand side are understood to be 0 if the r-labels do not define a weight in $\mathsf{P}^{\mathsf{u}-3}_{\geqslant}$. It follows that

$$\left[\mathcal{H}\left[\begin{smallmatrix} \mathsf{u}-4 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]\right] \boxtimes \left[\mathcal{H}\left[\begin{smallmatrix} r_0' & r_1' & r_2' \\ 1 & -1 & 0 \end{smallmatrix}\right]\right] = \left[\mathcal{H}\left[\begin{smallmatrix} r_0'-1 & r_1'+1 & r_2' \\ 1 & -1 & 0 \end{smallmatrix}\right]\right] + \left[\mathcal{H}\left[\begin{smallmatrix} r_0'+1 & r_1' & r_2'-1 \\ 1 & -1 & 0 \end{smallmatrix}\right]\right] + \left[\mathcal{H}\left[\begin{smallmatrix} r_0' & r_1'-1 & r_2'+1 \\ 1 & -1 & 0 \end{smallmatrix}\right]\right], \tag{3.3.35}$$

where again, the modules appearing on the right-hand side are understood to be 0 if the r-labels do not define a weight in $\mathsf{P}^{\mathsf{u}-3}_{\geqslant}$. To see that (3.3.35) can be lifted to a genuine fusion rule, it suffices to show that the $\widetilde{L}_0$-eigenvalues of the highest-weight vectors of any two of the modules appearing on the right-hand side differ by nonintegers. That is, that these modules admit no nonsplit extensions. A straightforward calculation shows that this is the case, and therefore

$$\mathcal{H}\left[\begin{smallmatrix} \mathsf{u}-4 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right] \times \mathcal{H}\left[\begin{smallmatrix} r_0' & r_1' & r_2' \\ 1 & -1 & 0 \end{smallmatrix}\right] \simeq \mathcal{H}\left[\begin{smallmatrix} r_0'-1 & r_1'+1 & r_2' \\ 1 & -1 & 0 \end{smallmatrix}\right] \oplus \mathcal{H}\left[\begin{smallmatrix} r_0'+1 & r_1' & r_2'-1 \\ 1 & -1 & 0 \end{smallmatrix}\right] \oplus \mathcal{H}\left[\begin{smallmatrix} r_0' & r_1'-1 & r_2'+1 \\ 1 & -1 & 0 \end{smallmatrix}\right]. \tag{3.3.36}$$

An identical approach yields an analogous result for $\mathsf{r}=[\mathsf{u}-4,0,1]$. It can then be shown that $\mathcal{H}\left[\begin{smallmatrix} \mathsf{u}-4 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ and $\mathcal{H}\left[\begin{smallmatrix} \mathsf{u}-4 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ generate the desired semisimple fusion subring of $\mathsf{BP}(\mathsf{u},3)$. ∎

3.3.3. Examples. We conclude our $\mathsf{v} = 3$ studies by revisiting one of the examples from Section 2.3.5.

Example ($\mathsf{BP}(4,3)$). *Recall that the minimal model* $\mathsf{BP}(4,3)$ *has level* $\mathsf{k} = -\frac{5}{3}$ *and central charge* $\mathsf{c}^{\mathsf{BP}}_{4,3} = -1$ *with respect to the conformal vector* L.

There are 9 *untwisted highest-weight modules that are arranged into* 3 *spectral flow orbits as follows:*

$$\begin{gathered} \cdots \xrightarrow{\sigma} \mathcal{H}\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{smallmatrix}\right] \xrightarrow{\sigma} \mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right] \xrightarrow{\sigma} \mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right] \xrightarrow{\sigma} \cdots, \\ \cdots \xrightarrow{\sigma} \mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \end{smallmatrix}\right] \xrightarrow{\sigma} \mathcal{H}\left[\begin{smallmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right] \xrightarrow{\sigma} \mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \xrightarrow{\sigma} \cdots, \\ \cdots \xrightarrow{\sigma} \mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \end{smallmatrix}\right] \xrightarrow{\sigma} \mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix}\right] \xrightarrow{\sigma} \mathcal{H}\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \xrightarrow{\sigma} \cdots. \end{gathered} \tag{3.3.37}$$

The first line in (3.3.37) *consists of spectral flows of the vacuum module, while the remaining modules are spectral flows of the simple currents from Proposition 3.3.9. Therefore the fusion rules of highest-weight modules are determined by* (3.3.32) *(and the* $[0,0,\mathsf{u}-3]$ *version) and the fact that fusion with the vacuum does not change the module. For example,*

$$\begin{aligned} \mathcal{H}\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{smallmatrix}\right] \times \mathcal{H}\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] &\simeq \sigma^{-1}\Big(\mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]\Big) \times \sigma\Big(\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix}\right]\Big) \\ &\simeq \mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right] \times \mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix}\right] \\ &\simeq \mathcal{H}\left[\begin{smallmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]. \end{aligned} \tag{3.3.38}$$

There is a single family of twisted relaxed highest-weight $\mathsf{BP}(4,3)$*-modules* $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$*, where* $[\lambda] = [\Gamma\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]] \in \Gamma_{4,3}/\mathbb{Z}_3$ *and* $[j] \in \mathbb{R}/\mathbb{Z}$*. We shall simplify the notation by dropping the dependence on* $[\lambda]$*:* $\mathcal{R}^{\mathrm{tw}}_{[j]} \equiv \mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$*. The condition on* $[j]$ *for* $\mathcal{R}^{\mathrm{tw}}_{[j]}$ *to be simple is* $j \neq \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \pmod 1$*, by Theorem 2.3.25. When* j *assumes one of these values,* $\mathcal{R}^{\mathrm{tw}}_{[j]}$ *is nonsemisimple with* G^-_0 *acting injectively.*

The highest-weight-by-relaxed Grothendieck fusion rules are easily obtained as every highest-weight $\mathsf{BP}(4,3)$*-module is the spectral flow of the vacuum module or a simple current:*

$$\Big[\mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]\Big] \boxtimes \big[\mathcal{R}^{\mathrm{tw}}_{[j']}\big] = \big[\mathcal{R}^{\mathrm{tw}}_{[j'+\mathsf{u}/3]}\big], \tag{3.3.39}$$

using (3.3.25)*. If the relaxed module on the left-hand side is simple, then so is that on the right. We therefore obtain the following genuine fusion rule:*

$$\mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right] \times \mathcal{R}^{\mathrm{tw}}_{[j']} \simeq \mathcal{R}^{\mathrm{tw}}_{[j'+\mathsf{u}/3]}, \qquad j' \notin \tfrac{1}{3}\mathbb{Z} + \tfrac{1}{6}. \tag{3.3.40}$$

The remaining Grothendieck fusion rules are the relaxed-by-relaxed rules, which are given by

$$\big[\mathcal{R}^{\mathrm{tw}}_{[j]}\big] \boxtimes \big[\mathcal{R}^{\mathrm{tw}}_{[j']}\big] = \big[\sigma^{3/2}\big(\mathcal{R}^{\mathrm{tw}}_{[j+j'+1/6]}\big)\big] + \big[\sigma^{-3/2}\big(\mathcal{R}^{\mathrm{tw}}_{[j+j'-1/6]}\big)\big]. \tag{3.3.41}$$

By comparing conformal weights for the summands on the right-hand side, we conclude that this can be uplifted to a genuine fusion rule for almost all j:

$$\mathcal{R}^{\mathrm{tw}}_{[j]} \times \mathcal{R}^{\mathrm{tw}}_{[j']} \simeq \sigma^{3/2}\big(\mathcal{R}^{\mathrm{tw}}_{[j+j'+1/6]}\big) \oplus \sigma^{-3/2}\big(\mathcal{R}^{\mathrm{tw}}_{[j+j'-1/6]}\big), \qquad j+j' \notin \tfrac{1}{3}\mathbb{Z}. \tag{3.3.42}$$

When $j+j' \in \frac{1}{3}\mathbb{Z}$, we conjecture that the fusion product is nonsemisimple. In fact, we expect that the fusion products are staggered $\mathsf{BP}(4,3)$*-modules in the sense of [120].*

Recall that $\mathsf{BP}(4,3)$ is known to be a $\mathbb{Z}_3$-orbifold of the $\beta\gamma$ ghost vertex algebra B. The field identifications were given in (2.3.58). In the opposite direction, the simple current extension of $\mathsf{BP}(4,3)$ *corresponding to $\mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ and $\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ is the vertex operator algebra B', whose vacuum module decomposes as*

$$\mathsf{B}' = \mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right] \oplus \mathcal{H}\left[\begin{smallmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right] \oplus \mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix}\right]. \tag{3.3.43}$$

It is easy to check that the field $\beta(z)$ of weight $(\frac{1}{3},\frac{1}{2})$ and the field $\gamma(z)$ of weight $(-\frac{1}{3},\frac{1}{2})$ generate a copy of the bosonic ghosts vertex operator algebra in B'. As the generating fields of $\mathsf{BP}(4,3)$ can be expressed in terms of β and γ, $\mathsf{B}' \simeq \mathsf{B}$.

The simple current orbits

$$\mathcal{H}\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{smallmatrix}\right] \oplus \mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \end{smallmatrix}\right] \oplus \mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \end{smallmatrix}\right] \quad \text{and} \quad \mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right] \oplus \mathcal{H}\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \oplus \mathcal{H}\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \tag{3.3.44}$$

are not B'-modules as they are not $\frac{1}{2}\mathbb{Z}$-graded by conformal weight. In fact, $\sigma^\ell(\mathsf{B}')$ is an untwisted B'-module (is $\frac{1}{2}\mathbb{Z}$-graded) if and only if $\ell \in 3\mathbb{Z}$ and is a twisted B'-module (is $\mathbb{Z}$-graded) if and only if $\ell \in 3(\mathbb{Z}+\frac{1}{2})$. This reflects the fact that the natural unit of ghost spectral flow is σ^3, not σ (see [144, (2.2)]).

Consider the simple current orbit of the twisted relaxed highest-weight $\mathsf{BP}(4,3)$-module:

$$\mathcal{B}_{[\![j]\!]} = \mathcal{R}^{\mathrm{tw}}_{[j-1/3]} \oplus \mathcal{R}^{\mathrm{tw}}_{[j]} \oplus \mathcal{R}^{\mathrm{tw}}_{[j+1/3]}, \quad [\![j]\!] \in \mathbb{R}/\tfrac{1}{3}\mathbb{Z}. \tag{3.3.45}$$

Conformal weight considerations show that it is a simple twisted B-module for all $[\![j]\!] \neq [\![\frac{1}{6}]\!]$. Thus we have constructed two classes of (untwisted or twisted) B'-modules: $\sigma^\ell(\mathsf{B}')$ and $\sigma^\ell(\mathcal{B}_{[\![j]\!]})$ with either $\ell \in 3\mathbb{Z}$ or $\ell \in 3(\mathbb{Z}+\frac{1}{2})$.

Fusion rules for these B'-modules may be obtained from the $\mathsf{BP}(4,3)$ fusion rules by induction [146], see also [48]. Those involving the simple current extension B' (and its spectral flows) are obvious, so the only nontrivial fusion rule is

$$\mathcal{B}_{[\![j]\!]} \times \mathcal{B}_{[\![j']\!]} \simeq \sigma^{3/2}\big(\mathcal{B}_{[\![j+j'+1/6]\!]}\big) \oplus \sigma^{-3/2}\big(\mathcal{B}_{[\![j+j'-1/6]\!]}\big), \qquad [\![j+j']\!] \neq [\![0]\!]. \tag{3.3.46}$$

Unsurprisingly, this fusion rule is identical to the bosonic ghosts fusion rule computed in [144, App. A], up to rescaling charges and spectral flow indices by a factor of 3.

3.3.4. One-Point Functions for Highest-Weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-Modules. We now turn to the generalisation of our $\mathsf{v}=3$ modularity results to $\mathsf{v}>3$. Here, there are always highest-weight modules of every type. The same strategy as before largely works in this context too: construct resolutions that express the one-point functions of the highest-weight modules in terms of those of the standard modules and use these to obtain modular transformations. The technical complexity of the computations increases considerably and so we shall not be exhaustive in our investigations.

The first order of business is obtaining resolutions for highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules. Since spectral flow is an exact functor, it suffices to choose a representative highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module in each orbit. We therefore take $\Gamma(\mathsf{r},\mathsf{s}) \in \Sigma_{\mathsf{u},\mathsf{v}}$ to be as in Corollary 2.3.23, thus the leftmost in its orbit (as pictured in Figure 1). Then, $s_2 \neq 0$ and Proposition 3.3.1 gives the following short exact sequence:

$$0 \longrightarrow \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+1 & s_2-1 \end{smallmatrix}\right]^1 \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+1 & s_2-1 \end{smallmatrix}\right] \longrightarrow \mathcal{H}(\mathsf{r},\mathsf{s}) \longrightarrow 0. \tag{3.3.47}$$

Note that $\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+1 & s_2-1 \end{smallmatrix}\right]$ is rightmost in its orbit. As long as $s_2 \neq 1$, it is type-1 and thus also leftmost. The sequence (3.3.47) can therefore be spliced until we reach a highest-weight module with $s_2 = 0$ which is no longer type-1.

What results from this procedure is the exact sequence

$$\begin{aligned} 0 \to \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+s_2 & 0 \end{smallmatrix}\right]^{s_2} \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+s_2 & 0 \end{smallmatrix}\right]^{s_2-1} \to \dots \\ \dots \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+2 & s_2-2 \end{smallmatrix}\right]^1 \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+1 & s_2-1 \end{smallmatrix}\right] \to \mathcal{H}(\mathsf{r},\mathsf{s}) \to 0. \end{aligned} \tag{3.3.48}$$

Resolving highest-weight modules therefore reduces to resolving those with $s_2 = 0$. Indeed the module $\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+s_2 & 0 \end{smallmatrix}\right] = \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & \mathsf{v}-3-s_0 & 0 \end{smallmatrix}\right]$ is type-2 if $s_0 \neq 0$, and type-3 if $s_0 = 0$. Being rightmost in its orbit, this module is therefore obtained from the leftmost by applying one or two units of spectral flow, respectively. By Theorem 2.3.21,

$$\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+s_2 & 0 \end{smallmatrix}\right]^{s_2} \simeq \begin{cases} \mathcal{H}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ \mathsf{v}-2-s_0 & -1 & s_0 \end{smallmatrix}\right]^{s_2+1} & \text{if } s_0 \neq 0, \\ \mathcal{H}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & -1 & \mathsf{v}-2 \end{smallmatrix}\right]^{s_2+2} & \text{if } s_0 = 0. \end{cases} \tag{3.3.49}$$

The modules on the right-hand side are now leftmost in their orbits and are therefore fit into a short exact sequence of the form (3.3.47). We can therefore iteratively splice spectral flows of (3.3.48) to obtain the desired resolution for $s_2 = 0$ highest-weight modules.

If we start with a highest-weight module having $s_0 = 0$, then all the sequences (3.3.48) to be spliced together will have $s_0 = 0$ and the resolutions will only involve type-1 and type-3 highest weights.

Similarly, if we start with $s_0 \neq 0$, then the sequences being spliced will all have $s_0 \neq 0$ because $\mathcal{H}(\mathsf{r}, \mathsf{s})$ was chosen to be leftmost in its orbit and so we cannot have $s_0 = \mathsf{v} - 2$. Thus the resolutions will only involve type-1 and type-2 highest weights.

The easiest resolution obtained in this manner are those involving type-3 $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules. Of course when $\mathsf{v} = 3$, these are the only highest-weight modules in the spectrum. The following proposition (along with its spectral flows) is therefore a generalisation of Proposition 3.3.2. Note as well that the the vacuum module is always type-3.

Proposition 3.3.11. *Let* k *be nondegenerate-admissible. Then, the* $\mathsf{BP}(\mathsf{u}, \mathsf{v})$*-modules of the form* $\mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & s_1 & s_2 \end{smallmatrix}\right]$ *(chosen as in Figure 1 to be leftmost in its spectral flow orbit) is resolved by the nonsimple standard modules as follows:*

(3.3.50)
$$\begin{aligned}
\cdots \to \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & \mathsf{v}-3 & 0 \end{smallmatrix}\right]^{s_2-1+3\mathsf{v}} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 1 & \mathsf{v}-4 \end{smallmatrix}\right]^{s_2+3+2\mathsf{v}} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & 0 & \mathsf{v}-3 \end{smallmatrix}\right]^{s_2+2+2\mathsf{v}} \\
\longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & \mathsf{v}-3 & 0 \end{smallmatrix}\right]^{s_2-1+2\mathsf{v}} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & 1 & \mathsf{v}-4 \end{smallmatrix}\right]^{s_2+3+\mathsf{v}} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & 0 & \mathsf{v}-3 \end{smallmatrix}\right]^{s_2+2+\mathsf{v}} \\
\longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & \mathsf{v}-3 & 0 \end{smallmatrix}\right]^{s_2-1+\mathsf{v}} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & 1 & \mathsf{v}-4 \end{smallmatrix}\right]^{s_2+3} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & 0 & \mathsf{v}-3 \end{smallmatrix}\right]^{s_2+2} \\
\longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & \mathsf{v}-3 & 0 \end{smallmatrix}\right]^{s_2-1} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & s_1+2 & s_2-2 \end{smallmatrix}\right]^{1} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & s_1+1 & s_2-1 \end{smallmatrix}\right] \longrightarrow \mathcal{H}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & s_1 & s_2 \end{smallmatrix}\right] \to 0.
\end{aligned}$$

The resolution for $s_0 \neq 0$ is somewhat more complicated.

Proposition 3.3.12. *Let* k *be nondegenerate-admissible with* $\mathsf{v} > 3$ *and suppose that* $s_0 \neq 0$*. Then, the* $\mathsf{BP}(\mathsf{u}, \mathsf{v})$*-module* $\mathcal{H}(\mathsf{r}, \mathsf{s})$ *(chosen as in Figure 1 to be leftmost in its spectral flow orbit) is resolved by the nonsimple standard modules as follows:*

(3.3.51)
$$\begin{aligned}
\cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & \mathsf{v}-3-s_0 & 0 \end{smallmatrix}\right]^{3\mathsf{v}+s_2-1} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & 1 & \mathsf{v}-4-s_0 \end{smallmatrix}\right]^{3\mathsf{v}-s_1} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & 0 & \mathsf{v}-3-s_0 \end{smallmatrix}\right]^{3\mathsf{v}-s_1-1} \\
\longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ \mathsf{v}-2-s_0 & s_0-1 & 0 \end{smallmatrix}\right]^{3\mathsf{v}-s_1-3} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ \mathsf{v}-2-s_0 & 1 & s_0-2 \end{smallmatrix}\right]^{2\mathsf{v}+s_2+2} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ \mathsf{v}-2-s_0 & 0 & s_0-1 \end{smallmatrix}\right]^{2\mathsf{v}+s_2+1} \\
\longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ s_0 & \mathsf{v}-3-s_0 & 0 \end{smallmatrix}\right]^{2\mathsf{v}+s_2-1} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ s_0 & 1 & \mathsf{v}-4-s_0 \end{smallmatrix}\right]^{2\mathsf{v}-s_1} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ s_0 & 0 & \mathsf{v}-3-s_0 \end{smallmatrix}\right]^{2\mathsf{v}-s_1-1} \\
\longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2-s_0 & s_0-1 & 0 \end{smallmatrix}\right]^{2\mathsf{v}-s_1-3} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2-s_0 & 1 & s_0-2 \end{smallmatrix}\right]^{\mathsf{v}+s_2+2} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2-s_0 & 0 & s_0-1 \end{smallmatrix}\right]^{\mathsf{v}+s_2+1} \\
\longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ s_0 & \mathsf{v}-3-s_0 & 0 \end{smallmatrix}\right]^{\mathsf{v}+s_2-1} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ s_0 & 1 & \mathsf{v}-4-s_0 \end{smallmatrix}\right]^{\mathsf{v}-s_1} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_2 & r_0 & r_1 \\ s_0 & 0 & \mathsf{v}-3-s_0 \end{smallmatrix}\right]^{\mathsf{v}-s_1-1} \\
\longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ \mathsf{v}-2-s_0 & s_0-1 & 0 \end{smallmatrix}\right]^{\mathsf{v}-s_1-3} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ \mathsf{v}-2-s_0 & 1 & s_0-2 \end{smallmatrix}\right]^{s_2+2} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_1 & r_2 & r_0 \\ \mathsf{v}-2-s_0 & 0 & s_0-1 \end{smallmatrix}\right]^{s_2+1} \\
\longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & \mathsf{v}-3-s_0 & 0 \end{smallmatrix}\right]^{s_2-1} &\longrightarrow \cdots \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+2 & s_2-2 \end{smallmatrix}\right]^{1} \longrightarrow \widetilde{\mathcal{R}}\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+1 & s_2-1 \end{smallmatrix}\right] \longrightarrow \mathcal{H}(\mathsf{r}, \mathsf{s}) \longrightarrow 0.
\end{aligned}$$

The type-3 resolution (3.3.50) may be recovered from (3.3.51) by setting $s_0 = 0$, i.e. by deleting every second line. As for $\mathsf{v} = 3$, the one-point functions for spectral flows of highest-weight $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules are easily obtained from the resolution (3.3.51).

Corollary 3.3.13. *For* k *nondegenerate-admissible, the one-point function of* $\mathcal{H}(\mathsf{r}, \mathsf{s})$ *(chosen as in Figure 1 to be leftmost in its spectral flow orbit) is given by*

(3.3.52)
$$
\begin{aligned}
\widetilde{\operatorname{ch}}\big[\mathcal{H}(\mathsf{r},\mathsf{s})\big] = {} & \sum_{m=0}^{s_2-1} (-1)^m \, \widetilde{\operatorname{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & s_1+m+1 & s_2-m-1 \end{smallmatrix}\big]^m\Big] \\
& + \sum_{n=0}^{\infty} \sum_{m=0}^{s_0-1} (-1)^{s_2+m+n\mathsf{v}} \bigg(\widetilde{\operatorname{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_1 & r_2 & r_0 \\ \mathsf{v}-2-s_0 & m & s_0-m-1 \end{smallmatrix}\big]^{m+3n\mathsf{v}+s_2+1}\Big] \\
& \qquad + (-1)^{\mathsf{v}} \, \widetilde{\operatorname{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2-s_0 & m & s_0-m-1 \end{smallmatrix}\big]^{m+(3n+1)\mathsf{v}+s_2+1}\Big] \\
& \qquad + \widetilde{\operatorname{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_2 & r_0 & r_1 \\ \mathsf{v}-2-s_0 & m & s_0-m-1 \end{smallmatrix}\big]^{m+(3n+2)\mathsf{v}+s_2+1}\Big] \bigg) \\
& - \sum_{n=1}^{\infty} \sum_{m=0}^{\mathsf{v}-3-s_0} (-1)^{s_1+m+n\mathsf{v}} \bigg(\widetilde{\operatorname{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_2 & r_0 & r_1 \\ s_0 & m & \mathsf{v}-3-s_0-m \end{smallmatrix}\big]^{m+(3n-2)\mathsf{v}-s_1-1}\Big] \\
& \qquad + (-1)^{\mathsf{v}} \, \widetilde{\operatorname{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_1 & r_2 & r_0 \\ s_0 & m & \mathsf{v}-3-s_0-m \end{smallmatrix}\big]^{m+(3n-1)\mathsf{v}-s_1-1}\Big] \\
& \qquad + \widetilde{\operatorname{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ s_0 & m & \mathsf{v}-3-s_0-m \end{smallmatrix}\big]^{m+3n\mathsf{v}-s_1-1}\Big] \bigg).
\end{aligned}
$$

The one-point function of the spectrally flow module $\mathcal{H}(\mathsf{r}, \mathsf{s})^\ell$ with $\ell \neq 0$ is obtained from (3.3.52) by simply adding ℓ to all spectral flow indices on the right-hand side.

The most important highest-weight module is the type-3 vacuum module which has $\mathsf{r} = [\mathsf{u}-3, 0, 0]$ and $\mathsf{s} = [\mathsf{v}-2, -1, 0]$. The one-point function formula (3.3.52) simplifies greatly for the vacuum module. In fact, it simplifies considerably for all type-3 $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules. Rather than choosing the representative of highest-weight type-3 modules to be the leftmost one in Figure 1, we shall find it convenient to choose the middle module as the representative of the type-3 spectral flow orbits. In particular, the vacuum module is such a type-3 representative.

Corollary 3.3.14. *For* k *be nondegenerate-admissible, the one-point function of the type-*3 *module* $\mathcal{H}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2 & -1 & 0 \end{smallmatrix}\big]$ *is given by*

(3.3.53)
$$
\widetilde{\operatorname{ch}}\Big[\mathcal{H}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2 & -1 & 0 \end{smallmatrix}\big]\Big] = \sum_{n=0}^{\infty} \sum_{m=0}^{\mathsf{v}-3} (-1)^{m+n\mathsf{v}} \bigg(\widetilde{\operatorname{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_1 & r_2 & r_0 \\ 0 & m & \mathsf{v}-3-m \end{smallmatrix}\big]^{m+3n\mathsf{v}+1}\Big]
$$

$$+ (-1)^{\mathsf{v}}\, \widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ 0 & m & \mathsf{v}-3-m \end{smallmatrix}\big]^{m+(3n+1)\mathsf{v}+1}\Big] + \widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}\big[\begin{smallmatrix} r_2 & r_0 & r_1 \\ 0 & m & \mathsf{v}-3-m \end{smallmatrix}\big]^{m+(3n+2)\mathsf{v}+1}\Big]\bigg)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\mathsf{v}-3}\sum_{i=0}^{2}(-1)^{m+(n+i)\mathsf{v}}\, \widetilde{\mathrm{ch}}\Big[\widetilde{\mathcal{R}}(\nabla^{i-1}(\mathsf{r}), [0, m, \mathsf{v}-3-m])^{m+(3n+i)\mathsf{v}+1}\Big].$$

Before turning to the modular transforms of the type-3 one-point functions, we generalise the notation (3.3.10) to $\mathsf{v} \geqslant 3$:

$$j(\mathsf{r},\mathsf{s}) = j\big(\Gamma(\mathsf{r},\mathsf{s})\big) = \tfrac{1}{3}(r_1 - r_2) - \tfrac{\mathsf{u}}{3\mathsf{v}}(s_1 - s_2 + 1),$$
$$j^{\mathrm{tw}}(\mathsf{r},\mathsf{s}) = j(\mathsf{r},\mathsf{s}) + \kappa = \tfrac{1}{3}(r_1 - r_2) - \tfrac{\mathsf{u}}{3\mathsf{v}}(s_1 - s_2) - \tfrac{1}{2}. \tag{3.3.54}$$

For each $\lambda = \Gamma\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2 & -1 & 0 \end{smallmatrix}\big] \in \Sigma_{\mathsf{u},\mathsf{v}}$, we define the convenient notation $\underline{\lambda} = \lambda + \frac{\mathsf{u}}{\mathsf{v}}(\omega_0 - \omega_1)$, noting that

$$\lambda = \Gamma\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2 & -1 & 0 \end{smallmatrix}\big] \in \Sigma_{\mathsf{u},\mathsf{v}} \quad \Rightarrow \quad \underline{\lambda} = \Gamma\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-3 & 0 & 0 \end{smallmatrix}\big] \in \Gamma_{\mathsf{u},\mathsf{v}}. \tag{3.3.55}$$

The S-transforms of the type-3 one-point functions when $\mathsf{v} > 3$ are found in a similar way to those for $\mathsf{v} = 3$ in Theorem 3.3.4. The main additional complication is the presence of one-point functions of standard modules with differing s labels. The saving grace here is the relationship between the W_3 S-matrix and characters of simple highest-weight $\mathfrak{sl}_3$-modules described in Section 1.3.3.

Theorem 3.3.15. *Let k be nondegenerate-admissible and let $\lambda = \Gamma\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2 & -1 & 0 \end{smallmatrix}\big] \in \Sigma_{\mathsf{u},\mathsf{v}}$. Then for all $\ell \in \mathbb{Z}$, the S-transform of the one-point function of $\sigma^{\ell}(\mathcal{H}_\lambda) = \mathcal{H}\big[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2 & -1 & 0 \end{smallmatrix}\big]^{\ell}$ is given by*

$$\mathsf{S}\Big\{\widetilde{\mathrm{ch}}\big[\sigma^{\ell}(\mathcal{H}_\lambda)\big]\Big\} = \frac{|\tau|}{-\mathrm{i}\tau}\sum_{\ell'\in\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}}\sum_{[\lambda']\in\Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3}\mathsf{S}_{\ell,\lambda}^{\ell',[j'],[\lambda']}\,\widetilde{\mathrm{ch}}\big[\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}\big]\,\mathrm{d}[j'], \tag{3.3.56}$$

where the entries of the 'highest-weight S-matrix' are given by

$$\mathsf{S}_{\ell,\lambda}^{\ell',[j'],[\lambda']} = \mathsf{S}^{\mathsf{W}_3}_{[\underline{\lambda}],[\lambda']}\frac{\mathrm{e}^{-2\pi\mathrm{i}\left(2\kappa(\ell-1/2)\ell'+(\ell-1/2)(j'-\kappa)+j(\lambda)\ell'\right)}}{2\cos\big(3\pi(j'-\kappa)\big) - \sum_{i\in\mathbb{Z}_3}2\cos\big(\pi a_i(j',\lambda')\big)} \tag{3.3.57}$$

and $a_i(j,\lambda) = (j-\kappa) + 2j^{\mathrm{tw}}\big(\nabla^i(\lambda)\big)$.

