The Structure Constants of the Minimal Models

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Abstract

Rational conformal field theories (RCFTs) are perhaps the most well studied class of conformal field theories. RCFTs where the Virasoro algebra $\mathfrak{Vir}$ is the only symmetry algebra of the theory are called the minimal models. This work serves as an introduction to conformal field theory via the minimal models, with a focus on computing structure constants. After giving a general introduction to conformal field theory, we will focus on exploiting Virasoro algebra symmetry to obtain differential equations to constrain correlation functions. We finish by showing how one may use these equations to derive quadratic equations for the structure constants, and solving these for arbitrary minimal models, with applications to many examples.

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Chapter 1

Introduction

Symmetry is perhaps one of the most fundamental properties in nature and mathematics. It is the art of pattern recognition, study of structure preservation, and it is paramount in solving many modern problems in mathematical physics. It is almost a universal property that symmetry supplies solutions, a property that we will explore in this thesis.

When studying the universe at the most fundamental level, it is natural to look for symmetries that will aid in admitting solutions. Specifically, we will consider symmetries of quantum field theories. A field is, classically, a function on space and time. *Quantum field theories* study operator valued functions, that act on a quantum state space.

Quantum field theories describe and unify the fundamental forces of nature, with gravity perhaps being the most elusive exception [Shi12]. We expect a generic quantum field theory to exhibit certain symmetries, such as rotation and translation invariance. We will however consider field theories with symmetry that is a bit stronger: conformal invariance.

Conformal Symmetry

Conformal symmetry in a field theory is, in a nutshell, invariance under angle preserving transformations, which is a larger set of conditions than one expects from a generic quantum field theory. Conformal quantum field theories (or *conformal field theories*) therefore demand more symmetry than just the Lorentz and translation invariance of quantum field theories. In two dimensions, the algebra of conformal transformations is infinite-dimensional. A 2D conformal field theory (CFT) is therefore very strongly constrained by its symmetries and, in the case of the minimal models we consider, this conformal symmetry can be used to exactly solve the theory.

Due to its ability to exactly solve the theory, conformal field theory has been applied to many other fields of mathematics other than just quantum fields. Most impressively, it has been a bridge between finite groups and modular forms, through the monstrous moonshine conjectures [CN79], later proved in [Bor92]. It also has applications to topology, geometry, number theory, tensor categories, and many other areas of mathematics. One worth mentioning in more detail is the application to string theory, which aims to unify gravity with the other fundamental forces in nature. The Polyakov formulation of string theory can be found in [Pol81a, Pol81b]. We will not consider string theory applications here, but the reader may see [Kak99, Kir07, BP09] for more details.

The study of conformal field theory erupted in 1984 primarily from [BPZ84]. Belavin, Polyakov and Zamolodchikov studied the representation theory of the Virasoro algebra from Kac, and combined it with an algebra of local operators from Feigin and Fuchs [Kac79, FF82], to show these field theories can be constructed from finitely many representations of the Virasoro algebra. They named these field theories *minimal models*, and it will be these theories we consider in this thesis. Another paper relevant to the explosion of conformal field theory is [Wit84].

A primary goal of quantum field theory, and therefore of conformal field theory, is to compute observables through *correlation functions*. These measure how fields interact with each other in a field theory, and are mathematically modelled as scalar products of conformal fields acting on the vacuum state. In a general quantum field theory, one computes correlation functions perturbatively. In favourable cases, conformal field theories have the benefit of not needing a Lagrangian description to compute correlation functions, as invariance under conformal transformations forces the correlation functions to obey partial differential equations, which one can solve.

Why the Minimal Models?

As mentioned previously, the Virasoro minimal models are a class of conformal field theories that are made of finitely many representations of the Virasoro algebra. Correlation functions of minimal models obey partial differential equations, beyond the usual Ward identities from conformal invariance, known as Belavin-Polyakov-Zamolodchikov (BPZ) equations.

Due to this, one can find a set of solutions to these BPZ equations for correlation functions of 4 fields called *conformal blocks*. Correlation functions are built from linear combinations of finitely many conformal blocks, where the coefficients are products of the structure constants of the operator product algebra. Combining these two sets of differential equations, and imposing that correlators are physically measurable, we are able to solve for the correlation functions and the structure constants. This is called exactly solving the theory.

Belavin, Polyakov and Zamolodchikov introduced these minimal models to describe statistical systems, as these minimal models were identified with the scaling limits of various two-dimensional statistical systems at their critical points, such as the Ising Model and tricritical Ising model. These are described as minimal models in Chapter 4 and solved.

Structure of the thesis

We begin this thesis by discussing the operator formalism for a general conformal field theory in Chapter 2. We derive the conformal algebra in a classical and quantum field theory, then describe the quantum state space as a direct sum of highest weight irreducible representations. This chapter also introduces the operator product expansion, and correlation functions as scalar products of conformal fields. Finally, we utilise conformal invariance to constrain correlation functions, particularly the 3-point correlation function is solved up to a structure constant, which this thesis aims to compute.

Next, we further explore Virasoro algebra symmetry in Chapter 3. We will show how quotienting out singular vectors from representations will lead to ordinary differential equations called Belavin-Polyakov-Zamolodchikov equations for 4-point functions. We also show how fusion limits the fields that can appear in a non-zero correlation function, and how these correlation functions can also be built from functions called conformal blocks. We describe these functions explicitly as products of powers of polynomials and hypergeometric functions.

Finally, in Chapter 4, we explore the how to use the formalism established in Chapters 2 and 3 to compute these structure constants for some well known systems. Beginning with the Yang-Lee singularity and the Ising model, we finish with a general algorithm for computing a subset of structure constants for the minimal models, illustrating this with an example.

There is another standard method that can be used to compute these structure constants called the *Coulomb Gas* formalism, as found in [DMS97]. This method relies on the integral representation of conformal blocks of Feigin and Fuchs, summarised in [FF90]. This was developed for the minimal models, and the structure constants were found by Dotsenko and Fateev [DF84, DF85]. This formalism will not be considered in this thesis.

Chapter 2

Conformal Field Theory

In this chapter, we will outline the basic foundations of conformal field theory and several tools that will be required to obtain our results in Chapter 4. The focus of this chapter will be to establish notation for the reader, accentuate important aspects, and explore why conformal field theory in two dimensions is of specific interest. This will not be an exhaustive background on conformal field theory, as many other references already exist. The standard reference is [DMS97], [Sch08] provides a mathematical introduction and [BP09] provides an introduction with applications to string theory. If the reader is unfamiliar with conformal field theory, [Rid13] provides an introduction to the simplest conformal field theory: the free boson.

2.1 Field Theories

We will quickly outline the terms and notation that will be used throughout this thesis. The reader is not expected to have knowledge of quantum field theory, and we will largely avoid it in this thesis, but may see [Wei95] for an introduction.

Classically, a field theory refers to a construction of the dynamics of a field, i.e., a specification of how a field changes with time or with respect to other independent physical variables on which the field depends. Fields are modelled as vector (or tensor) valued functions on space-time. Usually this is done by writing an action for the field, and treating it as a classical mechanical system with an infinite number of degrees of freedom. In a quantum field theory, these functions become operators on the quantum state space $\mathcal{S}$. Conformal field theories are quantum field theories which are invariant under angle-preserving transformations. These transformations preserve the angle between two non-zero vectors in the field theory. Due to this invariance, conformal field theories do not require a Lagrangian or Hamiltonian description, as we will see in Section 2.5.2.

The objects of concern in this thesis are *conformal fields* $\phi(x^\mu)$, which are mathematically modelled are functions on space-time, whose values are operators on the quantum state space. In the case of two dimensions, $\phi(x^\mu) = \phi(z, \bar{z})$ and $z, \bar{z} \in \mathbb{C}$. The primary goal of this thesis will be to compute correlation functions of conformal fields, which are defined to be scalar products of conformal fields acting on the vacuum: $\langle \phi_1(x^\mu)\phi_2(x^\nu)\ldots\phi_n(x^\rho)\rangle$. A field theory is described by its correlation functions, i.e. physically measurable quantities are constructed from correlation functions, so computing these is a primary goal in this chapter.

2.1.1 Notation

The *metric tensor* $g_{\mu\nu}$ is a matrix of dimension d that measures the infinitesimal distance between space-time points in a field theory. For $d = 2$, The infinitesimal distance squared is therefore

$$ds^2 = g_{11}(dx^1)^2 + g_{12}dx^1dx^2 + g_{21}dx^2dx^1 + g_{22}(dx^2)^2 \tag{2.1}$$

If we choose the Euclidean metric $\delta_{\mu\nu}$ we obtain the expected Pythagorean theorem

$$ds^2 = (dx^1)^2 + (dx^2)^2$$

In this thesis, we will use Einstein summation notation, so (2.1) will be expressed as

$$ds^2 = \sum_{\mu,\nu=1}^{2} g_{\mu\nu}dx^\mu dx^\nu \equiv g_{\mu\nu}dx^\mu dx^\nu.$$

The metric tensor also can take a vector from a vector space to its dual space: $x_\mu = g_{\mu\nu}x^\nu$. The dot product in this notation is therefore

$$x \cdot x = x^\mu g_{\mu\nu} x^\nu \tag{2.2}$$

We also denote the derivative $\frac{\partial}{\partial x^\mu}$ as ∂_μ and the norm as $\|x\| = (x \cdot x)^{1/2}$.

2.2 The Conformal Algebra

Before we can dive into a discussion of correlation functions of conformal fields, we need to discuss the algebra of conformal transformations that will be used to constrain our correlation functions. We will first classically derive how the conformal transformations infinitesimally affect a function on $\mathbb{R}^d$, then take $d = 2$, and using $\mathbb{R}^2 \cong \mathbb{C}$, show that these transformations become holomorphic and antiholomorphic transformations on the complex plane.

2.2.1 Conformal Transformations

To derive the conformal transformations, we will take our metric $g_{\mu\nu}$ to be a symmetric, non-degenerate matrix, and our manifold M to be smooth and semi-Riemannian.[1] We will keep the metric $g_{\mu\nu}$ general for now, later specialising to the Euclidean metric with $d = 2$, where d indicates the dimension.

We define a *conformal transformation* to be a transformation $x \mapsto x'$ which preserves the metric up to a strictly positive, non-zero scale factor

$$g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x), \tag{2.3}$$

[1] See [Sch08] for more details.

where the scale factor depends on x.

Consider two vectors x^μ and y^ν in the field theory over the field $\mathbb{R}^d$ where d is the dimension. If they are both transformed conformally to a point z so that (2.3) applies, then the angle between these vectors will transform as

$$\begin{aligned}\cos(\theta') &= \frac{x'\cdot y'}{\|x'\|\,\|y'\|}\\ &= \frac{g_{\mu\nu}(z')x'^\mu y'^\nu}{\left(g_{\mu\nu}(z')x'^\mu x'^\nu\right)^{1/2}\left(g_{\mu\nu}(z')y'^\mu y'^\nu\right)^{1/2}}\\ &= \frac{\Lambda(z)g_{\mu\nu}(z)x^\mu y^\nu}{\left(\Lambda(z)g_{\mu\nu}(z)x^\mu x^\nu\right)^{1/2}\left(\Lambda(z)g_{\mu\nu}(z)y^\mu y^\nu\right)^{1/2}}\\ &= \frac{x\cdot y}{\|x\|\,\|y\|}\\ &= \cos(\theta).\end{aligned} \tag{2.4}$$

Clearly the above (2.4) applies to a metric that depends on the coordinates x. For the remainder of this thesis, we will assume the metric now to be constant $g_{\mu\nu}(x) = g_{\mu\nu}$, however we will not assume that the transformed metric $g'_{\mu\nu}(x)$ is, in general.

To derive the conformal transformations infinitesimally, we will expand our vector $x'^\mu = x^\mu + \varepsilon^\mu(x)$ and observe that the infinitesimal length squared of a vector $\mathrm{d}s^2 = g'_{\mu\nu}\mathrm{d}x'^\mu\mathrm{d}x'^\nu$ between two points does not depend on our choice of coordinates. Therefore, $\mathrm{d}s^2 = g'_{\mu\nu}\mathrm{d}x'^\mu\mathrm{d}x'^\nu = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. This leads to

$$\begin{aligned}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = g'_{\mu\nu}\mathrm{d}x'^\mu\mathrm{d}x'^\nu &= g'_{\mu\nu}\left(\mathrm{d}x^\mu + \partial_\rho\varepsilon^\mu\mathrm{d}x^\rho\right)\left(\mathrm{d}x^\nu + \partial_\sigma\varepsilon^\nu\mathrm{d}x^\sigma\right)\\ &= g'_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \partial_\rho g'_{\mu\nu}\varepsilon^\mu\mathrm{d}x^\rho\mathrm{d}x^\nu + \partial_\sigma g'_{\mu\nu}\varepsilon^\nu\mathrm{d}x^\mu\mathrm{d}x^\sigma\\ &= g'_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \partial_\rho\varepsilon_\nu\mathrm{d}x^\rho\mathrm{d}x^\nu + \partial_\sigma\varepsilon_\mu\mathrm{d}x^\mu\mathrm{d}x^\sigma\\ &= \left(\Lambda(x)g_{\mu\nu} + \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu\right)\mathrm{d}x^\mu\mathrm{d}x^\nu,\end{aligned} \tag{2.5}$$

where we note $g'_{\mu\nu} = g'_{\nu\mu}$ since $g'_{\mu\nu}$ is symmetric, and we can let $\rho = \mu$ and $\sigma = \nu$, as these indices are summed over. We also safely ignored the infinitesimal squared term.[2]

If we let $\Lambda(x) = 1 - \Omega(x)$, the equality (2.5) becomes the *infinitesimal conformal equation*

$$\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu = \Omega g_{\mu\nu}. \tag{2.6}$$

Multiplying both sides of (2.6) by $g^{\nu\mu}$, we obtain

$$\begin{aligned}g^{\nu\mu}\partial_\mu\varepsilon_\nu + g^{\nu\mu}\partial_\nu\varepsilon_\mu &= \Omega g^{\nu\mu}g_{\mu\nu}\\ \partial^\nu\varepsilon_\nu + \partial^\mu\varepsilon_\mu &= \Omega d\\ \frac{2}{d}(\partial\cdot\varepsilon) &= \Omega,\end{aligned} \tag{2.7}$$

where $d = \delta^\mu_\mu$ is the dimension.

[2]The reader may also have noticed that we used the primed metric to lower an index on an unprimed quantity. The error in doing so is also infinitesimal squared, and can be safely ignored.

As we can see already from (2.7) that $d = 2$ appears to be of special interest. One could continue with these derivations for conformal transformations in $d > 2$, however as we do not consider these field theories, we will not do so here. The reader may see [Rid13, DMS97, Gin98] for the conformal transformations in $d > 2$.

