The Coulomb Gas Formalism

in Conformal Field Theory

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Abstract

Conformal field theory is an area of theoretical and mathematical physics that has applications in statistical physics, string theory, interacting quantum field theories, and many branches of mathematics. The Coulomb gas formalism is a celebrated technique in conformal field theory that provides effective machinery for describing critical phenomena that arise in the scaling limit of certain two-dimensional statistical lattice models.

In this work we develop the Coulomb gas formalism starting from a bosonic string theory. We demonstrate some applications in statistical field theory by constructing minimal model conformal field theories and relating these to three different critical lattice models, namely the Ising model, the tricritical Ising model, and the Yang-Lee model. We then use the Coulomb gas formalism to analytically compute all of the primary three-point constants and primary operator-product expansion coefficients for the minimal conformal field theories corresponding to these critical models. We compare Coulomb gas calculations with results obtained using other standard techniques, namely bosonisation of the free fermion conformal field theory, and the conformal bootstrap in Liouville theory. We also compare Coulomb gas calculations with a non-standard technique of our own construction, based on the monodromy theory of Fuchsian differential equations. All comparisons are consistent with the Coulomb gas calculations.

The main original contribution of this work is to conjecture and verify a connection between the Coulomb gas formalism and the monodromy theory of certain Fuchsian differential equations. The success of the verification suggests an opportunity to extend Coulomb gas methods beyond conformal field theory and, conversely, for greater application of the monodromy theory of differential equations in conformal field theory.

Declaration

This thesis is an account of research undertaken between February 2015 and October 2015 at the Department of Theoretical Physics, Research School of Physics and Engineering, College of Physical and Mathematical Sciences, The Australian National University, Canberra, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or in part for a degree at any university.

Matthew D. Geleta
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Dedicated to Peter and Sharon

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Introduction

"Seek first to understand..."

Statistical mechanics and critical phenomena, string theory and quantum gravity, monstrous moonshine and modular forms, supersymmetry, condensed matter physics, number theory, representation theory, quantum groups... this is the realm of conformal field theory.

The 1970s and early 1980s were particularly prosperous times for fundamental physics, rife with influential research such as the work of Polyakov [106–109] and Witten [1, 128–130]. The foundations had been laid for another conceptual leap, and this leap was made in 1984 with [9], which made a simple observation — two-dimensional quantum field theories with a particular type of symmetry have *infinite-dimensional* symmetry algebras. The symmetry in question is conformal symmetry and its consequences are profound. This was the birth of conformal field theory as it is known today.

Conformal field theory

A conformal field theory is a two-dimensional quantum field theory with conformal symmetry.

The subsequent years saw conformal field theory ripen into a rich and mature area of theoretical physics. There were a number of drivers for progress. At the time, field theories were playing an increasingly prominent role in statistical physics [88], and here conformal field theories found great application. They were found to describe and classify critical phenomena [122], such as the onset of superconductivity in metals [68] and the spontaneous magnetisation of ferromagnets [78]. On the other hand there were its applications in string theory and the search for a quantum theory of gravity [29, 32, 110]. In later years conformal field theory even found application in describing the entropy of black holes [23, 69]. Moreover, conformal field theory is responsible for great developments in mathematics, such as the birth of vertex algebras and their role in proving [17] the infamous monstrous moonshine conjectures of Conway and Norton [26], for which Borcherds received his 1998 Fields Medal [62]. Conformal field theory still plays a leading role in each of these fields today. It is a true gem of theoretical and mathematical physics.

Our focus will be on applications to statistical physics. In particular, we construct a class of conformal field theories called the *minimal models*. These were conceived in [9], and describe the critical behaviour of many of the famous two-dimensional statistical models in their scaling limits [14] — the Ising model [22] and its variations [56, 86], the Potts models [8, 37], and the Yang-Lee model [21], to name a few. Theoretical physicists know all too well how difficult it can be to deal with these models analytically, a good example being Onsager's solution to the Ising model [100]. Often one defers to approximate numerical methods [11, 19]. We will not — one of the most attractive aspects of conformal field theory is that the problems it deals with are, in principle, exactly solvable.

Statistical field theory

The scaling limit of a statistical lattice model is the limit in which the lattice spacing is taken to zero in an appropriate way (for example by block spin renormalisation [81]) so that particular lattice observables are described by a statistical field theory — these observables are *scaling fields*. It is here that critical phase transitions arise — this is the domain of the renormalisation group and conformal field theory [12, 25].

Phase transitions are a treasured phenomenon in statistical physics. In the thermodynamic limit, a statistical phase transition is a discontinuous change in a thermodynamic quantity as some parameter is varied continuously across a critical value. Some phase transitions, such as the melting of ice, are associated with a latent heat. Here a finite amount of energy must be supplied for the phase transition to occur, and during this process the macroscopic properties do not change. These are the *first-order phase transitions*, so called because the free energy has a discontinuity in one of its first derivatives. There are also *critical phenomena* [91], or *second-order phase transitions* — here there is no latent heat and the first derivatives of the free energy remain continuous, although there is a discontinuity in one of the second derivatives. An example is the spontaneous onset of magnetisation in a ferromagnet surrounded by negligible external magnetic field, which occurs when the temperature is lowered past the *Curie temperature*. It is at these critical points that conformal invariance arises — Polyakov had already demonstrated this as early as 1970 [105], but it took over a decade for the consequences to become widely appreciated.

The thermodynamic properties of a generic statistical lattice model are described by a set of observables, for example the spin or energy density at each lattice site in a model of a ferromagnet. The values taken by observables at different locations (and times) on the lattice are usually not independent, and their inter-dependence is quantified through *correlation functions* — these are particular ensemble averages over lattice configurations. The common lore of critical phenomena is, roughly speaking, that information about the state of one lattice site is propagated to others via inter-site interactions, which causes the observables at each site to become correlated. At high temperatures, thermal fluctuations dominate over interactions between lattice sites. When the distance R_{ij} between two lattice sites i and j is much greater than some characteristic length scale $\xi_{A,B}$ for two observables A and B, the correlation between the observables decays exponentially with increasing distance:

$$\langle A_i B_j \rangle \sim e^{-R_{ij}/\xi_{A,B}}, \qquad R_{ij} \gg \xi_{A,B}. \tag{1}$$

The characteristic length scale $\xi_{A,B}$, or *correlation length*, decreases in the limit of high temperatures. At a critical point the correlation length diverges, the theory becomes scale-invariant, and the correlation functions decay not exponentially, but rather according to a *power law*. Remarkably, under a few reasonable assumptions [104], a locally scale-invariant theory in two dimensions exhibits a far stronger symmetry — conformal symmetry. Strong symmetry severely constrains the possible behaviour of the system, and opens up a range of mathematical tools for theoretical inquiry.

The Ising model

The most famous example of a statistical lattice model is the *Ising model* [93] — an ostensibly innocuous lattice model that exhibits surprisingly non-trivial behaviour. The Ising model describes the behaviour of a simple ferromagnet and, in the thermodynamic limit, exhibits spontaneous magnetisation as the temperature is varied across a critical value. The model possesses observables such as the spin density σ_i and energy density ϵ_j, local to the lattice sites i and j respectively. At the critical temperature, the Ising model correlation functions decay

according to the power laws [45]

$$\langle \sigma_i \sigma_j \rangle \sim \frac{1}{R_{ij}^{d-2+\eta}} \sim \frac{1}{R_{ij}^{2\Delta_\sigma}}, \qquad \langle \epsilon_i \epsilon_j \rangle \sim \frac{1}{R_{ij}^{2(d-1/\nu)}} \sim \frac{1}{R_{ij}^{2\Delta_\epsilon}}, \tag{2}$$

where d is the number of dimensions of the lattice, $\eta, \nu \in \mathbb{R}$ are called *critical exponents*, and $\Delta_\sigma, \Delta_\epsilon \in \mathbb{R}$ are called *scaling dimensions* of the observables σ and ϵ. These values characterise and classify the behaviour of the model at the critical point. For some lattice models, like the Ising model, the critical exponents and scaling dimensions are known exactly [100]; for others they are not, and are usually estimated numerically [11, 19].

One of the great successes of conformal field theory is that it can be used to analytically compute critical exponents and scaling dimensions permitted by conformal symmetry in two-dimensional field theories, such as the scaling limit of a critical lattice model, thereby providing a classification scheme for critical phenomena [55]. Critical exponents are determined by the nature of the interactions between lattice sites, and critical lattice models can be put into *universality classes* [20] based on their critical exponents. For example, in chapter 5 we relate the Ising model (and all models in the same universality class) to a conformal field theory called $\mathcal{M}(3,4)$, which predicts the aforementioned critical exponents, and we use it to compute correlation functions explicitly.

The tricritical Ising model

Some fascinating phenomena emerge when the conventional Ising model is modified by annealing, allowing lattice sites to be vacant (equivalently, permitting spin values of zero). There arises a *tricritical point* at which three different phases can coexist [87]. At this critical point the scaling fields are described by another of the conformal field theories that we will be constructing — a minimal model called $\mathcal{M}(4,5)$ [86]. Conformal field theory provides tools sufficient to determine the critical exponents and scaling dimensions of the scaling fields in this model, and to compute correlation functions analytically. On top of its importance in statistical physics, one of the most interesting features of $\mathcal{M}(4,5)$ is that it is endowed with *supersymmetry* — the conformal algebra that characterises $\mathcal{M}(4,5)$ can be extended to a *superconformal* algebra [56]. Hence not only is $\mathcal{M}(4,5)$ of interest in statistical physics, but it is a playground for studying the mathematics of supersymmetry, which plays a distinguished role in string theory, and might one day be a standard component of fundamental physics.

The Coulomb gas formalism

A powerful aspect of conformal field theory is that its string theory and statistical field theory divisions walk side by side — many statistical field theoretical problems can be translated into string theoretical language and solved with reduced effort, and vice versa. The key insight is to recognise the relationship between the two.

With this idea in mind, an overarching goal of this thesis is to construct and then use some tools from string theory to find critical exponents and compute correlation functions of observables in critical statistical models. In the series of papers [34–36], a technique was introduced which turns out to be perfectly suited to our aims. This is the *Coulomb gas formalism*. Deriving and then using this technique is the end goal of this work.

Thesis outline

Chapters 1 and 2 cover preliminaries. If a quantum field theory exhibits symmetry, then much of its behaviour can be deduced by studying the representation theory of its symmetry algebra (or group), and this is where we commence in chapter 1. In chapter 2 we review conformal

symmetry, and derive an infinite-dimensional symmetry algebra of conformal transformations. The representation theory of this symmetry algebra allows us to quantify the constraints of conformal symmetry, which we use later to solve problems in statistical physics. Chapter 3 deals with the fundamental principles of conformal field theory, and here we develop essential mathematical techniques that will guide us through subsequent chapters. In chapter 4 we put our background knowledge to good use in a first encounter with a true conformal field theory — a massless scalar free field theory called the *free boson*. This is a conformal field theory that features prominently in string theory. The free boson is used to construct *vertex operators*, which are essential in subsequent chapters. In chapter 5 we isolate the consequences of quantum conformal symmetry by examining the representation theory of the conformal algebra. Here we construct the *minimal models*, and discuss some of their salient features in relation to statistical field theory. Chapter 6 introduces the *free fermion* conformal field theory, another example from string theory. Here the vertex operators from chapter 4 and the representation theory of chapter 5 are combined with the free fermion theory, and used to solve the critical Ising model. Chapter 7 is the penultimate chapter, and here we take a slight detour from the algebraic approach of the previous chapters to consider some differential equations. Conformal symmetry imposes constraints on quantum systems, and these manifest as partial differential equations satisfied by correlation functions — with a few manipulations and some physical consistency conditions, we are able to transform these into ordinary differential equations and solve them.

Finally in chapter 8 everything is brought together. In the spirit of conformal field theory, the aim is to connect up knowledge from diverse areas; in particular we point out a close relationship between the Coulomb gas formalism and the differential equations of chapters 5 and 7. In this final chapter we draw on content from all of the previous chapters, culminating in the solution of three different statistical models.

Chapter 1

Representing Symmetry

The beauty of physics and mathematics stems in no small part from symmetry. Conformal field theory as applied to relativistic quantum field theory is essentially the study of the highly non-trivial consequences of a particular type of symmetry — conformal symmetry. To appreciate the power of conformal symmetry and to make the notion quantitative, we need some tools from the theory of Lie algebras[1] *and their representations, and this is where we commence our journey.*

This chapter covers mathematical preliminaries that will be used in subsequent chapters. Readers familiar with Lie algebras and Lie algebra modules may wish to skip this chapter in a first reading, and refer to it as necessary.

1.1 Lie groups and Lie algebras

Consider the equation for a circle of radius $r \in \mathbb{R}_{>0}$ centred at the origin of $\mathbb{R}^2$:

$$x^2 + y^2 = r^2. \tag{1.1}$$

The circle is the collection of points $(x, y) \in \mathbb{R}^2$ which satisfy eq. (1.1). The plane can be rotated:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.2}$$

and the new point (x', y') solves eq. (1.1) whenever (x, y) does, thus rotations of the plane are said to be a *symmetry* for the circle.[2] The collection of all 2×2 rotation matrices forms a *Lie group*.

The symmetries of a quantum system are also encoded by the action of a symmetry group (or algebra). A **group** is a set G equipped with a binary operation $\circ : G \times G \to G$ which satisfies

- **Associativity**: $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$.
- **Identity**: There is an element $\mathbf{1} \in G$ which satisfies $a \circ \mathbf{1} = \mathbf{1} \circ a = a$ for every $a \in G$.
- **Invertibility**: For every $a \in G$ there is an inverse element denoted by a^{-1} which satisfies $a \circ a^{-1} = a^{-1} \circ a = \mathbf{1}$.

It follows from the group axioms that the identity element is unique, that each $a \in G$ has a unique inverse, and that $(a^{-1})^{-1} = a$. A **Lie group** is a group that is also a differentiable manifold with the property that the group operations of multiplication and inversion are differentiable

[1] "*Lie*" is pronounced "*Lee*", named after Sophus Lie.
[2] This example is borrowed from [95].

maps. If a subset of a group also satisfies the group axioms, and contains the same identity element as the group, then this subset is called a **subgroup**.

Example: General linear group
The set of real $n \times n$ invertible matrices, together with matrix multiplication and inversion as the group operations, is called the real general linear group of degree n, denoted by $GL(n, \mathbb{R})$. If we allow complex matrices, then we have the complex general linear group of degree n, denoted by $GL(n, \mathbb{C})$. The group $GL(n, \mathbb{R})$ (respectively $GL(n, \mathbb{C})$) is a Lie group because the matrices can be considered to be points on an $n \times n$-dimensional real (respectively complex) differentiable manifold, and because matrix multiplication and inversion are both differentiable with respect to the matrix coefficients.

Example: Special linear group
The set of real (respectively complex) $n \times n$ matrices with unit determinant is called the real (respectively complex) special linear group, denoted by $SL(n, \mathbb{R})$ (respectively $SL(n, \mathbb{C})$). It is a subgroup of the general linear group because any product of matrices with unit determinant is also a matrix with unit determinant. It is a Lie group.

Closely related to Lie groups are Lie algebras, which are infinitesimal linearisations of Lie groups — see [57, Ch. 9]. A **Lie algebra** is a vector space $\mathfrak{g}$ equipped with a bilinear pairing $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the **Lie bracket**. For all $x, y, z \in \mathfrak{g}$, the Lie bracket satisfies

- **Antisymmetry**: $[x, y] = -[y, x]$.
- **Jacobi identity**: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

Note that in general a Lie algebra is neither an associative nor a commutative algebra. If $x, y \in \mathfrak{g}$ satisfy $[x, y] = 0$, then x and y are said to **commute**, and if all elements of a Lie algebra commute, then the algebra is called **abelian**. The set of elements that commute with the whole algebra is called the **centre** of the algebra, denoted by $Z(\mathfrak{g})$. A **subalgebra** of a Lie algebra is a vector subspace that is closed under the Lie bracket. The centre of a Lie algebra is always a subalgebra.

Example: General linear Lie algebra
If a Lie algebra is over the field $\mathbb{R}$ then it is a *real Lie algebra*. The vector space of $n \times n$ matrices with real coefficients becomes a real Lie algebra if we take the commutator of matrices to be the Lie bracket. This algebra is called the real general linear Lie algebra of degree n, and is denoted by $\mathfrak{gl}(n, \mathbb{R})$. In this work we shall *always* take the field to be $\mathbb{C}$, in which case we have a *complex Lie algebra*. The analogous complex general linear Lie algebra of degree n is denoted by $\mathfrak{gl}(n, \mathbb{C})$.

Example: Virasoro algebra
The most important Lie algebra in conformal field theory is the Virasoro algebra — an infinite-dimensional (complex) Lie algebra $\mathfrak{Vir} := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$ with Lie bracket:[3]

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{C}{12} n(n^2 - 1)\delta_{n,-m}, \qquad [L_m, C] = 0. \tag{1.3}$$

The centre is $Z(\mathfrak{Vir}) = \mathbb{C}C$. The Virasoro algebra is contained in the symmetry algebra of every two-dimensional conformal field theory. It is obtained from another algebra called the *Witt algebra*, which arises when studying conformal transformations in a classical setting (see

[3]For readers unfamiliar with this notation, note that for v an element of a vector space over a field $\mathbb{F}$, the symbol $\mathbb{F}v$ denotes the one-dimensional vector space spanned by v.

section 2.2.2).[4] The Virasoro algebra is obtained from the Witt algebra by appending to it the element C. This process is called a central extension [57, Ch. 12]. The coefficient $1/12$ multiplying C in eq. (1.3) is merely conventional and can be set to any complex number by renormalising C, but apart from this factor the choice of Virasoro commutation relations is the only possible way to centrally extend the Witt algebra while satisfying the Lie algebra axioms [117, Ch. 5.3].[5] Physically, the central extension occurs naturally when quantising a classical conformal field theory (see section 3.3).

Example: Heisenberg algebra
The Heisenberg algebra is an infinite-dimensional Lie algebra $\mathfrak{H} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}a_n \oplus \mathbb{C}k$ with Lie bracket:

$$[a_n, a_m] = n\delta_{n,-m}k, \qquad [a_n, k] = 0. \tag{1.4}$$

The centre is $Z(\mathfrak{H}) = \mathbb{C}a_0 \oplus \mathbb{C}k$. The Heisenberg algebra appears remarkably often in many areas of physics. For example, it describes an infinite number of independent harmonic oscillators — [57, Ch. 2.5]. It is important for us because it can be used to construct the Virasoro algebra. In chapter 4 we will encounter the Heisenberg algebra in the context of string theory, and in chapters 6 and 8 we will use it to solve statistical field theory problems.

1.2 Representations

For modelling the symmetries of a quantum system, it will be convenient to consider the symmetry Lie algebra not in abstract form as merely a vector space with a Lie bracket, but rather as a collection of linear operators acting on the quantum state space.

For a vector space V, the collection of linear maps from V to itself is a vector space denoted by $\operatorname{End} V$. Such maps are called *endomorphisms* of V. Linear maps can be composed, and composition is associative, making $\operatorname{End} V$ into an associative algebra. Any associative algebra can be turned into a Lie algebra by taking the commutator with respect to the associative product to be the Lie bracket. In this way, $\operatorname{End} V$ becomes a Lie algebra denoted by $\mathfrak{gl}(V)$.

A **representation** of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ that replaces Lie brackets by commutators:

$$\rho([x, y]) = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x). \tag{1.5}$$

for all $x, y \in \mathfrak{g}$. The map ρ is called the *representation*, and the vector space V is called the *representation space*. The dimension of the representation is the dimension of V, which may be infinite. A representation of $\mathfrak{g}$ together with a representation space V is called a **$\mathfrak{g}$-module**. For $x \in \mathfrak{g}$ and $|v\rangle \in V$ it is common practice to write $x\,|v\rangle$ instead of $\rho(x)\,|v\rangle$, leaving the map ρ implicit, so that the Lie bracket takes the form of a commutator bracket:

$$[x, y]\,|v\rangle = x(y\,|v\rangle) - y(x\,|v\rangle). \tag{1.6}$$

Since the existence of a representation implies the existence of a representation space, we shall use the terms "representation" and "module" interchangeably — the former term is more common in the physics literature, and the latter is more common in the mathematics literature.

A subspace of V that is closed under the action of $\mathfrak{g}$ is called a **submodule**. Every $\mathfrak{g}$-module V has itself and the zero module $\{0\}$ as submodules. If V has another submodule, then V is said to be **reducible**, otherwise it is **irreducible**. A submodule of V is called **proper** if it is not equal to V itself. A submodule of V is called **maximal** if it is a proper submodule of V that is not properly contained in another proper submodule of V.

[4] The Witt algebra is the (complexification of) the Lie algebra of polynomial vector fields on the circle [124].

[5] The Virasoro algebra is the unique non-trivial central extension of the Witt algebra [64]. There is no complex "Virasoro group" that has the Virasoro algebra as its Lie algebra [117, Thm. 5.4].

1.2.1 Triangular decomposition

Some Lie algebras admit a special decomposition into a direct sum of three subalgebras $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_a$. This is called a **triangular decomposition** if the following axioms are satisfied [57, Ch. 13.11]:

- For $x \in \mathfrak{g}_c$ and $y \in \mathfrak{g}_0$, the bracket $[x, y]$ is in $\mathfrak{g}_c$.
- For $x \in \mathfrak{g}_a$ and $y \in \mathfrak{g}_0$, the bracket $[x, y]$ is in $\mathfrak{g}_a$.
- $\mathfrak{g}_0$ is abelian. That is, $[x, y] = 0$ for all $x, y \in \mathfrak{g}_0$.
- There is an *adjoint map*.

We define an **adjoint map** to be a $\mathbb{C}$-linear map $\dagger : \mathfrak{g} \to \mathfrak{g}$ that is its own inverse and is a one-to-one pairing between $\mathfrak{g}_a$ and $\mathfrak{g}_c$, and that maps $\mathfrak{g}_0$ to $\mathfrak{g}_0$, and reverses the order of the Lie bracket: $[x, y]^\dagger = [y^\dagger, x^\dagger]$.[6]

Triangular decompositions are very common in physics, although seldom formally defined — the elements of $\mathfrak{g}_c$ are called **creation operators**, those in $\mathfrak{g}_a$ are called **annihilation operators**, and those in $\mathfrak{g}_0$ are called **zero modes**, which accounts for the choice of subscripts. In Lie theory, the subalgebra $\mathfrak{g}_0$ is called the **Cartan subalgebra**.[7] It is the Cartan subalgebra that contains the **quantum observables**.[8]

Example: General linear Lie algebra
Consider the general linear Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of $n \times n$ complex matrices. If we define the creation operators $\mathfrak{g}_c$, annihilation operators $\mathfrak{g}_a$, and zero modes $\mathfrak{g}_0$ to be the upper-triangular, lower-triangular, and diagonal matrices respectively, and define the adjoint map to be matrix transposition, then $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{g}_c \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_a$ is a triangular decomposition.

1.2.2 Highest weight modules

A very important example of a $\mathfrak{g}$-module is a highest weight module, which can be constructed for any Lie algebra that admits a triangular decomposition. Highest weight modules arise naturally in quantum theories — they combine the notion of quantised energy with the requirement that the energy of a quantum system be bounded from below. A highest weight module is constructed from a vector $|v\rangle$ called the **vacuum vector**, or highest weight vector, in the following way.

Define $|v\rangle$ to be a simultaneous eigenvector of each element x in the Cartan subalgebra, and to be annihilated by every annihilation operator. The vector space formed by acting on $|v\rangle$ with all finite linear combinations of products[9] of a finite number of creation operators is called a *highest weight $\mathfrak{g}$-module*.

Every $\mathfrak{g}$-module constructed by acting on a highest weight vector with creation operators is called a highest weight module. One can modify highest weight modules by taking quotients by submodules (chapter 5). A highest weight module with no modifications is called a **Verma module**, which is in a sense the largest and most general type of highest weight module.

[6] Adjoint maps are often defined differently in other areas of physics. For example, instead of $\mathbb{C}$-linear the adjoint map is often defined to be Hermitian, so that for $B \in \mathbb{C}$ and $x \in \mathfrak{g}$ one has $(Bx)^\dagger = B^*(x)^\dagger$. The adjoint map is also sometimes defined to be anti-self-inverse. In conformal field theory we use a self-inverse linear adjoint merely for mathematical convenience.

[7] Technically $\mathfrak{g}_0$ is the Cartan subalgebra only if $\mathfrak{g}$ is semi-simple.

[8] In quantum theories that take the adjoint map to be Hermitian, the zero modes that are self-adjoint are the quantum observables — they necessarily have real eigenvalues. When using a $\mathbb{C}$-linear adjoint map, self-adjoint operators can have complex eigenvalues.

[9] Products of Lie algebra elements are in the sense explained in section 1.2.3.

1.2.3 Universal enveloping algebra

We have seen that any associative algebra can be used to construct a Lie algebra by taking the commutator with respect to the associative product to be the Lie bracket. Alternatively, starting from a Lie algebra $\mathfrak{g}$, one may wish to construct an associative algebra that contains $\mathfrak{g}$ as a subspace and for which the commutator on this subspace reproduces the Lie bracket on $\mathfrak{g}$. For each Lie algebra $\mathfrak{g}$ that we shall deal with there is a canonical way to construct such an algebra — see [57, Ch. 14.1]. This algebra is denoted by $U(\mathfrak{g})$, and is called the *universal enveloping algebra* of $\mathfrak{g}$. We mention this only for clarity — the only "multiplication" operation defined on a Lie algebra is the Lie bracket, so when we write products of Lie algebra elements, really they are elements of $U(\mathfrak{g})$. With this notation the highest weight $\mathfrak{g}$-module defined in section 1.2.2 can be written succinctly as $U(\mathfrak{g}_c)\,|v\rangle$, which is the same vector space as $U(\mathfrak{g})\,|v\rangle$.

To appreciate how this construction works, let us turn to an example very well known in the physics literature — Fock spaces.

1.3 Fock spaces

Fock spaces are good examples of physically relevant highest weight modules, and play a fundamental role in constructing the conformal field theories of chapters 4, 6 and 8. Fock spaces are highest weight representations for the Heisenberg algebra.

The Heisenberg algebra admits a triangular decomposition. Let $\mathfrak{H}_c := \bigoplus_{n<0} \mathbb{C}a_n$ be the creation operators, let $\mathfrak{H}_a := \bigoplus_{n>0} \mathbb{C}a_n$ be the annihilation operators, and denote the Cartan subalgebra by $\mathfrak{H}_0 := \mathbb{C}a_0 \oplus \mathbb{C}k$. We define the linear map $\dagger$ such that $a_n^\dagger := a_{-n}$, and $k^\dagger := k$. Clearly $\dagger$ is its own inverse, and from eq. (1.4) we can confirm that $\dagger$ satisfies $[x,y]^\dagger = [y^\dagger, x^\dagger]$ for all $x, y \in \mathfrak{H}$. $\dagger$ is an adjoint map.

With these definitions it follows that the decomposition $\mathfrak{H} = \mathfrak{H}_c \oplus \mathfrak{H}_0 \oplus \mathfrak{H}_a$ is a triangular decomposition. Define the vacuum vector $|p\rangle$ with $p \in \mathbb{R}$ to satisfy

$$a_0\,|p\rangle = p\,|p\rangle\,, \qquad k\,|p\rangle = |p\rangle\,, \qquad a_n\,|p\rangle = 0 \text{ for } n > 0. \tag{1.7}$$

The vector space

$$\mathcal{F}(p) := U(\mathfrak{H}_c)\,|p\rangle = \mathrm{span}_{\mathbb{C}}\{\ldots a_{-3}^{j_3} a_{-2}^{j_2} a_{-1}^{j_1}\,|p\rangle : j_i \in \mathbb{Z}_{\geq 0},\ \sum_{i=1}^{\infty} j_i < \infty\} \tag{1.8}$$

is a highest weight module for the Heisenberg algebra. $\mathcal{F}(p)$ is called a (bosonic) **Fock space**. It is an *infinite-dimensional* vector space, but each vector in $\mathcal{F}(p)$ consists of finite sums of products of a finite number of Heisenberg generators acting on $|p\rangle$. For every $p \in \mathbb{R}$, $\mathcal{F}(p)$ is an irreducible Heisenberg module, that is, $\mathcal{F}(p)$ contains no non-zero proper submodule — see [117, Ch. 7.2].

Each element in the centre $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ commutes with all of $\mathfrak{g}$, and so acts a scalar multiple of the identity in any highest weight $\mathfrak{g}$-module. Here we take $k \in Z(\mathfrak{H})$ to act as multiplication by 1 on $|p\rangle$, and therefore on every vector in $\mathcal{F}(p)$. This can be viewed as fixing a normalisation for the Heisenberg generators, since $[a_n, a_m] \propto k$. The a_0-eigenvalue of a vector is called the **momentum** of that vector, since it has the string theoretic interpretation of the momentum of a closed string propagating through flat Minkowski space. Since $a_0 \in Z(\mathfrak{H})$, it follows that every vector in $\mathcal{F}(p)$ has momentum p.

1.4 Scalar products

Our main goal is to construct probabilistic quantum field theories that can be used to solve problems in statistical field theory. We need a way to relate theory to experiment, and the

standard method of doing so is by defining a scalar product. The scalar product that we define here is called the **Shapovalov form**, named after [119]. The following construction works for any Verma module over a Lie algebra with a triangular decomposition, but here we shall demonstrate the construction specifically for Fock spaces. In chapters 3 and 5 we shall repeat the construction for Verma modules over the Virasoro algebra.

Define the $\mathbb{C}$-bilinear[10] form $(\cdot,\cdot) : \mathcal{F}(p) \times \mathcal{F}(q) \to \mathbb{C}$ to be invariant with respect to the adjoint map:

$$(x^\dagger \left|\alpha\right\rangle, \left|\beta\right\rangle) := (\left|\alpha\right\rangle, x \left|\beta\right\rangle) \tag{1.9}$$

for all $\left|\alpha\right\rangle \in \mathcal{F}(p)$, $\left|\beta\right\rangle \in \mathcal{F}(q)$, and all $x \in \mathfrak{H}$, and fix the normalisation:

$$(\left|p\right\rangle, \left|p\right\rangle) := 1. \tag{1.10}$$

Since a_0 is self-adjoint, vectors with different momenta are orthogonal with respect to this form. Moreover, because $a_0 \in Z(\mathfrak{H})$, it acts as multiplication by p on every vector in $\mathcal{F}(p)$, from which it follows that every vector in $\mathcal{F}(p)$ is orthogonal to every vector in $\mathcal{F}(q)$ if $p \neq q$.

In Dirac notation it is customary to write

$$\langle\alpha|\beta\rangle := (\left|\alpha\right\rangle, \left|\beta\right\rangle), \tag{1.11}$$

where the covector $\langle\alpha|$ lives in a covector space $\mathcal{F}^\vee(p)$ of linear maps:

$$\langle\alpha| : \mathcal{F}(q) \to \mathbb{C}, \qquad \left|\beta\right\rangle \mapsto (\left|\alpha\right\rangle, \left|\beta\right\rangle), \tag{1.12}$$

with the action of Heisenberg algebra on covectors defined by the adjoint:

$$\langle\alpha| x \left|\beta\right\rangle = (x^\dagger \left|\alpha\right\rangle, \left|\beta\right\rangle). \tag{1.13}$$

Hence, the highest weight covector $\langle p|$ satisfies

$$\langle p| a_0 = p \langle p|, \qquad \langle p| k = \langle p|, \qquad \langle p| a_n = 0 \text{ for } n < 0. \tag{1.14}$$

A scalar product is called *non-degenerate* if the vanishing of $(\left|\alpha\right\rangle, \left|\beta\right\rangle)$ for all $\left|\beta\right\rangle$ implies that $\left|\alpha\right\rangle$ is zero. The scalar product we have constructed here is non-degenerate — [117, Lem. 7.7]. The study of non-degeneracy is an important theme in chapter 5.[11]

1.5 Partition basis

It will be useful for subsequent chapters to specify a basis for our highest weight modules. In the case of the Heisenberg algebra there is a particularly convenient basis for $\mathcal{F}(p)$ called the **partition basis**. A similar basis will be used in chapter 5 for Virasoro modules.

A **partition** $Y = [y_1, y_2, \ldots, y_k]$ is a weakly decreasing finite sequence of positive integers $y_1 \geq y_2 \geq \cdots \geq y_k > 0$ with $y_i \in \mathbb{Z}_{>0}$. The elements y_i are called *parts* and the sum of the parts is denoted by $|Y|$. We say that Y is a partition of $|Y|$. We define the empty sequence [] to be the (unique) partition of 0. Two partitions are equal if all of their parts are equal, otherwise they are distinct. The number of distinct partitions of N is denoted by $p(N)$, and is finite for each non-negative integer N.

[10] This is in contrast to the commonly used *sesquilinear* scalar products that complex conjugate one of the two arguments (usually the one on the left-hand side in the physics literature: $(\left|\alpha\right\rangle, B\left|\beta\right\rangle) = (B^* \left|\alpha\right\rangle, \left|\beta\right\rangle) = B(\left|\alpha\right\rangle, \left|\beta\right\rangle)$ for $B \in \mathbb{C}$, and the one on the right-hand side in the mathematics literature).

[11] As is well known in quantum mechanics, for a scalar product to have a standard probabilistic interpretation requires that this product be positive-definite. A scalar product with this property is called an inner-product. A $\mathbb{C}$-bilinear scalar product can never be positive-definite, since multiplying both vectors in a positive scalar product by the imaginary unit i results in a negative scalar product. This is one downside to using a bilinear scalar product instead of a sesquilinear one.

To each partition $Y = [y_1, y_2, \ldots, y_k]$ we associate vectors in $U(\mathfrak{H})$:

$$a_Y := a_{y_k} \cdots a_{y_1}, \qquad a_{-Y} := a_{-y_1} \cdots a_{-y_k}, \tag{1.15}$$

and define $a_Y = a_{-Y} := \mathbf{1}$ if $|Y| = 0$. The set $\{a_{-Y}\,|p\rangle : |Y| \in \mathbb{Z}_{\geq 0}\}$ is a basis for $\mathcal{F}(p)$.

$\mathcal{F}(p)$ is a *graded* vector space, which means that it decomposes into a direct sum of subspaces $\mathcal{F}_N(p)$ called **homogeneous spaces** of grade N:

$$\mathcal{F}_N(p) := \mathrm{span}_{\mathbb{C}}\{a_{-Y}\,|p\rangle : |Y| = N\} \implies \mathcal{F}(p) = \bigoplus_{N=0}^{\infty} \mathcal{F}_N(p), \tag{1.16}$$

The dimension of $\mathcal{F}_N(p)$ is equal to the number of distinct partitions $p(N)$. Since $p(N)$ is finite for each $N \in \mathbb{Z}_{\geq 0}$ it follows that each homogeneous space has finite dimension.

We have now covered the essential background material in representation theory needed for the rest of this work. Most important is the construction of Fock spaces and the representation theory of the Heisenberg algebra, for as we shall see in chapter 4, the conformal algebra resides within the Heisenberg algebra and can also be represented on Fock spaces. The building blocks for chapters 4, 6 and 8 all stem from the Heisenberg algebra. Scalar products feature prominently throughout this work, and computing them will be a primary goal.

Chapter 2

Conformal Invariance

Symmetry transformations of a field theory are realised by the action of a symmetry group or Lie algebra. With the necessary Lie theory under our control, we turn now to the study of conformal symmetry. The power of conformal field theory arises from a very simple observation about two-dimensional conformally invariant quantum field theories — the symmetry algebra is infinite-dimensional. In this chapter we will derive this infinite-dimensional algebra. Let us begin in the natural place, by discussing what it means to be conformal.

2.1 Conformal transformations

A *semi-Riemannian manifold* (also called a pseudo-Riemannian manifold) is a pair (M, g), where M is a real smooth manifold and g is a metric on M — see [117, Def. 1.1]. A *metric* is a smooth symmetric non-degenerate bilinear form that lets us define lengths and angles. A d-dimensional manifold can be charted with coordinates in $\mathbb{R}^d$, and the metric can be represented by a coordinate-dependent symmetric $d \times d$-dimensional real matrix $(g_{\mu\nu})$, denoted by its matrix coefficients $g_{\mu\nu}$. The matrix depends smoothly on the coordinates and may be constant. We denote coordinate vectors x by their components $x^\mu \in \mathbb{R}$, with $\mu \in \{1, \dots, d\}$. Since the metric is symmetric it can be diagonalised.

A flat space metric can be put into the form

$$(g_{\mu\nu}) = \begin{pmatrix} -\mathbf{1}_m & \\ & \mathbf{1}_n \end{pmatrix}, \tag{2.1}$$

where $\mathbf{1}_p$ is the $p \times p$ identity matrix. Flat space with a metric of this form is denoted by $\mathbb{R}^{m,n}$. Some examples of semi-Riemannian manifolds are **Minkowski 4-space** $\mathbb{R}^{1,3}$ (or $\mathbb{R}^{3,1}$) and the **Euclidean plane** $\mathbb{R}^{0,2}$.

We use Einstein summation notation so that repeated indices are summed over. The dimension of the manifold is given by

$$d = g_{\mu\nu} g^{\mu\nu}, \tag{2.2}$$

where $(g^{\mu\nu})$ is the matrix inverse of $(g_{\mu\nu})$. The metric can then be used to raise and lower indices of coordinate vectors:

$$x_\mu := g_{\mu\nu} x^\nu. \tag{2.3}$$

The scalar product defined by the metric is:

$$x \cdot y := x_\mu y^\mu = g_{\mu\nu} x^\nu y^\mu, \tag{2.4}$$

which is not necessarily positive-definite. A *Riemannian manifold* is a semi-Riemannian manifold for which this scalar product is positive-definite, and this is what we will assume we are dealing

with. In this case the length of a vector x is the non-negative real number $\|x\|$ defined by

$$\|x\| := \sqrt{x \cdot x}. \tag{2.5}$$

The angle θ between two non-zero vectors x and y is defined by

$$\cos\theta := \frac{x \cdot y}{\|x\|\|y\|}. \tag{2.6}$$

We do not define an angle if either x or y is the zero vector.

A **conformal transformation** is a smooth coordinate transformation that results in a strictly positive rescaling of the metric:

$$x \mapsto x', \qquad g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x), \tag{2.7}$$

with $\Lambda(x)$ real and positive. The factor $\Lambda(x)$ is called the *conformal factor* of the transformation. Such transformations preserve angles. To see this, let u and v be tangent vectors to curves intersecting at x, so that the transformed angle between them is given by

$$\cos\theta' = \frac{\Lambda g_{\mu\nu} u^\mu v^\nu}{(\Lambda g_{\mu\nu} u^\mu u^\nu)^{1/2} (\Lambda g_{\mu\nu} v^\mu v^\nu)^{1/2}} = \frac{g_{\mu\nu} u^\mu v^\nu}{(g_{\mu\nu} u^\mu u^\nu)^{1/2} (g_{\mu\nu} v^\mu v^\nu)^{1/2}} = \cos\theta. \tag{2.8}$$

The angle is preserved.[1]

Given a conformal transformation, the conformal factor can be calculated using the defining transformation rules for tensors. The metric tensor is a rank 2 covariant tensor and transforms under a general smooth coordinate transformation $x \mapsto x'$ as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}. \tag{2.9}$$

If the transformation is conformal then the conformal factor can be found by equating the expressions in eqs. (2.7) and (2.9).

2.1.1 Global transformations

Working on the manifold $\mathbb{R}^{m,n}$ including points at infinity (section 2.1.2), the **global conformal transformations** are those which are defined for all points on the manifold — see [117, Def. 2.10]. These transformations are [65, Sec. 1.1]

1. **Rotations**: All vectors are rotated about the origin through the same angle. Rotations generate the orthogonal group $O(m,n)$ of isometries of $\mathbb{R}^{m,n}$, and the conformal factor is $\Lambda = 1$. The metric is therefore unchanged, which is what it means to be an isometry. In Minkowski 4-space this is the *Lorentz group* $O(1,3)$.

2. **Translations**: All points are shifted by a constant vector $b \in \mathbb{R}^{m+n}$. The conformal factor is $\Lambda = 1$. In Minkowski 4-space the group of translations and rotations together make up the *Poincaré group* of $\mathbb{R}^{1,3}$ isometries.

3. **Dilations**: All vectors are scaled by the same non-zero real number. If this number is k then the conformal factor is $\Lambda = k^2$.

4. **Special conformal transformations**: These are described below. They are not generally isometries of $\mathbb{R}^{m,n}$. In Minkowski 4-space the group obtained by extending the Poincaré group to include dilations and special conformal transformations is the *conformal group* of globally defined conformal transformations of $\mathbb{R}^{1,3}$.

[1] Refer to [117, Ch. 1.2] for a more rigorous treatment.

The global transformations defined above all have the property that they can be continuously deformed to the identity map. In the terminology of Lie groups, a transformation with this property is said to be in the *connected component of the identity*. For example, for translations one can take the translation vector b continuously to zero, thus recovering the identity. Similarly one can take the rotation angle to zero, or the dilation factor to unity. For $d \geq 2$, every global conformal transformation connected to the identity can be written as a composition of rotations, translations, dilations, and special conformal transformations [117, Thm. 1.9].

There is another conformal transformation called *inversion* which is not in the connected component of the identity, defined by

$$x^\mu \mapsto x'^\mu = \frac{x^\mu}{\|x\|^2}. \tag{2.10}$$

Inversion is not defined at the origin of $\mathbb{R}^{m,n}$. It is conformal, which can be checked by calculating the conformal factor using eq. (2.9). The result is $\Lambda(x) = 1/\|x\|^4$. The inversion transformation has no parameter that can be continuously tuned to recover the identity transformation, but by composing inversions with translations one can construct *special conformal transformations*, which are connected to the identity. The sequence of compositions is: invert, then translate by b, then invert again, resulting in the transformation

$$x^\mu \mapsto \frac{x^\mu + b^\mu \|x\|^2}{1 + 2b \cdot x + \|b\|^2 \|x\|^2}. \tag{2.11}$$

Taking each component b^μ continuously to 0 recovers the identity map.

2.1.2 Conformal group

The special conformal transformation in eq. (2.11) is not defined on $\mathbb{R}^{m,n}$ where the denominator vanishes. To get a truly global transformation one needs to conformally compactify the space to include points at infinity. We will do this in the two-dimensional case by declaring that we are working not on the complex plane, but on its conformal compactification, the **Riemann sphere**, which we will denote by $\mathbb{S}^2 := \mathbb{C} \cup \{\infty\}$.[2] See [117, Ch. 2.1] for details on conformal compactification.

The **conformal group** of $\mathbb{R}^{m,n}$ is the connected component containing the identity in the group of global conformal transformations on the conformal compactification of $\mathbb{R}^{m,n}$ — see [117, Def. 2.1]. For statistical field theoretic purposes, the choice of Riemannian manifold used in the literature is the Euclidean plane $\mathbb{R}^{0,2}$. It is sometimes claimed (for example, in the abstract of [10]) that the conformal group of $\mathbb{R}^{0,2}$ is infinite dimensional. We will see in section 2.2 where this misunderstanding comes from.[3]

2.1.3 Infinitesimal transformations

We define a coordinate transformation to be an **infinitesimal conformal transformation** if it satisfies eq. (2.7) but does not belong to the conformal group, and has the form

$$x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu(x), \tag{2.12}$$

where $\epsilon(x)$ is formally infinitesimal. To say that $\epsilon(x)$ is formally infinitesimal means that terms proportional to $\mathcal{O}(\epsilon^2)$ or higher are to be ignored.

[2] The Riemann sphere $\mathbb{S}^2$ is actually obtained from the augmented complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by inverse stereographic projection, which is an embedding: $\mathbb{R}^2 \cong \mathbb{C} \hookrightarrow \widehat{\mathbb{C}} \cong \mathbb{S}^2 \subset \mathbb{R}^3$. The sphere and the plane do not share the same metric (the sphere is not flat), but we will not worry about this technicality. It is sufficient to mention that stereographic projection is conformal, as is inverse stereographic projection.

[3] Interestingly, for the Minkowski plane $\mathbb{R}^{1,1}$ the conformal group is infinite dimensional — see [117, remark. 2.16].

Under the transformation in eq. (2.12), the metric transforms as per eq. (2.9). The derivatives can be calculated using a property of infinitesimals — from the transformation in eq. (2.12) we have

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \frac{\partial}{\partial x^{\alpha}}(x^{\mu} + \epsilon^{\mu}) = \delta^{\mu}_{\alpha} + \partial_{\alpha}\epsilon^{\mu}, \tag{2.13}$$

and since we are dealing with infinitesimals the derivative with respect to x' is given by a sign change:

$$\frac{\partial x^{\alpha}}{\partial x'^{\mu}} = \delta^{\alpha}_{\mu} - \partial_{\mu}\epsilon^{\alpha}, \tag{2.14}$$

where ∂_{μ} is a derivative with respect to the *unprimed* coordinate x^{μ}. This works because the difference between the primed and unprimed derivative is $\mathcal{O}(\epsilon^2)$, which we are allowed to ignore.