Proof. Let $\mathsf{r} = [r_0, r_1, r_2]$ and $\mathsf{s} = [\mathsf{v}-2, -1, 0]$, so that $\lambda = \Gamma(\mathsf{r},\mathsf{s})$. The relaxed modules appearing in (3.3.53) have linearly independent one-point functions, so the 'highest-weight' S-matrix element corresponding to $\sigma^{\ell}(\mathcal{H}_\lambda)$ and $\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}$ is

(3.3.58)

$$\mathsf{S}_{\ell,\lambda}^{\ell',[j'],[\lambda']} = \sum_{n=0}^{\infty}\sum_{m=0}^{\mathsf{v}-3}(-1)^{m+n\mathsf{v}}\Big(\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+m+3n\mathsf{v}+1,[j^{\mathrm{tw}}(\nabla^{-1}(\mathsf{r}),\mathsf{s}_m)+\kappa],[\Gamma(\nabla^{-1}(\mathsf{r}),\mathsf{s}_m)]}$$
$$+ (-1)^{\mathsf{v}}\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+m+(3n+1)\mathsf{v}+1,[j^{\mathrm{tw}}(\mathsf{r},\mathsf{s}_m)+\kappa],[\Gamma(\mathsf{r},\mathsf{s}_m)]}$$

$$+\, \mathsf{S}^{\ell',[j'],[\lambda']}_{\ell+m+(3n+2)\mathsf{v}+1,[j^{\mathrm{tw}}(\nabla(\mathsf{r}),\mathsf{s}_m)+\kappa],[\Gamma(\nabla(\mathsf{r}),\mathsf{s}_m)]}\Big),$$

where $\mathsf{s}_m = [0, m, \mathsf{v}-3-m]$. Extracting the n dependent factors on the right-hand side using (3.2.31) and summing over n gives

$$\sum_{n=0}^{\infty}(-1)^{n\mathsf{v}}\mathsf{e}^{-2\pi\mathsf{i}\left(3\mathsf{v}(j'-\kappa)+6\mathsf{v}\kappa\ell'\right)n} = \frac{1}{1-(-1)^{\mathsf{v}}\mathsf{e}^{-6\pi\mathsf{i}\mathsf{v}(j'-\kappa)}}, \tag{3.3.59}$$

using that $6\mathsf{v}\kappa = 2\mathsf{u}-3\mathsf{v}$. To simplify the n-independent part, first note that

$$[j^{\mathrm{tw}}(\nabla(\mathsf{r}),\mathsf{s})] = [j^{\mathrm{tw}}(\mathsf{r},\mathsf{s})+\tfrac{\mathsf{u}}{3}], \quad [6\mathsf{v}j^{\mathrm{tw}}(\lambda')] = [\mathsf{v}] \tag{3.3.60}$$

for all $\lambda' \in \Gamma_{\mathsf{u},\mathsf{v}}$. The $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix entries in (3.3.58) (after expanding the BP S-matrix elements according to (3.2.31)) are related by (1.3.26). Putting this all together, the n-independent part of (3.3.58) reduces to

(3.3.61)

$$\frac{1-(-1)^{\mathsf{v}}\mathsf{e}^{-6\pi\mathsf{i}\mathsf{v}(j'-\kappa)}}{1-\mathsf{e}^{-2\pi\mathsf{i}\mathsf{v}(j'-\kappa)}\mathsf{e}^{2\pi\mathsf{i}\mathsf{v}j^{\mathrm{tw}}(\lambda')}} \cdot \sum_{m=0}^{\mathsf{v}-3}(-1)^m\mathsf{e}^{-2\pi\mathsf{i}(\ell+m+1)(j'-\kappa)}\mathsf{e}^{-4\pi\mathsf{i}\kappa(\ell+m+1)\ell'}\mathsf{e}^{-2\pi\mathsf{i}j^{\mathrm{tw}}(\nabla^{-1}(\mathsf{r}),\mathsf{s}_m)\ell'}\mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\nabla^{-1}(\mathsf{r}),\mathsf{s}_m)],[\lambda']}.$$

What remains to evaluate is the tricky sum over m. Note that when $\mathsf{v}=3$, only the $m=0$ term remains and we get the S-matrix element (3.3.16).

To attack this sum when $\mathsf{v}>3$, firstly note that

$$[j^{\mathrm{tw}}(\mathsf{r},\mathsf{s}_m)] = [j^{\mathrm{tw}}(\mathsf{r},\mathsf{s}_0)-2\kappa m]. \tag{3.3.62}$$

The $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix entries in (3.3.61) can also be simplified using Proposition 1.3.5:

$$\begin{aligned}\frac{\mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\nabla^{-1}(\mathsf{r}),\mathsf{s}_m)],[\lambda']}}{\mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r},\nabla(\mathsf{s}_0))],[\lambda']}} &= \frac{\mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r},\nabla(\mathsf{s}_m))],[\Gamma(\mathsf{r}',\mathsf{s}')]}}{\mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r},0)],[\Gamma(\mathsf{r}',\mathsf{s}')]}} \\ &= \mathsf{e}^{2\pi\mathsf{i}\langle\overline{\mathsf{r}'}+\overline{\rho},\overline{\nabla(\mathsf{s}_m)}\rangle}\chi_{\overline{\nabla(\mathsf{s}_m)}}(\xi_{\overline{\mathsf{s}'}}) \\ &= \mathsf{e}^{2\pi\mathsf{i}m\langle\overline{\mathsf{r}'}+\overline{\rho},\omega_2\rangle}\chi_{m\omega_2}(\xi_{\overline{\mathsf{s}'}}),\end{aligned} \tag{3.3.63}$$

where $\lambda' = \Gamma(\mathsf{r}',\mathsf{s}')$, $0 = [\mathsf{v}-3,0,0]$ and $\chi_{m\omega_2}(\xi_{\overline{\mathsf{s}'}})$ is the character of the simple highest-weight $\mathfrak{sl}_3$-module $\overline{\mathcal{L}}_{m\omega_2}$ evaluated at the $\mathfrak{sl}_3$ weight

$$\xi_{\overline{\mathsf{s}'}} = -2\pi\mathsf{i}\tfrac{\mathsf{u}}{\mathsf{v}}(\overline{\mathsf{s}'}+\overline{\rho}). \tag{3.3.64}$$

The sum over m in (3.3.61) thus simplifies to, using that $[\underline{\lambda}] = [\Gamma(\mathsf{r}, \nabla(\mathsf{s}_0))]$,

$$\mathrm{e}^{-2\pi\mathrm{i}(\ell+1)(j'-\kappa)}\mathrm{e}^{-4\pi\mathrm{i}\kappa(\ell+1)\ell'}\mathrm{e}^{-2\pi\mathrm{i}j^{\mathrm{tw}}(\nabla^{-1}(\mathsf{r}),\mathsf{s}_0)\ell'}\mathsf{S}^{\mathsf{W}_3}_{[\underline{\lambda}],[\lambda']}\sum_{m=0}^{\mathsf{v}-3} x^m \chi_{m\omega_2}(\xi_{\overline{\mathsf{s}'}}), \tag{3.3.65}$$

where $x = -\mathrm{e}^{2\pi\mathrm{i}\left(\langle\overline{\mathsf{r}'}+\overline{\rho},\omega_2\rangle-(j'-\kappa)\right)}$. The remaining sum over m can be evaluated (see Section 3.3.5) and is given by

$$\sum_{m=0}^{\mathsf{v}-3} x^m \chi_{m\omega_2}(\xi_{\overline{\mathsf{s}'}}) = \mathrm{e}^{3\pi\mathrm{i}(j'-\kappa)}\frac{1-\mathrm{e}^{-2\pi\mathrm{i}\mathsf{v}(j'-\kappa)}\mathrm{e}^{2\pi\mathrm{i}\mathsf{v}j^{\mathrm{tw}}(\lambda')}}{8\sin(\pi c_1)\sin(\pi c_2)\sin(\pi c_3)}, \tag{3.3.66}$$

where $c_i = (j'-\kappa) - j^{\mathrm{tw}}\big(\nabla^i(\lambda')\big)$. Finally, putting (3.3.59), (3.3.61), (3.3.65) and (3.3.66) together, we obtain

$$\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell,\lambda} = \frac{\mathrm{e}^{-2\pi\mathrm{i}(\ell-1/2)(j'-\kappa)}\mathrm{e}^{-4\pi\mathrm{i}\kappa(\ell+1)\ell'}\mathrm{e}^{-2\pi\mathrm{i}j^{\mathrm{tw}}(\nabla^{-1}(\mathsf{r}),\mathsf{s}_0)\ell'}}{8\sin(\pi c_1)\sin(\pi c_2)\sin(\pi c_3)}\mathsf{S}^{\mathsf{W}_3}_{[\underline{\lambda}],[\lambda']}. \tag{3.3.67}$$

The proof is completed by applying the trigonometric product-to-sum formula twice:

$$8\sin(\pi c_1)\sin(\pi c_2)\sin(\pi c_3) = 2\cos\big(3\pi(j'-\kappa)\big) - \sum_{i\in\mathbb{Z}_3} 2\cos\big(\pi a_i(j',\lambda')\big), \tag{3.3.68}$$

where $a_i(j,\lambda) = (j-\kappa) + 2j^{\mathrm{tw}}\big(\nabla^i(\lambda)\big)$. ■

Observe that the denominator of the S-matrix entries (3.3.57) only depends on j' and λ': the dependence of $\mathsf{S}^{\ell',[j'],[\lambda']}_{\ell,\lambda}$ on the type-3 module $\mathcal{H}^{\ell}_{\lambda}$ is confined entirely to the exponential term and the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ S-matrix element. This will prove useful when calculating Grothendieck fusion rules involving type-3 modules.

As always, the S-matrix elements involving the vacuum module $\mathcal{H}_{\mathsf{k}\omega_0} = \mathcal{H}\left[\begin{smallmatrix}\mathsf{u}-3 & 0 & 0\\ \mathsf{v}-2 & -1 & 0\end{smallmatrix}\right]$ are of particular importance in Verlinde computations. These will again be given the special notation $\mathsf{S}^{\ell',[j'],[\lambda']}_{\mathrm{vac.}} = \mathsf{S}^{\ell',[j'],[\lambda']}_{0,\mathsf{k}\omega_0}$.

Corollary 3.3.16. *Let* k *be nondegenerate-admissible. Then,*

$$\mathsf{S}^{\ell',[j'],[\lambda']}_{\mathrm{vac.}} = \mathsf{S}^{\mathsf{W}_3}_{\mathrm{vac.},[\lambda']}\frac{\mathrm{e}^{2\pi\mathrm{i}\kappa\ell'}\mathrm{e}^{\pi\mathrm{i}(j'-\kappa)}}{2\cos\big(3\pi(j'-\kappa)\big)-\sum_{i\in\mathbb{Z}_3}2\cos\big(\pi a_i(j',\lambda')\big)}, \qquad \mathsf{S}^{\mathsf{W}_3}_{\mathrm{vac.},[\lambda']} = \mathsf{S}^{\mathsf{W}_3}_{[\underline{\mathsf{k}\omega_0}],[\lambda']}. \tag{3.3.69}$$

As the denominator of (3.3.69) is proportional to $\sin(c_1\pi)\sin(c_2\pi)\sin(c_3\pi)$ (see (3.3.66)) it vanishes if and only if one of the c_i is an integer. This is equivalent to having

$$[j'] = [j^{\mathrm{tw}}\big(\nabla^i(\lambda')\big)+\kappa] \tag{3.3.70}$$

for some $i \in \mathbb{Z}_3$. Therefore the vacuum S-matrix elements again diverge precisely when $\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}$ is nonsimple.

3.3.5. A Character Identity. This section is devoted to the proof of the character identity (3.3.66). The techniques used here are largely independent of the previous sections.

To begin, given that $\chi_{\omega_2} = \mathrm{e}^{\omega_2} + \mathrm{e}^{\omega_1-\omega_2} + \mathrm{e}^{-\omega_1}$ and $\overline{\mathcal{L}}_{m\omega_2}$ is isomorphic to the m-th symmetric product of $\overline{\mathcal{L}}_{\omega_2}$, the character of $\overline{\mathcal{L}}_{m\omega_2}$ is

$$\chi_{m\omega_2} = h_m \left(\mathrm{e}^{\omega_2}, \mathrm{e}^{\omega_1-\omega_2}, \mathrm{e}^{-\omega_1}\right), \tag{3.3.71}$$

where h_m is the m-th complete symmetric polynomial. The following proposition evaluates the required weighted sum for general arguments.

Lemma 3.3.17. *Let $x, X_1, X_2, X_3 \in \mathbb{C}$ be such that the X_i are distinct and $x \neq X_i^{-1}$, for all $i = 1, 2, 3$. Suppose further that $X_1^{\mathsf{v}} = X_2^{\mathsf{v}} = X_3^{\mathsf{v}}$ for some $\mathsf{v} \in \mathbb{Z}_{\geqslant 3}$. Then,*

$$\sum_{m=0}^{\mathsf{v}-3} x^m h_m (X_1, X_2, X_3) = \frac{1 - x^{\mathsf{v}} X_2^{\mathsf{v}}}{(1 - xX_1)(1 - xX_2)(1 - xX_3)}. \tag{3.3.72}$$

Proof. By computing a partial fraction decomposition for the standard generating function of the complete symmetric polynomials, we arrive at the identity

$$h_m (X_1, X_2, X_3) = \frac{X_1^{m+2}}{(X_1 - X_2)(X_1 - X_3)} + \frac{X_2^{m+2}}{(X_2 - X_1)(X_2 - X_3)} + \frac{X_3^{m+2}}{(X_3 - X_1)(X_3 - X_2)}. \tag{3.3.73}$$

Since $X_1^{\mathsf{v}} = X_2^{\mathsf{v}} = X_3^{\mathsf{v}}$, explicit calculation now gives

$$\begin{aligned} \sum_{m=0}^{\mathsf{v}-3} x^m h_m (X_1, X_2, X_3) &= \frac{X_1^2(1 - x^{\mathsf{v}-2}X_1^{\mathsf{v}-2})}{(X_1 - X_2)(X_1 - X_3)(1 - xX_1)} + \frac{X_2^2(1 - x^{\mathsf{v}-2}X_2^{\mathsf{v}-2})}{(X_2 - X_1)(X_2 - X_3)(1 - xX_2)} \\ &\quad + \frac{X_3^2(1 - x^{\mathsf{v}-2}X_3^{\mathsf{v}-2})}{(X_3 - X_1)(X_3 - X_2)(1 - xX_3)} \\ &= \frac{X_1^2 - x^{\mathsf{v}-2}X_2^{\mathsf{v}}}{(X_1 - X_2)(X_1 - X_3)(1 - xX_1)} + \frac{X_2^2 - x^{\mathsf{v}-2}X_2^{\mathsf{v}}}{(X_2 - X_1)(X_2 - X_3)(1 - xX_2)} \\ &\quad + \frac{X_3^2 - x^{\mathsf{v}-2}X_2^{\mathsf{v}}}{(X_3 - X_1)(X_3 - X_2)(1 - xX_3)} \\ &= \frac{1 - x^{\mathsf{v}}X_2^{\mathsf{v}}}{(1 - xX_1)(1 - xX_2)(1 - xX_3)}. \end{aligned} \tag{3.3.74}$$

■

Proposition 3.3.18. *Let k be nondegenerate-admissible, $[\lambda'] = [\Gamma(\mathsf{r}', \mathsf{s}')] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$, $[j'] \in \mathbb{R}/\mathbb{Z}$, $\xi_{\overline{\mathsf{s}'}}$ be as in (3.3.64) and $x = -\mathrm{e}^{2\pi\mathrm{i}\left(\langle \overline{\mathsf{r}'} + \overline{\rho}, \omega_2\rangle - (j' - \kappa)\right)}$. Then,*

$$\sum_{m=0}^{\mathsf{v}-3} x^m \chi_{m\omega_2}(\xi_{\overline{\mathsf{s}'}}) = \left(1 - \mathrm{e}^{-2\pi\mathrm{i}\mathsf{v}(j'-\kappa)}\mathrm{e}^{2\pi\mathrm{i}\mathsf{v}j^{\mathrm{tw}}(\lambda')}\right)\frac{\mathrm{e}^{3\pi\mathrm{i}(j'-\kappa)}}{8\sin(\pi c_1)\sin(\pi c_2)\sin(\pi c_3)}, \tag{3.3.75}$$

where $c_i = (j' - \kappa) - j^{\mathrm{tw}}\big(\nabla^i(\lambda')\big)$.

Proof. For each $\mathfrak{sl}_3$-weight ω, we have

$$\left(\mathrm{e}^{\langle \omega, \xi_{\overline{\mathsf{s}'}}\rangle}\right)^{\mathsf{v}} = \mathrm{e}^{-2\pi \mathrm{i}\mathsf{u}\langle \omega, \overline{\mathsf{s}'} + \overline{\rho}\rangle}, \tag{3.3.76}$$

which is clearly invariant under shifting ω by elements of $\overline{\mathsf{Q}}$. It follows that if we set

$$X_1 = \mathrm{e}^{\langle \omega_2, \xi_{\overline{\mathsf{s}'}}\rangle}, \quad X_2 = \mathrm{e}^{\langle \omega_1 - \omega_2, \xi_{\overline{\mathsf{s}'}}\rangle} \quad \text{and} \quad X_3 = \mathrm{e}^{\langle -\omega_1, \xi_{\overline{\mathsf{s}'}}\rangle}, \tag{3.3.77}$$

then $X_1^{\mathsf{v}} = X_2^{\mathsf{v}} = X_3^{\mathsf{v}}$. This allows us to apply Lemma 3.3.17, which results in

$$\sum_{m=0}^{\mathsf{v}-3} x^m \chi_{m\omega_2}(\xi_{\overline{\mathsf{s}'}}) = \sum_{m=0}^{\mathsf{v}-3} x^m h_m(X_1, X_2, X_3) = \frac{1 - x^{\mathsf{v}} X_2^{\mathsf{v}}}{(1 - xX_1)(1 - xX_2)(1 - xX_3)}. \tag{3.3.78}$$

Since

$$\langle \overline{\mathsf{r}'} + \overline{\rho} - \tfrac{\mathsf{u}}{\mathsf{v}}(\overline{\mathsf{s}'} + \overline{\rho}), \omega_2\rangle = \tfrac{1}{3}(r_1' + 2r_2' + 3) - \tfrac{\mathsf{u}}{3\mathsf{v}}(s_1' + 2s_2' + 3) = j^{\mathrm{tw}}\big(\nabla^2(\lambda')\big) + \tfrac{1}{2}, \tag{3.3.79a}$$

$$\langle \overline{\mathsf{r}'} + \overline{\rho} - \tfrac{\mathsf{u}}{\mathsf{v}}(\overline{\mathsf{s}'} + \overline{\rho}), \omega_1 - \omega_2\rangle = j^{\mathrm{tw}}(\lambda') + \tfrac{1}{2}, \tag{3.3.79b}$$

$$\langle \overline{\mathsf{r}'} + \overline{\rho} - \tfrac{\mathsf{u}}{\mathsf{v}}(\overline{\mathsf{s}'} + \overline{\rho}), -\omega_1\rangle = j^{\mathrm{tw}}\big(\nabla(\lambda')\big) + \tfrac{1}{2}, \tag{3.3.79c}$$

we obtain

$$xX_i = \mathrm{e}^{-2\pi \mathrm{i}(j' - \kappa)}\mathrm{e}^{2\pi \mathrm{i} j^{\mathrm{tw}}\left(\nabla^{i+1}(\lambda')\right)}. \tag{3.3.80}$$

Substituting into (3.3.78), we arrive at the desired result by rearranging the exponentials and observing that $\sum_{i \in \mathbb{Z}_3} j^{\mathrm{tw}}\big(\nabla^i(\lambda')\big) = -\frac{3}{2}$. ∎

3.3.6. Grothendieck Fusion Rules for $\mathsf{BP}(\mathsf{u}, \mathsf{v})$**.** We now have all the information necessary to apply the standard Verlinde formula (3.3.2b) and obtain the standard-by-standard Grothendieck fusion rules for $\mathsf{BP}(\mathsf{u}, \mathsf{v})$.

Theorem 3.3.19. *Let* k *be nondegenerate-admissible. Then for* $\ell, \ell' \in \frac{1}{2}\mathbb{Z}$, $[j], [j]' \in \mathbb{R}/\mathbb{Z}$ *and* $[\lambda], [\lambda'] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$, *the Grothendieck fusion rules of the standard* $\mathsf{BP}(\mathsf{u}, \mathsf{v})$*-modules are*

$$\begin{aligned}
\big[\widetilde{\mathcal{R}}^{\ell}_{[j],[\lambda]}\big] \boxtimes \big[\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}\big] = &\sum_{[\lambda''] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3} \mathcal{N}^{\mathsf{W}_3\,[\lambda'']}_{[\lambda],[\lambda']}\Big(\big[\widetilde{\mathcal{R}}^{\ell+\ell'+2}_{[j+j'-4\kappa],[\lambda'']}\big] + \big[\widetilde{\mathcal{R}}^{\ell+\ell'-1}_{[j+j'+2\kappa],[\lambda'']}\big]\Big) \\
&+ \sum_{[\lambda''] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3} \sum_{i \in \mathbb{Z}_3} \Big(\mathcal{N}^{\mathsf{W}_3\,[\lambda'']}_{[\lambda],[\Gamma(\mathsf{r}',\mathsf{s}'-\omega_i+\omega_{i+1})]}\big[\widetilde{\mathcal{R}}^{\ell+\ell'+1}_{[j+j'-2\kappa],[\lambda'']}\big] \\
&\qquad + \mathcal{N}^{\mathsf{W}_3\,[\lambda'']}_{[\lambda],[\Gamma(\mathsf{r}',\mathsf{s}'+\omega_i-\omega_{i+1})]}\big[\widetilde{\mathcal{R}}^{\ell+\ell'}_{[j+j'],[\lambda'']}\big]\Big),
\end{aligned} \tag{3.3.81}$$

where $\lambda' = \Gamma(\mathsf{r}', \mathsf{s}')$.

Proof. As in the $\mathsf{v} = 3$ case, we apply the standard Verlinde formula (3.3.2b) with $\ell = \ell' = 0$ using (3.2.31) and (3.3.69):

(3.3.82)
$$\begin{pmatrix} & \ell'', [j''], [\lambda''] & \\ 0, [j], [\lambda] & & 0, [j'], [\lambda'] \end{pmatrix} = \sum_{[\mu] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3} \frac{\mathsf{S}^{\mathsf{W}_3}_{[\lambda],[\mu]} \mathsf{S}^{\mathsf{W}_3}_{[\lambda'],[\mu]} \left(\mathsf{S}^{\mathsf{W}_3}_{[\lambda''],[\mu]}\right)^*}{\mathsf{S}^{\mathsf{W}_3}_{\mathrm{vac.},[\mu]}} \sum_{m \in \mathbb{Z}} \mathrm{e}^{-2\pi\mathrm{i}(j+j'-j''-2\kappa\ell'')m}$$
$$\cdot \int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{2\pi\mathrm{i}(\ell''-1/2)(k-\kappa)} \left(2\cos\big(3\pi(k-\kappa)\big) - \sum_{i \in \mathbb{Z}_3} 2\cos\big(\pi a_i(k,\mu)\big)\right) \mathrm{d}[k].$$

The Grothendieck fusion coefficient thus naturally splits as a sum of two contributions. That which involves the term $2\cos\big(3\pi(k-\kappa)\big)$ is identical to the $\mathsf{v} = 3$ coefficient computed in Theorem 3.3.6:

$$\mathcal{N}^{\mathsf{W}_3\,[\lambda'']}_{[\lambda],[\lambda']} \Big(\delta\big([j''] - [j + j' - 4\kappa]\big)\delta_{\ell'',2} + \delta\big([j''] - [j + j' + 2\kappa]\big)\delta_{\ell'',-1}\Big). \tag{3.3.83}$$

The remaining contributions involving $2\cos\big(\pi a_i(k,\mu)\big)$ simplify to

(3.3.84)
$$-\sum_{[\mu] \in \Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3} \sum_{\varepsilon=\pm 1} \sum_{i \in \mathbb{Z}_3} \mathrm{e}^{2\pi\mathrm{i}\varepsilon j^{\mathrm{tw}}(\nabla^i(\mu))} \frac{\mathsf{S}^{\mathsf{W}_3}_{[\lambda],[\mu]} \mathsf{S}^{\mathsf{W}_3}_{[\lambda'],[\mu]} \left(\mathsf{S}^{\mathsf{W}_3}_{[\lambda''],[\mu]}\right)^*}{\mathsf{S}^{\mathsf{W}_3}_{\mathrm{vac.},[\mu]}} \delta\big([j''] - [j + j' - (1-\varepsilon)\kappa]\big)\delta_{\ell'',\frac{1}{2}(1-\varepsilon)}.$$

This contribution can be evaluated with help from (3.3.79) and the W_3 S-matrix identity Proposition 1.3.6 (with $\bar{\mathsf{t}} = \omega_2$):

$$\sum_{i \in \mathbb{Z}_3} \mathrm{e}^{2\pi\mathrm{i}j^{\mathrm{tw}}(\nabla^i(\mu))} \mathsf{S}^{\mathsf{W}_3}_{[\lambda'],[\mu]} = -\mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r}',\mathsf{s}'\otimes\omega_2)],[\mu]} = -\sum_{i \in \mathbb{Z}_3} \mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r}',\mathsf{s}'+\omega_i-\omega_{i+1})],[\mu]}, \tag{3.3.85}$$

where $\lambda' = \Gamma(\mathsf{r}', \mathsf{s}')$. Similarly, $\bar{\mathsf{t}} = \omega_1$ results in

$$\sum_{i \in \mathbb{Z}_3} \mathrm{e}^{-2\pi\mathrm{i}j^{\mathrm{tw}}(\nabla^i(\mu))} \mathsf{S}^{\mathsf{W}_3}_{[\lambda'],[\mu]} = -\mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r}',\mathsf{s}'\otimes\omega_1)],[\mu]} = -\sum_{i \in \mathbb{Z}_3} \mathsf{S}^{\mathsf{W}_3}_{[\Gamma(\mathsf{r}',\mathsf{s}'-\omega_i+\omega_{i+1})],[\mu]}. \tag{3.3.86}$$

As $\mathsf{s}' \in \mathsf{P}^{\mathsf{v}-3}_{\geqslant}$, the weight $\mathsf{s}' + \varepsilon(\omega_i - \omega_{i+1})$ is either in $\mathsf{P}^{\mathsf{v}-3}_{\geqslant}$ or it lies on a boundary of a shifted affine alcove, in which case the corresponding S-matrix entry is 0 (see (1.3.28) and the surrounding discussion). The $[\mu]$-sum in (3.3.84) therefore evaluates to

$$\sum_{\varepsilon=\pm 1} \sum_{i \in \mathbb{Z}_3} \mathcal{N}^{\mathsf{W}_3\,[\lambda'']}_{[\lambda],\Gamma(\mathsf{r}',\mathsf{s}'+\varepsilon(\omega_i-\omega_{i+1}))} \delta\big([j''] - [j + j' - (1-\varepsilon)\kappa]\big)\delta_{\ell'',\frac{1}{2}(1-\varepsilon)}, \tag{3.3.87}$$

where the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ fusion coefficient is understood to be 0 whenever $\mathsf{s}' + \varepsilon(\omega_i - \omega_{i+1}) \notin \mathsf{P}^{\mathsf{v}-3}_{\geqslant}$. ∎

Reassuringly, all the standard-by-standard Grothendieck fusion coefficients are nonnegative integers.

As in the $\mathsf{v}=3$ case, the asymmetry in spectral flow indices and J_0-eigenvalues can be remedied by recasting the (3.3.81) in terms of the twisted modules $\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]}$:

$$\begin{aligned}(3.3.88)\quad &\big[\sigma^{\ell}(\mathcal{R}^{\mathrm{tw}}_{[j],[\lambda]})\big] \boxtimes \big[\sigma^{\ell'}(\mathcal{R}^{\mathrm{tw}}_{[j'],[\lambda']})\big] \\ &= \sum_{[\lambda'']\in\Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3} \mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\lambda],[\lambda']}\Big(\big[\sigma^{\ell+\ell'+3/2}(\mathcal{R}^{\mathrm{tw}}_{[j+j'-3\kappa],[\lambda'']})\big] + \big[\sigma^{\ell+\ell'-3/2}(\mathcal{R}^{\mathrm{tw}}_{[j+j'+3\kappa],[\lambda'']})\big]\Big) \\ &\quad + \sum_{[\lambda'']\in\Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3}\sum_{i\in\mathbb{Z}_3}\Big(\mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\lambda],[\Gamma(\mathsf{r}',\mathsf{s}'-\omega_i+\omega_{i+1})]}\big[\sigma^{\ell+\ell'+1/2}(\mathcal{R}^{\mathrm{tw}}_{[j+j'-\kappa],[\lambda'']})\big] \\ &\qquad + \mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\lambda],[\Gamma(\mathsf{r}',\mathsf{s}'+\omega_i-\omega_{i+1})]}\big[\sigma^{\ell+\ell'-1/2}(\mathcal{R}^{\mathrm{tw}}_{[j+j'+\kappa],[\lambda'']})\big]\Big).\end{aligned}$$

In principle, all Grothendieck fusion rules involving a highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module can now be derived using the resolutions of Section 3.3.4. The general approach being: identify the highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module $\mathcal{M}$ as a spectral flow of a highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-module leftmost in its spectral flow orbit. Then, (a spectral flow of) (3.3.52) gives an equality between the Grothendieck image $[\mathcal{M}]$ and a weighted sum of Grothendieck images of standard modules. The desired fusion rule can then be obtained by applying the rule (3.3.81) term-by-term.

The fact that we were able to derive the type-3 S-matrix coefficients in Theorem 3.3.15 means that we can avoid this process for the type-3-by-standard Grothendieck fusion rule, by way of the standard Verlinde formula (3.3.2b).

Corollary 3.3.20. *Let k be nondegenerate-admissible. Then for $\ell,\ell'\in\frac{1}{2}\mathbb{Z}$, $[j']\in\mathbb{R}/\mathbb{Z}$, $\lambda=\Gamma\big[\begin{smallmatrix} r_0 & r_1 & r_2\\ \mathsf{v}-2 & -1 & 0\end{smallmatrix}\big]\in\Sigma_{\mathsf{u},\mathsf{v}}$ and $[\lambda']\in\Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3$, the type-3-by-standard Grothendieck fusion rules are*

$$(3.3.89)\qquad \big[\sigma^{\ell}(\mathcal{H}_{\lambda})\big]\boxtimes\big[\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}\big] = \sum_{[\lambda'']\in\Gamma_{\mathsf{u}}\mathsf{v}/\mathbb{Z}_3}\mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\underline{\lambda}],[\lambda']}\big[\widetilde{\mathcal{R}}^{\ell+\ell'}_{[j(\lambda)+j'],[\lambda'']}\big].$$

Proof. By (3.3.2b), the coefficients for the Grothendieck fusion of $\sigma^{\ell}(\mathcal{H}_{\lambda})$ and $\widetilde{\mathcal{R}}^{\ell'}_{[j'],[\lambda']}$ are given by

$$(3.3.90)\qquad \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}}\sum_{[\mu]\in\Gamma_{\mathsf{u},\mathsf{v}}/\mathbb{Z}_3}\frac{\mathsf{S}^{m,[k],[\mu]}_{\ell,\lambda}\ \mathsf{S}^{m,[k],[\mu]}_{\ell',[j'],[\lambda']}\ \Big(\mathsf{S}^{m,[k],[\mu]}_{\ell'',[j''],[\lambda'']}\Big)^{*}}{\mathsf{S}^{m,[k],[\mu]}_{\mathrm{vac.}}}\,\mathrm{d}[k].$$

Substituting (3.2.31), (3.3.57) and (3.3.69) and noting that the trigonometric factors all cancel, this evaluates to

$$\begin{aligned}(3.3.91)\qquad &\mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\underline{\lambda}],[\lambda']}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}}\mathsf{e}^{-2\pi\mathsf{i}\left(2\kappa(\ell+\ell'-\ell'')m+(j(\lambda)+j'-j'')m+(\ell+\ell'-\ell'')(k-\kappa)\right)}\,\mathrm{d}[k] \\ &= \mathcal{N}^{\mathsf{W}_3\ [\lambda'']}_{[\underline{\lambda}],[\lambda']}\delta\big([j'']-[j(\lambda)+j']\big)\delta_{\ell'',\ell+\ell'}.\end{aligned}$$

∎

More difficult are the type-3-by-type-3 Grothendieck fusion rules. Here, applying (3.3.2b) is technically demanding due to the abundance of trigonometric factors that must be handled. So we shall compute this fusion rule using the resolutions from Proposition 3.3.12.