2.2.2 The Two-Dimensional Conformal Algebra

We will now derive the conformal transformations in two dimensions to create an algebra, namely the *conformal algebra*. In two dimensions with the Euclidean metric, this algebra is two commuting copies of the *Witt algebra*.

For $d = 2$, the constraints (2.6) and (2.7) heavily constrain the infinitesimal conformal transformations, and combining them will lead to the two-dimensional conformal algebra. We will now solve these for the Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$. Take the infinitesimal conformal equation (2.6) and substitute our calculation for the scale factor Ω as in (2.7). This gives us

$$\begin{aligned} \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu &= \delta_{\mu\nu} \frac{2}{d} (\partial \cdot \varepsilon) \\ &= \delta_{\mu\nu} \delta^{\rho\sigma} \partial_\rho \varepsilon_\sigma . \end{aligned}$$

Expanding out our indices, we get

$$\begin{aligned} 2\partial_1 \varepsilon_1 = \partial_1 \varepsilon_2 + \partial_2 \varepsilon_1 = 2\partial_2 \varepsilon_2 &\quad \text{for } \mu = \nu, \\ \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1 &\quad \text{for } \mu \neq \nu. \end{aligned}$$

These are the Cauchy-Riemann equations for ε^1 and ε^2. We now make a change of variables, since $\mathbb{R}^2$ is homeomorphic to $\mathbb{C}$:

$$\begin{aligned} \varepsilon = \varepsilon^1 + \mathrm{i}\varepsilon^2, &\qquad \bar{\varepsilon} = \varepsilon^1 - \mathrm{i}\varepsilon^2, \\ z = x^1 + \mathrm{i}x^2, &\qquad \bar{z} = x^1 - \mathrm{i}x^2, \\ \partial_1 = \partial + \overline{\partial}, &\qquad \partial_2 = \partial - \overline{\partial}. \end{aligned} \tag{2.8}$$

Where we have used the notation $\partial = \partial_z$ and $\overline{\partial} = \partial_{\bar{z}}$. In these coordinates the Cauchy-Riemann equations, now for ε and $\bar{\varepsilon}$, read $\partial\bar{\varepsilon} = 0$ and $\overline{\partial}\varepsilon = 0$. Therefore, we conclude that $\varepsilon = \varepsilon(z)$ is holomorphic and $\overline{\varepsilon} = \overline{\varepsilon}(\bar{z})$ is antiholomorphic.

Using this, we can now define the generators of these conformal transformations. These will be differential operators, and they generate a Lie algebra called the *conformal algebra*. To construct these generators, we take a classical field $\phi(x'^\mu)$ (scalar function), which is now a function of two variables: z and $\bar{z}$. Under the change of coordinates, $x'^\mu = x^\mu + \varepsilon^\mu$, we perform an infinitesimal Taylor expansion to obtain

$$\phi(x'^\mu) = \phi(x^\mu + \varepsilon^\mu) = \phi(x^\mu) + \varepsilon^\mu \partial_\mu \phi(x^\mu) + \cdots . \tag{2.9}$$

The generators are given as $\varepsilon^\mu \partial_\mu$. From here we will denote the field $\phi(x^\mu)$ as $\phi(z, \bar{z})$.

To derive a basis for these generators, note that any infinitesimal holomorphic transformation may be expressed as

$$z' = z + \epsilon(z), \quad \varepsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1} \tag{2.10}$$

where the infinitesimal variation $\varepsilon(z)$ is expanded in a Laurent series around $z = 0$. Suppose that we had an arbitrary dimensionless field $\phi(z', \bar{z}')$, where we adopt our coordinates as specified in (2.8). Making an infinitesimal transformation (2.10) in (2.9) gives

$$\phi(z', \bar{z}') = \phi(z, \bar{z}) - \varepsilon(z)\partial\phi(z, \bar{z}) - \bar{\varepsilon}(\bar{z})\,\bar{\partial}\phi(z, \bar{z}).$$

Setting $\delta\phi(z, \bar{z}) = \phi(z', \bar{z}') - \phi(z, \bar{z})$, we see

$$\begin{aligned} \delta\phi(z, \bar{z}) &= -\varepsilon(z)\partial\phi(z, \bar{z}) - \bar{\varepsilon}(\bar{z})\,\bar{\partial}\phi(z, \bar{z}) \\ &= \sum_{n=-\infty}^{\infty} \left(c_n \ell_n \phi(z, \bar{z}) + \bar{c}_n \bar{\ell}_n \phi(z, \bar{z})\right), \end{aligned} \tag{2.11}$$

where we identify the generators

$$\ell_n = -z^{n+1}\partial, \quad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial}. \tag{2.12}$$

Since these generators (2.12) involve differential operators, two generators ℓ_m and ℓ_n for $m \neq n$ may not commute. We can compute the commutators of these generators:

$$\begin{aligned} [\ell_m, \ell_n] &= [-z^{m+1}\partial, -z^{n+1}\partial] \\ &= z^{m+1}\partial\left(z^{n+1}\partial\right) - z^{n+1}\partial\left(z^{m+1}\partial\right) \\ &= z^{m+1}\left((n+1)z^n\partial + z^{n+1}\partial^2\right) - z^{n+1}\left((m+1)z^m\partial + z^{m+1}\partial^2\right) \\ &= (n-m)z^{m+n+1}\partial \\ &= (m-n)\ell_{m+n}. \end{aligned} \tag{2.13}$$

The anti-holomorphic calculation is identical, and the reader can see from inspection that the holomorphic and anti-holomorphic generators commute, hence $[\ell_m, \bar{\ell}_n] = 0$.

This gives us the conformal algebra in two dimensions as the direct sum of two infinite dimensional Lie algebras, each known as the *Witt Algebra*. In other words, the Witt algebra has the generators $(\ell_n)_{n\in\mathbb{Z}}$ as a complex vector space basis, satisfying (2.13). If the reader is unfamiliar with Lie algebras, they may see [Hal03, FS09, Rid19].

The algebra of infinitesimal conformal transformations with a Euclidean metric in two dimensions is infinite dimensional. Note that the Witt algebra (2.13) contains a finite dimensional subalgebra generated by $\{\ell_{-1}, \ell_0, \ell_1\}$.

There is a much longer story to be told here about this finite dimensional subalgebra. However, for the purposes of this thesis, we need only realise that the set of conformal transformations $\{\ell_{-1}, \bar{\ell}_{-1}, \ell_0, \bar{\ell}_0, \ell_1, \bar{\ell}_1\}$ encode the following:

- $-\left(\ell_{-1} + \bar{\ell}_{-1}\right)$, $-\mathfrak{i}\left(\ell_{-1} + \bar{\ell}_{-1}\right)$ generate classical infinitesimal translations
- $-\left(\ell_0 + \bar{\ell}_0\right)$, $-\mathfrak{i}\left(\ell_0 + \bar{\ell}_0\right)$ generate classical infinitesimal dilations and rotations
- $-\left(\ell_1 + \bar{\ell}_1\right)$, $-\mathfrak{i}\left(\ell_1 + \bar{\ell}_1\right)$ generate classical infinitesimal special conformal transformations

These transformations are called the *global conformal transformations*, which can be exponentiated to give the *Möbius Transformations*.

The global conformal transformations defined on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ are the Möbius transformations:

$$z \mapsto \frac{az+b}{cz+d} \quad \text{with} \quad a, b, c, d \in \mathbb{C}, \tag{2.14}$$

such that $ad - bc \neq 0$.

- Global translations are given by $c = 0$ and $a, d = 1$: $z \mapsto z + b$
- Global rotations and dilations are given by $b, c = 0$ and $d = 1$: $z \mapsto az$
- Global special conformal transformation (SCT) is given by $a = 0$ and $b, d = 1$: $z \mapsto \frac{1}{cz+1}$

Now that we know the two-dimensional *conformal algebra*, which is two commuting copies of the *Witt algebra*, we need to quantise these symmetry generators so that they may act upon our quantum field theory.

2.2.3 The Virasoro Algebra

Thus far we have worked in a classical field theory, however in a quantum field theory, the symmetry generators become operators on the quantum state space $\mathcal{S}$. The goal here is now to take our algebra of conformal transformations in two dimensions, the Witt algebra (2.13), and derive the quantised version: the Virasoro algebra $\mathfrak{Vir}$.

In a conformally invariant quantum field theory, instead of the quantum state space admitting a representation of two copies of the Witt algebra, the projective state space will admit such a representation. As we wish to work in the state space $\mathcal{S}$, we will lift this projective representation to a representation of the central extension of the Witt algebra, giving the Virasoro algebra $\mathfrak{Vir}$.

For a general Lie algebra $\mathfrak{g}$, we define the *central extension* of $\mathfrak{g}$ to be $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}$ as specified in [BP09], with commutation relations:

$$\begin{aligned} [x', y']_{\mathfrak{g}'} &= [x, y]_{\mathfrak{g}} + c\, p(x, y) \quad x', y' \in \mathfrak{g}',\ x, y \in \mathfrak{g}, \\ [x', c]_{\mathfrak{g}'} &= 0, \quad x' \in \mathfrak{g}',\ c \in \mathbb{C}, \end{aligned} \tag{2.15}$$

where $x' = x \oplus 0$, $c = 0 \oplus c$ in $\mathfrak{g}'$ for $x \in \mathfrak{g}, c \in \mathbb{C}$. Furthermore, $p : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a bilinear function.

We denote the generators of the quantum conformal symmetry algebra by L_n. In terms of (2.15), $x', y' = L_m, L_n$ with $\mathfrak{g}' = \mathfrak{Vir}$ will be the Virasoro algebra modes, with $x, y = \ell_m, \ell_n$ as the Witt algebra modes. Using (2.15), the L_n modes will have the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + c\, p(m, n), \tag{2.16}$$

for some function p. Since the Lie bracket is antisymmetric, $p(m, n) = -p(n, m)$. We can show that by appropriately redefining the generators $L_m \mapsto \widehat{L}_m$, we can arrange for $p(n,0)$ and $p(1,-1)$ to be zero by setting

$$\begin{aligned}\widehat{L}_m &= L_m + \frac{c\,p(n,0)}{n}, \text{ for } n \neq 0,\\ \widehat{L}_0 &= L_0 + \frac{c\,p(1,-1)}{2}.\end{aligned}$$

This is to provide the commutation relations

$$\begin{aligned}[\widehat{L}_m, \widehat{L}_0] &= mL_m + c\,p(m,0) = \widehat{L}_m\\ [\widehat{L}_1, \widehat{L}_{-1}] &= 2L_0 + c\,p(1,-1) = 2\widehat{L}_0.\end{aligned} \tag{2.17}$$

We will drop the hats, but the algebra from this point on in this thesis can be understood with this redefinition in mind. To determine $p(m,n)$, we will consider two Jacobi identities.

First consider the Jacobi identity:

$$0 = [[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] = (m+n)p(n,m).$$

When $m+n \neq 0$, we have $p(m,n) = 0$. Therefore, the only values to fix are $p(n,-n)$ for $n \geq 2$, since we set $p(1,-1) = 0$, and p is antisymmetric.

Now consider a second Jacobi identity:

$$\begin{aligned}[[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n] &= 0\\ c\,p(n-1,-n+1) + (n-2)c\,p(-n,n) &= 0.\end{aligned} \tag{2.18}$$

where we have made use of our redefinition (2.17). Rearranging (2.18), we obtain a recursion relation for $p(n,-n)$:

$$\begin{aligned}p(n,-n) &= \frac{n+1}{n-2}p(n-1,-n+1)\\ &= \frac{(n+1)n(n-1)\cdots 5\cdot 4}{(n-2)(n-3)(n-4)\cdots 2\cdot 1}p(2,-2)\\ &= \binom{n+1}{3}p(2,-2).\end{aligned}$$

Normalising $p(2,-2) = \frac{1}{2}$, we have obtained

$$p(m,n) = \frac{1}{12}(m+1)m(m-1)\delta_{m+n,0},$$

which completes the calculation.

We have now arrived at the *Virasoro algebra* $\mathfrak{Vir}$ as the central extension of the Witt algebra. This algebra has basis $\{L_n\} \cup \{c\}$ for $n \in \mathbb{Z}$ and $c \in \mathbb{C}$ with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m+1)m(m-1)\delta_{m+n,0}, \quad [L_m, c] = 0. \tag{2.19}$$

This calculation informs us that we have two commuting copies of the Virasoro algebra in our quantum conformal algebra. A slightly different proof showing that $H^2(W,\mathbb{C}) \cong \mathbb{C}$ is in [Sch08].

Remark 1. The number c in (2.19) is called the *central charge* of the conformal field theory.

Remark 2. In the quantum state space, the global conformal transformations are now generated by $L_{-1}, \bar{L}_{-1}, L_0, \bar{L}_0, L_1$ and $\bar{L}_1$.

Let us summarise what we have achieved so far. We started by defining a conformal transformation to be one that leaves the metric invariant up to a scale (2.3). This allowed us to derive the equation (2.6), and specialising to two dimensions, we found that the Witt algebra is an infinite dimensional Lie algebra that encodes infinitesimal conformal transformations. We finished by extending this algebra to a quantum field theory, where our algebra for infinitesimal conformal transformations is now the Virasoro algebra (2.19). Understanding these operators that encode the conformal invariance, we now move onto conformal fields.

2.3 The Quantum State Space

The quantum state space $\mathcal{S}$ is a complex vector space admitting a representation of two copies of the Virasoro algebra $\mathfrak{Vir}$. We say that a representation V is *irreducible* if the only sub-representations are 0 and V, otherwise, V is *reducible*. For the conformal field theories considered in this thesis, the quantum state space decomposes into a direct sum of tensor products of holomorphic and antiholomorphic representations,

$$\mathcal{S} \cong \bigoplus_{h,\bar{h}} \left(V_h \otimes \overline{V}_{\bar{h}}\right), \tag{2.20}$$

where V_h and $\overline{V}_{\bar{h}}$ are irreducible representations of the Virasoro algebra. The space is infinite dimensional in all non-trivial minimal models.

Remark 3. The only minimal model that is finite dimensional is the trivial model.

Remark 4. In (2.20), h and $\bar{h}$ indicate independent quantities, not complex conjugates of each other.

From (2.20), we conclude that the commuting holomorphic and antiholomorphic algebras can be treated separately. We will take advantage of this by looking only at the holomorphic representations, tensoring them with the antiholomorphic representations later to obtain the complete conformal field theory.