From eqs. (2.9) and (2.14) we find that

$$g'_{\mu\nu} = g_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu}. \tag{2.15}$$

For a conformal map, this should be the same as an infinitesimal rescaling. Writing the conformal factor as $\Lambda(x) = 1 - \Omega(x)$ and substituting this into eq. (2.15) we get

$$\Omega(x) g_{\mu\nu} = (\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}). \tag{2.16}$$

This constrains how the metric transforms on the infinitesimal level. Contracting both sides with $g^{\mu\nu}$ and using eq. (2.2) for the dimension of the manifold we get

$$\Omega(x) = \frac{2}{d}\partial_{\rho}\epsilon^{\rho}. \tag{2.17}$$

2.2 Conformal invariance in two dimensions

The case of interest is the two-dimensional case. Working with $\mathbb{R}^{0,2}$, which has $d = 2$, and substituting eq. (2.17) into eq. (2.16), we find that

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = g_{\mu\nu}\partial_{\rho}\epsilon^{\rho} = g_{\mu\nu}g^{\rho\sigma}\partial_{\rho}\epsilon_{\sigma}. \tag{2.18}$$

The $\mathbb{R}^{0,2}$ metric is the 2×2 identity matrix $\mathbf{1}_2$, which is its own inverse. For notational clarity we write $\epsilon^1 := u$, $\epsilon^2 := v$, $x^1 := x$ and $x^2 := y$. The following two equations can be read off from eq. (2.18):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.19}$$

These are the **Cauchy-Riemann equations**,[4] written in conventional notation of complex analysis. Recall that these equations define what it means for a function to be **holomorphic** (equivalently *analytic*). The appearance of the Cauchy-Riemann equations suggests that working on $\mathbb{C}$ might be more convenient.

To map $\mathbb{R}^{0,2}$ to $\mathbb{C}$, let $z, \bar{z} \in \mathbb{C}$ be defined by $z := x + iy$, and $\bar{z} := x - iy$.[5] With u and v defined as above, the Cauchy-Riemann equations state that the functions $\omega(z, \bar{z}) := u(x, y) + iv(x, y)$ and $\bar{\omega}(z, \bar{z}) := u(x, y) - iv(x, y)$ are holomorphic and anti-holomorphic respectively, which is equivalent to the statement that

$$\frac{\partial \omega(z, \bar{z})}{\partial \bar{z}} = 0, \qquad \frac{\partial \bar{\omega}(z, \bar{z})}{\partial z} = 0. \tag{2.20}$$

[4] See any text on complex analysis, for example [116].

[5] Formally, z and $\bar{z}$ are independent because x and y are independent. They become conjugates only when we define complex conjugation. The space in $\mathbb{C}^2$ formed by taking z and $\bar{z}$ to be conjugates is sometimes called the "*real surface*".

Since $\omega(z,\bar{z})$ is independent of $\bar{z}$ and $\bar{\omega}(z,\bar{z})$ is independent of z we may write these functions as $\omega(z)$ and $\bar{\omega}(\bar{z})$ respectively.

We should make an important remark regarding the terminology "holomorphic" and "anti-holomorphic" as used in conformal field theory. These terms should be interpreted in the local sense, that is in neighbourhoods away from poles where the derivatives exist. There may be poles elsewhere, and strictly speaking the correct term to use is *"meromorphic"*, but this is almost never done in the conformal field theory literature.[6]

From the Cauchy-Riemann equations it follows that a transformation on $\mathbb{C} \cong \mathbb{R}^{0,2}$ is conformal if and only if it is a holomorphic or anti-holomorphic map [117, Thm. 1.11].

This implies that there exists an infinite number of linearly independent infinitesimal conformal transformations. To prove this simply consider the collection of maps $\omega_n(z) = z^n$ and $\bar{\omega}_n(\bar{z}) = \bar{z}^n$, which are respectively holomorphic and anti-holomorphic for every $n \in \mathbb{Z}$. It is for this reason that the conformal group of $\mathbb{R}^{0,2}$ is sometimes misunderstood to be infinite-dimensional. As we will discuss in section 2.2.2, only a finite subset of the generators of these transformations integrate up to globally defined transformations. See [117, Ch. 2.4] for a detailed discussion on this point.

2.2.1 The Möbius group

The conformal group of $\mathbb{S}^2 := \mathbb{C} \cup \{\infty\}$ is the **Möbius group** Mb [117, Thm. 2.11]. The Möbius group is a quotient group of $SL(2,\mathbb{C})$, the special linear group of complex 2×2 matrices with unit determinant which we encountered in section 1.1. We will focus on the holomorphic sector for now, since the working for the anti-holomorphic sector is similar.

A *Möbius transformation* is a map γ of the form[7]

$$\gamma : z \mapsto \gamma(z) = \frac{az+b}{cz+d}, \tag{2.21}$$

where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc = 1$ — [117, Def. 2.12].[8] The map $\gamma(z)$ satisfies the Cauchy-Riemann equations, and is therefore conformal. Setting $c = 0$ and $a = d$ recovers the translation map $z \mapsto z + b/d$; setting $c = b = 0$ gives dilations and rotations $z \mapsto az/d$;[9] setting $a = d = 0$ and $b = c$ gives the inversion map $z \mapsto 1/z$, and composing with translations gives special conformal transformations. This set of transformations exhausts the possible global conformal transformations on $\mathbb{S}^2$. Note that the transformation is defined even where the denominator of $\gamma(z)$ vanishes, since the point at infinity is included in the compactified space $\mathbb{S}^2$.

With four complex parameters a, b, c, and d, and one complex relation between them, the Möbius group is a 6-dimensional Lie group (3 complex, or 6 real parameters), so the group of globally defined conformal transformations on $\mathbb{S}^2$ is not infinite-dimensional.

The transformation γ can be identified with the complex matrix

$$M_\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}). \tag{2.22}$$

Since they form a group, Möbius transformations are invertible. The inverse of eq. (2.21) is

$$\gamma^{-1} : z \mapsto \gamma^{-1}(z) = \frac{dz-b}{-cz+a}, \tag{2.23}$$

[6] [59] is a notable exception.

[7] Möbius transformations have the geometric interpretation of inverse stereographically projecting the plane into the Riemann sphere, then translating and rotating the sphere and subsequently stereographically projecting it back onto the plane.

[8] Some authors require only that $ad - bc \neq 0$, but then the Möbius group cannot be identified with a quotient of the special linear group.

[9] Multiplying by a complex number both rotates and rescales. Choosing $a/d \in \mathbb{R}$ gives pure rescaling, and choosing $|a/d| = 1$ gives pure rotation.

and the inverse transformation corresponds to the matrix inverse of M_γ. Since both M_γ and $-M_\gamma$ correspond to the same Möbius transformation, we should consider them as equivalent: $M_\gamma \sim -M_\gamma$. This quotient is the Möbius group

$$Mb := SL(2,\mathbb{C})/\{\mathbf{1}_2, -\mathbf{1}_2\}. \tag{2.24}$$

The Möbius group acts **transitively** on triplets (z_1, z_2, z_3) of distinct points $z_i \in \mathbb{S}^2$, which means that any three distinct points on $\mathbb{S}^2$ can be mapped to any other. Mb plays an important role in this thesis. In chapter 3 it is instrumental in deriving the *conformal Ward identities*, which are constraints on physically measurable quantities called *correlation functions*; in chapter 7 it is used to transform partial differential equations into ordinary differential equations; and in chapter 8 it is used to simplify contour integrals by mapping them to real integrals.[10]

2.2.2 The classical conformal algebra

Here we will show that although the Möbius group is finite-dimensional, the Lie algebra of infinitesimal conformal transformations is infinite-dimensional.

Focusing once more on the holomorphic sector and using complex coordinates, an infinitesimal conformal transformation is a holomorphic map of the form

$$z \mapsto z' = z + \epsilon\omega(z), \tag{2.25}$$

with $\epsilon \in \mathbb{R}$ formally infinitesimal.

Let us motivate the following by making an analogy to Lie groups. A *near identity element* of a Lie group G is an element of the form $\mathbf{1} + \epsilon X$, where $\mathbf{1}$ is the identity in G, X is an element of the corresponding Lie algebra, and ϵ is a formally infinitesimal scalar. The inverse of such an element given by a change of sign:[11]

$$g = \mathbb{I} + \epsilon X \leftrightarrow g^{-1} = \mathbb{I} - \epsilon X. \tag{2.26}$$

The (left) action of a group element g on a function f is defined by the inverse:

$$f(z) \mapsto (g * f)(z) := f(g^{-1}z). \tag{2.27}$$

The inverse is used so that composition of transformations respects the group multiplication rules.

Using eq. (2.26) and Taylor expanding eq. (2.27) about $\epsilon = 0$ we have

$$f(z) \mapsto f(z - \epsilon\omega(z)) = f(z) - \epsilon\omega(z)\frac{df}{dz}. \tag{2.28}$$

The *infinitesimal generator* of the transformation is the differential operator $X_\omega = -\omega(z)\frac{d}{dz}$. Expanding $\omega(z)$ in a Laurent series gives a mode expansion for X_ω:

$$X_\omega = -\omega(z)\frac{d}{dz} = \sum_{n\in\mathbb{Z}} -\omega_n z^{n+1}\frac{d}{dz}, \tag{2.29}$$

which is interpreted as an expansion in elements of a Lie algebra. If we label the modes:

$$l_n := -z^{n+1}\frac{d}{dz}, \tag{2.30}$$

[10] Yet another reason that the Möbius group is so important in conformal field theory is its close relationship *modular group* — see [53, Ch. 10].

[11] To see why this is sensible, note that $(\mathbf{1} + \epsilon X)(\mathbf{1} - \epsilon X) = \mathbf{1} + \mathcal{O}(\epsilon^2)$.

we find that the set $\{l_n : n \in \mathbb{Z}\}$ spans an infinite-dimensional Lie algebra with the Lie bracket given by the commutator

$$[l_n, l_m] = (n - m)l_{n+m}. \tag{2.31}$$

This is the *Witt algebra* $\mathfrak{W}$ from chapter 1 — the *classical conformal algebra*. Of course there is also the anti-holomorphic sector to consider, with generators $\bar{l}_n := -\bar{z}^{n+1}\frac{d}{d\bar{z}}$ which satisfy the same commutation relations as in eq. (2.31). This sector defines a second copy $\bar{\mathfrak{W}}$ of the Witt algebra that commutes with the first, and the full classical conformal algebra is the direct sum $\mathfrak{W} \oplus \bar{\mathfrak{W}}$.

Global and infinitesimal transformations

Not all of the transformations generated by $\mathfrak{W} \oplus \bar{\mathfrak{W}}$ are globally defined. The Möbius group is generated by the subalgebra spanned by the set $\{l_{-1}, l_0, l_1, \bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$. From eq. (2.30), it follows that l_{-1} and $\bar{l}_{-1}$ generate translations in z and $\bar{z}$ respectively, that $l_0 + \bar{l}_0$ generates dilations, that $i(l_0 - \bar{l}_0)$ generates rotations, and that l_1 and $\bar{l}_1$ generate special conformal transformations [65, Sec. 1.2]. The generators that are not in this subalgebra correspond to infinitesimal transformations which are not defined globally.

The commutation relations in eq. (2.31) are those of the Virasoro algebra from eq. (1.3) with the central element C removed. As we will see in chapter 4, the central element appears naturally when moving from a classical to a quantum theory. The Witt algebra is the classical limit of the Virasoro algebra; the quantum version of the conformal algebra is the Virasoro algebra.

We have now laid down the mathematical foundations of conformal symmetry in two dimensions — we have defined what conformal transformations are, we know that the conformal group of globally defined conformal transformations is the Möbius group, and we know that infinitesimal conformal transformations are generated by an infinite-dimensional Lie algebra with ($C = 0$) Virasoro commutation relations. Let us now transition into the quantum world, and begin our studies of conformal field theory.

Chapter 3

Fundamentals and Techniques

The foundations of conformal field theory consist of a collection of mathematical objects and physically motivated assumptions that quantify the profound consequences of two-dimensional conformal symmetry. In this chapter we cover the fundamental notions and techniques that will guide us through to the end of this work.

3.1 Fields and states

3.1.1 Fields

The most fundamental objects in conformal field theory are fields $\Phi_i(z)$. Fields are maps from subsets of the Riemann sphere $\mathbb{S}^2$ or some other Riemann surface [118] to a space of linear operators on a *space of states* $\mathcal{S}$. The set of fields spans a complex vector space. Some mathematical details concerning quantum fields can be found in [117, Ch. 8].

The above set of fields $\{\Phi_i(z)\}$ actually represents only half of the full field content. Fields are written in full as $\Phi_i(z,\bar{z})$, where z and $\bar{z}$ are formally independent. Such fields are called **non-chiral** fields. In all of the theories that we will deal with the non-chiral fields factorise into sums of independent holomorphic and anti-holomorphic halves of the form $\Phi_i(z)\otimes\bar{\Phi}_i(\bar{z})$, a phenomenon called **holomorphic factorisation**.[1] Each tensor factor is called a **chiral field**. We will often treat the tensor factors independently in calculations, and recombine the results afterwards by imposing physical consistency conditions (chapters 7 and 8).

We assume there is a basis of non-chiral fields $\{\Phi_i\}$ such that every field in the theory can be expanded as a linear combination of these basic fields. The index i runs over some infinite set. Whether or not this set is countable depends on which conformal field theory we are dealing with. We will not worry about ordering the fields. Each non-chiral field $\Phi_i(z,\bar{z})$ is assigned two numbers $h_i,\bar{h}_i\in\mathbb{R}$ called **conformal weights**, which are related to the scaling properties of the field under conformal transformations. For example, there is a subset of the space of fields called the *quasi-primary fields* characterised by how they transform under global conformal transformations — for any Möbius transformation $\omega\in Mb$, the **quasi-primary fields** are those which transform as

$$\Phi_i(\omega(z),\bar{\omega}(\bar{z}))=\left(\frac{d\omega}{dz}\right)^{-h_i}\left(\frac{d\bar{\omega}}{d\bar{z}}\right)^{-\bar{h}_i}\Phi_i(z,\bar{z}). \tag{3.1}$$

The conformal weights h_i and $\bar{h}_i$ are both real and need not be conjugates of one another, despite

[1] Holomorphic factorisation is not a general feature of all conformal field theories, for example the logarithmic conformal field theories that appear in [27]. If a quantum space of states $\mathcal{S}$ decomposes as a direct sum of tensor products of irreducible modules for the symmetry algebra, then every state in $\mathcal{S}$ can be expressed as a linear combination of tensor states, which implies (by the state-field correspondence of section 3.1.3) that every field can be too. This is the case for all of the theories in this thesis.

the suggestive notation. The sum of the conformal weights

$$\Delta_i := h_i + \bar{h}_i \tag{3.2}$$

is called the **scaling dimension** of the field, and the difference

$$s_i := h_i - \bar{h}_i \tag{3.3}$$

is called the *conformal spin*, or simply the **spin**.

3.1.2 States

For convenience, we shall usually restrict our attention to chiral fields. The corresponding chiral space of states is a complex vector space $\mathcal{S}$, and states will be denoted using Dirac notation: $|v\rangle \in \mathcal{S}$. As in standard quantum field theory, vectors in $\mathcal{S}$ represent the possible configurations of a quantum system.

The conformal algebra is the Virasoro algebra $\mathfrak{Vir}$, which admits a triangular decomposition $\mathfrak{Vir} = \mathfrak{Vir}_c \oplus \mathfrak{Vir}_0 \oplus \mathfrak{Vir}_a$ into creation operators $\mathfrak{Vir}_c := \bigoplus_{n<0} \mathbb{C}L_n$, annihilation operators $\mathfrak{Vir}_a := \bigoplus_{n>0} \mathbb{C}L_n$, and zero modes $\mathfrak{Vir}_0 := \mathbb{C}L_0 \oplus \mathbb{C}C$. The linear map †, defined so that $L_n^\dagger := L_{-n}$ and $C^\dagger := C$, is an adjoint map.

The space of states $\mathcal{S}$ is a representation space for $\mathfrak{Vir}$ built out of a collection of highest weight representations. In every highest weight representation the central element C acts as some scalar multiple of the identity $c\mathbf{1}$. The number c is called the **central charge** of the representation. We will only consider representations with $c \in \mathbb{R}$. In some cases the central element C has the physical interpretation of a Casimir operator [57, Ch. 14.7], and the central charge has the interpretation of the corresponding Casimir energy.[2]

A vacuum vector (or highest weight vector — section 1.2.2) in $\mathcal{S}$ with L_0-eigenvalue h will be denoted by $|h\rangle$. We shall only consider $h \in \mathbb{R}$. The L_0-eigenvalue is the conformal weight of $|h\rangle$.[3] We assume the existence of a unique vacuum vector $|0\rangle \in \mathcal{S}$ with L_0-eigenvalue $h = 0$, called the **true vacuum** [96, Def. 2.1]. The assumption of a unique true vacuum is omitted in some formulations of conformal field theory, for example in [112, Ch. 3].

3.1.3 The state-field correspondence

An axiom of conformal field theory is that there is a one-to-one correspondence between states and fields — this is called the **state-field correspondence** [96, Def. 2.2].

In one direction, the state-field correspondence is particularly easy to formulate. To each field $\Phi(z)$ we associate a state $|\Phi\rangle \in \mathcal{S}$ given by

$$|\Phi\rangle := \lim_{z\to 0} \Phi(z)\,|0\rangle\,. \tag{3.4}$$

The operation in eq. (3.4) is defined by acting first with the field, and then taking the limit. This order of operations is needed for the limit to be well defined, as we will see in chapter 4. The conformal weight of a field is the same as its corresponding state.

If the state $|\Phi\rangle$ is a vacuum vector for the Virasoro algebra, then the corresponding field is called a Virasoro **primary field**. A vacuum vector is also called a **primary state**. When

[2] The central charge is sometimes regarded as an anomaly due to symmetry breaking, but this interpretation is model-dependent and often ambiguous. A more universally applicable treatment can be found in [117, Ch. 6.5], where it is shown that the central charge is a consequence of lifting a projective representation to a conventional linear representation on a vector space of states.

[3] The highest weight vector could also be labelled by its C-eigenvalue, the central charge $c \in \mathbb{R}$, but in all spaces of states that we construct, each highest weight representation will have the same central charge, and so C acts as multiplication by the same central charge on every vector in $\mathcal{S}$.

referring specifically to primary fields, we will denote them by uncapitalised symbols: $\phi_i(z)$. Primary fields are also quasi-primary, but the converse is not necessarily true.

In conformal field theory the time coordinate τ is mapped to the radial direction on the complex plane, and the $|z| \to 0$ limit corresponds to $\tau \to -\infty$. A vector $|\Phi\rangle$ therefore corresponds to a state created in the infinite past. Such a state is called an *asymptotic in-state* in the theory of particle scattering.

A state obtained by acting on a primary state $|\phi\rangle$ with a product of creation operators and that is not proportional to $|\phi\rangle$ is called a **descendent state**. These states are in one-to-one correspondence with **descendent fields**:

$$\dots L_{-2}^{j_2} L_{-1}^{j_1} |\phi_i\rangle \leftrightarrow (\dots L_{-2}^{j_2} L_{-1}^{j_1} \phi_i)(z), \qquad j_i \in \mathbb{Z}_{\geq 0}. \tag{3.5}$$

The precise meaning of the notation on the right-hand side will be specified in section 3.7.3. The descendent field $(\dots L_{-2}^{j_2} L_{-1}^{j_1} \phi_i)(z)$ is another field $\Phi_j(z)$ in the theory, which may in fact satisfy the definition of a primary field. We explore this possibility in chapter 5.

3.2 Operator-product expansions (OPEs)

As in standard quantum field theory, multiplication of fields can be defined as composition of operators. The set of fields is assumed to be closed under operator composition, and so defining the multiplication operation elevates the space of fields to an associative algebra. In [9] it is assumed that every product of fields decomposes into a sum of basic fields Φ_i. The decomposition is called the **operator-product expansion**, or **OPE**.[4] The operator-product expansion is *associative*. OPEs were originally proposed in [127] in the context of standard quantum field theory — see [24, Ch. 10] for a review.

Using the notation $z_{12} := z_1 - z_2$ we have

$$\Phi_i(z_1)\Phi_j(z_2) = z_{12}^{-h_i-h_j} \sum_k C_{ij}^k z_{12}^{h_k} \Phi_k(z_2). \tag{3.6}$$

The sum includes all fields in the theory, the number of which is infinite.[5] The structure constants $C_{ij}^k \in \mathbb{C}$ are called **OPE-coefficients**. If $C_{ij}^k \neq 0$ we say that the OPE of Φ_i and Φ_j *"contains"* Φ_k. The exponents of z_{12} in eq. (3.6) are required so that each side transforms in the same way under conformal transformations. For example, eq. (3.1) is satisfied when Φ_i, Φ_j, and Φ_k are all quasi-primary.

The **primary OPE-coefficients** are those C_{ij}^k for which Φ_i, Φ_j, and Φ_k are all primary fields. Non-primary OPE-coefficients can be determined from primary ones [53, Ch. 6.6.3], so that in principle it is sufficient to know the primary OPE-coefficients in order to determine any OPE-coefficient. However, actually performing the necessary calculations can be very difficult in general.

Usually when referring to the OPE of two fields we specify only the terms that are singular as $z_1 \to z_2$. This is denoted by the equivalence relation '$\sim$' indicating equality up to regular terms:

$$\Phi_i(z_1)\Phi_j(z_2) \sim z_{12}^{-h_i-h_j} \sum_{\substack{k \\ -h_i-h_j+h_k<0}} C_{ij}^k z_{12}^{h_k} \Phi_k(z_2). \tag{3.7}$$

[4] Sometime authors also refer to OPEs as *short distance expansions*. The reason is that, when more than two fields are involved in an operator product, convergence requires that the distance between the fields being expanded is less than the distance between either of those fields and the next closest other field in the product.

[5] In general the sum could be an integral over a continuum of fields.

3.2.1 Radial ordering

As in conventional quantum field theory the fields are assumed (usually implicitly) to be time-ordered, with operators acting at earlier times placed to the right of operators acting at later times. Operator composition is not defined for products that violate time-ordering, consistent with the physical requirement that causes do not precede their effects. In conformal field theory the time coordinate is mapped to the radial coordinate of the complex plane in such a way that time order $\tau_1 > \cdots > \tau_n$ corresponds to $|z_1| > \cdots > |z_n|$. Time-ordering is therefore called **radial ordering** in conformal field theory. Sometimes radial ordering is written explicitly using the symbol $\mathcal{R}\{\Phi_{i_1}(z_1)\dots\Phi_{i_n}(z_n)\}$, but most of the time we will omit this symbol unless it helps with clarity — products of fields are always implicitly radially ordered.

There is an additional subtlety in defining the product of fields with the same argument — the radially ordered product of such fields is only defined if their OPE contains no singular terms. If the OPE does contain singular terms an alternative definition of the product is required, as is the case in chapter 4, where we define a *normal ordering* prescription.

3.3 The energy-momentum tensor and the Virasoro algebra

The conformal algebra is the Virasoro algebra $\mathfrak{Vir}$, and is generated by a holomorphic field called the (quantum) **energy-momentum tensor** $L(z)$.[6] A second copy of the Virasoro algebra that commutes with the first copy is generated by an anti-holomorphic counterpart of the energy-momentum tensor denoted by $\bar{L}(\bar{z})$. In chapter 4 we will see how these relate to the classical energy-momentum tensor. The fields $L(z)$ and $\bar{L}(\bar{z})$ have conformal weights $(h, \bar{h})$ equal to $(2, 0)$ and $(0, 2)$ respectively, and have Fourier expansions

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad \bar{L}(\bar{z}) \sum_{n\in\mathbb{Z}} \bar{L}_n = \bar{z}^{-n-2}. \tag{3.8}$$

The Virasoro modes can be expressed as residues:

$$L_n = \oint_0 L(w) w^{n+1} \frac{dw}{2\pi i}, \qquad \bar{L}_n = \oint_0 \bar{L}(w) w^{n+1} \frac{dw}{2\pi i}. \tag{3.9}$$

Both contours are taken to be positively oriented simple loops about the origin. Each residue is really an integral operator, and when acting on another field the radius of the loop is defined to respect the radial ordering.

Focusing on the holomorphic sector, the **defining OPE** for the energy-momentum tensor is

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4}\mathbf{1} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w}, \tag{3.10}$$

where c is the central charge of the Virasoro representation. The OPE is *defining* because it is equivalent to the Virasoro commutation relations in eq. (1.3), with C replaced by $c\mathbf{1}$. This equivalence is derived using an important contour manipulation technique that we will use frequently, described in section 3.7.1.

There is also a defining OPE for a primary field with the energy-momentum tensor, derived in section 3.7.2:

$$L(z)\phi_i(w) \sim \frac{h_i\phi_i(w)}{(z-w)^2} + \frac{\partial\phi_i(w)}{z-w}. \tag{3.11}$$

The same contour manipulations from section 3.7.1 can be used to derive the equivalent *defining commutation relations* for primary fields [96, Def. 2.3]:

$$[L_n, \phi_i(z)] = h_i(n+1)z^n\phi_i(z) + z^{n+1}\partial\phi_i(z). \tag{3.12}$$

[6] In the literature the energy-momentum tensor is usually denoted by $T(z)$. See [123, Ch. 1] for a review on Noether's theorem and the energy-momentum tensor in classical field theories.

How to compute commutation relations between modes and fields is demonstrated in section 3.7.3.

3.4 Correlation functions

Theory is connected to experiment in the traditional quantum field theoretic manner by defining a scalar product on the space of states $\mathcal{S}$. Scalar products represent (not-yet-normalised) probability amplitudes whose squared moduli are probabilities.

As per section 1.4 the scalar product we use is a bilinear form $(\cdot,\cdot) : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$, defined to be invariant under the adjoint map. As usual we use Dirac notation, so that we may write $(|u\rangle, |v\rangle) := \langle u|v\rangle$. We may think of $\langle u|$ as a covector in a space $\mathcal{S}^\vee$ dual to $\mathcal{S}$:

$$\langle u| : \mathcal{S} \to \mathbb{C}, \qquad |v\rangle \mapsto (|u\rangle, |v\rangle) := \langle u|v\rangle, \tag{3.13}$$

with the action of the Virasoro algebra on $\langle u|$ defined by the adjoint.

According to the state-field correspondence one can associate to each vector a corresponding field. There is also a sense in which the state-field correspondence generalises to allow one to associate to each covector a corresponding field. For a primary field $\phi_i(z)$ the definition is

$$\langle \phi_i| := \lim_{z\to\infty} z^{2h_i} \langle 0|\, \phi_i(z). \tag{3.14}$$

Section 3.5.2 shows why this is a consistent definition.[7] We considered $|\phi_i\rangle$ to be an asymptotic in-state, because $z \to 0$ corresponds to the infinitely distant past. It is therefore natural to refer to $\langle \phi_i|$ as an *asymptotic out-state*, because $z \to \infty$ corresponds to the infinitely distant future.

By replacing vectors and covectors by their corresponding fields, one should be able to write every scalar product (up to taking limits) in the form:[8]

$$\langle 0|\, \Phi_1(z_1) \dots \Phi_n(z_n)\, |0\rangle. \tag{3.15}$$

This is called a **correlation function**, or **correlator**. A correlator containing n fields is also called an **n-point function**. A correlation function involving only primary fields is called a **primary correlator**. General correlation functions can, in principle, be reduced to closed form expressions involving only primary correlators (see section 3.7.3 for an example) so that if all primary correlators are known then there is sufficient information to determine any correlation function. For this reason computing primary correlators is one of the main goals of conformal field theory.

3.4.1 Chiral and physical correlators

As mentioned in section 3.1.2 the full space of fields consists of both a holomorphic sector and an anti-holomorphic sector, sometimes called *chiral sectors* in the literature. Accordingly, the correlation function in eq. (3.15) is more correctly referred to as a **chiral correlator**. The corresponding anti-holomorphic fields also give rise to a chiral correlator.[9] Chiral correlators are not necessarily single-valued. For example, the value of a chiral correlator might change under the replacement $z_j \mapsto e^{2\pi i} z_j$ — see section 4.7.

However, physically measurable quantities must be single-valued, and so chiral correlators alone cannot correspond directly to such quantities. The physically meaningful quantities such as

[7] The prescription for non-primary fields seems poorly understood. See [53, Ch. 6.1] for example.

[8] Note that the indices here do not refer to specific fields, they are merely labels — we do not define an ordering prescription for the fields.

[9] Chiral correlators corresponding to the anti-holomorphic sector are sometimes referred to as *anti-chiral* correlators, but we do not use this terminology.

probability amplitudes are described by correlation functions that include both sectors, written in the form

$$\langle \Phi_1(z_1, \bar{z}_1) \dots \Phi_n(z_n, \bar{z}_n) \rangle \,. \tag{3.16}$$

This is called a **physical correlator**. We will usually suppress the qualifiers "*chiral*" and "*physical*" when speaking of correlation functions, unless it is not obvious from the context which we are referring to.

In calculations it is usually more convenient to work with chiral correlators, and then use the results to construct physical correlators. Examples of how this can be done are discussed in section 4.7, and in greater detail in chapters 7 and 8. For the moment we mention only that chiral correlators decompose into linear combinations of functions called **conformal blocks**. Conformal blocks are not necessarily single-valued, but by combining conformal blocks from the holomorphic and anti-holomorphic sectors one may be able to construct single-valued physical correlators.

All physically measurable information that can be described by a conformal field theory is represented by way of correlation functions, or equivalently by scalar products. If a scalar product is positive-definite then the corresponding theory is called **unitary**, and the space of states is a Hilbert space.[10] Unlike in conventional quantum field theories, unitarity is not always necessary, although unitary theories do have many desirable properties. See the discussions in [55] and [57, Ch. 13.4] for details.

3.5 The conformal Ward identities

The identity field should remain the identity field under global conformal transformations, so the corresponding state, the true vacuum vector, must be annihilated by the set $\{L_{-1}, L_0, L_1\}$.[11] This set is closed under the adjoint $L_n^\dagger = L_{-n}$, from which it follows that the vacuum covector $\langle 0|$ must also be annihilated by the set $\{L_{-1}, L_0, L_1\}$.

From these vanishing conditions, three partial differential equations can be derived that are satisfied by all primary correlators. Firstly, using L_{-1} we have

$$0 = \langle 0 | \, L_{-1}\phi_1(z_1) \dots \phi_n(z_n) \, |0\rangle \,. \tag{3.17}$$

Commuting L_{-1} across until it annihilates the vacuum at the other end, we have

$$0 = \sum_{i=1}^{n} \langle 0 | \, \phi_1(z_1) \dots [L_{-1}, \phi_i(z_i)] \dots \phi_n(z_n) \, |0\rangle \,. \tag{3.18}$$

From the commutation relations of eq. (3.12) we know that $[L_{-1}, \phi_i(z)] = \partial_z \phi_i(z)$, and so we have the identity

$$0 = \sum_{i=1}^{n} \partial_i \, \langle 0 | \, \phi_1(z_1) \dots \phi_n(z_n) \, |0\rangle \,, \tag{3.19}$$

which is the first differential equation. This equation is a manifestation of *translational invariance*, since ∂ is the generator of translations. Repeating the process for L_0, for which the commutator is $[L_0, \phi_i(z)] = h_i \phi_i(z) + z\partial\phi_i(z)$, we find

$$0 = \langle 0 | \, L_0\phi_1(z_1) \dots \phi_n(z_n) \, |0\rangle = \sum_{i=1}^{n} (h_i + z_i\partial_i) \, \langle 0 | \, \phi_1(z_1) \dots \phi_n(z_n) \, |0\rangle \,, \tag{3.20}$$

[10] Technically the space of states is a pre-Hilbert space, but can be canonically completed to give a genuine Hilbert space.

[11] Recall that the set $\{L_{-1}, L_0, L_1\}$ spans a subalgebra of $\mathrm{span}_{\mathbb{C}}\{L_{-1}, L_0, L_1, \bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$, which generates the Möbius group of global conformal transformations on $\mathbb{S}^2$ (section 2.2.1).

which gives a second differential equation. This equation is a manifestation of symmetry under *complex scaling*, since L_0 generates complex rescaling of the complex plane. Finally, the operator L_1 with the commutator $[L_1, \phi_i(z)] = 2h_i z\phi_i(z) + z^2\partial\phi_i(z)$ gives

$$0 = \langle 0| L_1\phi_1(z_1)\dots\phi_n(z_n)|0\rangle = \sum_{i=1}^{n}(2h_i z_i + z_i^2\partial_i)\langle 0|\phi_1(z_1)\dots\phi_n(z_n)|0\rangle . \tag{3.21}$$

This equation corresponds to symmetry under *special conformal transformations*. Every primary n-point function is annihilated by the differential operators

$$\sum_{i=1}^{n}\partial_i, \qquad \sum_{i=1}^{n}(z_i\partial_i + h_i), \qquad \sum_{i=1}^{n}(z_i^2\partial_i + 2h_i z_i). \tag{3.22}$$

These are the conformal **Ward identities**. They are modified slightly when the fields are not all primary — see [112, Sec. 2.2.3]. Alternative derivations can be found in [117, Ch. 9.2] and [53, Ch. 4.3].

3.5.1 Consequences of the Ward identities

One-point functions

The first identity in eq. (3.22) gives

$$\partial_z\langle 0|\phi(z)|0\rangle = 0, \tag{3.23}$$

so primary one-point functions are all constant. The second identity then gives

$$(z\partial_z + h)\langle 0|\phi(z)|0\rangle = h\langle 0|\phi(z)|0\rangle = 0, \tag{3.24}$$

so a primary one-point function can be non-vanishing if and only if the field has conformal weight $h = 0$.

The identity field is the unique primary field with conformal weight $h = 0$. In this case the correlation function is already normalised by the definition of the scalar product: $\langle 0|\mathbf{1}|0\rangle = \langle 0|0\rangle := 1$. So one-point functions satisfy[12]

$$\langle 0|\phi(z)|0\rangle = \delta_{\phi,\mathbf{1}}. \tag{3.25}$$

For one-point functions the third identity in eq. (3.22) automatically satisfied if the first and second are.

Two-point functions

Consider the two-point function $\langle 0|\phi_1(z_1)\phi_2(z_2)|0\rangle$. Make the change of variables $z := z_1 + z_2$ and $z_{12} := z_1 - z_2$. The first identity becomes

$$\partial_z\langle 0|\phi_1(z_1)\phi_2(z_2)|0\rangle = 0, \tag{3.26}$$

so the two-point function depends only on the difference z_{12}. The second identity becomes

$$(z_{12}\partial_{12} + h_1 + h_2)\langle 0|\phi_1(z_1)\phi_2(z_2)|0\rangle , \tag{3.27}$$

[12] There is a subtlety here that we are suppressing. This is almost certainly too much detail for the moment but, for correctness, we should mention that the formula in eq. (3.25) should contain the *conjugate* (section 3.5.2) of the identity field, and not the identity field itself. Very often the identity field is self-conjugate, so the formula in eq. (3.25) is correct as written. In some cases, like the theory we construct in chapter 8, the identity field is not self-conjugate, and the symbol $\mathbf{1}$ in eq. (3.25) should be replaced by $\mathbf{1}^\vee$.

which is an ordinary differential equation with solution

$$\langle 0|\, \phi_1(z_1)\phi_2(z_2)\, |0\rangle = \frac{C_{1,2}}{z_{12}^{h_1+h_2}}, \tag{3.28}$$

where $C_{1,2} \in \mathbb{C}$ is a constant, called a primary **two-point constant**. Substituting this into the third identity gives

$$(h_1 - h_2)\frac{C_{1,2}}{z_{12}^{h_1+h_2-1}} = 0, \tag{3.29}$$

so the two-constant $C_{1,2}$ can be non-zero only if $h_1 = h_2$.

Three-point functions

For a generic primary n-point function with $n \geq 3$, the three Ward identities are relations that reduce the number of independent complex variables to $n-3$. The functional dependence on the arguments is completely determined for three-point functions. However, the Ward identities are linear, and so the best that we can do for any n-point function is to find a solution up to a multiplicative constant.

For three-point functions the general solution to the system in eq. (3.22) is[13]

$$\langle 0|\, \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\, |0\rangle = \frac{C_{1,2,3}}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1+h_3-h_2} z_{23}^{h_2+h_3-h_1}}, \tag{3.30}$$

where $C_{1,2,3} \in \mathbb{C}$ is a constant, called a primary **three-point constant**. These constants are not arbitrary, but the Ward identities are not sufficient to fix them. Determination of three-point constants is one of the main objectives in conformal field theory. We will see why this is so shortly.

For n-point functions with $n \geq 4$ the solution to the system in eq. (3.22) is determined up to an arbitrary differentiable function of $n-3$ Möbius invariant cross-ratios (also called anharmonic ratios). General formulae can be found in [53, Ch. 8.3]. Four-point functions will be considered (without reference to the general formulae) in chapters 6 to 8.

3.5.2 Unphysical fields

Given a two-point function, one can insert the OPE of the two fields, thereby reducing it to a sum of one-point functions. From the general expression for one-point functions in eq. (3.25) it follows that the two-point function is non-zero if and only if the two fields contain the identity in their OPE.[14] Two fields with non-vanishing two-point function are said to be **conjugates** of one another [96, Def. 2.2]. Every n-point function can be reduced to two-point functions by inserting OPEs, and it follows that if there exists a field without a conjugate, then every correlation function containing that field vanishes. All physically measurable information in a conformal field theory is represented by correlation functions, and a field without a conjugate is therefore indistinguishable from zero by any measurement — such a field is unphysical. We are justified in identifying all such fields with zero,[15] and so may assume that in every conformal field theory that we deal with, every non-zero field has a conjugate.

We denote the conjugate of a field Φ_i by $\Phi_{i^\vee}$.[16] It follows that the two-point constant $C_{i,j}$ is equal to the OPE-coefficient $C_{ij}^{\mathbf{1}}$ of the identity field, and this vanishes unless the fields are

[13] Solving the Ward identities is a rite of passage for learners of conformal field theory, but the solution takes some space — see [53, Ch. 5] or [112, Sec. 2.2.3] for a derivation.

[14] Strictly speaking the conjugate of the identity.

[15] Fields are set to zero by performing a quotient — see chapter 5.

[16] We assume that conjugate pairs are unique. In some cases this is demonstrably true, and is true for all the theories we shall encounter. For others, refer to the discussion in [96], which relates the existence of conjugate pairs to CPT symmetry [83].

conjugate, which we will denote in shorthand by writing $j = i^\vee$. In this case the OPE-coefficient is conventionally normalised to unity: $C_{ij}^{\mathbf{1}} := \delta_{i,j^\vee}$. The normalised general solution to two-point functions is therefore

$$\langle 0| \phi_i(z_1)\phi_j(z_2) |0\rangle = \frac{\delta_{i,j^\vee}}{z_{12}^{2h_i}}. \tag{3.31}$$

Using the definitions in eqs. (3.4) and (3.14) for the state-field correspondence, the correlator in eq. (3.31) leads to

$$\langle \phi_i | \phi_j \rangle = \delta_{i,j^\vee}. \tag{3.32}$$

3.5.3 Operator-product expansions and correlation functions

Now consider a primary three-point function $\langle 0| \phi_i(z_1)\phi_j(z_2)\phi_k(z_3) |0\rangle$. By inserting the OPE

$$\phi_i(z_1)\phi_j(z_2) = z_{12}^{-h_i-h_j} \sum_k C_{ij}^k z_{12}^{h_k} \phi_k(z_2) + \text{descendants}, \tag{3.33}$$

and using the general solutions of eqs. (3.30) and (3.31), we find that the three-point constant $C_{i,j,k}$ coincides with the OPE-coefficient $C_{ij}^{k^\vee}$ of the field $\phi_{k^\vee}$ conjugate to ϕ_k. Hence OPEs determine primary three-point functions, and vice versa.

3.6 The main objective of this thesis

We have now accumulated sufficient background knowledge to understand and appreciate the primary goals of this thesis and the motivation for these goals.

First we aim to construct conformal field theories called the *minimal models*, whose primary fields have conformal weights that can be related to critical exponents of statistical models — these fields will model statistical observables at criticality, and in particular should describe the scaling behaviour of these observables in correlation functions.

Next we aim to determine the primary OPE-coefficients for our conformal field theories. For this we shall use some tools from string theory, and these tools will be revealed in subsequent chapters. Since every correlation function can be expressed in terms of primary correlation functions, and primary n-point functions can be reduced to three-point functions by inserting OPEs, it follows that if primary OPE-coefficients are known, then primary three-point functions are known, and so in principle enough information is known for any correlation function to be determined.

Finding all of the primary three-point constants, or equivalently the primary OPE-coefficients, is what it means to solve a conformal field theory. This is our main objective, and we shall attain it for three different minimal models describing the scaling limits of certain statistical lattice models at criticality.

3.7 Fundamental techniques

Here we develop fundamental tools that will be used very frequently in almost every subsequent chapter.

3.7.1 Commutation relations from generating functions

In this section we demonstrate how the defining OPE for the energy-momentum tensor gives rise to Virasoro commutation relations. However, the following technique is not limited to the

energy-momentum tensor only — the process can be adapted to other quantum fields, and we will use this technique again in chapters 4 and 6.

Virasoro generators are extracted as residues from the energy-momentum tensor using eq. (3.9). The commutation relations of the modes are given by

$$\begin{aligned}[L_n, L_m] &= [\oint_0 L(z) z^{n+1} \frac{dz}{2\pi i}, \oint_0 L(w) w^{m+1} \frac{dw}{2\pi i}] \\ &= \oint_0 \oint_0 L(z)L(w) z^{n+1} w^{m+1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} - \oint_0 \oint_0 L(w)L(z) z^{n+1} w^{m+1} \frac{dz}{2\pi i}\frac{dw}{2\pi i}, \\ &\qquad |z|>|w| \qquad\qquad\qquad |w|>|z|\end{aligned} \tag{3.34}$$

where the contours respect the radial ordering. The contours of integration can be deformed as per fig. 3.1.

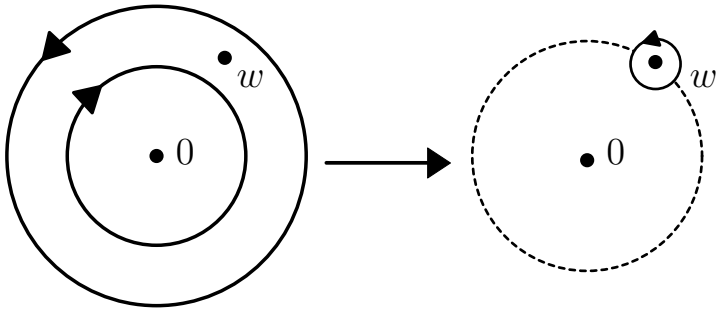


Figure 3.1: *Two oppositely oriented z-contours are deformed to a tight single contour enclosing the point w.*

From the contour deformation we find that the integral in eq. (3.34) is equal to an integral with respect to z along a tight contour about w, followed by an integral with respect to w along a contour about the origin, with the integrand given by the radially ordered product of the two fields:

$$= \oint_0 \left(\oint_w \mathcal{R}\{L(z)L(w)\} z^{n+1} w^{m+1} \frac{dz}{2\pi i} \right) \frac{dw}{2\pi i}. \tag{3.35}$$

Inserting the defining OPE of eq. (3.10), all regular terms in the integral vanish by Cauchy's theorem, leaving only the singular terms:

$$= \oint_0 \left(\oint_w \left\{ \frac{c/2}{(z-w)^4}\mathbf{1} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)} \right\} z^{n+1} w^{m+1} \frac{dz}{2\pi i} \right) \frac{dw}{2\pi i}. \tag{3.36}$$

The z-integral can be computed using Cauchy's integral formula:

$$= \oint_0 \left(\frac{c}{12}(n+1)(n)(n-1)\mathbf{1} w^{n+m-1} + 2L(w)(n+1) w^{n+m+1} + \partial L(w) w^{m+n+2} \right) \frac{dw}{2\pi i}. \tag{3.37}$$

Using the series expansion for $L(z)$, and the residue theorem, we have

$$\begin{aligned} &= \frac{c}{12} n(n^2-1)\mathbf{1}\delta_{n,-m} + 2(n+1)L_{m+n} + \mathrm{Res}_{w=0}\left\{ \sum_{r\in\mathbb{Z}} L_r(-r-2) w^{m+n-r-1} \right\} \\ &= \frac{c}{12} n(n^2-1)\mathbf{1}\delta_{n,-m} + (n-m)L_{m+n}. \end{aligned} \tag{3.38}$$

These are the commutation relations in eq. (1.3) for the Virasoro algebra. The commutation relation $[C, L_n] = 0$ is included in $C = c\mathbf{1}$.