Noting that all standard modules in the resolution (3.3.51) are nonsimple, it is useful to consider the type-3-by-nonsimple standard fusion rule more carefully:

$$\begin{aligned}[\sigma^{\ell}(\mathcal{H}_{\lambda})] \boxtimes [\widetilde{\mathcal{R}}^{\ell'}_{\lambda'}] &= [\sigma^{\ell}(\mathcal{H}_{\lambda})] \boxtimes [\widetilde{\mathcal{R}}^{\ell'}_{[j^{\mathrm{tw}}(\lambda')+\kappa],[\lambda']}] \\ &= \sum_{[\lambda'']\in\Gamma_{\mathsf{u}\mathsf{v}}/\mathbb{Z}_3} \mathcal{N}^{\mathsf{W}_3\,[\lambda'']}_{[\underline{\lambda}],[\lambda']} [\widetilde{\mathcal{R}}^{\ell+\ell'}_{[j(\lambda)+j^{\mathrm{tw}}(\lambda')+\kappa],[\lambda'']}].\end{aligned} \tag{3.3.92}$$

Our first task is to show that the standard modules appearing on the right-hand side are also nonsimple.

Without loss of generality, let $\lambda'' = \Gamma(\mathsf{r}'', \mathsf{s}'') \in [\lambda'']$ be a representative satisfying the conditions required by Theorem 1.3.7. Since $\underline{\lambda}, \lambda' \in \Gamma_{\mathsf{u},\mathsf{v}}$ are fixed in the (Grothendieck) fusion of $\mathcal{H}_{\lambda}$ and $\widetilde{\mathcal{R}}_{\lambda'}$, the corresponding representatives of $[\underline{\lambda}]$ and $[\lambda']$ will have the form $\nabla^m(\underline{\lambda})$ and $\nabla^n(\lambda')$, respectively, for some $m, n \in \mathbb{Z}_3$. Write $\underline{\lambda} = \Gamma(\mathsf{r}', 0)$, where $0 \equiv [\mathsf{v}-3, 0, 0]$ as usual, and $\lambda' = \Gamma(\mathsf{r}', \mathsf{s}')$. Then, by (1.3.37) and (1.3.36), the W_3 fusion coefficient in the summand of (3.3.92) decomposes according to

$$\begin{aligned}\mathcal{N}^{\mathsf{W}_3\,[\lambda'']}_{[\underline{\lambda}],[\lambda']} &= \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\nabla^m(\mathsf{r}),\nabla^n(\mathsf{r}')} \mathcal{N}^{\mathsf{v}-3\ \mathsf{s}''}_{\nabla^m(0),\nabla^n(\mathsf{s}')} \\ &= \mathcal{N}^{\mathsf{u}-3\ \nabla^{-m-n}(\mathsf{r}'')}_{\mathsf{r},\mathsf{r}'} \mathcal{N}^{\mathsf{v}-3\ \nabla^{-m-n}(\mathsf{s}'')}_{0,\mathsf{s}'} \\ &= \mathcal{N}^{\mathsf{u}-3\ \nabla^{-m-n}(\mathsf{r}'')}_{\mathsf{r},\mathsf{r}'} \delta_{\mathsf{s}',\nabla^{-m-n}(\mathsf{s}'')}.\end{aligned} \tag{3.3.93}$$

Hence, the W_3 fusion coefficient is only nonzero when $\lambda'' = \nabla^{m+n}(\Gamma(\mathsf{t}'', \mathsf{s}'))$, for some $\mathsf{t}'' \in \mathsf{P}^{\mathsf{u}-3}_{\geqslant}$. On the other hand, if $\mathcal{N}^{\mathsf{u}-3\ \nabla^{-m-n}(\mathsf{r}'')}_{\mathsf{r},\mathsf{r}'} = \mathcal{N}^{\mathsf{u}-3\ \mathsf{t}''}_{\mathsf{r},\mathsf{r}'}$ is nonzero, then by the Kac–Walton formula (1.2.14) we must have that $\overline{t''} = \overline{r} + \overline{r'} \bmod \mathsf{Q}$. Then,

$$[j(\lambda) + j^{\mathrm{tw}}(\lambda') + \kappa] = [j^{\mathrm{tw}}(\nabla^{-m-n}(\lambda'')) + \kappa] = [j^{\mathrm{tw}}(\mathsf{t}'', \mathsf{s}') + \kappa]. \tag{3.3.94}$$

Therefore the standard modules on the right-hand side of (3.3.92) are the nonsimple modules $\widetilde{\mathcal{R}}^{\ell+\ell'}_{\Gamma(\mathsf{t}'',\mathsf{s}')}$, where t'' satisfies the equation above. That is,

$$[\sigma^{\ell}(\mathcal{H}_{\lambda})] \boxtimes [\widetilde{\mathcal{R}}^{\ell'}_{\lambda'}] = \sum_{\substack{\mathsf{t}''\in\mathsf{P}^{\mathsf{u}-3}_{\geqslant} \\ [j(\mathsf{t}'')]=[j(\mathsf{r})+j(\mathsf{r}')]}} \mathcal{N}^{\mathsf{W}_3\,[\Gamma(\mathsf{t}'',\mathsf{s}')]}_{[\underline{\lambda}],[\lambda']} [\widetilde{\mathcal{R}}^{\ell+\ell'}_{\Gamma(\mathsf{t}'',\mathsf{s}')}]. \tag{3.3.95}$$

As in the proof of Corollary 3.3.8, the additional constraint on t'' may be removed by converting the $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ fusion coefficient to a $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$ one. Replacing t'' with r'', the final type-3-by-nonsimple standard Grothendieck fusion rule is

$$[\sigma^{\ell}(\mathcal{H}_{\lambda})] \boxtimes [\widetilde{\mathcal{R}}^{\ell'}_{\lambda'}] = \sum_{\mathsf{r}''\in\mathsf{P}^{\mathsf{u}-3}_{\geqslant}} \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\mathsf{r},\mathsf{r}'} [\widetilde{\mathcal{R}}^{\ell+\ell'}_{\Gamma(\mathsf{r}'',\mathsf{s}')}]. \tag{3.3.96}$$

Corollary 3.3.21. *Let* k *be nondegenerate-admissible. Then for all* $\ell,\ell'\in\frac{1}{2}\mathbb{Z}$*,* $\lambda=\Gamma\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2 & -1 & 0\end{smallmatrix}\right]\in\Sigma_{\mathsf{u},\mathsf{v}}$ *and* $\lambda'=\Gamma\left[\begin{smallmatrix} r'_0 & r'_1 & r'_2 \\ \mathsf{v}-2 & -1 & 0\end{smallmatrix}\right]\in\Sigma_{\mathsf{u},\mathsf{v}}$*, the Grothendieck fusion rules between type-*3 *highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules are*

$$[\sigma^{\ell}(\mathcal{H}_{\lambda})] \boxtimes [\sigma^{\ell}(\mathcal{H}_{\lambda'})] = \sum_{\mathsf{r}''\in\mathsf{P}^{\mathsf{u}-3}_{\geqslant}} \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\mathsf{r},\mathsf{r}'} \Big[\sigma^{\ell+\ell'}\Big(\mathcal{H}\left[\begin{smallmatrix} r''_0 & r''_1 & r''_2 \\ \mathsf{v}-2 & -1 & 0\end{smallmatrix}\right]\Big)\Big]. \tag{3.3.97}$$

Proof. As usual, it is sufficient to prove (3.3.97) with $\ell=\ell'=0$. Let $\mathsf{s}'_m=[0,m,\mathsf{v}-3-m]$. Substituting (3.3.53) and then (3.3.95) into the left-hand side of (3.3.97) gives

$$\begin{aligned}
[\mathcal{H}_{\lambda}] \boxtimes [\mathcal{H}_{\lambda'}] &= \sum_{n=0}^{\infty}\sum_{m=0}^{\mathsf{v}-3}\sum_{i=0}^{2}(-1)^{m+(n+i)\mathsf{v}}[\mathcal{H}_{\lambda}] \boxtimes [\widetilde{\mathcal{R}}^{m+(3n+i)\mathsf{v}+1}_{\Gamma(\nabla^{i-1}(\mathsf{r}'),\mathsf{s}'_m)}] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\mathsf{v}-3}\sum_{i=0}^{2}(-1)^{m+(n+i)\mathsf{v}} \sum_{r''\in\mathsf{P}^{\mathsf{u}-3}_{\geqslant}} \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\mathsf{r},\nabla^{i-1}(\mathsf{r}')} [\widetilde{\mathcal{R}}^{m+(3n+i)\mathsf{v}+1}_{\Gamma(\mathsf{r}'',\mathsf{s}'_m)}] \\
&= \sum_{r''\in\mathsf{P}^{\mathsf{u}-3}_{\geqslant}} \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\mathsf{r},\mathsf{r}'} \sum_{n=0}^{\infty}\sum_{m=0}^{\mathsf{v}-3}\sum_{i=0}^{2}(-1)^{m+(n+i)\mathsf{v}} [\widetilde{\mathcal{R}}^{m+(3n+i)\mathsf{v}+1}_{\Gamma(\nabla^{i-1}(\mathsf{r}''),\mathsf{s}'_m)}] \\
&= \sum_{r''\in\mathsf{P}^{\mathsf{u}-3}_{\geqslant}} \mathcal{N}^{\mathsf{u}-3\ \mathsf{r}''}_{\mathsf{r},\mathsf{r}'} \Big[\mathcal{H}\left[\begin{smallmatrix} r''_0 & r''_1 & r''_2 \\ \mathsf{v}-2 & -1 & 0\end{smallmatrix}\right]\Big],
\end{aligned} \tag{3.3.98}$$

where the final equality is (3.3.53) for the highest-weight module $\mathcal{H}\left[\begin{smallmatrix} r''_0 & r''_1 & r''_2 \\ \mathsf{v}-2 & -1 & 0\end{smallmatrix}\right]$. ∎

We therefore know all Grothendieck fusion rules between standard and type-3 $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules. Before discussing other fusion rules, we are in a position to generalise several of the results for $\mathsf{v}=3$ from Section 3.3.2.

Firstly, as the simple highest-weight $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$-modules of highest weights $[0,\mathsf{u}-3,0]$ and $[0,0,\mathsf{u}-3]$ are simple currents, Corollary 3.3.21 implies the existence of simple currents for $\mathsf{BP}(\mathsf{u},\mathsf{v})$.

Proposition 3.3.22. *Let* k *be nondegenerate-admissible with* $\mathsf{u}>3$*. Then,* $\mathcal{H}\left[\begin{smallmatrix} 0 & \mathsf{u}-3 & 0 \\ \mathsf{v}-2 & -1 & 0\end{smallmatrix}\right]$ *and* $\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & \mathsf{u}-3 \\ \mathsf{v}-2 & -1 & 0\end{smallmatrix}\right]$ *are simple currents of order* 3*, inverse to one another. Their highest weights (with respect to* J_0 *and* L_0*) are*

$$(j,\Delta)=\left(+\tfrac{\mathsf{u}-3}{3},\tfrac{(\mathsf{u}-3)(2\mathsf{v}-3)}{6}\right) \quad \text{and} \quad (j,\Delta)=\left(-\tfrac{\mathsf{u}-3}{3},\tfrac{(\mathsf{u}-3)(2\mathsf{v}-3)}{6}\right), \tag{3.3.99}$$

respectively.

This generalises Proposition 3.3.9 to admissible levels with $\mathsf{v} > 3$. Secondly, Proposition 3.3.10 also generalises to $\mathsf{v} > 3$. The proofs are largely unchanged so we omit them.

Proposition 3.3.23. *Let* k *be nondegenerate-admissible. Then, the fusion subring of* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules generated by the type-*3 *simple highest-weight* $\mathsf{BP}(\mathsf{u},\mathsf{v})$*-modules* $\mathcal{H}_\lambda$, $\lambda = \Gamma\left[\begin{smallmatrix} r_0 & r_1 & r_2 \\ \mathsf{v}-2 & -1 & 0 \end{smallmatrix}\right] \in \Sigma_{\mathsf{u},\mathsf{v}}$*, is isomorphic to the fusion ring of the affine vertex operator algebra* $\mathsf{L}_{\mathsf{u}-3}(\mathfrak{sl}_3)$.

Of course there are highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules of type-1 and type-2 to care about. The Grothendieck fusion rules of such modules with standard modules is manageable. The more general highest-weight-by-highest-weight fusion rules are considerably more difficult. The reason being that the sheer number of terms encountered when expanding highest-weight Grothendieck images makes the result of fusion difficult to identify.

There are small cases, such as type-1 weights with $s_2 = 1$, where it is possible to identify such cancellations but even this limited case involves handling a large number of terms.

Rather than writing down closed-form expressions for all Grothendieck fusion rules explicitly, our philosophy is that it is better to provide an algorithmic means to construct the desired rules in individual cases. In other words, the resolutions and standard-by-standard Grothendieck fusion rules should be enough to compute all Grothendieck fusion rules for $\mathsf{BP}(\mathsf{u},\mathsf{v})$ at a given nondegenerate admissible level.

3.3.7. Examples. To illustrate the aforementioned philosophy, we will determine all of the Grothendieck fusion rules for the 'smallest' BP minimal model with $\mathsf{v} > 3$.

Example ($\mathsf{BP}(3,4)$). *Consider the Bershadsky–Polyakov minimal model* $\mathsf{BP}(3,4)$ *with* $\mathsf{k} = -\frac{9}{4}$ *and* $\mathsf{c} = -\frac{23}{2}$*. This minimal model is denoted by* $\mathcal{B}_4$ *in [53]. By the results of Chapter 2 (also shown in [5]), there are* 6 *untwisted (with respect to* $L(z)$*) simple highest-weight modules. We arrange them as in Figure 1, also adding the action of* ∇ *to the spectral flow orbits:*

$$\begin{tikzcd}[column sep=small] \textit{type-1:} & & & \mathcal{H}\left[\begin{smallmatrix} 0&0&0\\0&0&1 \end{smallmatrix}\right] \arrow[dr, "\nabla"] & & \\ \textit{type-2:} & & \mathcal{H}\left[\begin{smallmatrix} 0&0&0\\1&-1&1 \end{smallmatrix}\right] \arrow[rr, "\sigma"] & & \mathcal{H}\left[\begin{smallmatrix} 0&0&0\\1&0&0 \end{smallmatrix}\right] \arrow[dr, "\nabla"] & \\ \textit{type-3:} & \mathcal{H}\left[\begin{smallmatrix} 0&0&0\\0&-1&2 \end{smallmatrix}\right] \arrow[rr, "\sigma"] & & \mathcal{H}\left[\begin{smallmatrix} 0&0&0\\2&-1&0 \end{smallmatrix}\right] \arrow[rr, "\sigma"] & & \mathcal{H}\left[\begin{smallmatrix} 0&0&0\\0&1&0 \end{smallmatrix}\right] \end{tikzcd} \tag{3.3.100}$$

As can be clearly seen, ∇ *only acts on highest-weight* $\mathsf{BP}(3,4)$*-modules with infinite-dimensional top spaces.*

In addition to these highest-weight modules, there is a single one-parameter family of untwisted relaxed highest-weight modules

$$\widetilde{\mathcal{R}}_{[j]} = \widetilde{\mathcal{R}}_{[j],[\Gamma([0,0,0],[1,0,0])]}. \tag{3.3.101}$$

These modules are simple for all $[j] \in \mathbb{R}/\mathbb{Z}$ *except* $[j] = [0]$, $[\frac{1}{2}]$ *or* $[\frac{3}{4}]$. *The nonsimple cases are labelled by elements of* $\Gamma_{3,4}$ *according to*

$$\widetilde{\mathcal{R}}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] = \widetilde{\mathcal{R}}_{[1/2]}, \quad \widetilde{\mathcal{R}}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right] = \widetilde{\mathcal{R}}_{[1/4]}, \quad \widetilde{\mathcal{R}}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right] = \widetilde{\mathcal{R}}_{[0]}. \tag{3.3.102}$$

By the short exact sequence (3.3.47)*, the Grothendieck images of the nonsimple relaxed highest-weight modules satisfy the equalities*

$$\begin{aligned} \left[\widetilde{\mathcal{R}}_{[1/2]}\right] &= \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]^{1}\right] + \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \end{smallmatrix}\right]^{-1}\right], \\ \left[\widetilde{\mathcal{R}}_{[1/4]}\right] &= \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]^{1}\right] + \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]^{-1}\right], \\ \left[\widetilde{\mathcal{R}}_{[0]}\right] &= \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \end{smallmatrix}\right]^{2}\right] + \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]\right]. \end{aligned} \tag{3.3.103}$$

From (3.3.100)*, all type-*3 *modules are spectral flows of the vacuum module so their Grothendieck fusion rules are easy to determine.*

To unpack the standard-by-standard Grothendieck fusion rule of Theorem 3.3.19, observe that $\Gamma_{3,4}$ *has only one* $\mathbb{Z}_3$*-orbit. Additionally, the representatives used in* $\mathsf{W}_3(3,4)$ *fusion coefficients as per Theorem 1.3.7 all have the form* $\lambda = \Gamma(\mathsf{r},\mathsf{s})$*, with* $\mathsf{r} = [0,0,0]$ *and* $\mathsf{s} = [1,0,0]$ *because* $\mathsf{u} = 3$.

The relevant fusion coefficients in (3.3.81) *for* $(\mathsf{u},\mathsf{v}) = (3,4)$ *are then*

$$\mathcal{N}^{\mathsf{W}_3\,[\lambda]}_{[\lambda],[\lambda]} = \mathcal{N}^{1\ [0,0]}_{[0,0],[0,0]} = 1, \qquad \begin{aligned} \mathcal{N}^{\mathsf{W}_3\,[\lambda]}_{[\lambda],[\Gamma(\mathsf{r},\mathsf{s}-\omega_i+\omega_{i+1})]} &= \delta_{i,0}\mathcal{N}^{1\ [0,0]}_{[0,0],[0,0]} = \delta_{i,0}, \\ \mathcal{N}^{\mathsf{W}_3\,[\lambda]}_{[\lambda],[\Gamma(\mathsf{r},\mathsf{s}+\omega_i-\omega_{i+1})]} &= \delta_{i,2}\mathcal{N}^{1\ [0,0]}_{[0,0],[0,0]} = \delta_{i,2}, \end{aligned} \tag{3.3.104}$$

since, for example, $[\Gamma(\mathsf{r},\mathsf{s}-\omega_i+\omega_{i+1})] = [\Gamma(\mathsf{r},[0,1,0])] = [\lambda]$ *when* $i = 0$ *and* $\mathsf{s}-\omega_i+\omega_{i+1} \notin \mathsf{P}^1_{\geqslant}$ *otherwise. The standard-by-standard Grothendieck fusion rule is therefore*

$$\left[\widetilde{\mathcal{R}}^{\ell}_{[j]}\right] \boxtimes \left[\widetilde{\mathcal{R}}^{\ell'}_{[j']}\right] = \left[\widetilde{\mathcal{R}}^{\ell+\ell'-1}_{[j+j'+1/2]}\right] + \left[\widetilde{\mathcal{R}}^{\ell+\ell'}_{[j+j']}\right] + \left[\widetilde{\mathcal{R}}^{\ell+\ell'+1}_{[j+j'+1/2]}\right] + \left[\widetilde{\mathcal{R}}^{\ell+\ell'+2}_{[j+j']}\right]. \tag{3.3.105}$$

*We now move on to fusion rules involving type-*1 *modules. Using* (3.3.47)*, the type-*1*-by-standard Grothendieck fusion rule can be easily computed and is*

$$\left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]^{\ell}\right] \boxtimes \left[\widetilde{\mathcal{R}}^{\ell'}_{[j']}\right] = \left[\widetilde{\mathcal{R}}^{\ell+\ell'-1}_{[j'+1/2]}\right] + \left[\widetilde{\mathcal{R}}^{\ell+\ell'}_{[j]}\right] + \left[\widetilde{\mathcal{R}}^{\ell+\ell'+1}_{[j'+1/2]}\right]. \tag{3.3.106}$$

*The type-*1*-by-type-*1 *Grothendieck fusion rule follows from this using* (3.3.103)*:*

$$\left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]^{\ell}\right] \boxtimes \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]^{\ell'}\right] = \left[\widetilde{\mathcal{R}}^{\ell+\ell'-1}_{[1/2]}\right] + \left[\widetilde{\mathcal{R}}^{\ell+\ell'}_{[0]}\right] + \left[\widetilde{\mathcal{R}}^{\ell+\ell'+1}_{[1/2]}\right] - \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]^{\ell+\ell'+2}\right] \tag{3.3.107}$$

$$= 2\left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]^{\ell+\ell'}\right] + \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \end{smallmatrix}\right]^{\ell+\ell'-2}\right]$$
$$+ \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \end{smallmatrix}\right]^{\ell+\ell'}\right] + \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \end{smallmatrix}\right]^{\ell+\ell'+2}\right].$$

*Unlike the type-*1 *case where* (3.3.103) *involves type-*1 *modules and type-*3 *modules, type-*2 *cases only involve type-*2 *modules. So rather than using* (3.3.47) *to compute the type-*2*-by-standard Grothendieck fusion rule, we must use the full type-*2 *resolution* (3.3.51) *and the corresponding Grothendieck image identity:*

$$\left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \end{smallmatrix}\right]^{\ell}\right] = \sum_{n=0}^{\infty}(-1)^n\left[\widetilde{\mathcal{R}}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]^{2n+\ell}\right]. \tag{3.3.108}$$

Grothendieck fusing both sides with $\widetilde{\mathcal{R}}^{\ell'}_{[j']}$ *results in an alternating sum that simplifies nicely to*

$$\left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \end{smallmatrix}\right]^{\ell}\right] \boxtimes \left[\widetilde{\mathcal{R}}^{\ell'}_{[j']}\right] = \left[\widetilde{\mathcal{R}}^{\ell+\ell'-1}_{[j'-1/4]}\right] + \left[\widetilde{\mathcal{R}}^{\ell+\ell'}_{[j'+1/4]}\right]. \tag{3.3.109}$$

*As for type-*1*, the type-*2*-by-type-*2 *Grothendieck fusion rule follows from the type-*2*-by-standard one and* (3.3.103)*:*

$$\left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]^{\ell}\right] \boxtimes \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \end{smallmatrix}\right]^{\ell'}\right] = \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]^{\ell+\ell'}\right] + \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \end{smallmatrix}\right]^{\ell+\ell'}\right]. \tag{3.3.110}$$

*All that remains is the type-*1*-by-type-*2 *Grothendieck fusion rules. This can be computed in a number of ways, each involving expanding in terms of standard modules and cancelling terms until one obtains:*

$$\left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]^{\ell}\right] \boxtimes \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \end{smallmatrix}\right]^{\ell'}\right] = \left[\mathcal{H}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \end{smallmatrix}\right]^{\ell+\ell'}\right] + \left[\widetilde{\mathcal{R}}^{\ell+\ell'-1}_{[3/4]}\right]. \tag{3.3.111}$$

One nice feature of this example is that these fusion rules can actually be checked: The coset of $\mathsf{BP}(3,4)$ *by the Heisenberg subalgebra generated by* J *is the singlet algebra* $W^0(1,4)$ *[53]. So the representation theory of the latter may then be constructed from that of the former, using the results of [46].*

The triplet algebra $W(1,4)$ *of central charge* $-\frac{25}{2}$ *[111] is an infinite-order simple current extension of* $W^0(1,4)$ *[143] and again, the representation theory of the latter may be constructed from that of the former. The fusion rules of* $W(1,4)$ *are well known (see [79, 86, 155]) and can, in principle, be compared against those obtained from* $\mathsf{BP}(3,4)$ *by the aforementioned coset and simple current extension construction. We do not perform this check here. Exploring this construction of fusion rules for* $W(1,4)$ *is an interesting direction of future study.*

Chapter 4

Subregular W-Algebras

4.1. $\mathfrak{sl}_{n+1}$ W-Algebras

As we have seen in the previous chapter, inverse quantum hamiltonian reduction is a powerful tool for unpacking the representation theory of Bershadsky–Polyakov algebras, particularly with respect to vertex operator algebraic data important in logarithmic conformal field theory. Motivated by the success of this approach for $\mathsf{BP}(\mathsf{u}, \mathsf{v})$, we now consider more general cases for which inverse quantum hamiltonian reductions can be defined and analysed.

The examples of inverse quantum hamiltonian reduction we have mentioned so far involve W-algebras related to $\mathfrak{sl}_2$ (Section 3.1.2) and $\mathfrak{sl}_3$ (Section 3.2.2). In both of these cases, there is a path in the ordering of W-algebras from the 'bottom-most' W-algebra to the corresponding affine vertex operator algebra consisting of known inverse quantum hamiltonian reductions.
The other inverse quantum hamiltonian reduction in the path for $\mathfrak{sl}_3$ is the one between $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ and BP^{k}. This was explicitly defined by Adamović, Creutzig and Genra in [3], and takes the form

$$
\begin{tikzcd}
\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_2, 0) \simeq \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \arrow[d] \\
\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_2, f) \simeq \mathsf{Vir}^{\mathsf{k}} \arrow[u, bend right, "{[2, 147]}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, 0) \simeq \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \arrow[d] \arrow[dd, bend right=60] \\
\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, f_\theta) \simeq \mathsf{BP}^{\mathsf{k}} \arrow[u, bend right, "{[3]}"'] \arrow[d, dashed] \\
\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_3, f_{\alpha_1} + f_{\alpha_2}) \simeq \mathsf{W}_3^{\mathsf{k}} \arrow[u, bend right, "{[4]}"']
\end{tikzcd}
$$

FIGURE 1. The partial ordering of W-algebras for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$. Solid downward arrows represent quantum hamiltonian reduction, dashed downward arrows represent partial quantum hamiltonian reduction and blue upwards arrows represent inverse quantum hamiltonian reduction.

of an embedding $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \hookrightarrow \mathsf{BP}^{\mathsf{k}} \otimes \Pi \otimes \mathsf{B}$ where B is the $\beta\gamma$ ghost vertex algebra and Π is the half-lattice vertex algebra of Section 3.1.1.

For $\mathfrak{sl}_{n+1}$ with $n > 2$, there are many more W-algebras to consider. Recall that the partial ordering on $\mathfrak{sl}_{n+1}$ W-algebras of interest is that induced by the partial ordering on $\mathfrak{sl}_{n+1}$ nilpotent orbits.

Nilpotent orbits of $\mathfrak{sl}_{n+1}$ are indexed by partitions of $n+1$; the partition of $n+1$ corresponding to the nilpotent orbit containing some $M \in \mathfrak{sl}_{n+1}$ is the unique non-increasing sequence of Jordan block sizes in the Jordan normal form of M. For this reason, we will denote the nilpotent orbit in $\mathfrak{sl}_{n+1}$ corresponding to the partition λ of $n+1$ by $\mathbb{O}_\lambda$

The partial ordering on nilpotent orbits of $\mathfrak{sl}_{n+1}$ is given by the dominance ordering on the partitions of $n+1$ [**91**]. For $n < 5$, this partial ordering is actually a total ordering. For $n \geqslant 5$, the structure of the partial ordering of nilpotent orbits is complicated but there are generic features that are noteworthy.

For example, the largest nilpotent orbit is always $\mathbb{O}_{\text{reg}} = \mathbb{O}_{(n+1)}$ (the *regular* nilpotent orbit), while the smallest is $\mathbb{O}_{(1^{n+1})} = \{0\}$. The *subregular* nilpotent orbit $\mathbb{O}_{\text{sub}} = \mathbb{O}_{(n,1)}$ is always smaller than $\mathbb{O}_{\text{reg}}$ and larger than all other nilpotent orbits. Similarly, the *minimal* nilpotent orbit $\mathbb{O}_{\text{min}} = \mathbb{O}_{(2,1^{n-1})}$ is always larger than $\mathbb{O}_{(1^{n+1})}$ but smaller than all other nilpotent orbits.

Sprinkled throughout this partial ordering are nilpotent orbits of the form $\mathbb{O}_{(A^B)}$ where (A^B) is the partition consisting of B copies of A (subject to $AB = n+1$). The corresponding W-algebras are known as *rectangular* W-algebras and have interesting connections to theories of higher spin gravity [**45**].

Owing to the structure of the partial ordering of $\mathfrak{sl}_{n+1}$ W-algebras induced by the partial ordering of nilpotent orbits, the 'simplest' class of $\mathfrak{sl}_{n+1}$ W-algebras for which we might expect there to be an inverse quantum hamiltonian reduction are the universal regular W-algebras $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}})$ and the universal subregular W-algebras $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$ where $f_{\text{reg}} \in \mathbb{O}_{\text{reg}}$ and $f_{\text{sub}} \in \mathbb{O}_{\text{sub}}$. This can be seen as the first step in constructing a path from the bottom-most $\mathfrak{sl}_{n+1}$ W-algebra to the affine vertex operator algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$.

Much recent work in physics and mathematics has subregular W-algebras $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_{\text{sub}})$ playing a central role. For example, subregular W-algebras appear in the Schur index of $4D$ superconformal field theories known as Argyres–Douglas theories [**24**, **29**, **41**]. The nilpotent orbit $\mathbb{O}_{\text{sub}}$ plays a crucial role in singularity theory: the ADE classification of simple surface singularities connects to the ADE classification of simply-laced Lie algebras through the geometry of the Slodowy slice corresponding to $\mathbb{O}_{\text{sub}}$ [**148**]. In light of these and many more motivations, much recent work has been done to improve our understanding of the structure and representation theory of subregular W-algebras [**7**, **43**, **44**, **49**, **59**, **89**].

$$
\begin{tikzcd}
 & \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, 0) \simeq \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_6) \arrow[d] & \\
 & \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{\text{min}}) \arrow[d, dashed] & \\
 & \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(2,2,1,1)}) \arrow[dl, dashed, bend right] \arrow[dr, dashed, bend left] & \\
\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(3,1,1,1)}) \arrow[dr, dashed, bend right] & & \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(2,2,2)}) \arrow[dl, dashed, bend left] \\
 & \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(3,2,1)}) \arrow[dl, dashed, bend right] \arrow[dr, dashed, bend left] & \\
\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(4,1,1)}) \arrow[dr, dashed, bend right] & & \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(3,3)}) \arrow[dl, dashed, bend left] \\
 & \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(4,2)}) \arrow[d, dashed] & \\
 & \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{\text{sub}}) \arrow[d, dashed] & \\
 & \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{\text{reg}}) &
\end{tikzcd}
$$

Figure 2. The partial ordering of W-algebras for $\mathfrak{sl}_6$. Here we choose an element f_λ from each nilpotent orbit $\mathbb{O}_\lambda$. The nilpotent orbits in $\mathfrak{sl}_6$ associated to the W-algebras increase in size from top to bottom. The W-algebras appearing at the same height (e.g. $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(4,1,1)})$ and $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(3,3)})$) are not related by the partial ordering however.