Vectors in the quantum state space will be represented as $|v\rangle$ while $\langle v|$ denote linear functionals that act on the state space $|v\rangle$, i.e. $\langle v|$ is in the dual of $\mathcal{S}$. This state space's vectors $|v\rangle$ are acted upon by the Virasoro modes subject to (2.19). L_0 is the energy operator, and its eigenvalue is the *conformal dimension* h. The Virasoro L_n for $n > 0$ are called *annihilation operators,* while L_{-n} are known as creation operators. We define a *highest weight state* $|h\rangle$ of conformal dimension h to be a state satisfying

$$L_0 |h\rangle = h |h\rangle, \quad L_n |h\rangle = 0 \text{ for all } n > 0. \tag{2.21}$$

A representation V generated from $|h\rangle$ by acting with creation operators L_{-n} with $n > 0$ is a *highest weight representation.*

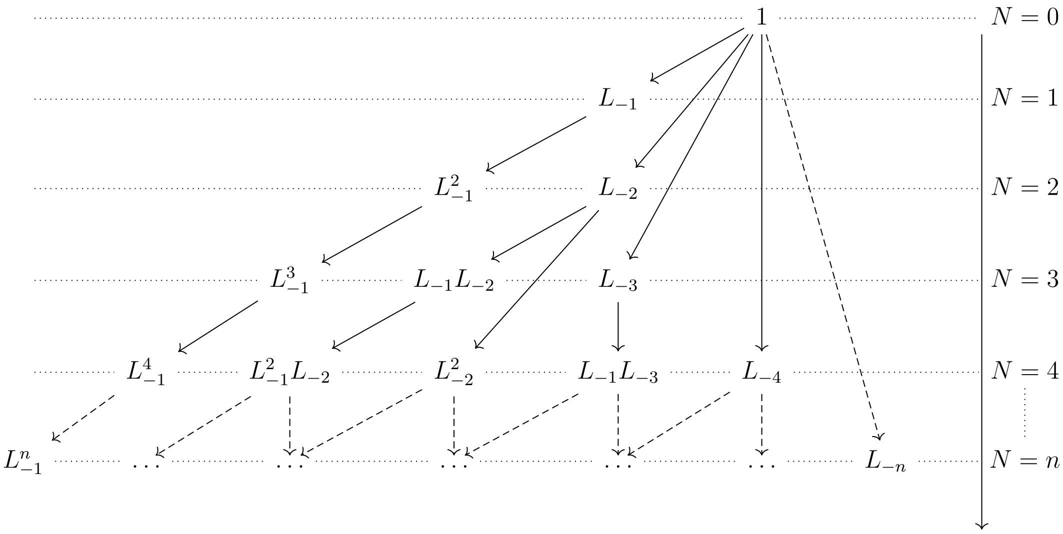

Figure 2.1: Basis for a Verma Module

In the theories considered in this thesis, since we require that the spectrum (2.20) is bounded from below by an eigenvalue, the representations V_i and $\overline{V}_j$ will be highest weight representations.

A *Verma module* $V(c,h)$ with conformal dimension h and central charge c as in (2.19), is a highest weight representation (2.21) with the *basis* given by

$$L_{n_1}L_{n_2}\cdots L_{n_k}|h\rangle \quad (n_1 \leq n_2 \leq \ldots \leq n_k \leq -1)\,. \tag{2.22}$$

This basis is illustrated in Fig. 2.1 for the first few *levels* N, and illustrating how the basis continues to level $N = n$. States that are of the form (2.22) that are created by acting upon a highest weight state $|h\rangle$ with creation operators are known as *descendant states* or *secondary states*.

The quantum state space $\mathcal{S}$ also admits a scalar product $\langle\ ,\ \rangle$. To construct this, we start by proposing an adjoint (hermitian conjugate) for each element of $\mathfrak{Vir}$. The adjoint must respect the Lie bracket $[x,y]^\dagger = [y^\dagger, x^\dagger]$ for $x, y \in \mathfrak{Vir}$. This condition leads us to propose the adjoint $L_n^\dagger = L_{-n}$, $c^\dagger = c$. The scalar product is denoted $\langle |v\rangle, |v\rangle\rangle \equiv \langle v|v\rangle$. As in traditional bra-ket notation, the ket $|v\rangle$ in the first entry of the scalar product is identified with the bra $\langle v|$.

We declare that this scalar product is normalised, so that for a vacuum state $|p\rangle$, we have $\langle p|p\rangle = 1$. This scalar product is also bilinear and invariant with respect to the adjoint

$$\langle L_n\,|u\rangle\,, |v\rangle\rangle = \langle |u\rangle\,, L_n^\dagger\,|v\rangle\rangle = \langle |u\rangle\,, L_{-n}\,|v\rangle\rangle,$$

for $|u\rangle\,, |v\rangle \in \mathcal{S}$ and $L_n \in \mathfrak{Vir}$.

2.3.1 Conformal Fields

A *conformal field* $\phi(x,t)$ in two dimensions embeds the degrees of freedom of position x and time t with a *Lorentzian* metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1)$. Conformal fields are classically real-valued functions living on a cylinder of circumference L:

$$\phi(x,t) = \phi(x+L,t). \tag{2.23}$$

However, our analysis of conformal transformations relied on a *Euclidean* metric = diag$(1,1)$ in the complex plane. To change our metric from Lorentzian to Euclidean, we perform a *Wick rotation* $t = \mathfrak{i}\tau$.

Now to map our conformal field $\phi(x,\mathfrak{i}\tau)$ to the complex plane, we make the change of variables

$$z = e^{2\pi(\tau+\mathfrak{i}x)/L}, \quad \bar{z} = e^{2\pi(\tau-\mathfrak{i}x)/L}. \tag{2.24}$$

This transformation is illustrated in Fig. 2.2. Our conformal field $\phi(z,\bar{z})$ is now well-defined on the complex plane.

Remark 5. A conformal field $\phi_i(z)$ has an associated conformal dimension h_i.

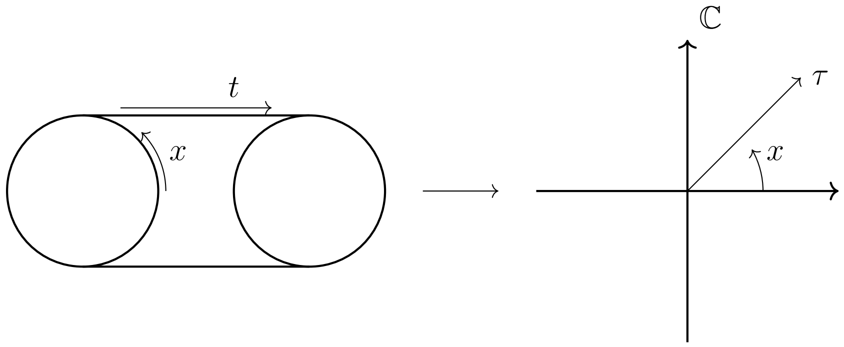

Figure 2.2: Conformal mapping from cylinder to complex plane

Recall in (2.20) that our quantum state space is a direct sum over the tensor products of holomorphic and antiholomorphic representations (2.20). So once again, we will work entirely in the holomorphic representation, and tensor it with the antiholomorphic half later.

From (2.24), we can identify that $\phi(z,\bar{z})$ is periodic in the arguments of z and $\bar{z}$. This motivates a Fourier decomposition, thus a holomorphic conformal field $\phi(z)$ of conformal dimension h such that $-n-h \in \mathbb{Z}$ may be expanded as

$$\phi(z) = \sum_n \phi_n z^{-n-h} \tag{2.25}$$

where the Fourier modes ϕ_n are operators on the quantum state space.

Remark 6. The power $-n-h$ in (2.25) is assumed to be an integer so that the field is single valued.

These modes may be calculated as follows

$$\phi_n = \oint_0 \phi(z) z^{n+h-1} \frac{\mathrm{d}z}{2\pi\mathfrak{i}}. \tag{2.26}$$

The subscript "0" on the contour integral indicates that the contour should be taken such that it encloses just the singularity of $\phi(z)$ at the origin. The direction of the integration is assumed to be taken anticlockwise. These modes will also have commutation relations, which we delay computing until the next section.

We define the energy-momentum field[3] to be

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad \overline{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}, \tag{2.27}$$

where the Fourier modes are the Virasoro operators satisfying (2.19). The Virasoro modes can be calculated by a similar contour integral

$$L_n = \oint_0 T(z) z^{n+1} \frac{\mathrm{d}z}{2\pi\mathfrak{i}}. \tag{2.28}$$

Remark 7. The conformal dimension of $T(z)$ is $h = 2$.

Using our change of coordinates (2.24), the radial coordinate in the complex plane becomes the time direction and the angular coordinate is the spatial direction. Using this map, we can take the origin to be the infinite past $z \to 0$.

Remark 8. The state $|0\rangle$ is the true vacuum; it is a highest weight state of conformal dimension $h = 0$.

Our asymptotic in state will be a conformal field $\phi(z)$ acting on the vacuum at time $t \to -\infty$:

$$|\phi_{\text{in}}\rangle = \lim_{z\to 0} \phi(z) |0\rangle .$$

This leads us to define our *state-field correspondence*. A conformal field $\phi(z)$ acting on the true vacuum $|0\rangle$ as $z \to 0$ corresponds to the state $|\phi\rangle$ by

$$|\phi\rangle = \lim_{z\to 0} \phi(z) |0\rangle . \tag{2.29}$$

Using this state field correspondence, we can calculate the state that the energy-momentum field corresponds to:

$$\begin{aligned}
|T\rangle &= \lim_{z\to 0} T(z)|0\rangle = \lim_{z\to 0} \sum_{n\in\mathbb{Z}} L_n|0\rangle z^{-n-2} \\
&= \lim_{z\to 0} \left(\sum_{n\leq -3} L_n|0\rangle z^{-n-2} + L_{-2}|0\rangle + L_{-1}|0\rangle z^{-1} + \sum_{n\geq 0} L_n|0\rangle z^{-n-2} \right) \\
&= L_{-2} |0\rangle .
\end{aligned}$$

Here, evaluating the limit gives us $|T\rangle = L_{-2} |0\rangle$ because $L_{-1} |0\rangle = 0$, as we assume the vacuum is translation invariant.

[3]This is classically derived from varying the action of the classical field theory S. The conformal field theories considered in this thesis will not have an action description, so we will just define $T(z)$ as above.

2.4 Operator Product Expansions

Before we consider correlation functions, we need to understand how to describe conformal fields acting upon a state:

$$\phi(z)\psi(w)\,|\theta\rangle\,. \tag{2.30}$$

where θ is a field at $z = 0$ (i.e. time $= -\infty$) acting on the true vacuum $|0\rangle$.

We interpret (2.30) as $|\theta\rangle$ is a state inserted at time $z = 0$, we act on this state with $\psi(w)$ at time $|w|$, then we act on this with $\phi(z)$ at time $|z|$. This only makes sense if we have $|z| > |w|$, if $|z| < |w|$ we would write

$$\psi(w)\phi(z)\,|\theta\rangle\,. \tag{2.31}$$

If we then act on (2.30) with $\langle\chi|$, this then evaluates the result at time $t \to \infty$, $\langle\chi|\,\psi(w)\phi(z)\,|\theta\rangle$ outputting a value. How this is modelled as a scalar product is detailed in Section 2.5, and is a correlation function. This motivates an ordering of fields, depending on their inputs.

2.4.1 Radial Ordering

Time ordering in a quantum field theory is referred to as *radial ordering* when mapped to the complex plane. It provides a framework for ordering products of operations in a two-dimensional conformal field theory.

Motivated by (2.30) and (2.31), we define the radially ordered product of two bosonic fields $\phi(z)$ and $\psi(w)$ to be

$$\mathcal{R}\{\phi(z)\psi(w)\} = \begin{cases} \phi(z)\psi(w) & \text{if } |z| > |w|, \\ \psi(w)\phi(z) & \text{if } |z| < |w|. \end{cases} \tag{2.32}$$

Remark 9. In this thesis, all fields will be bosonic fields.

Now, as z may be taken arbitrarily close to w in (2.32), it may be expanded out as a formal Laurent series around w in terms of z. This series as a product of two conformal field fields is called an *operator product expansion*. The operator product expansion was first proposed by [Wil69] for quantum field theories. A method of axiomising a conformal field theory via the "bootstrap approach" of operator product expansions was proposed by Polyakov [Pol74], and is used is in [Rib14].

As this expansion is a Laurent series, we will for now assume for two arbitrary conformal fields $\phi(z)$ and $\psi(w)$, the operator product expansion has the form

$$\mathcal{R}\{\phi(z)\psi(w)\} = \sum_{j=-\infty}^{\infty} \frac{A_j(w)}{(z-w)^{j+1}}, \tag{2.33}$$

for fields $A_j(w)$. Let the field $\phi(z)$ have conformal weight h_ϕ and $\psi(w)$ have conformal weight h_ψ. Using radial ordering (2.32), and our operator product expansion

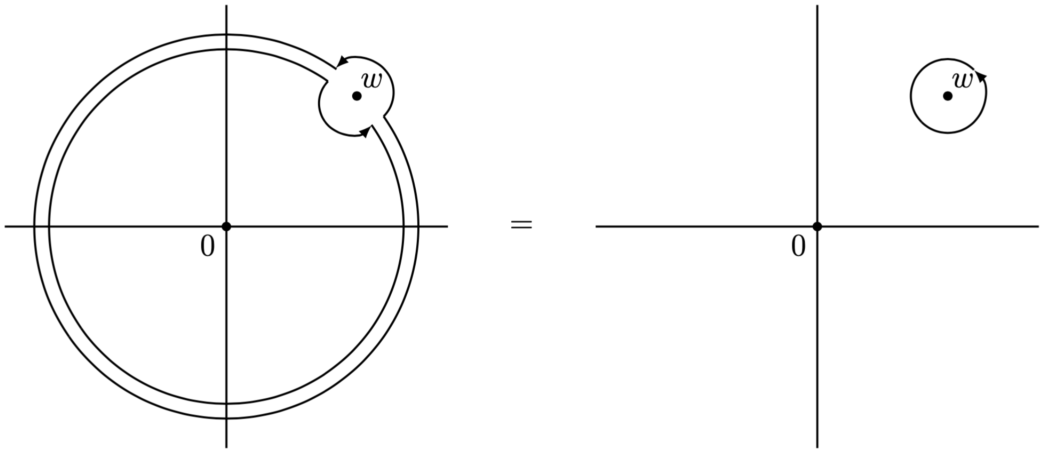

Figure 2.3: Contour addition/subtraction used in (2.35)

(2.33), we can compute the commutation relations of Fourier modes:

$$
\begin{aligned}
[\phi_n, \psi_m] &= \oint_0 \oint_0 \phi(z)\psi(w) z^{n+h_\phi-1} w^{n+h_\psi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \\
&\qquad - \oint_0 \oint_0 \psi(w)\phi(z) z^{n+h_\phi-1} w^{n+h_\psi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \\
&= \left[\underset{|z|>|w|}{\oint_0 \oint_0} - \underset{|z|<|w|}{\oint_0 \oint_0} \right] \mathcal{R}\{\phi(z)\psi(w)\} z^{n+h_\phi-1} w^{n+h_\psi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}}.
\end{aligned} \tag{2.34}
$$

Using the contour integral technique illustrated in Fig. 2.3, we can express (2.34) as

$$
\begin{aligned}
[\phi_n, \psi_m] &= \oint_0 \oint_w \mathcal{R}\{\phi(z)\psi(w)\} z^{n+h_\phi-1} w^{n+h_\psi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \\
&= \oint_0 \oint_w \sum_{j=-\infty}^{\infty} \left(\frac{A_j(w)}{(z-w)^{j+1}} \right) z^{n+h_\phi-1} w^{n+h_\psi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}},
\end{aligned} \tag{2.35}
$$

noting that the regular terms in the operator product expansion do not contribute. Evaluating the singular terms using Cauchy's integral theorem:

$$
\begin{aligned}
[\phi_n, \psi_m] &= \oint_0 \oint_w \sum_{j=0}^{\infty} \left(\frac{A_j(w)}{(z-w)^{j+1}} \right) z^{n+h_\phi-1} w^{n+h_\psi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \\
&= \sum_{j=0}^{n+h_\phi-1} \binom{n+h_\phi-1}{j} \oint_0 A_j(w) w^{m+n+h_\phi+h_\psi-2-j} \frac{\mathrm{d}w}{2\pi\mathrm{i}}.
\end{aligned} \tag{2.36}
$$

This integral (2.36) will be a mode of the field $A_j(w)$ in its Fourier decomposition. It is the $(m+n)$-th term in the expansion, with the conformal weight of $A_j(w)$ given by $h_{A_j} = h_\phi + h_\psi - j - 1$. This calculation tells us, if we know the commutation relations, we can read off the singular coefficients of the operator product expansion.