3.7.2 Defining OPEs

In this section we show how to calculate OPEs using the state-field correspondence, and how the defining OPE of eq. (3.11) for a Virasoro primary field is implied by the definition of a Virasoro primary state once the state-field correspondence is taken into account. These methods will be used frequently in chapters 4, 6 and 8.

Let $\phi(w)$ be a Virasoro primary field with conformal weight h. Make an ansatz for the OPE with the energy-momentum tensor: $L(z)\phi(w) = \sum_n (z-w)^{-n-2}\phi_n(w)$, where $\phi_n(w)$ are unknown fields and the values taken by n are to be decided. We apply both sides to $|0\rangle$ and use the state-field correspondence to get

$$\lim_{w\to 0} L(z)\phi(w)\,|0\rangle = L(z)\,|\phi\rangle = \sum_n |\phi_n\rangle\, z^{-n-2}. \tag{3.39}$$

Inserting the mode expansion in eq. (3.8) for $L(z)$ in terms of Virasoro modes and comparing powers of z we find that

$$L_n\,|\phi\rangle = |\phi_n\rangle\,. \tag{3.40}$$

From the definition in section 3.1.3 of a Virasoro primary state we have

$$|\phi_n\rangle = \begin{cases} 0 & n > 0 \\ h\,|\phi\rangle & n = 0 \\ L_{-1}\,|\phi\rangle & n = -1. \end{cases} \tag{3.41}$$

The remaining terms are regular in z and do not contribute to the singular part of the OPE. Since L_{-1} corresponds to the generator of translations ∂ we can read of the OPE

$$L(z)\phi(w) \sim \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}, \tag{3.42}$$

which confirms eq. (3.11) as a consequence of the definition of a primary state and the state-field correspondence.

3.7.3 Descendent correlators from primary correlators

In this section we demonstrate how correlation functions containing descendent fields can be expressed in terms of primary correlators. On the one hand it is important to show that this is possible, because the rest of this thesis focuses on primary correlators only. On the other hand we will need to use this technique in chapter 5 when dealing with unphysical fields in correlation functions.

Let $\Phi(w)$ be any non-zero field and consider the corresponding descendent state

$$L_{-n}\,|\Phi\rangle = L_{-n} \lim_{w\to 0} \Phi(w)\,|0\rangle\,. \tag{3.43}$$

The action of the Virasoro algebra can be extracted from the energy-momentum tensor:

$$\lim_{w\to 0} L_{-n}\Phi(w)\,|0\rangle = \lim_{w\to 0} \oint_0 \frac{L(z)\Phi(w)}{(z-w)^{n-1}} \frac{dz}{2\pi i}\,|0\rangle := \lim_{w\to 0} (L_{-n}\Phi)(w)\,|0\rangle\,. \tag{3.44}$$

As always, the contour respects the radial ordering. The corresponding field is written as

$$(L_{-n}\Phi)(w) := \oint_w \frac{L(z)\Phi(w)}{(z-w)^{n-1}} \frac{dz}{2\pi i}, \tag{3.45}$$

which is an integral operator that can be applied to fields and states.

Now consider a correlator of the form

$$\langle 0| \left(L_{-n}\Phi\right)(z_1) S(z_2, z_3, \dots) |0\rangle \,, \tag{3.46}$$

where $\Phi(z_1)$ is not necessarily primary, and where $S(z_2, z_3, \dots) := \phi_2(z_2)\phi_3(z_3)\dots$ is a product of primary fields. Inserting the contour integral of eq. (3.44) we have

$$= \langle 0| \oint_{z_1} \frac{L(z)\Phi(z_1)}{(z-z_1)^{n-1}} \frac{dz}{2\pi i} S(z_2, z_3, \dots) |0\rangle \,. \tag{3.47}$$

The contour about z_1 may be deformed as shown in fig. 3.2.

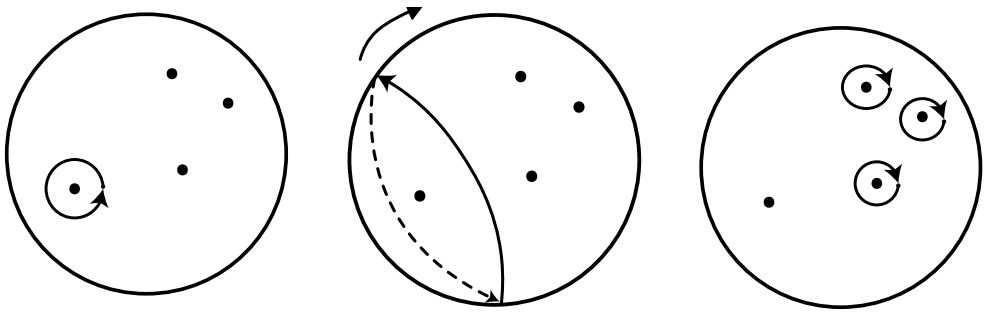


Figure 3.2: *The tight contour oriented positively about one pole is equivalent to tight contours negatively oriented about the other poles.*

$$\oint_{z_1} \text{“}=\text{”} \oint_{\infty} - \oint_{0} - \sum_{j\neq 1} \oint_{z_j} \,. \tag{3.48}$$

There is no contribution from the origin because there is no pole there, unless one of the z_j is at the origin. In fact there is also no contribution from the point at infinity. To confirm this we insert the mode expansion for the energy-momentum tensor into eq. (3.47) — the modes L_{-n} with $n \geq -1$ annihilate $\langle 0|$ (via the adjoint action). For the remaining possible values of n, one can confirm that the integral about infinity vanishes by making the change of variables $z = 1/w$, and integrating with respect to w over a tight contour about the origin (remembering to change the integration measure: $dz = -w^{-2}dw$).

The energy-momentum tensor can now be commuted through the fields $S(z_2, z_3, \dots)$, and since the fields are all primary, the contribution from each field $\phi_j(z_j)$ can be determined from the defining OPE of eq. (3.11):

$$\begin{aligned} \oint_{z_j} \frac{L(z)\phi_j(z_j)}{(z-z_1)^{n-1}} \frac{dz}{2\pi i} &= \oint_{z_j} \left(\frac{h_j \phi_j(z_j)}{(z-z_1)^{n-1}(z-z_j)^2} + \frac{\partial_j \phi_j(z_j)}{(z-z_1)^{n-1}(z-z_j)} \right) \frac{dz}{2\pi i} \\ &= \frac{-(n-1)h_j\phi_j(z_j)}{(z_j-z_1)^n} + \frac{\partial_j\phi_j(z_j)}{(z_j-z_1)^{n-1}}. \end{aligned} \tag{3.49}$$

The calculation in eq. (3.49) defines the commutator $[L_{-n}, \phi_j(z_j)]$.

Summing the negatives of all of these terms as per eq. (3.48) gives the desired result:

$$\begin{aligned} &\langle 0| \left(L_{-n}\Phi\right)(z_1) S(z_2, z_3, \dots) |0\rangle \\ &= \sum_{j\neq 1} \left\{ \frac{(n-1)h_j}{(z_j-z_1)^n} - \frac{1}{(z_j-z_1)^{n-1}} \partial_j \right\} \langle 0| \Phi(z_1) S(z_2, z_3, \dots) |0\rangle \end{aligned} \tag{3.50}$$

The differential operator out the front is useful enough to deserve a label:

$$\sum_{j\neq i}\left\{\frac{(n-1)h_j}{(z_j-z_i)^n}-\frac{1}{(z_j-z_i)^{n-1}}\partial_j\right\}:=\mathcal{L}_{-n}(z_i). \tag{3.51}$$

This result shows that a non-primary correlator can be expressed as a differential operator acting on a primary correlator. For the case where $S(z_2,z_3,\dots)$ contains fields that are not primary, see [53, Ch. 6.6.3].

This chapter has introduced conformal field theory in an abstract form. The abstraction makes clearer the distinction between model-dependent data, such as the physical interpretation of the central charge in a particular model, from universal data pertaining to all conformal field theories, such as the assumption of a state-field correspondence and the existence of an operator product expansion. In chapter 4 we shall consider a specific model and observe how these abstract notions arise. The starting point is a classical Lagrangian formulation of a massless scalar field theory known as the *free boson*.

Chapter 4

Vertex Operators and the Free Boson

The most illustrative and important example of a conformal field theory is undoubtedly the free boson. It is tractable, but still versatile, and contains the essential building blocks for constructing more complicated conformal field theories. Here we make concrete the content of chapters 1 to 3 by demonstrating how these ideas naturally emerge from a Lagrangian formalism. The main goal of this chapter is to develop the formalism of vertex operators, using the free boson as a physically motivated backdrop inspired by string theory. We will return to the free boson and its vertex algebra[1] time and time again — vertex operators are the most important objects in this thesis.

4.1 The classical free boson

The free boson is a scalar field $\varphi(x,t)$ whose domain is an infinite cylinder $S^1 \times \mathbb{R}$ and whose image is in $\mathbb{R}$:[2]

$$\varphi : S^1 \times \mathbb{R} \to \mathbb{R}, \qquad (x,t) \mapsto \varphi(x,t). \tag{4.1}$$

The "spatial coordinate" is x, and the spatial dimension is compactified to a circle S^1 of circumference $L \in \mathbb{R}$, on which the boson is assigned periodic boundary conditions:

$$\varphi(x+L,t) = \varphi(x,t). \tag{4.2}$$

Working in flat Minkowski space $\mathbb{R}^{1,1}$ with diagonal metric $\mathrm{diag}(-1,1)$, the dynamics of the free boson are described in the Lagrangian formalism by the Lagrange density[3]

$$\mathcal{L}(\varphi, \partial_\mu\varphi, x^\nu) = \frac{1}{2g}\partial_\mu\varphi\partial^\mu\varphi, \tag{4.3}$$

where g is a coupling constant, which we will fix for convenience later.[4] The action is given by the functional integral of the Lagrange density:

$$S[\varphi] = \frac{1}{2g}\int_{S^1\times\mathbb{R}} \partial_\mu\varphi\partial^\mu\varphi \, dxdt. \tag{4.4}$$

The equations of motion are obtained by extremising the action, leading to the *Euler-Lagrange equations*:

$$0 = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = \frac{1}{g}(\partial_x^2\varphi - \partial_t^2\varphi). \tag{4.5}$$

[1] Vertex algebras are sometimes confused with vertex operator algebras. Vertex algebras were defined in [17]. Vertex operator algebras are special cases of vertex algebras, and were defined in [54].

[2] The field $\varphi(x,t)$ is actually just one component of the full string theoretic version. In string theory the free bosonic string is usually defined to take values in 26-dimensional Minkowski spacetime $\mathbb{R}^{1,25}$, the target space of the string — see [134].

[3] See [123, Ch. 1] for review of the Lagrangian formalism and calculus of variations.

[4] The fact that the Lagrange density depends only on derivatives of φ, but not on φ explicitly, makes the free boson a *massless* scalar field theory.

The negative sign in front of the time derivative is from the choice of Minkowski metric. From eq. (4.5) we see that the classical equation of motion is the wave equation

$$\partial_t^2 \varphi = \partial_x^2 \varphi. \tag{4.6}$$

4.1.1 Wick rotation

The conformal algebra we derived in chapter 2 was for Euclidean space $\mathbb{R}^{0,2}$, not Minkowski space $\mathbb{R}^{1,1}$. To convert between the two one performs a *Wick rotation*. The prescription is to complexify the time variable $t \mapsto i\tau$, where τ is formally treated as real valued.[5] The cylinder is then mapped to $\mathbb{C}$ by the exponential map:

$$z := \exp\left(\frac{2\pi}{L}(\tau + ix)\right), \qquad \bar{z} := \exp\left(\frac{2\pi}{L}(\tau - ix)\right). \tag{4.7}$$

Differential operators transform under the chain rule:

$$\partial_x = \frac{2\pi i}{L}(z\partial_z - \bar{z}\partial_{\bar{z}}), \qquad \partial_t = -\frac{2\pi i}{L}(z\partial_z + \bar{z}\partial_{\bar{z}}). \tag{4.8}$$

In terms of the new variables the equations of motion become

$$\partial_t^2 \varphi = \partial_x^2 \varphi \implies \partial_z \partial_{\bar{z}} \varphi = 0. \tag{4.9}$$

Since the partial derivatives commute we learn that $\partial_z \varphi$ is independent of $\bar{z}$, and that $\partial_{\bar{z}} \varphi$ is independent of z. The fields $\partial_z \varphi$ and $\partial_{\bar{z}} \varphi$ are thus (locally) holomorphic and anti-holomorphic respectively, and we will therefore display their arguments explicitly by writing $\partial\varphi(z)$ and $\bar{\partial}\varphi(\bar{z})$.

4.1.2 Conformal invariance

A symmetry of the action is a symmetry of the dynamics, so to show that we are dealing with a conformal field theory if suffices to check that the action functional is invariant under a general infinitesimal conformal transformation (to first order of course). From chapter 2 we know that such a transformation is implemented by holomorphic or anti-holomorphic maps ω and $\bar{\omega}$ and infinitesimals $\epsilon, \bar{\epsilon} \in \mathbb{R}$:

$$z \mapsto z' = z + \epsilon\omega(z), \qquad \bar{z} \mapsto \bar{z}' = \bar{z} + \bar{\epsilon}\bar{\omega}(\bar{z}). \tag{4.10}$$

Under this transformation the derivatives transform via the chain rule:

$$\partial_z = \frac{\partial z'}{\partial z}\partial_{z'} = (1 + \epsilon\partial_z\omega)\partial_{z'} \qquad \partial_{\bar{z}} = \frac{\partial \bar{z}'}{\partial \bar{z}}\partial_{\bar{z}'} = (1 + \bar{\epsilon}\partial_{\bar{z}}\bar{\omega})\partial_{\bar{z}'} \tag{4.11}$$

and the differentials transform contravariantly:

$$dz = (1 - \epsilon\partial_z\omega)dz', \qquad d\bar{z} = (1 - \bar{\epsilon}\partial_{\bar{z}}\bar{\omega})d\bar{z}'. \tag{4.12}$$

The free boson $\varphi(z, \bar{z})$ is a *scalar field* and therefore is invariant under coordinate transformations, so $\varphi'(z', \bar{z}') = \varphi(z, \bar{z})$. Substituting these into the action functional and simplifying we find

$$S'[\varphi'] = \frac{1}{g}\int \partial_{z'}\varphi'\partial_{\bar{z}'}\varphi' dz' d\bar{z}' = S[\varphi] + \mathcal{O}(\epsilon^2, \bar{\epsilon}^2, \epsilon\bar{\epsilon}). \tag{4.13}$$

The action is invariant under infinitesimal conformal transformations, which confirms that the free boson is a (classical) conformal field theory.

[5] The motivation here is to perform calculations in Euclidean space, and later Wick rotate back to Minkowski space. This technique is poorly motivated physically, but as a calculational tool our intentions are clear. In any case this is standard practice. See [117, Ch. 8.5] for mathematical details and [24, Ch. 2.1] for Wick rotations in standard quantum field theory.

4.2 The quantum free boson

The idea behind quantising a classical field is to construct a field theory that incorporates features of quantum mechanics, such as non-commuting observables, while preserving important classical features, such as symmetries. In classical Hamiltonian mechanics, if a system has degrees of freedom q_i with conjugate momenta p_i, then these satisfy the Poisson brackets:

$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{i,j}. \tag{4.14}$$

Dirac introduced the *canonical quantisation* prescription for quantising a classical field theory, which consists of promoting the degrees of freedom and their conjugate momenta to quantum operators and replacing the Poisson brackets in eq. (4.14) by commutator brackets:[6]

$$\{\cdot, \cdot\} \mapsto \frac{1}{i}[\cdot, \cdot]. \tag{4.15}$$

To quantise the free boson we Fourier expand the free boson field, using the fact that the x-direction is periodic:

$$\varphi(x,t) = \sum_{n\in\mathbb{Z}} \phi_n(t) e^{2\pi i n x/L}. \tag{4.16}$$

The time dependent Fourier modes ϕ_n are the degrees of freedom. Substituting the Fourier expansion into the Lagrange density, and denoting $\partial_t\phi$ by $\dot{\phi}$, we have

$$\mathcal{L} = \frac{1}{2g}\left(-(\partial_t\phi)^2 + (\partial_x\phi)^2\right) = \frac{1}{2g}\sum_{n,m\in\mathbb{Z}} \left\{-\dot{\phi}_n(t)\dot{\phi}_m(t) - \frac{4\pi^2 nm}{L^2}\phi_n(t)\phi_m(t)\right\} e^{2\pi i(n+m)x/L}. \tag{4.17}$$

The Lagrangian $\mathbb{L}$ is the integral of the Lagrange density over one period in x. Due to the exponential factor, all terms with $n \neq -m$ vanish, leaving

$$\mathbb{L} := \frac{1}{2g}\int_0^L \mathcal{L}dx = \frac{-L}{2g}\sum_{n\in\mathbb{Z}} \left\{\dot{\phi}_n(t)\dot{\phi}_{-n}(t) - \frac{4\pi^2 n^2}{L^2}\phi_n(t)\phi_{-n}(t)\right\}. \tag{4.18}$$

The momentum $\pi_n(t)$ conjugate to $\phi_n(t)$ is defined as

$$\pi_n(t) := \frac{\partial\mathbb{L}}{\partial(\dot{\phi}_n(t))} = \frac{-L}{2g}\sum_{m\in\mathbb{Z}} \frac{\partial}{\partial(\dot{\phi}_n(t))}\left\{\dot{\phi}_n(t)\dot{\phi}_{-n}(t) - \frac{4\pi^2 n^2}{L^2}\phi_n(t)\phi_{-n}(t)\right\} = \frac{-L}{g}\dot{\phi}_{-n}(t). \tag{4.19}$$

We now replace the Poisson brackets of eq. (4.14) with commutator brackets, as per eq. (4.15):

$$[\phi_m(t), \phi_n(t)] := 0, \qquad [\pi_m(t), \pi_n(t)] := 0, \qquad [\phi_m(t), \pi_n(t)] := i\delta_{m,n}. \tag{4.20}$$

These are the *canonical equal-time commutation relations*. Note that the final commutation relation can be written as

$$[\phi_m(t), \dot{\phi}_n(t)] = \frac{ig}{L}\delta_{m,-n}. \tag{4.21}$$

4.2.1 The Heisenberg algebra

Let us translate the canonical commutation relations into complex coordinates. Focusing on the holomorphic sector, the Fourier expansion can be written as

$$\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \implies a_n = \oint_0 \partial\varphi(z) z^n \frac{dz}{2\pi i}. \tag{4.22}$$

[6] Natural units, $\hbar = 1$.

To figure out what the a_n should be from canonical quantisation, we can compute this integral explicitly in the original variables x and t. We choose a circular integration contour so that t is fixed and x varies over one period. The differential becomes

$$\frac{dz}{2\pi i} = \frac{1}{2\pi i}\frac{\partial z}{\partial x}dx = z dx. \tag{4.23}$$

Then using the change of variables in eq. (4.8) the integral becomes

$$a_n = \oint_0 \partial\varphi(z)\frac{dz}{2\pi i} = \int_0^L \frac{L}{4\pi i}(\partial_x\varphi(x,t) - \partial_t\varphi(x,t))e^{2\pi i(x-t)n/L}dx. \tag{4.24}$$

Inserting the Fourier expansion in eq. (4.22) we find that

$$a_n = -\left\{\frac{n}{2}\phi_{-n}(t) + \frac{L}{4\pi i}\dot{\phi}_{-n}(t)\right\}e^{-2\pi i t n/L}. \tag{4.25}$$

It is now straightforward to compute the commutation relations of the modes a_n using the canonical commutation relations in eqs. (4.20) and (4.21):

$$[a_n, a_m] = [\left\{\frac{n}{2}\phi_{-n}(t) + \frac{L}{4\pi i}\dot{\phi}_{-n}(t)\right\}, \left\{\frac{m}{2}\phi_{-m}(t) + \frac{L}{4\pi i}\dot{\phi}_{-m}(t)\right\}]e^{-2\pi i(m+n)t/L} \tag{4.26}$$

$$= n\frac{g}{4\pi^2}\delta_{n,-m}. \tag{4.27}$$

If we work in units such that the coupling constant takes the value $g = 4\pi^2$, then the commutation relations take the familiar form

$$[a_n, a_m] = n\delta_{n,-m}, \tag{4.28}$$

which we recognise as the commutation relations of the Heisenberg algebra from eq. (1.4).

4.2.2 The Virasoro algebra and the energy-momentum tensor

From section 3.3 we know that the generating function for the Virasoro algebra is the (quantum) energy-momentum tensor $L(z)$, which expands in Virasoro modes as

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{4.29}$$

Classically, the energy-momentum tensor $T^{\mu\nu}$ encodes the first-order change in the action under general (not necessarily conformal) infinitesimal transformations $x^\mu \mapsto x'^\mu = x^\mu + \epsilon w^\mu(x)$, defined by

$$\delta S := S' - S = \epsilon \int T^{\mu\nu}\partial_\rho w^\rho \, d^d x, \tag{4.30}$$

where d is the number of dimensions. For the free boson in two-dimensional complex coordinates, substituting into the action functional the general infinitesimal transformation

$$z \mapsto z' = z + \epsilon w(z,\bar{z}), \qquad \bar{z} \mapsto \bar{z}' = \bar{z} + \epsilon\bar{w}(z,\bar{z}) \tag{4.31}$$

results in

$$\delta S = \epsilon \int \frac{-1}{g}\left(\partial\varphi\partial\varphi\partial_{\bar{z}}w + \bar{\partial}\varphi\bar{\partial}\varphi\partial_z\bar{w}\right) dz d\bar{z}. \tag{4.32}$$

The components (in complex coordinates) of the energy-momentum tensor can be read off as

$$T^{zz} = 0 = T^{\bar{z}\bar{z}}, \qquad T^{\bar{z}z} = \frac{-1}{g}\partial\varphi\partial\varphi, \qquad T^{z\bar{z}} = \frac{-1}{g}\bar{\partial}\varphi\bar{\partial}\varphi. \tag{4.33}$$

The final two terms define holomorphic and anti-holomorphic fields respectively. Conventionally the coupling constant is absorbed and the energy-momentum tensor is written as

$$T(z) = -\frac{g}{2}T^{\bar{z}z} = \frac{1}{2}\partial\varphi(z)\partial\varphi(z), \qquad \bar{T}(\bar{z}) = -\frac{g}{2}T^{z\bar{z}} = \frac{1}{2}\bar{\partial}\varphi(\bar{z})\bar{\partial}\varphi(\bar{z}). \tag{4.34}$$

The expressions in eq. (4.34) are rather subtle. In the classical setting it makes perfect sense to multiply two fields together — fields are just functions. But, having implemented canonical quantisation, these objects are definitely not functions. They are operator valued *quantum fields*,[7] and we know from section 3.2.1 that multiplying quantum fields with the same argument requires care. To properly define these products we need to first define what space of states these operators are going to act on.

Given that the energy-momentum tensor is some kind of product of free bosons, which are generating functions for the Heisenberg algebra $\mathfrak{H}$, the natural choice is to act on a Fock space $\mathcal{F}(p)$. The object $\partial\varphi(z)\partial\varphi(z)$ is well defined if it has well defined action on every vector in $\mathcal{F}(p)$. If the free boson Fourier expansions are substituted directly into eq. (4.34), one quickly finds that the result is divergent when acting with L_0 on the vacuum vector $|p\rangle$ as if the vacuum had infinite energy. This is of course unphysical. It is instructive to give this a try and observe the divergent result.

Divergent products are well known to quantum field theorists — see [24, Ch. 6] a review. The resolution, if possible, is often simply to subtract off the divergent part from a product. For the free boson this amounts to a **normal ordering** prescription[8] — the product of quantum fields with the same argument needs to be normally ordered to make sense. See any textbook on quantum field theory or renormalisation for a review on this, for example [103, Part II]. For the free boson, normal ordering amounts to putting creation operators to the left of annihilation operators:[9]

$$:a_n a_m: \;:= \begin{cases} a_n a_m & \text{if } n \leq -1 \\ a_m a_n & \text{if } n \geq 0. \end{cases} \tag{4.35}$$

This prescription is really only necessary when $n = -m \neq 0$, since otherwise the modes commute. The quantum energy-momentum tensor is defined as the normally ordered product

$$L(z) := \frac{1}{2}:\partial\varphi(z)\partial\varphi(z):. \tag{4.36}$$

Substituting eqs. (3.9) and (4.22) into eq. (4.36) and comparing powers of z, we can extract the modes:

$$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} :a_{n-r}a_r: = \frac{1}{2}\sum_{r\leq -1} a_r a_{n-r} + \frac{1}{2}\sum_{r\geq 0} a_{n-r}a_r. \tag{4.37}$$

The normal ordering makes no difference when $n \neq 0$ since the Heisenberg modes commute if their indices do not sum to zero. However, for L_0 we have

$$L_0 = \sum_{r>0} a_{-r}a_r + \frac{1}{2}a_0^2. \tag{4.38}$$

These modes satisfy the Virasoro commutation relations in eq. (1.3) with central charge $c = 1$.[10] To prove this, a good starting point is to show that

$$[a_n, L_m] = n a_{m+n} \tag{4.39}$$

[7] In more mathematical terminology, one might refer to quantum fields as *operator valued formal distributions*.

[8] Sometimes quantum field theorists refer to normal ordering as *Wick ordering*.

[9] Note that the transition from the classical field theory to the quantum field theory does not specify any particular ordering prescription for infinite products of Heisenberg generators. The choice of normal ordering in eq. (4.35) is not the unique way to regularise such products, and there is no guarantee that this choice is physically relevant (for example, other definitions are used in string theory).

[10] Showing this is another rite of passage for conformal field theory learners — it is tedious, but can be done in a page or two of algebra.

using eqs. (4.37) and (4.38). See [117, Thm. 8.8] for a full proof. This confirms that the free boson is a conformal field theory with central charge $c = 1$.

Note that the action of L_0 on a vacuum vector $|p\rangle \in \mathcal{F}(p)$ with momentum p gives

$$L_0 |p\rangle = \frac{1}{2} a_0^2 |p\rangle = \frac{1}{2} p^2 |p\rangle . \tag{4.40}$$

Since p represents momentum this is perfectly consistent with L_0 being the energy operator.[11] We will still refer to the L_0-eigenvalue $h_p = \frac{1}{2}p^2$ as the conformal weight. The fact that p is real ensures that h_p is non-negative for every momentum, which is a good thing — the energy of a physical system should be bounded from below.

It remains to check that the action of every mode L_n is well defined on every vector in every Fock space. This is indeed true, and the proof is straightforward. First note that, for any $p \in \mathbb{R}$, every partition basis vector in $\mathcal{F}(p)$ is annihilated by all a_m with sufficiently large m. It follows that when acting on any partition basis vector with L_n, only a finite number of terms from each sum in eq. (4.38) are non-zero. Since every vector in $\mathcal{F}(p)$ is a linear combination of partition basis vectors, L_n has well defined action on every vector in $\mathcal{F}(p)$.

4.3 The state-field correspondence

Having derived mode expansions for our quantum fields we can check explicitly the state-field correspondence as per eq. (3.4). For example, the state corresponding to the field $\partial\varphi(z)$ is

$$|\partial\varphi\rangle := \lim_{z\to 0} \partial\varphi(z) |0\rangle = \lim_{z\to 0} (\cdots + a_{-2} |0\rangle z + a_{-1} |0\rangle z^0 + a_0 |0\rangle z^{-1} + \dots) = a_{-1} |0\rangle . \tag{4.41}$$

We see why the order of operations in eq. (3.4) is important — there are poles at the origin, but their coefficients annihilate the vacuum and the limit is well defined. Using the mode expansion in eq. (4.37) for the Virasoro algebra one can confirm that $|\partial\varphi\rangle$ is a Virasoro primary state (vacuum vector) with conformal weight $h = 1$ by showing that it is an L_0-eigenvector that is annihilated by L_n with $n > 0$. It follows that $\partial\varphi(z)$ satisfies the defining OPE in eq. (3.11) for a Virasoro primary field:

$$L(z)\partial\varphi(w) \sim \frac{\partial\varphi(z)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}, \tag{4.42}$$

which can be derived explicitly using the techniques in section 3.7.2 and the commutation relations in eq. (4.39).

4.3.1 Virasoro modules in Fock spaces

Despite being a Virasoro highest weight state, $|\partial\varphi\rangle$ is not a highest weight state for the Heisenberg algebra — it is not annihilated by a_1 (in fact, $a_1 |\partial\varphi\rangle = |0\rangle$). Whether or not a state is primary depends on which symmetry algebra we are referring to. All Heisenberg primaries are Virasoro primaries — it follows from eq. (4.37) that all states that are annihilated by a_n with $n > 0$ are also annihilated by L_n with $n > 0$ and, of such states, it is clear from eq. (4.38) that a_0-eigenvectors are also L_0-eigenvectors. The converse is not true, a counterexample being the state $|\partial\varphi\rangle$.

The Fock space $\mathcal{F}(p)$ is constructed first and foremost as a highest weight $\mathfrak{H}$-module — the Virasoro algebra is realised as a subalgebra of the Heisenberg algebra (technically as a subalgebra of a completion of the universal enveloping algebra $U(\mathfrak{H})$) by defining normally ordered infinite sums, and $\mathcal{F}(p)$ becomes a $\mathfrak{Vir}$-module after the fact. The vacuum vector $|p\rangle$ has conformal weight $h_p = \frac{1}{2}p^2$, and becomes a highest weight vector for the $\mathfrak{Vir}$-module $U(\mathfrak{Vir}) |p\rangle$ with $c = 1$.

[11] Actually there is also the anti-holomorphic sector to consider — the full Hamiltonian is given by $L_0 + \bar{L}_0$. This is also the generator of dilations on $\mathbb{C}$, which is consistent with the Hamiltonian being the generator of time translation in quantum field theory, since the time direction was mapped to the radial direction on the plane.

The Heisenberg modes a_n shift conformal weights in the same way as the Virasoro modes. Using the commutation relations in eqs. (1.3) and (4.39) one can show that if $|v\rangle$ is an L_0-eigenvector with eigenvalue h then both $L_{-n}|v\rangle$ and $a_{-n}|v\rangle$ are also L_0-eigenvectors with eigenvalue $h+n$.

However, $\mathfrak{H}$-modules and $\mathfrak{Vir}$-modules are not quite the same thing. For example, the Heisenberg vacua $|p\rangle$ and $|-p\rangle$ have the same conformal weight $h_p = \frac{1}{2}p^2$ and are indistinguishable by the Virasoro algebra, but the Heisenberg algebra can distinguish them from their a_0-eigenvalues.

4.4 Scalar products

In section 1.4 we defined the Shapovalov form on a Fock space to be a $\mathbb{C}$-bilinear form invariant with respect to a Heisenberg adjoint map: $a_n^\dagger = a_{-n}$ and $k^\dagger = k$. The construction of correlation functions in section 3.4 was similar, but for the scalar product to respect conformal symmetry required invariance under the Virasoro adjoint map: $L_n^\dagger = L_{-n}$ and $C^\dagger = C$.

Equation (4.38) for the Virasoro algebra is quadratic in Heisenberg generators, providing two possible choices for the corresponding Heisenberg adjoint — both $a_n^\dagger = a_{-n}$ and $a_n^\dagger = -a_{-n}$ result in $L_n^\dagger = L_{-n}$. The conventional choice of adjoint is the former, but we will instead choose that latter because this connects up well with chapter 8. We should point out that the scalar product defined here is not positive-definite.

The vacuum covector $\langle p|$ and vector $|p\rangle$ have the same conformal weight $h_p = \frac{1}{2}p^2$ since L_0 is self-adjoint. However, a_0 is *anti*-self-adjoint, and so we have

$$\langle p|\, a_0 = (-a_0\,|p\rangle)^\dagger = -p\,\langle p|\,, \tag{4.43}$$

which means that $\langle p|$ has momentum $-p$. With the standard normalisation we have the scalar product

$$(|p\rangle\,, |q\rangle) = \langle p|q\rangle = \delta_{p+q=0}. \tag{4.44}$$

Since L_0 is self-adjoint, scalar products vanish if the vectors have different conformal weights, consistent with the Ward identities and eq. (3.31). Since a_0 is anti-self-adjoint, scalar products also vanish if the vectors do not have momenta that sum to zero — a *conservation of momentum* rule.

Correlation functions can now be constructed as per section 3.4 by replacing vectors in a scalar product by their corresponding fields. A powerful result called Wick's theorem allows us to compute some of them in a very simple way.

4.5 Wick's theorem for bosons

A conformal field theory is said to be *free* if the identity field is the only non-zero field that appears in the singular terms of each defining OPE. Unsurprisingly, the free boson is a free field. We can show this by explicitly computing the OPE using the mode expansion in eq. (4.22). We start by considering the radially ordered product of two free bosons, into which we substitute the mode expansions:

$$\partial\varphi(z)\partial\varphi(w) = \sum_{r,s\in\mathbb{Z}} a_r a_s z^{-r-1} w^{-s-1} = \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} a_r a_{n-r} z^{-r-1} w^{r-n-1}. \tag{4.45}$$

Now we put the sum into normal order. For $n \neq 0$ the modes in the sum commute, and so are already normally ordered. For $n = 0$ the terms are normally ordered for $r \leq -1$, and can be

commuted into normal order for $r \geq 0$ using $a_r a_{-r} = a_{-r} a_r + r$. Doing so we find that

$$\begin{aligned}\partial\varphi(z)\partial\varphi(w) &= \sum_{r=0}^{\infty} r z^{-r-1} w^{-r-1} + \sum_{r,s\in\mathbb{Z}} :a_r a_s: z^{-r-1} w^{-s-1} \\ &= \frac{1}{(z-w)^2} + :\partial\varphi(z)\partial\varphi(w):, \end{aligned} \tag{4.46}$$

where we recognise the geometric series that gives rise to the first term, which converges due to the radial ordering $|z| > |w|$. The normally ordered product of fields with different arguments in the second term is defined as a Laurent expansion about $z = w$:

$$:\partial\varphi(z)\partial\varphi(w): = :\partial\varphi(w)\partial\varphi(w): + :\partial^2\varphi(w)\partial\varphi(w):(z-w) + \mathcal{O}(z-w)^2. \tag{4.47}$$

As all of these terms are regular, we arrive at the *defining OPE* for the free boson

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{\mathbf{1}}{(z-w)^2}. \tag{4.48}$$

Deriving the OPE took a bit of work for just two fields, and it gets much more tedious when more fields are involved. This is made easier using a standard result in quantum field theory known as **Wick's theorem**. See any quantum field theory textbook for a review, for example [103]. Refer to [6, app. A] for a more general procedure. Wick's theorem is a prescription for computing the OPE of a radially ordered product of an arbitrary number of free fields, requiring only that the OPE of each pair is known.

A *contraction* of two fields is the sum of the singular terms in their OPE. Wick's theorem says that the OPE of a product of free fields is given by the sum over all possible contractions of the normally ordered product of the fields (including the term with no contractions).

Using Wick's theorem we can efficiently confirm the OPE

$$L(z)L(w) \sim \frac{1/2}{(z-w)^4}\mathbf{1} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w}, \tag{4.49}$$

which is indeed the defining OPE for a $c = 1$ Virasoro generating function. Example calculations (for more general cases) are provided in sections 6.A and 8.A. We can also use Wick's theorem to verify the OPE in eq. (4.42), confirming that the free boson is a Virasoro primary field with conformal weight $h = 1$. We will use Wick's theorem again in chapters 6 and 8.

4.6 Vertex operators

We are now in a position to construct the most important objects in this thesis — vertex operators.

4.6.1 Extending the Heisenberg algebra

Consider a formal antiderivative of the holomorphic free boson $\partial\varphi(z)$:[12]

$$\phi(z) = \widehat{a} + a_0 \log(z) - \sum_{n\neq 0} \frac{a_n}{n} z^{-n}. \tag{4.50}$$

The integration constant $\widehat{a}$ is an operator that is independent of z and $\bar{z}$, and we will shortly determine what its commutation relations with the Heisenberg algebra must be. First let us determine some OPEs involving $\phi(z)$ by integrating the free boson defining OPE

$$\partial\phi(z)\partial\phi(w) = \partial\varphi(z)\partial\varphi(w) \sim \frac{\mathbf{1}}{(z-w)^2}. \tag{4.51}$$

[12] Note that there is also the anti-holomorphic sector to consider, and we will do so in section 4.7.

We find that

$$\partial\varphi(z)\phi(w) \sim \frac{\mathbf{1}}{z-w}, \qquad \phi(z)\partial\varphi(w) \sim \frac{-\mathbf{1}}{z-w}, \qquad \phi(z)\phi(w) \sim \mathbf{1}\log(z-w). \tag{4.52}$$

To work out how $\widehat{a}$ should behave we need to determine its commutation relations with the other modes. After subtracting the logarithmic term in eq. (4.50) we can express $\widehat{a}$ as a residue integral:

$$\widehat{a} = \oint_0 \Big(\phi(w) - a_0\log(w)\Big) w^{-1}\frac{dw}{2\pi i}. \tag{4.53}$$

From this we can work out the commutation relations with a_m using the techniques of section 3.7.1:

$$[a_m, \widehat{a}] = \oint_0 [a_m, \phi(w) - a_0\log(w)] w^{-1}\frac{dw}{2\pi i} = \oint_0 [a_m, \phi(w)] w^{-1}\frac{dw}{2\pi i}. \tag{4.54}$$

Inserting a residue integral for the field $\partial\varphi(z)$ to extract the mode a_m, using the leftmost OPEs in eq. (4.52), and deforming contours as per fig. 3.1, we find that

$$[a_m, \widehat{a}] = \mathbf{1}\delta_{m,0}. \tag{4.55}$$

A lot can be said for the physics behind the commutation relations we have just derived. In string theory the mode a_0 is the momentum operator for the centre of mass of the string, and $\widehat{a}$ has the interpretation of a position operator. The commutation relation $[a_0, \widehat{a}] = \mathbf{1}$ is analogous to the commutation relation in quantum mechanics that defines the momentum operator $\widehat{p}$ as the generator of spatial translations: $[\widehat{q}, \widehat{p}] = i\mathbf{1}$, where $\widehat{q}$ is the position operator.[13] For this reason $\widehat{a}$ is sometimes said to be *conjugate* to a_0. The creation operators a_m with $m < 0$ represent vibrational excitations on the string. Each mode can be assigned a different amount of energy. That excitations come in quanta and not a continuum is consistent with the classical notion of quantised waves on a closed string. In fact there is more — there are also the modes $\bar{a}_n$ from the anti-holomorphic sector. These correspond to excitations moving around the string in the opposite direction to those of a_m. The fact that a_m and $\bar{a}_n$ commute reflects the fact that the vibrational modes are independent — the *left-* and *right-moving* excitations pass through one another without interfering.

The exponential of $\widehat{a}$ shifts the momentum. For example, let $|q\rangle$ be a Heisenberg primary with a_0-eigenvalue $q \in \mathbb{R}$ and consider the state

$$e^{p\widehat{a}}\,|q\rangle := \sum_{k=0}^{\infty} \frac{p^k}{k!}\widehat{a}^k\,|q\rangle\,. \tag{4.56}$$

This state is annihilated by a_n with $n > 0$ because they commute with $\widehat{a}$. However, acting with a_0 we find that

$$a_0 e^{p\widehat{a}}\,|q\rangle = (p+q)e^{p\widehat{a}}\,|q\rangle\,. \tag{4.57}$$

The state $e^{p\widehat{a}}\,|q\rangle$ satisfies the definition of the Heisenberg vacuum state $|p+q\rangle$. This means that $e^{p\widehat{a}}$ maps vacuum vectors to vacuum vectors. Starting from the true vacuum $|0\rangle$ this lets us construct Fock spaces $\mathcal{F}(p)$ for all momenta $p \in \mathbb{R}$. The operator $e^{p\widehat{a}}$ with $p = 0$ is the identity operator. Momentum shifting operators can also be composed and inverted:

$$e^{p\widehat{a}}e^{q\widehat{a}} = e^{(p+q)\widehat{a}}, \qquad \left(e^{p\widehat{a}}\right)^{-1} = e^{-p\widehat{a}}, \qquad e^{p\widehat{a}}e^{-p\widehat{a}} = \mathbf{1}. \tag{4.58}$$

[13] The additional factor of i compared to eq. (4.55) goes right back to the very start of this chapter, when we defined the free boson scalar field. Some authors define the free boson to be $i\varphi(z,\bar{z})$. If we had done this, then we would be in the position of having conformal weights that were not bounded from below, unless we decided to restrict the momentum p of the Fock space $\mathcal{F}(p)$ to be pure imaginary: $p \in i\mathbb{R}$. On the other hand, we are working in imaginary time anyway (due to the Wick rotation), so perhaps permitting purely imaginary momenta actually makes *more* sense from a physical perspective. Suffice it to say that $\widehat{a}$ is subtle.

4.6.2 Vertex operators

According to the state-field correspondence, to each state corresponds a field. The Heisenberg primary fields are called **vertex operators**. We denote the field corresponding to $|p\rangle$ by the symbol $V_p(z)$:

$$\lim_{z\to 0} V_p(z)\,|0\rangle := |p\rangle\,. \tag{4.59}$$

This seems to be doing the same thing as the momentum shifting operator $e^{p\widehat{a}}$. Indeed vertex operators can be realised explicitly in terms of the momentum shifting operator by taking the "normally ordered exponential" of the field $\phi(z)$:

$$V_p(z) \text{ “}=\text{”} :e^{p\phi(z)}:. \tag{4.60}$$

The normal ordering is defined so that $\widehat{a}$ is placed to the left of a_0, so that in terms of the mode expansions we have

$$V_p(z) := e^{p\widehat{a}} e^{pa_0\log(z)} \prod_{n=1}^{\infty} \exp\left(pa_{-n}\frac{z^n}{n}\right)\exp\left(-pa_n\frac{z^{-n}}{n}\right). \tag{4.61}$$

Using this expansion and the action of the momentum shifting operator it is easy to verify that the field in eq. (4.61) satisfies eq. (4.59).

4.6.3 Vertex operators in correlation functions

Using the techniques of section 3.7.2, the mode expansion in eq. (4.61) for vertex operators, and the commutation relations in eq. (4.55), the OPE of two vertex operators can be determined up to any number of terms:

$$V_{p_1}(z_1)V_{p_2}(z_2) = z_{12}^{p_1p_2}V_{p_1+p_2}(z_2) + p_1 z_{12}^{p_1p_2+1}{:}\partial\varphi(z_2)V_{p_1+p_2}(z_2){:} + \dots \tag{4.62}$$

where $z_{ij} := z_i - z_j$. However, we can do better — the mode expansion in eq. (4.61) can be used to compose an arbitrary number of vertex operators, with the result

$$\begin{aligned} &V_{p_1}(z_1)\dots V_{p_n}(z_n) = \\ &e^{\widehat{a}\sum_{i=1}^n p_i}\prod_{1\le i<j\le n} z_{ij}^{p_ip_j}\prod_{i=1}^n z_i^{p_ia_0}\prod_{m\ge 1}\exp\left(\frac{a_{-m}}{m}\sum_{i=1}^n z_i^m\right)\prod_{m\ge 1}\exp\left(-\frac{a_m}{m}\sum_{i=1}^n z_i^{-m}\right), \end{aligned} \tag{4.63}$$

where we have written $z_i^{p_ia_0}$ for $\exp(p_ia_0\log z_i)$. This expansion can be derived using the commutation relations of the modes, and can be found in [115].