Another notable feature of the regular-subregular case of inverse quantum hamiltonian reduction is that the simple regular W-algebra $\mathsf{W}_{\mathsf{k}}(\mathfrak{g}, f_{\text{reg}})$ is rational [17] for nondegenerate admissible levels. At these levels, one goal is to construct relaxed $\mathsf{W}_{\mathsf{k}}(\mathfrak{g}, f_{\text{sub}})$-modules out of the (finitely many) $\mathsf{W}_{\mathsf{k}}(\mathfrak{g}, f_{\text{reg}})$-modules and eventually use these to construct 'logarithmic minimal models' with admissible-level $\mathsf{W}_{\mathsf{k}}(\mathfrak{g}, f_{\text{sub}})$ symmetry. This has been done for $\mathfrak{g} = \mathfrak{sl}_2$ [52], as well as for $\mathfrak{sl}_3$ (Section 2.3).

As explained in Section 3.1 regarding the existence of inverse quantum hamiltonian reductions for W-algebras, what we expect concretely is the existence of an embedding of the subregular W-algebra into the regular W-algebra tensored with some vertex operator algebra. The hope is that the mysterious subregular W-algebra can be understood in terms of the regular W-algebra whose representation theory is comparatively well-understood.

It is easy to see that $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$ and BP^{k} are the universal subregular W-algebras for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ respectively. In these cases, the lattice vertex algebra part of the inverse reduction embedding is the half-lattice vertex algebra from Section 3.1.1.

What is therefore desired is an embedding $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}}) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}}) \otimes \Pi$ for noncritical k generalising the embeddings (3.1.10) and (3.2.1). The representation theory of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$ can then be analysed in terms of that of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{reg}})$ and Π, as has been done for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$

Before describing such an embedding, we outline some definitions and facts relating to the regular and subregular $\mathfrak{sl}_{n+1}$ W-algebras.

4.1.1. Regular. By far the most studied and well-understood W-algebras are the regular (or *principal*) W-algebras $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_{\text{reg}})$. This vertex operator algebra has been at the forefront of many developments in mathematics and physics. See [**10**, **20**, **31**, **73**, **169**] for example. Consequentially, there is much one can say about these vertex operator algebras (see for example the reviews [**37**,**38**]). We restrict our discussions to what is needed in the quest to unravel subregular W-algebras.

Definition 4.1.1. *Let $\mathfrak{g}$ be a simple, finite-dimensional Lie algebra over $\mathbb{C}$ and $\mathsf{k} \neq -h^\vee$. The (universal)* regular W-algebra *$\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_{reg})$ is defined as the quantum hamiltonian reduction of the level-k universal affine vertex operator algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ corresponding to the regular nilpotent orbit in $\mathfrak{g}$. Denote its unique simple quotient by $\mathsf{W}_{\mathsf{k}}(\mathfrak{g}, f_{reg})$.*

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ (i.e. type A_n) and use the notation

$$\mathsf{W}^{\mathsf{k}}_{n+1} = \mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_{\text{reg}}), \quad \mathsf{W}_{n+1,\mathsf{k}} = \mathsf{W}_{\mathsf{k}}(\mathfrak{g}, f_{\text{reg}}). \tag{4.1.1}$$

In this case, the nilpotent element $f_{\text{reg}} \in \mathfrak{sl}_{n+1}$ can be taken to be $f_{\alpha_1} + f_{\alpha_2} + \cdots + f_{\alpha_n}$, where α_i denotes the i'th simple root of $\mathfrak{sl}_{n+1}$ and $\{h_{\alpha_i}, e_{\alpha_i}, f_{\alpha_i}\}$ is the corresponding $\mathfrak{sl}_2$ triple in the Chevalley basis of $\mathfrak{sl}_{n+1}$. The partition corresponding to the regular nilpotent orbit is $(n+1)$. This W-algebra was first defined in the $\mathfrak{g} = \mathfrak{sl}_3$ case (the Zamolodchikov algebra of Section 1.3.2) [**170**] and later for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ [**123**]. $\mathsf{W}^{\mathsf{k}}_{n+1}$ has strong generators denoted by $\{W_2(z), \dots, W_{n+1}(z)\}$ that we will describe in Section 4.2.1 by way of a free-field realisation. The field $T(z) = \frac{1}{\mathsf{k}+n+1} W_2(z)$ is an energy-momentum field with central charge [**123**]

$$\mathsf{c}^{n+1}_{\mathsf{k}} = -\frac{n\left((n+1)(\mathsf{k}-1) + n^2 + 2n\right)\left((n+2)\mathsf{k} + (n+1)^2\right)}{\mathsf{k}+n+1}. \tag{4.1.2}$$

The generating fields $W_i(z)$ have conformal weight i with repect to $T(z)$. It is known that the vertex operator algebra $\mathsf{W}^{\mathsf{k}}_{n+1}$ is reducible if k is a *nondegenerate admissible level* [14]. That is,

$$\mathsf{k} + n + 1 = \frac{\mathsf{u}}{\mathsf{v}}, \quad \text{where } \mathsf{u}, \mathsf{v} \in \mathbb{Z}_{\geqslant n+1} \text{ and } \gcd\{\mathsf{u}, \mathsf{v}\} = 1. \tag{4.1.3}$$

At these levels, $\mathsf{W}_{n+1,\mathsf{k}}$ is rational [17]. For this reason, we use the special notation $\mathsf{W}_{n+1}(\mathsf{u}, \mathsf{v}) = \mathsf{W}_{n+1,\mathsf{k}}$ when k is nondegenerate-admissible. The modules of $\mathsf{W}_{n+1}(\mathsf{u}, \mathsf{v})$ are all highest-weight modules and admit a parametrisation in terms of $\mathfrak{sl}_{n+1}$ weights.

Following Section 8.3 in [23], let Pr^{k} be the set of principal admissible $\mathfrak{sl}_{n+1}$ weights of level k. Each weight $\lambda \in \mathrm{Pr}^{\mathsf{k}}$ defines a central character $\gamma^{\lambda} : Z(\mathfrak{sl}_{n+1}) \to \mathbb{C}$ by evaluation. Let $\mathrm{Pr}^{\mathsf{k}}_{\mathcal{W}}$ be the set of all such central characters with λ ranging over Pr^{k}. Associated to each $\gamma \in \mathrm{Pr}^{\mathsf{k}}_{\mathcal{W}}$ is a simple $\mathsf{W}_{n+1}(\mathsf{u}, \mathsf{v})$-module $\mathcal{W}_{\gamma}$. The simple $\mathsf{W}_{n+1}(\mathsf{u}, \mathsf{v})$-modules have been classified [17] and consist only of the set of modules

$$\{\mathcal{W}_{\gamma} \mid \gamma \in \mathrm{Pr}^{\mathsf{k}}_{\mathcal{W}}\}. \tag{4.1.4}$$

All of these modules are highest-weight with one-dimensional top spaces and are mutually non-isomorphic. Let v_{γ} be the highest-weight vector of $\mathcal{W}_{\gamma}$. Without loss of generality and owing to the fields $W_i(z)$ being strong generators of $\mathsf{W}^{\mathsf{k}}_{n+1}$, we can view γ as an element of $\mathbb{C}^n$ with $\gamma \leftrightarrow (\gamma_2, \dots, \gamma_{n+1})$ defined by

$$(W_i)_0\, v_{\gamma} = \gamma_i v_{\gamma}. \tag{4.1.5}$$

For example, the vacuum module is $\mathsf{W}_{n+1}(\mathsf{u}, \mathsf{v}) \simeq \mathcal{W}_0$.

4.1.2. Subregular. We now move on to defining the main vertex operator algebras of interest and establishing some useful notation. A more detailed account of the strong generators of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$ following [89] will be deferred to Section 4.2.2.

Definition 4.1.2. *Let $\mathfrak{g}$ be a simple, finite-dimensional Lie algebra over $\mathbb{C}$ and $\mathsf{k} \neq -h^{\vee}$. The (universal)* subregular W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f_{sub})$ *is the quantum hamiltonian reduction of the level-k universal affine vertex operator algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ corresponding to the subregular nilpotent orbit in $\mathfrak{g}$. Denote its unique simple quotient by $\mathsf{W}_{\mathsf{k}}(\mathfrak{g}, f_{sub})$.*

While not much is known about subregular W-algebras of type A, even less is known about other types. The most studied, non-type A, non-super example is the type C_2 ($\mathfrak{g} = \mathfrak{sp}_4 \simeq \mathfrak{so}_5$) subregular W-algebra whose operator product expansions are listed in Section 5 of [59], where the representation theory of $\mathsf{W}_{\mathsf{k}}(\mathfrak{sp}_4, f_{\mathrm{sub}})$ at certain levels is explored. Work on the inverse quantum hamiltonian reduction question for $\mathsf{W}^{\mathsf{k}}(\mathfrak{sp}_4, f_{\text{sub}})$ has been done from a 4D superconformal field theoretic perspective [30].

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and use the notation

$$\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} = \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}}), \quad \overline{\mathsf{W}}_{n+1,\mathsf{k}} = \mathsf{W}_{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}}). \tag{4.1.6}$$

The nilpotent element $f_{\text{sub}} \in \mathfrak{sl}_{n+1}$ can be taken to be $f_{\alpha_2} + \cdots + f_{\alpha_n}$. The partition corresponding to the subregular nilpotent orbit is $(n, 1)$. The vertex algebra $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ was long suspected to be isomorphic to the *Feigin–Semikhatov vertex algebra* $\mathcal{W}^{(2)}_{n+1}$ [68]. This was proven recently (Theorem 6.9 in [88]) utilising a certain free-field realisation of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. We will return to free-field realisations of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ in Section 4.2.2 as they provide us with convenient explicit formulae for strong generators of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$.

Certain choices for n in $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ give vertex operator algebras that are well-known. Indeed the main motivation for applying the approach described in this chapter to $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ is the success of this approach in small n cases.

Example ($n = 1$). *$\overline{\mathsf{W}}^{\mathsf{k}}_2$ is the universal affine vertex algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$. It has strong generators denoted by $h(z)$, $e(z)$ and $f(z)$. Their operator product expansions are well-known and are given in Section 3.1.2.*

That this is an affine vertex algebra and not something more exotic is due to the subregular nilpotent orbit of $\mathfrak{sl}_2$ being equal to $\{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\}$. This is the only n for which the affine vertex algebra and subregular W-algebra coincide.

Example ($n = 2$). *$\overline{\mathsf{W}}^{\mathsf{k}}_3$ is isomorphic to the Bershadsky–Polyakov algebra BP^{k} discussed in detail in Chapter 2. Interestingly the Bershadsky–Polyakov algebra is also isomorphic to the* minimal *W-algebra corresponding to $\mathfrak{sl}_3$. This helps in, for example, the classification of highest-weight BP_{k}-modules as we saw in Section 2.3.1. This is the only n for which the subregular and minimal nilpotent orbits/W-algebras coincide.*

In general there exists a set of strongly generating fields

$$\left\{G^+(z), J(z), L(z), U_3(z), \dots, U_n(z), G^-(z)\right\} \subset \overline{\mathsf{W}}^{\mathsf{k}}_{n+1}, \tag{4.1.7}$$

where we omit the fields $L(z)$, $U_i(z)$ when $n = 1$ and omit just the fields $U_i(z)$ when $n = 2$. For $n > 1$, we can take L to be a conformal vector. The conformal vector of $\overline{\mathsf{W}}^{\mathsf{k}}_2$ is given by the usual Sugawara construction for $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$.

That such strongly generating fields of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ exist is a consequence of Theorem 4.1 in [106]. We can choose the conformal field $L(z)$ such that conformal weights of the strong generators of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ above are $1, 1, 2, 3, \dots, n, n$ respectively, and the central charge is

$$\bar{\mathsf{c}}^{n+1}_{\mathsf{k}} = -\frac{(n(\mathsf{k}+n)-1)\left(\mathsf{k}(n-1)(n^2+5n-2)+(n+1)(n^3+3n^2-9n+2)\right)}{(n+1)(\mathsf{k}+n+1)}. \tag{4.1.8}$$

This can be seen explicitly with the free-field realisation and corresponding strong generators of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ described in [89]. The details of this construction are recounted in Sections 4.2.2 and 4.3.2.

As mentioned earlier, the case when k is an admissible level for $\mathfrak{sl}_{n+1}$ is of particular importance for applications to logarithmic conformal field theory. Admissible levels are those k satisfying

$$\mathsf{k}+n+1 = \frac{\mathsf{u}}{\mathsf{v}}, \quad \text{where } \mathsf{u} \in \mathbb{Z}_{\geqslant n+1}, \mathsf{v} \in \mathbb{Z}_{\geqslant 1} \text{ and } \gcd\{\mathsf{u},\mathsf{v}\} = 1. \tag{4.1.9}$$

At such levels, we use the special notation $\overline{\mathsf{W}}_{n+1}(\mathsf{u},\mathsf{v}) = \overline{\mathsf{W}}_{n+1,\mathsf{k}}$. When $\mathsf{v} = n$, $(\mathfrak{sl}_{n+1}, \mathsf{k})$ forms an *exceptional pair* [107]. The simple vertex operator algebra $\overline{\mathsf{W}}_{n+1}(\mathsf{u}, n)$ is rational and the modular transformations of characters and fusion rules are in principle known [22].

Operator product expansions for $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ can be worked out on a case-by-case basis in principle but only a handful are required in what follows. For example, with $\ell_n(\mathsf{k}) = \frac{n\mathsf{k}}{n+1} + n - 1$:

$$\begin{aligned} J(z)G^{\pm}(w) &\sim \frac{\pm G^{\pm}(w)}{z-w}, \quad J(z)J(w) \sim \frac{\ell_n(\mathsf{k})\mathbb{1}}{(z-w)^2}, \\ L(z)G^{\pm}(w) &\sim \frac{(n+1\pm(1-n))\,G^{\pm}(w)}{2(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w}, \\ L(z)J(w) &\sim \frac{-(n-1)\ell_n(\mathsf{k})\mathbb{1}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}. \end{aligned} \tag{4.1.10}$$

Another important operator product expansion is that between the fields $G^+(z)$ and $G^-(z)$. The other strong generators of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ in (4.1.7) all appear somewhere in this expansion. That is, G^+ and G^- actually generate $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ [68]. The complexity of each successive singular term grows rather quickly so we only show the first few terms here. The ellipsis contains all singular terms of order $j < n-1$.

$$\begin{aligned} G^+(z)G^-(w) \sim{}& \frac{\lambda_n(n,\mathsf{k})\mathbb{1}}{(z-w)^{n+1}} + \frac{(n+1)\lambda_{n-1}(n,\mathsf{k})J(w)}{(z-w)^n} \\ &+ \lambda_{n-2}(n,\mathsf{k})(z-w)^{n-1}\Bigg(\frac{n}{2}(n+1){:}JJ{:}(w) - (\mathsf{k}+n+1)L(z) \\ &+ \frac{1}{2}\left((n+1)(n^2-2)+\mathsf{k}(n+2)(n-1)\right)\partial J(w)\Bigg) + \dots, \end{aligned} \tag{4.1.11}$$

where

$$\lambda_j(n,\mathsf{k}) = \prod_{m=1}^{j} (m(k+n)-1). \tag{4.1.12}$$

Additional terms of this operator product expansion are presented in Appendix A of [68] albeit with respect to a slightly different set of strong generators. To appreciate the complexity of the full set of operator product expansions of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ for $n > 2$, one needs only to refer to the operator product expansions for $\overline{\mathsf{W}}^{\mathsf{k}}_{4}$ presented in Appendix A.3 of [61].

Choosing the conformal structure defined by L has the drawback of making $J(z)$ not quasiprimary and introducing an asymmetry in the conformal weights of G^+ and G^-. This is not a problem *a priori* but can be rectified by choosing the conformal field to be

$$\tilde{L}(z) = L(z) - \frac{n-1}{2}\partial J(z). \tag{4.1.13}$$

With respect to $\tilde{L}$, G^+ and G^- have conformal weight $(n+1)/2$ and $J(z)$ is a primary field of conformal weight 1. The $\tilde{L}$ conformal structure has the drawback of requiring the consideration of twisted modules when n is even owing to the half-integer conformal weight of $G^\pm$.

An important extension of the results in this chapter is to consider the modularity of conjectured standard $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$-modules and compute their Grothendieck fusion rules. The presence of twisted modules complicates such computations as care must be given to which sector one is working in. Therefore unless otherwise indicated we will keep L as the conformal vector.

This is the same phenomenon encountered for BP^{k} in Section 3.2.3 that prompted a change in conformal structure. Note that the fields $L(z), \tilde{L}(z) \in \mathsf{BP}^{\mathsf{k}}$ from Section 2.1.2 have been renamed to $\tilde{L}(z), L(z)$ respectively in the basis (4.1.7) for $n = 2$ and (4.1.13). This is simply to emphasise that the 'asymmetric' conformal structure on $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ is best suited for modularity considerations.

Another reason for sidestepping twisted modules is the existence of a *spectral flow* automorphism of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ that exchanges twisted and untwisted sectors when n is even. To construct this automorphism, as before we expand homogeneous fields of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ as

$$A(z) = \sum_{m\in\mathbb{Z}} A_{(m)} z^{-m-1} = \sum_{m\in\mathbb{Z}} A_m z^{-m-\Delta_A}, \tag{4.1.14}$$

where Δ_A is the conformal weight of A with respect to $L(z)$. The spectral flow automorphism is constructed using certain intertwining operators [122].

Proposition 4.1.3. *Let $\ell \in \mathbb{Z}$. The map $\sigma^\ell : \overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \to \overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ defined by*

$$\sigma^\ell\big(A(z)\big) = Y\left(\Lambda(\ell J, z)A, z\right), \tag{4.1.15}$$

where

$$\Lambda(\ell J, z) = z^{-\ell J_0} \prod_{m=1}^{\infty} \exp\left(\frac{(-1)^m}{m} \ell J_m z^{-m}\right), \tag{4.1.16}$$

is a vertex algebra automorphism, where Y is the vertex map for $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$.

That this a vertex algebra automorphism is a straightforward application of Proposition 3.2 in [122]. Direct computation shows that

$$\begin{aligned} \sigma^\ell\big(G^\pm(z)\big) &= z^{\mp\ell} G^\pm(z), \quad \sigma^\ell\big(J(z)\big) = J(z) - \ell_n(\mathsf{k})\ell z^{-1}, \\ \sigma^\ell\big(\tilde{L}(z)\big) &= \tilde{L}(z) - \ell z^{-1} J(z) + \frac{1}{2}\ell_n(\mathsf{k})\ell^2 z^{-2}. \end{aligned} \tag{4.1.17}$$

The action of spectral flow can also be written in terms of modes as follows:

$$\begin{aligned} \sigma^\ell\big(G^\pm_m\big) &= G^\pm_{m\mp\ell}, \quad \sigma^\ell\big(J_m\big) = J_m - \ell_n(\mathsf{k})\ell\delta_{m,0}\mathbb{1}, \\ \sigma^\ell\big(\tilde{L}_m\big) &= \tilde{L}_m - \ell J_m + \frac{1}{2}\ell_n(\mathsf{k})\ell^2\delta_{m,0}\mathbb{1}. \end{aligned} \tag{4.1.18}$$

These formulae reproduce the spectral flow automorphism of BP^{k} (2.1.18) when $n = 2$. In principle, for any fixed n, one could also compute the action of spectral flow on the the fields $U_3(z), \dots, U_n(z)$ given complete information about the relevant operator product expansions. In the first case where one gets such fields ($n = 3$), spectral flow acts on the field $U_3(z)$ as the identity automorphism. However, for general n where the operator product expansions are more involved, the action of spectral flow on the fields $U_i(z)$ is more difficult to determine.

What can be shown using the free-field expansions described in Section 4.2.2 is that the fields $\{U_3(z), \dots, U_n(z)\}$ have 'J-charge' 0, i.e. that $J_0 U_i = 0$ for all i. Therefore by (4.1.15), spectral flow acts on the modes of the form $(U_i)_m$ as

$$\sigma^\ell\big((U_i)_m\big) = (U_i)_m + \dots. \tag{4.1.19}$$

As the characters we will eventually define here for $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules only keep track of J_0- and L_0-eigenvalues, the formulae (4.1.17) and the fact that the $U_i(z)$ have J-charge 0 is sufficient for our purposes.

From the definition of spectral flow (4.1.15) it is clear that the inverse of σ^ℓ is $\sigma^{-\ell}$. Moreover, spectral flow is only a vertex operator algebra automorphism for $\ell = 0$ by (4.1.17).

As in the $n = 1$ and 2 cases, we restrict attention to a particular subclass of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules called *weight modules*. The corresponding category $\mathscr{W}_{\mathsf{k}}$ of weight modules for the simple quotient $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$ is expected to have the modular properties desirable for defining a logarithmic conformal field theory for certain k.

To define weight modules, let U be the mode algebra of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. That is, U is the unital associative $\mathbb{C}$-algebra spanned by the modes A_n for $A(z) \in \overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ subject to the generalised commutation relations defined by the operator product expansions. The grading on U by $[L_0, \cdot]$-eigenvalue gives a generalised triangular decomposition [106]

$$\mathsf{U} = \mathsf{U}_{>} \otimes \mathsf{U}_0 \otimes \mathsf{U}_{<}, \tag{4.1.20}$$

where $\mathsf{U}_{>}$, U_0 and $\mathsf{U}_{<}$ denote the unital subalgebras generated by A_m for all homogeneous $A(z) \in \overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ with $m > 0$, $m = 0$ and $m < 0$ respectively.

Definition 4.1.4. • *A vector v in a $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$-module $\mathcal{M}$ is a* weight vector *of* weight (j, Δ) *where j, $\Delta \in \mathbb{C}$ if it is a simultaneous eigenvector of J_0 and L_0 with eigenvalues j (*charge*) and Δ (*conformal weight*) respectively. The nonzero simultaneous eigenspaces of J_0 and L_0 are called* weight spaces *of $\mathcal{M}$. If $\mathcal{M}$ has a basis of weight vectors and each weight space is finite-dimensional, then $\mathcal{M}$ is a* weight module.

- *A vector in a $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module is a* highest-weight vector *if it is a weight vector that is annihilated by the action of $\mathsf{U}_{>}$ and G^+_0. A* highest-weight module *is a $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module generated by a highest-weight vector.*
- *A vector in a $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module is a* relaxed highest-weight vector *if it is a weight vector that is annihilated by the action of $\mathsf{U}_{>}$. A* relaxed highest-weight module *is a $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module generated by a relaxed highest-weight vector.*

Observe that we do not require that the zero modes $(U_i)_0$ act semisimply on weight modules.

The definition of conjugate highest-weight vectors and modules is identical except with G^+_0 replaced with G^-_0. From the actions (4.1.17) of the spectral flow automorphism, we see that if $v \in \mathcal{M}$ is a weight vector of charge j and conformal weight Δ, $\sigma^\ell(v) \in \sigma^\ell(\mathcal{M})$ is a weight vector with charge and conformal weight

$$j' = j + \ell_n(\mathsf{k})\ell, \quad \Delta' = \Delta + j\ell + \frac{1}{2}\ell_n(\mathsf{k})\left(\ell^2 - \ell(n-1)\right) \tag{4.1.21}$$

respectively.

4.2. Free-Field Realisations

An inverse quantum hamiltonian reduction embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ for $n = 1$ and 2 can be obtained by fairly direct methods; In these cases, the operator product expansions are known on both sides and are straightforward to work with. The desired map can be obtained by making some reasonable assumptions (the embedding is conformal, the Heisenberg field gets mapped to a linear combination of $c(z)$ and $d(z)$, ...) and imposing the operator product expansions of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$.

Once this is done, the work of showing that the image of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ is isomorphic to $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ and not one of its quotients remains. In both the $n = 1$ and 2 cases, this can be shown directly using suitable bases of the vertex algebras involved.

When $n > 2$, the complexity of the operator product expansions makes this approach exceedingly difficult. Therefore to prove the existence of this embedding in general, we need some more information about $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. This information takes the form of the free-field realisations of W-algebras obtained as the kernel of certain screening operators [88].

An additional benefit of the free-field approach is that the map $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \to \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ we get is automatically injective. This is because the map is a composition of the free-field realisation of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ (which is injective [88]), the FMS bosonisation of the $\beta\gamma$ ghost vertex algebra (which is injective [77]) and a vertex algebra isomorphism.

4.2.1. Free-Field Realisation for Regular W-Algebras. In [123] the regular W-algebra is described as the intersection of kernels of certain screening operators: Let H_α be the Heisenberg vertex algebra (see Section 1.1.2) strongly generated by n fields $\alpha_1(z), \dots, \alpha_n(z)$ with operator product expansions

$$\alpha_i(z)\alpha_j(w) \sim \frac{A_{i,j}(\mathsf{k}+n+1)\mathbb{1}}{(z-w)^2}, \tag{4.2.1}$$

where $A = [A_{i,j}]$ is the Cartan matrix of $\mathfrak{sl}_{n+1}$.

Proposition 4.2.1. *Let* $\mathsf{k} \neq -n-1$. *The regular W-algebra* $\mathsf{W}^{\mathsf{k}}_{n+1}$ *embeds into* H_α. *Additionally, for generic* k,

$$\mathsf{W}^{\mathsf{k}}_{n+1} \simeq \bigcap_{i=1}^{n} \ker \int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_i}(z)\, \mathrm{d}z \subset \mathsf{H}_\alpha. \tag{4.2.2}$$

Here, a level k is called *generic* (for a given $f \in \mathfrak{g}$) if the homology of certain sub-complexes of the quantum hamiltonian reduction complex for $f \in \mathfrak{g}$ are isomorphic (see [88, Def. 4.5]). It is known that the set of generic levels is Zariski dense in $\mathbb{C}$ [88, Lem. 4.4].

At non-generic levels, isomorphisms such as (4.2.2) need not exist. For example, if $n = 1$ and $\mathsf{k}+2 \in \mathbb{Z}_{\geqslant 2}$, the intersection of kernels of screening operators in (4.2.2) defines the singlet vertex algebra [1] that contains $\mathsf{W}^{\mathsf{k}}_2$ as a proper sub-vertex algebra. The precise details of what makes a level generic or not is not important for our purposes.

The identification in (4.2.2) gives a useful free-field description of strong generators for the regular W-algebra using the *Miura transform*.

Following the presentation in Section 6.3.3 of [38], let ε_s, $s = 1, \dots, n+1$ denote the weights of the defining representation of $\mathfrak{sl}_{n+1}$ ordered so that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for all simple roots α_i of $\mathfrak{sl}_{n+1}$, and $\sum_s \varepsilon_s = 0$. The relationship between the simple roots α_i and weights ε_s can be inverted and upgraded to a relationship between fields given by

$$\varepsilon_s(z) = -\sum_{j=1}^{s-1} \frac{j}{n+1}\alpha_j(z) + \sum_{j=s}^{n} \frac{n+1-j}{n+1}\alpha_j(z). \tag{4.2.3}$$

Define the generating function of a set of fields $\{W_0(z), W_1(z), \dots, W_{n+1}(z)\} \subset \mathsf{H}_\alpha$ by

$$\begin{aligned} R_n(z) &= -\sum_{s=0}^{n+1} W_s(z)\left((\mathsf{k}+n)\partial\right)^{n+1-s} \\ &= \left((k+n)\partial - \varepsilon_{n+1}(z)\right)\cdots\left((k+n)\partial - \varepsilon_1(z)\right). \end{aligned} \tag{4.2.4}$$

It can be shown that the singular part of the operator product expansion of $R_n(z)$ and the screening operators in (4.2.2) is a total derivative, so the component fields $W_s(z)$ are all $\mathsf{W}^{\mathsf{k}}_{n+1}$ fields. Additionally, the fields $W_2(z), \dots, W_{n+1}(z)$ strongly generate $\mathsf{W}^{\mathsf{k}}_{n+1}$ for all k [123]. These fields are not in general quasi-primary but one can usually take appropriate linear combinations of them and their derivatives to obtain primary fields. Fortunately this is not necessary for our purposes.

To find convenient closed-form expressions for the fields $W_s(z)$ in terms of the fields of H_α, we recall the noncommutative elementary symmetric polynomials described in [132, Ch. 12].

Definition 4.2.2. *Let $\omega_1, \dots, \omega_N$ be N mutually associative operators. The* m-th noncommutative elementary symmetric polynomial *in $\omega_1, \dots, \omega_N$ is*

$$E_m(\omega_1, \dots, \omega_N) = \sum_{i_1 > \cdots > i_m} \omega_{i_1} \cdots \omega_{i_m}. \tag{4.2.5}$$

Rewriting R_n in terms of modes from H_α,

$$\begin{aligned} &\left((k+n)\partial - (\varepsilon_{n+1})_{-1}\right)\cdots\left((k+n)\partial - (\varepsilon_1)_{-1}\right) \\ &\qquad = E_{n+1}\left((k+n)\partial - (\varepsilon_1)_{-1}, \dots, (k+n)\partial - (\varepsilon_{n+1})_{-1}\right) \\ &\qquad = -\sum_{s=0}^{n+1} (W_s)_{(-1)}\left((\mathsf{k}+n)\partial\right)^{n+1-s}. \end{aligned} \tag{4.2.6}$$

By Proposition 12.4.4 of [132],

$$W_s = -E_s\left((k+n)\partial - (\varepsilon_1)_{-1}, \dots, (k+n)\partial - (\varepsilon_{n+1})_{-1}\right)\mathbb{1}, \tag{4.2.7}$$

where $\mathbb{1}$ denotes the vacuum of H_α. For example, $W_0(z) = -\mathbb{1}(z)$, $W_1(z) = 0$ and

$$W_2(z) = (k+n)\sum_{j=1}^{n+1}(n+1-j)\partial\varepsilon_j(z) - \sum_{i>j} {:}\varepsilon_i(z)\varepsilon_j(z){:}. \tag{4.2.8}$$

We will often refer to the set of fields $\{W_2(z), \dots, W_{n+1}(z)\}$ as the *Miura basis* of $\mathsf{W}^{\mathsf{k}}_{n+1}$

4.2.2. Free-Field Realisations for Subregular W-Algebras. Screening operators for general W-algebras were obtained in [88]. For $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$, the screening operators described therein are superficially similar to those for $\mathsf{W}^{\mathsf{k}}_{n+1}$ from Section 4.2.1. Let B be the $\beta\gamma$ ghost vertex algebra from Section 1.1.2 and H_α be the Heisenberg vertex algebra from the free-field realisation of $\mathsf{W}^{\mathsf{k}}_{n+1}$.