2.4.2 Normal Ordering

In a classical field theory, the ordering of products of fields is irrelevant. Upon quantisation, the fields become non-commuting operators on states. Looking at the singular terms in an operator product expansion (2.33), we can see that radially ordered operator product expansions may not be defined at $z = w$.

We therefore introduce the notion of *normal ordering*. The normally ordered product of two fields $\phi(w)$ and $\psi(w)$ at the same coordinate w is denoted by

$$:\phi(w)\psi(w):.$$

We will assume the normally ordered product has no singularities, so we can represent this as

$$:\phi(w)\psi(w): = \lim_{z\to w} :\phi(z)\psi(w):. \tag{2.37}$$

It is therefore natural to consider the radially ordered operator product expansion to be a collection of singular terms and normally ordered (non-singular terms). This gives an OPE the form

$$\mathcal{R}\{\phi(z)\psi(w)\} = \sum_{j=0}^{\infty} \frac{A_j(w)}{(z-w)^{j+1}} + :\phi(z)\psi(w):. \tag{2.38}$$

So the normally ordered product of two fields $:\phi(z)\psi(w):$ is defined to be the regular terms in the operator product expansion.

Since the operator product expansion is a Laurent series as $z \to w$, we can consider (2.37) to be the first regular term (i.e. the coefficient of $(z-w)^0$) in the operator product expansion. Therefore, we can express (2.37) as

$$:\phi(w)\psi(w): = \oint_w \frac{\mathcal{R}\{\phi(z)\psi(w)\}}{z-w}\frac{\mathrm{d}z}{2\pi\mathfrak{i}},$$

and we can use this to derive the normal ordering for the modes of the conformal fields (2.25).

First we expand out our normally ordered product using the radial ordering definition

$$:\phi(w)\psi(w): = \oint_{|z|>|w|} \frac{\phi(z)\psi(w)}{z-w}\frac{\mathrm{d}z}{2\pi\mathfrak{i}} - \oint_{|z|<|w|} \frac{\psi(w)\phi(z)}{z-w}\frac{\mathrm{d}z}{2\pi\mathfrak{i}}. \tag{2.39}$$

Using the fact that $(z-w)^{-1} = \sum_{n=0}^{\infty} w^n/z^{n+1}$ is a convergent geometric series whenever $|z| > |w|$, we can expand both denominators in (2.39) into convergent series and arrive at

$$\begin{aligned} :\phi(w)\psi(w): = &\oint_{|z|>|w|} \sum_{j=0}^{\infty} \frac{w^j}{z^{j+1}}\phi(z)\psi(w)\frac{\mathrm{d}z}{2\pi\mathfrak{i}} \\ &+ \oint_{|z|<|w|} \sum_{j=0}^{\infty} \frac{z^j}{w^{j+1}}\psi(w)\phi(z)\frac{\mathrm{d}z}{2\pi\mathfrak{i}}. \end{aligned}$$

Evaluating the contour integrals, and expanding the conformal fields using (2.25), we arrive at the result

$$:\phi(w)\psi(w): = \left(\sum_{n} \sum_{m+h_\phi \leq 0} \phi_m \psi_n + \sum_{n} \sum_{m+h_\psi > 0} \phi_n \psi_m \right) w^{-m-n-h_\phi-h_\psi}.$$

Therefore, the natural way to define normal ordering of modes is

$$:\phi_m \psi_n: = \begin{cases} \phi_m \psi_n & \text{if } m + h_\phi \leq 0, \\ \psi_n \phi_m & \text{if } m + h_\phi > 0. \end{cases} \tag{2.40}$$

We now have a mathematical way to describe normal ordering, but why do we care? Well, two reasons. First, by construction in (2.38), a normally ordered product will not contain any singularities. This means when substituting operator product expansions into contour integrals, the normally ordered terms will evaluate to zero.

Second, (2.40) tells us that all the positive modes in the Fourier expansion are pushed to the left of the product, and all the negative modes to the right. So the creation operators annihilate $\langle 0|$, and the annihilation operators annihilate $|0\rangle$. Therefore, if we have a normally ordered product inside a correlation function, it gives zero:

$$\langle 0| :\phi(z)\psi(w): |0\rangle = 0.$$

This tells us that if we have an operator product expansion inside a correlation function, we only need to consider the singular terms.

2.4.3 Example: Virasoro Algebra

We will now illustrate how to utilise these tools using the example of the Virasoro algebra (2.19). Recall the commutation relations are given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m+1)m(m-1)\delta_{m+n,0}.$$

We can use the fact that these are the Fourier modes of (2.27), and use (2.28) to express this as

$$\begin{aligned} [L_m, L_n] &= \left[\oint_0 T(z) z^{m+1} \frac{\mathrm{d}z}{2\pi \mathrm{i}}, \oint_0 T(w) w^{n+1} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \right] \\ &= \left[\underset{|z|>|w|}{\oint_0 \oint_0} - \underset{|z|<|w|}{\oint_0 \oint_0} \right] \mathcal{R}\{T(z)T(w)\} z^{m+1} w^{n+1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{\mathrm{d}w}{2\pi \mathrm{i}}. \end{aligned}$$

Then once again using the contour integral technique in Fig. 2.3 to rewrite this as

$$[L_m, L_n] = \oint_0 \oint_w \mathcal{R}\{T(z)T(w)\} z^{m+1} w^{n+1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{\mathrm{d}w}{2\pi \mathrm{i}}, \tag{2.41}$$

and assuming the OPE will have the form (2.33), substituting this form into (2.41) gives

$$[L_m, L_n] = \oint_0 \oint_w \sum_{j=0}^{\infty} \left(\frac{A_j(w)}{(z-w)^{j+1}} + :T(z)T(w): \right) z^{m+1} w^{n+1} \frac{\mathrm{d}z}{2\pi\mathfrak{i}} \frac{\mathrm{d}w}{2\pi\mathfrak{i}}.$$

The normally ordered terms do not contain any singularities by construction, and using (2.36), this will simply become

$$\begin{aligned}[L_m, L_n] &= \sum_{j=0}^{m+1} \binom{m+1}{j} \oint_0 A_j(w) w^{m+n+2-j} \frac{\mathrm{d}w}{2\pi\mathfrak{i}} \\ &= \oint_0 \Bigg(A_0(w) w^{m+n+2} + (m+1) A_1(w) w^{m+n+1} \\ &\quad + \binom{m+1}{2} A_2(w) w^{m+n} + \binom{m+1}{3} A_3(w) w^{m+n-1} + \dots \Bigg) \frac{\mathrm{d}w}{2\pi\mathfrak{i}} \end{aligned} \tag{2.42}$$

as the conformal dimension of $T(z)$ is 2. The commutation relations (2.19) can also be expressed in a contour integral, this allows us to read off the coefficients $A_j(w)$.

$$\begin{aligned}[L_m, L_n] &= (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\ &= \oint_0 \left[(m-n) T(w) w^{n+m+1} + \frac{c}{12}(m^3 - m) w^{n+m-1} \right] \frac{\mathrm{d}w}{2\pi\mathfrak{i}} \\ &= \oint_0 \left[(m-n) T(w) w^{n+m+1} + \binom{m+1}{3} \frac{c}{2} w^{n+m-1} \right] \frac{\mathrm{d}w}{2\pi\mathfrak{i}}, \end{aligned}$$

where we have used (2.28). We can rewrite $m - n = 2(m+1) - (m+n+2)$ and $(m+n+2)w^{m+n+1} = \partial w^{m+n+2}$, giving us

$$\begin{aligned}[L_m, L_n] = \oint_0 \Bigg[& 2(m+1) T(w) w^{n+m+1} \\ & - \partial w^{m+n+2} T(w) + \binom{m+1}{3} \frac{c}{2} w^{n+m-1} \Bigg] \frac{\mathrm{d}w}{2\pi\mathfrak{i}}, \end{aligned}$$

and then integrate by parts to give

$$\begin{aligned}[L_m, L_n] = \oint_0 \Bigg[& 2(m+1) T(w) w^{n+m+1} \\ & + w^{m+n+2} \partial T(w) + \binom{m+1}{3} \frac{c}{2} w^{n+m-1} \Bigg] \frac{\mathrm{d}w}{2\pi\mathfrak{i}}. \end{aligned} \tag{2.43}$$

Comparing both (2.42) and (2.43), we can read off the coefficient in the operator product expansion

$$A_0(w) = \partial T(w), \quad A_1(w) = 2T(w), \quad A_2(w) = 0, \quad A_3(w) = \frac{c}{2},$$

with $A_j(w) = 0$ for $j > 3$. Finally, we have arrived at the operator product expansion for the energy-momentum field

$$\mathcal{R}\{T(z)T(w)\} = \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + :T(z)T(w):. \tag{2.44}$$

Remark 10. From this point on we will not use the notation $\mathcal{R}\{\phi_1(z_1)\dots\phi(z_n)\}$, to indicate fields are radially ordered. All products of fields from this point will be assumed to be radially ordered.

2.5 Correlation Functions

In a quantum field theory, the physically measurable quantities are called correlation functions, which are modelled mathematically as scalar products of states on which fields are acting. Using the state-field correspondence (2.29), we can replace any state with a field acting on the true vacuum $|0\rangle$. Therefore, without loss of generality, we define a *correlation function* (also called a *correlator*) to be

$$\langle\psi_1(z_1)\dots\psi_m(z_m)\,|0\rangle\,,\phi_1(z_1)\dots\phi_k(z_k)\,|0\rangle\rangle.$$

Using our adjoint of a field, we can move the fields on the right-hand side of the scalar product, then redefine $\psi_1^\dagger(z_1)\dots\psi_m^\dagger(z_m)\phi_1(z_1)\dots\phi_k(z_k) = \phi_1(z_1)\dots\phi_n(z_n)$. This gives our general correlation function the form

$$\langle 0|\,\phi_1(z_1)\dots\phi_n(z_n)\,|0\rangle \equiv \langle\phi_1(z_1)\dots\phi_n(z_n)\rangle. \tag{2.45}$$

Consistent with our previous discussion of states acting on the vacuum, (2.45) is interpreted as ϕ_n acting on $|0\rangle$ at time $|z_n|$, followed by ϕ_{n-1} acting on $\phi_n\,|0\rangle$ at time $|z_{n-1}|$ and so on. After all states have acted, we evaluate the result with the linear functional $\langle 0|$ at $t\to\infty$, giving physically measurable observables.

2.5.1 Primary and Secondary Fields

Our spectrum (2.20) is made up of highest weight irreducible representations, generated from the states $|h\rangle$. From this, we act upon with creation operators to obtain any state in the field theory. The natural question arises, does this methodology correspond to fields?

The answer is yes! We know that a state and field relate under our state field correspondence (2.29). This motivates primary and secondary fields. In short, primary fields have a state-field correspondence to a highest weight state, and secondary fields have a correspondence to descendent states.

A field $\phi_p(z)$ is a *primary field* if it corresponds to a highest weight state (2.21) $|p\rangle$, via the state-field correspondence

$$\lim_{z\to 0}\phi_p(z)\,|0\rangle = |p\rangle\,. \tag{2.46}$$

Remark 11. We note that the energy-momentum field $T(z)$ is not a primary field, as $\lim_{z\to 0}T(z)\,|0\rangle = L_{-2}\,|0\rangle$, which is not a highest weight state, but a descendant of the true vacuum $|0\rangle$.

Using primary fields (2.46) and the state-field correspondence (2.29) we are able to compute the operator product expansion for $T(z)\phi(w)$, where $\phi(w)$ is primary

$$\begin{aligned}\lim_{w\to 0}\left(T(z)\phi(w)\right)|0\rangle &= T(z)\,|\phi\rangle \\ &= \sum_{n\in\mathbb{Z}} L_n z^{n-2}\,|\phi\rangle \\ &= L_0\,|\phi\rangle\, z^{-2} + L_{-1}\,|\phi\rangle\, z^{-1} + \dots \\ &= h\,|\phi\rangle\, z^{-2} + |\partial\phi\rangle\, z^{-1} + \dots \\ &= \lim_{w\to 0}\left(\frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + \dots\right)|0\rangle\,.\end{aligned}$$

This leaves us with the operator product expansion

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + :T(z)\phi(w):, \tag{2.47}$$

where the non-singular terms are normally ordered as per (2.38).

A field $\psi(z)$ is a *secondary field* or a *descendent field* if it can be expressed as a primary field acted upon by linear combinations of Virasoro modes. A secondary field $\psi(w)$ therefore corresponds to a linear combination of states of the form

$$L_{n_1}L_{n_2}\cdots L_{n_k}\,|\phi\rangle \quad (n_1 \le n_2 \le \ldots \le n_k \le -1)\,. \tag{2.48}$$

from the basis of Virasoro states (2.22). For example, $\psi(w) = (L_{-3}^2\phi)(w) + (L_{-2}\phi)(w)$ is a secondary field.