Inserting eq. (4.63) into a correlation function of the form

$$\langle 0|\,V_{p_1}(z_1)\dots V_{p_n}(z_n)\,|0\rangle \tag{4.64}$$

allows us to compute this correlator in a very simple way. Using the fact that creation operators a_m with $m<0$ all commute with a_0 and $\widehat{a}$ we can reorder the mode expansion in eq. (4.63) by placing the creation operators to the far left:

$$\langle 0|\prod_{m\ge 1}\exp\left(\frac{a_{-m}}{m}\sum_{i=1}^n z_i^m\right)e^{\widehat{a}\sum_{i=1}^n p_i}\prod_{1\le i<j\le n} z_{ij}^{p_ip_j}\prod_{i=1}^n z_i^{p_ia_0}\prod_{m\ge 1}\exp\left(-\frac{a_m}{m}\sum_{i=1}^n z_i^{-m}\right)|0\rangle\,. \tag{4.65}$$

The leftmost exponentials contain only creation operators, which annihilate the covector $\langle 0|$, and the rightmost exponentials contain only annihilation operators, which annihilate the vector $|0\rangle$. Hence, when exponentiated, these act as the identity in the correlator. We are left with

$$\langle 0|\,V_{p_1}(z_1)\dots V_{p_n}(z_n)\,|0\rangle = \langle 0|\,e^{\widehat{a}\sum_{i=1}^n p_i}\prod_{1\le i<j\le n} z_{ij}^{p_ip_j}\prod_{i=1}^n z_i^{p_ia_0}\,|0\rangle\,. \tag{4.66}$$

Since a_0 annihilates $|0\rangle$ the rightmost product acts as the identity. The momentum operator then acts to shift the momentum of $|0\rangle$ and we are left with

$$\langle 0|\, V_{p_1}(z_1)\dots V_{p_n}(z_n)\,|0\rangle = \langle 0|p_1+\dots+p_n\rangle \prod_{1\le i<j\le n} z_{ij}^{p_ip_j} = \delta_{p_1+\dots+p_n,0}\prod_{1\le i<j\le n} z_{ij}^{p_ip_j}. \tag{4.67}$$

Remarkably, in just a few lines of work we have derived the general solution to all vertex operator chiral correlation functions. Equation (4.67) is a very important result.

Conservation of momentum

That $\langle 0|p_1+\dots+p_n\rangle$ vanishes if $\sum_i p_i \neq 0$ is a conservation of momentum condition; the in-state $|0\rangle$ arrives from the infinite past ($|z|\to-\infty$) with zero momentum and undergoes time-ordered interactions, which are modelled by vertex operators. The vacuum covector $\langle 0|$ projects the resulting state onto the true vacuum $|0\rangle$. If the interactions do not conserve momentum then the scalar product vanishes. On the level of the representation theory this is a consequence of the fact that the momentum operator a_0 is anti-self-adjoint. In chapter 8 we modify the representation theory, and derive an analogous condition called the *neutrality condition*.

4.6.4 Solving the free boson

In section 3.6 we stated the main objectives of this thesis. One objective is to use tools from string theory to compute primary three-point constants for conformal field theories that describe two-dimensional critical models in statistical physics. Although the free boson conformal field theory does not correspond directly to such a statistical model, the result of eq. (4.67) does achieve the objective of determining the three-point constants for the free boson.

Recall from eq. (3.30) that the Ward identities give the general solution to a given primary three-point function up to an undetermined three-point constant:[14]

$$\langle 0|\, V_{p_1}(z_1)V_{p_2}(z_2)V_{p_3}(z_3)\,|0\rangle = \frac{C_{p_1,p_2,p_3}}{z_{12}^{h_1+h_2-h_3}z_{13}^{h_1+h_3-h_2}z_{23}^{h_2+h_3-h_1}}, \tag{4.68}$$

where $h_i = \frac{1}{2}p_i^2$. It follows from the conservation of momentum condition that the three-point constant C_{p_1,p_2,p_3} vanishes if $p_1+p_2+p_3\neq 0$. On the other hand if $p_1+p_2+p_3=0$ then the exponents in eq. (4.68) can be written as

$$-(h_1+h_2-h_3) = -\frac{1}{2}(p_1^2+p_2^2-p_3^2) = -\frac{1}{2}\left(p_1^2+p_2^2-(p_1+p_2)^2\right) = p_1p_2, \tag{4.69}$$

and so on. The exponents in eq. (4.68) are equal to those in eq. (4.67), and so by equating eq. (4.68) and eq. (4.67) the three-point constant can be immediately read off as

$$C_{p_1,p_2,p_3} = \delta_{p_1+p_2+p_3,0}. \tag{4.70}$$

This *solves* the free boson conformal field theory (in the sense defined in section 3.6).

4.7 Physical correlation functions

So far we have worked only with the holomorphic sector and its chiral correlators, but we know from section 3.4.1 that to get physically meaningful quantities we need to construct *physical* correlators. To see why this is necessary, note that unless the momenta in eq. (4.67) take special

[14] Note that here the fields are indexed by their momenta, which run over a continuum. Hence there is a continuum of three-point constants.

values (if they are all integers, for example), then the chiral correlation function is not single-valued. As a demonstration, consider in eq. (4.67) taking one of the variables z_i anti-clockwise about another point z_j and back to its starting point, encircling none of the other variables z_k. The chiral correlation function picks up a phase of $e^{2\pi i p_i p_j}$, and is therefore not single-valued for generic p_i and p_j. Physically meaningful correlation functions must be single-valued, and physical correlators can be constructed by including the anti-holomorphic sector.

The modes $\bar{a}_n$ of the anti-holomorphic sector also generate a copy of $\mathfrak{H}$ that commutes with that generated by $\{a_n\}$. It follows that the anti-holomorphic component of the energy-momentum tensor generates a copy of the Virasoro algebra $\bar{\mathfrak{Vir}}$ that commutes with $\mathfrak{Vir}$. The full symmetry algebra decomposes into two mutually commuting halves $\mathfrak{H} \oplus \bar{\mathfrak{H}}$ (and therefore $\mathfrak{Vir} \oplus \bar{\mathfrak{Vir}}$). Since $\mathfrak{H}$ and $\bar{\mathfrak{H}}$ commute, the representation space $\mathcal{H}$ factorises into a tensor product of representation spaces of the form[15]

$$\mathcal{H} = \bigoplus_{p,\bar{p}\in\mathbb{R}} \mathcal{F}(p) \otimes \mathcal{F}(\bar{p}) \tag{4.71}$$

on which the two copies of the Heisenberg algebra act as

$$\mathfrak{H} \to \mathfrak{H} \otimes \mathbf{1}, \qquad a_n \mapsto a_n \otimes \mathbf{1}, \tag{4.72}$$

$$\bar{\mathfrak{H}} \to \mathbf{1} \otimes \mathfrak{H}, \qquad \bar{a}_n \mapsto \mathbf{1} \otimes a_n. \tag{4.73}$$

In order to decide how to pair up the momenta p and $\bar{p}$ in eq. (4.71) we need to impose physical consistency conditions. Firstly, if we simultaneously integrate up both fields $\partial\varphi(z)$ and $\bar{\partial}\varphi(\bar{z})$ then we find that the non-chiral free boson field $\varphi(z,\bar{z})$ has the mode expansion

$$\varphi(z,\bar{z}) = \widehat{a} + a_0 \log z + \bar{a}_0 \log \bar{z} - \sum_{n\neq 0} \frac{a_n}{n} z^{-n} - \sum_{n\neq 0} \frac{\bar{a}_n}{n} \bar{z}^{-n}. \tag{4.74}$$

The two chiral halves share the same copy of the integration constant $\widehat{a}$. To construct physical correlators we take z and $\bar{z}$ to be complex conjugates. Note that if we take z anticlockwise about the origin (that is, $z \mapsto e^{2\pi i} z$) then $\bar{z}$ moves clockwise about the origin (that is, $\bar{z} \mapsto e^{-2\pi i}\bar{z}$), and due to the logarithmic terms in eq. (4.74) this results in

$$\varphi(z,\bar{z}) \mapsto \varphi(e^{2\pi i} z, e^{-2\pi i}\bar{z}) = \varphi(z,\bar{z}) + (a_0 - \bar{a}_0) 2\pi i. \tag{4.75}$$

If $\varphi(z,\bar{z})$ is to represent something physical it must be single-valued, and it is indeed single-valued when acting on each state $|u\rangle\otimes|v\rangle \in \mathcal{H}$ for which $(a_0 - \bar{a}_0)(|u\rangle\otimes|v\rangle) = (a_0 |u\rangle)\otimes|v\rangle - |u\rangle\otimes(a_0 |v\rangle) = 0$. Therefore the momentum in the holomorphic and anti-holomorphic tensor factor should be the same.[16]

Now since a_0 belongs to the centre $Z(\mathfrak{H})$, every vector in $\mathcal{F}(p)$ has the same momentum p, so it follows that the momenta p and $\bar{p}$ should be paired up *diagonally* in $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_{p\in\mathbb{R}} \mathcal{F}(p) \otimes \mathcal{F}(p). \tag{4.76}$$

On this space, the momentum shifting operator (which is not in the Heisenberg algebra), acts as

$$e^{p\widehat{a}} \mapsto e^{p\widehat{a}} \otimes e^{p\widehat{a}}, \tag{4.77}$$

which includes the identity map by setting $p = 0$.

[15] Technically the direct sum should be a *direct integral*, but such considerations lead us into to subtle issues of analysis — see [13].

[16] Note that in general there are other possible choices. For example, one can *compactify* $\varphi(z,\bar{z})$ to values on a circle by considering the equivalence class: $\varphi \sim \varphi + 2m\pi i$, where $m \in \mathbb{Z}$ is a winding number. In this case the momenta of the holomorphic and anti-holomorphic sector need only differ by an integer multiple of m for the field to be single-valued (up to equivalence). We will not consider such cases here.

That the two tensor factors of each vector $|u\rangle \otimes |v\rangle \in \mathcal{H}$ share the same momentum is physically sensible if we remember what the operators represent. The modes a_n and $\bar{a}_n$ represent non-interacting vibrational excitations moving in opposite directions on a closed string. Since these do not interact the algebras that define them must commute with one another. But these non-interacting excitations are confined to the same string as it propagates through space, and any shift of the momentum of the string is the same in both holomorphic and anti-holomorphic sectors. The momentum shifting operator must be the same in both sectors.

The scalar product generalises to a map $(\cdot,\cdot) : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ given by the product of Shapovalov forms for each factor:

$$(|a\rangle \otimes |b\rangle\,, |c\rangle \otimes |d\rangle) := (|a\rangle\,, |c\rangle) \cdot (|b\rangle\,, |d\rangle) = \langle a|c\rangle \cdot \langle b|d\rangle\,. \tag{4.78}$$

The state-field correspondence has the natural generalisation:

$$\mathbf{1}\otimes\mathbf{1} \leftrightarrow |0\rangle \otimes |0\rangle\,, \qquad \partial\varphi(z) \leftrightarrow (a_{-1}\,|0\rangle) \otimes |0\rangle\,, \qquad \bar{\partial}\varphi(\bar{z}) \leftrightarrow |0\rangle \otimes (a_{-1}\,|0\rangle), \tag{4.79}$$

and so on. In this framework the non-chiral vertex operators on $\mathcal{H}$ are given by

$$V_p(z,\bar{z}) = V_p(z) \otimes V_p(\bar{z}). \tag{4.80}$$

From eq. (4.61) we can write down the mode expansion:

$$\begin{aligned} V_p(z,\bar{z}) = &\left(e^{p\hat{a}} z^{pa_0} \prod_{n=1}^{\infty} \exp\left(pa_{-n}\frac{z^n}{n}\right) \exp\left(-pa_n\frac{z^{-n}}{n}\right)\right) \otimes \\ &\left(e^{p\hat{a}} \bar{z}^{pa_0} \prod_{n=1}^{\infty} \exp\left(pa_{-n}\frac{\bar{z}^n}{n}\right) \exp\left(-pa_n\frac{\bar{z}^{-n}}{n}\right)\right). \end{aligned} \tag{4.81}$$

Physical correlators of the non-chiral vertex operators follow immediately from eq. (4.67) and the modified scalar product in eq. (4.78) by simply multiplying the scalar products in each sector and taking z and $\bar{z}$ to be complex conjugates, giving

$$\langle V_p(z_1,\bar{z}_1)\dots V_{p_n}(z_n,\bar{z}_n)\rangle = \delta_{p_1+\dots+p_n,0} \prod_{1\le i<j\le n} |z_{ij}|^{2p_ip_j}. \tag{4.82}$$

Equation (4.82) is the main result of this chapter, and will be important in chapters 6 and 8.

The result in eq. (4.82) is clearly single-valued. To see this explicitly consider as before taking one variable z_i anti-clockwise about another z_j, encircling none of the other variables z_k. This results in a phase $e^{2\pi i p_i p_j}$. The conjugate variable $\bar{z}_i$ traverses $\bar{z}_j$ in the opposite direction, as depicted in fig. 4.1, and the phase gained is $e^{-2\pi i p_i p_j}$. The fact that the momenta are paired up diagonally means that the two phases cancel, and the physical correlation function is single-valued.

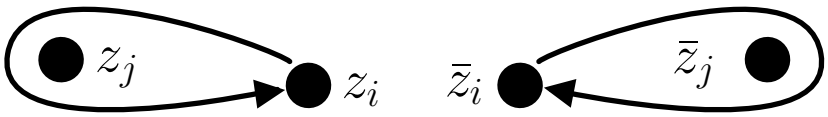


Figure 4.1: *On the "real surface" where z_i and $\bar{z}_i$ are taken to be complex conjugates, if z_i is taken in a loop about a point z_j, then the conjugate $\bar{z}_i$ moves in an oppositely oriented loop about the conjugate point $\bar{z}_j$.*

In chapters 6 and 8 we will return to the formalism of vertex operators in order to determine OPE-coefficients for a number of statistical models. Before doing so we need to construct the conformal field theories that describe these models. For this we shall now depart from the familiar Fock space representations of the Heisenberg algebra and the Virasoro representations contained within. Instead we turn to the representation theory of the Virasoro algebra in its own right.

Chapter 5

Virasoro Representations and the Minimal Models

The free boson conformal field theory exhibits more symmetry than conformal symmetry alone — the Virasoro algebra is not the only symmetry algebra around, but resides within the Heisenberg algebra. The additional structure makes the free boson particularly easy to work with. For one thing the Heisenberg algebra can be extended to construct vertex operators, for which all correlation functions can be explicitly determined.

The Heisenberg algebra is exceptional in this regard, and most conformal field theories do not have this additional structure to play with. In fact we will not even refer to a Lagrangian in the theories we deal with in this chapter. Here we study the consequences of conformal invariance in its own right, which comes down to studying the representation theory of the Virasoro algebra. Virasoro symmetry is very general — it is something that all conformal field theories possess, and warrants close study.

The breadth of physical applications of this approach to conformal field theory is remarkable. The theories we construct here are the *minimal models*, first introduced in [9]. They describe the critical scaling limits of all of the statistical lattice models mentioned in the introduction, and can be used to model their correlation functions.

5.1 Verma modules

Recall from section 3.1.2 that the Virasoro algebra admits a triangular decomposition $\mathfrak{Vir} = \mathfrak{Vir}_0 \oplus \mathfrak{Vir}_c \oplus \mathfrak{Vir}_a$, where the Cartan subalgebra $\mathfrak{Vir}_0$, the creation subalgebra $\mathfrak{Vir}_c$, and the annihilation subalgebra $\mathfrak{Vir}_a$ are given by

$$\mathfrak{Vir}_0 := \mathbb{C}L_0 \oplus \mathbb{C}C, \qquad \mathfrak{Vir}_c := \bigoplus_{n<0} \mathbb{C}L_n, \qquad \mathfrak{Vir}_a := \bigoplus_{n>0} \mathbb{C}L_n. \tag{5.1}$$

Let us begin by constructing Verma modules for $\mathfrak{Vir}$. Label the vacuum vector $|h\rangle$ by its L_0-eigenvalue $h \in \mathbb{R}$. It is customary to leave implicit its C eigenvalue, the central charge $c \in \mathbb{R}$. Being a vacuum vector requires that $L_n |h\rangle = 0$ if $n > 0$. The Virasoro Verma module $V(h, c)$ constructed from $|h\rangle$ is defined by

$$V(h, c) := U(\mathfrak{Vir}_c) |h\rangle . \tag{5.2}$$

A partition basis for $V(h, c)$ can be constructed in an analogous manner to the partition basis for Fock spaces that we constructed in section 1.5. Using the same notation, for $Y = [y_1, \ldots, y_k]$ a partition of a positive integer $|Y|$ we define products of modes in $U(\mathfrak{Vir})$ as

$$L_{-Y} := L_{-y_1} \ldots L_{-y_k}, \tag{5.3}$$

and for $|Y| = 0$ we define Y to be the empty sequence with $L_{-Y}|h\rangle = |h\rangle$. The set $\{L_{-Y}|h\rangle : |Y| \in \mathbb{Z}_{\geq 0}\}$ is a basis for $V(h,c)$. The Verma module decomposes into homogeneous subspaces $V_N(h,c)$:

$$V_N(h,c) := \mathrm{span}_{\mathbb{C}}\{L_{-Y}|h\rangle : |Y| = N\} \implies V(h,c) = \bigoplus_{N=0}^{\infty} V_N(h,c). \tag{5.4}$$

The integer N is called the *level* or the *grade* of homogeneous space $V_N(h,c)$, and the vectors it contains are called *level N vectors*. $V_N(h,c)$ is an L_0-eigenspace with eigenvalue $h+N$, and so $V(h,c)$ is L_0-graded. An L_0-graded vector space is said to be **conformally graded**. Virasoro generators are maps between grades:

$$L_n : V_N(h,c) \to \begin{cases} V_{N-n}(h,c) & \text{if } n \leq N \\ 0 & \text{otherwise .} \end{cases} \tag{5.5}$$

The above statement includes the requirement that the vacuum vector is annihilated by L_n if $n > 0$.

5.1.1 Scalar product

We define the scalar product as per chapters 3 and 4 as the $\mathbb{C}$-bilinear form $(\cdot,\cdot) : V(h,c) \times V(h,c) \to \mathbb{C}$, normalised so that $(|h\rangle, |h\rangle) := 1$, and defined to be invariant with respect to an adjoint map:

$$(x^\dagger |\alpha\rangle, |\beta\rangle) = (|\alpha\rangle, x|\beta\rangle) \tag{5.6}$$

for all $x \in \mathfrak{Vir}$. The adjoint map is the $\mathbb{C}$-linear pairing between creation and annihilation that leaves the centre invariant: $L_n^\dagger = L_{-n}$ and $C^\dagger = C$, as defined in section 3.1.2.

Since L_0 is self-adjoint, it follows that different homogeneous subspaces are orthogonal.

5.2 Singular vectors

For particular values of the conformal weight and central charge it may happen that $V(h,c)$ is *reducible*, that is $V(h,c)$ might contain a non-zero proper Verma submodule. Such a module is generated by highest weight vectors called *singular vectors* [74].

A **singular vector** in $V(h,c)$ is an L_0-eigenvector at level $N > 0$ that is annihilated by all annihilators, L_n with $n > 0$.[1]

Note that a level N singular vector $|X_N\rangle$ in $V(h,c)$ satisfies the definition of a vacuum vector of conformal weight $h+N$, which means that $|X_N\rangle$ generates a proper Verma submodule $V(h+N,c)$ in $V(h,c)$. Every Verma submodule of a Virasoro Verma module is generated by its singular vectors [40, Thm. 2.1]. In a given Verma module a singular vector with a given conformal weight is unique (up to proportionality) [74, Ch. 5.4.2]. It follows that, in a given Verma module, any Verma submodule is specified by the level of the singular vector, so the notation $V(h+N,c)$ for a Verma submodule is unambiguous. $V(h,c)$ is irreducible if and only if it contains no singular vectors [40, Prop. 1.1].

Before we go looking for singular vectors we can derive some very important properties. Firstly, *if there exists a singular vector in $V(h,c)$ then it is orthogonal to every vector in $V(h,c)$.*

The proof is straightforward. Since vectors in different grades are orthogonal it suffices to consider vectors in the same grade. Let $L_{-Y}|h\rangle = L_{-y_1}\dots L_{-y_k}|h\rangle \in V_N(h,c)$ be a grade $N > 0$

[1] We include the qualifier "non-zero" in the definition of the term "eigenvector", so singular vectors are non-zero.

partition basis vector, and let $|X_N\rangle$ be a singular vector in the same grade. Then by the adjoint action we have

$$(|X_N\rangle, L_{-Y}|h\rangle) = (L_{y_1}|X_N\rangle, L_{-y_2}\dots L_{-y_k}|h\rangle) = 0, \tag{5.7}$$

which vanishes because $L_{y_1}|X_N\rangle = 0$. Since every vector in $V_N(h,c)$ is a linear combination of partition basis vectors, and the scalar product is bilinear, it follows that $(|X_N\rangle, |v\rangle) = 0$ for every $|v\rangle \in V(h,c)$.

This implies also that every *descendent* of a singular vector in $V(h,c)$ is orthogonal to every vector in $V(h,c)$. To see this consider a generic descendant $|v_X\rangle$ of a singular vector $|X\rangle$. The vector $|v_X\rangle$ is given by some linear combination of Virasoro modes A acting on $|X\rangle$:

$$|v_X\rangle = A|X\rangle \in V(h,c). \tag{5.8}$$

The scalar product with any vector $|u\rangle$ in $V(h,c)$ gives

$$(|v_X\rangle, |u\rangle) = (A|X\rangle, |u\rangle) = (|X\rangle, A^\dagger|u\rangle) = 0, \tag{5.9}$$

which vanishes because $A^\dagger|u\rangle$ is in $V(h,c)$, and therefore is orthogonal to $|X\rangle$.

5.2.1 Unphysical states

From a physical point of view, having singular vectors in the space of states is a bad thing for a quantum field theory. All physically measurable information that can be described by the theory should be represented by scalar products (or correlation functions), and so as far as measurability is concerned a singular vector is indistinguishable from the zero vector — singular vectors, and all of their descendants, are not physically meaningful states. For this reason singular vectors can be justifiably referred to as *unphysical states*.[2] These are very closely related to the *unphysical fields* of section 3.5.2, and we will soon make use of this relationship.

If a Verma module contains a singular vector then the space of states is too large. In addition to all of the physical states there exists an infinite number of unphysical states (at least one singular vector and all its descendants). These should be removed if the theory is to be physically meaningful. Vectors are "removed" from a Verma module by constructing a **quotient module**. A quotient of a Verma module V by a Verma submodule $\mathcal{I} \subset V$ is a vector space $\mathcal{Q}V$ in which vectors which differ only by terms in $\mathcal{I}$ are considered to be equivalent: for all $v \in V$ and for all $i \in \mathcal{I}$, we have $v + i \sim v$. This is what it means to "set to zero" vectors in the submodule $\mathcal{I}$. As far as measurability is concerned, singular vectors *are* zero. If there are no unphysical states then $\mathcal{Q}V$ and V coincide. The quotient is denoted by $\mathcal{Q}V := V/\mathcal{I}$.

Let $\mathcal{I}(h,c)$ denote the maximal proper Verma submodule in $V(h,c)$. It is generated by its singular vectors, and may be zero. $\mathcal{I}(h,c)$ contains all of the unphysical states and no physical ones. The physical space of states $\mathcal{Q}V(h,c)$ is the quotient module

$$\mathcal{Q}V(h,c) := V(h,c)/\mathcal{I}(h,c). \tag{5.10}$$

$\mathcal{Q}V(h,c)$ is the unique irreducible $\mathfrak{Vir}$-quotient module with highest weight h and central charge c [40, Prop. 1.1].

In fact a model for a physical system generically contains not just one quotient module, but a collection of them built from vacuum vectors of different conformal weights. There are some things to clarify, such as which weights appear, and how to handle the ever-present anti-holomorphic sector. We will come to this in section 5.5, but first let us go looking for some singular vectors.

[2] Actually, singular vectors are not always unphysical in general conformal field theories — singular vectors appear with physical relevance in logarithmic conformal field theories [46].

5.3 Finding singular vectors

Given a Verma module $V(h,c)$ the task of actually finding singular vectors (if any exist) may seen rather difficult. It seems that one has to look for vectors that are annihilated by every L_n with $n>0$, which is an infinite number of cases to check. Fortunately it is easier than that, due to the following result:

If a vector is annihilated by L_1 and L_2 then it is annihilated by all L_n with $n>0$.

The proof is a simple exercise in induction. Let $|X\rangle \in V(h,c)$ with $L_1|X\rangle = L_2|X\rangle = 0$ and suppose that $L_m|X\rangle = 0$ for some $m \geq 2$. Using the Virasoro commutation relations in eq. (1.3), we can show that $L_{m+1}|X\rangle = 0$. This allows for an equivalent definition of a singular vector:

A singular vector in $V(h,c)$ is an L_0-eigenvector at level $N>0$ that is annihilated by L_1 and L_2.

Now let us look for singular vectors in $V(h,c)$. The requirement that a singular vector be an L_0-eigenvector means that it can be written as a linear combination of partition basis vectors in the same grade. The most general grade 1 vector (up to normalisation) is $|X_1\rangle = L_{-1}|h\rangle$, which is an L_0-eigenvector with eigenvalue $h+1$. Annihilation by L_1 requires that

$$0 = L_1|X_1\rangle = [L_1, L_{-1}]|h\rangle = 2L_0|h\rangle = 2h|h\rangle\,, \tag{5.11}$$

which holds if and only if $h=0$, so the only possible level 1 singular vector is $L_{-1}|0\rangle \in V(0,c)$.[3] This is a singular vector for every central charge c.

Now let us look for a level 2 singular vector. The most general form is

$$|X_2\rangle = \left(\eta_{1,1}L_{-1}^2 + \eta_2 L_{-2}\right)|h\rangle\,, \tag{5.12}$$

where $\eta_{1,1}, \eta_2 \in \mathbb{C}$ are to be determined. According to [5, Thm. 1.2], for any grade N singular vector, the coefficient of L_{-1}^N is non-zero. Hence we can choose the normalisation so that $\eta_{1,1}=1$. Clearly $|X_2\rangle$ is an L_0-eigenvector, with eigenvalue $h+2$. The next requirement is that it be annihilated by L_1, which requires that

$$\begin{aligned} 0 = L_1|X_2\rangle &= \left([L_1, L_{-1}L_{-1}] + \eta_2[L_1, L_{-2}]\right)|h\rangle \\ &= \left(L_{-1}[L_1,L_{-1}] + [L_1,L_{-1}]L_{-1} + 3\eta_2 L_{-1}\right)|h\rangle \\ &= (4h+2+3\eta_2)L_{-1}|h\rangle\,. \end{aligned} \tag{5.13}$$

We must have

$$\eta_2 = \frac{-1}{3}(4h+2). \tag{5.14}$$

Similarly for $|X_2\rangle$ to be annihilated by L_2 requires that

$$0 = L_2|X_2\rangle = \left((6+4\eta_2)h + \frac{c}{2}\eta_2\right)|h\rangle\,, \tag{5.15}$$

which gives the further constraint

$$(6+4\eta_2)h + \frac{c}{2}\eta_2 = 0. \tag{5.16}$$

Substituting eq. (5.14) into (5.16) and solving the quadratic equation for the conformal weight h gives a relationship between h and the central charge c:

$$h_\pm = \frac{5-c \pm \sqrt{(c-25)(c-1)}}{16}. \tag{5.17}$$

[3] Setting $L_{-1}|0\rangle$ to zero is consistent with the requirement that the true vacuum be *translationally invariant*, since L_{-1} corresponds to ∂, the generator of translations. Contrast this to chapter 4 where $L_{-1}|0\rangle$ *really is zero* by the definition of the highest weight $\mathfrak{H}$-module, whereas here it is *identified with zero* by performing a quotient.

This relationship defines a curve in the (h, c) plane along which $V(h, c)$ has a singular vector at level 2. For values of (h, c) not on this curve, $V(h, c)$ has no singular vector at level 2.

A similar search can be performed for a level 3 singular vector by starting with the general form

$$|X_3\rangle = \left(L_{-1}^3 + \eta_{1,2} L_{-1} L_{-2} + \eta_3 L_{-3}\right) |h\rangle \,, \tag{5.18}$$

with the result

$$h_\pm = \frac{7 - c \pm \sqrt{(c-25)(c-1)}}{6}. \tag{5.19}$$

We find that $V(h, c)$ has a level 3 singular vector if and only if h and c satisfy eq. (5.19).

We could continue in this way by looking for singular vectors at each level, but the number of levels to check is infinite. Fortunately there is a more sophisticated alternative.

5.3.1 The Kac determinant

Each L_0-eigenspace $V_N(h, c)$ has the partition basis vectors $L_{-Y} |h\rangle$ with $|Y| = N$. The number of basis vectors in $V_N(h, c)$ is the number of distinct partitions $p(N)$ of the integer N. The first few homogeneous spaces are shown in table 5.1.

L_0-eigenspace	partition vectors	partitions $p(N)$
$V_0(h, c)$	$\|h\rangle$	1
$V_1(h, c)$	$L_{-1} \|h\rangle$	1
$V_2(h, c)$	$L_{-1}^2 \|h\rangle$, $L_{-2} \|h\rangle$,	2
$V_3(h, c)$	$L_{-1}^3 \|h\rangle$, $L_{-2} L_{-1} \|h\rangle$, $L_{-3} \|h\rangle$	3
$\vdots$	$\vdots$	$\vdots$

Table 5.1: *Conformal grading of $V(h, c)$.*

Since different grades are mutually orthogonal it suffices to study each grade in isolation. For each grade N we define the index set $I = \{1, \ldots, p(N)\}$, and label the partition basis vectors (in any order) by $|v_i\rangle$ with $i \in I$. Now we form the following $p(N) \times p(N)$ matrix of scalar products for each level:

$$M_{ij}^{(N)} := \left(|v_i\rangle \,, |v_j\rangle\right) \qquad \text{for} \qquad i, j \in I. \tag{5.20}$$

These are called *Gram matrices*. If there is a singular vector in $V_N(h, c)$ then at level N and at all subsequent levels there is a linear combination of partition basis vectors that is orthogonal to the whole Verma module.[4] In this case rank $M^{(N)} < p(N)$, and so $\det M^{(N)} = 0$. On the other hand if $\det M^{(N)} \neq 0$ then there is no singular vector at any level $\leq N$.

There is an explicit formula for the determinant of the Gram matrices, conjectured by V. Kac in [80] and proven in [39]: for positive integers r and s we have

$$\det M^{(N)} = \xi_N \prod_{r,s \geq 1}^{rs \leq N} \Big(h - h_{r,s}(c)\Big)^{p(N - rs)}, \tag{5.21}$$

where ξ_N is a non-zero constant independent of h and c, and where the numbers $h_{r,s}(c)$ are

$$h_{r,s}(c) = \frac{(r\alpha_+ + s\alpha_-)^2}{4} - \alpha_0^2, \tag{5.22}$$

where

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}, \qquad \alpha_0 = \frac{\sqrt{1-c}}{\sqrt{24}} = \frac{1}{2}(\alpha_+ + \alpha_-). \tag{5.23}$$

[4] The reason that this holds for levels $\geq N$, and not just level N, is that all descendants of singular vectors are also orthogonal to the whole Verma module.

It is standard practice to leave the c dependence implicit and write $h_{r,s}$ in place of $h_{r,s}(c)$. The conformal weights $h_{r,s}$ will be called **Kac weights**, and the numbers $\alpha_\pm$ will be called **Kac momenta**.

The Kac determinant vanishes in the Verma module $V(h_{r,s}, c)$ at level $N = rs$, and there is a singular vector at level rs in $V(h_{r,s}, c)$. Note that, depending on the value of c, there may also exist a singular vector at a *lower level* than rs. For example, there might be another pair of indices $r', s' \in \mathbb{Z}$ such that $h_{r',s'} = h_{r,s}$, with $r's' < rs$. Values of c for which this can happen will be discussed in section 5.4.

Let us compare the predictions of eq. (5.21) with the singular vectors we found explicitly by brute force computation. We determined that the only possible level 1 singular vector occurs in $V(0, c)$, built from $|0\rangle$. The Kac determinant formula predicts a level 1 singular vector in $V(h_{1,1}, c)$, and indeed $h_{1,1} = 0$. The same is true for the level 2 and 3 singular vectors — the roots h_+ and h_- in eq. (5.17) are equal to $h_{2,1}$ and $h_{1,2}$ respectively, and in eq. (5.19) the roots are $h_+ = h_{3,1}$ and $h_- = h_{1,3}$.

The Kac weights play a crucial role in this chapter and in the remainder of this thesis. Note that they exhibit reflection symmetry:

$$h_{r,s} = h_{-r,-s}. \tag{5.24}$$

Symmetries of the Kac weights have important consequences that will soon become apparent.

5.4 Singular fields in correlation functions

By the state-field correspondence, to each singular vector $|X\rangle$ there corresponds a singular field $X(z)$. The vanishing of the scalar product of $|X\rangle$ with any state in the theory requires the vanishing of every correlator containing the field $X(z)$ and any other fields in the theory. These singular fields are precisely the unphysical fields of section 3.5.2 — a singular field has no conjugate field in the theory.

Imposing this vanishing condition on all correlators containing a singular field constrains which fields are allowed in the theory. For instance, consider the level 2 singular vector $|X_2\rangle = (L_{-1}^2 + \eta_2 L_{-2})\,|h_1\rangle$ that we derived in section 5.3, where $\eta_2 = -1(4h_1 + 2)/3$. From the Kac determinant, existence of a level 2 singular vector requires that h_1 take one of the two values $h_{2,1}$ or $h_{1,2}$. The corresponding singular field $X_2(z)$ descends from the primary field $\phi_1(z)$ of conformal weight h_1.

In section 3.7.3 we demonstrated how correlators containing descendent fields can be transformed into primary correlators acted upon by differential operators. Using the same method and notation as in section 3.7.3, a primary field correlation function containing $X_2(z_1)$ is reduced to

$$\langle 0|\, X_2(z_1) S(z_2, z_3, \dots)\, |0\rangle = \left\{\mathcal{L}_{-1}^2(z_1) + \eta_2 \mathcal{L}_{-2}(z_1)\right\} \langle 0|\, \phi_1(z_1) S(z_2, z_3, \dots)\, |0\rangle\,, \tag{5.25}$$

where the differential operators $\mathcal{L}_n(z_1)$ are defined in eq. (3.51). Since every correlator containing a singular vector must vanish, eq. (5.25) gives a partial differential equation that is satisfied by all primary correlators containing $\phi_1(z)$:

$$\left\{\partial_1^2 + \eta_2 \sum_{j \neq 1} \left[\frac{h_j}{(z_j - z_1)^2} - \frac{1}{z_j - z_1}\partial_j\right]\right\} \langle 0|\, \phi_1(z_1) S(z_2, z_3, \dots)\, |0\rangle = 0. \tag{5.26}$$

The Ward identities of section 3.5 fix three-point functions up to multiplicative constants. Equation (5.26) is linear, so it is of no help in fixing the constant, but it does impose additional

constraints on the allowed conformal weights. For example, the general form (3.30) of a primary three-point function is

$$\langle 0|\, \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\, |0\rangle \propto z_{12}^{-h_1-h_2+h_3} z_{13}^{-h_1-h_3+h_2} z_{23}^{-h_2-h_3+h_1}. \tag{5.27}$$

To simplify matters, write the right-hand side as

$$z_{12}^{A_{12}} z_{13}^{A_{13}} z_{23}^{A_{23}}, \qquad \text{with} \quad A_{12} := -h_1 - h_2 + h_3, \quad \text{and so on.} \tag{5.28}$$

The derivatives in eq. (5.26) act on eq. (5.27) as

$$\begin{aligned} \partial_1^2 &\leftrightarrow \frac{A_{12}(A_{12}-1)}{z_{12}^2} + \frac{2A_{12}A_{13}}{z_{12}z_{13}} + \frac{A_{13}(A_{13}-1)}{z_{13}^2}, \\ \partial_2 &\leftrightarrow -\frac{A_{12}}{z_{12}} + \frac{A_{23}}{z_{23}}, \qquad \partial_3 \leftrightarrow -\frac{A_{23}}{z_{23}} - \frac{A_{13}}{z_{13}}, \end{aligned} \tag{5.29}$$

and so eq. (5.26) now reads

$$\begin{aligned} &\frac{A_{12}(A_{12}-1)}{z_{12}^2} + \frac{2A_{12}A_{13}}{z_{12}z_{13}} + \frac{A_{13}(A_{13}-1)}{z_{13}^2} \\ &+\eta_2 \left\{ \frac{h_2}{z_{12}^2} + \frac{h_3}{z_{13}^2} + \frac{1}{z_{12}}\left[\frac{-A_{12}}{z_{12}} + \frac{A_{23}}{z_{23}}\right] + \frac{1}{z_{13}}\left[\frac{-A_{13}}{z_{13}} - \frac{A_{23}}{z_{23}}\right] \right\} = 0. \end{aligned} \tag{5.30}$$

Equation (5.30) is greatly simplified by performing the map $(z_1, z_2, z_3) \mapsto (0, 1, \infty)$, which gives[5]

$$A_{12}(A_{12}-1) + \eta_2(h_2 - A_{12}) = 0. \tag{5.31}$$

Making the substitution $A_{12} = -h_1 - h_2 + h_3$ and solving the resulting quadratic equation for h_3 in terms of h_1 and h_2, we find that

$$h_3 = \frac{1}{6}\left(1 + 2h_1 + 6h_2 \pm \sqrt{(1-4h_1)^2 + 24(1+2h_1)h_2}\right). \tag{5.32}$$

5.4.1 Fusion rules

Recall that one of the main goals of conformal field theory is to compute primary three-point constants, or equivalently to compute primary OPE-coefficients (section 3.5.3) — constraints on correlation functions are also constraints on OPE-coefficients. In fact, eq. (5.32) tells us that some OPE-coefficients necessarily vanish in a consistent theory.

We start off by writing the conformal weight corresponding to the momentum α as

$$h(\alpha) := \alpha(\alpha - 2\alpha_0), \tag{5.33}$$

with the Kac momenta $\alpha_\pm$ as in eq. (5.23). We define another momentum variable

$$\alpha_{r,s} := \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_-. \tag{5.34}$$

With this notation the Kac weights take the particularly clean form

$$h_{r,s} = h(\alpha_{r,s}). \tag{5.35}$$

Substituting $h(\alpha_i) = h_i$ into eq. (5.32) and solving for α_3 we find the following.

[5]Note that the choice of map violates the implicit radial ordering $|z_1| > |z_2| > |z_3|$, but this will not influence whether or not the differential equation is satisfied.

For $h_1 = h_{2,1}$ we get

$$\alpha_3 = \alpha_2 \pm \alpha_+ \tag{5.36}$$

and for $h_1 = h_{1,2}$ we get

$$\alpha_3 = \alpha_2 \pm \alpha_-. \tag{5.37}$$

If h_2 is the Kac weight $h_{r,s}$ then eq. (5.36) gives

$$\alpha_3 = \alpha_{r,s} \pm \alpha_+ \implies h_3 = h_{r\pm1,s} \tag{5.38}$$

and eq. (5.37) gives

$$\alpha_3 = \alpha_{r,s} \pm \alpha_- \implies h_3 = h_{r,s\pm1}. \tag{5.39}$$

Let us think about what this means. As we know from eq. (3.30) the primary three-point function in eq. (5.27) is proportional to the three-point constant $C_{1,2,3}$, which is equal to the OPE-coefficient $C_{1,2}^{3\vee}$ (which will turn out to be equal to $C_{1,2}^3$ — see section 5.6.2). If $\phi_1(z_1)$ has conformal weight $h_{2,1}$ (respectively $h_{1,2}$), and $\phi_2(z_2)$ has weight $h_{r,s}$, then the only way eq. (5.26) can be satisfied is if $\phi_3(z_3)$ has weight $h_{r\pm1,s}$ (respectively $h_{r,s\pm1}$), or if the three-point constant $C_{1,2,3}$ vanishes. Equation (5.26) restricts which fields can appear in the OPE of two others. This leads us to the concept of *fusion*.

A Virasoro primary field $\phi_i(z_i)$ with conformal weight h_i corresponds to an irreducible representation $\mathcal{QV}(h_i, c)$ for the Virasoro algebra. Denote by $[\phi_i]$ the set of fields corresponding to vectors in this representation. A **fusion rule**

$$[\phi_i] \bowtie [\phi_j] = \sum_k N_{ij}^k [\phi_k] \tag{5.40}$$

is a representation theoretic notion that describes some of the universal properties of OPEs. Each set of fields $[\phi_i]$ is called a *conformal family*, and is an irreducible representation for the Virasoro algebra. The operation $\bowtie$ is called the *fusion product*. A collection of conformal families together with the fusion product generates a *fusion algebra* [31]. The coefficients N_{ij}^k are non-negative integers called *fusion multiplicities*. N_{ij}^k is equal to the number of times that the primary field $\phi_k(w)$ appears in the OPE $\phi_i(z)\phi_j(w)$, and so N_{ij}^k vanishes if the OPE-coefficient C_{ij}^k vanishes. For the minimal models, the fusion multiplicities turn out to be either zero or one [126].[6]

Equations (5.38) and (5.39) imply the following fusion rules:

$$[\phi_{2,1}] \bowtie [\phi_{r,s}] = N_{(2,1),(r,s)}^{(r+1,s)} [\phi_{r+1,s}] + N_{(2,1),(r,s)}^{(r-1,s)} [\phi_{r-1,s}] \tag{5.41}$$

$$[\phi_{1,2}] \bowtie [\phi_{r,s}] = N_{(1,2),(r,s)}^{(r,s+1)} [\phi_{r,s+1}] + N_{(1,2),(r,s)}^{(r,s-1)} [\phi_{r,s-1}]. \tag{5.42}$$

Associativity and commutativity of the OPE (in correlation functions) carry over to associativity and commutativity of the fusion product, and this implies restrictions on the fusion multiplicities. For example, consider the fusion product:

$$[\phi_{2,1}] \bowtie [\phi_{1,2}] = N_{(2,1),(1,2)}^{(2,2)} [\phi_{2,2}] + N_{(2,1),(1,2)}^{(0,2)} [\phi_{0,2}]. \tag{5.43}$$

By commutivitity this is equal to

$$[\phi_{1,2}] \bowtie [\phi_{2,1}] = N_{(1,2),(2,1)}^{(2,2)} [\phi_{2,2}] + N_{(1,2),(2,1)}^{(2,0)} [\phi_{2,0}]. \tag{5.44}$$

Equating the expressions one finds that

$$N_{(2,1),(1,2)}^{(0,2)} = 0, \qquad N_{(1,2),(2,1)}^{(2,0)} = 0. \tag{5.45}$$

[6] One of the most celebrated and beautiful results in conformal field theory is the *Verlinde formula* [126], which relates fusion multiplicities to the *modular group*.

This is a an example of a *truncation phenomenon* — no families $[\phi_{r,s}]$ with $r, s \leq 0$ can be generated by fusions of $[\phi_{2,1}]$ and $[\phi_{1,2}]$.

From the uniqueness of highest weight vectors of a given conformal weight [40] we have

$$[\phi_{r,s}] = [\phi_{r',s'}] \quad \text{if and only if} \quad h_{r,s} = h_{r',s'}. \tag{5.46}$$

The indices (r, s) and (r', s') are therefore considered to be equivalent if $h_{r,s} = h_{r',s'}$. Given a representation $[\phi_{r,s}]$ at generic central charge, repeated fusion with $[\phi_{2,1}]$ and $[\phi_{1,2}]$ allows one to increase r and s to arbitrarily large values, corresponding to infinitely many inequivalent Kac indices, and hence an infinite number of inequivalent Virasoro representations.

However, consider the following discrete collection of values for the central change:

$$c = c(p', p) := 1 - 6\frac{(p - p')^2}{pp'} \tag{5.47}$$

where $p, p' \geq 2$ are coprime integers. For the values in eq. (5.47), the ratio of Kac momenta $\alpha_+/\alpha_- = -p/p'$ is rational. Substituting eq. (5.47) into eq. (5.22) for the Kac weights we find that

$$h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'}. \tag{5.48}$$

On top of the reflection symmetry of eq. (5.24), these weights have translational symmetry:

$$h_{r,s} = h_{r+p',s+p}. \tag{5.49}$$

Combining the reflection symmetry (5.24) and translation symmetry (5.49), we find that the fusion rules truncate to families built from a finite set $\mathcal{T}_{p',p}$ of inequivalent indices [9]:

$$\mathcal{T}_{p',p} := \{(r, s) \in \mathbb{Z}^2 \ : 1 \leq r \leq p' - 1, 1 \leq s \leq p - 1\}/\sim . \tag{5.50}$$

The set of indices $\mathcal{T}_{p',p}$ is called the **Kac table** for pair (p', p). The equivalence relation is: $(r, s) \sim (p' - r, p - s)$, expressing the symmetries of the Kac weights in eq. (5.48). The set $\mathcal{T}_{p',p}$ contains $(p - 1)(p' - 1)/2$ inequivalent pairs (r, s), and corresponds to exactly $(p - 1)(p' - 1)/2$ inequivalent irreducible highest weight representations of the Virasoro algebra.