To minimise notational clutter, we will suppress the tensor product symbol when it is clear on which vertex algebra the involved fields are acting. We will also suppress tensor products involving vacuum fields when possible.

Proposition 4.2.3 (Theorem 3.2 [43])**.** *Let* $\mathsf{k} \neq -n-1$. *The subregular W-algebra* $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ *embeds into* $\mathsf{H}_\alpha \otimes \mathsf{B}$. *Additionally, for generic* k*,*

$$\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \simeq \left(\ker \int \beta(z)\mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_1}(z)\,\mathrm{d}z\right) \cap \left(\bigcap_{i=2}^{n} \ker \int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_i}(z)\,\mathrm{d}z\right) \subset \mathsf{H}_\alpha \otimes \mathsf{B}. \tag{4.2.9}$$

This identification defines a free-field realisation of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ in terms of the fields from B and H_α. The exact decomposition of the strong generators (4.1.7) of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ in terms of B and H_α is not important for our purposes.

As can be seen, the key difference between the screening operators in (4.2.9) and (4.2.2) is the $\beta(z)$ factor in the screening operator involving $\alpha_1(z)$. It is conceivable that there is some chain complex involving $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ with associated differential whereupon taking the zeroth homology has the effect of setting $\beta(z) = 1$, reminiscent of the usual quantum hamiltonian reduction of affine vertex algebras from Section 1.3. Indeed this is exactly what one would expect a partial quantum hamiltonian reduction from $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ to $\mathsf{W}^{\mathsf{k}}_{n+1}$ to do. Our interest here however is the 'inverse' of this as-of-yet undefined partial reduction.

The free-field realisation defined by Proposition 4.2.3 is not the only free-field realisation of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ available to us. Owing to the fact that $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ is isomorphic to the Feigin–Semikhatov algebra $\mathcal{W}^{(2)}_{n+1}$, a free-field realisation of the latter is automatically a free-field realisation of the former. In [68], $n+2$ different free-field realisations of $\mathcal{W}^{(2)}_{n+1}$ are described. For our immediate purposes, only the 'maximally asymmetric' realisation described in Section 2.1.1 of [68] is required.

The screening operators of $\mathcal{W}^{(2)}_{n+1}$(and therefore $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$) from this realisation act on a vertex algebra $\mathcal{H}^{\mathsf{k}}$ generated by fields $A_1(z), \dots, A_n(z), Q(z), Y(z)$ and $\mathsf{e}^{mY}(z)$ where $m \in \mathbb{Z}$. To define the operator product expansions of these fields, let $\mathbb{V} = \mathbb{C}A_n \oplus \cdots \oplus \mathbb{C}A_1 \oplus \mathbb{C}Q \oplus \mathbb{C}Y$ and define a symmetric bilinear form $(\cdot, \cdot)$ on $\mathbb{V}$ by

$$(A_i, A_j) = A_{i,j}(\mathsf{k} + n + 1), \quad (A_1, Q) = -(\mathsf{k} + n + 1), \quad (Q, Q) = 1, \quad (Q, Y) = 1, \tag{4.2.10}$$

with all omitted evaluations yielding zero. The operator product expansions of $\mathcal{H}^{\mathsf{k}}$ can be written as, for all $A, B \in \mathbb{V}$,

$$A(z)B(w) \sim \frac{(A, B)}{(z-w)^2}, \quad \mathsf{e}^{mY}(z)\mathsf{e}^{nY}(w) \sim 0, \quad A(z)\mathsf{e}^{mY}(w) \sim \frac{(A, mY)\mathsf{e}^{mY}(w)}{z-w}. \tag{4.2.11}$$

Similar to $\mathsf{e}^{mY}(z)$, we also have vertex operators $\mathsf{e}^{C}(z)$ where $C \in \{A_1, \ldots, A_n, Q\}$. The zero modes of these operators are interpreted as intertwining maps for certain representations of $\mathcal{H}^{\mathsf{k}}$. Their operator product expansions are

$$\begin{gathered} A(z)\mathsf{e}^{C}(w) \sim \frac{(A, C)\mathsf{e}^{C}(w)}{z-w}, \quad \mathsf{e}^{A_i}(z)\mathsf{e}^{mY}(w) \sim 0, \\ e^{Q}(z)\mathsf{e}^{mY}(w) \sim (z-w)^m \mathsf{e}^{mY+Q}(w). \end{gathered} \tag{4.2.12}$$

Using the presentation in [89, Prop. 2.1], we obtain an embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathcal{H}^{\mathsf{k}}$ at non-critical level with

$$\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \simeq \left(\ker \int \mathsf{e}^{Q}(z)\,\mathrm{d}z\right) \cap \left(\bigcap_{i=1}^{n} \int \mathsf{e}^{A_i}(z)\,\mathrm{d}z\right) \tag{4.2.13}$$

for generic k.

4.3. From Regular W-Algebras to Subregular W-Algebras

4.3.1. Inverse Quantum Hamiltonian Reduction. To be able to recognise the $\mathsf{W}^{\mathsf{k}}_{n+1}$ screening operators in (4.2.9), we need to deal with the factor of $\beta(z)$ appearing in the first screening operator. This is done by 'absorbing' it into the vertex operator it sits next to in the free-field-realisation of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ from Proposition 4.2.3. In order to do this, we use the *Friedan-Martinec-Shenker (FMS) bosonisation* of the $\beta\gamma$ ghost vertex algebra B with operator product expansions (1.1.21) [77].

Proposition 4.3.1. *The vertex algebra homomorphism* $\varphi : \mathsf{B} \to \Pi$ *defined by*

$$\beta(z) \mapsto \mathsf{e}^{c}(z), \ \gamma(z) \mapsto \frac{1}{2}{:}(c(z) + d(z))\,\mathsf{e}^{-c}(z){:} \tag{4.3.1}$$

is an embedding whose image is specified by

$$\varphi(\mathsf{B}) \simeq \ker \int \mathsf{e}^{\frac{1}{2}c + \frac{1}{2}d}(z)\,\mathrm{d}z. \tag{4.3.2}$$

We can therefore compose the embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{H}_\alpha \otimes \mathsf{B}$ with FMS bosonisation φ tensored with the identity map $\mathrm{id}_{\mathsf{H}_\alpha}$ to obtain an embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{H}_\alpha \otimes \Pi$. To describe this embedding using screening operators, let $S(z)$ be one of the screening fields in (4.2.9). Let F_S be the Fock space of H_α that $\int S(z)\, \mathrm{d}z$ maps H_α to. Then,

$$\int S(z)\, \mathrm{d}z : \mathsf{H}_\alpha \otimes \mathsf{B} \to F_S \otimes \mathsf{B}. \tag{4.3.3}$$

What we would like is a screening operator $\int S'(z)\, \mathrm{d}z : \mathsf{H}_\alpha \otimes \Pi \to F_S \otimes \Pi$ such the following diagram commutes:

$$\begin{tikzcd}
\mathsf{H}_\alpha \otimes \mathsf{B} \arrow[rr, "\int S(z)\, \mathrm{d}z"] \arrow[d, hook, "\mathrm{id}_{\mathsf{H}_\alpha} \otimes \varphi"'] & & F_S \otimes \mathsf{B} \arrow[d, hook, "\mathrm{id}_{F_S} \otimes \varphi"] \\
\mathsf{H}_\alpha \otimes \Pi \arrow[rr, dashed, "\int S'(z)\, \mathrm{d}z"] & & F_S \otimes \Pi
\end{tikzcd} \tag{4.3.4}$$

If the screening operator $\int S'(z)\, \mathrm{d}z$ exists, it follows that

$$\left(\mathrm{id}_{\mathsf{H}_\alpha} \otimes \varphi\right) \left(\ker \left(\int S(z)\, \mathrm{d}z\right)\right) = \ker \left(\int S'(z)\, \mathrm{d}z\right) \cap \mathrm{im} \left(\mathrm{id}_{\mathsf{H}_\alpha} \otimes \varphi\right). \tag{4.3.5}$$

For $S(z) = \mathsf{e}^{-1/(\mathsf{k}+n+1)\alpha_i}(z)$ with $i = 2, \dots, n$, we can take $S'(z) = S(z)$ as $S(z)$ doesn't act on B. For $S(z) = \beta(z)\mathsf{e}^{-1/(\mathsf{k}+n+1)\alpha_1}(z)$, since all arrows in (4.3.4) are vertex algebra homomorphisms it is sufficient to replace $\beta(z)$ with its image under φ to obtain $S'(z) = \mathsf{e}^{c(z)}\mathsf{e}^{-1/(\mathsf{k}+n+1)\alpha_1}(z)$.

Therefore the image of the embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{H}_\alpha \otimes \Pi$ is specified by

$$\begin{aligned}
\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} &\simeq \left(\ker \int \mathsf{e}^{c}(z)\mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_1}(z)\, \mathrm{d}z\right) \cap \left(\bigcap_{i=2}^{n} \ker \int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_i}(z)\, \mathrm{d}z\right) \cap \mathrm{im}\left(\mathrm{id}_{\mathsf{H}_\alpha} \otimes \varphi\right) \\
&\simeq \left(\bigcap_{i=1}^{n} \ker \int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\tilde{\alpha}_i}(z)\, \mathrm{d}z\right) \cap \left(\ker \int \mathsf{e}^{\frac{1}{2}c+\frac{1}{2}d}(z)\, \mathrm{d}z\right),
\end{aligned} \tag{4.3.6}$$

where $\tilde{\alpha}_1 = \alpha_1 - (\mathsf{k} + n + 1)c$ and $\tilde{\alpha}_i = \alpha_i$ otherwise. As the conformal structure on Π defined by $\frac{1}{2}{:}c(z)d(z){:}$ gives $\mathsf{e}^{c}(z)$ conformal weight zero and $\mathsf{e}^{-1/(\mathsf{k}+n+1)\alpha_1}(z)$ has conformal weight one with respect to $T(z)$, $\mathsf{e}^{-1/(\mathsf{k}+n+1)\tilde{\alpha}_1}(z)$ is a screening field on $\mathsf{H}_\alpha \otimes \Pi$. The embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{H}_\alpha \otimes \Pi$ defined by (4.3.6) was first considered by Creutzig, Genra and Nakatsuka where it is used to analyse the coset of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ by the Heisenberg vertex subalgebra generated by $J(z)$ [43].

The first n screening operators in (4.3.6) superficially appear to be the screening operators for $\mathsf{W}^{\mathsf{k}}_{n+1}$. The difference between these sets of operators is that the $\tilde{\alpha}_i$ screening operators act on both H_α and Π non-trivially while the α_i screening operators only act on H_α. So the corresponding kernels of vertex operators need not agree.

To decouple the $\tilde{\alpha}_i$ fields from the rest of $\mathsf{H}_\alpha \otimes \Pi$, let $\mathsf{H}_{\tilde{\alpha}} \subset \mathsf{H}_\alpha \otimes \Pi$ be the vertex subalgebra generated by $\tilde{\alpha}_1(z), \dots, \tilde{\alpha}_n(z)$. It is easy to see that $\mathsf{H}_{\tilde{\alpha}} \simeq \mathsf{H}_\alpha$. Let $\tilde{\Pi}$ be the vertex subalgebra of $\mathsf{H}_\alpha \otimes \Pi$ generated by

$$\tilde{c}(z) = c(z), \quad \mathsf{e}^{m\tilde{c}}(z) = \mathsf{e}^{mc}(z), \quad \tilde{d}(z) = d(z) - \frac{n}{n+1}(\mathsf{k}+n+1)c(z) + 2\omega_1(z), \tag{4.3.7}$$

where $\omega_1(z) = \frac{1}{n+1}\sum_{i=1}^{n}(n-i+1)\alpha_i(z)$. That is, $\omega_1(z)$ is the field associated to the first fundamental coweight of $\mathfrak{sl}_{n+1}$. As before it is easy to show that $\tilde{\Pi} \simeq \Pi$.

A direct computation shows that the operator product expansion $A(z)B(w)$ of any fields $A(z) \in \mathsf{H}_{\tilde{\alpha}}$ and $B(z) \in \tilde{\Pi}$ is nonsingular. Moreover, the expressions in (4.3.7) along with those for $\tilde{\alpha}_i(z)$ can be inverted to express the strong generators of $\mathsf{H}_\alpha \otimes \Pi$ in terms of linear combinations of fields in $\mathsf{H}_{\tilde{\alpha}}$ and $\tilde{\Pi}$. Therefore $\mathsf{H}_{\tilde{\alpha}} \otimes \tilde{\Pi} = \mathsf{H}_\alpha \otimes \Pi$. By performing this 'change of basis', we see that the $\tilde{\alpha}_i$ screening operators are the screening operators for $\mathsf{W}^{\mathsf{k}}_{n+1}$ with respect to the Heisenberg vertex algebra $\mathsf{H}_{\tilde{\alpha}}$.

Before seeing how this change of basis leads to the embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$, it is useful to define the following Heisenberg fields $a(z), b(z) \in \Pi$ by

$$a(z) = -\frac{\ell_n(\mathsf{k})}{2}c(z) + \frac{1}{2}d(z) \quad \text{and} \quad b(z) = \frac{\ell_n(\mathsf{k})}{2}c(z) + \frac{1}{2}d(z). \tag{4.3.8}$$

Note that $-\langle a, a\rangle = \langle b, b\rangle = \ell_n(\mathsf{k})$, while $\langle a, b\rangle = 0$. Substituting $n = 1$ and $n = 2$ into (4.3.8) reproduces the basis of the Heisenberg fields of Π used in the inverse quantum hamiltonian reductions for $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2)$ and BP^{k} in Theorem 3.1.2 and Theorem 3.2.1 respectively.

Theorem 4.3.2. *Let* k *be generic. There exists an embedding* $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ *with*

$$\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \simeq \ker \int \mathsf{e}^{a-\omega_1}(z)\,\mathrm{d}z, \tag{4.3.9}$$

where $\mathsf{e}^{-\omega_1}(z)$ *acts on fields in* $\mathsf{W}^{\mathsf{k}}_{n+1}$ *by way of the strong generators given in the Miura basis* (4.2.6).

Proof. By (4.3.6), the fields of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ must be of the form $F(z) = \sum_m A_m(z) \otimes B_m(z)$ for some $A_m(z) \in \mathsf{H}_\alpha$ and $B_m(z) \in \Pi$. By the above discussion, we can also write $F(z) = \sum_m \tilde{A}_m(z)\tilde{B}_m(z)$ for some fields $\tilde{A}_m(z) \in \mathsf{H}_{\tilde{\alpha}}$, $\tilde{B}_m(z) \in \tilde{\Pi}$ and inserting tensor products when necessary.

For convenience, suppose that the fields $\tilde{B}_m(z)$ are linearly independent. If they are not we can simply redefine the fields $\tilde{A}_m(z)$ and reduce the range of m such that this is the case. Since $\tilde{\alpha}_i(z)\tilde{B}_m(w) \sim 0$ for all $i \in \{1, \dots, n\}$ and m, $F(z)$ satisfying

$$\int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\tilde{\alpha}_i}(z)F(w)\,\mathrm{d}z = 0 \tag{4.3.10}$$

for all $i \in \{1, \dots, n\}$ is equivalent to

$$\int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\tilde{\alpha}_i}(z)\tilde{A}_m(w)\,\mathrm{d}z = 0 \tag{4.3.11}$$

for all $i \in \{1, \dots, n\}$ and m by the linear independence of the fields $\tilde{B}_m(z)$. Therefore if $F(z) \in \overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$,

$$\tilde{A}_m(w) \in \bigcap_{i=1}^{n} \ker \int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\tilde{\alpha}_i}(z)\,\mathrm{d}z \quad \text{for all } m. \tag{4.3.12}$$

As $\tilde{A}_m(z) \in \mathsf{H}_{\tilde{\alpha}}$, this means that $\tilde{A}_m(z) \in \mathsf{W}^{\mathsf{k}}_{n+1}$ by (4.2.2). More precisely, $\tilde{A}_m(z)$ is a normally ordered product of the fields $\tilde{\alpha}_i(z)$ and their derivatives. Imposing (4.3.11) for $i = 1, \dots, n$ on $\tilde{A}_m(z)$ constrains this expansion to be a normally ordered product of the fields (4.2.6) (replacing α_i with $\tilde{\alpha}_i$) and their derivatives, which strongly generate a vertex operator algebra isomorphic to $\mathsf{W}^{\mathsf{k}}_{n+1}$.

Hence we may treat the fields $\tilde{A}_m(z)$ as fields in $\mathsf{W}^{\mathsf{k}}_{n+1}$. The fields $\tilde{B}_m(z) \in \tilde{\Pi}$ are unaffected by the $\tilde{\alpha}_i$ screening operators and are therefore unconstrained up to this point. The screening operator from FMS bosonisation (4.3.2) present in (4.3.6) therefore dictates how the fields $\tilde{A}_m(z) \in \mathsf{W}^{\mathsf{k}}_{n+1}$ and $\tilde{B}_m(z) \in \tilde{\Pi}$ are combined to form a field in $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$.

The FMS bosonisation screening field $\mathsf{e}^{\frac{1}{2}c+\frac{1}{2}d}(z)$ can be written in terms of the tilded fields using (4.3.7) and the definition of $\mathsf{H}_{\tilde{\alpha}}$, and its exponent becomes

$$\frac{1}{2}c(z) + \frac{1}{2}d(z) = \left(-\frac{\ell_n(\mathsf{k})}{2}\tilde{c}(z) + \frac{1}{2}\tilde{d}(z)\right) - \frac{1}{n+1}\sum_{i=1}^{n}(n-i+1)\tilde{\alpha}_i(z). \tag{4.3.13}$$

The corresponding map $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \to \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \tilde{\Pi}$ is an embedding since its image is equal to the image of the embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{H}_{\alpha} \otimes \Pi$ defined by (4.3.6), treating $\mathsf{W}^{\mathsf{k}}_{n+1}$ as a vertex subalgebra of $\mathsf{H}_{\tilde{\alpha}} \subset \mathsf{H}_{\alpha} \otimes \Pi$. The desired result and screening field description (4.3.9) follow from the isomorphisms $\mathsf{H}_{\tilde{\alpha}} \simeq \mathsf{H}_{\alpha}$ and $\tilde{\Pi} \simeq \Pi$. ∎

4.3.2. Explicit Expressions. While Theorem 4.3.2 proves the existence of an embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ at generic levels k and gives an associated screening operator, one might also want expressions for the strongly generating fields (4.1.7) in terms of fields from $\mathsf{W}^{\mathsf{k}}_{n+1}$ and Π. Some fields of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ can be readily extracted using the screening operator in (4.3.9). For example, since $\langle a, b \rangle = 0$,

$$b(z) \in \ker \int \mathsf{e}^{a-\omega_1}(z)\,\mathrm{d}z. \tag{4.3.14}$$

The corresponding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ field is denoted by $J(z)$. Similarly, the field $\mathsf{e}^{c}(z)$ has nonsingular operator product expansion with the screening field so we can treat it as a field $G^{+}(z)$ in $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. One can quickly check that these assignments reproduce the relevant operator product expansions in (4.1.10).

Since we are free to choose a conformal structure on Π, choose the conformal structure furnished by the Π field

(4.3.15) $$t(z) = \frac{1}{2}{:}c(z)d(z){:} + \frac{n}{2}\ell_n(\mathsf{k})\partial c(z) - \frac{1}{2}\partial d(z).$$

With respect to $t(z)$, both $a(z)$ and $b(z)$ have conformal weight 1 (though neither are quasiprimary) whilst that of $\mathsf{e}^{mc}(z)$ is m. The associated central charge $\mathsf{c}_{\mathsf{k}}^{\Pi}$ satisfies

(4.3.16) $$\mathsf{c}_{\mathsf{k}}^{\Pi} + \mathsf{c}_{\mathsf{k}}^{n+1} = \bar{\mathsf{c}}_{\mathsf{k}}^{n+1}.$$

The energy-momentum field $L(z)$ of $\overline{\mathsf{W}}_{n+1}^{\mathsf{k}}$ can be decomposed according to

(4.3.17) $$L(z) = T(z) + t(z),$$

as the operator product expansion of the right-hand-side with the screening operator in (4.3.9) is a total derivative. In particular, this shows that the embedding $\overline{\mathsf{W}}_{n+1}^{\mathsf{k}} \hookrightarrow \mathsf{W}_{n+1}^{\mathsf{k}} \otimes \Pi$ is conformal (with the chosen conformal structure on Π).

To find similar expressions for the remaining strong generators of $\overline{\mathsf{W}}_{n+1}^{\mathsf{k}}$, we use explicit expressions for the aforementioned strong generators of $\overline{\mathsf{W}}_{n+1}^{\mathsf{k}}$ [89]. These expressions rely on the fact that the subregular W-algebra of type A $\overline{\mathsf{W}}_{n+1}^{\mathsf{k}}$ is isomorphic to the Feigin–Semikhatov algebra $\mathcal{W}_{n+1}^{(2)}$ [88, Thm. 6.9] and screening operators for the latter are known [68]. These screening operators for $\mathcal{W}_{n+1}^{(2)}$ (and therefore $\overline{\mathsf{W}}_{n+1}^{\mathsf{k}}$) were recounted in Section 4.2.2 and act on the vertex algebra $\mathcal{H}^{\mathsf{k}}$.

Importantly, expressions for the strong generators $L, G^{+}, J, U_3, \dots, U_n, G^{-}$ in terms of the fields in $\mathcal{H}^{\mathsf{k}}$ are known [89]. So we have two different free-field realisations of $\overline{\mathsf{W}}_{n+1}^{\mathsf{k}}$, one of which has been analysed further to obtain expressions for strong generators of $\overline{\mathsf{W}}_{n+1}^{\mathsf{k}}$. In order to use the $\mathcal{H}^{\mathsf{k}}$ expressions in our present setting of $\mathsf{H}_{\alpha} \otimes \Pi$, we must understand how $\mathcal{H}^{\mathsf{k}}$ and (4.2.13) relate to $\mathsf{H}_{\alpha} \otimes \Pi$ and (4.3.6).

Proposition 4.3.3. *Define $\psi : \mathcal{H}^{\mathsf{k}} \to \mathsf{H}_{\alpha} \otimes \Pi$ to be the vertex algebra map defined by*

(4.3.18) $$A_i(z) \mapsto \alpha_i(z), \quad Y(z) \mapsto c(z), \quad \mathsf{e}^{mY}(z) \mapsto \mathsf{e}^{mc}(z), \quad Q(z) \mapsto a(z) - \omega_1(z).$$

Then ψ is a vertex algebra isomorphism.

As this is an isomorphism, the kernel of the screening operator $\int \mathsf{e}^{Q}(z)\,\mathrm{d}z$ is equal to ψ^{-1} applied to the kernel of the screening operator in (4.3.9). The kernel of the screening operator $\int \mathsf{e}^{A_i}(z)\,\mathrm{d}z$

is equal to ψ^{-1} applied to the kernel of $\int \mathrm{e}^{\alpha_i}(z)\, \mathrm{d}z$. By Feigin–Frenkel duality for the Virasoro vertex algebra (see [72, Ch. 15]),

$$\ker \int \mathrm{e}^{\alpha_i}(z)\, \mathrm{d}z = \ker \int \mathrm{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_i}(z)\, \mathrm{d}z \tag{4.3.19}$$

for $i = 1, \dots, n$. So the isomorphism ψ maps the intersection of kernels in (4.2.13) to that in (4.3.6). In other words, ψ intertwines the action of the screening operators on $\mathcal{H}^{\mathsf{k}}$ and $\mathsf{H}_\alpha \otimes \Pi$ that define embeddings of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$.

To summarise what we have shown thus far, applying ψ to the strong generators of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ in terms of the fields of $\mathcal{H}^{\mathsf{k}}$ presented in [89] gives strong generators of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ in terms of fields of $\mathsf{H}_\alpha \otimes \Pi$. The latter set of strong generators also belong to (and therefore strongly generate) the intersection of kernels in (4.3.6), and by Theorem 4.3.2 must consist of fields of $\mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ only, treating $\mathsf{W}^{\mathsf{k}}_{n+1}$ as a subalgebra of H_α by way of the Miura transformation.

To write down the generators obtained by applying ψ to those in [89], let $\mathbb{1}_R$ and $\mathbb{1}_\Pi$ be the vacuum states of $\mathsf{W}^{\mathsf{k}}_{n+1}$ and Π respectively. Then the vacuum of $\mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ is $\mathbb{1} = \mathbb{1}_R \otimes \mathbb{1}_\Pi$. We will frequently omit tensor product symbols in what follows when it is clear which modes act on which component of $\mathsf{H}_\alpha \otimes \Pi$. Define operators $\rho_0, \rho_1, \dots, \rho_{n+1}$ on $\mathsf{H}_\alpha \otimes \Pi$ by

$$\rho_0 = (\mathsf{k}+n)(\partial + c_{-1}), \qquad \rho_i = (\mathsf{k}+n)\partial + b_{-1} + \frac{\mathsf{k}+n+1}{n+1} c_{-1} - (\varepsilon_i)_{-1}. \tag{4.3.20}$$

As $T + t$ is a conformal vector, we may write $\partial = T_{-1} + t_{-1}$. Observe that

$$\rho_i \mathrm{e}^{-c} = a_{-1}\mathrm{e}^{-c} + \left((\mathsf{k}+n)T_{-1} - (\varepsilon_i)_{-1}\right) \mathrm{e}^{-c}. \tag{4.3.21}$$

Proposition 4.3.4. *Let* $\mathsf{k} \neq -n-1$. *Define the fields* $L(z)$, $G^+(z)$, $J(z)$, $U_3(z), \dots, U_n(z)$, $G^-(z) \in \mathsf{H}_\alpha \otimes \Pi$ *by*

$$\begin{gathered} L = T + t, \quad J = b, \quad G^+ = \mathrm{e}^c, \\ G^- = -E_{n+1}(\rho_1, \dots, \rho_{n+1})\mathrm{e}^{-c}, \\ U_i = \sum_{j=0}^{i} (-1)^{i+j} \left(\prod_{m=1}^{j} \frac{m(\mathsf{k}+n)+1}{m(\mathsf{k}+n)} \right) E_{i-j}(\rho_1, \dots, \rho_{n+1}) \rho_0^j \mathbb{1}. \end{gathered} \tag{4.3.22}$$

The associated fields strongly generate $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ *and satisfy the operator product expansions* (4.1.10) *and* (4.1.11).

Proof. These fields are ψ applied to the strong generators described in Remark 3.14 in [89] except for $L(z)$. The relationship between $L(z)$ and the field $U_2(z)$, which is defined using the above formula for $U_i(z)$, is a straightforward calculation and is of the form

$$L(z) = \frac{1}{\mathsf{k}+n+1}\left(-U_2(z) + a_1\partial J(z) + a_2{:}J(z)J(z){:}\right), \tag{4.3.23}$$

with $a_1, a_2 \in \mathbb{R}[\mathsf{k}]$ whose precise form is not important. Since the set $\{G^+(z), J(z), U_2(z), U_3(z), \ldots, U_n(z), G^-(z)\}$ strongly generates $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$, the same set with $U_2(z)$ replaced with $L(z)$ also strongly generates $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. ■

The apparent singularity at $\mathsf{k} = -n$ in the formula (4.3.22) for U_i is taken care of by the factor of $(\mathsf{k}+n)$ in (4.3.20) for ρ_0. To see how the fields above decompose in terms of $\mathsf{W}^{\mathsf{k}}_{n+1}$ fields, notice that the only place where fields of H_α appear (outside of T in L) is in the symmetric polynomials involving the operators ρ_i: Since the vacuum is translation invariant, $\rho_0\mathbb{1} = (\mathsf{k}+n)c$ and successive applications of ρ_0 do not introduce any additional modes/states from H_α as T_{-1} commutes with all modes in Π and $T_{-1}\mathbb{1} = (T_{-1}\mathbb{1}_R) \otimes \mathbb{1}_\Pi = 0$. Therefore all that remains is to determine which $\mathsf{W}^{\mathsf{k}}_{n+1}$ modes appear in $E_m(\rho_1, \ldots, \rho_{n+1})$.

Lemma 4.3.5. *Let $m \in \{1, \ldots, n+1\}$. Then,*

$$\begin{aligned} E_m&(\rho_1, \ldots, \rho_{n+1}) \\ &= \sum_{j=0}^{m} \binom{n+1-j}{m-j} E_j(\sigma_1, \ldots, \sigma_{n+1}) \left((\mathsf{k}+n)t_{-1} + b_{-1} + \frac{\mathsf{k}+n+1}{n+1}c_{-1}\right)^{m-j}, \end{aligned} \tag{4.3.24}$$

where $\sigma_i = (\mathsf{k}+n)T_{-1} - (\varepsilon_i)_{-1}$.

Proof. This is Proposition 12.4.4 in [132] with $N = n+1$. In terms of the notation used therein,

$$\tau + \mu_i[-1] = \sigma_i, \qquad u = (\mathsf{k}+n)t_{-1} + b_{-1} + \frac{\mathsf{k}+n+1}{n+1}c_{-1} \tag{4.3.25}$$

and under this identification, $\rho_i = u + \tau + \mu_i[-1]$. ■

The above lemma shows that the only H_α fields that appear in the free-field realisation of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ in Proposition 4.3.4 are those given by (4.2.6), i.e. fields of $\mathsf{W}^{\mathsf{k}}_{n+1}$ as expected. The '$\mathsf{W}^{\mathsf{k}}_{n+1}$ content' of the strong generators $L(z)$, $G^+(z)$, $J(z)$ is clear from the formulae given earlier. For the fields $U_3(z), \ldots, U_n(z), G^-(z)$, we have the following structural result.