Say we have a descendent state of the form

$$|\psi\rangle = L_{-n}\,|\phi\rangle \;\text{ for } n > 0,$$

with $|\phi\rangle = |h\rangle$ corresponding to a primary field. This secondary field is denoted by $\psi(w) = (L_{-n}\phi)(w)$. So $(L_{-n}\phi)(w)$ is a primary field acted upon by the Virasoro mode L_{-n}. We can use the operator product expansion (2.47) to compute,

$$\begin{aligned}|\psi\rangle = L_{-n}\,|\phi\rangle &= \lim_{w\to 0} L_{-n}\phi(w)\,|0\rangle \\ &= \lim_{w\to 0}\oint_w T(z)\phi(w)\,|0\rangle\, z^{-n+1}\frac{\mathrm{d}z}{2\pi\mathfrak{i}} \\ &= \lim_{w\to 0}\oint_w \left(\frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + :T(z)\phi(w):\right)|0\rangle\, z^{-n+1}\frac{\mathrm{d}z}{2\pi\mathfrak{i}}.\end{aligned}$$

This allows us to read off the descendent fields $(L_{-n}\phi)(w)$. The singular terms give $(L_0\phi)(w) = h\phi(w)$ and $(L_{-1}\phi)(w) = \partial\phi(w)$. So $L_{-1} \leftrightarrow \partial_z$.

Remark 12. From the operator product expansion (2.47), we deduce $L_{-1}^n \leftrightarrow \partial_z^n$

We can expand out the normally ordered product $:T(z)\phi(w):$ to read off higher order secondary fields

$$:T(z)\phi(w): = :T(w)\phi(w): + :\partial T(w)\phi(w):(z-w) + \dots.$$

So we read off coefficients as $(L_{-2}\phi)(w) = :T(w)\phi(w):$, $(L_{-3}\phi)(w) = :\partial T(w)\phi(w):$ and $(L_{-n-2}\phi)(w) = \frac{1}{n!}:\partial^n T(w)\phi(w):$. These terms can be read off via the state-field correspondence as in Remark 11.

2.5.2 Ward Identities

Now we will finally show how conformal invariance constrains correlation functions (2.51). We will do so by inserting a mode of the Virasoro algebra L_m, which encodes a conformal transformation into a correlation function of primary fields. A primary field can be decomposed into a field with Fourier modes as in (2.25), therefore we expect a primary field and a mode of the Virasoro algebra to have commutation relations.

Using the operator product expansion (2.47), we can compute how a Virasoro mode L_m commutes with a primary field $\phi_i(z)$ of conformal dimension h_i:

$$\begin{aligned}[L_n, \phi_i(w)] &= \oint_w T(z)\phi_i(w) z^{n+1} \frac{\mathrm{d}z}{2\pi \mathfrak{i}} \\ &= \oint_w \left(\frac{h_i \phi_i(w)}{(z-w)^2} + \frac{\partial \phi_i(w)}{z-w} + :T(z)\phi(w): \right) z^{n+1} \frac{\mathrm{d}z}{2\pi \mathfrak{i}} \\ &= h_i(n+1)w^n \phi_i(w) + w^{n+1}\partial\phi_i(w). \end{aligned} \tag{2.49}$$

Here the normally ordered terms $:T(z)\phi(w):$ contain no singularities (2.38), hence do not contribute.

Now we will see why the global conformal transformations L_{-1}, L_0 and L_1 are special. The correlator (2.45) will have $\langle 0|$ and $|0\rangle$ annihilated by L_{-1}, L_0 and L_1. In other words, $L_m |0\rangle = 0$ and $\langle 0| L_m = 0$ for $m = -1, 0, 1$. Using this fact, and inserting L_m into our correlator (2.45) where $\phi_i(z_i)$ are all Virasoro primary with conformal dimension h_i, we get

$$\langle 0| \phi_1(z_1) \dots \phi_n(z_n) L_m |0\rangle = 0.$$

Commuting this L_m through our correlator using (2.49) we see

$$\begin{aligned} &\sum_{i=1}^{n} \langle 0| \phi_1(z_1) \dots [L_m, \phi_i(z_i)] \dots \phi_n(z_n) |0\rangle = 0 \\ \implies &\sum_{i=1}^{n} \left[(m+1)h_i z_i^m + z_i^{m+1}\partial_i\right] \langle 0| \phi_1(z_1)\phi_2(z_2) \dots \phi_n(z_n) |0\rangle = 0. \end{aligned}$$

Substituting $m = -1, 0, 1$, we can derive 3 partial differential equation constraints on our n-point correlation functions. We call these constraints on our correlators *Ward identities.*

$$\sum_{i=1}^{n} \partial_i \langle 0| \phi_1(z_1) \dots \phi_n(z_n) |0\rangle = 0 \tag{2.50a}$$

$$\sum_{i=1}^{n} (z_i\partial_i + h_i) \langle 0| \phi_1(z_1) \dots \phi_n(z_n) |0\rangle = 0 \tag{2.50b}$$

$$\sum_{i=1}^{n} (z_i^2\partial_i + 2h_i z_i) \langle 0| \phi_1(z_1) \dots \phi_n(z_n) |0\rangle = 0. \tag{2.50c}$$

These partial differential equations will heavily constrain our correlation functions for primary fields.

Solving the Ward identities (2.50) for correlations of up to 4 primary fields can be found in Appendix A, here we will just present the following solutions:

$$\langle 0|\,\phi_1(z_1)\,|0\rangle = \delta_{\phi_1=\mathbb{1}}, \tag{2.51a}$$

$$\langle 0|\,\phi_1(z_1)\phi_2(z_2)\,|0\rangle = \frac{C_{12}\delta_{h_1=h_2}}{(z_1-z_2)^{h_1+h_2}}, \tag{2.51b}$$

$$\langle 0|\,\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\,|0\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}z_{13}^{h_1-h_2+h_3}z_{23}^{-h_1+h_2+h_3}}, \tag{2.51c}$$

$$\langle 0|\,\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\,|0\rangle = F(\eta)\prod_{i<j} z_{ij}^{\frac{h}{3}-h_i-h_j}. \tag{2.51d}$$

We have used the notation $z_{ij} = z_i - z_j$ and $h = h_1 + h_2 + h_3 + h_4$ in (2.51d). Additionally, η is the *cross ratio*, given by

$$\eta = \frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_2)(z_3-z_4)}. \tag{2.52}$$

The function $F(\eta)$ is an undetermined function of the cross ration (2.52). Other cross ratios exist, but all cross ratios can be expressed as functions of η (2.52), such as

$$1+\eta = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)}, \qquad \frac{\eta}{1+\eta} = \frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_2-z_4)}.$$

C_{12} and C_{123} are undetermined constants from the Ward identities. The constants C_{123} are the *structure constants* that we aim to compute in the minimal models. To compute this, we will solve for $F(\eta)$ in (2.51d) in Chapter 3.

Remark 13. (2.51a) tells us that the only non-zero correlation function of a primary field is the *identity field* $\mathbb{1}$.

One could continue solving the Ward identities and attempt to derive some form of 5-point function, however having only 3 partial differential equation this becomes difficult, so we will stop here. We will discuss how one can reduce n-point functions to 3-point functions using operator product expansions later in this chapter.

Now, due to global conformal invariance, one can take a correlation function of four fields evaluated at z_1, z_2, z_3 and z_4, and use conformal transformations L_{-1}, L_0 and L_1 to send $z_1 \to 0, z_2 \to 1$ and $z_3 \to \infty$ respectively. Doing so will send $z_4 \to -\eta$ for the cross ratio (2.52)

To see this, first one applies two translation operators to move z_1 to the origin. Then applying a rotation until z_2 touches the positive real axis, followed by a dilation will send $z_2 \to 1$. Finally, applying a special conformal transformation will send $z_3 \to \infty$, while keeping z_1 and z_2 fixed in place. This series of global conformal transformations is illustrated in Fig. 2.4.

Under these transformations, the cross ratio (2.52) then becomes $\eta = -z_4$. So now a four point correlation function will depend only on z_4. In this thesis we will largely work with 4 point correlation functions dependent only on the cross ratio, where in principle we can recover the solution for a correlator of 4 points through the parameters a, b, c, d in the Möbius transformations (2.14) that we used to send $(z_1, z_2, z_4, z_4) \mapsto (0, 1, \infty, -\eta)$.

Remark 14. Since correlators measure the observables in a quantum system, we assume non-chiral correlators are single valued.

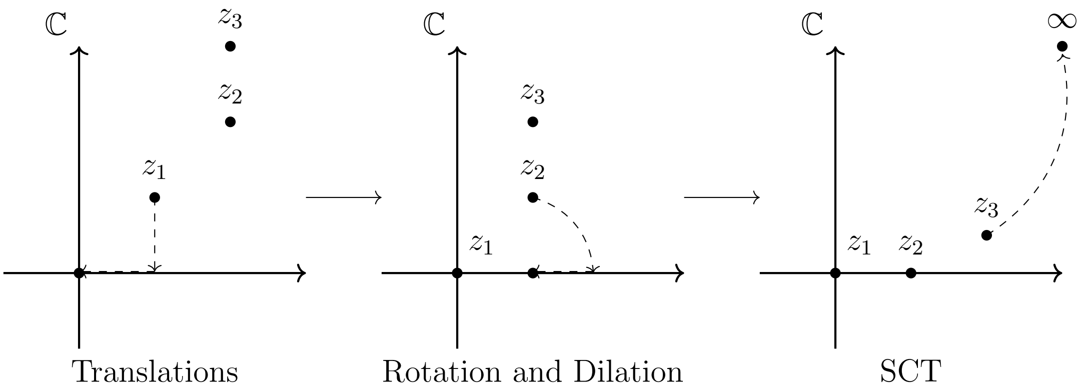

Figure 2.4: Global conformal invariance sending $z_1 \to 0, z_2 \to 1$ and $z_3 \to \infty$.

2.5.3 Correlators of Secondary Fields

So far, we have only considered how correlation functions of primary fields. Now we will consider correlators that contain secondary fields. If we substitute a secondary field $(L_{-n}\phi)(w)$ into a correlator, we find

$$\begin{aligned}&\langle 0| (L_{-n}\phi)(w)\phi(w_1)\dots\phi_n(w_m)|0\rangle \\ &\qquad= \oint_w (z-w)^{1-n}\langle 0| T(z)\phi(w)\phi(w_1)\dots\phi_n(w_m)|0\rangle \frac{\mathrm{d}z}{2\pi \mathrm{i}}.\end{aligned}$$

The radially ordered product will have singularities at $z = w, w_1, \dots, w_m$, so we can replace the contour around w, $\oint_w$, with a huge contour $\oint_\infty$ that will encircle all w_i and the point w as well. There will be poles around each w_i so we can substitute

$$\oint_w = \oint_\infty - \sum_{i=1}^{m}\oint_{w_i}.$$

This large contour $\oint_\infty$ is taken so that $|z| > |w|, |w_1|, \dots, |w_n|$ to allow the $T(z)$ to be taken all the way to the left in the correlator and then expanded out into its Fourier expansion (2.28) to give us

$$\begin{aligned}&\oint_w (z-w)^{1-n}\langle 0| T(z)\phi(w)\phi(w_1)\dots\phi_n(w_m)|0\rangle \frac{\mathrm{d}z}{2\pi \mathrm{i}} \\ &= \sum_{j=-\infty}^{\infty}\oint_\infty (z-w)^{1-n}z^{-j-2}\langle 0| L_j\phi(w)\phi_1(w_1)\dots\phi_m(w_m)|0\rangle \frac{\mathrm{d}z}{2\pi \mathrm{i}} \\ &- \sum_{i=1}^{m}\oint_{w_i}(z-w)^{1-n}\langle 0|\phi(w)\phi_1(w_1)\dots T(z)\phi_i(w)\dots\phi_m(w_m)|0\rangle \frac{\mathrm{d}z}{2\pi \mathrm{i}}. \end{aligned} \tag{2.53}$$

We can take the j-sum in (2.53) from $j = 2$ since L_j would annihilate $\langle 0|$ for $j \le 1$. However, upon further inspection of (2.53) we can see for $j > 1$ in the contour $\oint_\infty$

that the integrand is regular as $z \to \infty$ and so will be zero. This leads to the relation

$$\begin{aligned}&\langle 0|(L_{-n}\phi)(w)\phi(w_1)\dots\phi_n(w_m)|0\rangle\\&\quad=-\sum_{i=1}^{m}\oint_{w_i}(z-w)^{1-n}\langle 0|T(z)\phi(w)\phi_1(w_1)\dots\phi_m(w_m)|0\rangle\frac{\mathrm{d}z}{2\pi\mathrm{i}}.\end{aligned}$$

Once again we can substitute in our operator product expansion (2.47) and evaluate the integral to give

$$\begin{aligned}&\langle 0|(L_{-n}\phi)(w)\phi(w_1)\dots\phi_n(w_m)|0\rangle\\&\quad=\sum_{i=1}^{m}\left[\frac{(n-1)h_i}{(w_i-w)^n}-\frac{1}{(w_i-w)^{n-1}}\partial_{z_i}\right]\langle 0|\,\phi(w)\phi_1(w_1)\dots\phi_m(w_m)|0\rangle\,,\end{aligned}\tag{2.54}$$

where h_i is the conformal dimension of $\phi_i(w_i)$. To simplify notation, later in this thesis we will define the operator

$$\mathcal{L}_{-n}=\sum_{i=1}^{m}\left\{\frac{(n-1)h_i}{(w_i-w)^n}-\frac{1}{(w_i-w)^{n-1}}\partial_{z_i}\right\},\tag{2.55}$$

giving (2.54) to be

$$\langle 0|\,(L_{-n}\phi)(w)\phi(w_1)\dots\phi_n(w_m)|0\rangle=\mathcal{L}_{-n}\langle 0|\,\phi(w)\phi_1(w_1)\dots\phi_m(w_m)|0\rangle\,.\tag{2.56}$$

This calculation informs us that if we know the primary field correlators, we can calculate the presence of secondary fields using (2.56). This conclusion generalises beyond the presence of one secondary field, which leads us to conclude that all correlation functions in a conformal field theory can be obtained from the correlators of primary fields by repeatedly applying (2.55).

Remark 15. In this thesis, we will only consider correlation functions of primary fields, as in principle we can derive correlation functions with secondary fields through (2.56).

2.5.4 Structure Constants

Computing the structure constants in (2.51c) is the goal of this thesis. We will now show that if we choose a specific normalisation, we can conclude that the constants in an operator product expansion of primary fields are the same as the constants from the Ward identities in the 3-point function (2.51c).

Recall that the 2-point function (2.51b) has fields that are radially ordered. For any two point function, ϕ_1 and ϕ_2 can range over all possible primary fields. The function will be zero unless the conformal weights of these two fields match as per (2.51b). This is commutative, therefore a matrix of all 2-point constants C_{12} will be symmetric and diagonalizable.

We can assume that this matrix has no zero eigenvalues, since if any were 0, say the i-th one, then any 2-point function involving the corresponding field $\phi_i(z_i)$ will vanish. Using operator product expansions, any n-point correlation function can be reduced to a linear combination of 2-point correlation functions, so it follows that every n-point function involving $\phi_i(z_i)$ would vanish, meaning the field is unphysical. This is why we set all unphysical fields to zero, any correlator of them is zero, hence they are zero.