By repeated fusions with $[\phi_{2,1}]$ and $[\phi_{1,2}]$, and imposing associativity, commutativity, and the equivalence relation in eq. (5.46), one can build up a complete set of fusion rules.[7] For $(r, s), (m, n) \in \mathcal{T}_{p',p}$ we have

$$[\phi_{r,s}] \bowtie [\phi_{m,n}] \ = \sum_{k=|m-r|+1,\, l=|n-s|+1}^{k^+,\, l^+} [\phi_{k,l}], \tag{5.51}$$

where each summation increments in *steps of two*, and the terminals are

$$\begin{aligned} k^+ &:= \min(m + r - 1; 2p' - 1 - m - r), \\ l^+ &:= \min(n + s - 1; 2p - 1 - n - s). \end{aligned} \tag{5.52}$$

The summation formula in eq. (5.51) first appeared in [9]. The Kac table $\mathcal{T}_{p',p}$ is closed under fusion for each value of the central charge $c(p', p)$ in eq. (5.47).

To summarise, for generic values of the central charge the number of inequivalent Virasoro representations is infinite. For the discrete collection of values in eq. (5.47) and weights corresponding to $\mathcal{T}_{p',p}$, the number is finite and equal to $(p - 1)(p' - 1)/2$.

[7] The families $[\phi_{2,1}]$ and $[\phi_{1,2}]$ are said to polynomially generate the fusion ring.

5.5 The minimal models

As demonstrated in section 4.7, for a conformal field theory to be a useful model for a physical system, the theory should include both copies of the Virasoro algebra, and the space of states should be a representation space for $\mathfrak{Vir} \otimes \mathfrak{Vir}$. Accordingly we define a **minimal model** to be a conformal field theory for which the number of inequivalent representations of $\mathfrak{Vir} \otimes \mathfrak{Vir}$ is finite.

We showed that if $c = c(p', p)$ as per eq. (5.47), and highest weight representations are chosen according to the Kac table in eq. (5.50), then the number of inequivalent $\mathfrak{Vir}$ representations is indeed finite. In fact, combining the holomorphic and anti-holomorphic sectors, one can show that having a finite number of inequivalent $\mathfrak{Vir} \otimes \mathfrak{Vir}$ representations is possible if and only if the central charge is of the form $c(p', p)$. This follows from considerations such as modular invariance — see [53, Ch. 10.5].

We will denote by $\mathcal{M}(p', p)$ the $c = c(p', p)$ minimal model with diagonal space of states $\mathcal{H}(p', p)$, given by

$$\mathcal{H}(p', p) := \bigoplus_{(r,s) \in \mathcal{T}_{p',p}} \mathcal{QV}(h_{r,s}, c) \otimes \mathcal{QV}(h_{r,s}, c), \tag{5.53}$$

where $\mathcal{QV}(h_{r,s}, c)$ is the (unique) irreducible quotient module defined in eq. (5.10). $\mathcal{M}(p', p)$ is called *diagonal*[8] because the quotient modules $\mathcal{QV}(h_{r,s}, c)$ for each sector are paired up with equal conformal weights. Diagonal pairing is often required for reasons such as modular invariance of partition functions, or for single-valued correlation functions — we will deal with the latter issue in chapters 7 and 8, and we have already encountered such a problem when combining the two sectors for the free boson in section 4.7.[9] For diagonal minimal models the non-chiral primary fields are of the form $\Phi_{r,s}(z, \bar{z}) = \phi_{r,s}(z) \otimes \phi_{r,s}(\bar{z})$, and therefore have zero spin, since each factor has the same conformal weight (recall the definition of spin in eq. (3.3)).

The $\mathcal{M}(p', p)$ theories are minimal not only in the sense that they contain a finite number of inequivalent Virasoro representations, but also in the sense that the space of states $\mathcal{H}(p', p)$ contains no unphysical states. However, for some p', p the space $\mathcal{H}(p', p)$ might still contain states of negative norm — $\mathcal{M}(p', p)$ is not necessarily unitary. In fact $\mathcal{M}(p', p)$ is unitary if and only if $|p - p'| = 1$. This was proven in [66, Sec. 2], but the "only if" part had been announced without proof in [55]. In a unitary theory, scalar products are positive-definite, and so correlation functions are non-negative. From eq. (3.30) we know that three-point functions are proportional to three-point constants, which are equal to OPE-coefficients. It follows that in a unitary theory the OPE-coefficients and three-point constants are non-negative.

By the symmetry of the central charge $c(p', p) = c(p, p')$ we may without loss of generality assume that $p > p'$, which we will always do. Minimal models of the form $\mathcal{M}(p, p+1)$ with $p \geq 2$ are called **main series** and are the only unitary minimal models.

5.5.1 Physical relevance of minimal model theories

Our end goal is to solve problems in statistical physics. It is reasonable to question the purpose of all this formality and representation theory.

Recall that we are trying to build probabilistic models that describe the critical behaviour of physical systems. From a mathematical perspective the fact that the number of inequivalent primary fields is finite makes the representation theory far more tractable — with infinitely many representations come technicalities, such as issues of analysis. The next key observation is that the minimal models contain no unphysical states, which is essential if scalar products are to represent probabilities. The main series minimal models are especially important because unitarity is

[8] Some authors call the diagonal minimal models the "*A-series*" models — [112, Ch. 3.2.1].

[9] It is also possible to construct non-diagonal minimal models such as the "*D-series*" and "*E-series*" models — see [53, Ch. 10.7] and [112, Ch. 3.2.1]. We shall only consider diagonal models.

required for conservation of probability — inner products may represent probabilities, but scalar products with negative-norm states violate the essential additivity property of probabilities.

Most important is the reason given in the introduction — the minimal models describe the scaling limit of critical lattice models in statistical physics. For example, according to [55] the scaling limit of the *Ising model*, *tricritical Ising model*, *3-state Potts model*, and *tricritical 3-state Potts model* are described respectively by the main series minimal models $\mathcal{M}(3,4), \mathcal{M}(4,5), \mathcal{M}(5,6)$, and $\mathcal{M}(6,7)$. Another application of the minimal models is in the study of *solid-on-solid* (or rather *restricted-solid-on-solid*) statistical models — these were originally devised in [2], and their critical points were related to the minimal models in [70]. The relationship was expanded upon in [101].

More recently it has been shown that certain modifications of the minimal models describe numerous statistical phenomena pertaining to *non-local observables*. These are observables that cannot be determined by considering a single lattice site (or a single instance in time perhaps) independently of others. Often such observables involve averaging over sites (or times), and are very difficult to deal with, be it analytically or numerically. Theories that are closely related to the minimal models are turning out to be very useful here. Some references on this topic are given in section 5.6.2.

5.6 Example: the Ising model and $\mathcal{M}(3,4)$

5.6.1 The Ising model

The Ising model is probably the most famous of the critical lattice models due to its simplicity and scope. The connection between the Ising model and $\mathcal{M}(3,4)$ dates back to antiquity — it was already suggested in [9]. A thorough treatment of the Ising lattice model can be found in [93]. Its scaling limit is discussed in [20]. As described in the introduction, the lattice model possesses a spin observable σ_i and an energy observable ϵ_j, indexed by their positions on a lattice. In the scaling limit there is a second-order phase transition at a critical temperature. If R_{ij} is the distance between sites i and j then at the critical temperature, correlation functions decay as

$$\langle \sigma_i \sigma_j \rangle \sim \frac{1}{R_{ij}^{d-2+\eta}} \sim \frac{1}{R_{ij}^{2\Delta_\sigma}}, \qquad \langle \epsilon_i \epsilon_j \rangle \sim \frac{1}{R_{ij}^{2(d-1/\nu)}} \sim \frac{1}{R_{ij}^{2\Delta_\epsilon}}, \tag{5.54}$$

where d is the number of dimensions of the lattice, $\eta, \nu \in \mathbb{R}$ are critical exponents, and $\Delta_\sigma, \Delta_\epsilon \in \mathbb{R}$ are scaling dimensions of the observables σ and ϵ. For the two-dimensional lattice model the critical exponents are exactly known [44, Sec. D]:

$$\eta = \frac{1}{4}, \qquad \nu = 1, \tag{5.55}$$

so the correlators decay as

$$\langle \sigma_i \sigma_j \rangle \sim \frac{1}{R_{ij}^{1/4}}, \qquad \langle \epsilon_i \epsilon_j \rangle \sim \frac{1}{R_{ij}^{2}}. \tag{5.56}$$

5.6.2 The minimal model $\mathcal{M}(3,4)$

$\mathcal{M}(3,4)$ is a diagonal $c = 1/2$ main series minimal model. That the representations in $\mathcal{H}(3,4)$ must be paired up diagonally follows from imposing modular invariance of the partition function — [53, Ch. 10.5]. There are exactly three inequivalent primary fields, and they are of the form $\Phi_{r,s}(z,\bar{z}) = \phi_{r,s}(z) \otimes \phi_{r,s}(\bar{z})$ with $(r,s) \in \mathcal{T}_{3,4}$. The primary fields are all spinless, and their conformal weights $(h, \bar{h})$ are:

$$\Phi_{1,1} \leftrightarrow (0,0), \qquad \Phi_{2,1} \leftrightarrow (\frac{1}{2}, \frac{1}{2}), \qquad \Phi_{1,2} \leftrightarrow (\frac{1}{16}, \frac{1}{16}). \tag{5.57}$$

Minimal model primary fields are all *self-conjugate*. In each minimal model there is only up to one primary field of a given conformal weight. It follows that the only candidate for the conjugate of a primary field is that primary field itself. Every minimal model field has a conjugate, because fields without conjugates are unphysical fields (singular fields), which have been removed by taking quotients. We can therefore immediately write down normalised minimal model two-point functions using the general formula in eq. (3.31). For $(r,s),(r',s') \in \mathcal{T}_{p',p}$, we have

$$\langle 0|\, \phi_{r,s}(z_i)\phi_{r',s'}(z_j)\, |0\rangle = \frac{\delta_{(r,s)\sim(r',s')}}{z_{ij}^{2h_{r,s}}}. \tag{5.58}$$

There is also the anti-holomorphic sector to consider. Using eq. (4.78) for combining holomorphic and anti-holomorphic scalar products we have the following $\mathcal{M}(3,4)$ two-point physical correlators

$$\begin{aligned}
\langle 0|\, \Phi_{1,2}(z_i,\bar{z}_i)\Phi_{1,2}(z_j,\bar{z}_j)\, |0\rangle &= \frac{1}{|z_{ij}|^{2(h_{1,2}+\bar{h}_{1,2})}} = \frac{1}{|z_{ij}|^{1/4}}, \\
\langle 0|\, \Phi_{2,1}(z_i,\bar{z}_i)\Phi_{2,1}(z_j,\bar{z}_j)\, |0\rangle &= \frac{1}{|z_{ij}|^{2(h_{2,1}+\bar{h}_{2,1})}} = \frac{1}{|z_{ij}|^{2}}.
\end{aligned} \tag{5.59}$$

The $\mathcal{M}(3,4)$ two-point functions have the same scaling behaviour as the critical Ising model correlation functions in eq. (5.56), which suggests the correspondence

$$\Phi_{1,1} \leftrightarrow \mathbf{1}, \qquad \Phi_{2,1} \leftrightarrow \epsilon, \qquad \Phi_{1,2} \leftrightarrow \sigma. \tag{5.60}$$

With these identifications we are almost ready to compute Ising model correlation functions, but first we should mention a technicality that is often ignored, particularly in the early conformal field theory literature.

The locally defined observables we have encountered so far are not the only observable quantities that can be described by the Ising lattice model. Other *non-local observables* can also be defined. For example, one can define the expectation value of the *fractal dimension* of spin-clusters (fractal structures arise because of scale invariance at criticality). Such observables turn out to have critical exponents that do not correspond to the conformal weights of the Kac table, and are not described by $\mathcal{M}(3,4)$. However, rather amazingly, such critical exponents do correspond to Kac indices in the boundary of the Kac table — see [4] for details regarding the Ising model and the boundary of the Kac table. The boundary of the Kac table is relevant for many other statistical phenomena as well, such as certain stochastic processes [7, 84] and percolation models [113]. When describing non-local observables, the conformal field theory that we have seen up until now is generalised to a *logarithmic conformal field theory* — see [27, 92] for details and references. The identifications in eq. (5.60) are insufficient for the computation of correlation functions involving non-local observables as described above.

With this caveat out of the way we turn now to chapter 6, where we shall combine the vertex operators of chapter 4 with yet another conformal field theory borrowed from string theory — the free fermion. When brought together the free boson and free fermion conformal field theories provide a simple and efficient method for finding all of the $\mathcal{M}(3,4)$ primary OPE-coefficients, and hence for computing Ising model correlation functions.

Chapter 6

Bosons, Fermions, and the Ising Model

In chapter 5 we linked the critical Ising model to the minimal model $\mathcal{M}(3,4)$. It just so happens that $\mathcal{M}(3,4)$ is closely related to another conformal field theory — the free fermion. This suggests a realisation of Ising model observables in terms of free fermion fields. Actually we can go one step further — bosons and fermions are also intimately connected. The aim of this chapter is to unify these perspectives by constructing a free field realisation of $\mathcal{M}(3,4)$ primary correlators in terms of bosonic vertex operators, using the free fermion as an intermediate step. This chapter provides a first example of the synergistic relationship between string theory and statistical field theory, and will later serve to verify Coulomb gas calculations.

6.1 The free fermion

As we did for the free boson in chapter 4, we could commence here with a Lagrange density and a conformally invariant action functional for the free fermion field $\psi(z)$ on an infinite cylinder, find the equations of motion by the principle of stationary action, Wick rotate from Minkowski metric to Euclidean metric, map the cylinder onto $\mathbb{C}$, and canonically quantise to get a Lie algebra of modes. Instead let us emphasise the equivalence between (the singular terms of) OPEs and the mode algebra by starting with an alternative definition:

The **free fermion** is the field $\psi(z)$ with defining OPE[1]

$$\psi(z)\psi(w) \sim \frac{\mathbf{1}}{z-w}. \tag{6.1}$$

The anti-symmetry under the exchange of fields that characterises fermionic statistics is immediately apparent. The energy-momentum tensor $L_F(z)$ is given by the normally ordered product

$$L_F(z) = \frac{1}{2}{:}\partial\psi(z)\psi(z){:}. \tag{6.2}$$

Using the fermionic version of Wick's theorem one can calculate the OPE of the energy-momentum tensor with itself. This is demonstrated in section 6.A. The result is

$$L_F(z)L_F(w) \sim \frac{1/4}{(z-w)^4}\mathbf{1} + \frac{2L_F(w)}{(z-w)^2} + \frac{\partial L_F(w)}{z-w}, \tag{6.3}$$

which is the defining OPE for a Virasoro symmetry field with central charge $c = 1/2$. *The free fermion is a $c = 1/2$ conformal field theory.* The OPE of $L_F(z)$ with $\psi(w)$ can also be calculated

[1] See any introductory text on conformal field theory or string theory, for example [65, Ch. 5-7].

using Wick's theorem, with the result

$$L_F(z)\psi(w) \sim \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}, \tag{6.4}$$

from which it follows that $\psi(w)$ is a Virasoro primary field with conformal weight $h = 1/2$. The free fermion is a generating function for modes ψ_n defined by

$$\psi(z) = \sum_n \psi_n z^{-n-1/2} \implies \psi_n = \oint_0 z^{n-1/2}\psi(z)\frac{dz}{2\pi i}. \tag{6.5}$$

The values taken by the index n will be specified shortly. The algebra of modes can be determined from the OPE using contour subtraction techniques as in section 3.7.1, but first we should mention that radial ordering is modified for non-bosonic fields.

Two fields $A(z)$ and $B(w)$ are said to be *mutually local* if commuting one field over the other results in a phase:

$$\mathcal{R}\{A(z)B(w)\} = \begin{cases} A(z)B(w), & \text{if } |z| > |w| \\ \mu_{AB}B(w)A(z), & \text{if } |w| > |z|, \end{cases} \tag{6.6}$$

where the factor $\mu_{AB} \in \mathbb{C}$ is called the *mutual locality index*, equal to 1 for mutually bosonic fields, and -1 for mutually fermionic fields. Free fermions are of course mutually fermionic.

Using the contour subtraction method of section 3.7.1 and the defining OPE in eq. (6.1), we can derive the algebra of the modes. Unlike the analogous free boson calculation, a minus sign is gained due to the mutual locality by commuting $\psi(z)$ and $\psi(w)$ into radial order in one of the integrals, and we derive not commutation relations, but *anti-commutation relations*:

$$\{\psi_n, \psi_m\} = \delta_{m+n,0}\mathbf{1}. \tag{6.7}$$

We shall denote this algebra by $\widehat{\mathfrak{ff}}$. The bracket implies the relationship

$$\{\psi_n, \psi_n\} = 2\psi_n^2 = \delta_{n,0}\mathbf{1}, \tag{6.8}$$

which vanishes if $n \neq 0$. This is an instance of the **Pauli exclusion principle** — two identical fermionic particles cannot simultaneously occupy the same quantum state. On level of the algebra, any operator that creates a fermion in a particular state must therefore square to zero.

Being fermionic also has some important implications on the level of fields. Suppose that $|z| > |w|$ and consider the product $\psi(z)\psi(w)$, which can be obtained from $\psi(w)\psi(z)$ by either commuting the fields into radial order and picking up a minus sign from the mutual locality, or by declaring the radial ordering to be implicit and interchanging z and w without gaining an overall sign. It follows that only terms that are anti-symmetric under the interchange of z and w can be non-vanishing in the OPE. Consider the next-to-leading order term of the free fermion OPE:

$$\psi(z)\psi(w) = \frac{\mathbf{1}}{z-w} + :\psi(w)\psi(w): + \mathcal{O}(z-w). \tag{6.9}$$

The normally ordered term $:\psi(w)\psi(w):$ is not anti-symmetric under interchange of z and w, and so vanishes identically.

What makes the free fermion conformal field theory particularly interesting is that $\psi(z)$ must admit both periodic and anti-periodic boundary conditions.[2] The $\widehat{\mathfrak{ff}}$ generating function $\psi(z)$ actually stands for one of two possible generating functions with different periodicities: $\psi_p(z)$ is

[2]The reason that both types of boundary conditions must be admitted comes down to the requirement of *modular invariance* of the free fermion partition function on a torus. Under the modular group, the characters of representations in each sector transform amongst one another, and there is no way to achieve a modular invariant partition function without summing characters from *both sectors*. This is a fascinating subject, but it is not relevant for us. See [53, Ch. 10,12] or [16, Ch. 4] for details.

periodic and said to be in the **Neveu-Schwarz sector**, and $\psi_a(z)$ is anti-periodic and in the **Ramond sector**:

$$\psi(e^{2\pi i}z) = \begin{cases} +\psi(z), & \text{if } \psi(z) = \psi_p(z) \\ -\psi(z), & \text{if } \psi(z) = \psi_a(z). \end{cases} \tag{6.10}$$

The Neveu-Schwarz and Ramond sectors are named after [98] and [111] respectively. In the Neveu-Schwarz sector the modes take half-integer indices, and in the Ramond sector they take integer indices:

$$\psi_p(z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} \psi_n z^{-n-1/2}, \qquad \psi_a(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1/2}. \tag{6.11}$$

The algebra spanned by modes with half-integer indices is denoted by $\widehat{\mathrm{NS}}$, and that spanned by modes with integer indices is denoted by $\widehat{\mathrm{R}}$ — these are called the *Neveu-Schwarz* and *Ramond algebras* respectively. These algebras decompose into creation operators, annihilation operators, and zero modes, and so highest weight modules can be constructed for each. For $\widehat{\mathrm{NS}}$, let ψ_n with $n > 0$ be annihilation operators and let ψ_n with $n < 0$ be creation operators. The Cartan subalgebra is spanned by the central element $\mathbf{1}$. Let $|0\rangle$ be the highest weight vector. Then the vector space defined by

$$\mathrm{NS} := \mathrm{span}_{\mathbb{C}}\{\cdots \psi_{-3/2}^{j_{3/2}} \psi_{-1/2}^{j_{1/2}} |0\rangle : j_i \in \{0,1\},\ \sum_k j_k < \infty\} \tag{6.12}$$

is a highest weight $\widehat{\mathrm{NS}}$-module.

For $\widehat{\mathrm{R}}$ things are more subtle. From eq. (6.8) we find that $\psi_0^2 = \mathbf{1}/2$, so ψ_0 cannot be treated as a zero-mode, since these must (anti-)commute. ψ_0 must be treated as a creation operator.[3] Let ψ_n with $n \leq 0$ be creation operators and let ψ_n with $n > 0$ be annihilation operators. Again the Cartan subalgebra is spanned by $\mathbf{1}$. Let $|\sigma\rangle$ be the highest weight vector. The vector space defined by

$$\mathrm{R} := \mathrm{span}_{\mathbb{C}}\{\cdots \psi_{-2}^{j_2} \psi_{-1}^{j_1} \psi_0^{j_0} |\sigma\rangle : j_i \in \{0,1\},\ \sum_k j_k < \infty\}$$

is a highest weight $\widehat{\mathrm{R}}$-module.

The Virasoro algebra is constructed in terms of $\widehat{\mathfrak{ff}}$ from the energy-momentum tensor. In this construction it turns out that the $\widehat{\mathrm{R}}$ mode ψ_0 commutes with L_0 and therefore does not alter conformal weights — this can be checked using the defining OPE in eq. (6.4) for $L_F(z)$ with $\psi(w)$, and the contour subtraction technique of section 3.7.1:

$$[L_n, \psi_m] = \oint_0 \oint_w \left\{ \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w} \right\} z^{n+1} w^{m-1/2} \frac{dz}{2\pi i}\frac{dw}{2\pi i}. \tag{6.13}$$

Setting $n = m = 0$, we find that the Lie bracket vanishes, confirming that ψ_0 does not change conformal weights. The vectors $|\sigma\rangle$ and $\psi_0|\sigma\rangle$ are both annihilated by all Virasoro annihilation operators L_n with $n > 0$, and so the Ramond sector has a *doubly-degenerate vacuum* — the $\widehat{\mathfrak{ff}}$ vacuum vectors $|\sigma\rangle$ and $\psi_0|\sigma\rangle$ are Virasoro vacuum vectors with the same conformal weight.[4]

There is now the question of the state-field correspondence. The field corresponding to $|0\rangle$ is the identity field, but which Ramond field corresponds to $|\sigma\rangle$? Let us label the fields corresponding to the two Ramond vacua by

$$\sigma(z) \leftrightarrow |\sigma\rangle\,, \qquad \mu(z) \leftrightarrow \psi_0|\sigma\rangle\,. \tag{6.14}$$

[3] ψ_0 cannot be an annihilation operator. To see this, let $|\sigma\rangle$ be a non-zero vacuum vector. For ψ_0 to be an annihilation operator would require that $\psi_0|\sigma\rangle = 0$, but acting again with ψ_0 retrieves $1/2\,|\sigma\rangle$, which is non-zero. Furthermore ψ_0 cannon be a zero-mode — we cannot have $\psi_0|\sigma\rangle = \lambda|\sigma\rangle$ for a scalar λ, since fermions carry *parity* information, and each side of the equation $\psi_0|\sigma\rangle = \lambda|\sigma\rangle$ has different parity.

[4] The two Ramond vacua $|\sigma\rangle$ and $\psi_0|\sigma\rangle$ become distinguishable if one extends $\widehat{\mathfrak{ff}}$ to include parity operators — see [65, Ch. 6,7].

These fields are called **twist-fields** in the string theory literature. It is well known in conformal field theory and string theory that they have conformal weight $h_\sigma = h_\mu = 1/16$ — see [65, Ch. 6] for a derivation.

6.1.1 Statistical mechanics

We now have sufficient knowledge of the free fermion to relate it to the minimal model $\mathcal{M}(3,4)$, and therefore to the Ising model.

Firstly, we note that the free fermion conformal field theory has central charge $c = 1/2$, which is equal to the minimal model value $c(3,4)$ from eq. (5.47). The Virasoro representation theory should therefore be the same for both theories, provided that the same collection of irreducible highest weight Virasoro representations appear. We check this by noting which Virasoro primary fields are present in each theory, and what conformal weights they have. In the free fermion conformal field theory these are[5]

$$\mathbf{1} \leftrightarrow h_{\mathbf{1}} = 0, \qquad \psi(z) \leftrightarrow h_\psi = 1/2, \qquad \sigma(z) \leftrightarrow h_\sigma = 1/16. \tag{6.15}$$

These are precisely the conformal weights of the $\mathcal{M}(3,4)$ primary fields

$$h_{1,1} = h_{\mathbf{1}} = 0 \qquad h_{2,1} = h_\epsilon = 1/2 \qquad h_{1,2} = h_\sigma = 1/16. \tag{6.16}$$

This suggests a correspondence[6] between the Virasoro primary fields in the free fermion theory, and the Ising model observables described by $\mathcal{M}(3,4)$ primary fields

$$\mathbf{1} \leftrightarrow \phi_{1,1}, \qquad \psi(z) \leftrightarrow \phi_{2,1}(z), \qquad \sigma(z) \leftrightarrow \phi_{1,2}(z). \tag{6.17}$$

Section 5.6.2 provides a link between $\mathcal{M}(3,4)$ and the Ising model. We now also have a link between $\mathcal{M}(3,4)$ and the free fermion conformal field theory. The twist-field $\sigma(z)$ of the free fermion corresponds to the spin-density of the Ising model from section 5.6.1, and the free fermion field $\psi(z)$ corresponds to the energy operator. There is still one more link to unveil — that which connects the free fermion to the free boson.

6.2 Bosonisation of the free fermion

The $c = 1$ free boson is connected to the $c = 1/2$ free fermion by taking tensor products of free fermion representations, and therefore of Virasoro representations.

6.2.1 Tensor products of representations

We define the action of $\mathfrak{Vir}$ on a tensor product of $\mathfrak{Vir}$-modules $V(h_1,c_1) \otimes V(h_2,c_2)$ in the standard manner [57, Ch. 15]:

$$\Delta(L_n) := L_n \otimes \mathbf{1} + \mathbf{1} \otimes L_n, \qquad \Delta(C) := C \otimes \mathbf{1} + \mathbf{1} \otimes C. \tag{6.18}$$

The linear map $\Delta : \mathfrak{Vir} \to \text{End}\left(V(h_1,c_1) \otimes V(h_2,c_2)\right)$ is a representation of the Virasoro algebra with central charge $c_1 + c_2$ and vacuum conformal weight $h_1 + h_2$. To see this consider as an example the action of $\Delta(L_0)$ on the vector $|h_1\rangle \otimes |h_2\rangle$:

$$\Delta(L_0)(|h_1\rangle \otimes |h_2\rangle) = (L_0 |h_1\rangle) \otimes \mathbf{1}\,|h_2\rangle + \mathbf{1}\,|h_1\rangle \otimes (L_0\,|h_2\rangle) = (h_1 + h_2)(|h_1\rangle \otimes |h_2\rangle). \tag{6.19}$$

[5] There is also the twist-field $\mu(z)$ to consider, but our derivation of OPE-coefficients will circumvent the need for this field. The reason is that the fields $\sigma(z)$ and $\mu(z)$ are not distinguishable by the Virasoro algebra, and here it is the Virasoro algebra that dictates the OPE-coefficients. From chapter 3 we know that OPE-coefficients completely determine correlation functions, which in turn encode all the physical information.

[6] Technically one must also check that there are no other Virasoro primary fields in the free fermion theory. This can be confirmed by comparing characters of $\mathcal{M}(3,4)$ and free fermion representations — see [53, Ch. 10,12,15].

The generalisation to a greater number of Virasoro modules is clear — simply include more tensor products.[7] The central charge and conformal weight are additive over tensor products of representations. The additivity of the central charge suggests a link between the representations constructed from the $c = 1$ Heisenberg vertex algebra, and two free fermions, each with $c = 1/2$. Let us explore this link.

For $A(z)$ a generic quantum field we define the notation

$$A^{(1)}(z) := A(z) \otimes \mathbf{1}, \qquad A^{(2)}(z) := \mathbf{1} \otimes A(z). \tag{6.20}$$

We denote the identity by $\mathrm{id} := \mathbf{1} \otimes \mathbf{1}$. Given that conformal weights are additive over tensor products, a candidate for the $h = 1$ free boson field is

$$B(z) := \beta\psi(z) \otimes \psi(z) = \beta\psi^{(1)}(z)\psi^{(2)}(z), \tag{6.21}$$

where $\beta \in \mathbb{C}$ will be specified momentarily. The OPE of $B(z)$ with itself can be determined:

$$B(z)B(w) = -\beta\psi(z)\psi(w) \otimes \beta\psi(z)\psi(w), \tag{6.22}$$

where the minus sign arises from commuting the two fermions $\psi(z)$ and $\psi(w)$ using mutual locality.

Inserting the defining OPE for $\psi(z)\psi(w)$ we have

$$\begin{aligned} B(z)B(w) &= -\beta^2 \left(\frac{\mathbf{1}}{z-w} + :\psi(w)\psi(w): + \mathcal{O}(z-w) \right) \otimes \left(\frac{\mathbf{1}}{z-w} + :\psi(w)\psi(w): + \mathcal{O}(z-w) \right) \\ &\sim -\beta^2 \left(\frac{\mathbf{1}}{z-w} \otimes \frac{\mathbf{1}}{z-w} \right) - \beta^2 \left(:\psi(w)\psi(w): \otimes \frac{\mathbf{1}}{z-w} \right) - \beta^2 \left(\frac{\mathbf{1}}{z-w} \otimes :\psi(w)\psi(w): \right). \end{aligned} \tag{6.23}$$

From eq. (6.9) we have $:\psi(w)\psi(w): = 0$, and so we are left with

$$B(z)B(w) \sim -\beta^2 \frac{\mathbf{1}}{z-w} \otimes \frac{\mathbf{1}}{z-w} = -\beta^2 \frac{\mathrm{id}}{(z-w)^2}. \tag{6.24}$$

If we choose either $\beta = i$ or $\beta = -i$ then we recover the defining OPE for the free boson — for each choice of β the contour subtraction method from section 3.7.1 can be used to verify that $B(z)$ is a generating function for the Heisenberg algebra.

We can also construct the free boson energy-momentum tensor $L_B(z)$. We make the ansatz

$$L_B(z) := \Delta\Big(L_F(z)\Big) = L_F^{(1)}(z) + L_F^{(2)}(z). \tag{6.25}$$

We can compute the OPE:

$$\begin{aligned} &L_B(z)L_B(w) \sim L_F(z)L_F(w) \otimes \mathbf{1} + \mathbf{1} \otimes L_F(z)L_F(w) \\ &\sim \left(\frac{1/4}{(z-w)^4}\mathbf{1} + \frac{2L_F(w)}{(z-w)^2} + \frac{\partial L_F(w)}{z-w} \right) \otimes \mathbf{1} + \mathbf{1} \otimes \left(\frac{1/4}{(z-w)^4}\mathbf{1} + \frac{2L_F(w)}{(z-w)^2} + \frac{\partial L_F(w)}{z-w} \right) \\ &\sim \frac{1/2}{(z-w)^4}\mathrm{id} + \frac{2L_B(w)}{(z-w)^2} + \frac{\partial L_B(w)}{z-w}, \end{aligned} \tag{6.26}$$

which is the defining OPE for the free boson energy-momentum tensor. This process of constructing free bosons out of two non-interacting free fermions is called **bosonisation** of the free fermion.

[7] Technically we are dealing with a *Hopf algebra*, which has extensive application in conformal field theory [18].

6.3 Fermionisation of the free boson

In conformal field theory the process can be reversed — the free fermion can be constructed using free boson vertex operators.

With $\psi^{(1)}$ and $\psi^{(2)}$ defined as in eq. (6.20) we have the OPEs

$$\psi^{(i)}(z)\psi^{(j)}(w) \sim \frac{\delta_{i,j}}{z-w}\mathrm{id}. \tag{6.27}$$

The tensor product construction defines two non-interacting free fermion fields.

In eq. (4.37) the Virasoro algebra was realised by taking normally ordered quadratic expressions of Heisenberg generators. The conformal weight of the Heisenberg vacuum $|p\rangle$ and its corresponding field, the vertex operator $V_p(z)$, was found in eq. (4.40) to be $h_p = p^2/2$. The vertex operators $V_1(z)$ and $V_{-1}(z)$ both have conformal weight $h = 1/2$, and this is the same weight as the free fermion $\psi(z)$. The OPE for vertex operators is given in eq. (4.62), from which we find that

$$V_{\pm 1}(z)V_{\mp 1}(w) \sim \frac{\mathbf{1}}{z-w}, \qquad V_{\pm 1}(z)V_{\pm 1}(w) \sim 0. \tag{6.28}$$

Now consider the following linearly independent linear combinations of vertex operators

$$\Psi_1(z) := \frac{1}{\sqrt{2}}\Big(V_1(z) + V_{-1}(z)\Big), \qquad \Psi_2(z) := \frac{i}{\sqrt{2}}\Big(V_1(z) - V_{-1}(z)\Big). \tag{6.29}$$

We can explicitly compute the OPEs of these fields, for example

$$\Psi_1(z)\Psi_1(w) = \frac{1}{2}\Big(V_1(z)V_1(w) + V_1(z)V_{-1}(w) + V_{-1}(z)V_1(w) + V_{-1}(z)V_{-1}(w)\Big) \sim \frac{\mathbf{1}}{z-w}, \tag{6.30}$$

and similarly for the remaining fields, giving the full set of relations

$$\Psi_i(z)\Psi_j(w) \sim \frac{\delta_{i,j}}{z-w}\mathbf{1}. \tag{6.31}$$

These are precisely the relations in eq. (6.27) for two non-interacting fermions. Hence not only can free boson theories be constructed from free fermions, but the converse is also true. This process is called **fermionisation** of the free boson.

That bosons and fermions can be constructed from one another is called the *boson-fermion correspondence*.

6.4 Solving the Ising model

Bosonisation and fermionisation allow for efficient determination of $\mathcal{M}(3,4)$ primary OPE-coefficients with relative ease. The idea is to construct correlation functions for tensor products of $\mathcal{M}(3,4)$ fields using bosonic vertex operators, whose correlation functions we already know from eqs. (4.67) and (4.82).

6.4.1 Ising model correlation functions

The Ising model spin-field $\sigma(z)$ corresponds to the free fermion twist-field, and the tensor product of two twist-fields lives in a $c = 1$ conformal field theory that can be realised in terms of vertex operators. With some foresight to constructing physically relevant solutions we do not want to confine ourselves to the representation space of the holomorphic sector only. Instead we work in the full representation space $\mathcal{H}$ defined in section 4.7, and use the non-chiral vertex

operators $V_p(z,\bar{z})$. On this representation space the holomorphic factorisation for the twist-field is: $\sigma(z,\bar{z}) = \sigma(z) \otimes \sigma(\bar{z})$. This field has scaling dimension $\Delta_\sigma = 1/16 + 1/16 = 1/8$.

We are interested in the tensor product of free fermion representations and so we want to consider fields of the form $\sigma^{(1)}(z,\bar{z})\sigma^{(2)}(z,\bar{z})$. Since conformal weights (and thus scaling dimensions) are additive over tensor products of representations, this operator has scaling dimension $1/8 + 1/8 = 1/4$. The scaling dimension of $V_p(z,\bar{z})$ is $\Delta = h_p + \bar{h}_p = p^2$, and so the non-chiral vertex operators of scaling dimension $1/4$ are $V_{1/2}(z,\bar{z})$ and $V_{-1/2}(z,\bar{z})$.

The aim is to replicate the conformal properties of $\sigma^{(1)}(z,\bar{z})\sigma^{(2)}(z,\bar{z})$ in correlation functions using the vertex operators $V_{1/2}(z,\bar{z})$ and $V_{-1/2}(z,\bar{z})$. The conservation of momentum conditions from section 4.6.3 and eq. (4.82) limit how this can done. For example, replacements such as

$$\sigma^{(1)}(z,\bar{z})\sigma^{(2)}(z,\bar{z}) \overset{?}{\leftrightarrow} V_{1/2}(z,\bar{z}), \qquad \sigma^{(1)}(z,\bar{z})\sigma^{(2)}(z,\bar{z}) \overset{?}{\leftrightarrow} V_{-1/2}(z,\bar{z}) \tag{6.32}$$

do not work. Although the scaling dimensions are matched up correctly, all correlation functions containing only $V_{1/2}$ or only $V_{-1/2}$ vanish, since the momenta cannot sum to zero. However, since minimal model primaries are self-conjugate, we know from eq. (3.31) that the two-point function containing two spin-fields does not vanish.

On the other hand linear combinations as per eq. (6.29) do work — an appropriate correspondence between twist-fields and vertex operators that gives the correct scaling behaviour in correlation functions and satisfies the conservation of momentum condition is[8]

$$\sigma^{(1)}(z,\bar{z})\sigma^{(2)}(z,\bar{z}) \leftrightarrow \frac{1}{\sqrt{2}}\Big(V_{1/2}(z,\bar{z}) + V_{-1/2}(z,\bar{z})\Big). \tag{6.33}$$

The factor of $1/\sqrt{2}$ multiplying the vertex operators is required for correctly normalised two-point functions, as we will demonstrate shortly. We can also express the energy-field $\epsilon(z,\bar{z}) = \epsilon(z)\otimes\epsilon(\bar{z})$ in terms of vertex operators that match the scaling dimension of $\epsilon^{(1)}(z,\bar{z})\epsilon^{(2)}(z,\bar{z})$, but we will not need to do so in order to solve $\mathcal{M}(3,4)$.[9]

6.4.2 Ising three-point constants by bosonisation

We use the notation $\sigma_i := \sigma(z_i,\bar{z}_i)$ and $\epsilon_i := \epsilon(z_i,\bar{z}_i)$ in the following. Squares of Ising model physical correlators can be easily computed using the vertex operator correspondence in eq. (6.33):

$$\langle \prod_{i=1}^{N} \sigma_i^{(1)}\sigma_i^{(2)} \rangle = \langle \prod_{i=1}^{N} \frac{1}{\sqrt{2}}\Big(V_{1/2}(z_i,\bar{z}_i) + V_{-1/2}(z_i,\bar{z}_i)\Big) \rangle \,. \tag{6.34}$$

Observe that each N-point function vanishes if N is odd. For the free fermion this is a manifestation of the square root branch cut of the twist-field — correlation functions must be single-valued and so branch cuts must cancel, which requires that there be an even number of them. For the vertex operators on the right-hand side of eq. (6.34) this is a consequence of the conservation of momentum condition — terms whose momenta sum to zero in the product of vertex operators can exist only if the number of factors in the product is even.[10] Hence we can restrict attention to the case where $N = 2n$ is even.

[8] An alternative choice is the other linearly independent combination $\frac{i}{\sqrt{2}}(V_{1/2}(z,\bar{z}) - V_{-1/2}(z,\bar{z}))$. There is no reason to favour one combination over the other — either can be used in the calculations following calculations, and both choices result in the same OPE-coefficients and correlations functions.

[9] For the sake of interest, the two vertex operators are $V_{\pm\sqrt{2}}(z,\bar{z})$ with scaling dimension $\Delta = 2 = 2\Delta_\epsilon$. The association $\epsilon^{(1)}(z,\bar{z})\epsilon^{(2)}(z,\bar{z}) \leftrightarrow \frac{1}{\sqrt{2}}\left(V_{\sqrt{2}}(z,\bar{z}) + V_{-\sqrt{2}}(z,\bar{z})\right)$ can be used to calculate energy-field correlators.

[10] Note that for the Ising lattice model the vanishing condition follows from physical considerations alone — it follows from the refection symmetry of the lattice under the reversal of all spins in the limit of vanishing external magnetic field. We never imposed such a condition on our conformal field theory, but fortunately it is naturally respected.

In section 4.7 we defined correlation functions for tensor products of fields to be given by the product of the correlation functions for each tensor factor. Hence the left-hand side of eq. (6.34) is the square of the $2n$-point function of spins:

$$\langle \prod_{i=1}^{2n} \sigma_i^{(1)} \sigma_i^{(2)} \rangle = \left(\langle \prod_{i=1}^{2n} \sigma_i \rangle \right)^2 . \tag{6.35}$$

The right-hand side of eq. (6.34) simplifies by expanding the product and keeping only terms whose momenta sum to zero. For these terms the physical correlators are given by eq. (4.82), and we find that

$$\langle \prod_{i=1}^{2n} \frac{1}{\sqrt{2}} \Big(V_{1/2}(z_i, \bar{z}_i) + V_{-1/2}(z_i, \bar{z}_i) \Big) \rangle = \frac{1}{2^n} \sum_{\substack{i=1 \\ \mu_i = \pm 1/2 \\ \sum_k \mu_k = 0}}^{2n} \prod_{i<j} |z_{ij}|^{2\mu_i \mu_j} . \tag{6.36}$$

Let us now check the normalisation of these expressions. From section 5.6.2, we know that minimal model fields are all self-conjugate, and so from the general solution to the Ward identities in eq. (3.31) we have the conventionally normalised two-point function

$$\langle \sigma_1 \sigma_2 \rangle^2 = \left(\frac{1}{|z_{12}|^{2\Delta_\sigma}} \right)^2 = \frac{1}{|z_{12}|^{1/2}} . \tag{6.37}$$

Indeed the right-hand side of the formula in eq. (6.36) reads

$$\frac{1}{2} \left(\frac{1}{|z_{12}|^{1/2}} + \frac{1}{|z_{12}|^{1/2}} \right) = \frac{1}{|z_{12}|^{1/2}} , \tag{6.38}$$

so the expressions in eqs. (6.34) and (6.36) are correctly normalised.

We are now in a position to determine the primary three-point constants and OPE-coefficients for $\mathcal{M}(3,4)$. This can be done by inserting OPEs into a four-point function and performing an asymptotic analysis. Consider the squared four-point function

$$\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle^2 . \tag{6.39}$$

The fusion rules of eq. (5.51) for the $\mathcal{M}(3,4)$ fields $\sigma = \phi_{1,2} \sim \phi_{2,2}$, $\epsilon = \phi_{2,1} \sim \phi_{1,3}$ and $\mathbf{1} = \phi_{1,1} \sim \phi_{2,3}$ give

$$[\sigma] \bowtie [\sigma] = [\mathbf{1}] + [\epsilon], \qquad [\sigma] \bowtie [\epsilon] = [\sigma], \qquad [\epsilon] \bowtie [\epsilon] = [\mathbf{1}]. \tag{6.40}$$

From these fusion rules and eq. (3.6) for the general form of the OPE, we have the decomposition[11]

$$\sigma(z_1, \bar{z}_1) \sigma(z_2, \bar{z}_2) = |z_{12}|^{-2\Delta_\sigma} \mathbf{1} + C_{\sigma\sigma}^{\epsilon} |z_{12}|^{-2\Delta_\sigma + \Delta_\epsilon} \epsilon(z_2, \bar{z}_2) + \mathcal{O}(|z_{12}|^{-2\Delta_\sigma + 2}), \tag{6.41}$$

where the scaling dimensions are $\Delta_\sigma = 1/8$ and $\Delta_\epsilon = 1$. The neglected terms contain descendent fields such as $(L_{-2}\mathbf{1})(z_2, \bar{z}_2) = L(z_2)$ and $(L_{-1}\epsilon)(z_2, \bar{z}_2) = \partial\epsilon(z_2, \bar{z}_2)$, both with scaling dimension 2, as well as the anti-holomorphic counterparts $(\bar{L}_{-2}\mathbf{1})(z_2, \bar{z}_2) = \bar{L}(\bar{z}_2)$ and $(\bar{L}_{-1}\epsilon)(z_2, \bar{z}_2) = \bar{\partial}\epsilon(z_2, \bar{z}_2)$, also with scaling dimension 2.[12] All other neglected terms have higher scaling dimension, and therefore have coefficients that are less singular as $|z_{12}| \to 0$. The neglected terms will not contribute in the following calculations, and we will continue to denote them by $(\dots)$.

[11] Note that these OPE-coefficients are the *squares* of the coefficients in eq. (3.6) for the holomorphic sector only, because we are working with non-chiral fields.

[12] Note that the fields $(L_{-1}\mathbf{1})$ and $(\bar{L}_{-1}\mathbf{1})$ vanish — the state corresponding to $\mathbf{1}$ is the true vacuum, and the true vacuum is translationally invariant.