Theorem 4.3.6. *For each $i \in \{3, \ldots, n\}$, there exist i fields $\pi^{i,j}(z)$, $j = 0, \ldots, i-1$, in the vertex subalgebra of Π generated by $c(z)$ and $d(z)$ such that*

$$U_i(z) = (-1)^{i+1}W_i(z) + \sum_{j=0}^{i-1} W_j(z) \otimes \pi^{i,j}(z). \tag{4.3.26}$$

The field $G^-(z)$ can be written as

$$G^-(z) = W_{n+1}(z) \otimes \mathsf{e}^{-c}(z) + \sum_{j=0}^{n} W_j(z) \otimes \left(\pi^{-,j}_{(-1)}\mathsf{e}^{-c}(z)\right), \tag{4.3.27}$$

where $\pi^{-,j} \in \Pi$ is given by

$$\pi^{-,j} = \left((\mathsf{k}+n)t_{-1} + a_{-1}\right)^{n+1-j} \mathbb{1}_{\Pi}. \tag{4.3.28}$$

Proof. Substituting (4.3.24) into the expression for G^- in Proposition 4.3.4 gives

$$\begin{aligned} G^- &= -\sum_{j=0}^{n+1} \left(E_j(\sigma_1, \dots, \sigma_{n+1})\mathbb{1}_R\right) \otimes \left(\left((\mathsf{k}+n)t_{-1} + b_{-1} + \frac{\mathsf{k}+n+1}{n+1}c_{-1}\right)^{n+1-j} \mathsf{e}^{-c}\right) \\ &= \sum_{j=0}^{n+1} W_j \otimes \left(\tilde{\pi}^{-,j}_{(-1)}\mathsf{e}^{-c}\right) \end{aligned} \tag{4.3.29}$$

for some field $\tilde{\pi}^{-,j}(z) \in \Pi$. That $\tilde{\pi}^{-,j} = \pi^{-,j}$ follows from the action $t_{-1}\mathsf{e}^{-c} = -c_{-1}\mathsf{e}^{-c}$ and the fact that negative modes in Π commute. The decomposition of $U_i(z)$ in terms of $W_j(z)$ for $j \leqslant i$ is similarly obtained by substituting (4.3.24) into the expression for U_i given in Proposition 4.3.4. ■

A consequence of these formulae is that the embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ exists for all non-critical k. This is because the strong generators $\{G^+, J, L, U_3, \dots, U_n, G^-\}$ and $\{T, W_3, \dots, W_{n+1}\}$ of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ and $\mathsf{W}^{\mathsf{k}}_{n+1}$ respectively exist for all non-critical k, and the decompositions of the former in terms of the latter and states in Π are well-defined for all non-critical k. Injectivity of the corresponding vertex operator algebra homomorphism $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \to \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ follows from the injectivity of the free-field realisations $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{H}_{\alpha} \otimes \Pi$ [89] and $\mathsf{W}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{H}_{\tilde{\alpha}}$ [14].

When k is generic, Proposition 4.3.3 shows that the explicit embedding described in this section is indeed the one specified by the screening operator in (4.3.9).

The critical level embedding of vertex algebras $\overline{\mathsf{W}}^{-n-1}_{n+1} \hookrightarrow \mathsf{W}^{-n-1}_{n+1} \otimes \Pi$, as described in [89], is obtained using the same formulae as in the non-critical case except for multiplying $L = T + t$ by $(\mathsf{k}+n+1)$.

An important feature of these decompositions is that the Miura basis $\mathsf{W}^{\mathsf{k}}_{n+1}$ fields appear undisturbed in the sense that their derivatives and normally ordered products do not appear. Moreover, all Miura basis fields of $\mathsf{W}^{\mathsf{k}}_{n+1}$ appear somewhere in these expansions. These properties will allow us to prove the almost-simplicity of a large class of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules that we construct using the inverse reduction embedding in an analogous way to that used in Section 5 of [4].

The identifications $J(z) = b(z)$, $G^+(z) = \mathsf{e}^c(z)$ and $L(z) = T(z) + t(z)$ with the decompositions in Theorem 4.3.6 reproduce the embeddings $\mathsf{BP}^{\mathsf{k}} \hookrightarrow \mathsf{W}_3^{\mathsf{k}} \otimes \Pi$ and $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{Vir}^{\mathsf{k}} \otimes \Pi$. Composing the former with certain isomorphisms of Π and BP^{k} gives precisely the embedding used in Section 3.2.2 as mentioned therein.

The next simplest example is the embedding $\overline{\mathsf{W}}_4^{\mathsf{k}} \hookrightarrow \mathsf{W}_4^{\mathsf{k}} \otimes \Pi$ for $\mathsf{k} \neq -4$. The regular W-algebra $\mathsf{W}_4^{\mathsf{k}}$ is also known as the spin-4 extended conformal algebra [109]. As usual, denote the Miura basis fields of $\mathsf{W}_4^{\mathsf{k}}$ by $T(z)$, $W_3(z)$ and $W_4(z)$. Define the Π fields

$$a(z) = -\frac{3\mathsf{k}+8}{8}c(z) + \frac{1}{2}d(z), \quad b(z) = \frac{3\mathsf{k}+8}{8}c(z) + \frac{1}{2}d(z), \tag{4.3.30}$$

and give Π the conformal structure furnished by the energy-momentum field

$$t(z) = \frac{1}{2}{:}c(z)d(z){:} + \frac{3(3\mathsf{k}+8)}{8}\partial c(z) - \frac{1}{2}\partial d(z). \tag{4.3.31}$$

The embedding $\overline{\mathsf{W}}_4^{\mathsf{k}} \hookrightarrow \mathsf{W}_4^{\mathsf{k}} \otimes \Pi$ for $\mathsf{k} \neq -4$ given by Theorem 4.3.2 is

$$\begin{aligned} &J(z) \mapsto b(z), \quad L(z) \mapsto T(z) + t(z), \quad G^+(z) \mapsto \mathsf{e}^c(z), \\ &G^-(z) \mapsto W_4(z) \otimes \mathsf{e}^{-c}(z) + W_3(z) \otimes {:}a(z)\mathsf{e}^{-c}(z){:} \\ &\qquad + W_2(z) \otimes {:}\Big((\mathsf{k}+3)\partial a(z) + a(z)^2\Big)\mathsf{e}^{-c}(z){:} \\ &\qquad - {:}\Big(a(z)^4 + (\mathsf{k}+3)^3\partial^3 a(z) + 3(\mathsf{k}+3)\partial a(z)^2 \\ &\qquad + 4(\mathsf{k}+3)^2 a(z)\partial^2 a(z) + 6a(z)^2\partial a(z)\Big)\mathsf{e}^{-c}(z){:}, \end{aligned} \tag{4.3.32}$$

$$\begin{aligned} U_3(z) \mapsto\ & W_3(z) + 2W_2(z) \otimes m(z) - 4(\mathsf{k}+3)^2\partial^2 m(z) - 12(\mathsf{k}+3){:}m(z)\partial m(z){:} - 4{:}m(z)^3{:} \\ & + (\mathsf{k}+4)\Big(- W_2(z) \otimes c(z) + 6(\mathsf{k}+3)^2\partial^2 c(z) + 6(\mathsf{k}+3){:}\partial m(z)c(z){:} \\ & \qquad + 12(\mathsf{k}+3){:}m(z)\partial c(z){:} + 6{:}m(z)^2 c(z){:}\Big) \\ & - 2(\mathsf{k}+4)(2\mathsf{k}+7)\Big((\mathsf{k}+3)(\partial^2 c(z) + {:}c(z)\partial c(z){:}) \\ & \qquad + {:}m(z)\partial c(z){:} + {:}m(z)c(z)^2{:}\Big) \\ & + \frac{(\mathsf{k}+4)(2\mathsf{k}+7)(3\mathsf{k}+10)}{6}\Big(\partial^2 c(z) + 3{:}c(z)\partial c(z){:} + {:}c(z)^3{:}\Big), \end{aligned}$$

where $m(z) = b(z) + \frac{\mathsf{k}+4}{4}c(z) \in \Pi$. Define the $\overline{\mathsf{W}}_4^{\mathsf{k}}$ field

$$\begin{aligned} W(z) = &-\frac{1}{\mathsf{k}+2}U_3(z) - \frac{2(2\mathsf{k}+5)}{3}\partial^2 J(z) + \frac{4(\mathsf{k}+4)}{3\mathsf{k}+8}{:}J(z)L(z){:} \\ & - 6{:}J(z)\partial J(z){:} + \frac{\mathsf{k}+4}{2}\partial L(z) - \frac{8(11\mathsf{k}+32)}{3(3\mathsf{k}+8)^2}{:}J(z)^3{:}. \end{aligned} \tag{4.3.33}$$

This is a primary field of conformal weight 3. The fields $\{J(z), L(z), W(z), G^{\pm}(z)\}$ also strongly generate $\overline{\mathsf{W}}^{\mathsf{k}}_4$. The operator product expansions of these fields are given in [**49**], albeit with $L(z)$ replaced with $\tilde{L}(z)$. Checking that the above embedding correctly reproduces the desired operator product expansions can be done tediously by hand or quickly using a computer using tools such as the Mathematica package *OPEdefs* [**152**].

4.3.3. Relaxed Modules for Subregular W-Algebras. In Chapter 2, we saw that the representation theory of $\overline{\mathsf{W}}^{\mathsf{k}}_3$ (the Bershadsky–Polyakov algebra) can be largely understood in terms of representations of its untwisted and twisted Zhu algebrs. For $n > 2$, the complexity of the operator product expansion of $G^{+}(z)$ and $G^{-}(z)$ makes determining these algebras difficult.

However, the existence of an embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ for any non-critical level allows us to construct infinite families of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules by taking appropriate tensor products of $\mathsf{W}^{\mathsf{k}}_{n+1}$- and Π-modules. Particularly important amongst modules for subregular W-algebras are relaxed modules.

These play a central role in computing modular transformations and fusion rules for admissible-level $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ and admissible-level BP_{k}. In both cases, this is precisely because the relaxed modules for the simple subregular W-algebra can be realised in terms of modules for the corresponding simple regular W-algebra and for Π using inverse quantum hamiltonian reduction. This is described in detail in Section 7.1 of [**2**] for $n = 1$ and Section 3.2.2 for $n = 2$. It is therefore reasonable to anticipate that some of the $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules that we will construct by way of our embedding will play a central role in the representation theory of $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$ and the construction of 'logarithmic minimal models' with $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$ symmetry.

Much of this section follows the approach taken for $n = 2$ described in [**4**]. As we are interested in extracting information out of this embedding, we assume that k is non-critical for the remainder of this section. We will also frequently identify $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ with its image under the embedding obtained in Proposition 4.3.4.

Before constructing $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules, we recall some general properties and definitions of modules over vertex operator algebras as used in the analysis of relaxed modules for Bershadsky–Polyakov algebras in [**4**].

Definition 4.3.7. *Let V be a vertex operator algebra, M a $\mathbb{Z}_{\geqslant 0}$-graded V-module and denote the top space of M by $M^{\mathrm{top}} = M_0$.*

- *M is* top-generated *if M is generated by M^{top}.*
- *M has* only top-submodules *if every nonzero submodule of M has nonzero intersection with M^{top}.*
- *M is* almost-irreducible *if it is top-generated and has only top-submodules.*

By Zhu's theorem [171], M^{top} is a module for the associative algebra $\mathsf{Zhu}[V]$. When looking to construct irreducible V-modules, it is convenient to consider V-modules M whose submodules are all generated by $\mathsf{Zhu}[\mathsf{V}]$-submodules of M^{top}. This is because checking if such a V-module M is irreducible requires only checking if M^{top} is an irreducible $\mathsf{Zhu}[\mathsf{V}]$-module:

Proposition 4.3.8 (Proposition 5.2 [4])**.** *If M is almost-irreducible and M^{top} is irreducible as a $\mathsf{Zhu}[V]$-module then M is irreducible.*

A key ingredient for constructing relaxed $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules is relaxed Π-modules as we saw for BP^{k} in Section 3.2.2. The relaxed Π-modules we require for $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ are defined in the same way as the modules $\Pi_{[j]}$ but are slightly modified to be positive-energy with respect to the conformal structure defined by $t(z) \in \Pi$. That is, we induce Π-modules from the $\mathbb{Z}c$-modules generated by (certain) elements $\mathsf{e}^h \in \mathbb{C}[\mathfrak{h}]$ on which $h' \in \mathfrak{h}$ acts as $h' \cdot \mathsf{e}^h = \langle h', h\rangle\, \mathsf{e}^h$ [33].

Proposition 4.3.9. *The weight Π-module generated from $\mathsf{e}^{rb+\lambda c}$ is positive-energy if and only if $r = -1$. In this case, the Π-module is $\mathbb{Z}$-graded, simple and the minimal t_0-eigenvalue is $\frac{n}{2}\ell_n(\kappa)$.*

Denote the Π-module generated from $\mathsf{e}^{rb+\lambda c}$ by $\Pi_r(\lambda)$. These Π-modules satisfy a number of nice properties: By direct computation, the eigenvalue of b_0 on $\mathsf{e}^{-b+\lambda c}$ is $\lambda - \ell_n(\mathsf{k})$. The zero modes $\mathsf{e}^{\pm c}_0$ act injectively on the top space of $\Pi_{-1}(\lambda)$. In general, $\Pi_r(\lambda)$ is $\mathbb{Z}$-graded as long as $r \in \mathbb{Z}$ and $\mathbb{Z} + \frac{1}{2}$-graded when $r \in \mathbb{Z} + \frac{1}{2}$. In either case, $\Pi_r(\lambda) \simeq \Pi_r(\lambda + n)$ as Π-modules for all $n \in \mathbb{Z}$.

As for $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ we can use the Heisenberg field $b(z) \in \Pi$ to define an vertex algebra automorphism of Π called *spectral flow* ρ^ℓ for all $\ell \in \mathbb{Z}$.

Proposition 4.3.10. *Let $\ell \in \mathbb{Z}$. The map $\rho^\ell : \Pi \to \Pi$ defined by*

$$\rho^\ell\left(A(z)\right) = Y\left(\Lambda(\ell b, z)A, z\right), \quad \text{where } \Lambda(\ell b, z) = z^{-\ell b_0} \prod_{m=1}^{\infty} \exp\left(\frac{(-1)^m}{m} \ell b_m z^{-m}\right), \tag{4.3.34}$$

is a vertex algebra automorphism, where Y is the vertex map for Π.

The action of spectral flow on the fields $a(z)$, $b(z)$ and $\mathsf{e}^{mc}(z)$ where $m \in \mathbb{Z}$ can be explicitly computed with help from (3.1.4) and is given by

$$\rho^\ell(a(z)) = a(z), \quad \rho^\ell(b(z)) = b(z) - \ell_n(\mathsf{k})\ell z^{-1}, \quad \rho^\ell(\mathsf{e}^{mc}(z)) = z^{-m\ell}\mathsf{e}^{mc}(z). \tag{4.3.35}$$

This is only a vertex operator algebra automorphism when the conformal vector $t(z)$ is preserved i.e. when $\ell = 0$. As usual, for any Π-module and any nonzero $\ell \in \mathbb{Z}$ we get a new Π-module by applying spectral flow $\rho^\ell(M)$.

Here as before, the range of ℓ can be extended to include $\mathbb{Z} + \frac{1}{2}$ in which case a $\mathbb{Z}$-graded Π-module becomes $(\mathbb{Z} + \frac{1}{2})$-graded upon twisting with respect to ρ^ℓ and vice-versa.

By the embedding (4.3.22), there are two strong generators of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ living only in Π: $G^+ \to \mathsf{e}^c$ and $J \to b$. Let U be the vertex subalgebra of Π generated by b and e^c. As a U-module, $\Pi_{-1}(\lambda)$ satisfies the many of the same properties that the $n = 2$ version was shown to satisfy in [4]:

Proposition 4.3.11. $\Pi_{-1}(\lambda)$ *is almost-irreducible as a* U*-module.*

Proof. The proof for all n is identical to that for the $n = 2$ case described in Section 5.1 of [4]. Adapting it to the case of general n simply requires replacing j and i with b and a respectively and keeping in mind the different conformal structures. ∎

Again, under the identification described in Section 4.3.1, the generators of $\mathbb{1}_R \otimes U$ are elements of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. So $\mathbb{1}_R \otimes U$ is also a vertex subalgebra of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$.

Given a $\mathsf{W}^{\mathsf{k}}_{n+1}$-module M, the $(\mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi)$-module $M(r, \lambda) = M \otimes \Pi_r(\lambda)$, where $r \in \mathbb{Z}$ to ensure $\mathbb{Z}$-grading, is also a $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module by restriction. As $\Pi_r(\lambda) \simeq \Pi_r(\lambda + n)$ as Π-modules for all $n \in \mathbb{Z}$, $M(r, \lambda) \simeq M(r, \lambda + n)$ as $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules. Additionally, $J(z) \in \overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ is identified with $b(z)$ under the embedding of Theorem 4.3.2 so applying the $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ version of spectral flow (4.1.15) to $M(r, \lambda)$ can be performed purely in terms of the Π version of spectral flow:

$$\sigma^\ell\big(M(r,\lambda)\big) = \sigma^\ell\big(M \otimes \Pi_r(\lambda)\big) = M \otimes \rho^\ell\left(\Pi_r(\lambda)\right) = M \otimes \Pi_{r+\ell}(\lambda) = M(r+\ell, \lambda). \tag{4.3.36}$$

We therefore interpret the label 'r' in $M(r, \lambda)$ as a spectral flow index. Owing again to the simplicity of the expressions for $J(z)$ and $L(z)$ in terms of $\mathsf{W}^{\mathsf{k}}_{n+1}$ and Π fields, character formulae for $M(r, \lambda)$ are immediate from their construction:

As $\Pi_r(\lambda)$ are all relaxed modules for a lattice-like vertex algebra, their characters are straightforward to compute. Here, we define the character of a Π-module M to be

$$\mathrm{ch}\big[M\big]\,(\mathsf{z},\mathsf{q}) = \mathrm{tr}_M\left(\mathsf{z}^{b_0}\mathsf{q}^{t_0 - \frac{\mathsf{c}^\Pi_\mathsf{k}}{24}}\right). \tag{4.3.37}$$

A computation using a PBW basis for Π shows that

$$\mathrm{ch}\big[\Pi_{-1}(\lambda)\big]\,(\mathsf{z},\mathsf{q}) = \frac{\mathsf{z}^{\lambda - \ell_n(\mathsf{k})}}{\eta(\mathsf{q})^2}\sum_{i\in\mathbb{Z}}\mathsf{z}^i, \tag{4.3.38}$$

which also gives the characters of $\Pi_r(\lambda)$ for all $r \in \frac{1}{2}\mathbb{Z}$ as

$$\begin{aligned}\mathrm{ch}\big[\Pi_r(\lambda)\big]\,(\mathsf{z},\mathsf{q}) &= \mathrm{tr}_{\rho^{r+1}(\Pi_{-1}(\lambda))}\left(\mathsf{z}^{b_0}\mathsf{q}^{t_0 - \frac{\mathsf{c}^\Pi_\mathsf{k}}{24}}\right)\\ &= \mathrm{tr}_{\Pi_{-1}(\lambda)}\left(\mathsf{z}^{\rho^{-r-1}(b_0)}\mathsf{q}^{\rho^{-r-1}(t_0) - \frac{\mathsf{c}^\Pi_\mathsf{k}}{24}}\right)\end{aligned} \tag{4.3.39}$$

$$= \mathsf{z}^{(r+1)\ell_n(\mathsf{k})}\mathsf{q}^{(r+1)(r+2-n)\frac{\ell_n(\mathsf{k})}{2}}\,\mathrm{ch}\left[\Pi_{-1}(\lambda)\right]\left(\mathsf{z}\mathsf{q}^{r+1},\mathsf{q}\right).$$

Corollary 4.3.12. *Suppose that M is a $\mathsf{W}^{\mathsf{k}}_{n+1}$-module with q-character*

$$\mathrm{ch}\left[M\right](\mathsf{q}) = \mathrm{tr}_M\left(\mathsf{q}^{T_0-\mathsf{c}^{n+1}_{\mathsf{k}}/24}\right). \tag{4.3.40}$$

Then the $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module $M(r,\lambda)$ has character

$$\begin{aligned}\mathrm{ch}\left[M(r,\lambda)\right](\mathsf{z},\mathsf{q}) &= \mathrm{tr}_{M(r,\lambda)}\left(\mathsf{z}^{J_0}\mathsf{q}^{L_0-\frac{\bar{\mathsf{c}}^{n+1}_{\mathsf{k}}}{24}}\right)\\ &= \mathrm{ch}\left[M\right](\mathsf{q})\,\mathrm{ch}\left[\Pi_r(\lambda)\right](\mathsf{z},\mathsf{q}),\end{aligned} \tag{4.3.41}$$

where $\mathrm{ch}\left[\Pi_r(\lambda)\right](\mathsf{z},\mathsf{q})$ is given by (4.3.39).

It is useful to know what properties of M are inherited by $M(-1,\lambda)$ (recall that by Proposition 4.3.9, $\Pi_r(\lambda)$ is positive-energy with respect to $t(z)$ only when $r=-1$). For example, only having top-submodules and being top-generated. Fortunately, $\Pi_{-1}(\lambda)$ being an almost-irreducible U-module is strong enough to require fairly mild constraints on the M for which such properties are inherited by $M(-1,\lambda)$. As in the $n=2$ case, it is convenient to introduce $U_{n+1}=(-1)^nG^-_{(-1)}G^+\in\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. This field can be expanded as

$$U_{n+1}(z) = (-1)^nW_{n+1}(z) + \sum_{j=0}^{n} W_j(z)\otimes\pi^{n+1,j}(z) \tag{4.3.42}$$

for some fields $\pi^{n+1,j}(z)$ in the vertex subalgebra of Π generated by $c(z)$ and $d(z)$. The following theorems are generalisations of Theorems 5.9 and 5.10 in [4]. The main difference when $n>2$ is the existence of strong-generating fields $U_i(z)$. This would be an issue if not for the structure of the decompositions (4.3.26) of such fields in terms of fields in $\mathsf{W}^{\mathsf{k}}_{n+1}$ (i.e. the lack of derivatives or normally ordered products).

Theorem 4.3.13. *If M is a weight $\mathsf{W}^{\mathsf{k}}_{n+1}$-module that has only top-submodules, then the weight $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module $M(-1,\lambda)$ has only top-submodules for all $\lambda\in\mathbb{C}$.*

Proof. The proof used here follows the same approach used for $n=2$ in Theorem 5.9 of [4]. Assume that N is a nonzero $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-submodule of $M(-1,\lambda)$ and let $w\in N$ be a weight vector. As $\Pi_{-1}(\lambda)$ has only top-submodules as a U-module, w can be sent to a nonzero element of $M\otimes\Pi_{-1}(\lambda)^{\mathrm{top}}$ under the action of modes from $\mathbb{1}_R\otimes U\subset\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. The result is therefore an element $w_0=u_0\otimes v^{\mathrm{top}}\in N$ where $u_0\in M$ and $v^{\mathrm{top}}\in\Pi_{-1}(\lambda)^{\mathrm{top}}$.

If we can show that applying suitable modes from $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ to w_0 results in an element of the top space $M(-1,\lambda)^{\text{top}} = M^{\text{top}} \otimes \Pi_{-1}(\lambda)^{\text{top}}$ then we are done. We will do this recursively by defining $w_1, \dots, w_k \in N$ with $w_p = u_p \otimes v^{\text{top}}$ such that the conformal weight strictly decreases at each step and $w_k \in M(-1,\lambda)^{\text{top}}$ for some $k \in \mathbb{Z}_{\geqslant 0}$. In other words, we recursively move up the submodule until we reach the top space and do so in such a way that v^{top} is untouched at each step.

To define this recursion, let $w_p = u_p \otimes v^{\text{top}} \in N$ be nonzero. If $(W_j)_m u_p = 0$ for all $j \in \{2, \dots, n+1\}$ and $m > 0$, then u_p generates a highest-weight submodule of M. Since M only has top-submodules, it follows that $u_p \in M^{\text{top}}$ and therefore that $w_p \in M(-1,\lambda)^{\text{top}}$.

Otherwise, let $i \in \{2, ..., n+1\}$ be minimal satisfying

$$(W_j)_\ell u_p = 0 \quad \text{for all } \ell \in \mathbb{Z}_{>0} \text{ and } j < i, \tag{4.3.43}$$

and there exists $m \in \mathbb{Z}_{>0}$ such that $(W_i)_m u_p \neq 0$. That is, u_p is annihilated by all positive modes of the $\mathsf{W}^{\mathsf{k}}_{n+1}$ fields $W_2(z), \dots, W_{i-1}(z)$ and is not annihilated by all positive modes of $W_i(z)$.

Define $w_{p+1} = (U_i)_m w_p$. As w_{p+1} is obtained from w_p by the action of a mode from $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$, it follows that $w_{p+1} \in N$. Additionally, conformal weight of w_{p+1} is strictly smaller than that of w_p. By the decompositions (4.3.6),

$$\begin{aligned} w_{p+1} &= \left((-1)^{i+1} W_i(z) + \sum_{j=0}^{i-1} W_j(z) \otimes \pi^{i,j}(z) \right)_m u_p \otimes v^{\text{top}} \\ &= (-1)^{i+1} (W_i)_m u_p \otimes v^{\text{top}} + \sum_{j=0}^{i-1} \sum_{r=0}^{\infty} (W_j)_{m+r} u_p \otimes \pi^{i,j}_{-r} v^{\text{top}} \\ &= (-1)^{i+1} (W_i)_m u_p \otimes v^{\text{top}} \neq 0. \end{aligned} \tag{4.3.44}$$

As the conformal weight of u_p decreases at each iteration and M is positive-energy, applying this procedure sufficiently many times will yield a nonzero $u_k \in M^{\text{top}}$. That is, $w_k \in M(-1,\lambda)^{\text{top}} \cap N$ and therefore $M(-1,\lambda)$ has only top-submodules. ∎

Theorem 4.3.14. *If M is a top-generated weight $\mathsf{W}^{\mathsf{k}}_{n+1}$-module, then $M(-1,\lambda)$ is a top-generated weight $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module for all $\lambda \in \mathbb{C}$.*

Proof. The proof used here follows the same approach used for $n = 2$ in Theorem 5.10 of [4]. Let N be the submodule of $M(-1,\lambda)$ generated by the top space $M(-1,\lambda)^{\text{top}} = M^{\text{top}} \otimes \Pi_{-1}(\lambda)^{\text{top}}$. We begin by showing that $M \otimes \Pi_{-1}(\lambda)^{\text{top}} \subset N$. Once established, it then follows that $M(-1,\lambda) \subset N$. This is because $\Pi_{-1}(\lambda)$ being top generated as a U-module means that any $u \otimes v \in M(-1,\lambda)$ can be written as a collection of modes from $\mathbb{1}_R \otimes U \subset \overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ acting on $u \otimes v^{\text{top}}$ for some $v^{\text{top}} \in \Pi_{-1}(\lambda)^{\text{top}}$.

With this in mind, let $u \otimes v^{\mathrm{top}} \in M \otimes \Pi_{-1}(\lambda)^{\mathrm{top}}$. If $u \in M^{\mathrm{top}}$, then it is clear that $u \otimes v^{\mathrm{top}} \in N$. Otherwise, as M is top-generated, u is obtained from some $u^{\mathrm{top}} \in M^{\mathrm{top}}$ by the application of modes of the fields $W_2(z), \dots, W_{n+1}(z)$.

It therefore suffices to show that if $u \otimes v^{\mathrm{top}} \in N$ for all $v^{\mathrm{top}} \in \Pi_{-1}(\lambda)^{\mathrm{top}}$, then so is $(W_i)_{-m} u \otimes v^{\mathrm{top}}$ for all $i = 2, \dots, n+1$ and $m > 0$. Starting with $i = 2$,

$$(U_2)_{-m}\left(u \otimes v^{\mathrm{top}}\right) = -(W_2)_{-m} u \otimes v^{\mathrm{top}} - u \otimes \pi^{2,0}_{-m} v^{\mathrm{top}} \in N. \tag{4.3.45}$$

As $\Pi_{-1}(\lambda)$ is top-generated as a U-module, $\pi^{2,0}_{-m} v^{\mathrm{top}}$ can be obtained from some $v'^{\,\mathrm{top}}$ by the action of modes from U. So the right most term above can be written as modes from $\mathbb{1}_R \otimes U \subset \overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ acting on $u \otimes v'^{\,\mathrm{top}} \in N$ and is therefore also in N. Hence $(W_2)_{-m} u \otimes v^{\mathrm{top}} \in N$ for all $m > 0$. For $i = 3$,

$$\begin{aligned}(U_3)_{-m}\left(u \otimes v^{\mathrm{top}}\right) &= (W_3 + W_2 \otimes \pi^{3,2} - \pi^{3,0})_{-m} u \otimes v^{\mathrm{top}} \\ &= (W_3)_{-m} u \otimes v^{\mathrm{top}} + \sum_{r=0}^{\infty} (W_2)_{-m+r} u \otimes \pi^{3,2}_{-r} v^{\mathrm{top}} - u \otimes \pi^{3,0}_{-m} v^{\mathrm{top}} \in N.\end{aligned} \tag{4.3.46}$$

An identical argument as in the $i = 2$ case shows that $u \otimes \pi^{3,0}_{-m} v^{\mathrm{top}} \in N$. That the summands $(W_2)_{-m+r} u \otimes \pi^{3,2}_{-r} v^{\mathrm{top}} \in N$ follows from $\Pi_{-1}(\lambda)$ being top-generated, also using a similar argument to the $i = 2$ case. Hence $(W_3)_{-m} u \otimes v^{\mathrm{top}} \in N$ for all $m > 0$.

That $(W_i)_{-m} u \otimes v^{\mathrm{top}} \in N$ can be established in the same way as in the $i = 3$ case, where we use the expansions (4.3.26) to reduce to the $j < i$ cases and use that $\Pi_{-1}(\lambda)$ is top-generated. Hence we conclude that $M \otimes \Pi_{-1}(\lambda)^{\mathrm{top}} \subset N$ as required. ∎

Corollary 4.3.15. *If M is an almost-irreducible weight $\mathsf{W}^{\mathsf{k}}_{n+1}$-module then $M(-1,\lambda)$ is an almost-irreducible weight $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module for all $\lambda \in \mathbb{C}$.*

We conclude this section with the case of M being an irreducible $\mathsf{W}^{\mathsf{k}}_{n+1}$-module. This is particularly important when we eventually want to discuss $\mathsf{W}_{n+1,\mathsf{k}}$ and its modules: $\mathsf{W}_{n+1,\mathsf{k}}$ is irreducible as a $\mathsf{W}^{\mathsf{k}}_{n+1}$-module and, for nondegenerate admissible k, $\mathsf{W}_{n+1,\mathsf{k}}$-modules are all direct sums of irreducible $\mathsf{W}^{\mathsf{k}}_{n+1}$-modules [17].