We can therefore conclude that C_{12} has no zero eigenvalues. This means we can choose our basis of conformal fields $\{\phi_i(z)\}$ to diagonalise and rescale our matrix of two point constants, so it becomes the identity matrix $\mathbb{1}$. Hence, $C_{12} = 1$ when the two fields are *conjugate* to each other, and zero otherwise. Therefore, for a field $\phi(z)$, the conjugate is the unique field $\phi^*(z)$ such that $C_{12} = 1$ in (2.51b).

This motivates the following definition. For a Virasoro primary field $\phi(z)$ with conformal dimension h, there will be a state field correspondence $\langle\phi|$ via

$$\langle\phi^*| = \lim_{z\to\infty} z^{2h} \langle 0| \phi^*(z), \tag{2.57}$$

Where denote $\phi^*(z)$ is the field conjugate to $\phi(z)$. One can now check

$$\begin{aligned}
\langle\phi^*|\phi\rangle &= \lim_{z\to\infty}\lim_{w\to 0} z^{2h} \langle 0| \phi^*(z)\phi(w) |0\rangle \\
&= \lim_{z\to\infty}\lim_{w\to 0} z^{2h} \frac{C_{\phi^*\phi}}{(z-w)^{2h}} \\
&= C_{\phi^*\phi} = 1.
\end{aligned} \tag{2.58}$$

Therefore $\langle\phi^*|\phi\rangle$ is just the scalar product. From this, we conclude that a field without a conjugate is unphysical, i.e. all scalar products will be zero.

Now considering a 3-point function of primary fields, we can use our operator product expansion Ansatz of two primary fields to be

$$\begin{aligned}
\phi_i(z_1)\phi_j(z_2) = \sum_l C_{ij}{}^l (z_1-z_2)^{h_l-h_i-h_j} \Big(&\phi_l(z_2) + c_1(z_1-z_2)\cdot(L_{-1}\phi_l)(z_2) \\
&+ c_2(z_1-z_2)^2\cdot\left(L_{-1}^2\phi_l\right)(z_2) \\
&+ c_3(z_1-z_2)^2\cdot(L_{-2}\phi_l)(z_2) + \dots\Big).
\end{aligned} \tag{2.59}$$

where l ranges over all primary fields. These descendent field constants c_m depend on the primary fields $c_m \equiv c_m(i,j,l)$. These constants can be calculated by requiring that the left-hand side (LHS) and right-hand side (RHS) of (2.59) describe the same quantum state space, so $\langle v|LHS\rangle = \langle v|RHS\rangle$ for all states v will give a system of equations to solve.

Remark 16. We will not need to consider descendent terms in operator product expansions for the remainder of this thesis.

Therefore, take an operator product expansion of primary fields to be of the form

$$\phi_i(z_1)\phi_j(z_2) = \sum_l \frac{C_{ij}{}^l\phi_l(z_2)}{(z_1-z_2)^{h_i+h_j-h_l}} + \dots, \tag{2.60}$$

where the dots represent the descendent fields. Substituting this into a 3-point function

$$\begin{aligned}
\langle 0|\, \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\,|0\rangle &= \sum_l \frac{C_{ij}{}^l \langle 0|\, \phi_l(z_2)\phi_k(z_3)\,|0\rangle}{z_{12}^{h_i+h_j-h_l}} + \dots \\
&= \sum_l \frac{C_{ij}{}^l \delta_{l=k^*}}{z_{12}^{h_i+h_j-h_l} z_{23}^{2h_k}} + \dots \\
&= \frac{C_{ij}{}^{k^*}}{z_{12}^{h_i+h_j-h_l} z_{23}^{2h_k}} + \dots, \qquad (2.61)
\end{aligned}$$

where $j = k^*$ indicates the fields ϕ_j and ϕ_k must be conjugates to ensure the two point function is non-zero (2.58). On the other hand, from the Ward identities we know the 3-point function (2.51c) up to the structure constant C_{123}. If we write $z_{13} = z_{23}(1 + z_{12}/z_{23})$, we see

$$\begin{aligned}
\langle 0|\, \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\,|0\rangle &= \frac{C_{ijk}}{z_{12}^{h_i+h_j-h_k} z_{13}^{h_i-h_j+h_k} z_{23}^{-h_i+h_j+h_k}} \\
&= \frac{C_{ijk}}{z_{12}^{h_i+h_j-h_k} \left(z_{23}(1+\frac{z_{12}}{z_{23}})\right)^{h_i-h_j+h_k} z_{23}^{-h_i+h_j+h_k}} \\
&= \frac{C_{ijk}}{z_{12}^{h_i+h_j-h_k} \left(1+\frac{z_{12}}{z_{23}}\right)^{h_i-h_j+h_k} z_{23}^{2h_k}}.
\end{aligned}$$

If we now expand $(1 + z_{12}/z_{23})$ as a geometric series, we find

$$\langle 0|\, \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\,|0\rangle = \frac{C_{ijk}}{z_{12}^{h_i+h_j-h_k} z_{23}^{2h_k}} + \dots. \qquad (2.62)$$

Comparing (2.61) and (2.62), we are left with

$$C_{ijk} = C_{ij}{}^{k^*}. \qquad (2.63)$$

We conclude that the 3-point functions and operator product expansions of primary fields have the same constants. This calculation highlights that if we know the 2-point functions, we know which fields are conjugate, and if we know the 3-point functions, we can compute the operator product expansions. Inserting operator product expansions into a n-point function allows us to reduce any n-point function to a 3-point function in principle. Due to this, we claim that a theory is *solved* if we can determine the structure constants C_{ijk}.

Let's summarise what we've achieved so far. We derived the algebra of conformal transformations in a conformally invariant quantum field theory and found it to be the Virasoro algebra (2.19). Then we analysed the quantum state space of our field theory that is made up of highest weight irreducible representations (2.20). These states have correspondence to conformal fields via (2.29), and we were able to construct correlation functions of these fields (2.45) as scalar products. Using the Ward identities (2.50), we have our correlators of up to 4 fields fixed to be the functions (2.51), specifically the 3-point function up to the structure constant. Having just learnt that these are the same constants in the operator product expansions of primary fields, we will now look to compute.

Chapter 3

Virasoro Algebra Symmetry

This chapter will further explore Virasoro algebra symmetry and derive additional constraints on our correlation functions. It is worth remarking that this is a somewhat arbitrary break from the last chapter, as the Ward identities constrains correlation functions in a conformal field theory purely as a result of Virasoro algebra symmetry.

We will begin by exploring irreducible representations, then construct constraints on correlation functions from singular vectors and null fields. After deducing fusion rules from singular vectors and introducing the Belavin-Polyakov-Zamolodchikov (BPZ) equations [BPZ84, Gin98], we will show the solutions to these are hypergeometric functions called conformal blocks [MS89, MS90].

3.1 Irreducible Representations

Recall that the spectrum of the quantum state space (2.20) was made up of irreducible highest weight representations. We also defined Verma modules $V(c,h)$ to be highest weight representations with the basis (2.22). These Verma modules however are not always irreducible, and may contain another highest weight state $|\chi\rangle$ that generates a highest-weight representation (*submodule*), descended from the highest-weight state $|h\rangle$. Therefore, the representations in the quantum state space (2.20) will be quotients of Verma modules: Verma modules with all proper submodules set to 0. This section will outline this procedure.

3.1.1 Reducibility

We will now look for other highest weight states that may appear inside a Verma module. We call such a state $|\chi\rangle$ a *singular vector*. This $|\chi\rangle$ is orthogonal to every basis state in the Verma module (2.22). Clearly $|\chi\rangle$ is orthogonal to any descendant of $|h\rangle$, as

$$\begin{aligned}
&(|\chi\rangle, L_{-n_1}L_{-n_2}\cdots L_{-n_k}|h\rangle) \quad (n_k \geq \ldots \geq n_2 \geq n_1 \geq 1)\\
&= (L^\dagger_{-n_1}L^\dagger_{-n_2}\cdots L^\dagger_{-n_k}|\chi\rangle, |h\rangle)\\
&= (L_{n_1}L_{n_2}\cdots L_{n_k}|\chi\rangle, |h\rangle)\\
&= (0, |h\rangle) = 0,
\end{aligned}$$

where L_n annihilates $|\chi\rangle$ for $n > 0$ as this is a highest weight state. Since $|\chi\rangle$ is a linear combination on creation operators acting of the highest weight state, as with secondary fields (2.48), taking $(|\chi\rangle, |h\rangle)$ and using our adjoint gives $(|\chi\rangle, |h\rangle) = 0$. So the singular vector is orthogonal to every state in the representation. This tells us that $|\chi\rangle$ is a *null state*.

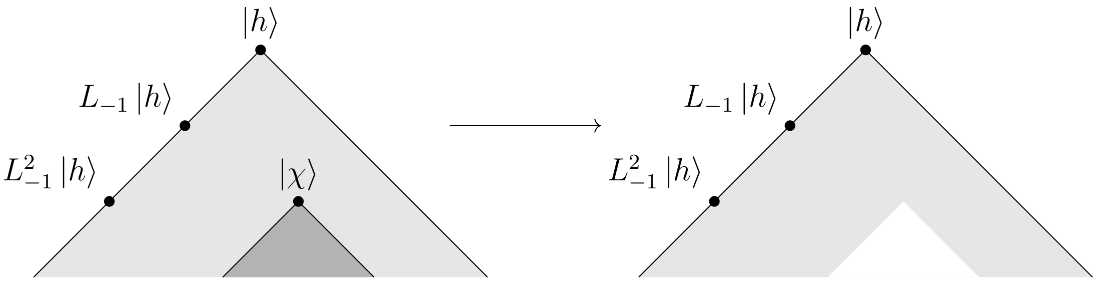

Figure 3.1: Quotienting a submodule $|\chi\rangle$ out of a highest weight representation $|h\rangle$.

If a Verma module contains a singular vector, then it is *reducible*. To maintain the existence of only irreducible representations in (2.20), we set all singular vectors to 0, along with their descendants. Doing so however means that the highest weight state $|h\rangle$ no longer generates a Verma module, as it will not have a complete basis (2.22). This is due to the fact we just removed a large chunk of it when we removed $|\chi\rangle$ and its descendants, as illustrated in Fig. 3.1. The quantum state space with the Virasoro algebra as the symmetry algebra is now consistent with (2.20).

It is worth mentioning when a conformal field theory is *unitary*. This occurs when the scalar product on every representation appearing in the state space is an inner product. We can derive a constraint to determine if a representation is non-unitary by considering

$$\langle h| L_nL_{-n}|h\rangle = \langle h| \Big(L_{-n}L_n + 2nL_0 + \frac{c}{12}(n-1)n(n+1)\Big)|h\rangle,$$

using the definition of the Virasoro algebra (2.19). We know that L_{-n} annihilates $\langle h|$ and L_n annihilates $|h\rangle$, and $L_0|h\rangle = h|h\rangle$. This gives us

$$\langle h| L_nL_{-n}|h\rangle = \Big(2nh + \frac{c}{12}(n^3-n)\Big)\langle h|h\rangle = 2nh + \frac{c}{12}(n^3-n). \tag{3.1}$$

Looking at (3.1), we can see that if we have a negative central change, for sufficiently large n we will obtain a negative norm. Therefore, all representations with a negative central charge are *non-unitary*. It turns out that some representations with $c > 0$ can also be non-unitary. We will nevertheless use this result to conclude the first minimal model we consider is non-unitary.

3.1.2 The Kac Determinant

The natural question now arises as to when a Verma module $V(c,h)$ is reducible. To address this, we need to consider a matrix called the *Gram matrix* M.

The Gram matrix M of a set of vectors $v_1, v_2, \dots v_n \in V$ is the hermitian matrix, whose entries are $G_{ij} = \langle v_i | v_j \rangle$. We will take our vectors to be basis states (2.22) of fixed grade k, calling this the Gram matrix $M^{(k)}$, so each vector can be represented as a combination of creation operators acting on the highest weight state $|h\rangle$.

Consider a Verma module $V(c,h)$ generated from the state $|h\rangle$. A basis of states for this Verma module was given in (2.22). There will be negative norm states at grade k, if and only if $M^{(k)}$ has one or more negative eigenvalues, and there will be a null state if an eigenvalue is zero. The dimension of the Gram matrix M will be given by the partition number p of level k in basis (2.22), $n = p(k)$. See Fig. 2.1 for a visual representation.

We now compute the first few Gram matrices,

$$\begin{aligned}
M^{(0)} &= \langle h|h\rangle = (1), \\
M^{(1)} &= \langle h| L_1 L_{-1} |h\rangle = 2h\langle h|h\rangle = (2h), \\
M^{(2)} &= \begin{pmatrix} \langle h| L_1^2 L_{-1}^2 |h\rangle & \langle h| L_1^2 L_{-2} |h\rangle \\ \langle h| L_2 L_{-1}^2 |h\rangle & \langle h| L_2 L_{-2} |h\rangle \end{pmatrix} = \begin{pmatrix} 8h^2+4h & 6h \\ 6h & 4h + c/2 \end{pmatrix}.
\end{aligned} \tag{3.2}$$

The commutation relations rapidly become non-trivial. An example of how $M_{11}^{(2)}$ in (3.2) is computed is

$$\begin{aligned}
M_{11}^{(2)} &= \langle h| L_1^2 L_{-1}^2 |h\rangle \\
&= \langle h| L_1 L_1 L_{-1} L_{-1} |h\rangle \\
&= \langle h| L_1 (L_{-1} L_1 + 2L_0) L_{-1} |h\rangle \\
&= \langle h| 2L_0 L_1 L_{-1} + 2L_1 L_{-1} L_0 + 2L_1 L_{-1} |h\rangle \\
&= \langle h| 4h^2 + 4h^2 + 4h |h\rangle \\
&= (8h^2 + 4h)\langle h|h\rangle \\
&= 8h^2 + 4h,
\end{aligned} \tag{3.3}$$

using the Virasoro commutation relations (2.19).

Looking at these Gram matrices, we derive some necessary conditions for these secondary states to be null states. $M^{(0)}$ tells us nothing, $M^{(1)}$ tells us that if $h = 0$, then $L_{-1}|h\rangle$ is a null state. We calculate the determinant of $M^{(2)}$:

$$\begin{aligned}
\det M^{(2)} &= (8h^2 + 4h)(4h + \tfrac{c}{2}) - 36h^2 \\
&= 32h^3 - 20h^2 + 4h^2 c + 2hc \\
&= 32(h - h_{1,1})(h - h_{1,2})(h - h_{2,1}),
\end{aligned}$$

where we let

$$\begin{aligned}
h_{1,1} &= 0 \\
h_{1,2} &= \frac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right) \\
h_{2,1} &= \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right).
\end{aligned} \tag{3.4}$$

The states at level two $L_{-2}|h\rangle$ and $L_{-1}^2|h\rangle$ will have a null linear combination whenever the determinant is zero, so when $h = h_{1,1}, h_{2,1}$ or $h_{1,2}$.