Inserting the OPEs $\sigma_1\sigma_2$ and $\sigma_3\sigma_4$ into eq. (6.39) we have

$$\begin{aligned}\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle^2 &= \Big\langle\left(\frac{\mathbf{1}}{|z_{12}|^{1/4}} + C^{\epsilon}_{\sigma\sigma}|z_{12}|^{3/4}\epsilon_2 + \dots\right)\left(\frac{\mathbf{1}}{|z_{34}|^{1/4}} + C^{\epsilon}_{\sigma\sigma}|z_{34}|^{3/4}\epsilon_4 + \dots\right)\Big\rangle^2 \\ &= \left(\frac{1}{|z_{12}z_{34}|^{1/4}} + |z_{12}z_{34}|^{3/4}(C^{\epsilon}_{\sigma\sigma})^2\,\langle\epsilon_2\epsilon_4\rangle\right)^2 + \dots \end{aligned} \tag{6.42}$$

Note that $\langle\mathbf{1}\epsilon_i\rangle$, $\langle(L_{-2}\mathbf{1})\mathbf{1}\rangle$, $\langle(L_{-2}\mathbf{1})\epsilon_i\rangle$, $\langle\mathbf{1}(L_{-1}\epsilon_i)\rangle$, $\langle(\bar{L}_{-2}\mathbf{1})\mathbf{1}\rangle$, $\langle(\bar{L}_{-2}\mathbf{1})\epsilon_i\rangle$, $\langle\mathbf{1}(\bar{L}_{-1}\epsilon_i)\rangle$, and $\langle\mathbf{1}(L_{-1}\bar{L}_{-1}\epsilon_i)\rangle$ all vanish, which follows from eq. (3.31) because the two fields that appear in each correlator are not conjugates. All descendent terms in eq. (6.42) containing these correlators can be omitted. All other terms omitted from eq. (6.42) are less singular than the ones shown, and for this reason even those descendent terms which might be non-vanishing can safely be neglected in the following calculation.

From eq. (3.31) we have the normalised two-point function

$$\langle\epsilon_2\epsilon_4\rangle = \frac{C^{\mathbf{1}}_{\epsilon\epsilon}}{|z_{24}|^{2\Delta_\epsilon}} = \frac{1}{|z_{24}|^2}, \tag{6.43}$$

and so the four-point function in eq. (6.42) can be expanded as

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle^2 = \frac{1}{|z_{12}z_{34}|^{1/2}} + 2\frac{|z_{12}z_{34}|^{1/2}}{|z_{24}|^2}(C^{\epsilon}_{\sigma\sigma})^2 + \frac{|z_{12}z_{34}|^{3/2}}{|z_{24}|^4}(C^{\epsilon}_{\sigma\sigma})^4 + \dots \tag{6.44}$$

On the other hand we have eq. (6.36), which reads

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle^2 = \frac{1}{4}\sum_{\substack{i=1\\ \mu_i=\pm1/2\\ \sum_k\mu_k=0}}^{4}\prod_{i<j}|z_{ij}|^{2\mu_i\mu_j}. \tag{6.45}$$

The valid sign choices for $(\mu_1,\mu_2,\mu_3,\mu_4)$ are all permutations of $(+,+,-,-)$, counting multiplicities.[13] Summing these we arrive at the expression

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle^2 = \frac{1}{4}\times 2\left(\frac{|z_{12}z_{34}|^{1/2}}{|z_{13}z_{14}z_{23}z_{24}|^{1/2}} + \frac{|z_{13}z_{24}|^{1/2}}{|z_{12}z_{14}z_{23}z_{34}|^{1/2}} + \frac{|z_{14}z_{23}|^{1/2}}{|z_{12}z_{13}z_{24}z_{34}|^{1/2}}\right). \tag{6.46}$$

By equating eqs. (6.44) and (6.46) we can explicitly extract three-point constants by considering asymptotics. Consider the regime where $|z_{12}|, |z_{34}| \approx r \ll R \approx |z_{13}|, |z_{23}|, |z_{14}|, |z_{24}|$, which corresponds to the Feynman type diagram in figure 6.1.

In this asymptotic regime eq. (6.44) becomes

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle^2 = \frac{1}{r} + 2\frac{r}{R^2}(C^{\epsilon}_{\sigma\sigma})^2 + \mathcal{O}(r^3/R^2), \tag{6.47}$$

and eq. (6.46) becomes

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle^2 = \frac{1}{2}\frac{R}{rR}\left(1 + \frac{r^2}{R^2} + \frac{R^2}{R^2}\right) = \frac{1}{r} + \frac{r}{2R^2}. \tag{6.48}$$

Comparing coefficients of the leading term $1/r$ in each expression confirms again that we have the correct normalisation for eqs. (6.33) and (6.36). The next to leading term $r/2R^2$ gives the

[13] With four elements, two pairs of which are equal, there are $4!/(2!2!) = 6$ permutations. But $(\mu_1,\mu_2,\mu_3,\mu_4)$ and $(-\mu_1,-\mu_2,-\mu_3,-\mu_4)$ give identical terms, so in fact there are only three distinct terms, each with multiplicity two.

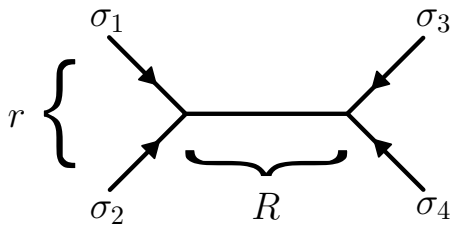


Figure 6.1: *Feynman type diagrams are used often in conformal field theory to illustrate OPEs. A vertex between two fields represents an insertion of the OPE of those fields. Here the spin-fields "fuse" to give the identity field and the energy-field, as per the fusion rules of eq.* (6.40).

correctly normalised OPE-coefficient:[14]

$$2(C^{\epsilon}_{\sigma\sigma})^2 = \frac{1}{2} \implies C^{\epsilon}_{\sigma\sigma} = \frac{1}{2}. \tag{6.49}$$

To verify eq. (6.49), another possible approach is to insert just one OPE into $\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle$, and to then use the general formula in eq. (3.30) for the resulting three-point functions. From the self-conjugacy of minimal model primary fields we know that the three-point constant $C_{\sigma\sigma\epsilon}$ and the OPE-coefficient $C^{\epsilon}_{\sigma\sigma}$ coincide. By looking in the same asymptotic regime as above we should arrive at the same result for $C^{\epsilon}_{\sigma\sigma}$. Although not shown here, we have performed this calculation, and the results confirm eq. (6.49).

Equation (6.49) actually gives more than just one OPE-coefficient; we have yet to use the associativity of OPEs. Taking this into account gives all of the non-trivial primary OPE-coefficients, and since OPE-coefficients and three-point constants coincide, we have in fact calculated the full set of $\mathcal{M}(3,4)$ primary three-point constants:

$$C_{\sigma\sigma\mathbf{1}} := C_{\epsilon\epsilon\mathbf{1}} := 1, \qquad C_{\sigma\sigma\epsilon} = \frac{1}{2}, \tag{6.50}$$

where the indices can be cyclically permuted. All other primary three-point constants vanish according to the fusion rules (5.51). This is the complete set of primary three-point constants for the Ising model.

Realising $\mathcal{M}(3,4)$ correlators in terms of bosonic vertex operators was a relatively straightforward task. The key observation was that the tensor product of two $c = c(3,4) = 1/2$ free fermion representations is a $c = 1$ free boson representation. However, this simple construction does not work for generic $\mathcal{M}(p',p)$ models. A generalisation is needed, and this motivates the question: *"can the free boson be modified to produce $c = c(p',p)$ theories in such a way that vertex operators can still be constructed?"*. The answer is *"yes"*, but some work is required, and this is the topic of chapter 8. Before turning to this construction we shall first reconsider the differential equations derived in chapter 5. The theory and results of chapter 7 will be found hiding in the techniques of chapter 8.

[14] The choice of positive branch of the root is justified by the fact that $\mathcal{M}(3,4)$ is unitary (section 5.5), which means that scalar products are positive-definite, and so non-chiral three-point constants and OPE-coefficients are non-negative.

6.A Appendix: Fermionic version of Wick's theorem

Differentiating the defining OPE in 6.1 we have

$$\partial\psi(z)\psi(w) = \frac{\mathbf{-1}}{(z-w)^2}, \qquad \psi(z)\partial\psi(w) = \frac{\mathbf{1}}{(z-w)^2}, \qquad \partial\psi(z)\partial\psi(w) = \frac{-2}{(z-w)^3}\mathbf{1}. \tag{6.51}$$

Wick's theorem is modified for free fermions — the prescription in section 4.5 still holds, except that when two fields are being contracted we take them to have first been commuted into adjacent positions, which results in the gain of a minus sign each time two mutually fermionic fields are commuted with one another. Refer to [6, App. A] for more details. The free fermion energy-momentum tensor OPE is

$$\begin{aligned} L_F(z)L_F(w) =& \frac{1}{4}\Big(:\partial\psi(z)\psi(z):\Big)\Big(:\partial\psi(w)\psi(w):\Big) = \ :(\text{regular term}): + \\ & \frac{1}{4}\Big\{:\partial\psi(z)\overbrace{\psi(z}\overbrace{\partial\psi(w}\psi(w):\, - \,:\partial\psi(z)\overbrace{\psi(z}\overbrace{\partial\psi(w)\psi(w)}: \\ & -\,:\partial\overbrace{\psi(z)\psi(z}\overbrace{\partial\psi(w)}\psi(w):\, + \,:\partial\overbrace{\psi(z)\psi(z)\partial\psi(w)\psi(w)}: \\ & -\,:\partial\overbrace{\psi(z)\underbrace{\psi(z}\partial\psi(w)\underbrace{\psi(w)}}:\, + \,:\partial\underbrace{\psi(z)\overbrace{\psi(z}\partial\psi(w)\psi(w)}:\Big\}. \end{aligned} \tag{6.52}$$

Inserting the OPEs and Laurent expanding the normally ordered terms containing different arguments, we arrive at the result in eq. (6.2).

Chapter 7

Differential Equations and Monodromy

Minimal model chiral correlation functions satisfy singular vector differential equations, as derived in chapter 5. Particular solutions to these differential equations are the conformal blocks discussed in section 3.4.1 — the building blocks for physical correlators. Without boundary conditions indicating how conformal blocks should be combined, one might wonder how physical correlators can be constructed. Imposing consistency conditions is key — a physical correlator must be single-valued, leading to a constraint called monodromy invariance. This constraint specifies how conformal blocks from the holomorphic and anti-holomorphic sectors should be combined, which allows for the construction of physical correlators even without a set of boundary conditions.

7.1 Four-point functions

In chapter 5 we derived the level 2 singular vector differential equation (5.26). We considered the case of a primary three-point function, from which we derived fusion rules. Now let us consider the equation for a primary four-point function:

$$\left\{\partial_1^2 + \eta_2 \sum_{j=2}^{4} \left[\frac{h_j}{(z_j - z_1)^2} - \frac{1}{z_j - z_1}\partial_j\right]\right\} \langle 0|\, \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\, |0\rangle = 0, \tag{7.1}$$

where η_2 is defined in eq. (5.14). Equation (7.1) is satisfied if ϕ_1 has Kac weight $h_{2,1}$ or $h_{1,2}$. The Ward identities from section 3.5 determine four-point functions up to an arbitrary differentiable function of a single cross-ratio, say

$$x = \frac{z_{12}z_{34}}{z_{14}z_{32}}. \tag{7.2}$$

This fact allows one to transform the four-variable partial differential equation in eq. (7.1) into a one-variable ordinary differential equation. See [53, Ch. 8.3] for expressions of N-point functions involving $N-3$ cross-ratios. Instead of referring to these general expressions, we will appeal to the definition of primary fields from eq. (3.1).

The Möbius transformation ω that produces the cross-ratio x is

$$\omega : z \mapsto \omega(z) = \frac{(z - z_2)(z_3 - z_4)}{(z - z_4)(z_3 - z_2)}, \tag{7.3}$$

which performs the mapping $(z_1, z_2, z_3, z_4) \mapsto (x, 0, 1, \infty)$. Since ω is a conformal transformation the primary fields transform as

$$\phi_i(z_i) = \left(\frac{d\omega}{dz}\right)^{h_i}_{|z=z_i} \phi_i(\omega(z_i)). \tag{7.4}$$

Writing $\omega_i := \omega(z_i)$ we have the corresponding scaling of the correlator

$$\langle 0|\,\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\,|0\rangle = \prod_{i=1}^{4}\left(\frac{d\omega}{dz}\right)^{h_i}_{|z=z_i} \langle 0|\,\phi_1(\omega_1)\phi_2(\omega_2)\phi_3(\omega_3)\phi_4(\omega_4)\,|0\rangle\,. \tag{7.5}$$

Denote is as

$$\langle 0|\,\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\,|0\rangle := FH(x), \tag{7.6}$$

where $H(x)$ is defined by[1]

$$H(x) = \langle \phi_4|\,\phi_3(1)\phi_1(x)\,|\phi_2\rangle\,, \tag{7.7}$$

and the factor F is

$$F = z_{14}^{-2h_1} z_{32}^{-h_1-h_2-h_3+h_4} z_{24}^{-h_2-h_4+h_3+h_1} z_{34}^{-h_3-h_4+h_2+h_1}. \tag{7.8}$$

There is a minor comment to make about deriving this factor. Note that the derivative with respect to z_4 cannot be computed directly from eq. (7.4), because $(d\omega/dz)$ diverges at $z = z_4$. The subtlety arises from the definition of $\langle\phi_4|$, which we defined in eq. (3.14) to be $\langle\phi_4| := \lim_{R\to\infty} R^{2h_4}\langle 0|\,\phi_4(R)$. Taking this into account gives a well defined factor. The other derivatives can be computed directly from eq. (7.4).

The function on the right-hand side of eq. (7.6) satisfies the same singular vector equation (7.1) as the four-point function. Note that $H(x)$ depends on only one free parameter x, with the dependence on each individual z_i factored out. This enables us to transform the partial differential equation in eq. (7.1) into an ordinary differential equation for $H(x)$, which we can then solve. This transformation takes a bit of work — see App. 7.A. The result is

$$\left\{\frac{d^2}{dx^2} - \eta_2\left(\frac{1}{x}+\frac{1}{x-1}\right)\frac{d}{dx} - \eta_2\left[\frac{h_1+h_2+h_3-h_4}{x(x-1)} - \frac{h_3}{(x-1)^2} - \frac{h_2}{x^2}\right]\right\} H(x) = 0. \tag{7.9}$$

Equation (7.9) appears without a detailed derivation in [9].

7.2 The hypergeometric class

Let us classify eq. (7.9) before attempting to solve it. Refer to [121, Ch. 1,2] for a review of the following theory. Equation (7.9) is a linear homogeneous second-order ordinary differential equation, written in the following standard form:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \tag{7.10}$$

where $x \in \mathbb{C}$ and the coefficients $P(x)$ and $Q(x)$ are ratios of polynomials in x. For eq. (7.10) a point x_0 is called **ordinary** if $P(x)$ and $Q(x)$ are analytic there, otherwise x_0 is called a **singular point**. A singular point x_0 is called **regular** if $Q(x)$ has at most a second-order pole at x_0 and $P(x)$ at most a first-order pole, or equivalently if $(x-x_0)^2Q(x)$ and $(x-x_0)P(x)$ remain finite as $x \to x_0$. If this is not the case then the singular point is **irregular**.

From these definitions we find that both $x = 0$ and $x = 1$ are regular singular points of eq. (7.9). To classify the point at infinity we make the change of variables $\xi = 1/x$ and repeat the checks for $\xi \to 0$. The point at infinity is also a regular singular point. So eq. (7.9) is a linear homogeneous second-order ordinary differential equation with *three distinct regular singular points*.

Having only singular points that are regular on $\mathbb{S}^2 := \mathbb{C} \cup \{\infty\}$ (and a few other minor conditions [121, Sec. 1.1]) puts eq. (7.9) into a class of equations called the *Fuchsian class*. In particular, Fuchsian equations with *three* singular points constitute the **hypergeometric class**

[1] The fields have been rearranged as a reminder of the implicit radial ordering.

— every equation in this class can be transformed into the famous **hypergeometric differential equation**. Make the substitution [67, Ch. 10]

$$H(x) = x^{\mu}(1-x)^{\nu}G(x) \tag{7.11}$$

into eq. (7.9), where μ and ν satisfy the indicial equations:[2]

$$\mu^2 - (\eta_2+1)\mu + \eta_2 h_2 = 0, \qquad \nu^2 - (\eta_2+1)\nu + \eta_2 h_3 = 0. \tag{7.12}$$

Substituting eq. (7.11) into eq. (7.9) we get the following differential equation for $G(x)$:

$$x(1-x)G''(x) + [c - (a+b+1)x]\,G'(x) - abG(x) = 0, \tag{7.13}$$

where a, b, and c satisfy

$$\begin{cases} a+b+1 = 2(\mu+\nu-\eta_2) \\ ab = 2\mu\nu - (\mu+\nu+h_1+h_2+h_3-h_4)\eta_2 \\ c = 2\nu - \eta_2. \end{cases} \tag{7.14}$$

7.2.1 Hypergeometric functions

For brevity, when performing calculations in the remainder of this chapter we shall not present a detailed discussion of convergence issues. App. 7.C contains sufficient convergence information to verify the calculations in this chapter.

Equation (7.13) is the hypergeometric differential equation.[3] Solutions to this equation are known in terms of the **hypergeometric function** ${}_2F_1(a,b;c;x)$.[4] The hypergeometric function admits integral representations. For example, we have the following integral expressions [30, eq. 3,3']:

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt, \tag{7.15}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-x)^{-a}dt. \tag{7.16}$$

The integral in eq. (7.16) is obtained by making the change of integration variable $t \mapsto 1/t$ in eq. (7.15). Both eqs. (7.15) and (7.16) will appear in chapter 8. The hypergeometric function is normalised so that ${}_2F_1(a,b;c;0) = 1$ — the ratio of gamma functions multiplying the integrals is the reciprocal of Euler's beta function

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{7.17}$$

Equations (7.15) to (7.17) will be important in chapter 8.

A solution to the differential equation in eq. (7.10) is said to be **local** to a regular singular point x_0 if it is analytic at x_0. The solution in eq. (7.15) to the hypergeometric equation is local to the singular point at $x = 0$ (under some convergence conditions given in App. 7.C).

[2] The exponents μ and ν are called *characteristic exponents* for the singular points at $x = 0$ and $x = 1$ respectively.

[3] The hypergeometric differential equation is the unique second-order linear ordinary differential equation with precisely three singular points $0, 1$, and ∞, all of which are regular, with characteristic exponents $(0, 1-c)$, $(0, c-a-b)$ and (a, b) respectively [30, Sec. 1].

[4] The original solution to the hypergeometric equation dates back to Euler in 1778. The hypergeometric function has found outstandingly diverse applications since then — the book [132] on hypergeometric functions deserves special mention on this point.

${}_2F_1(a, b; c; x)$ is analytic within the unit disc centred at $x = 0$ — whether the disc is open or closed depends on a, b, and c (App. 7.C).

For a second-order ordinary differential equation we should expect another linearly independent solution, with the general solution given by a linear combination of the two. Another linearly independent solution is [121, Ch. 2]

$$x^{1-c}{}_2F_1(a + 1 - c, b + 1 - c; 2 - c; x). \tag{7.18}$$

The function in eq. (7.18) is also local to $x = 0$ provided that $c < 1$ and the hypergeometric function converges as per the conditions in App. 7.C.

So, we have a differential equation for a chiral four-point function with $\phi_1(z_1) = \phi_{1,2}(z_1)$ or $\phi_{2,1}(z_1)$, and we have found particular solutions. However, we have no boundary conditions to tell us how these should be combined to give the physical correlator. The answer to this problem is a consistency condition called monodromy invariance.

7.3 Monodromy and the two-point connection problem

7.3.1 Monodromy

A solution to an ordinary differential equation is often obtained locally about a singular point, for example by series solutions or integrals. If a solution is analytic on its domain then one can try to analytically continue the solution into a larger domain. If analytic continuation is possible, one can then try to continue the solution back to the original domain. The resulting continuation, if possible, will still solve the ordinary differential equation, but it might be a different solution to what we started with — the value taken by the original solution might differ from the value taken by continued solution even where the arguments coincide. How can one be sure that two analytic continuations result in the same solution?

A result from complex analysis called the **monodromy theorem** [85, Thm. 2.3] says that for an analytic function, if two paths of analytic continuation with the same endpoints can be continuously deformed into one another (that is, without passing over a singular point) then the endpoint values of the continuations will agree. The monodromy theorem does not apply to paths that are not continuously deformable to one another, such as those depicted in fig. 7.1.

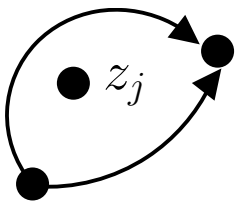


Figure 7.1: *If z_j is a singular point, then the two paths illustrated are not continuously deformable into one another. If an analytic function is continued along these paths, then the values at the endpoints might not agree.*

Continuing a solution to a differential equation about a loop is called a **monodromy transformation**, as depicted in fig. 7.2. If every monodromy transformation acts as the identity operation on a solution to an equation then that solution is said to be **monodromy invariant**. In particular a non-zero solution is single-valued if and only if it is monodromy invariant. *Physical correlators must be monodromy invariant.*

Particular solutions to a differential equation might not be monodromy invariant, but it may be possible to construct a monodromy invariant solution by taking certain combinations of

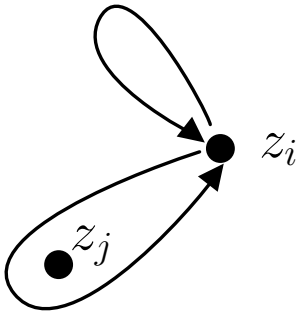


Figure 7.2: *Both loops depict monodromy transformations. The upper loop is continuously contractible to a point, so the monodromy theorem guarantees that the monodromy transformation for this loop acts as the identity on an analytic function. If z_j is a singular point then the same cannot be said for the lower loop.*

particular solutions. To do so we need to determine how each particular solution to the differential equation transforms under monodromy transformations.[5] Understanding this procedure is the goal of section 7.3.2.

7.3.2 The two-point connection problem

Local to each singular point (all of which are regular) of eq. (7.13) one can construct standard *fundamental sets* of solutions.[6] Two standard solutions local to $x = 0$ are given by

$$ {}_2F_1(a, b; c; x), \qquad x^{1-c}{}_2F_1(a + 1 - c, b + 1 - c; 2 - c; x). \tag{7.19}$$

These are linearly independent, and analytic within the unit disc about $x = 0$. There is also a pair of fundamental solutions local to $x = 1$ given by [121, Ch. 2]

$$ {}_2F_1(a, b; a + b + 1 - c; 1 - x), \qquad (1 - x)^{c-a-b}{}_2F_1(c - a, c - b; 1 + c - a - b; 1 - x). \tag{7.20}$$

These are linearly independent, and analytic within the unit disc about $x = 1$. Another pair of solutions local to $x = \infty$ can be found by making the change of variables $\xi = 1/x$ and searching for solutions local to $\xi = 0$. For reasons that will be explained later we do not need to consider solutions local to $x = \infty$.

For a second-order linear differential equation, how do we end up with 6 solutions? The answer of course is that they are not all linearly independent. Any pair of solutions can be analytically continued into any other by a linear transformation[7] — we say that the transformation *connects* one solution to another. Finding the transformation that connects pairs of solutions is called a **connection problem**.

For example, we write the solutions local to $x = 0$ as a column vector:

$$ \mathbf{F_0}(x) := \begin{bmatrix} f_1^{(0)}(x) \\ f_2^{(0)}(x) \end{bmatrix} := \begin{bmatrix} {}_2F_1(a, b; c; x) \\ x^{1-c}{}_2F_1(a + 1 - c, b + 1 - c; 2 - c; x) \end{bmatrix}, \tag{7.21}$$

and local to $x = 1$ as

$$ \mathbf{F_1}(x) := \begin{bmatrix} f_1^{(1)}(x) \\ f_2^{(1)}(x) \end{bmatrix} := \begin{bmatrix} {}_2F_1(a, b; a + b + 1 - c; 1 - x) \\ (1 - x)^{c-a-b}{}_2F_1(c - a, c - b; 1 + c - a - b; 1 - x) \end{bmatrix}. \tag{7.22}$$

[5] This has a nice interpretation as finding a representation for the *monodromy group* [30].

[6] Sometimes these are called *Frobenius* solutions. They can be found using the well known *Frobenius method*, which consists of looking for series solutions and deriving recursion relations for the series coefficients.

[7] For the hypergeometric differential equation there are 24 such pairings, known as *Kummer's relations* [120].

We do not need to consider the solutions $\mathbf{F}_\infty(x)$ local to ∞. The solutions $\mathbf{F_0}(x)$ and $\mathbf{F_1}(x)$ are connected by a 2×2 matrix T called a **connection matrix**:

$$\mathbf{F_0}(x) = T\mathbf{F_1}(x). \tag{7.23}$$

The connection matrix T is a constant matrix whose coefficients depend on the parameters a, b, and c. The coefficients can be determined by considering the values taken by the fundamental solutions in eqs. (7.21) and (7.22) at specific values of x, which we can compute using the integral representation in eq. (7.15).

Firstly, setting $x = 1$, the integral in eq. (7.15) becomes an integral representation for Euler's beta function from eq. (7.17), and we obtain

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(c-a-b)\Gamma(b)}{\Gamma(c-a)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}. \tag{7.24}$$

The real integral converges if and only if b and $c - a - b$ are positive, but the final ratio in eq. (7.24) is the unique analytic continuation defined wherever the ratio of gamma functions is finite [121, Ch. 2.4].

Using the result in eq. (7.24), and the fact that the hypergeometric function is normalised so that ${}_2F_1(a, b; c; 0) = 1$, we can compute $\mathbf{F_0}(0), \mathbf{F_0}(1), \mathbf{F_1}(0)$, and $\mathbf{F_1}(1)$ explicitly. For example, assuming we have $0 < c < 1$ it follows that $f_2^{(0)}(0) = 0$ due to the factor of x^{1-c} in eq. (7.21) and finiteness of ${}_2F_1(a, b; c; 0) = 1$. Similarly, assuming we have $0 < c - a - b < 1$ it follows that $f_2^{(1)}(1) = 0$. Substituting these values of x into eq. (7.23) we obtain the following linear system of equations

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = T \begin{bmatrix} f_1^{(1)}(0) \\ f_2^{(1)}(0) \end{bmatrix}, \qquad \begin{bmatrix} f_1^{(0)}(1) \\ f_2^{(0)}(1) \end{bmatrix} = T \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{7.25}$$

The remaining values of $f_i^{(j)}(0)$ and $f_i^{(j)}(1)$ can all be determined from eq. (7.24) and the normalisation ${}_2F_1(a, b; c; 0) = 1$. The connection coefficients can then be found by inverting the linear system in eq. (7.25). Solving the system we find that

$$\begin{aligned} T_{11} &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, & T_{12} &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \\ T_{21} &= \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}, & T_{22} &= \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}. \end{aligned} \tag{7.26}$$

A derivation of eq. (7.26) is given in App. 7.B.

We are now in a position to compute monodromy transformations for the solutions to the equation. Our differential equation has $\mathbb{S}^2 \backslash \{0, 1, \infty\}$ as its domain, and the solution space is closed under continuations in this domain. A monodromy transformation about any point will therefore act as a 2×2 transformation matrix on the vectors of solutions $\mathbf{F_0}(x)$ and $\mathbf{F_1}(x)$.

Monodromy about 0

For the solutions $\mathbf{F_0}(x)$, which are analytic within the unit disc centred at $x = 0$, we can explicitly compute the monodromy transformation about $x = 0$. Denote by M_0 the monodromy operator that acts on x, taking it once about the origin in an anticlockwise loop that encloses no singular points except for the origin. The associated monodromy transformation about the origin is

$$\mathbf{F_0}(x) \mapsto \mathbf{F_0}(M_0 \cdot x) = (M_0)\mathbf{F_0}(x), \tag{7.27}$$

where (M_0) is a 2×2 matrix called the **monodromy matrix** of the operator M_0.

We can parametrise the action of M_0 explicitly. We choose a circular loop $\gamma_0 : [0,1] \to \mathbb{C}\backslash\{0,1\}$ given by

$$\gamma_0(\kappa) := xe^{2\pi i\kappa}, \tag{7.28}$$

which starts and ends at x, and encircles the origin once, avoiding all singular points. The only singular point it encloses is at the origin, because the radial ordering we have assumed is $0 < |x| < 1$.

We can find the matrix (M_0) by substituting $\gamma_0(\kappa)$ for x in eq. (7.21) and taking κ continuously from 0 to 1. The hypergeometric function ${}_2F_1(a,b;c;x)$ is unchanged by M_0, since ${}_2F_1(a,b;c;x)$ is analytic within the unit disc centred at the origin. However, we do gain a phase from the factor x^{1-c}, and we derive the matrix

$$(M_0) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi ic} \end{pmatrix}. \tag{7.29}$$

Note that (M_0) is *diagonal* and the solutions in $\mathbf{F_0}(x)$ are unchanged up to phase — this is important, but we will have to wait until chapter 8 to appreciate why.

Monodromy about 1

The monodromy transformation M_1 which takes x about 1 corresponds to the transformation $\mathbf{F_0}(x) \mapsto \mathbf{F_0}(M_1 \cdot x) = (M_1)\mathbf{F_0}(x)$. To determine (M_1) one might try to explicitly parametrise the transformation, as we did before using $\gamma_0(\kappa)$, but then one quickly runs into a problem — the integrals in eqs. (7.15) and (7.16) that define $\mathbf{F_0}(x)$ diverge if x is taken out of the unit disc centred at the origin. However, the transformation can be easily computed for $\mathbf{F_1}(x)$, which is convergent within the unit disc about $x = 1$. Denote the corresponding monodromy matrix by (M_1') so that we have $\mathbf{F_1}(M_1 \cdot x) = (M_1')\mathbf{F_1}(x)$. If we make the substitution $x' = 1 - x$ in $\mathbf{F_1}(x)$, and compare the resulting expressions to the monodromy transformation about 0 for $\mathbf{F_0}(x)$, then from eq. (7.29) we read off

$$\mathbf{F_1}(x) \mapsto \mathbf{F_1}(M_1 \cdot x) = (M_1')\mathbf{F_1}(x) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(c-a-b)} \end{pmatrix} \mathbf{F_1}(x). \tag{7.30}$$

Now we come to the point of all of this. The matrix (M_1) can be determined by first connecting $\mathbf{F_0}(x)$ to $\mathbf{F_1}(x)$, then performing the monodromy transformation M_1 using the known monodromy matrix (M_1'), and then connecting the result back to $\mathbf{F_0}(x)$:

$$(M_1)\mathbf{F_0}(x) = \mathbf{F_0}(M_1 \cdot x) = T\mathbf{F_1}(M_1 \cdot x) = T(M_1')\mathbf{F_1}(x) = T(M_1')T^{-1}\mathbf{F_0}(x). \tag{7.31}$$

The monodromy matrix (M_1) is therefore given by

$$(M_1) = T(M_1')T^{-1}. \tag{7.32}$$

With a few gamma function manipulations similar to those in App. 7.B and section 8.8 we find that the determinant of the connection matrix is

$$\det T = T_{11}T_{22} - T_{12}T_{21} = \frac{1-c}{a+b-c}. \tag{7.33}$$

Our previous assumptions that $0 < c < 1$ and $0 < c - a - b < 1$ therefore guarantee invertibility of T. The inverse connection matrix takes the simple form:

$$T^{-1} = \frac{a+b-c}{1-c} \begin{pmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{pmatrix}. \tag{7.34}$$

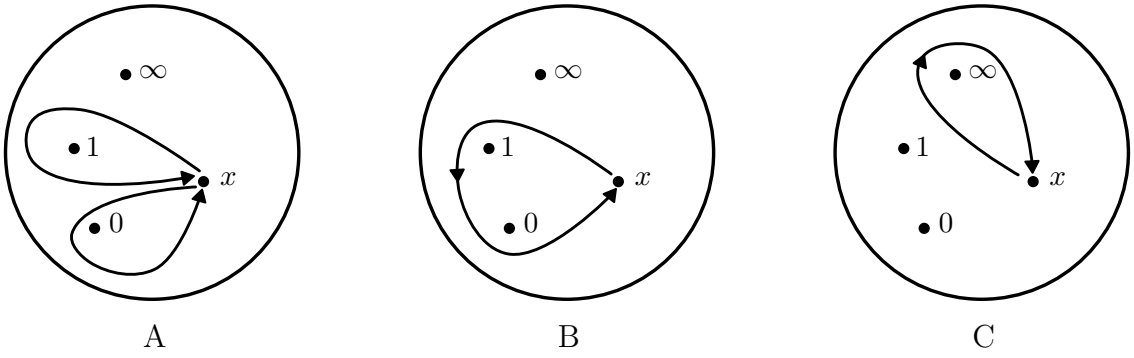


Figure 7.3: *Monodromy transformations on the Riemann Sphere.* ***A:*** *Monodromy transformations M_0 and M_1.* ***B:*** *Invariance with respect to M_0 and M_1 implies invariance with respect to the transformation enclosing both points 0 and 1.* ***C:*** *The transformation in* ***B*** *is equal to M_∞^{-1}, the inverse transformation about ∞. Hence invariance with respect to both M_0 and M_1 implies that M_∞^{-1} acts as the identity, and therefore so does M_∞.*

From eqs. (7.32) and (7.34) we can calculate the monodromy matrix (M_1):

$$(M_1) = \frac{a+b-c}{1-c} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(c-a-b)} \end{pmatrix} \begin{pmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{pmatrix}. \tag{7.35}$$

From these expressions, and by considering the anti-holomorphic sector, we can construct the physical correlation functions we have been seeking, assuming that such functions exist.

7.3.3 The anti-holomorphic sector

In the anti-holomorphic sector we derive expressions that are the complex conjugates of those in the holomorphic sector. The physical correlation function can be obtained by combining chiral conformal blocks from the holomorphic and anti-holomorphic sectors in a monodromy invariant way. We will demonstrate this procedure in detail in section 8.5.1 (although from a different starting point) and so for now we shall just briefly outline the corresponding method using connection and monodromy matrices.

The transformed chiral correlation function $G(x)$ from eq. (7.11) and the corresponding anti-holomorphic function $\bar{G}(\bar{x})$ are to be combined into a function $G(x, \bar{x})$, which is expanded in terms of particular solutions to eq. (7.13) as

$$G(x, \bar{x}) = \sum_{i,j=1}^{2} X_{ij} f_i^{(0)}(x) \bar{f}_j^{(0)}(\bar{x}), \tag{7.36}$$

where (X_{ij}) is a 2×2 constant matrix whose coefficients can be fixed by imposing monodromy invariance. As will be discussed in section 8.5.1 the corresponding physical correlation function is single-valued if and only if the matrix coefficients X_{ij} are selected so that $G(x, \bar{x})$ is monodromy invariant.

Note from fig. 7.3 that if $G(x, \bar{x})$ is invariant with respect to M_0 and M_1 then it is automatically invariant with respect to M_∞. This is the reason that we have not considered the solutions $\mathbf{F}_\infty(x)$, nor have we calculated the monodromy matrix (M_∞).

The monodromy transformation M_0 can be calculated explicitly using the monodromy matrix (M_0) from eq. (7.29). When acting with M_0 on the conjugate variable $\bar{x}$, the orientation of the

loop is reversed, and the corresponding monodromy matrix $(\bar{M}_0)$ is the complex conjugate of (M_0). We therefore find that $G(x,\bar{x})$ transforms as

$$G(x,\bar{x}) \mapsto G(M_0 \cdot x, M_0 \cdot \bar{x}) = \sum_{i,j,k,l=1}^{2} X_{ij}(M_0)_{ik}(\bar{M}_0)_{jl} f_k^{(0)}(x)\bar{f}_l^{(0)}(\bar{x}). \tag{7.37}$$

Similarly, the monodromy transformation M_1 can be calculated using the monodromy matrix (M_1) from eq. (7.35), and we find that

$$G(x,\bar{x}) \mapsto G(M_1 \cdot x, M_1 \cdot \bar{x}) = \sum_{i,j,k,l=1}^{2} X_{ij}(M_1)_{ik}(\bar{M}_1)_{jl} f_k^{(0)}(x)\bar{f}_l^{(0)}(\bar{x}). \tag{7.38}$$

If we demand that M_0 and M_1 act as the identity on the function in eq. (7.36) so that the transformed expansions in eqs. (7.37) and (7.38) are equal to the untransformed expansion in eq. (7.36), then the matrix coefficients X_{ij} satisfy a linear system of equations involving the matrix coefficients of the monodromy matrices (M_0) and (M_1). Solving this system for (X_{ij}) allows one to construct a single-valued physical correlator.[8]

Instead of continuing with this calculation here we shall turn now to the Coulomb gas formalism. The problem of combining conformal blocks to construct a physical correlation function will be revisited and completed using the Coulomb gas formalism in section 8.5. The the work we have already done in this chapter will not go unrewarded, for in section 8.8 the connection transformations for hypergeometric functions will be revealed to be lurking within the Coulomb gas formalism.

7.A Appendix: From PDE to ODE

This appendix explains the transformation of the partial differential equation eq. (7.1) *into the ordinary differential equation eq.* (7.9). Write the factor F from eq. (7.6) as

$$F = z_{14}^{A_{14}} z_{32}^{A_{32}} z_{24}^{A_{24}} z_{34}^{A_{34}}, \tag{7.39}$$

with the exponents given by

$$\begin{aligned} A_{14} &= -2h_1, \qquad A_{32} = -h_1 - h_2 - h_3 + h_4, \\ A_{24} &= -h_2 - h_4 + h_1 + h_3, \qquad A_{34} = -h_3 - h_4 + h_1 + h_2. \end{aligned} \tag{7.40}$$

The transformation of eq. (7.1) begins by application of the chain rule:

$$\frac{\partial}{\partial z_i} FH(x) = \frac{\partial F}{\partial z_i} H(x) + F \frac{\partial x}{\partial z_i} \frac{dH}{dx}. \tag{7.41}$$

The double derivative with respect to z_1 is

$$\frac{\partial^2}{\partial z_1^2} FH(x) = \frac{\partial^2 F}{\partial z_1^2} H(x) + 2\frac{\partial F}{\partial z_1}\frac{\partial x}{\partial z_1}\frac{dH}{dx} + F\frac{\partial^2 x}{\partial z_1^2}\frac{dH}{dx} + F\left(\frac{\partial x}{\partial z_1}\right)^2 \frac{d^2H}{dx^2}. \tag{7.42}$$

[8] Actually the overall normalisation is not fixed by solving the linear system, but in section 8.5 we will demonstrate how the normalisation can be fixed.

Although perhaps tedious, the derivatives are all easy to calculate. Substituting them into the singular vector partial differential equation (7.1) we find that

$$\begin{aligned}
0 &= \left\{ \frac{\partial^2}{\partial z_1^2} + \eta_2 \sum_{j=2}^{4} \left[\frac{h_j}{z_{1j}^2} + \frac{1}{z_{1j}} \frac{\partial}{\partial z_j} \right] \right\} F H(x) \\
&= F \left\{ \frac{A_{14}(A_{14}-1)}{z_{14}^2} + \left(2\frac{A_{14}}{z_{14}} \left[\frac{z_{34}}{z_{14}z_{32}} - \frac{z_{12}z_{34}}{z_{14}^2 z_{32}} \right] - 2\frac{z_{34}}{z_{14}^2 z_{32}} + 2\frac{z_{12}z_{34}}{z_{14}^3 z_{32}} \right) \frac{d}{dx} \right. \\
&+ \left(\frac{z_{34}}{z_{14}z_{32}} - \frac{z_{12}z_{34}}{z_{14}^2 z_{32}} \right)^2 \frac{d^2}{dx^2} + \eta_2 \left(\frac{h_2}{z_{12}^2} + \frac{h_3}{z_{13}^2} + \frac{h_4}{z_{14}^2} \right. \\
&+ \frac{1}{z_{12}} \left[\frac{-A_{32}}{z_{32}} + \frac{A_{24}}{z_{24}} \left(-\frac{z_{34}}{z_{14}z_{32}} + \frac{z_{12}z_{34}}{z_{14}z_{32}^2} \right) \frac{d}{dx} \right] \\
&+ \frac{1}{z_{13}} \left[\frac{A_{32}}{z_{32}} + \frac{A_{34}}{z_{34}} + \left(\frac{z_{12}}{z_{14}z_{32}} - \frac{z_{12}z_{34}}{z_{14}z_{32}^2} \right) \frac{d}{dx} \right] \\
&\left. \left. + \frac{1}{z_{14}} \left[-\frac{A_{14}}{z_{14}} - \frac{A_{24}}{z_{24}} - \frac{A_{34}}{z_{34}} + \left(-\frac{z_{12}}{z_{14}z_{32}} + \frac{z_{12}z_{34}}{z_{14}^2 z_{32}} \right) \frac{d}{dx} \right] \right) \right\} H(x).
\end{aligned} \tag{7.43}$$

The factor F has now been factored out, and so what remains must vanish. Performing the Möbius transformation in eq. (7.3) we are left with

$$0 = \left\{ \frac{d^2}{dx^2} + \eta_2 \left[\frac{h_2}{x^2} + \frac{h_3}{(x-1)^2} + \frac{1}{x} \left[-A_{32} + (x-1)\frac{d}{dx} \right] + \frac{1}{x-1} \left[A_{32} - x\frac{d}{dx} \right] \right] \right\} H(x) \tag{7.44}$$

Replacing A_{32} by its value $-h_1 - h_2 - h_3 + h_4$, we arrive at the ordinary differential equation given in eq. (7.9).

7.B Appendix: Connection coefficients

In this appendix we solve the linear system in eq. (7.25) and derive the connection coefficients in eq. (7.26). The linear system in eq. (7.25) for the connection coefficients can be written as

$$1 = T_{11} f_1^{(1)}(0) + T_{12} f_2^{(1)}(0), \qquad 0 = T_{21} f_1^{(1)}(0) + T_{22} f_2^{(1)}(0), \tag{7.45}$$

$$f_1^{(0)}(1) = T_{11}, \qquad f_2^{(0)}(1) = T_{21}. \tag{7.46}$$

The coefficients T_{11} and T_{21} are easily computed using eq. (7.24). We shall demonstrate the calculation of T_{22}. The coefficient T_{12} is calculated in a similar manner.

Firstly, using eq. (7.24) we find that

$$T_{21} = {}_2F_1(a+1-c, b+1-c; 2-c; 1) = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(1-a)}, \tag{7.47}$$

$$f_1^{(1)}(0) = {}_2F_1(a, b; a+b+1-c; 1) = \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}, \tag{7.48}$$

$$f_2^{(1)}(0) = {}_2F_1(c-a, c-b; 1+c-a-b; 1) = \frac{\Gamma(1+c-a-b)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}. \tag{7.49}$$

Solving eq. (7.45) for T_{22} gives

$$T_{22} = -T_{21} \frac{f_1^{(1)}(0)}{f_2^{(1)}(0)} = \frac{-\Gamma(2-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \left[\frac{\Gamma(c-a-b)\Gamma(a+b+1-c)}{\Gamma(1+c-a-b)} \right]. \tag{7.50}$$

The second factor in square brackets can be simplified using Euler's reflection identity

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}, \tag{7.51}$$

which makes the calculation of $\det T$ easier, and will be convenient for proving some identities in section 8.8. The factor becomes

$$\left[\frac{\pi}{\sin\pi(c-a-b)}\frac{\sin\pi(-c+a+b)}{\pi}\Gamma(a+b-c)\right] = -\Gamma(a+b-c). \tag{7.52}$$

Substituting this back into eq. (7.50), we arrive at the result in eq. (7.26).

7.C Appendix: Convergence information

The following convergence information is from [121]. The necessary convergence conditions are satisfied in the example that we encounter in chapter 8. In particular we will not need to worry about the taking the real and imaginary parts of a, b, and c, because they will be real. However, for the sake of completeness we still state the convergence information in its general form below.

- The integrals in eqs. (7.15) and (7.16) converge in the Cauchy principal value sense if $\mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0$ and $|x| < 1$, or $|x| = 1$ and $\mathrm{Re}\,(c-b-a) > 0$ and $\mathrm{Re}\,(b) > 0$. We frequently used the phrase "within the unit disc" for domains of convergence — that disc is open in the former case and closed in the latter.
- The integral in eq. (7.17) for Euler's beta function converges as a principal value integral for $\mathrm{Re}\,(x), \mathrm{Re}\,(y) > 0$.
- For the local sets of solutions in eqs. (7.21) and (7.22) we have assumed that $c \notin \mathbb{Z}$.

Chapter 8

The Coulomb Gas Formalism

Vertex operators are spectacularly useful in conformal field theory. For one thing their correlation functions are all known. Moreover they have explicit mode expansions in terms of the Heisenberg algebra, making them easy to manipulate. Translating Ising model correlation functions into problems involving vertex operators allowed us to compute them in a few pages of work.

Herein lies the power of the Coulomb gas formalism. It allows one to realise minimal model correlation functions in terms of vertex operators and, on top of that, provides a way to systematically stitch together monodromy-invariant solutions to singular vector differential equations using techniques of contour integral manipulation. Lurking beneath the contour manipulations are links to many ostensibly unrelated areas, one such area being the hypergeometric-type connection problem from chapter 7.