Proposition 4.3.16. *Let M be an irreducible $\mathsf{W}^{\mathsf{k}}_{n+1}$-module. Then*

- *$M(-1,\lambda)$ is an indecomposable relaxed highest-weight $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module for all $\lambda \in \mathbb{C}$.*
- *$M(-1,\lambda)$ is irreducible for almost all $\lambda \in \mathbb{C}$.*

Proof. As M is irreducible, M^{top} is a module for $\mathsf{Zhu}\left[\mathsf{W}^{\mathsf{k}}_{n+1}\right]$ which is abelian [17]. Therefore, M^{top} is one-dimensional and spanned by a vector v_γ with $\gamma = (\gamma_2, \dots, \gamma_{n+1})$ defined by

$$(W_i)_0\, v_\gamma = \gamma_i v_\gamma. \tag{4.3.47}$$

The top space of $M(-1,\lambda)$ is spanned by the set $\{v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m)c} | m \in \mathbb{Z}\}$. To show that $M(-1,\lambda)$ is a relaxed highest-weight $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module, we need to find a relaxed highest-weight vector that generates $M(-1,\lambda)$. As $M(-1,\lambda)$ is top-generated by Theorem 4.3.14, it suffices to find an m for which $v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m)c}$ generates $M(-1,\lambda)^{\mathrm{top}}$. The modes G^-_0 and G^+_0 act on $v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m)c}$ according to

$$\begin{aligned} G^+_0\left(v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m)c}\right) &= v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m+1)c}, \\ G^-_0\left(v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m)c}\right) &= p(\gamma,\lambda+m) v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m-1)c}, \end{aligned} \tag{4.3.48}$$

where $p(\gamma,x)$ is a polynomial in x of order at most $n+1$ by Theorem 4.3.6. Choose $m' \in \mathbb{Z}$ such that $\lambda + m'$ is strictly less than the real parts of all roots of $p(\gamma,x)$. As G^+_0 and G^-_0 act injectively on $v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m')c}$, $v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m)c}$ is a relaxed highest-weight vector of $M(-1,\lambda)$ that generates $M(-1,\lambda)^{\mathrm{top}}$ and therefore $M(-1,\lambda)$.

That $M(-1,\lambda)$ is indecomposable follows from $M(-1,\lambda)$ being uniserial as in the $n=2$ case described in the proof of Theorem 5.12 of [4].

Finally, recall that $M(-1,\lambda)$ has only top-submodules so any submodule of $N \subset M(-1,\lambda)$ must contain an element of the form $\mathsf{e}^{-b+(\lambda+m)c}$. As G^+_0 acts injectively on $M(-1,\lambda)^{\mathrm{top}}$, there exists $m' \leqslant m$ such $G^-_0 \mathsf{e}^{-b+(\lambda+m')c} = 0$. That is, $\lambda + m'$ is a root of $p(\gamma,x)$. As $p(\gamma,x)$ is polynomial in x for fixed γ, there are finitely many $[\lambda] \in \mathbb{C}/\mathbb{Z}$ such that $[\lambda]$ contains a root of $p(\gamma,x)$. ∎

Corollary 4.3.17. *Let M be an irreducible $\mathsf{W}^{\mathsf{k}}_{n+1}$-module. Conjugate highest-weight vectors in $M(-1,\lambda)$ are of the form $v_\gamma \otimes \mathsf{e}^{-b+(\lambda+m)c}$ with $p(\gamma,\lambda+m) = 0$. If conjugate highest-weight vectors are present (i.e. when $M(-1,\lambda)$ is reducible), let $m' \in \mathbb{Z}$ be the maximal m satisfying $p(\gamma,\lambda+m) = 0$. Then the submodule of $M(-1,\lambda)$ generated by $v_\gamma \otimes \mathsf{e}^{-j+(\lambda+m')c}$ is an irreducible conjugate highest-weight $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module.*

Given $\gamma \in \mathbb{C}^n$, it is not immediately clear what the roots of the polynomial $p(\gamma,x)$ are. For $n=1$ and 2, the roots of the polynomials $p(\gamma,x)$ corresponding to $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$-modules of the form $M(-1,\lambda)$ can be described using data from quantum hamiltonian reductions of certain highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$- and $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$-modules respectively. The $n=2$ case is essentially the third point in Theorem 2.3.28.

It is expected that such a description holds for general n. That is, for any $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$-module of the form $M(-1,\lambda)$, the roots of the corresponding polynomial $p(\gamma,x)$ should be related to the eigenvalues of various $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ zero modes on the highest-weight vector of quantum hamiltonian reductions of certain highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_{n+1})$-modules. Moreover the $\mathbb{Z}_{n+1}$ symmetry of $\mathsf{W}^{\mathsf{k}}_{n+1}$ should relate these roots as seen in $n = 2$ (Lemma 2.3.27).

Knowledge about the subregular quantum hamiltonian reduction functor is not currently sufficient to address this problem at present.

4.3.4. Simple Quotients. A natural question to ask is when the inverse quantum hamiltonian reduction embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ descends to an embedding of simple quotients. As we will see, this is almost-always true and depends on the level k. The restrictions for k are known for $n = 1$ [2] and $n = 2$ [4]:

$$\begin{aligned} \mathsf{L}_{\mathfrak{sl}_2}(\mathsf{k}) \simeq \overline{\mathsf{W}}_{2,\mathsf{k}} \hookrightarrow \mathsf{W}_{2,\mathsf{k}} \otimes \Pi \simeq \mathsf{Vir}_{\mathsf{k}} \otimes \Pi \quad &\Leftrightarrow \quad \mathsf{k}+1 \notin \mathbb{Z}_{\geqslant 1}, \\ \mathsf{BP}_{\mathsf{k}} \simeq \overline{\mathsf{W}}_{3,\mathsf{k}} \hookrightarrow \mathsf{W}_{3,\mathsf{k}} \otimes \Pi \quad &\Leftrightarrow \quad \mathsf{k}+2,\ 2\mathsf{k}+4 \notin \mathbb{Z}_{\geqslant 1}. \end{aligned} \tag{4.3.49}$$

A nice feature of (4.3.49) is that when k is admissible, these conditions exclude precisely the degenerate admissible levels. Then, results similar to Proposition 4.3.16 allow for the construction of continuous families of almost-always simple relaxed $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$-modules for $n = 1$ and 2.

As we saw in the $n = 2$ case in Section 3.2, the modules constructed in this manner and their spectral flows are referred to as *standard modules* and play a fundamental role in the determination of modular transformations and Grothendieck fusion rules for the simple subregular W-algebra at nondegenerate admissible levels. This is because the corresponding information for the simple regular W-algebra at these levels (also known as W_{n+1} *minimal models*) is known and the standard modules allow us to 'lift' this information using the relaxed modules defined by inverse quantum hamiltonian reduction.

Generalising this story to the $n > 2$ case fully is out of the scope of this thesis but is expected to follow the same lines as in Section 3.2.

Explicit formulae for singular vectors in $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ are only known for particular pairs of n and k. When $n = 1$, the Malikov–Feigin–Fuchs formula for singular vectors in Verma modules for $\widehat{\mathfrak{sl}}_2$ [127] can be used to describe singular vectors in admissible-level $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ [7]. Singular vectors for $\overline{\mathsf{W}}^{\mathsf{k}}_{3}$ are known for $\mathsf{k} = -\frac{5}{3}$, $\mathsf{k} = -\frac{9}{4}$ and $\mathsf{k} \in \mathbb{Z}_{\geqslant -1}$ [5], in addition to admissible levels of the form $\mathsf{k} = -3 + \frac{\mathsf{u}}{2}$ where $\mathsf{u} > 2$ is odd [15].

Little is known about singular vectors and the corresponding ideals of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ for general n and k. Fortunately, determining when embeddings of simple quotients exist only requires the knowledge we have about relaxed $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules from Section 4.3.3 and an understanding of singular vectors of a particular form.

Proposition 4.3.18. *The vector* $(G^+_{-1})^m \mathbb{1}$, $m > 0$, *is singular in* $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ *if and only if* $i(\mathsf{k}+n) = m$ *for some* $i \in \{1, \dots, n\}$.

Proof. By charge and conformal weight considerations, it is clear that J_s, L_s, $(U_i)_s$, G^+_{s-1} and G^-_{s+1} annihilate $(G^+_{-1})^m \mathbb{1}$ for all $s > 0$. All that remains to check is when $G^-_1 (G^+_{-1})^m \mathbb{1} = G^-_{(n)} (G^+)^m = 0$. By the embedding, we can identify $(G^+)^m$ with e^{mc} and G^- with $-\rho_{n+1} \cdots \rho_1 \mathsf{e}^{-c}$. Again using charge and conformal weight considerations, $((G^+)^m)_{(n')} G^- = 0$ for all $n' > n$ and therefore

$$G^-_{(n)} (G^+_{-1})^m = (-1)^{n+1} (\mathsf{e}^{mc})_{(n)} \left(-\rho_{n+1} \cdots \rho_1 \mathsf{e}^{-c}\right). \tag{4.3.50}$$

While this appears to have superficially made things more complicated, the above form allows us to explicitly compute the n-th product by performing the computation in $\mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$. The following technical lemma is the most tedious part of this proof but the result gives the desired conditions on k automatically.

The idea is to not compute the n'th product directly but to sneak up on it by gradually inserting more ρ_j operators while simultaneously raising the mode index on e^{mc}.

Lemma 4.3.19. *Let* $j \in \{0, 1, \dots, n\}$. *Then*

$$(\mathsf{e}^{mc})_{(j)} \left(-\rho_{j+1} \cdots \rho_1 \mathsf{e}^{-c}\right) = m \prod_{i=1}^{j} \left(i(\mathsf{k}+n) - m\right) \mathsf{e}^{(m-1)c}. \tag{4.3.51}$$

Proof. We proceed by induction. The $j = 0$ case can be checked directly:

$$\begin{aligned} (\mathsf{e}^{mc})_{(0)} \left(-\rho_1 \mathsf{e}^{-c}\right) &= -(\mathsf{e}^{mc})_{(0)} \left((\mathsf{k}+n) t_{-1} + b_{-1} + \frac{\mathsf{k}+n+1}{n+1} c_{-1} \right) \mathsf{e}^{-c} \\ &= -(\mathsf{e}^{mc})_{(0)} \left(a_{-1} \mathsf{e}^{-c}\right) \\ &= \left[a_{-1}, (\mathsf{e}^{mc})_{(0)}\right] \mathsf{e}^{-c} \\ &= m \mathsf{e}^{(m-1)c}. \end{aligned} \tag{4.3.52}$$

For the inductive step, suppose that (4.3.51) holds for some j. By the same charge and conformal weight considerations used earlier, $(\mathsf{e}^{mc})_{(j')} \left(-\rho_{j+1} \dots \rho_1 \mathsf{e}^{-c}\right) = 0$ for all $j' > j$. To see that (4.3.51) being true for j implies that (4.3.51) is true for $j+1$, we simply need to expand ρ_{j+2} and reduce back to the j case. That is,

$$\begin{aligned} &(\mathsf{e}^{mc})_{(j+1)} \big(-\rho_{j+2} \rho_{j+1} \dots \rho_1 \mathsf{e}^{-c} \big) \\ &\quad = (\mathsf{e}^{mc})_{(j+1)} \left(- \left((\mathsf{k}+n)\partial + b_{-1} + \frac{\mathsf{k}+n+1}{n+1} c_{-1} - (\varepsilon_{j+2})_{-1} \right) \rho_{j+1} \dots \rho_1 \mathsf{e}^{-c} \right) \end{aligned} \tag{4.3.53}$$

$$= (\mathsf{e}^{mc})_{(j+1)} \left(- \left((\mathsf{k}+n)t_{-1} + b_{-1} + \frac{\mathsf{k}+n+1}{n+1} c_{-1} \right) \rho_{j+1} \dots \rho_1 \mathsf{e}^{-c} \right)$$
$$= -(\mathsf{k}+n)(\mathsf{e}^{mc})_{(j+1)} \left(t_{-1}\rho_{j+1} \dots \rho_1 \mathsf{e}^{-c} \right) - m(\mathsf{e}^{mc})_{(j)} \left(-\rho_{j+1} \dots \rho_1 \mathsf{e}^{-c} \right)$$

using the commutation relation, for $p, q \in \mathbb{Z}$,

$$\left[Ac_{(p)} + Bd_{(p)}, \mathsf{e}^{mc}_{(q)} \right] = 2Bm\, \mathsf{e}^{mc}_{(p+q)}. \tag{4.3.54}$$

The term involving t_{-1} can be simplified using standard identities involving derivatives and i'th products described in, for example, [108]. Here we attack this head-on using the operator product expansions of $\mathsf{e}^{mc}(z)$ and $\left(-\rho_{j+1} \cdots \rho_1 \mathsf{e}^{-c}\right)(z)$. This is of course equivalent to working with i'th products.

By the inductive hypothesis and the observation earlier that the j''th product vanishes for $j' > j$,

$$\mathsf{e}^{mc}(z) \left(-\rho_{j+1} \cdots \rho_1 \mathsf{e}^{-c}\right)(w) \sim \frac{m \prod_{i=1}^{j} \left(i(\mathsf{k}+n) - m\right) \mathsf{e}^{(m-1)c}(w)}{(z-w)^{j+1}} + \dots. \tag{4.3.55}$$

where the ellipses contains singular terms with an order $\leqslant j$ pole in $z - w$. Applying ∂_w to both sides and extracting the coefficient field of the order $j+2$ pole finally gives

$$-(\mathsf{k}+n)(\mathsf{e}^{mc})_{(j+1)} \left(t_{-1}\rho_{j+1} \dots \rho_1 \mathsf{e}^{-c} \right) = (j+1)(\mathsf{k}+n)m \prod_{i=1}^{j} \left(i(\mathsf{k}+n) - m\right) \mathsf{e}^{(m-1)c}. \tag{4.3.56}$$

Combining this with (4.3.53) shows that (4.3.51) being true for j implies that (4.3.51) is also true for $j+1$ and therefore, by induction, we have our desired result. ■

Continuing the proof of Proposition 4.3.18, substituting $j = n$ into (4.3.51) gives

$$G^{-}_{(n)} (G^{+}_{-1})^m = (-1)^{n+1} m \prod_{i=1}^{n} \left(i(\mathsf{k}+n) - m\right) (G^{+}_{-1})^{m-1}, \tag{4.3.57}$$

and from here it is clear that the right-hand-side vanishes exactly when $i(\mathsf{k}+n) = m$ for some $i \in \{1, \dots, n\}$ as required. ■

Substituting $n = 1$ and 2 reproduces the conditions for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ and BP_{k} respectively. With this result in hand, we are in a position to answer the question of when our embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ descends to an embedding of simple quotients. Let ψ_{k} denote the composition of the embedding $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi$ with the projection from $\mathsf{W}^{\mathsf{k}}_{n+1}$ to its simple quotient $\mathsf{W}_{n+1,\mathsf{k}}$. That is,

$$\psi_{\mathsf{k}} : \overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{n+1} \otimes \Pi \twoheadrightarrow \mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi. \tag{4.3.58}$$

This is clearly non-zero as image of the vacuum of $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ is the vacuum of $\mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi$. In terms of ψ_{k}, the question at hand then becomes: for which k is $\psi_{\mathsf{k}}(\overline{\mathsf{W}}^{\mathsf{k}}_{n+1})$ simple, i.e. $\psi_{\mathsf{k}}(\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}) \simeq \overline{\mathsf{W}}_{n+1,\mathsf{k}}$?

Theorem 4.3.20. *The simple quotient* $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$ *embeds into* $\mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi$ *if and only if* $i(\mathsf{k}+n) \notin \mathbb{Z}_{\geqslant 1}$ *for all* $i \in \{1,\dots,n\}$.

Proof. The proof of this statement is very similar to the $n = 2$ case presented in Theorem 6.2 of [4]. Indeed the only modification in our present context is the slightly different definition of the relevant Π-modules and having to care about the action of modes from a few more fields.

Following the proof provided therein, suppose $\psi_{\mathsf{k}}(\overline{\mathsf{W}}^{\mathsf{k}}_{n+1})$ is not simple as a $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module and therefore has a non-zero proper ideal I. As $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-modules,

$$I \subset \mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi = \mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi_0(0). \tag{4.3.59}$$

Applying the spectral flow $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-automorphism σ^{-1} to both sides and observing that spectral flow can be realised using $\mathsf{W}^{\mathsf{k}}_{n+1}$- and Π-automorphisms according to $\sigma^{-1} = \mathrm{id}_r \otimes \rho^{-1}$,

$$\sigma^{-1}(I) \subset \mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi_{-1}(0) = \mathsf{W}_{n+1,\mathsf{k}}(-1,0). \tag{4.3.60}$$

As $\mathsf{W}_{n+1,\mathsf{k}}$ is an irreducible weight $\mathsf{W}^{\mathsf{k}}_{n+1}$-module, $\mathsf{W}_{n+1,\mathsf{k}}(-1,0)$ is an almost-irreducible $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$-module by Corollary 4.3.15. The top space of $\mathsf{W}_{n+1,\mathsf{k}}(-1,0)$ is spanned by the vectors $\mathbb{1}_R \otimes \mathsf{e}^{-j+mc}$. Hence there exists $m \in \mathbb{Z}$ such that $\mathbb{1}_R \otimes \mathsf{e}^{-b+mc} \in \sigma^{-1}(I)$ and therefore after applying spectral flow again $\mathbb{1}_R \otimes \mathsf{e}^{mc} \in I$.

As $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ contains only fields of nonnegative conformal weight, it must be that $m \geqslant 0$. Even better, if $m = 0$ then I contains the vacuum of $\mathsf{W}^{\mathsf{k}}_{n+1}$ and therefore $I = \psi_{\mathsf{k}}(\overline{\mathsf{W}}^{\mathsf{k}}_{n+1})$ which is a contradiction. Take $m > 0$ to be minimal satisfying $\mathbb{1}_R \otimes \mathsf{e}^{mc} \in I$. In particular, $\mathbb{1}_R \otimes \mathsf{e}^{(m-1)c} \notin I$.

As $\psi_{\mathsf{k}}((G^+)^m) = \mathbb{1}_R \otimes \mathsf{e}^{mc}$ is annihilated by all J_s, L_s, $(U_i)_s$, G^+_{s-1} and G^-_s with $s > 0$, it is a singular vector in I. As the embedding of Theorem 4.3.2 sends $(G^+)^\ell$ to $\mathbb{1}_R \otimes \mathsf{e}^{\ell c}$ for all $\ell \geqslant 0$ and composing with the projection map $\mathsf{W}^{\mathsf{k}}_{n+1} \twoheadrightarrow \mathsf{W}_{n+1,\mathsf{k}}$ leaves Π untouched, $\psi_{\mathsf{k}}((G^+)^{m-1}) = \mathbb{1}_R \otimes \mathsf{e}^{(m-1)c}$ is non-zero. Hence $(G^+)^m$ is singular in $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. By Proposition 4.3.18, if $i(\mathsf{k}+n) \neq m$ for some $i \in \{1,\dots,n\}$ then this cannot occur and therefore $\psi_{\mathsf{k}}(\overline{\mathsf{W}}^{\mathsf{k}}_{n+1})$ is simple.

For the converse, if $i(\mathsf{k}+n) = m$ for some $i \in \{1,\dots,n\}$ then $(G^+)^m$ is singular in $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$. Then $\psi_{\mathsf{k}}((G^+)^m) \neq 0$ is singular in $\psi_{\mathsf{k}}(\overline{\mathsf{W}}^{\mathsf{k}}_{n+1})$ and therefore $\psi_{\mathsf{k}}(\overline{\mathsf{W}}^{\mathsf{k}}_{n+1})$ is not simple. ∎

Anticipating that admissible levels are both interesting and important from a logarithmic conformal field theory point of view, it is useful to know when admissible-level $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$ is related to admissible-level $\mathsf{W}_{n+1,\mathsf{k}}$ using the inverse quantum hamiltonian reduction of Theorem 4.3.20. An admissible level k for $\mathfrak{sl}_{n+1}$ is one that satisfies

$$\mathsf{k}+n+1 = \frac{\mathsf{u}}{\mathsf{v}}, \text{ where } \mathsf{u} \in \mathbb{Z}_{\geqslant n+1}, \mathsf{v} \in \mathbb{Z}_{\geqslant 1} \text{ and } \gcd\{\mathsf{u},\mathsf{v}\} = 1. \tag{4.3.61}$$

Corollary 4.3.21. *Let* $\mathsf{k} = -n - 1 + \frac{\mathsf{u}}{\mathsf{v}}$ *be admissible.* $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$ *embeds into* $\mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi$ *if and only if* $\mathsf{v} > n$ *(i.e.* k *is nondegenerate admissible). That is,*

$$\overline{\mathsf{W}}_{n+1}(\mathsf{u},\mathsf{v}) \hookrightarrow \mathsf{W}_{n+1}(\mathsf{u},\mathsf{v}) \otimes \Pi. \tag{4.3.62}$$

Proof. Suppose that $\mathsf{k} = -n - 1 + \frac{\mathsf{u}}{\mathsf{v}}$ with $\mathsf{v} \leqslant n$. Then $v(\mathsf{k}+n) = \mathsf{u} - \mathsf{v} \in \mathbb{Z}_{\geqslant 1}$ as $\mathsf{u} \in \mathbb{Z}_{\geqslant n+1}$. So for such k, $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$ does not embed into $\mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi$. If $\mathsf{v} > n$, $i(\mathsf{k}+n) = i\frac{\mathsf{u}}{\mathsf{v}} - i$ is not an integer for all $i \in \{1, \dots, n\}$ so $\overline{\mathsf{W}}_{n+1}(\mathsf{u},\mathsf{v}) \hookrightarrow \mathsf{W}_{n+1}(\mathsf{u},\mathsf{v}) \otimes \Pi$ by Theorem 4.3.20. ■

Theorem 4.3.20 also shows there exists an embedding $\overline{\mathsf{W}}_{n+1,\mathsf{k}} \hookrightarrow \mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi$ for certain non-admissible levels. This includes fractional levels of the form $\mathsf{k} = -n - 1 + \frac{n}{n+1}$, at which $\mathsf{W}^{\mathsf{k}}_{n+1}$ is reducible [14].

Mirroring the construction of relaxed modules for the universal subregular W-algebra $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ using the inverse reduction embedding, relaxed modules for the simple subregular W-algebra $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$ can be constructed when the embedding $\overline{\mathsf{W}}_{n+1,\mathsf{k}} \hookrightarrow \mathsf{W}_{n+1,\mathsf{k}} \otimes \Pi$ exists.

At non-admissible level, the representation theory of the affine vertex operator algebras and W-algebras is largely mysterious. Despite this, Proposition 4.3.16 shows that $\overline{\mathsf{W}}_{n+1,\mathsf{k}}$ admits infinitely many irreducible modules of the form $\mathsf{W}_{n+1,\mathsf{k}}(-1,\lambda)$ for $\mathsf{k} = -n-1+\frac{n}{n+1}$, and likely for many other non-admissible levels.

Returning to a nondegenerate admissible level $\mathsf{k} = -n - 1 + \frac{\mathsf{u}}{\mathsf{v}}$, $\mathsf{W}_{n+1,\mathsf{k}} = \mathsf{W}_{n+1}(\mathsf{u},\mathsf{v})$ is rational and hence has finitely many irreducible modules denoted by $\mathcal{W}_\gamma$ for $\gamma \in \mathrm{Pr}^k_{\mathcal{W}}$ as described in Section 4.1.1. Therefore by Corollary 4.3.21 and Proposition 4.3.16, the $\overline{\mathsf{W}}_{n+1}(\mathsf{u},\mathsf{v})$-module $\mathcal{W}_\gamma(-1,\lambda)$ is an indecomposable relaxed highest-weight module that is almost-always irreducible. This shows that $\overline{\mathsf{W}}_{n+1}(\mathsf{u},\mathsf{v})$ is nonrational in the category of weight modules.

The relaxed modules $\mathcal{W}_\gamma(-1,\lambda)$ play the role of 'standard modules' in the computation of modular transformations and fusion rules of $\overline{\mathsf{W}}_{n+1}(\mathsf{u},\mathsf{v})$ when $n = 1$ [52] and $n = 2$ (Section 3.2). Much of the representation-theoretic structure present in those cases is present for all n. It is therefore expected that the relaxed $\overline{\mathsf{W}}_{n+1}(\mathsf{u},\mathsf{v})$-modules $\mathcal{W}_\gamma(-1,\lambda)$ will be essential in computing logarithmic conformal field theoretic data when $n > 2$ as well.

4.4. Beyond Subregular W-Algebras

In Section 4.3, we saw how to define an inverse quantum hamiltonian reduction from $\mathfrak{sl}_{n+1}$ regular W-algebras to $\mathfrak{sl}_{n+1}$ subregular W-algebras. A useful way of thinking about this example is that by bosonising the $\beta\gamma$ ghost vertex algebra, we have enough 'room' to change a nontrivial screening operator $(\beta(z)\mathrm{e}^{-1/(\mathsf{k}+n+1)\alpha_1}(z))$ into a regular-like screening operator $(\mathrm{e}^{-1/(\mathsf{k}+n+1)\tilde{\alpha}_1}(z))$.

Another place where the screening operator $\beta(z)\mathsf{e}^{-1/(\mathsf{k}+n+1)\alpha_1}(z)$ appears is in the *Wakimoto realisation* of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ [64, 157]. This is an embedding $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1}) \hookrightarrow \mathsf{H}_\alpha \otimes \mathsf{B}^{\otimes \frac{n(n+1)}{2}}$ for noncritical level k.

The Wakimoto realisation of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ can be described in terms of n screening operators $\int S_i(z)\mathrm{d}z$ acting on $\mathsf{H}_\alpha \otimes \mathsf{B}^{\otimes \frac{n(n+1)}{2}}$: Following [71], let N_+ be the Lie subgroup of $\mathsf{SL}(n+1)$ corresponding to the upper nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{sl}_{n+1}$. That is, N_+ consists of all matrices of the form

$$\begin{bmatrix} 1 & x_1 & x_{n+1} & \cdots & x_{\frac{1}{2}(n^2+n-4)} & x_{\frac{n(n+1)}{2}} \\ 0 & 1 & x_2 & & & x_{\frac{1}{2}(n+2)(n-1)} \\ \vdots & & & \ddots & & \vdots \\ & & & & x_{n-1} & x_{2n-1} \\ 0 & & & & 1 & x_n \\ 0 & 0 & & \cdots & 0 & 1 \end{bmatrix}, \tag{4.4.1}$$

where $x_1, \dots, x_{\frac{n(n+1)}{2}} \in \mathbb{C}$. The right action of N_+ on N_+ by matrix multiplication induces an antihomomorphism $\rho^R : \mathfrak{n}_+ \longrightarrow \mathcal{D}(N_+)$ where $\mathcal{D}(N_+)$ is the ring of differential operators on N_+. Denote the image of $a \in \mathfrak{n}_+$ by $\rho^R(a) \in \mathbb{C}[x_j, \frac{\partial}{\partial x_j}]$ and let e_{α_i} be the positive root vector in $\mathfrak{n}_+ \subset \mathfrak{sl}_{n+1}$ corresponding to the i'th simple root of $\mathfrak{sl}_{n+1}$. We can write

$$\rho^R(e_{\alpha_i}) = \frac{\partial}{\partial x_i} + \sum_{j=1}^{\frac{n(n+1)}{2}} P_j^{R,i}\left(x_1, \dots, x_{\frac{n(n+1)}{2}}\right)\frac{\partial}{\partial x_j} \tag{4.4.2}$$

for some polynomials $P_j^{R,i} \in \mathbb{C}[x_1, \dots, x_{\frac{n(n+1)}{2}}]$ [71, Sec. 1.5]. The screening fields $S_i(z)$, $i \in \{1, \dots, n\}$, of the Wakimoto realisation of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ are given by

$$S_i(z) = :\left(\beta_i(z) + \sum_{j=1}^{\frac{n(n+1)}{2}} P_j^{R,i}\left(\gamma_1(z), \dots, \gamma_{\frac{n(n+1)}{2}}(z)\right)\beta_j(z)\right)\mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_i}(z):, \tag{4.4.3}$$

where $\beta_j(z), \gamma_j(z)$ are the fields of the $j'th$ copy of B in $\mathsf{B}^{\otimes \frac{n(n+1)}{2}}$. Here, the fields $\beta_j(z)$ and $\gamma_j(z)$ satisfy the operator product expansions of $\beta(z)$ and $\gamma(z)$ in (2.1.3), rather than (1.1.21) in order to be consistent with [71, 119]. So, the image of the embedding $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1}) \hookrightarrow \mathsf{H}_\alpha \otimes \mathsf{B}^{\otimes \frac{n(n+1)}{2}}$ of the Wakimoto realisation is specified, for generic k, by

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1}) \simeq \bigcap_{i=1}^{n} \ker \int S_i(z)\,\mathrm{d}z \subset \mathsf{H}_\alpha \otimes \mathsf{B}^{\frac{n(n+1)}{2}}. \tag{4.4.4}$$

With respect to the choice of (homogeneous) coordinates in (4.4.1), the first few screening fields of the Wakimoto realisation are

$$
\begin{aligned}
S_1(z) &= :\beta_1(z)\mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_1}(z):, \\
S_2(z) &= :\big(\beta_2(z) + \gamma_1(z)\beta_{n+1}(z)\big)\mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_2}(z):, \\
S_3(z) &= :\big(\beta_3(z) + \gamma_2(z)\beta_{n+2}(z) + \gamma_{n+1}(z)\beta_{2n}(z)\big)\mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_3}(z):.
\end{aligned} \tag{4.4.5}
$$

Explicit formulae for the other screening operators can be found in [119].

Choosing different coordinates on N_+ (as long as they are homogeneous in the sense defined in [71]) might result in different expressions for the screening operators $S_i(z)$. However the image of the screening operators obtained from $S_i(z)$ is independent, up to canonical isomorphism, of the choice of homogeneous coordinates. The Wakimoto realisation can be defined in a coordinate-independent way, but this level of generality is not required for our purposes.

We can rewrite the free-field realisation (4.2.9) of the type-A subregular W-algebra $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ in the notation of the Wakimoto realisation as

$$
\overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \simeq \left(\ker \int S_1(z)\, \mathrm{d}z\right) \cap \left(\bigcap_{i=2}^{n} \ker \int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_i}(z)\, \mathrm{d}z\right) \subset \mathsf{H}_\alpha \otimes \mathsf{B}. \tag{4.4.6}
$$

The regular and subregular W-algebras of type A are contained in a larger, more mysterious class of W-algebras known as *hook-type* W-algebras of type A. These are the $\mathfrak{sl}_{n+1}$ W-algebras specified by nilpotent orbits whose partitions are of the form $(A, 1^B)$ where $A + B = n + 1$. Let $m \in \mathbb{Z}$ such that $n + 1 \geqslant m \geqslant 1$.

Define the nilpotent element

$$
f^{(m)} = \sum_{i=m}^{n} f_{\alpha_i}. \tag{4.4.7}
$$

and $f^{(n+1)} = 0$. The nilpotent orbit of $\mathfrak{sl}_{n+1}$ containing $f^{(m)}$ is $\mathbb{O}_{(n-m+2,1^{m-1})}$.

Definition 4.4.1. *The* hook-type W-algebra *corresponding to the partition* $(n-m+2, 1^{m-1})$ *of* $n+1$ *is the quantum hamiltonian reduction* $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(m)})$ *of* $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$

For example, the hook-type W-algebras $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(1)})$ and $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(2)})$ are the regular W-algebra $\mathsf{W}^{\mathsf{k}}_{n+1}$ and subregular W-algebra $\overline{\mathsf{W}}^{\mathsf{k}}_{n+1}$ respectively. The affine vertex algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ is isomorphic to $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(n+1)})$.