For this thesis we will only need to make use of the results (3.4), however the formula for calculating the determinant of an arbitrarily large Gram matrix is known as *Kac's determinant formula*

$$\det M^{(l)} = \alpha_l \prod_{r,s\geq 1}^{rs\leq l} [h - h_{r,s}(c)]^{p(l-rs)}, \tag{3.5}$$

where $p(l-rs)$ is the number of partitions of the integer $l-rs$ and α_l is a constant strictly greater than zero. This was first conjectured by Kac [Kac79], and later proved by Feigin and Fuchs [FF82]. As the proof is rather involved, it is further explained in [KR87, ID89, IK03].

It is useful to parameterise these $h_{r,s}(c)$, in terms of a parameter t, as the formulas can become congested. This gives the central charge c and the conformal dimensions $h_{r,s}$ as functions of t:

$$\begin{aligned} c &= 13 - 6\left(t + \frac{1}{t}\right), \\ h_{r,s}(t) &= \frac{1}{4}\left(r^2-1\right)t + \frac{1}{4t}\left(s^2-1\right) - \frac{1}{2}(rs-1). \end{aligned} \tag{3.6}$$

Using this parameterisation, (3.4) becomes

$$h_{1,1} = 0, \quad h_{2,1} = -\frac{1}{2} + \frac{3t}{4}, \quad h_{1,2} = -\frac{1}{2} + \frac{3}{4t}. \tag{3.7}$$

One could continue in this fashion to calculate determinants of higher dimension Gram matrices, for this thesis however we will only need those considered so far. We will see in the next subsection how to make use of these conformal dimensions to explicitly create singular vectors.

3.1.3 Singular Vectors

In the Kac determinant formula (3.5), conformal dimensions $h_{r,s}$ are parameterised by two natural numbers r and s, so it is natural to parameterise our singular vectors in the same manner. This gives a singular vector $|\chi\rangle$ the notation $|\chi_{r,s}\rangle$ for the unique null state $|\chi_{r,s}\rangle$ of grade rs that is descended from the highest weight state $|h_{r,s}\rangle$ [Ast97]. We will introduce the notation $L_{r,s}$ to represent a linear combination of basis Virasoro modes (2.22) to represent the creation operators used to generate the singular vector, $|\chi_{r,s}\rangle$, so

$$|\chi_{r,s}\rangle = L_{r,s}|h_{r,s}\rangle. \tag{3.8}$$

To check that $|\chi\rangle$ is a singular vector, we need to check that $L_n|\chi_{r,s}\rangle = 0$ for all L_n where $n > 0$. Actually, we only need to check

$$L_1|\chi_{r,s}\rangle = 0, \quad L_2|\chi_{r,s}\rangle = 0, \tag{3.9}$$

as L_n for $n \geq 2$ can be expressed as combinations of L_1 and L_2 using the Virasoro algebra. For example, $L_3 = [L_2, L_1] = L_2 L_1 - L_1 L_2$. A level one null state can only be of the form

$$|\chi_{1,1}\rangle = \alpha L_{-1} |h\rangle . \tag{3.10}$$

where $|h\rangle = |h_{1,1}\rangle$ and $\alpha \neq 0$. Applying L_1 gives

$$L_1 |\chi_{1,1}\rangle = 0 \implies \alpha L_1 L_{-1} |h\rangle = 2\alpha h |h\rangle = 0.$$

This is consistent with our Gram matrix calculation in Section 3.1.2, $h = h_{1,1} = 0$ is a sufficient condition for reducibility, and for $|\chi_{1,1}\rangle$ to be a singular vector.

For a state at level two, our singular vector $|\chi_{2,1}\rangle$ will be a linear combination of two basis vectors:

$$|\chi_{2,1}\rangle = \left(\alpha L_{-2} + \beta L_{-1}^2\right) |h\rangle ,$$

where $|h_{2,1}\rangle = |h\rangle$ and both α and β cannot be zero. Since the equations we are considering are $L_1 |\chi_{2,1}\rangle = 0$ and $L_2 |\chi_{2,1}\rangle = 0$, we can let $\eta = \beta/\alpha$ assuming $\alpha \neq 0$. This gives our singular vector the form

$$|\chi_{2,1}\rangle = \left(L_{-2} + \eta L_{-1}^2\right) |h\rangle . \tag{3.11}$$

Applying our constraint (3.9), and using our commutator relations from (2.19), we see that

$$\begin{aligned} L_1 |\chi_{2,1}\rangle &= (3 + 2\eta + 4h\eta) L_{-1} |h\rangle = 0, \\ L_2 |\chi_{2,1}\rangle &= \left(\frac{c}{2} + 4h + 6h\eta\right) |h\rangle = 0. \end{aligned}$$

Solving these two linear equations, we get the solutions

$$\eta = -\frac{3}{2(2h+1)}, \qquad h = \frac{5 - c \pm \sqrt{(1-c)(25-c)}}{16}. \tag{3.12}$$

Rewriting these in terms of the parameterisation (3.6), we see that $h = h_{2,1}(t), h_{1,2}(t)$ (3.7). Choosing $h = h_{2,1}(t)$, we find $\eta = -1/t$. This gives our linear combination $L_{r,s}$ for $r = 2$ and $s = 1$ to be

$$\begin{aligned} L_{2,1} &= L_{-2} - \frac{1}{t} L_{-1}^2 \\ \implies |\chi_{2,1}\rangle &= \left(L_{-2} - \frac{1}{t} L_{-1}^2\right) |h\rangle . \end{aligned} \tag{3.13}$$

Remark 17. Letting $h = h_{1,2}(t)$ gives $\eta = -t$ and $|\chi_{1,2}\rangle = \left(L_{-2} - t L_{-1}^2\right) |h\rangle$.

We can use the state-field correspondence (2.29) for $|\chi_{2,1}\rangle$, remembering $L_{-1} \leftrightarrow \partial_z$. In this way, obtain, the *singular field* from $|\chi_{2,1}\rangle$ to be

$$\chi_{2,1}(z) = (L_{-2}\phi)(z) - \frac{1}{t} \partial_z^2 \phi(z). \tag{3.14}$$

where $\partial_z^2 = \partial^2/\partial z^2$. In general, a singular vector $|\phi_{r,s}\rangle$ will have a state-field correspondence to a *singular field* $\chi_{r,s}(z)$ which will obey the relationship

$$\chi_{2,1}(z) = (L_{r,s}\phi_{r,s})(z) = 0,$$

as we set singular fields to zero just like the singular vectors.

3.2 Fusion

Recall that, in principle, any correlation function (2.45) can be computed once we know the correlation functions of primary fields. In Chapter 2, we used the Ward identities to constrain correlators up to the 4-point function, and claimed we could use operator product expansion to reduce higher point correlators to 3-point functions. Taking the operator product expansion in the first two terms in an n-point correlator will reduce it into a $n-1$-point function. Therefore, knowing which fields will appear in the operator product expansion (2.60) will help considerably. This idea of reducing n point correlators to $n-1$ point correlators with operator product expansions leads to the concept of *fusion*, which controls the fields appearing in the operator product expansion.

Consider the 3-point function of primary fields, using our operator product expansion (2.38) we can express this as

$$\langle 0| \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) |0\rangle = \sum_{j=-\infty}^{\infty} \frac{\langle 0| A_j(w)\phi_3(z_3) |0\rangle}{(z_1 - z_2)^{j+1}}.$$

This gives an infinite linear combination of 2-point functions.

Recall from (2.51b) that the two point function is non-zero if and only if $A_j(w)$ and $\phi_3(z_3)$ have the same conformal dimensions. If both fields are self-conjugate, this tells us that $A_j(w)$ must be either $\phi_3(z_3)$ or a descendant of it.

We call the collection of descendants for a primary field $\phi_3(z_3)$ a *family* of fields $[\phi_3]$, generated by $\phi_3(z_3)$. This tells us that the family of $\phi_3(z_3)$ must appear in the operator product expansion of $\phi_1(z_1)$ and $\phi_2(z_2)$ for the 3-point function (2.51c) to be non-zero. In terms of families, we say that $[\phi_3]$ must appear in the fusion of $[\phi_1] \times [\phi_2]$. We claim the generic fusion rule to be of the form

$$[\phi_i] \times [\phi_j] = \sum_k \mathcal{N}_{ij}{}^k [\phi_k], \tag{3.15}$$

where the *fusion coefficients* $\mathcal{N}_{ij}{}^k$ are non-negative integers. For the minimal models, these fusion coefficients will be either 0 or 1.

We will now derive more constraints on fusion rules by considering null fields $\phi_{r,s}$ inserted inside 3-point correlation functions. These fields inserted into correlation functions can be expressed as differential equations acting on the correlation functions.

Firstly, consider the singular vector $\chi_{1,1}(z_1) = (L_{-1}\phi_{1,1})(z_1)$. Setting this to zero, we obtain,

$$\chi_{1,1}(z_1) = (L_{-1}\phi_{1,1})(z_1) = \partial_{z_1}\phi_{1,1}(z_1) = 0.$$

So we conclude that $\phi_{1,1} = \mathbb{1}$ is the identity field.

Next, we will substitute the null field $\phi_{2,1}(z_1)$ into a 3-point correlation function (2.51c) to see which fields are in the operator product expansion that give a non-zero correlator. Using (3.13) for $L_{2,1}$ and (3.7) for $h_{2,1}$, we obtain

$$\langle 0| \left((L_{-2} - \tfrac{1}{t}L_{-1}^2)\phi_{2,1}\right)(z_1)\phi_2(z_2)\phi_3(z_3) |0\rangle = 0.$$

Using (2.56), this can be expressed as

$$\left(\sum_{i=2}^{3}\left[\frac{1}{z_1-z_i}\partial_{z_i}+\frac{h_i}{(z_1-z_i)^2}\right]-\frac{1}{t}\partial_{z_1}^2\right)\langle 0|\,\phi_{2,1}(z_1)\phi_2(z_2)\phi_3(z_3)\,|0\rangle = 0. \tag{3.16}$$

Utilising our results of the 3-point function from conformal invariance of the Ward identities (2.51c), this becomes

$$\begin{aligned}\left(\frac{1}{z_{12}}\partial_{z_2}+\frac{h_2}{z_{12}^2}+\frac{1}{z_{13}}\partial_{z_3}+\frac{h_3}{z_{13}^2}-\frac{1}{t}\partial_{z_1}^2\right)&\\ \cdot\frac{C_{123}}{z_{12}^{h_{2,1}+h_2-h_3}z_{13}^{h_{2,1}-h_2+h_3}z_{23}^{-h_{2,1}+h_2+h_3}}&=0,\end{aligned} \tag{3.17}$$

where we once again use z_{ij} to represent $z_i - z_j$. After computing the derivatives in (3.17), we can send $z_1 \to 0, z_2 \to \infty$ and $z_3 \to 1$, illustrated in Fig. 2.4. This gives the constraint

$$\begin{aligned}\Big(h_{2,1}-h_2+2h_3-&\frac{1}{t}\big(h_{2,1}^2-2h_{2,1}h_2+2h_{2,1}h_3\\ &+h_{2,1}+h_2^2-2h_2h_3-h_2+h_3^2+h_3\big)\Big)=0.\end{aligned}$$

Now we substitute in the conformal dimension of $h_{2,1}(t)$ (3.7) to derive

$$-\frac{3}{8}t^2+t-\frac{1}{2}-t(h_2+h_3)+2(h_2-h_3)^2=0. \tag{3.18}$$

Now, in order to solve this constraint, it is useful to make use of another parameterisation for $h_{r,s}$, this time in terms of a parameter b defined by $t=-b^2$. This can be found in [Rib14]. It gives the conformal dimension the form

$$h_{r,s}=\frac{1}{4}\left((b+b^{-1})^2-\left(rb+sb^{-1}\right)^2\right), \tag{3.19}$$

with $c = 1+6(b+b^{-1})^2$. Looking at (3.19) we can introduce a parameter $P_{r,s}$ to encode the r, s, called the *conformal momentum*

$$\begin{aligned}P_{r,s}&=\frac{1}{2}(rb+sb^{-1}),\\ h_{r,s}&=\frac{1}{4}(b+b^{-1})^2-P_{r,s}^2.\end{aligned} \tag{3.20}$$

Remark 18. $h_{r,s}$ is invariant under $P_{r,s} \leftrightarrow -P_{r,s}$.

Making use of this new parameterisation, we let $h_2 = (b+b^{-1})^2/4 - P_2^2$ and $h_3 = (b+b^{-1})^2/4 - P_3^2$. This will allow us to factorise (3.18) into

$$\begin{aligned}\frac{b^4}{16}+P_3^4-2P_2^2P_3^2+P_4^4-P_2b^2-P_3b^2&=0\\ \left(P_2+P_3+\frac{b}{2}\right)\left(P_2-P_3+\frac{b}{2}\right)\left(P_2+P_3-\frac{b}{2}\right)\left(P_2-P_3-\frac{b}{2}\right)&=0.\end{aligned}$$

Solving this constraint leads to $P_2 = \pm P_3 \pm \frac{b}{2}$, since $P_3 \leftrightarrow -P_3$, we may restrict to

$$P_2 = P_3 \pm \frac{b}{2}. \tag{3.21}$$

Recall a field $[\phi_1]$ is a family of primary fields of conformal dimension h_1. Now let $[\phi_{P_1}]$ be a family of primary fields of conformal momentum P_1. Using (3.21), we have the fusion rule

$$[\phi_{2,1}] \times [\phi_{P_2}] = [\phi_{P_2-\frac{b}{2}}] + [\phi_{P_2+\frac{b}{2}}]. \tag{3.22}$$

Similarly, we can derive the fusion rule for inserting a $\chi_{1,2}(z_1)$ null field into a 3-point correlation function (2.51c), with $L_{1,2} = L_{-2} - tL_{-1}^2$ and $h_{1,2} = -1/2 + 3/4t$. Making the same change of variables we get

$$P_2 = \pm P_3 \pm \frac{1}{2b},$$

with the fusion rule

$$[\phi_{1,2}] \times [\phi_{P_2}] = [\phi_{P_2-\frac{1}{2b}}] + [\phi_{P_2+\frac{1}{2b}}]. \tag{3.23}$$

We can now use the associativity of the operator product expansion (and hence the fusion) to show that $[\phi_{2,1}] \times [\phi_{2,1}]$ must be a sum of finitely many families of fields.

$$\begin{aligned}
([\phi_{2,1}] \times [\phi_{2,1}]) \times [\phi_{2,1}] &= [\phi_{2,1}] \times ([\phi_{2,1}] \times [\phi_{2,1}]) \\
&= [\phi_{2,1}] \times [\phi_{P_{2,1}-\frac{b}{2}}] + [\phi_{2,1}] \times [\phi_{P_{2,1}+\frac{b}{2}}] \\
&= [\phi_{P_{2,1}-b}] + 2[\phi_{P_{2,1}}] + [\phi_{P_{2,1}+b}].
\end{aligned}$$

We also can conclude from (3.22), where $\phi_{P_2} = \phi_{P_{2,1}}$, that $[\phi_{2,1}] \times [\phi_{2,1}]$ must be a sum of exactly 2 families of fields.[1]

$$[\phi_{2,1}] \times [\phi_{2,1}] = [\phi_{P_{2,1}-\frac{b}{2}}] + [\phi_{P_{2,1}+\frac{b}{2}}] \tag{3.24}$$

Using the parameterisation (3.20), we calculate that

$$\begin{aligned}
P_{2,1} + \frac{b}{2} &= \frac{3}{2}b + \frac{1}{2b} = P_{3,1} \\
P_{2,1} - \frac{b}{2} &= \frac{1}{2}\left(b + \frac{1}{b}\right) = P_{1,1}.
\end{aligned}$$

Therefore we have the fusion rule $[\phi_{2,1}] \times [\phi_{2,1}] = [\phi_{P_{1,1}}] + [\phi_{P_{3,1}}]$, which in terms of the conformal dimensions $h_{r,s}$ reads

$$[\phi_{2,1}] \times [\phi_{2,1}] = [\phi_{1,1}] + [\phi_{3,1}]. \tag{3.25}$$

[1]There are exceptions to this, such as the trivial minimal model and the critical Ising model, where there are not enough fields in the theory.