In this chapter we combine the knowledge from all previous chapters — the *minimal models* of chapter 5 are realised using *screening operators*, which are constructed from the bosonic vertex operators of chapter 4; the bosonisation calculations of chapter 6 are verified; we identify links to the singular vector differential equations from chapter 7; we relate contour manipulations to the monodromy problem and the two-point connection problem of section 7.3; and finally we demonstrate the power of the Coulomb gas formalism by providing complete sets of primary OPE-coefficients for three different minimal model theories, both main series and not.

Why is it called the "Coulomb gas" formalism?

In section 4.7, the OPE of the free boson anti-derivative $\phi(z)$ in eq. (4.52) was found to be $\phi(z)\phi(w) \sim \log(z-w)$, and in two dimensions the Coulomb electric potential between two charges separated by a displacement $|z-w|$ is proportional to $\log|z-w|$. Vertex operators, which play a crucial role in the Coulomb gas formalism, are given by normally ordered exponentials of the field $\phi(z)$. Moreover, many integral expressions that arise in the formalism closely resemble those of *log-gases* in statistical mechanics [51].

8.1 Modifying the free boson

Let us begin by playing some games with the representation theory of the free boson. Consider the field obtained if we modify the free boson energy-momentum tensor by appending to it the term $\partial^2\varphi(z)$:

$$L_A(z) = \frac{1}{2}{:}\partial\varphi(z)\partial\varphi(z){:} + A\partial^2\varphi(z), \tag{8.1}$$

where $A \in \mathbb{C}$ is for the moment unspecified. This modification is actually not so out-of-the-blue — the free boson energy-momentum tensor is modified in this way if the Lagrange density of eq. (4.3) is coupled to the scalar curvature of a curved manifold — see [53, Ch. 9.1.3]. This

Table 8.1: *Coulomb gas modifications*

	Free boson	**Coulomb gas**
c	1	$1-12A^2$
L_n	$\frac{1}{2}\sum_{r\in\mathbb{Z}} :a_{n-r}a_r:$	$\frac{1}{2}\sum_{r\in\mathbb{Z}} :a_{n-r}a_r: - A(n+1)a_n$
$L(z)\partial\varphi(w)$	$\sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}$	$\sim \frac{-2A}{(z-w)^3}\mathbf{1} + \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}$
$L(z)V_p(w)$	$\sim \frac{(\frac{1}{2}p^2)V_p(w)}{(z-w)^2} + \frac{\partial V_p(w)}{z-w}$	$\sim \frac{(\frac{1}{2}p^2 - Ap)V_p(w)}{(z-w)^2} + \frac{\partial V_p(w)}{z-w}$
$L_0\,\lvert p\rangle$	$\frac{1}{2}p^2\,\lvert p\rangle$	$(\frac{1}{2}p^2 - Ap)\,\lvert p\rangle$

modification can be interpreted as the influence of a "*background charge*" located at infinity, and the Coulomb gas theory can be interpreted as a perturbation of the free boson theory caused by this charge.

To check if this modification still defines a conformal field theory we need to check if $L_A(z)$ satisfies the defining OPE for a conformal energy-momentum tensor. An application of Wick's theorem, shown in App. 8.A, gives

$$L_A(z)L_A(w) \sim \frac{(1-12A^2)/2}{(z-w)^4}\mathbf{1} + \frac{2L_A(w)}{(z-w)^2} + \frac{\partial L_A(w)}{z-w}, \tag{8.2}$$

which is still the defining OPE of a Virasoro symmetry field, but with modified central charge. Using the range of techniques we have developed so far we can determine how the Coulomb gas theory is modified from the standard free boson theory. These modifications are shown in table 8.1.

In the Coulomb gas formalism the central charge is $c = 1 - 12A^2$ and the conformal weight of the Heisenberg vacuum $|p\rangle$ is $h_p = \frac{1}{2}p^2 - Ap$. For $A \neq 0$ the field $\partial\varphi(z)$ is not a Virasoro primary field, since it does not satisfy the defining OPE in eq. (3.11) for a Virasoro primary with the energy-momentum tensor, but vertex operators remain Virasoro primaries, although with modified conformal weight.

8.2 Screening fields

There exist special fields called *screening fields*, which are essential in constructing the minimal models and their correlation functions.

Consider a generic Virasoro primary field $S(w)$ with conformal weight h_S and with mode expansion of the form

$$S(w) = \sum_n S_n w^{-n-h_S}, \tag{8.3}$$

where the values taken by the index n depend on the field $S(w)$ and what it acts on. In the special case where $h_S = 1$, the residue S_0 of $S(w)$ commutes with the Virasoro algebra, and so constitutes a $\mathfrak{Vir}$-module *homomorphism* (an intertwiner), assuming that this residue is well defined. This is equivalent to the residue commuting with the energy-momentum tensor, which generates the Virasoro algebra. To show this we use the contour subtraction method from section 3.7.1 and fig. 3.1:[1]

$$[S_0, L_A(z)] = \oint_z \mathcal{R}\{L_A(z)S(w)\}w^{h_S-1}\frac{dw}{2\pi i} = \oint_z \left\{\frac{h_S S(w)}{(z-w)^2} + \frac{\partial S(w)}{z-w}\right\} w^{h_S-1}\frac{dw}{2\pi i}. \tag{8.4}$$

[1] We have used $\mathcal{R}\{S(w)L_A(z)\} = \mathcal{R}\{L_A(z)S(w)\}$, which is proven in [92, App. B].

If $h_S = 1$ then the integrand is a derivative, and the integral becomes

$$\oint_z \frac{\partial}{\partial w} \left\{ \frac{S(w)}{z-w} \right\} \frac{dw}{2\pi i}. \tag{8.5}$$

If the contour is closed then there are no boundary terms, and the integral vanishes, giving

$$[S_0, L_A(z)] = 0. \tag{8.6}$$

So S_0 is a $\mathfrak{Vir}$-module homomorphism.

There are some complications, because $S(w)$ is not a function, but some quantum field, and is not necessarily single-valued. Closure of the contour depends on which field $S(w)$ is and what it acts on. Detailed considerations can be technical — see [125, Sec. 5]. Suffice it to say that there will be an appropriate choice of integral so that eq. (8.6) holds in each case that we encounter.

Let us consider which primary fields are available in the theory that have conformal weight $h = 1$. The field $\partial\varphi(z)$ has conformal weight $h = 1$, but table 8.1 shows that, for $A \neq 1$, $\partial\varphi(z)$ is not a Virasoro primary field in the Coulomb gas theory. However, every vertex operator $V_p(z)$ is a Virasoro primary field. $V_p(z)$ has conformal weight $h_p = \frac{1}{2}p^2 - Ap$, and setting $h_p = 1$ we find two solutions for p:

$$\frac{1}{2}p^2 - Ap = 1 \implies p_\pm = A \pm \sqrt{A^2 + 2}. \tag{8.7}$$

The corresponding vertex operators are denoted by $\mathcal{V}_\pm(z)$ — we call them **screening fields**.

8.3 Realising the minimal models

When we first introduced the Heisenberg algebra we normalised the generators so that the central element k acted as the identity operator. For current purposes it will be convenient to work in a renormalised basis so that k acts as multiplication by 2. Make the change of notation

$$b_n := \sqrt{2}\, a_n, \qquad \widehat{b} := \sqrt{2}\, \widehat{a}, \qquad A := \sqrt{2}\, \alpha_0. \tag{8.8}$$

The commutation relations are

$$[b_n, b_m] = 2n\delta_{n,-m}, \qquad [b_n, k] = 0, \qquad [b_n, \widehat{b}] = 2\delta_{n,0}, \qquad [\widehat{b}, k] = 0, \tag{8.9}$$

and the Virasoro modes from table 8.1 take the form

$$L_n = \frac{1}{4} \sum_{m \in \mathbb{Z}} :b_{n-m} b_m: - \alpha_0 (n+1) b_n. \tag{8.10}$$

The corresponding vertex operators have mode expansions

$$\mathcal{V}_\alpha(z) := e^{\alpha\widehat{b}} z^{\alpha b_0} \prod_{n=1}^{\infty} \exp\left(\alpha b_n \frac{z^{-n}}{n}\right) \exp\left(-\alpha b_{-n} \frac{z^n}{n}\right). \tag{8.11}$$

The state $|\alpha\rangle$ corresponding to $\mathcal{V}_\alpha(z)$ is a highest weight state for $\mathfrak{H}$ that satisfies $b_0 |\alpha\rangle = 2\alpha |\alpha\rangle$. The b_0-eigenvalue 2α will be called the *charge* of $|\alpha\rangle$.

With the rescaling, table 8.1 determines the conformal weight of $|\alpha\rangle$ to be $h(\alpha) = \alpha(\alpha - 2\alpha_0)$, reminiscent of the value in eq. (5.33). The two screening fields with conformal weight $h(\alpha) = 1$ have charges:

$$h(\alpha) = 1 \implies \alpha = \alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}. \tag{8.12}$$

We call these *screening charges*. Table 8.1 determines α_0 in terms of the central charge c — it turns out to be precisely that of eq. (5.23) from the Kac determinant. The screening charges $\alpha_\pm$

are also determined in terms of c from eq. (8.12), and solving for them reproduces the values in eq. (5.23). Finally, if we define the charges $\alpha_{r,s}$ as per eq. (5.34), then we recover from table 8.1 the corresponding conformal weights $h(\alpha_{r,s}) = h_{r,s}$, which are the Kac weights from eqs. (5.22) and (5.35). Using the renormalised basis of Heisenberg generators, the vertex operators in the Coulomb gas formalism reproduce the values from the Kac determinant. Moreover, if the central charge takes a minimal model value $c = c(p', p)$ as per eq. (5.47), then the translation and reflection symmetries of the Kac table in eq. (5.50) are also realised:

$$h_{r,s} = h_{r+p',s+p} \leftrightarrow h(\alpha_{r,s}) = h(\alpha_{r+p',s+p}), \tag{8.13}$$

$$h_{r,s} = h_{-r,-s} \leftrightarrow h(\alpha_{r,s}) = h(\alpha_{-r,-s}) = h(2\alpha_0 - \alpha_{r,s}). \tag{8.14}$$

For a given $c(p', p)$ the Coulomb gas vertex operators with charges $\alpha_{r,s}$ corresponding to (r, s) in the Kac table (5.50) reproduce all of the $\mathcal{M}(p', p)$ conformal weights. In fact in [41] it is proven that for a given minimal model $\mathcal{M}(p', p)$ each irreducible representation $\mathcal{QV}(h_{r,s}, c(p', p))$ is isomorphic to a certain quotient of a submodule of the collection of bosonic Fock spaces in the $c = c(p', p)$ Coulomb gas theory.

Substituting $c(p', p)$ into eq. (8.12) we find that the screening charges $\alpha_\pm$ can be expressed in terms of the coprime integers p', p as

$$\alpha_+ = \sqrt{p/p'}, \qquad \alpha_- = -\sqrt{p'/p}. \tag{8.15}$$

Note that the ratio $\alpha_+/\alpha_- = -p/p'$ is rational.

The idea behind the Coulomb gas formalism

The vertex operator $\mathcal{V}_{\alpha_{r,s}}(z) := \mathcal{V}_{r,s}(z)$ has the same conformal weight as the minimal model field $\phi_{r,s}(z)$. When bosonising the Ising model in section 6.4 the idea was to compute Ising correlation functions by first relating to each $\mathcal{M}(3, 4)$ primary field a vertex operator with the same conformal weight, and then computing vertex operator correlators instead, which are all known from the general formulae in eqs. (4.67) and (4.82). The Coulomb gas formalism extends this prescription to all minimal models.

There is still an issue to address — by reflection symmetry the vertex operator $\mathcal{V}_{2\alpha_0-\alpha_{r,s}}(z) = \mathcal{V}_{-r,-s}(z)$ also has conformal weight $h_{r,s}$, so there seems to be some freedom in which vertex operator is chosen to represent $\phi_{r,s}(z)$. This is something we encountered in chapters 4 and 6, where the vertex operators $V_p(z)$ and $V_{-p}(z)$ had the same conformal weight. To construct meaningful correlation functions we chose combinations of vertex operators that satisfied the conservation of momentum condition of sections 4.4 and 4.6.1. The same idea applies to the Coulomb gas formalism, with some modification.

8.4 Neutrality

The conservation of momentum condition of section 4.4 has an analogue in the Coulomb gas formalism, here interpreted as conservation of charge. We can derive this condition from the expression for the Virasoro algebra in terms of the Heisenberg algebra in table 8.1.

8.4.1 Neutrality on the level of states

Virasoro modules are embedded in Fock spaces in the same way as in section 4.3.1. The Fock space $\mathcal{F}(\alpha) = U(\mathfrak{H}_c)\,|\alpha\rangle$ with highest weight vector $|\alpha\rangle$ has the conformal grading into L_0-eigenspaces: $\mathcal{F}(\alpha) = \bigoplus_{N=0}^{\infty} \mathcal{F}_N(\alpha)$, where $\mathcal{F}_N(\alpha)$ is an L_0-eigenspace with eigenvalue $h(\alpha) + N$. We construct the scalar product as per chapters 4 and 5 with an adjoint map $L_n^\dagger = L_{-n}$. The fact that L_0 is self-adjoint ensures that different grades are orthogonal.

Unlike the case in section 4.4, here we do not have a choice as to what the Heisenberg adjoint map can be — from eq. (8.10) for the Virasoro algebra in terms of $\mathfrak{H}$, we find that the only Heisenberg adjoint map that gives the correct Virasoro adjoint is:

$$b_n^\dagger = 4\alpha_0\delta_{n,0} - b_{-n}, \qquad k^\dagger = k. \tag{8.16}$$

The constant term $4\alpha_0\delta_{n,0}$ has non-trivial consequences for the scalar product. For example, for vacuum vectors $|\alpha\rangle$ and $|\beta\rangle$ labelled by their charges, we have

$$(b_0^\dagger |\alpha\rangle, |\beta\rangle) = (|\alpha\rangle, b_0 |\beta\rangle) \implies (4\alpha_0 - 2\alpha)(|\alpha\rangle, |\beta\rangle) = 2\beta(|\alpha\rangle, |\beta\rangle), \tag{8.17}$$

so $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal if $\beta \neq 2\alpha_0 - \alpha$. Moreover since b_0 belongs to the centre $Z(\mathfrak{H})$, it follows that every vector in $\mathcal{F}(\alpha)$ is orthogonal to every vector in $\mathcal{F}(\beta)$ if $\beta \neq 2\alpha_0 - \alpha$. The covector $\langle\alpha| \in \mathcal{F}^\vee(\alpha)$ is dual not to $|\alpha\rangle$, but to $|2\alpha_0 - \alpha\rangle \in \mathcal{F}(2\alpha_0 - \alpha)$, and from the reflection symmetry of eq. (8.14) it is clear that the vector and its dual do indeed have the same conformal weight.

Every covector in $\mathcal{F}^\vee(\alpha)$ maps to zero every vector that is not in $\mathcal{F}(2\alpha_0 - \alpha)$. With the conventional normalisation: $\langle\alpha|2\alpha_0 - \alpha\rangle := 1$, we have the condition

$$\langle\alpha|\beta\rangle = \delta_{\beta=2\alpha_0-\alpha} \implies \langle 0|\beta\rangle = \delta_{\beta=2\alpha_0}. \tag{8.18}$$

This is called the **neutrality condition**. The value $2\alpha_0$ is the aforementioned "*background charge*". Scalar products vanish if the sum of charges is not equal to the background charge, which is the origin of the name "neutrality condition".

8.4.2 Neutrality for correlation functions

There is a corresponding neutrality condition on the level of vertex operators. Using the general formula in eq. (4.67) for vertex operator correlation functions, and the neutrality condition in eq. (8.18), we find that

$$\langle 0| \mathcal{V}_{\alpha_1}(z_1) \dots \mathcal{V}_{\alpha_n}(z_n) |0\rangle = \delta_{\alpha_1+\dots+\alpha_n=2\alpha_0} \prod_{i<j} z_{ij}^{2\alpha_i\alpha_j}, \tag{8.19}$$

which vanishes if $\sum_{i=1}^n \alpha_i \neq 2\alpha_0$. This is the neutrality condition for correlation functions.

8.4.3 Neutrality and conservation of momentum

The neutrality condition is analogous to the conservation of momentum condition for the free boson in section 4.4. Indeed taking $\alpha_0 = 0$ (equivalently taking $c = 1$), eq. (8.18) becomes the conservation of momentum condition. This should be expected, because in this limit the Coulomb gas theory coincides with the free boson theory. This is the reason that we chose a_0 to be *anti*-self-adjoint rather than the conventional self-adjoint in section 4.4. Had we followed the conventional route and chosen a_0 to be self-adjoint then the $c = 1$ limit would not reproduce the familiar free boson theory, which would invalidate the interpretation of the Coulomb gas theory as a perturbation of the free boson theory.

8.4.4 Neutrality and screening operators

To construct minimal model correlation functions one associates to each minimal model field $\phi_{r,s}(z)$ one of the vertex operators $\mathcal{V}_{r,s}(z)$ or $\mathcal{V}_{-r,-s}(z)$, which have the same conformal weight $h_{r,s}$. For example, to construct a two-point function such as

$$\langle 0| \phi_{r,s}(z)\phi_{r,s}(w) |0\rangle, \tag{8.20}$$

one might try two-point functions of the form

$$\langle 0|\, \mathcal{V}_{r,s}(z)\mathcal{V}_{r,s}(w)\, |0\rangle\,, \qquad \langle 0|\, \mathcal{V}_{-r,-s}(z)\mathcal{V}_{-r,-s}(w)\, |0\rangle\,. \tag{8.21}$$

We encounter a problem that we have seen before in section 6.4 — although the vertex operators have the correct conformal weights, the correlation functions vanish because the neutrality condition is not satisfied. To see this explicitly, note from eq. (5.22) that $\alpha_{r,s}+\alpha_{r,s} = \alpha_{2r-1,2s-1} \neq 2\alpha_0$ for (r,s) in the Kac table $\mathcal{T}_{p',p}$. However, both combinations $\mathcal{V}_{\pm r,\pm s}(z)$ and $\mathcal{V}_{\mp r,\mp s}(w)$ do satisfy the neutrality condition since $\alpha_{r,s} + \alpha_{-r,-s} = \alpha_+ + \alpha_- = 2\alpha_0$. Using eq. (8.19) we have

$$\langle 0|\, \mathcal{V}_{r,s}(z)\mathcal{V}_{-r,-s}(w)\, |0\rangle = \frac{1}{(z-w)^{2\alpha_{r,s}(2\alpha_0-\alpha_{r,s})}} = \frac{1}{(z-w)^{2h_{r,s}}}, \tag{8.22}$$

which reproduces the correctly normalised general solution to the two-point function in eq. (3.31). The vertex operators $\mathcal{V}_{r,s}(z)$ and $\mathcal{V}_{-r,-s}(z)$ are *conjugate fields*. For valid two-point functions the neutrality condition can always be satisfied by selecting the vertex operators to be conjugates. When more fields are involved this is no longer true.

For instance, consider a chiral $\mathcal{M}(p',p)$ primary correlator of the form $\langle 0|\, \phi_{r,s}\phi_{r,s}\phi_{r,s}\phi_{r,s}\, |0\rangle$. Since minimal model primaries are self-conjugate it follows that this correlator is necessarily non-zero — this can be confirmed by inserting an OPE between any two adjacent fields and then using the general solution in eq. (3.30) for three-point functions. The available vertex operators for representing each field are $\mathcal{V}_{\alpha_{r,s}}$ and $\mathcal{V}_{2\alpha_0-\alpha_{r,s}}$. However, for $\alpha_0 \neq 0$ it seems that there is no combination of these such that the neutrality condition is satisfied. For example, both correlators $\langle 0|\, \mathcal{V}_{r,s}\mathcal{V}_{r,s}\mathcal{V}_{r,s}\mathcal{V}_{-r,-s}\, |0\rangle$ and $\langle 0|\, \mathcal{V}_{r,s}\mathcal{V}_{r,s}\mathcal{V}_{-r,-s}\mathcal{V}_{-r,-s}\, |0\rangle$ vanish if $\alpha_0 \neq 0$, since neither satisfies the neutrality condition.

The resolution to this issue comes in the form of the screening fields from section 8.2. The two screening fields $\mathcal{V}_+(z)$ and $\mathcal{V}_-(z)$ are vertex operators with conformal weight $h = 1$. The residues Q_+ and Q_- of these fields are integral operators called **screening operators**. They are $\mathfrak{Vir}$-module homomorphisms since they commute with the energy-momentum tensor — *they shift charges, but not conformal weights.* Q_+ has charge α_+, and Q_- has charge α_-, and neither alters the conformal weight of any vector it acts upon. The idea is to insert screening operators into correlation functions so that neutrality is satisfied, but OPE-coefficients are not altered.

There are various approaches to using screening operators, and we will use the prescription from the original works [34,35]. As was the case in section 6.4, the minimal model OPE-coefficients can be determined from four-point functions by inserting OPEs and considering asymptotics. To construct a minimal model primary four-point function of the form

$$\langle 0|\, \phi_{r_1,s_1}(z_1)\phi_{r_2,s_2}(z_2)\phi_{r_3,s_3}(z_3)\phi_{r_4,s_4}(z_4)\, |0\rangle \tag{8.23}$$

we replace the fields by their associated vertex operators and insert as many screening operators as required to satisfy the neutrality condition. Since the conjugate operator $\mathcal{V}_{-r,-s}(z) = \mathcal{V}_{2\alpha_0-\alpha_{r,s}}(z)$ has charge $2\alpha_0 - \alpha_{r,s}$, including just one conjugate field (conventionally the one labelled $\mathcal{V}_{-r_4,-s_4}(z_4)$) takes care of the background charge $2\alpha_0$, as required by the neutrality condition, and we arrive at a correlation function of the form

$$\langle 0|\, Q_-^{m_-} Q_+^{m_+} \mathcal{V}_{r_1,s_1}(z_1)\mathcal{V}_{r_2,s_2}(z_2)\mathcal{V}_{r_3,s_3}(z_3)\mathcal{V}_{-r_4,-s_4}(z_4)\, |0\rangle\,. \tag{8.24}$$

The integers m_+ and m_- are chosen in order to satisfy the neutrality condition:

$$\alpha_{r_1,s_1} + \alpha_{r_2,s_2} + \alpha_{r_3,s_3} + m_+\alpha_+ + m_-\alpha_- = \alpha_{r_4,s_4}. \tag{8.25}$$

Since α_+ and α_- are not independent there might be more than one charge-neutral choice for m_+ and m_-. For example, observe from eq. (8.15) that

$$m_+\alpha_+ + m_-\alpha_- = \frac{\alpha_+}{p}(m_+p - m_-p'), \tag{8.26}$$

and that this expression is invariant under the shift $(m_+, m_-) \mapsto (m_+ + p', m_- + p)$.[2] One way to determine a valid choice is by using the relations

$$\alpha_{r,s} + \alpha_{r',s'} = \alpha_{r+r'-1,s+s'-1}, \qquad \alpha_{r,s} + m\alpha_+ = \alpha_{r-2m,s}, \qquad \alpha_{r,s} + m\alpha_- = \alpha_{r,s-2m}, \tag{8.27}$$

which are straightforward to check using eq. (5.34). It follows that eq. (8.25) is satisfied if m_+ and m_- are chosen as

$$m_+ = \frac{r_1 + r_2 + r_3 - r_4 - 2}{2}, \qquad m_- = \frac{s_1 + s_2 + s_3 - s_4 - 2}{2}. \tag{8.28}$$

If this expression does not return integers for m_+ and m_- then the neutrality condition cannot be satisfied for that combination of vertex operators $\mathcal{V}_{\alpha_i}$, and the correlation function vanishes. If a negative integer is returned then one can use $(m_+, m_-) \mapsto (m_+ + p', m_- + p)$.

Screening operators are integral operators of the form

$$Q_\pm = \oint \mathcal{V}_\pm(u)du, \tag{8.29}$$

where the meaning of the integral will be specified in section 8.5. The symbols $Q_\pm^{m_\pm}$ are instructions to insert $m_\pm$ screening fields $\mathcal{V}_\pm$, and then perform $m_\pm$ integrals over the integrand

$$\langle 0| \mathcal{V}_-(u_1) \dots \mathcal{V}_-(u_{m_-}) \mathcal{V}_+(v_1) \dots \mathcal{V}_+(v_{m_+}) \mathcal{V}_{r_1,s_1}(z_1) \mathcal{V}_{r_2,s_2}(z_2) \mathcal{V}_{r_3,s_3}(z_3) \mathcal{V}_{-r_4,-s_4}(z_4) |0\rangle . \tag{8.30}$$

Equation (8.30) can be expanded using eq. (8.19). Writing $\alpha_i := \alpha_{r_i,s_i}$ for $1 \le i \le 3$ and $\alpha_4 := \alpha_{-r_4,-s_4}$, the function in eq. (8.30) expands as

$$\begin{aligned} &\prod_{1\le i<j\le m_-} (u_{ij})^{2\alpha_-^2} \prod_{i=1}^{m_-} \prod_{j=1}^{m_+} (u_i - v_j)^{2\alpha_-\alpha_+} \prod_{i=1}^{m_-} \prod_{j=1}^{4} (u_i - z_j)^{2\alpha_-\alpha_j} \\ &\cdot \prod_{1\le i<j\le m_+} (v_{ij})^{2\alpha_+^2} \prod_{i=1}^{m_+} \prod_{j=1}^{4} (v_i - z_j)^{2\alpha_+\alpha_j} \prod_{1\le i<j\le 4} (z_{ij})^{2\alpha_i\alpha_j}, \end{aligned} \tag{8.31}$$

where we use the usual notation $x_{ij} := x_i - x_j$, and where each u_i and v_i is to be integrated over.

Things can be simplified by using Möbius invariance to fix three points. As we shall soon see the Coulomb gas method fixes the overall normalisation of correlators only after all integrals have been computed, and the method is insensitive to non-zero constants multiplying the integrand. In particular this means that we may ignore phases gained by performing Möbius transformations that violate the implicit radial ordering, and this is what we will do. The choice most commonly used in the literature is the map $(z_1, z_2, z_3, z_4) \mapsto (0, x, 1, \infty)$.

It is convenient to relabel the exponents as

$$a_\pm = 2\alpha_\pm\alpha_1, \qquad b_\pm = 2\alpha_\pm\alpha_3, \qquad c_\pm = 2\alpha_\pm\alpha_2. \tag{8.32}$$

The integrand in eq. (8.31) now takes the form

$$\begin{aligned} &x^{2\alpha_1\alpha_2}(1-x)^{2\alpha_2\alpha_3} \prod_{1\le i<j\le m_-} (u_{ij})^{2\alpha_-^2} \prod_{i=1}^{m_-} \prod_{j=1}^{m_+} (u_i - v_j)^{2\alpha_-\alpha_+} \prod_{i=1}^{m_-} u_i^{a_-} (u_i - 1)^{b_-} (u_i - x)^{c_-} \\ &\cdot \prod_{1\le i<j\le m_+} (v_{ij})^{2\alpha_+^2} \prod_{i=1}^{m_+} v_i^{a_+} (v_i - 1)^{b_+} (v_i - x)^{c_+} \end{aligned} \tag{8.33}$$

[2] This should be compared to the translation symmetry of the weights and charges in eq. (8.13).

Before turning to an explicit example calculation we should make one final remark regarding the neutrality condition. Since each screening operation involves performing an integral it is convenient to minimise the number of screening operators used in a given calculation. There are a few options for doing this. Firstly, there is freedom in which minimal model field is chosen to be represented by the conjugate vertex operator $\mathcal{V}_{-r,-s}(z)$. One can also make use of the symmetries of the Kac table $\phi_{r,s} = \phi_{p'-r,p-s}$. Finally, one can use the translation symmetry $(m_+, m_-) \mapsto (m_+ + p', m_- + p)$ from eq. (8.26).

8.5 Computing four-point functions

The technique is best demonstrated by example, and we will chose examples that allow us to make contact with chapters 5 to 7. Consider a minimal model primary correlator

$$\langle 0 | \, \phi_{r_1,s_1}(z_1) \dots \phi_{r_4,s_4}(z_4) \, | 0 \rangle \, . \tag{8.34}$$

From chapter 5 we know that if the correlator contains either of the fields $\phi_{2,1}$ or $\phi_{1,2}$ then there is a level 2 singular vector in the corresponding Verma module and the correlator satisfies a second-order singular vector differential equation.

We will demonstrate a calculation involving a single screening operator, either Q_+ or Q_-. This will be sufficient to find all of the primary OPE-coefficients for $\mathcal{M}(3,4)$ and some other minimal models. We will leave the charge $\alpha_\pm$ of the operator unspecified for the moment. Let us consider the case in which (r_2, s_2) is either $(2,1)$ or $(1,2)$. Writing α_i for α_{r_i,s_i} if $1 \leq i \leq 3$ and α_4 for $\alpha_{-r_4,-s_4}$, the correlator in terms of vertex operators takes the form

$$\langle 0 | \, Q_\pm \mathcal{V}_{\alpha_1}(z_1) \mathcal{V}_{\alpha_2}(z_2) \mathcal{V}_{\alpha_3}(z_3) \mathcal{V}_{\alpha_4}(z_4) \, | 0 \rangle = \prod_{1 \leq i < j \leq 4} z_{ij}^{2\alpha_i \alpha_j} \oint \prod_{i=1}^{4} (u - z_i)^{2\alpha_\pm \alpha_i} du. \tag{8.35}$$

Performing the Möbius transformation and relabelling the exponents as per eq. (8.32), with $a_\pm, b_\pm$, and $c_\pm$ now written as a, b, and c respectively, we arrive at an integral of the form

$$x^{2\alpha_1\alpha_2}(1-x)^{2\alpha_2\alpha_3} \oint u^a (u-1)^b (u-x)^c du := x^{2\alpha_1\alpha_2}(1-x)^{2\alpha_2\alpha_3} G(x). \tag{8.36}$$

We recognise the integrand from the hypergeometric function in chapter 7, as well as the transformation in eq. (7.11). We will take a close look at this coincidence in section 8.8.

The precise meaning of a screening operation and the correct choice of integration "contour" seems to depend on who you ask. One possibility involving local systems and twisted-cycles is offered in [125, Sec. 5], and a another in [42, 43]. Still more alternatives can be found in [33, p. 123-170].[3] Many works, such as [3] and [71, 72], are dedicated to formalising integrals of this type.[4]

We will follow the original suggestion given in [34], which links up well with chapter 7 — these are standard contour integrals deformed to the real intervals $(0, x)$ and $(1, \infty)$, and define two linearly independent functions:

$$I_1(a,b,c;x) := \int_1^\infty u^a (u-1)^b (u-x)^c du, \qquad I_2(a,b,c;x) := \int_0^x u^a (u-1)^b (u-x)^c du. \tag{8.37}$$

We have actually seen these integrals before. The integral defining $I_1(x)$ has the same form as eq. (7.16), and upon the change of variables $u = xt$ in the integral defining $I_2(x)$, we recover

[3] The page numbers refer to V.S. Dotsenko's section, *Lectures on Conformal Field Theory*.

[4] It seems that there is some flexibility here, as one would expect, since contour integrals depend only on the homotopy class of the contours, and are insensitive to continuous deformations.

an integral of the same form as eq. (7.15). The integral representations for the hypergeometric function naturally emerge in the Coulomb gas formalism. Defining the factors

$$\lambda_1(a,b,c) = \frac{\Gamma(-a-b-c-1)\Gamma(b+1)}{\Gamma(-a-c)}, \qquad \lambda_2(a,b,c) = \frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)}, \tag{8.38}$$

we find that

$$I_1(a,b,c;x) = \lambda_1(a,b,c)\,{}_2F_1(-c,-a-b-c-1;-a-c;x), \tag{8.39}$$
$$I_2(a,b,c;x) = x^{1+a+c}\lambda_2(a,b,c)\,{}_2F_1(-b,a+1;a+c+2;x). \tag{8.40}$$

Since the correlation function satisfies a second-order singular vector equation it is not surprising that there are two linearly independent solutions to consider. What may be surprising is that these solutions emerge as a consequence of the neutrality condition and screening operations, without ever having written down the differential equation, and without the laborious transformations of chapter 7.[5]

In order to compute OPE-coefficients in section 8.5.3 we will need to know the asymptotic behaviour of these integrals as $x \to 0$. From section 7.2.1 we know that the normalisation of the hypergeometric function is:

$${}_2F_1(A,B;C;x) \overset{x\to 0}{=} 1 + \mathcal{O}(x), \tag{8.41}$$

so the small-x limits of the integrals $I_i(x)$ are

$$I_1(x) \overset{x\to 0}{=} \lambda_1(a,b,c)(1+\mathcal{O}(x)), \tag{8.42}$$
$$I_2(x) \overset{x\to 0}{=} x^{1+a+c}\lambda_2(a,b,c)(1+\mathcal{O}(x)). \tag{8.43}$$

The function $G(x)$ of eq. (8.36) is some linear combination of these integrals. As explained in section 7.3.3, the physical solution must include the anti-holomorphic sector, from which we obtain complex conjugate integral expressions $\{\bar{I}_1(\bar{x}), \bar{I}_2(\bar{x})\}$. As per eq. (7.36) the single-valued function $G(x,\bar{x})$ is some *sesquilinear* combination of the form

$$G(x,\bar{x}) = \sum_{i,j} X_{ij} I_i(x)\bar{I}_j(\bar{x}), \tag{8.44}$$

where (X_{ij}) is a 2×2 matrix. The matrix elements are constrained by imposing monodromy invariance.

8.5.1 Monodromy invariance

This section makes precise the statements made in section 7.3.3.

Note that the global factor $x^{2\alpha_1\alpha_2}(1-x)^{2\alpha_2\alpha_3}$ multiplying $G(x)$ in eq. (8.36) becomes the manifestly monodromy-invariant factor $|x|^{4\alpha_1\alpha_2}|1-x|^{4\alpha_2\alpha_3}$ multiplying $G(x,\bar{x})$ when both chiral sectors are taken into account. Therefore, monodromy-invariant combinations of the integrals $I_i(x)$ and $\bar{I}_j(\bar{x})$ can be studied independently of this global factor — the physical correlator is monodromy invariant if $G(x,\bar{x})$ is.

The functions $I_1(x)$ and $I_2(x)$ both have singular points at 0, 1, and ∞, and so we require that the monodromy transformations M_0, M_1, and M_∞ all act as the identity on $G(x,\bar{x})$. We know from fig. 7.3 that invariance with respect to both transformations M_0 and M_1 implies invariance with respect to M_∞. Hence it is sufficient for us to study the transformations M_0 and M_1.

[5] The connection between singular vectors and the Coulomb gas formalism runs deeper than we have the time to discuss here, but the papers [41, 42, 115, 125] are worthwhile reading on this topic.

Monodromy about 0

The monodromy transformations about 0 of the integrals $\{I_1(x), I_2(x)\}$ follow immediately from eqs. (8.39) and (8.40), and the monodromy matrix (M_0) in eq. (7.29):

$$\begin{bmatrix} I_1(M_0 \cdot x) \\ I_2(M_0 \cdot x) \end{bmatrix} = (M_0) \begin{bmatrix} I_1(x) \\ I_2(x) \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(a+c)} \end{pmatrix} \begin{bmatrix} I_1(x) \\ I_2(x) \end{bmatrix}. \tag{8.45}$$

On the basis $\{I_1(x), I_2(x)\}$ the transformation acts *diagonally* and at most the elements gain a phase, but are not transformed amongst each other. When multiplying through by the complex conjugate such phases cancel. The function $G(x, \bar{x})$ in eq. (8.44) is invariant with respect to this transformation if and only if the matrix (X_{ij}) is diagonal. Therefore, defining $X_j := X_{jj}$ (not using summation notation) we have

$$G(x, \bar{x}) = \sum_j X_j I_j(x) \bar{I}_j(\bar{x}), \tag{8.46}$$

which satisfies the invariance condition: $G(M_0 \cdot x, M_0 \cdot \bar{x}) = G(x, \bar{x})$. Equation (8.46) should be compared to eq. (7.37).

Monodromy about 1

Performing the monodromy transformation M_1 is more technical, as we already know from section 7.3, where we determined the monodromy matrix (M_1) in eq. (7.31) by using the sequence of connection transformations in eq. (7.31). The Coulomb gas formalism provides an alternative method.

Firstly, let us assume that the functions $\{I_1(x), I_2(x)\}$ can be transformed into a new basis of functions $\{\tilde{I}_1(x), \tilde{I}_2(x)\}$ on which the monodromy transformation about $x = 1$ acts diagonally:

$$I_j(x) = \sum_k A_{jk} \tilde{I}_k(x), \tag{8.47}$$

where (A_{jk}) is a 2×2 transformation matrix. The anti-holomorphic functions transform under the same matrix. In terms of the new basis functions eq. (8.46) becomes

$$G(x, \bar{x}) = \sum_{j,k,l} X_j A_{jk} A_{jl} \tilde{I}_k(x) \tilde{\bar{I}}_l(\bar{x}) := \sum_{k,l} \tilde{X}_{kl} \tilde{I}_k(x) \tilde{\bar{I}}_l(\bar{x}). \tag{8.48}$$

Equation (8.48) should be compared to eq. (7.38). The expansion in eq. (8.48) now has the same form as the calculation of the monodromy about 0, and $G(x, \bar{x})$ is monodromy-invariant about $x = 1$ if and only if $(\tilde{X}_{kl})$ is a diagonal matrix. This corresponds to the condition:

$$\sum_j X_j A_{jk} A_{jl} = 0 \text{ for } k \neq l. \tag{8.49}$$

Written out explicitly we have the constraint:

$$X_1 A_{11} A_{12} + X_2 A_{21} A_{22} = 0 \implies \frac{X_1}{X_2} = -\frac{A_{21} A_{22}}{A_{11} A_{12}}, \tag{8.50}$$

assuming that $A_{11} A_{12} \neq 0$ (which we will show to be true by explicitly finding the transformation matrix). This verifies the claim made in section 7.3.3 that $G(x, \bar{x})$ is monodromy invariant if and only if the coefficients X_{ij} solve a certain linear system, assuming that a suitable basis of functions $\{\tilde{I}_1(x), \tilde{I}_2(x)\}$ does in fact exist.

8.5.2 Contour manipulations

In [34] it is claimed that suitable functions $\tilde{I}_1(x)$ and $\tilde{I}_2(x)$ do exist and are given by integrating the same integrand as for $I_1(x)$ and $I_2(x)$, but over different contours. No justification is given as to why these integrals should define appropriate functions, but we will show it to be true in section 8.8.

The contours of integration $\Gamma_1, \Gamma_2, \tilde{\Gamma}_1$, and $\tilde{\Gamma}_2$ for the functions $I_1(x), I_2(x), \tilde{I}_1(x)$, and $\tilde{I}_2(x)$ respectively are illustrated in fig. 8.1. We use shorthand notation such that for constants A and B we have

$$\int_{\Gamma_1} := \int \Gamma_1 := I_1(x), \qquad \int (A\Gamma_1 + B\Gamma_2) := \int A\Gamma_1 + \int B\Gamma_2 := AI_1(x) + BI_2(x), \tag{8.51}$$

and so on. By continuously deforming the contours of integration the functions $\{I_1(x), I_2(x)\}$ can be analytically continued into $\{\tilde{I}_1(x), \tilde{I}_2(x)\}$. We shall demonstrate this technique in detail below.

When the complex plane is compactified to the Riemann sphere the point at infinity on the sphere corresponds to $|z| \to \infty$ in any direction on the plane. The contour Γ_1 can therefore be deformed so that infinity is approached along the negative real axis. We can do this in two different ways. The first way is to deform Γ_1 in the upper half-plane and avoid the singular points by pole-splitting with positive orientation — this results in the contour Γ_A depicted in fig. 8.1. The second way is to deform Γ_1 in the lower half-plane and avoid the singular points by pole-splitting with negative orientation — this results in the contour Γ_B, also depicted in fig. 8.1. Note that the integrals corresponding to Γ_A and Γ_B in are both equal to $I_1(x)$.

In the upper half-plane the tight half-circles about the singular points of the integrand in eq. (8.36) pick up positive phases each time a pole is split — a phase of $e^{i\pi b}$ for the point at $u = 1$, $e^{i\pi c}$ for the point at $u = x$, and $e^{i\pi a}$ for the point at $u = 0$. After splitting a pole the integral over the subsequent interval is weighted by the phase gained from that pole. In the lower half-plane negative phases are gained, because the half-circles are negatively oriented.

If we multiply $\int \Gamma_A$ by $e^{-i\pi(b+c)}$ and multiply $\int \Gamma_B$ by $e^{i\pi(b+c)}$ then the phases on the interval $(0, x)$ are the same in both expressions. Therefore when taking the combination $\int e^{i\pi(b+c)}\Gamma_B - \int e^{-i\pi(b+c)}\Gamma_A$ the contribution from the interval $(0, x)$ cancels. We are left with a sum of integrals over the contours $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ — using the notation $s(x) := \sin(\pi x)$ we have

$$\begin{aligned}\int \left(e^{i\pi(b+c)}\Gamma_B - e^{-i\pi(b+c)}\Gamma_A\right) &= \int \left(\left(e^{i\pi a} - e^{-i\pi a}\right)\tilde{\Gamma}_1 - \left(e^{i\pi c} - e^{-i\pi c}\right)\tilde{\Gamma}_2\right) \\ &= 2is(a)\tilde{I}_1(x) - 2is(c)\tilde{I}_2(x).\end{aligned} \tag{8.52}$$

Note that the signs of $\tilde{I}_1(x)$ and $\tilde{I}_2(x)$ in eq. (8.52) are due to the negative orientation of the contours after the subtraction, as compared to the definitions of $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ in fig. 8.1.

On the other hand since $\int \Gamma_A$ and $\int \Gamma_B$ are both equal to $I_1(x)$ we have

$$\int \left(e^{i\pi(b+c)}\Gamma_B - e^{-i\pi(b+c)}\Gamma_A\right) = \left(e^{i\pi(b+c)} - e^{-i\pi(b+c)}\right) I_1(x) = 2is(b+c)I_1(x). \tag{8.53}$$

Comparing eq. (8.52) with eq. (8.53) we find that

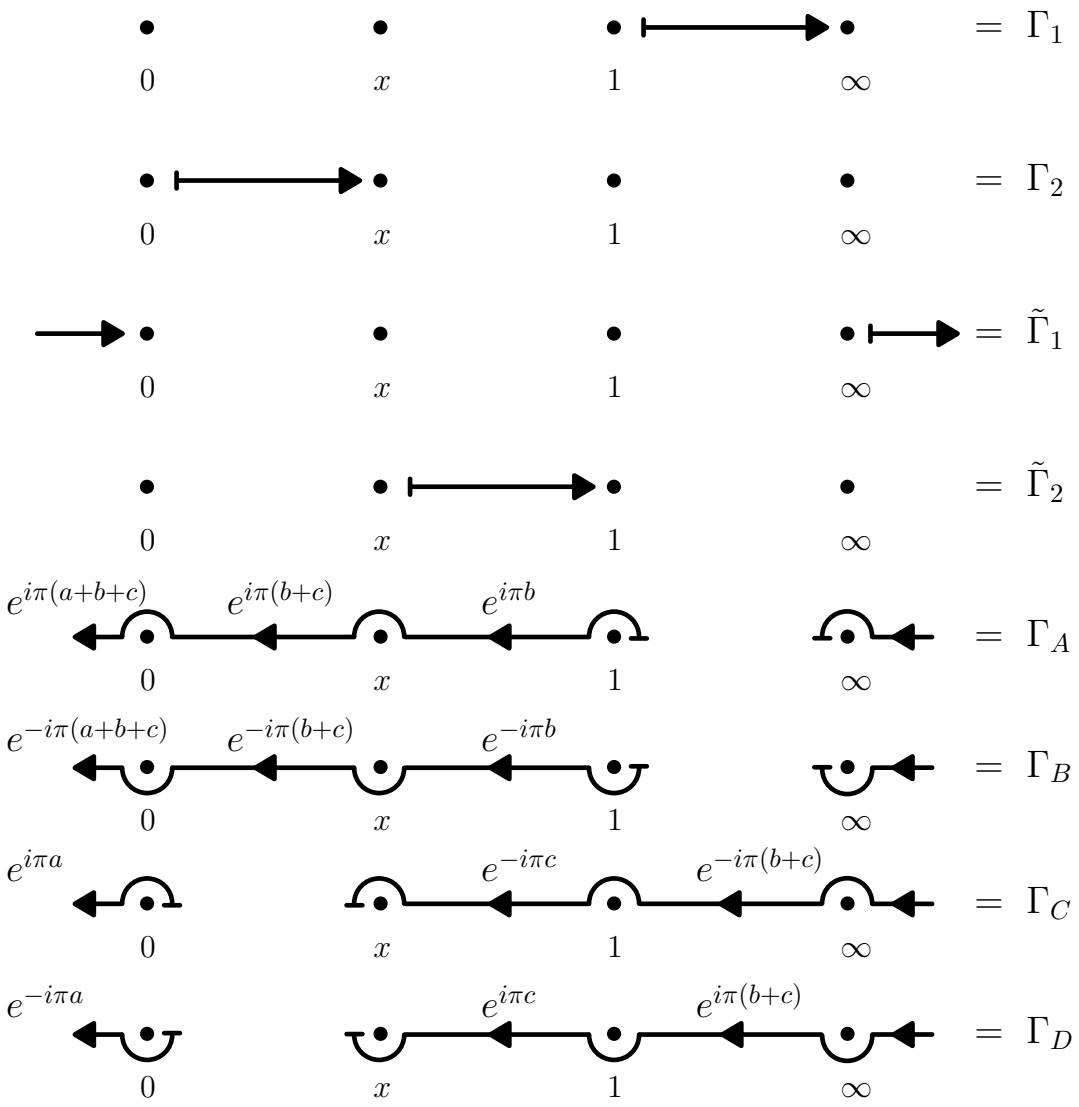
$$2is(b+c)I_1(x) = 2is(a)\tilde{I}_1(x) - 2is(c)\tilde{I}_2(x) \tag{8.54}$$

$$\implies I_1(x) = \frac{s(a)}{s(b+c)}\tilde{I}_1(x) - \frac{s(c)}{s(b+c)}\tilde{I}_2(x). \tag{8.55}$$

From this we read off coefficients A_{11} and A_{12} from eq. (8.47).