Hook-type W-algebras (at generic levels) all admit screening operator descriptions related to the Wakimoto realisation [42]:

$$
\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(m)}) \simeq \left(\bigcap_{i=1}^{m-1} \ker \int S_i(z)\, \mathrm{d}z\right) \cap \left(\bigcap_{j=m}^{n} \ker \int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_j}(z)\, \mathrm{d}z\right) \subset \mathsf{H}_\alpha \otimes \mathsf{B}^{\frac{m(m-1)}{2}}. \tag{4.4.8}
$$

In Section 4.3, we showed how to take the first step up the partial ordering of $\mathfrak{sl}_{n+1}$ W-algebras using inverse quantum hamiltonian reduction from regular W-algebras to subregular W-algebras. A natural question to ask is where to go from here. In the $n = 2$ case, the only remaining W-algebra is $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ and the inverse quantum hamiltonian reduction from the subregular W-algebra to $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3)$ is known [3].

For larger $n > 2$, there are a number of choices for the W-algebras involved in the inverse reduction. For $\mathfrak{sl}_4$, the affine $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_4)$, the minimal $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_4, f^{(3)})$ and the rectangular $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_4, f_{(2,2)})$ all have nilpotent orbits less than that of the subregular $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_4, f^{(2)})$ in the Chevalley ordering.

As the screening operators for hook-type W-algebras are particularly nice, we will focus on inverse quantum hamiltonian reduction from the subregular W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(2)})$ to the hook-type W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(3)})$. To ensure that such a hook-type W-algebra exists, assume that $n \geqslant 3$ in $\mathfrak{sl}_{n+1}$.

The existence argument is a simple generalisation of that used for $\mathfrak{sl}_3$ in [3] and is reminiscent of the argument used in Section 4.3.1.

4.4.1. From Subregular W-Algebras to Hook-Type W-Algebras. Substituting $m = 3$ into (4.4.8) and using (4.4.5), we have an embedding $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(3)}) \hookrightarrow \mathsf{H}_\alpha \otimes \mathsf{B}^{\otimes 3}$ for $\mathsf{k} \neq -n-1$ whose image is specified (for generic k) by

$$\begin{aligned} \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(3)}) \simeq &\left(\ker \int \beta_1(z) \mathrm{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_1}(z)\,\mathrm{d}z\right) \\ &\cap \left(\ker \int \left(\beta_2(z) + \gamma_1(z)\beta_3(z)\right) \mathrm{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_2}(z)\,\mathrm{d}z\right) \\ &\cap \left(\bigcap_{i=3}^{n} \ker \int \mathrm{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_i}(z)\,\mathrm{d}z\right) \subset \mathsf{H}_\alpha \otimes \mathsf{B}^{\otimes 3}. \end{aligned} \tag{4.4.9}$$

Guided by the approach taken in [3], we apply FMS bosonisation to $\beta_3(z), \gamma_3(z)$. The same argument used for the regular-subregular case in Section 4.3.1 allows us to describe the resulting embedding $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(3)}) \hookrightarrow \mathsf{H}_\alpha \otimes \Pi \otimes \mathsf{B}^{\otimes 2}$ for generic k as the intersection of kernels of screening operators according to

$$\begin{aligned} \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(3)}) \simeq &\left(\ker \int \beta_1(z) \mathrm{e}^{\frac{-1}{\mathsf{k}+n+1}\tilde{\alpha}_1}(z)\,\mathrm{d}z\right) \\ &\cap \left(\ker \int \left(\beta_2(z)\mathrm{e}^{-c}(z) + \gamma_1(z)\right) \mathrm{e}^{\frac{-1}{\mathsf{k}+n+1}\tilde{\alpha}_2}(z)\,\mathrm{d}z\right) \\ &\cap \left(\bigcap_{i=3}^{n} \ker \int \mathrm{e}^{\frac{-1}{\mathsf{k}+n+1}\tilde{\alpha}_i}(z)\,\mathrm{d}z\right) \cap \left(\ker \int \mathrm{e}^{\frac{1}{2}c+\frac{1}{2}d}(z)\,\mathrm{d}z\right), \end{aligned} \tag{4.4.10}$$

where $\tilde{\alpha}_2(z) = \alpha_2(z) - (\mathsf{k}+n+1)c(z)$ and $\tilde{\alpha}_i(z) = \alpha_i(z)$ for $i = 1$ and $i \geqslant 3$. Comparing the free-field content of (4.4.10) and (4.2.9), we expect the inverse quantum hamiltonian reduction to be of the form

$$\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(3)}) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(2)}) \otimes \Pi \otimes \mathsf{B}. \tag{4.4.11}$$

To obtain screening operators for the subregular W-algebra from (4.4.10), we use yet another free-field realisation for $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(2)})$ obtained by Feigin and Semikhatov in [68]: There is an embedding $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(2)}) \hookrightarrow \mathsf{H}_{\alpha} \otimes \mathsf{B}$ for $\mathsf{k} \neq -n-1$ whose image is specified by

$$\begin{aligned} \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(2)}) = \overline{\mathsf{W}}^{\mathsf{k}}_{n+1} \simeq &\left(\ker \int \beta(z)\mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_1}(z)\,\mathrm{d}z\right) \\ &\cap \left(\ker \int \gamma(z)\mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_2}(z)\,\mathrm{d}z\right) \\ &\cap \left(\bigcap_{i=2}^{n} \ker \int \mathsf{e}^{\frac{-1}{\mathsf{k}+n+1}\alpha_i}(z)\,\mathrm{d}z\right) \end{aligned} \tag{4.4.12}$$

for generic k. Let $\mathsf{H}_{\tilde{\alpha}} \subset \mathsf{H}_{\alpha} \otimes \Pi \otimes \mathsf{B}^{\otimes 2}$ be the vertex subalgebra generated by $\tilde{\alpha}_1(z), \dots, \tilde{\alpha}_n(z)$. It is easy to see that $\mathsf{H}_{\tilde{\alpha}} \simeq \mathsf{H}_{\alpha}$. Let $\tilde{\mathsf{B}}$ be the vertex subalgebra of $\mathsf{H}_{\alpha} \otimes \Pi \otimes \mathsf{B}^{\otimes 2}$ isomorphic to B generated by

$$\tilde{\beta}(z) = \beta_1(z), \quad \tilde{\gamma}(z) = \gamma_1(z) + \beta_2(z)\mathsf{e}^{-c}(z). \tag{4.4.13}$$

Finally, let $\widehat{\mathsf{B}}$ and $\widehat{\Pi}$ be the vertex subalgebras of $\mathsf{H}_{\alpha} \otimes \Pi \otimes \mathsf{B}^{\otimes 2}$ generated by

$$\widehat{\beta}(z) = \beta_2(z), \quad \widehat{\gamma}(z) = \gamma_2(z) + \beta_1(z)\mathsf{e}^{-c}(z) \tag{4.4.14}$$

and

$$\begin{aligned} \widehat{c}(z) = c(z), \quad &\mathsf{e}^{m\widehat{c}}(z) = \mathsf{e}^{mc}(z), \\ \widehat{d}(z) = d(z) - (\mathsf{k}+n+1)\frac{2(n-1)}{n+1}c(z) &- 2\beta_2(z)\beta_1(z)\mathsf{e}^{-c}(z) + 2\omega_2(z). \end{aligned} \tag{4.4.15}$$

respectively, where $\omega_2(z)$ is the field associated to the second fundamental coweight of $\mathfrak{sl}_{n+1}$. A straightforward calculation shows that $\mathsf{H}_{\alpha} \otimes \Pi \otimes \mathsf{B}^{\otimes 2} = \mathsf{H}_{\tilde{\alpha}} \otimes \tilde{\mathsf{B}} \otimes \widehat{\Pi} \otimes \widehat{\mathsf{B}}$. The definitions of $\mathsf{H}_{\tilde{\alpha}}$, $\tilde{\mathsf{B}}$, $\widehat{\mathsf{B}}$ and $\widehat{\Pi}$ for $n = 2$ can also be found in [3].

The first n screening operators of (4.4.10) act non trivially only on $\mathsf{H}_{\tilde{\alpha}} \otimes \tilde{\mathsf{B}}$. In fact, if we rewrite the first n screening operators in (4.4.10) in terms of the fields of $\mathsf{H}_{\tilde{\alpha}} \otimes \tilde{\mathsf{B}} \otimes \widehat{\Pi} \otimes \widehat{\mathsf{B}}$, we get exactly the tilded versions of the screening operators (4.4.12).

Theorem 4.4.2. *Let* k *be generic. There exists an embedding*

$$\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(3)}) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(2)}) \otimes \Pi \otimes \mathsf{B}, \tag{4.4.16}$$

whose image is specified by

$$\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(3)}) \simeq \ker \int \mathsf{e}^{A}(z)\, \mathrm{d}z, \tag{4.4.17}$$

where

$$A(z) = \frac{1}{2}\left(1 - 2(\mathsf{k}+n+1)\frac{n-1}{n+1}\right)c(z) + \frac{1}{2}d(z) + \beta^{(2)}(z)\beta(z)\mathsf{e}^{-c}(z) - \omega_2(z). \tag{4.4.18}$$

Here, $\beta(z)$ refers to the ghost field in B and the fields $\beta^{(2)}(z)$ and $\omega_2(z)$ act on the subregular W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(2)})$ by way of the free-field realisation (4.2.9). That is, $\beta^{(2)}(z)$ denotes the ghost field in the free-field realisation $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(2)}) \hookrightarrow \mathsf{H}_{\alpha} \otimes \mathsf{B}$.

Proof. The argument here is the same as that in the proof of Theorem 4.3.2. ∎

When $n = 2$, this is the embedding described in [3]. For $n = 3$, the embedding of Theorem 4.4.2 is an inverse quantum hamiltonian reduction from the subregular $\mathfrak{sl}_4$ W-algebra to the minimal $\mathfrak{sl}_4$ W-algebra.

It would be interesting to explore the representation theory of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(3)})$ using Theorem 4.4.2 in the same way as the subregular W-algebra was explored in Section 4.3.3. One would expect that the relaxed and logarithmic modules present in the $n = 2$ case from [3] are present for all n.

Chapter 5

Conclusion

5.1. Summary of Results

In this thesis, we have developed the representation theory of examples of nonrational W-algebras in various levels of detail.

The first W-algebras studied here were Bershadsky–Polyakov algebras. The first result of Chapter 2 was the explicit determination of their untwisted and twisted Zhu algebras. Then, the classification of simple modules for these algebras gave a classification of simple relaxed highest-weight modules of the universal Bershadsky–Polyakov algebra BP^{k} (Theorems 2.2.6 and 2.2.16) by Zhu's theorem and its twisted analogue. In addition to these, several families of reducible-but-indecomposable BP^{k}-modules were identified.

Determining when the above BP^{k}-modules are modules for the simple quotient BP_{k} is difficult in general. However we found that specialising to admissible levels of the form $\mathsf{k} = \frac{\mathsf{u}}{\mathsf{v}} - 3$, where $\mathsf{u} \geqslant 3$ and $\mathsf{v} \geqslant 2$ are coprime, allowed us to leverage results of Arakawa regarding the minimal quantum hamiltonian reduction functor [13]. In particular, we showed that all untwisted highest-weight $\mathsf{BP}_{\mathsf{k}} = \mathsf{BP}(\mathsf{u},\mathsf{v})$-modules are obtained from highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_3)$-modules by quantum hamiltonian reduction (Theorems 2.3.8 and 2.3.16).

Combining this surjectivity result with spectral flow of $\mathsf{BP}(\mathsf{u},\mathsf{v})$ and coherent families of $\mathsf{Zhu}^{\mathsf{tw}}\big[\mathsf{BP}(\mathsf{u},\mathsf{v})\big]$-modules allowed us to classify simple (untwisted and twisted) relaxed highest-weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules with finite-dimensional weight spaces (Theorem 2.3.25). A related argument proved the existence of reducible-but-indecomposable positive-energy $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules when $\mathsf{v} \geqslant 3$, and short exact sequences for these modules were found (Theorem 2.3.32).

These results in particular show that $\mathsf{BP}(\mathsf{u},\mathsf{v})$ is a *nonrational* W-algebra when $\mathsf{v} \geqslant 3$. The further analysis of $\mathsf{BP}(\mathsf{u},\mathsf{v})$ at these levels in Chapter 3 was greatly assisted by the inverse quantum hamiltonian reduction relating the nonrational $\mathsf{BP}(\mathsf{u},\mathsf{v})$, the rational Zamolodchikov minimal model $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ and the half lattice vertex algebra Π [4]. In particular, the characters of the (spectral flows of) simple and reducible-but-indecomposable $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules of Chapter 2 were expressed in terms of $\mathsf{W}_3(\mathsf{u},\mathsf{v})$ and Π characters (Proposition 3.2.8 and Corollary 3.3.13).

To determine modular transformations and Grothendieck fusion rules, we used the standard module formalism [51,146]. After modifying the conformal structure of $\mathsf{BP}(\mathsf{u},\mathsf{v})$ for convenience, identifying a suitable class of standard modules (3.2.16) and explaining how to define linearly independent one-point functions of these standard modules, we derived a modular S-matrix for the one-point functions of standard modules (Theorem 3.2.10). The modular transformations for the vacuum module were obtained using the aforementioned character formulae (Corollary 3.3.16).

The (conjectural) standard Verlinde formula (3.3.2b) was then applied to compute (conjectural) Grothendieck fusion rules for the standard modules (Theorem 3.3.19). This is quite a nontrivial calculation, requiring several identities involving W_3 minimal model fusion coefficients. That the conjectural Grothendieck fusion coefficients are nonnegative integers is strong evidence that they are indeed the structure constants of the Grothendieck ring of $\mathsf{BP}(\mathsf{u},\mathsf{v})$. As every simple weight BP_{k}-module is resolved in terms of standard modules (Proposition 3.3.12), this result implies the Grothendieck fusion rules for all $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules from Chapter 2. We also identified interesting simple currents in $\mathsf{BP}(\mathsf{u},\mathsf{v})$ (Proposition 3.3.22) and explored the example of $\mathsf{BP}(3,4)$.

The second family of W-algebras explored were the subregular W-algebras $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{sub}})$ in Chapter 4, examples of which are affine $\mathfrak{sl}_2$ vertex operator algebras and Bershadsky–Polyakov algebras BP^{k}. Utilising free-field realisations for the universal regular W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{reg}})$ [123] and subregular W-algebras of type A [88], and the FMS bosonisation of the $\beta\gamma$ ghost vertex algebra [77], we proved the existence of an inverse quantum hamiltonain reduction embedding generalising that used for BP^{k} (Theorem 4.3.2). Explicit formulae for strong generators for $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{sub}})$ in terms of fields in $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{reg}})$ and the half lattice Π were obtained in Section 4.3.2.

Recalling the importance of $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules obtained from inverse quantum hamiltonian reduction in Chapter 3, we constructed relaxed highest-weight $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{sub}})$-modules by taking the tensor products of irreducible $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{reg}})$-modules with relaxed Π-modules (Proposition 4.3.16). Such modules were shown to exhibit many of the properties present in the analogous modules for BP^{k}. The characters of these relaxed $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{sub}})$-modules were shown to be products of characters for $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{reg}})$- and Π-modules (Corollary 4.3.12).

When the aforementioned inverse quantum hamiltonian reduction embedding descends to an embedding $\mathsf{W}_{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{sub}}) \hookrightarrow \mathsf{W}_{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{reg}}) \otimes \Pi$ of simple quotients was determined (Theorem 4.3.20). This is highly nontrivial as expressions for singular vectors in both regular and subregular W-algebras are difficult to obtain in general. When k is a nondegenerate admissible level for $\mathfrak{sl}_{n+1}$, $\overline{\mathsf{W}}_{n+1}(\mathsf{u},\mathsf{v}) = \mathsf{W}_{\mathsf{k}}(\mathfrak{sl}_{n+1},f_{\text{sub}})$ and its relaxed modules can be realised in terms of the rational minimal model $\mathsf{W}_{n+1}(\mathsf{u},\mathsf{v})$ and Π (Corollary 4.3.21). A consequence of this is that $\overline{\mathsf{W}}_{n+1}(\mathsf{u},\mathsf{v})$ is *nonrational* in the category of weight modules at these levels.

The final family of W-algebras considered in this thesis were hook-type W-algebras of type A. Our final result was proving the existence of an inverse quantum hamiltonian reduction relating the subregular W-algebra of type A to a hook-type W-algebra of type A (Theorem 4.4.2) using free-field realisations related to the Wakimoto realisation of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$.

5.2. Future Directions

5.2.1. Bershadsky–Polyakov Algebras. While our studies of $\mathsf{BP}(\mathsf{u},\mathsf{v})$ were fairly comprehensive, there are many features of the category of weight $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules that remain unexplored. For example, identifying projective covers for the simple $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules defined in Chapter 2 and defining other reducible-but-indecomposable $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules. An example of the latter are staggered modules, which exist for $\mathsf{BP}(5,3)$ due to its relationship with an admissible level $\mathfrak{sl}_2$ minimal model known to exhibit such modules [2,9,62]. This is also the case for $\mathsf{BP}(4,3)$ and $\mathsf{BP}(3,4)$ due to their relationship to the $\beta\gamma$ ghost vertex algebra [5, Sec. 5.2] and triplet algebra [53, 143] respectively. More evidence for staggered $\mathsf{BP}(\mathsf{u},\mathsf{v})$-modules is in the form of the Grothendieck fusion of reducible standard modules (3.3.81).

There are also other levels in BP_{k} that are of interest. Outside of admissible levels, the structure of the maximal ideal of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ is more complicated and therefore so is addressing which of the simple relaxed highest-weight BP^{k}-modules are BP_{k}-modules. It is not immediately clear how much of the proof Theorem 2.3.8 can be adapted in this case but quantum hamiltonian reduction should play a role here too.

A classification of simple modules for the nonadmissible-level BP_{k} where $\mathsf{k} \in \mathbb{Z}_{\geqslant -1}$ is known but uses explicit formulae for singular vectors in BP^{k} at such levels [5, 6] rather than quantum hamiltonian reduction. It would be interesting to see how our results can be adapted to these nonadmissible levels and others.

5.2.2. Subregular W-Algebras. There are aspects of the analysis in Chapter 2 that might generalise to other subregular W-algebras of type A outside of what we have explored in this thesis. For example, the (twisted) Zhu algebra of a W-algebra is isomorphic to the finite W-algebra of the same type [55], and the finite W-algebra corresponding to the subregular nilpotent orbit in $\mathfrak{sl}_{n+1}$ is known to be a central extension of a Smith algebra for all n [140, Thm. 7.10]. As Smith algebras are '$\mathfrak{sl}_2$-like', the classification of simple weight modules for Smith algebras is an easy generalisation of that presented in Section 2.2.5 using constructions in [149].

Therefore, in principle, the simple relaxed highest-weight modules for universal subregular $\mathfrak{sl}_{n+1}$ W-algebras $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$ are known up to isomorphism. However these isomorphisms make it difficult to identify such relaxed highest-weight modules in terms of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$ fields, in the sense of the action of zero modes on the top space. Of course for Bershadsky–Polyakov algebras, we have a presentation (2.2.3) of the finite W-algebra in terms of the (images of) zero modes of BP^{k} fields. Such a presentation of the subregular $\mathfrak{sl}_{n+1}$ finite W-algebra in terms of the zero modes of the fields (4.1.7) may be achievable using a related presentation in [89].

Which of these simple relaxed highest-weight modules are also modules for the *simple* subregular $\mathfrak{sl}_{n+1}$ W-algebra $\mathsf{W}_{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$ is a difficult question in general. Here, the Bershadsky–Polyakov algebra being minimal comes in handy. A classification of highest-weight $\mathsf{W}_{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$ is unknown even in admissible level cases $\overline{\mathsf{W}}_{n+1}(\mathsf{u}, \mathsf{v})$.

One potential means for obtaining such a classification is through inverse quantum hamiltonian reduction: recall that simple relaxed highest-weight $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules are either standard modules (obtained by inverse quantum hamiltonian reduction and spectral flow) or their submodules and simple quotients. In this way, the classification of such $\mathsf{BP}(\mathsf{u}, \mathsf{v})$-modules reduces to constructing the standard modules by inverse quantum hamiltonian reduction and analysing their structure. Whether this strategy works more generally is an open question.

For the subregular W-algebras $\overline{\mathsf{W}}_{n+1}(\mathsf{u}, \mathsf{v})$, we have completed the first step. Determining precisely when the standard modules are reducible is difficult and requires more information about the polynomial $p(\gamma, x)$ in (4.3.48). This difficulty is related the complexity of the operator product expansion between $G^+(z)$ and $G^-(z)$. More detailed knowledge about how the subregular quantum hamiltonian reduction functor acts on highest-weight $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_{n+1})$-modules (analogous to Arakawa's results in the minimal case) will likely assist in this direction.

One area where such an understanding of highest-weight $\overline{\mathsf{W}}_{n+1}(\mathsf{u}, \mathsf{v})$-modules, in particular the vacuum module, is required is in computing modular transformations and Grothendieck fusion rules for $\overline{\mathsf{W}}_{n+1}(\mathsf{u}, \mathsf{v})$. Subject to being able to upgrade $\mathsf{W}_{n+1}(\mathsf{u}, \mathsf{v})$ characters to one-point functions in an appropriate manner (as in Section 3.2.4), a standard S-matrix for $\overline{\mathsf{W}}_{n+1}(\mathsf{u}, \mathsf{v})$ can be straightforwardly obtained from the character formula (4.3.12) and is of the expected form: the S-matrix of the corresponding $\mathsf{W}_{n+1}(\mathsf{u}, \mathsf{v})$-modules multiplied by an exponential containing the 'Π data' of the standard modules involved.

The natural next step from here would be to compute the Grothendieck fusion rules, which requires the S-matrix elements corresponding to the vacuum module $\overline{\mathsf{W}}_{n+1}(\mathsf{u}, \mathsf{v})$. While the structure of the vacuum module should be simpler than that of other highest-weight $\overline{\mathsf{W}}_{n+1}(\mathsf{u}, \mathsf{v})$-modules (c.f. the vacuum module being type-3 for $\mathsf{BP}(\mathsf{u}, \mathsf{v})$ from Section 2.3.3), much is still unknown in this direction.

One example of subregular minimal models of particular interest is $\overline{\mathsf{W}}_{n+1}(n+1, n+2)$, which is isomorphic to the logarithmic $\mathcal{B}_{n+1}$-algebra [25, 53]. Tensor categories related to $\mathcal{B}_{n+1}$ that are braided, rigid and non-semisimple have been constructed using a conjectural relationship between $\mathcal{B}_{n+1}$ and the unrolled restricted quantum groups of $\mathfrak{sl}_2$ [25]. It would be interesting to see if our preliminary representation-theoretic results for $\overline{\mathsf{W}}_{n+1}(n+1, n+2)$ are able to reproduce this categorical data, and if such data can shed light on the Grothendieck fusion rules of the standard modules of $\overline{\mathsf{W}}_{n+1}(n+1, n+2)$.

There are of course also non-type-A subregular W-algebras that might be accessible using inverse quantum hamiltonian reductions and the free-field approach. One such example is the case of the subregular $\mathfrak{sp}_4$ W-algebra. An embedding involving the the regular $\mathfrak{sp}_4$ W-algebra and the half lattice was proposed in [30]. An identical screening operator argument to that used in Theorem 4.3.2 provides an alternative construction of this embedding. The representation-theoretic content of the subregular $\mathfrak{sp}_4$ W-algebra inverse reduction remains to be explored, as well as looking at other subregular W-algebras.

In general types, subregular W-algebras are an important class of W-algebras from a number of perspectives. As mentioned previously, subregular W-algebras appear in the Schur index of 4D superconformal field theories known as Argyres–Douglas theories [24, 29, 41]. The ADE classification of simple surface singularities connects to the ADE classification of simply-laced Lie algebras through the geometry of the Slodowy slice corresponding to the subregular nilpotent orbit [148], and this connection is expressed beautifully in the associated variety of the subregular W-algebra [16]. Our new results on the structure and representation theory of subregular W-algebras therefore might have implications in these areas.

5.2.3. Other W-Algebras. Of course if we want to use inverse quantum hamiltonian reductions to learn about admissible-level affine vertex operator algebras, we eventually need to move on from subregular W-algebras.

The $n = 2$ version of the hook-type inverse quantum hamiltonian reduction in Theorem 4.4.2 was used in [3] to construct logarithmic and relaxed modules for $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_3)$. Obtaining such results for $\mathsf{W}^\mathsf{k}(\mathfrak{sl}_{n+1}, f^{(3)})$ for $n \geqslant 3$ requires an understanding of when the embedding of Theorem 4.4.2 descends to an embedding of simple quotients.

The difficulty here is that unlike in the subregular case, expressions for strong generators of $\mathsf{W}^\mathsf{k}(\mathfrak{sl}_{n+1}, f^{(3)})$ in terms of fields in $\mathsf{W}_\mathsf{k}(\mathfrak{sl}_{n+1}, f_{\text{sub}})$, Π and B are not known. Subject to determining the restrictions on the level k to get an embedding of simple quotients and making this inverse reduction more explicit, similar logarithmic and relaxed modules for $\mathsf{W}_\mathsf{k}(\mathfrak{sl}_{n+1}, f^{(3)})$ might be obtainable using a similar approach as [3].

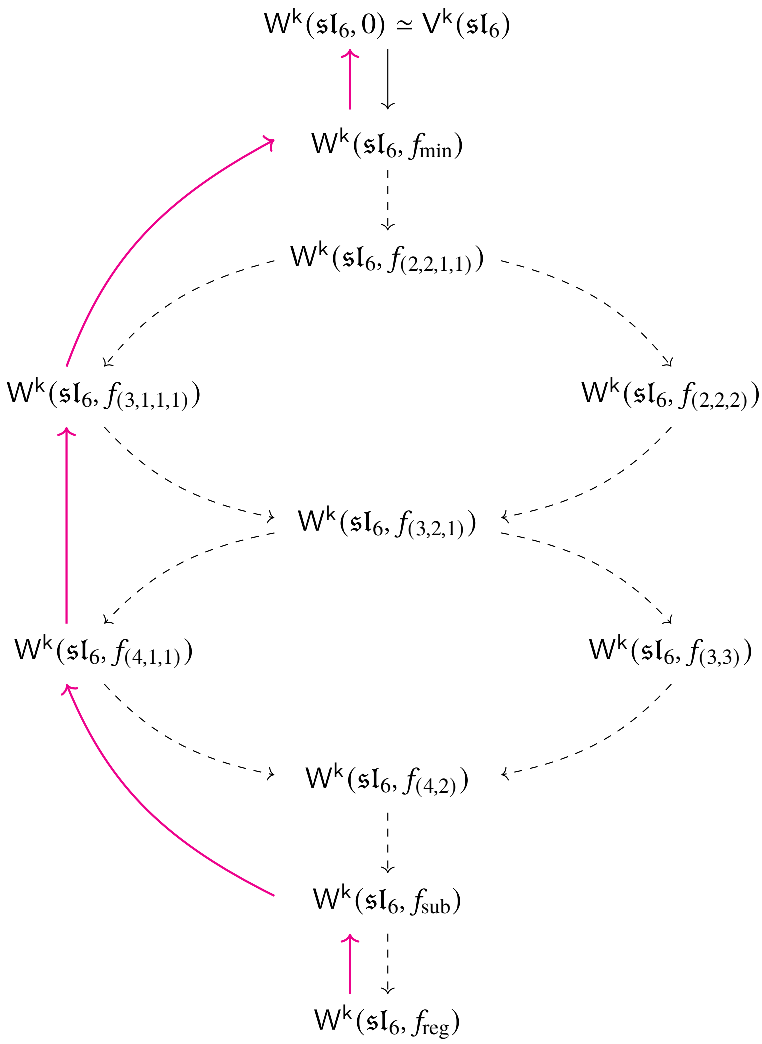

FIGURE 1. Inverse quantum Hamiltonian reductions for hook-type $\mathfrak{sl}_6$ W-algebras are depicted by the pink upwards arrows.

The systematic construction of the inverse quantum hamiltonian reductions in Sections 4.3.1 and 4.4.1 using screening operators suggests a potential path of inverse reductions starting from the regular $\mathfrak{sl}_{n+1}$ W-algebra to the affine vertex algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$.

Consider the $\mathfrak{sl}_6$ case, where the ordering of W-algebras is presented in Figure 1. Starting from $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{\mathrm{reg}})$, the first pink arrow represents the inverse quantum hamiltonian reduction described in Theorem 4.3.2, while the second represents that of Theorem 4.4.2. Interestingly, the latter skips the W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_6, f_{(4,2)})$.

It therefore appears that hook-type W-algebras define a traversable path from a regular W-algebra to an affine vertex algebra, along which inverse quantum hamiltonian reductions can be described using the Wakimoto realisation of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ and the screening operators of [88].

In order to better understand what the inverse quantum hamiltonian reductions along the path of hook-type W-algebras look like, the Wakimoto-type free-field realisation of hook-type W-algebras

[42] takes the form of embeddings

$$\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(m)}) \hookrightarrow \mathsf{H}_\alpha \otimes \mathsf{B}^{\frac{m(m-1)}{2}}. \tag{5.2.1}$$

Hence the free-field realisation of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(m+1)})$ requires m more copies of the $\beta\gamma$ ghost vertex algebra B than the free-field realisation of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(m)})$. In Theorem 4.3.2 and Theorem 4.4.2, it was only necessary to bosonise a single B in order to define an inverse quantum hamiltonian reduction embedding.

Conjecture. *Let* $\mathsf{k} \neq -n-1$. *There exists an embedding*

$$\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(m+1)}) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(m)}) \otimes \Pi \otimes \mathsf{B}^{\otimes(m-1)}. \tag{5.2.2}$$

The key claim in the Conjecture is that only one $\beta\gamma$ ghost vertex algebra needs to be bosonised in order to define an inverse quantum hamiltonian reduction embedding.

If the Conjecture were true, the embeddings (5.2.2) could be composed. It would therefore be possible to realise the affine vertex algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ in terms of any hook-type W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(m)})$ by way of an embedding, at noncritical k, of the form

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1}) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f^{(m)}) \otimes \Pi^{n-m+1} \otimes \mathsf{B}^{\otimes\frac{1}{2}(n+m-2)(n-m+1)}. \tag{5.2.3}$$

In light of the results of Section 3.2, such embeddings might assist in uncovering the highly sought-after modular transformations and fusion rules of admissible-level $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_{n+1})$.

Finally, there is also likely to be a finite W-algebra analogue of inverse quantum hamiltonian reduction. This is because finite W-algebras can be constructed as the (twisted) Zhu algebra corresponding to W-algebras [55]. Partial reduction for finite W-algebras, at least for type A, has been described [133]. It is likely that the inverse quantum hamiltonian reductions described here are the 'affinisation' of inverses to such partial reductions but there is much work to be done in this direction.

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