The coefficients A_{21} and A_{22} are calculated analogously by deforming the contour Γ_2 into the contours Γ_C and Γ_D. The only noteworthy difference here is that now the point at infinity

Figure 8.1: Coulomb gas integration contours. *The contours* $\Gamma_1, \Gamma_2, \tilde{\Gamma}_1$, *and* $\tilde{\Gamma}_2$ *correspond respectively to the functions* $I_1(x), I_2(x), \tilde{I}_1(x)$, *and* $\tilde{I}_2(x)$. *The integrals are interpreted as principal value integrals. The contours* Γ_A *and* Γ_B *correspond to two different analytic continuations for the function* $I_1(x)$, *which are implemented by deforming* Γ_1 *continuously into* Γ_A *and* Γ_B *respectively, with phases gained by pole splitting. The contours* Γ_C *and* Γ_D *correspond to two different analytic continuations for the function* $I_2(x)$. *On the Riemann sphere the point at infinity can be approached along either the positive or the negative real axis, which is why the contours in* $\tilde{\Gamma}_1, \Gamma_A, \Gamma_B, \Gamma_C$, *and* Γ_D *wrap around when leaving the page.*



contributes to the calculation. The phase gained from splitting the pole at infinity can be determined by making the change of variables $u = 1/\epsilon$ in integral for $I_2(x)$ and considering $\epsilon \to 0$. We find that the phase in the upper half-plane is $\exp(-i\pi(a+b+c+2)) = \exp(-i\pi(a+b+c))$.[6] The phase in the lower half-plane is therefore $\exp(i\pi(a+b+c))$. As in eq. (8.52), we multiply the integrals by phases and subtract so that the contribution from the interval $(1,\infty)$ cancels, and we obtain

$$\begin{aligned}\int \left(e^{i\pi(b+c)}\Gamma_C - e^{-i\pi(b+c)}\Gamma_D\right) &= \int \left(-\left(e^{i\pi(a+b+c)} - e^{-i\pi(a+b+c)}\right)\tilde{\Gamma}_1 - \left(e^{i\pi b} - e^{-i\pi b}\right)\tilde{\Gamma}_2\right) \\ &= -2is(a+b+c)\tilde{I}_1(x) - 2is(b)\tilde{I}_2(x). \end{aligned} \tag{8.56}$$

On the other hand we have

$$\int \left(e^{i\pi(b+c)}\Gamma_C - e^{-i\pi(b+c)}\Gamma_D\right) = \left(e^{i\pi(b+c)} - e^{-i\pi(b+c)}\right) I_2(x) = 2is(b+c)I_2(x), \tag{8.57}$$

and so we have the transformation

$$I_2(x) = -\frac{s(a+b+c)}{s(b+c)}\tilde{I}_1(x) - \frac{s(b)}{s(b+c)}\tilde{I}_2(x). \tag{8.58}$$

The matrix (A_{ij}) from eq. (8.47) is therefore

$$(A_{ij}) = \frac{1}{s(b+c)}\begin{pmatrix} s(a) & -s(c) \\ -s(a+b+c) & -s(b) \end{pmatrix} \tag{8.59}$$

From the coefficients in eq. (8.59) and the monodromy constraint in eq. (8.50) we find that appropriate matrix coefficients X_i are

$$X_1 = s(a+b+c)s(b), \qquad X_2 = s(a)s(c). \tag{8.60}$$

Letting ξ be a proportionality constant to be determined, the monodromy-invariant function $G(x,\bar{x})$ has the form

$$G(x,\bar{x}) = \xi\left[s(a+b+c)s(b)|I_1|^2 + s(a)s(c)|I_2|^2\right]. \tag{8.61}$$

The factor ξ will be determined in section 8.5.3.

8.5.3 Computing OPE-coefficients

The reason for focusing on four-point functions is two-fold. One reason is that Möbius invariance can be used to transform four-point functions into single-variable expressions. We used this fact in chapter 7 to transform a partial differential equation into an ordinary differential equation, and here to realise hypergeometric integrals. The second reason is that OPE-coefficients can be extracted from four-point functions by inserting OPEs and considering asymptotics.[7] We used this fact to solve the Ising model in section 6.4.

Consider the following OPE for non-chiral primary fields (with as yet unknown coefficients):

$$\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2) = |z_{12}|^{-\Delta_1-\Delta_2}\sum_k |z_{12}|^{\Delta_k} C_{1,2}^k \phi_k(z_2,\bar{z}_2) + \dots \tag{8.62}$$

[6] The additional -2 in the exponential is the result of changing integration measure: $du = -\epsilon^{-2}d\epsilon$.

[7] The same is not true for three-point functions, for which we derive only expressions that depend linearly on OPE-coefficients. As mentioned in section 3.5, without a non-linearity, we cannot fix the values of the OPE-coefficients.

where $\Delta_i = h_i + \bar{h}_i$ is the scaling dimension of the field ϕ_i, and where the indices on the fields and OPE-coefficients stand for Kac labels: $\phi_i := \phi_{r_i,s_i}$ and $C^k_{i,j} := C^{r_k,s_k}_{r_i,s_i;r_j,s_j}$. As usual the dots $(\dots)$ stand for descendent fields, which will not contribute here since the coefficients of the primary fields are the most singular. By inserting the OPE of eq. (8.62) into the four-point function

$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle := \langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4) \rangle \,, \tag{8.63}$$

we arrive at the expression

$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle = \sum_k \frac{C^k_{1,2}}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_k}} \langle \phi_k(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4) \rangle + \dots \tag{8.64}$$

Evaluating the three-point function using eq. (3.30) and the definition in section 4.7 for non-chiral correlation functions we get

$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle = \sum_k \frac{C^k_{1,2}C_{k,3,4}}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_k}|z_{23}|^{\Delta_k+\Delta_3-\Delta_4}|z_{24}|^{\Delta_k+\Delta_4-\Delta_3}|z_{34}|^{\Delta_3+\Delta_4-\Delta_k}} + \dots \tag{8.65}$$

From section 3.5.3 we know that three-point constants and OPE-coefficients are related via $C_{i,j,k} = C^{k^\vee}_{i,j}$, and since minimal model primary fields are self-conjugate we have $C^{k^\vee}_{i,j} = C^k_{i,j}$. Therefore we have

$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle = \sum_k \frac{C^k_{1,2}C^k_{3,4}}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_k}|z_{23}|^{\Delta_k+\Delta_3-\Delta_4}|z_{24}|^{\Delta_k+\Delta_4-\Delta_3}|z_{34}|^{\Delta_3+\Delta_4-\Delta_k}} + \dots \tag{8.66}$$

We can now relate the OPE-coefficients to $G(x,\bar{x})$ using the relationship

$$|x|^{4\alpha_1\alpha_2}|1-x|^{4\alpha_2\alpha_3}G(x,\bar{x}) = \langle \phi_{r_4,s_4}| \, \phi_{r_3,s_3}(1,1)\phi_{r_2,s_2}(x,\bar{x}) \, |\phi_{r_1,s_1}\rangle \,. \tag{8.67}$$

Performing the Möbius transformation $(z_1,z_2,z_3,z_4) \mapsto (0,x,1,\infty)$ in eq. (8.66) (after multiplying eq. (8.66) by $|z_4|^{2\Delta_4}$),[8] we find that the right-hand side of eq. (8.67) can be expressed as

$$\langle \phi_{r_4,s_4}| \, \phi_{r_3,s_3}(1,1)\phi_{r_2,s_2}(x,\bar{x}) \, |\phi_{r_1,s_1}\rangle = \sum_k \frac{C^k_{1,2}C^k_{3,4}}{|x|^{\Delta_1+\Delta_2-\Delta_k}|1-x|^{\Delta_k+\Delta_3-\Delta_4}} + \dots \tag{8.68}$$

Let us consider eq. (8.67) in the asymptotic regime $|x| \ll 1$. From eqs. (8.42) and (8.43) we know the small-x behaviour of the functions $I_1(x)$ and $I_2(x)$, and so we have the corresponding small-x behaviour for $G(x,\bar{x})$:

$$G(x,\bar{x}) = \xi \left[X_1\lambda_1^2 + X_2\lambda_2^2|x|^{2(1+a+c)} \right] \Big(1 + \mathcal{O}(x,\bar{x}) \Big). \tag{8.69}$$

Substituting eqs. (8.68) and (8.69) into eq. (8.67) we have the $|x| \ll 1$ relationship:

$$\xi|x|^{4\alpha_1\alpha_2} \left[X_1\lambda_1^2 + X_2\lambda_2^2|x|^{2(1+a+c)} \right] + \dots \approx \sum_k C^k_{1,2}C^k_{3,4}|x|^{\Delta_k-\Delta_1-\Delta_2} + \dots \tag{8.70}$$

We can now compare powers of $|x|$ on each side of eq. (8.70), and this lets us extract OPE-coefficients and fix the normalisation factor ξ, without having to consider the terms $(\dots)$. When deriving eq. (8.36) we allowed h_{r_2,s_2} to be either one of $h_{2,1}$ or $h_{1,2}$ — both choices result in a level 2 singular vector differential equation. Now we shall consider each case separately.

[8] To calculate the sum in eq. (8.68) from that in eq. (8.66) we use the fact that the non-chiral state $\langle \phi_i|$ is defined as $\lim_{z,\bar{z}\to\infty} |z|^{2\Delta_i} \langle 0| \, \phi_i(z,\bar{z})$.

Case 1: $h_{r_2,s_2} = h_{2,1}$

Consider the left-hand side of eq. (8.70). The term multiplying $X_1\lambda_1^2$ has $|x|$ to the power of $4\alpha_{2,1}\alpha_{r_1,s_1}$. This exponent can be written in terms of scaling dimensions using eq. (5.22) for the Kac weights:

$$4\alpha_{2,1}\alpha_{r_1,s_1} = \Delta_{r_1+1,s_1} - \Delta_{r_1,s_1} - \Delta_{2,1}. \tag{8.71}$$

Similarly the term multiplying $X_2\lambda_2^2$ has $|x|$ to the power of $4\alpha_{2,1}\alpha_{r_1,s_1} + 2(1+a+c)$. With a and c defined in eq. (8.32), this exponent can be written as

$$4\alpha_{2,1}\alpha_{r_1,s_1} + 2(1+a+c) = \Delta_{r_1-1,s_1} - \Delta_{r_1,s_1} - \Delta_{2,1}. \tag{8.72}$$

Comparing eqs. (8.71) and (8.72) to the fusion rules of eq. (5.51), which give

$$[\phi_{2,1}] \bowtie [\phi_{r_1,s_1}] = [\phi_{r_1+1,s_1}] + [\phi_{r_1-1,s_1}], \tag{8.73}$$

we see that eqs. (8.71) and (8.72) are compatible with non-zero primary OPE-coefficients.

On the right-hand side of eq. (8.70) the term with the exponent in eq. (8.71) multiplies the OPE-coefficient $C^{r_1+1,s_1}_{r_1,s_1;2,1}C^{r_1+1,s_1}_{r_3,s_3;r_4,s_4}$. On the left-hand side it multiplies $X_1\lambda_1^2$. Equating these we have

$$X_1\lambda_1^2 = \xi\left(C^{r_1+1,s_1}_{r_1,s_1;2,1}C^{r_1+1,s_1}_{r_3,s_3;r_4,s_4}\right). \tag{8.74}$$

Similarly on the right-hand side of eq. (8.70) the term with the exponent in eq. (8.72) multiplies the OPE-coefficient $C^{r_1-1,s_1}_{r_1,s_1;2,1}C^{r_1-1,s_1}_{r_3,s_3;r_4,s_4}$, and on the left-hand side it multiplies $X_2\lambda_2^2$. Equating these coefficients we have

$$X_2\lambda_2^2 = \xi\left(C^{r_1-1,s_1}_{r_1,s_1;2,1}C^{r_1-1,s_1}_{r_3,s_3;r_4,s_4}\right). \tag{8.75}$$

Case 2: $h_{r_2,s_2} = h_{1,2}$

The process may be repeated as above. Going through the same procedure we find that

$$4\alpha_{1,2}\alpha_{r_1,s_1} = \Delta_{r_1,s_1+1} - \Delta_{r_1,s_1} - \Delta_{1,2}, \tag{8.76}$$

$$4\alpha_{1,2}\alpha_{r_1,s_1} + 2(1+a+c) = \Delta_{r_1,s_1-1} - \Delta_{r_1,s_1} - \Delta_{1,2}, \tag{8.77}$$

and these are compatible with the fusion rules

$$[\phi_{1,2}] \bowtie [\phi_{r_1,s_1}] = [\phi_{r_1,s_1+1}] + [\phi_{r_1,s_1-1}]. \tag{8.78}$$

Comparing powers of $|x|$ in eq. (8.70) results in the coefficients

$$X_1\lambda_1^2 = \xi\left(C^{r_1,s_1+1}_{r_1,s_1;1,2}C^{r_1,s_1+1}_{r_3,s_3;r_4,s_4}\right), \qquad X_2\lambda_2^2 = \xi\left(C^{r_1,s_1-1}_{r_1,s_1;1,2}C^{r_1,s_1-1}_{r_3,s_3;r_4,s_4}\right). \tag{8.79}$$

Note that the global normalisation factor ξ in eqs. (8.74) and (8.75) is not the same factor as in eq. (8.79). In either case, ξ will be fixed using the normalisation of the two-point function as per eq. (3.31). For OPE-coefficients the normalisation gives $C^{1,1}_{r,s;r,s} := 1$, with cyclic permutations. Additional work is needed in more complicated cases, but often the above formulae are sufficient, as we shall demonstrate in the subsequent examples.

8.6 Solving statistical models

8.6.1 The Ising model

Let us now compute the same Ising model correlator that we computed using bosonisation techniques in section 6.4 in order to verify the calculations in this chapter. Consider the Ising model correlator

$$\langle \sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma(z_4,\bar{z}_4)\rangle \,. \tag{8.80}$$

From chapter 5 we know that the role of the spin-field $\sigma(z,\bar{z})$ is played by the $\mathcal{M}(3,4)$ field $\phi_{1,2}(z,\bar{z})$.[9] The corresponding vertex operator correlation function involves just one Q_- operator — this follows from eq. (8.28), which gives $m_- = 1$ and $m_+ = 0$.

Setting $(r_i, s_i) = (1,2)$ in eq. (8.79) gives

$$X_2\lambda_2^2 = \xi\left(C^{1,1}_{1,2;1,2}C^{1,1}_{1,2;1,2}\right) = \xi, \tag{8.81}$$

which fixes ξ, and so from eq. (8.79) we have

$$X_1\lambda_1^2 = X_2\lambda_2^2\left(C^{1,3}_{1,2;1,2}\right)^2 \implies \left(C^{1,3}_{1,2;1,2}\right)^2 = \frac{X_1\lambda_1^2}{X_2\lambda_2^2}. \tag{8.82}$$

The $\mathcal{M}(3,4)$ field $\phi_{1,3}$ corresponds to the Ising energy-field ϵ, and so the OPE-coefficient in eq. (8.82) is equal to $C^{\epsilon}_{\sigma\sigma}$. The parameters a, b, c are given in eq. (8.32). Using eq. (8.15) for the numerical values of $\alpha_\pm$ we find that[10]

$$a = b = c = 2\alpha_-\alpha_{1,2} = -\alpha_-^2 = -3/4. \tag{8.83}$$

With λ_1 and λ_2 defined in eq. (8.38) and X_1 and X_2 given in eq. (8.60), the OPE-coefficient can be calculated as

$$(C^{\epsilon}_{\sigma\sigma})^2 = \frac{X_1}{X_2}\left(\frac{\lambda_1}{\lambda_2}\right)^2 = \frac{\sin(-3\cdot 3\pi/4)}{\sin(-3\pi/4)}\left(\frac{\Gamma(-1+3\cdot 3/4)\Gamma(-2\cdot 3/4+2)}{\Gamma(2\cdot 3/4)\Gamma(-3/4+1)}\right)^2 = \frac{1}{4}, \tag{8.84}$$

in perfect agreement with our bosonisation calculations in eq. (6.49). As explained in section 6.4, by cyclic permutation of the indices we in fact have the full set of primary OPE-coefficients and three-point constants for $\mathcal{M}(3,4)$, and these are shown in eq. (6.50).

8.6.2 The Yang-Lee model

The Yang-Lee lattice model is a toy model used to study the high temperature behaviour of the Ising model partition function when the Ising lattice is placed in a purely imaginary magnetic field iH. The partition function acquires zeros which cluster together as H is increased towards a critical value H_c, and at H_c the clustering becomes dense.

Near the critical value H_c the density of zeros is described by the minimal model $\mathcal{M}(2,5)$. Note that $\mathcal{M}(2,5)$ is not main series — since $|p-p'| = |5-2| \neq 1$ it follows from section 5.5 that the model is non-unitary. In this instance non-unitarity is permissible, because Yang-Lee correlators do not describe physical observables — the Yang-Lee model is used study phase transitions abstractly. See [21] for details on $\mathcal{M}(2,5)$ and the Yang-Lee lattice model.

[9] Using the symmetries in eqs. (8.13) and (8.14) for the Kac weights, the spin-field could also be modelled by $\phi_{2,2}$, which has the same conformal weight. However, this would be a poor choice — one can confirm from eq. (8.28) that this choice results in at least *two* screening operators with different signs. We have only considered the case involving one screening operator — the Coulomb gas calculation resulting from the choice $\phi_{2,2}$ is very much more complicated.

[10] One can now verify that the hypergeometric functions in eqs. (8.39) and (8.40) do in fact satisfy the convergence conditions in App. 7.C, as claimed.

From the Kac table (5.50) we know that there are only two inequivalent primary fields to consider, the identity field $\phi_{1,1} = \phi_{1,4} = \mathbf{1}$ and the field $\phi_{1,2} = \phi_{1,3} := \Phi$, which represents the density of zeros of the partition function. The fusion rules (5.51) give

$$[\Phi] \bowtie [\Phi] = [\mathbf{1}] + [\Phi], \tag{8.85}$$

so the only OPE-coefficient not fixed by normalisation is $C^{\Phi}_{\Phi\Phi}$. Using the symmetry of the Kac table this can be written as

$$C^{\Phi}_{\Phi\Phi} = C^{1,3}_{1,2;1,2}, \tag{8.86}$$

which has the same form as our previous example in section 8.6.1, and can be determined by computing the four-point function $\langle\Phi\Phi\Phi\Phi\rangle$. Going through the same steps as in section 8.6.1 the Coulomb gas formalism gives (exact expression on the left, and written up to 5 decimal points on the right):

$$(C^{\Phi}_{\Phi\Phi})^2 = \frac{\sin(-3\cdot 2\pi/5)}{\sin(-2\pi/5)}\left(\frac{\Gamma(-1+3\cdot 2/5)\Gamma(-2\cdot 2/5+2)}{\Gamma(2\cdot 2/5)\Gamma(-2/5+1)}\right)^2 = -3.65312\ldots \tag{8.87}$$

The square root is purely imaginary:

$$C^{\Phi}_{\Phi\Phi} = \pm 1.91131\ldots i, \tag{8.88}$$

which is a reflection of the non-unitarity of $\mathcal{M}(2,5)$.

As a check of the validity of eq. (8.88), we compare with a result obtained by solving conformal bootstrap equations in *Liouville theory*, which gives [112, Ch. 3]

$$C^{\Phi}_{\Phi\Phi} = i\cdot\left(3\frac{\Gamma(-3/5)\Gamma(1/5)^3}{\Gamma(-1/5)^3\Gamma(3/5)}\right)^{1/2} = \pm 1.91131\ldots i. \tag{8.89}$$

Equation (8.87) and the square of eq. (8.89) are in fact identical. This can be proven using identities from section 8.8.

8.7 The general procedure

The generalisation to a greater number of screening operators is conceptually straightforward, but computationally far more demanding. Even just stating general expressions would require many pages, so instead we will just discuss how the method differs from the single screening operator case, before using it to solve the more complicated minimal model $\mathcal{M}(4,5)$.

The general procedure was first presented in [35]. For more than one screening operator the integrand is given by eq. (8.33). Integrands of this form are said to be of *hypergeometric-type* [28], and appear frequently in conformal field theory. Various authors suggest different approaches to screening operators and advocate a variety of different integral constructions. The prescription in [35] is to choose all possible combinations of contours on the intervals $(0,x)$ and $(1,\infty)$ in a suitably regularised way. There are $m_+ + 1$ and $m_- + 1$ different choices for the integration variables v_i and u_j respectively, leading to a total of $N = (m_+ + 1)(m_- + 1)$ different combinations, and each of these defines a different basis function $I_i(x)$. These basis functions are *conformal blocks*. The integrals are sometimes called *Dotsenko-Fateev integrals*, but more commonly they are referred to as *generalised Selberg integrals*, because when either m_+ or m_- is zero the integrals become the well known *Selberg integral* [52]. The hypergeometric integrals of eqs. (7.15) and (7.16) are examples of Selberg integrals. Closed form elementary expressions for Selberg integrals are known in general [51, Ch. 4], but not for generalised Selberg integrals. However, closed form expressions are known for the limit $x \to 0$ — [35, app. A]. This turns out to be sufficient for computing OPE-coefficients.

The physical correlation function is a monodromy-invariant combination of conformal blocks $I_i(x)$ and the conjugates $\bar{I}_i(\bar{x})$. Monodromy transformations about $x = 0$ still act diagonally on this basis of integrals, so the single-valued combination is of the form: $\sum_j X_j I_j(x)\bar{I}_j(\bar{x})$. Monodromy transformations about $x = 1$ do not act diagonally on this basis. The monodromy problem is solved by transforming to a new basis $\{\tilde{I}_i\}$ for which monodromy transformations about $x = 1$ do act diagonally. This basis is given by integrating the same integrand as in eq. (8.33), but with all contours on the intervals $(-\infty, 0)$ and $(x, 1)$. The basis transformation is implemented by an $N \times N$ matrix, whose coefficients are found by systematically manipulating integration contours as per section 8.5.2. The required manipulations are technical and the information in [35] alone is probably insufficient to repeat the derivation, but details can be found in [58, Sec. 3]. Once the transformation matrix is known the monodromy-invariant combination can be determined by solving a linear system, similar to that in eq. (8.49). At this stage the global normalisation is not yet determined — how to do this was first explained in [36]. The method is based on the associativity of OPEs, which manifests as *crossing symmetry* on the level of correlation functions. This consists of inserting OPEs into four-point function in two different ways, declaring the two results to be equal, and then taking ratios like that in eq. (8.82). When taking ratios, any spurious global normalisation factors cancel, resulting in expressions for OPE-coefficients that are completely fixed by the conventional normalisation $C_{i,j}^{\mathbf{1}} = \delta_{i,j^\vee}$.

8.7.1 The tricritical Ising model

Finally we are in a position to use the Coulomb gas formalism to compute the primary OPE-coefficients of the minimal model $\mathcal{M}(4,5)$ that we mentioned in the introduction. $\mathcal{M}(4,5)$ describes the tricritical Ising model — a generalisation of the Ising model where lattice sites are allowed to be vacant.[11]

The tricritical Ising model seems to have been conceived in [15]. Its connection to $\mathcal{M}(4,5)$ is detailed in [56]. The primary fields present in the theory are shown together with their physical interpretations in table 8.2. Justifications for the physical interpretations of the scaling fields can be found in [86, Sec. 3], which also considers different boundary conditions and perturbations away from criticality. The term *"irrelevant"* in table 8.2 is a technical term from renormalisation group theory.[12]

Kac indices (r,s)	Kac weight	Symbol	Interpretation
(1,1) or (3,4)	0	$\mathbf{1}$	identity
(1,2) or (3,3)	1/10	ϵ	energy
(1,3) or (3,2)	3/5	ϵ'	vacancy density
(1,4) or (3,1)	3/2	ϵ''	(irrelevant)
(2,2) or (2,3)	3/80	σ	magnetisation (spin)
(2,4) or (2,1)	7/16	σ'	"sub-magnetisation"

Table 8.2: *Scaling fields of the tricritical Ising model (with periodic boundary conditions) and the corresponding $\mathcal{M}(4,5)$ primary fields. Interpretations are from [86, table 2].*

The non-zero primary OPE-coefficients for the tricritical Ising model are shown, up to cyclic

[11] The minimal model $\mathcal{M}(4,5)$ actually describes the critical scaling limit of all models in the same universality class as the tricritical Ising model. The universality class in question is that of a scalar field theory called the *Landau-Ginzburg Φ^6-theory* [133].

[12] In renormalisation group theory, if $\mathcal{RG}$ is a renormalisation group operation that increases the length scale then an operator is called *relevant* if its magnitude always increases under $\mathcal{RG}$, and is called *irrelevant* if its magnitude always decreases under $\mathcal{RG}$, otherwise it is called *marginal*. For a statistical lattice model the irrelevant operators have negligible magnitude in the scaling limit. It is the relevant operators that correspond to macroscopic observables.

permutation, in table 8.3. To compute the OPE-coefficients we first determined all of the fusion rules for the fields in table 8.2 using eq. (5.51). The fusion rules specify which OPE-coefficients necessarily vanish and moreover specify the precise Kac indices that appear in the fusion of any two fields — although the weights have the symmetry $h_{r,s} = h_{p'-r,p-s}$, the expressions we used for the OPE-coefficients require that fields be input precisely as per the fusion rules. The non-zero OPE-coefficients were extracted by taking ratios of expressions in [36].

The Coulomb gas formalism actually gives only the squares of the OPE-coefficients, but from section 5.5 we know that $\mathcal{M}(4,5)$ is unitary and therefore that non-chiral OPE-coefficients and three-point constants are non-negative. We have therefore taken the positive branch of the square root of the Coulomb gas values in order to obtain the coefficients in table 8.3.

OPE-coefficient	**Numerical value**
$C_{\epsilon\epsilon}^{\epsilon'} = C_{\epsilon'\epsilon'}^{\epsilon'} := A$	$2^{-3/5}\sqrt{\dfrac{\Gamma(-3/10)\Gamma(2/5)\Gamma(7/5)}{\Gamma(-2/5)\Gamma(3/10)\Gamma(8/5)}}$
$C_{\epsilon\sigma}^{\sigma}$	$3A/2$
$C_{\epsilon'\sigma}^{\sigma}$	$A/4$
$C_{\epsilon\epsilon'}^{\epsilon''}$	$3/7$
$C_{\epsilon\sigma}^{\sigma'}$	$1/2$
$C_{\epsilon'\sigma}^{\sigma'}$	$3/4$
$C_{\epsilon''\sigma}^{\sigma}$	$1/56$
$C_{\epsilon''\sigma'}^{\sigma'}$	$7/8$

Table 8.3: *With cyclic permutations, this table presents numerical values for all non-vanishing primary OPE-coefficients for the tricritical Ising model* $\mathcal{M}(4,5)$.

8.8 Relationship to the connection problem

Let us end this chapter by unveiling a link between the Coulomb gas formalism and the two-point connection problem from chapter 7.

In [34, 35] it is claimed that the two bases of integrals $\{I_i(x)\}$ and $\{\tilde{I}_i(x)\}$ are sets of linearly independent solutions to the level 2 singular vector differential equation for which monodromy transformations about $x = 0$ and $x = 1$ respectively act diagonally. We confirmed that monodromy transformations about $x = 0$ act diagonally on the first set by explicitly finding the transformation matrix. That monodromy transformations about $x = 1$ do actually act diagonally on the second set of integrals has not been justified, nor has it been explained how exactly these integrals relate to the solutions of the singular vector equation.

The basis change suggested in [34] is:

$$\begin{bmatrix} I_1(a,b,c;x) \\ I_2(a,b,c;x) \end{bmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{bmatrix} \tilde{I}_1(a,b,c;x) \\ \tilde{I}_2(a,b,c;x) \end{bmatrix}, \tag{8.90}$$

where the transformation matrix elements are given in eq. (8.59).

We know from eqs. (8.39) and (8.40) that $I_1(x)$ and $I_2(x)$ can be expressed in terms of hypergeometric functions. Under the change of variables $x \mapsto 1 - x$, the integrals defined in fig. 8.1 for the functions $\tilde{I}_1(x)$ and $\tilde{I}_2(x)$ also take the form of the hypergeometric integrals in

eqs. (7.15) and (7.16). On the level of hypergeometric functions we find that the basis change in eq. (8.90) corresponds to

$$\begin{bmatrix} \lambda_1(a,b,c)\,{}_2F_1(-c,-a-b-c-1;-a-c;x) \\ x^{1+a+c}\lambda_2(a,b,c)\,{}_2F_1(-b,a+1;a+c+2;x) \end{bmatrix} = \\ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{bmatrix} \lambda_1(b,a,c)\,{}_2F_1(-c,-a-b-c-1;-b-c;1-x) \\ (1-x)^{1+b+c}\lambda_2(b,a,c)\,{}_2F_1(-a,b+1;b+c+2;1-x) \end{bmatrix}. \tag{8.91}$$

Let us compare this to chapter 7, where we found sets of fundamental solutions to the singular vector equation local to the singular points $x = 0$ and $x = 1$. Monodromy transformations about $x = 0$ were diagonal on the set in eq. (7.21), and those about $x = 1$ were diagonal on the set in eq. (7.22), and these solution sets were connected by the connection matrix (T_{ij}):

$$\begin{bmatrix} {}_2F_1(A,B;C;x) \\ x^{1-C}\,{}_2F_1(A+1-C,B+1-C;2-C;x) \end{bmatrix} = \\ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{bmatrix} {}_2F_1(A,B;A+B+1-C;1-x) \\ (1-x)^{C-A-B}\,{}_2F_1(C-A,C-B;1+C-A-B;1-x) \end{bmatrix}. \tag{8.92}$$

We conjecture that the integral manipulations of section 8.5.2 are equivalent to the connection transformations of section 7.3. Comparing the arguments on left-hand sides of eqs. (8.91) and (8.92) we make the identification

$$A = -c, \qquad B = -a-b-c-1, \qquad C = -a-c. \tag{8.93}$$

The right-hand sides of eqs. (8.91) and (8.92) are equal if and only if the following statements are identities:

$$\begin{aligned} \frac{\lambda_1(b,a,c)}{\lambda_1(a,b,c)}A_{11} &= T_{11}, & \frac{\lambda_2(b,a,c)}{\lambda_1(a,b,c)}A_{12} &= T_{12}, \\ \frac{\lambda_1(b,a,c)}{\lambda_2(a,b,c)}A_{21} &= T_{21}, & \frac{\lambda_2(b,a,c)}{\lambda_2(a,b,c)}A_{22} &= T_{22}. \end{aligned} \tag{8.94}$$

A bit of algebra confirms these identities. For example, let us prove explicitly the first statement in (8.94). The others can be proven similarly. From the connection matrix in eq. (7.26) we have

$$T_{11} = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-B)\Gamma(C-A)} = \frac{\Gamma(-a-c)\Gamma(1+b+c)}{\Gamma(1+b)\Gamma(-a)}, \tag{8.95}$$

and from contour manipulations we have

$$\begin{aligned} \frac{\lambda_1(b,a,c)}{\lambda_1(a,b,c)}A_{11} &= \frac{\Gamma(-a-b-c-1)\Gamma(a+1)}{\Gamma(-b-c)}\frac{\Gamma(-a-c)}{\Gamma(-a-b-c-1)\Gamma(b+1)}\frac{s(a)}{s(b+c)} \\ &= \frac{\Gamma(a+1)\Gamma(-a-c)}{\Gamma(b+1)\Gamma(-b-c)}\frac{s(a)}{s(b+c)}. \end{aligned} \tag{8.96}$$

Using Euler's reflection identity $\Gamma(1-x)\Gamma(x) = \pi/s(x)$, and the defining identity for the gamma function $x\Gamma(x) = \Gamma(1+x)$, we find that

$$\begin{aligned} \frac{\lambda_1(b,a,c)}{\lambda_1(a,b,c)}A_{11} &= \left(\frac{\Gamma(-a-c)}{\Gamma(1+b)}\right)\left(\frac{\Gamma(1+a)}{\Gamma(1-a)\Gamma(a)}\right)\left(\frac{\Gamma(1-b-c)\Gamma(b+c)}{\Gamma(-b-c)}\right) \\ &= \left(\frac{\Gamma(-a-c)}{\Gamma(1+b)}\right)\left(\frac{a\Gamma(a)}{-a\Gamma(-a)\Gamma(a)}\right)\left(\frac{(-b-c)\Gamma(-b-c)}{\Gamma(-b-c)}\frac{\Gamma(1+b+c)}{b+c}\right) \\ &= \left(\frac{\Gamma(-a-c)}{\Gamma(1+b)}\right)\left(\frac{-1}{\Gamma(-a)}\right)(-\Gamma(1+b+c)) \\ &= \frac{\Gamma(-a-c)\Gamma(1+b+c)}{\Gamma(1+b)\Gamma(-a)} = T_{11}, \end{aligned} \tag{8.97}$$

as claimed.

These formulae make explicit the link between the Coulomb gas formalism and the singular vector differential equation in eq. (5.26). The basis change in eq. (8.90) is a connection transformation between sets of solutions local to the singular points at 0 and 1 respectively. The contour manipulations therefore provide a method for implementing the procedure devised in eq. (7.31), and thus for finding monodromy-invariant combinations of particular solutions, without having to explicitly compute the monodromy matrices (M_0) and (M_1).

8.A Appendix: Coulomb gas energy-momentum tensor

Here the OPE of the Coulomb gas energy-momentum tensor is calculated using Wick's theorem.

Consider the OPE:

$$L_A(z)L_A(w) = \underbrace{\frac{1}{4}\Big(:\partial\varphi(z)\partial\varphi(z):\Big)\Big(:\partial\varphi(w)\partial\varphi(w):\Big)}_{1} + \underbrace{\frac{A}{2}\partial^2\varphi(z):\partial\varphi(w)\partial\varphi(w):}_{2} + \underbrace{\frac{A}{2}:\partial\varphi(z)\partial\varphi(z):\partial^2\varphi(w)}_{3} + \underbrace{A^2\partial^2\varphi(z)\partial^2\varphi(w)}_{4}. \tag{8.98}$$

The first term just the OPE of the unmodified free boson energy-momentum tensor. The additional terms are evaluated using Wick's theorem:

$$\begin{aligned} 2: &\qquad \partial^2\varphi(z):\partial\varphi(w)\partial\varphi(w): \sim 2\partial\varphi(w)\partial_z\frac{1}{(z-w)^2} = \frac{-4\partial\varphi(w)}{(z-w)^3}, \\ 3: &\qquad :\partial\varphi(z)\partial\varphi(z):\partial^2\varphi(w) \sim 2\partial\varphi(z)\partial_w\frac{1}{(z-w)^2} = \frac{4\partial\varphi(z)}{(z-w)^3}, \\ 4: &\qquad \partial^2\varphi(z)\partial^2\varphi(w) = \partial_z\partial_w\frac{1}{(z-w)^2} = \frac{-6}{(z-w)^4}. \end{aligned} \tag{8.99}$$

Putting this all together we have

$$\begin{aligned} &L_A(z)L_A(w) \sim \\ &\frac{1/2}{(z-w)^4}\mathbf{1} + \frac{:\partial\varphi(z)\partial\varphi(w):}{(z-w)^2} + 2A\frac{\partial\varphi(z)}{(z-w)^3} - 2A\frac{\partial\varphi(w)}{(z-w)^3} - 6A^2\frac{1}{(z-w)^4}. \end{aligned} \tag{8.100}$$

Laurent expanding normally ordered terms about $z = w$ we obtain

$$\begin{aligned} &\sim \frac{(1-12A^2)/2}{(z-w)^4}\mathbf{1} + \frac{:\partial\varphi(w)\partial\varphi(w):}{(z-w)^2} + \frac{:\partial^2\varphi(w)\partial\varphi(w):}{(z-w)} + \frac{2A\partial^2\varphi(w)}{(z-w)^2} + \frac{A\partial^3\varphi(w)}{(z-w)} \\ &\sim \frac{(1-12A^2)/2}{(z-w)^4}\mathbf{1} + \frac{2L_A(w)}{(z-w)^2} + \frac{\partial L_A(w)}{z-w} \end{aligned} \tag{8.101}$$

which is the result stated in eq. (8.2).

Conclusion

"..., then to be understood."
— Sean Covey

We have achieved our primary goal, which was to construct the Coulomb gas formalism and use it to compute critical exponents and OPE-coefficients for statistical models.

Conformal field theory is a mature discipline, having enjoyed over thirty years worth of active research by some of the true giants of theoretical physics. This thesis has been an attempt to stand on their shoulders and get a glimpse of what they see. As such, most of this work is not original; its content can be found scattered throughout the physics and mathematics literature, and I have made every effort to credit those who have contributed.

In this thesis we have focused on the interplay between physics and mathematics, and on how physical symmetries can be quantified and expressed mathematically. Chapter 1 laid down the essential preliminaries, such as how to represent the action of a symmetry algebra on a quantum space of states and how to construct a scalar product that respects the symmetry. The indispensable role that representation theory plays in conformal field theory cannot be overstated — deriving the Virasoro algebra was the topic of chapter 2, and in subsequent chapters some statistical physics problems were reduced to studying the representation theory of this algebra. Chapter 3 related conformal field theory to corresponding notions in conventional quantum field theory, and then built upon these. The aim was to make precise some fundamental ideas needed in subsequent chapters, and to reveal the connections between the foundational objects of a conformal field theory, such as the relationship between states and fields, between operator-product expansions and correlation functions, and between algebras and their generating functions. We demonstrated that primary OPE-coefficients provide sufficient information for computing correlation functions, thereby justifying the search for primary OPE-coefficients as our main goal.

Chapter 4 was our first encounter with a genuine conformal field theory. Here we put the notions of previous chapters on a concrete footing and witnessed those ideas come to life — holomorphic factorisation arose naturally from the equations of motion; the Virasoro algebra arose from canonical quantisation of the energy-momentum tensor; correlation functions arose from scalar products; the existence of an operator-product expansion followed from mode expansions for fields. The free boson was a physically motivated platform for constructing vertex operators, and showing how Virasoro modules can be embedded in bosonic Fock spaces — ideas that were essential for the bosonisation and Coulomb gas calculations of chapters 6 and 8. The connection linking conformal field theory to statistical physics arrived in chapter 5, where some basic notions in the representation theory of the conformal algebra gave rise to the minimal models. The remainder of the thesis was dedicated to solving these theories using the string theoretic vertex operators from chapter 4.

Of the original content in chapters 1 to 6, most of this takes the form of organisation and

exposition, selection of sources, and choice of notation. Some calculations and explanations are not readily found in the literature, and I have sometimes strayed from conventional approaches where I found it convenient.

Most of the original ideas occur in chapters 7 and 8. In chapter 7 we encountered hypergeometric-type differential equations. By imposing the physical requirement that correlation functions be single-valued we were led to the condition of monodromy invariance, and devised a technique for constructing monodromy-invariant solutions by solving connection problems. The sequence of analytic continuations for computing monodromy matrices that was proposed in eq. (7.31), and subsequently demonstrated explicitly, is not something I have seen in the literature. We related connection problems and monodromy problems to Coulomb gas integrals in chapter 8 by working through a specific example in detail, and we proved some identities in section 8.8 to verify the relationship. The connection problem for Fuchsian equations is a difficult problem in general,[13] and seems to be insufficiently studied. In particular the relationship between conformal blocks and connection problems is not something I have seen in the literature.

The prospects for generalisation and wider application of these ideas are immediate. The illustrative example we dealt with in chapters 7 and 8 was for a second-order differential equation on the sphere. In a sense, this is the most important case, because every minimal model has correlation functions that obey these second-order differential equations. But there are also infinitely many instances involving higher order differential equations and higher genera Riemann surfaces. This idea should not be limited to minimal models, to correlation functions, or even to conformal field theory. The concept of using connection transformations to find single-valued combinations of particular solutions to a differential equation should generalise to higher order differential equations with a different number of singularities, and on domains other than the sphere. These general structures are ubiquitous in physical and mathematical sciences, from engineering to fundamental physics and pure mathematics.

Recent developments

I have recently become aware of some newly published work with a similar idea to what I had believed to be an original contribution. In 2012 [60] pointed out that connection problems for *Painlevé VI* functions (perhaps more familiar are the related tau functions from inverse scattering theory and integrable systems) are closely related to monodromy-invariant constructions (some authors are calling these "isomonodromy constructions" [99]) of conformal blocks. Painlevé equations possess many of the same properties as hypergeometric equations [82]. In 2013 [76] derived Painlevé connection coefficients, and used these to give expressions for conformal blocks in the case of $c = 1$ (we did this for the minimal models with $c = c(p', p)$), and this work was continued in 2014 by [77]. Also in 2014 [75] used linear combinations of $c = 1$ conformal blocks to give formulae for tau functions (just as the Coulomb gas formalism gave solutions to the hypergeometric equation). In the same year [89] covered a more detailed version of the problem in chapter 7, but in greater generality, and using tau functions instead of hypergeometric functions. In 2015 (and at the time of writing still unpublished) works such as [63, 97, 102] are continuing down this avenue of research.

I have not referred to any of this work in the main body of my thesis, because I was not aware of it at the time of writing. It is not yet a standard component of the literature.

Future work

We began at the beginning, but have not ended at the end — the story is not yet over.

[13] Incidentally, finding monodromy-invariant solutions to hypergeometric equations was Riemann's initial motivation for developing the Riemann-Hilbert problem, and led to the formulation of Hilbert's twenty-first problem.

The most immediate step forward is to generalise the link between the Coulomb gas realisation of correlation functions and connection problems to cases involving higher order differential equations and other Riemann surfaces. The surge of recent work [38, 60, 61, 63, 75–77, 89, 97, 102] on this topic by physicists and mathematicians is a testament to the value of this research.

Papers such as [35] and [58] present integral manipulations for Coulomb gas integrals of all orders, with the claim that these are all integral representations of solutions to singular vector differential equations. This should definitely be true, although it has not yet been put on a rigorous footing (however, the recent series of papers [47–50] give a detailed affirmative proof for an infinite number of special cases). If it is true then these integrals are representations of solutions to generalised hypergeometric equations. Solving connection problems for these equations is usually very difficult. For instance, [94] is entirely dedicated to finding connection matrices for the function ${}_3F_2$, which is the next most simple case after ${}_2F_1$. Connection coefficients for hypergeometric and related functions are not known in general, and integral representations might provide tools sufficient to find them. Hypergeometric-type equations and integrals arise very frequently and have extensive applications in many mathematically based sciences. A general and effective technique (and one that is comprehensible) for finding monodromy representations is very much needed.

Finally, a good portion of time this year was spent on [41] and a number of subsequent related works, such as [42, 43], but, for lack of space and completeness, these made little appearance here. It seems that there are two distinct schools of thought regarding screening operators. The first follows closely the original works [34, 35] and is oriented towards theoretical and statistical physics. This is the approach presented in chapter 8. It is results-driven — integral manipulations and monodromy representations are used to compute correlation functions, with little focus *how and why* the method works on a mathematical level. The other approach is more representation theoretic and oriented towards mathematical physics and mathematics. Here the representation theory of vertex and screening operators plays a dominant role. The bridge connecting these schools of thought is somewhat lacking, as is clearly evident from the numerous different approaches mentioned in section 8.4.4. This is unfortunate, because both brands of physics stand to benefit from a better understanding. Clarifying this link is another future goal.

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