As we can see from (3.25), the first family $[\phi_{2,1}]$ appears to raise and lower the r index in the second family and sum them. Now, let's calculate $[\phi_{2,1}] \times [\phi_{r,s}]$. From (3.24) we receive

$$[\phi_{2,1}] \times [\phi_{r,s}] = [\phi_{P_{r,s}-\frac{b}{2}}] + [\phi_{P_{r,s}+\frac{b}{2}}].$$

Simplifying,

$$\begin{aligned} P_{r,s} - \frac{b}{2} &= \frac{1}{2}((r-1)b + sb^{-1}) = P_{r-1,s} \\ P_{r,s} + \frac{b}{2} &= \frac{1}{2}((r+1)b + sb^{-1}) = P_{r+1,s}. \end{aligned}$$

A similar analysis computes the fusion of $[\phi_{1,2}]$ with $[\phi_{r,s}]$.

Therefore, in terms of the conformal dimension, we are now able to summarise the fusion rules

$$\begin{aligned} [\phi_{2,1}] \times [\phi_{r,s}] &= [\phi_{r-1,s}] + [\phi_{r+1,s}] \\ [\phi_{1,2}] \times [\phi_{r,s}] &= [\phi_{r,s-1}] + [\phi_{r,s+1}]. \end{aligned} \tag{3.26}$$

This tells us that no other family of primary fields other than those on the right-hand side of (3.26) can appear in the operator product expansion or $\phi_{2,1}(z)$ or $\phi_{1,2}(z)$ with $\phi_{r,s}(w)$.

One last feature of fusion worth mentioning is that we can take advantage of the commutativity of the operator algebra $\mathfrak{Vir}$ to simplify some results even further. Using our fusion rules (3.26), we have

$$\begin{aligned} [\phi_{1,2}] \times [\phi_{2,1}] &= [\phi_{2,0}] + [\phi_{2,2}] \\ [\phi_{2,1}] \times [\phi_{1,2}] &= [\phi_{0,2}] + [\phi_{2,2}]. \end{aligned}$$

Since these two OPE's must be equivalent. The coefficients from $\phi_{2,0}$ and $\phi_{0,2}$ must vanish, Hence, leaving us with

$$[\phi_{1,2}] \times [\phi_{2,1}] = [\phi_{2,2}].$$

Other simplifications exist, such as when fields are not defined in the theory, these families will be omitted from any fusion result. There is a general fusion formula for the minimal models (4.4), which we will present in Chapter 4.

To summarise, fusion rules are a method of restricting which families of fields can appear in the operator product expansion when inserted into correlation functions. This will be very useful in solving for structure constants in the minimal models.

3.3 Belavin-Polyakov-Zamolodchikov Equations

We will now show how singular fields can be used to derive differential equations to allow us to solve for the unknown function $F(z)$ in the 4-point function (2.51d) where z is the cross ratio. Consider a 4-point function of fields, one of which is the singular field $\chi_{2,1}$, and the others being primary:

$$\langle 0| \phi_0(z_0)\phi_1(z_1)\chi_{2,1}(z_2)\phi_3(z_3) |0\rangle = 0. \tag{3.27}$$

Using (3.14), we can express this as

$$\begin{gathered}\langle 0|\, \phi_0(z_0)\phi_1(z_1)(L_{2,1}\phi_{2,1})(z_2)\phi_3(z_3)\,|0\rangle = 0\\ \left\{\mathcal{L}_{-2} - \tfrac{1}{t}\partial_{z_2}^2\right\} \langle 0|\, \phi_0(z_0)\phi_1(z_1)\phi_{2,1}(z_2)\phi_3(z_3)\,|0\rangle = 0.\end{gathered}$$

These differential equations are called *Belavin-Polyakov-Zamolodchikov equations*, or BPZ equations. These were first studied by [BPZ84]. Using our (2.56), we see this leads to[2]

$$\left(\frac{\partial^2}{\partial {z_2}^2} - t\sum_{j\in J}\left[\frac{1}{z_2 - z_j}\frac{\partial}{\partial z_j} + \frac{h_j}{(z_2 - z_j)^2}\right]\right) \cdot \langle 0|\, \phi_0(z_0)\phi_1(z_1)\phi_{2,1}(z_2)\phi_3(z_3)\,|0\rangle = 0. \tag{3.28}$$

where $J = \{0, 2, 3\}$. As we have shown in (2.51d), we know the four-point function takes the form

$$\langle \phi_0(z_0)\phi_1(z_1)\phi_{2,1}(z_2)\phi_3(z_3)\rangle = \prod_{0\le i<j\le 3} (z_i - z_j)^{\mu_{ij}}\, G(z), \tag{3.29}$$

where we let

$$\mu_{ij} = \frac{1}{3}\sum_{k=1}^{4} h_k - h_i - h_j. \tag{3.30}$$

Clearly $\mu_{ij} = \mu_{ji}$. We will use the notation $z_{ij} = z_i - z_j$ to simplify the calculations, taking z to be the cross ratio

$$z = \frac{z_{23}z_{10}}{z_{20}z_{13}}. \tag{3.31}$$

Taking (3.29) and substituting it for into (3.28), we can calculate derivatives with $z_0 \to \infty, z_1 \to 1, z_2 \to z$ and $z_3 \to 0$

$$\begin{aligned}\frac{\partial}{\partial z_2} &= \frac{\mu_{23}}{z} + \frac{\mu_{21}}{z-1} + \partial_z\\ \frac{\partial}{\partial z_3} &= -\frac{\mu_{23}}{z} - \mu_{31} + (z-1)\partial_z\\ \frac{\partial}{\partial z_1} &= -\frac{\mu_{21}}{z-1} + \mu_{31} - z\partial_z\\ \frac{\partial}{\partial z_0} &= 0\\ \frac{\partial^2}{\partial {z_2}^2} &= \frac{\mu_{23}(\mu_{23}-1)}{z^2} + \frac{\mu_{21}(\mu_{21}-1)}{(z-1)^2} + \frac{2\mu_{23}\mu_{21}}{z(z-1)}\\ &\quad + 2\left(\frac{\mu_{23}}{z} + \frac{\mu_{21}}{z-1}\right)\partial_z + \partial_z^2.\end{aligned}$$

[2]The choice of inserting $\phi_{2,1}(z_2)$ in the third position will be to preserve radial ordering, despite the awkward sum indices.

Substituting all these into (3.28), we arrive at the equation

$$\left[\frac{1}{t}\partial_z^2 + \frac{2\mu_{23}\mu_{21}}{tz(z-1)} + \left(\frac{2\mu_{23}}{tz} + \frac{2\mu_{21}}{t(z-1)} + \frac{2z-1}{z(z-1)}\right)\partial_z + \frac{\mu_{23}(\mu_{23}-1)}{tz^2} \right.$$
$$\left. + \frac{\mu_{21}(\mu_{21}-1)}{t(z-1)^2} + \frac{\mu_{23}-h_3}{z^2} + \frac{\mu_{21}-h_1}{(z-1)^2} - \frac{\mu_{31}}{z(z-1)}\right]G(z) = 0.$$

Letting $G(z) = z^{-\mu_{23}}(1-z)^{-\mu_{21}}H(z)$, gives us

$$\left[\frac{1}{t}\partial_z^2 + \frac{2z-1}{z(z-1)}\partial_z - \frac{(2z-1)\mu_{23}}{z^2(z-1)} - \frac{(2z-1)\mu_{21}}{z(z-1)^2} \right.$$
$$\left. + \frac{\mu_{23}-h_3}{z^2} + \frac{\mu_{21}-h_1}{(z-1)^2} - \frac{\mu_{31}}{z(z-1)}\right]H(z) = 0.$$

Substituting in our definitions for μ_{ij} in (3.30), we arrive the at the second order BPZ equation

$$\left[\frac{1}{t}\partial_z^2 + \frac{2z-1}{z(z-1)}\partial_z - \frac{h_3}{z^2} - \frac{h_1}{(z-1)^2} + \frac{h_2+h_3+h_1-h_0}{z(z-1)}\right]H(z) = 0. \tag{3.32}$$

This equation can be transformed into a hypergeometric equation, which has well known solutions called *conformal blocks*. Conformal blocks are the building blocks of correlation functions, and finitely many can be used to build correlation functions. More information on conformal blocks can be found in [Run00, MS89, MS90].

Since we have a second order ordinary differential equation (3.32) for a holomorphic correlation function (3.27) it will have two solutions $H_1(z)$ and $H_2(z)$. We can also take a purely antiholomorphic correlation function of the form

$$\langle 0|\,\phi_0(\bar{z}_0)\phi_1(\bar{z}_1)\chi_{2,1}(\bar{z}_2)\phi_3(\bar{z}_3)\,|0\rangle\,,$$

to derive a BPZ equation for $H(\bar{z})$, giving two solutions $H_1(\bar{z})$ and $H_2(\bar{z})$. This two-dimensional space of solutions, will give our correlation function formed from products of these holomorphic and antiholomorphic conformal blocks. In the minimal models, we will be working in a diagonal theory, meaning that the holomorphic and antiholomorphic conformal blocks are the same.

Using (3.29) and $G(z) = z^{-\mu_{23}}(1-z)^{-\mu_{21}}H(z)$, let $F_1(z)$ be the conformal block depending on $H_1(z)$ and $F_2(z)$ the conformal block depending on $H_2(z)$, similarly for the antiholomorphic blocks. This means we have the following expression for a generic 4-point function of primary fields

$$\langle 0|\,\phi_0(\infty,\infty)\phi_1(1,1)\phi_2(z,\bar{z})\phi_3(0,0)\,|0\rangle = \sum_{p,q=1}^{2} C_{p,q}F_p(z)F_q(\bar{z}), \tag{3.33}$$

where $C_{p,q}$ are undetermined constants and $F_p(z)$ and $F_q(\bar{z})$ are our conformal blocks.

Consider the chiral 4-point correlation function

$$\langle 0|\,\phi_0(\infty)\phi_1(1)\phi_2(z)\phi_3(0)\,|0\rangle\,,$$

with $0 < z < 1$. If we consider $z \to 0$, we can take the operator product expansion in the last two fields. On the other hand, if we consider $z \to 1$, we can take the operator product expansion in the second and third field.

Both of these are valid expansions for the correlation function, depending on the value z takes. This idea is illustrated in Fig. 3.2, where the regions of validity for the operator product expansions in the dashed circles. Being able to derive two valid equations for the correlation function will be paramount in deriving the structure constants of the 3 point functions (2.51).

This tells us that whether we evaluate the correlation function as an expansion around $z = 0$ or around $z = 1$, we are computing the same correlation function.

Remark 19. Taking the operator product expansion between any two adjacent fields in the correlator will not change the space of solutions.

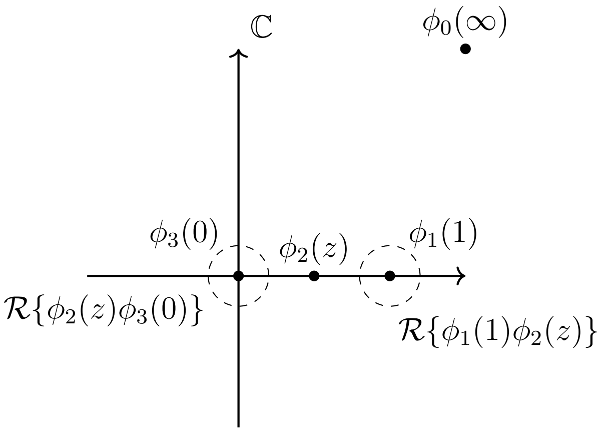

Figure 3.2: 4-point correlator on the complex plane, with regions for operator product expansions around $z = 0$ and $z = 1$ with $0 < z < 1$. Radial ordering has been included to indicate an OPE.

3.4 Hypergeometric Series

Now we know that for the minimal models, conformal blocks can entirely describe a 4-point correlation function, it is time to finally derive constraints for which functions can appear in the decomposition of conformal blocks (3.33). When the correlators obey the BPZ equation (3.32), we know from (2.51d) that a 4-point function can be determined up to an unknown function. This unknown function is transformed to satisfy the *hypergeometric equation*. There are many texts on hypergeometric series, including [GR90, Sea91].

The hypergeometric equation is

$$\left[z(1-z)\frac{\partial^2}{\partial z^2} + [C - (A+B+1)z]\frac{\partial}{\partial z} - AB\right] K(z) = 0, \tag{3.34}$$

for some parameters $A, B, C \in \mathbb{C}$. It has solution

$$F(A, B; C; z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(C)_n} \frac{z^n}{n!}, \tag{3.35}$$

and $z^{1-C}F(1+A-C,1+B-C;2-C;z)$ near $z=0$. See Appendix B for the derivation of these solutions and discussion of hypergeometric identities.

We will now take the equation (3.32) and transform it into a hypergeometric equation (3.34). Letting $H(z)=z^{\beta_1}(1-z)^{\beta_2}K(z)$ for $\beta_1,\beta_2\in\mathbb{R}$ and substituting into (3.32) gives

$$\left[\frac{1}{t}\partial_z^2+\frac{2z-1}{z(z-1)}\partial_z-\frac{h_3}{z^2}-\frac{h_1}{(z-1)^2}+\frac{h_2+h_3+h_1-h_0}{z(z-1)}\right]z^{\beta_1}(1-z)^{\beta_2}K(z)=0.$$

Expanding out the derivatives will give

$$\begin{aligned}&\frac{1}{t}z^{\beta_1-2}(1-z)^{\beta_2-2}\Bigg[\Big(\big(-\beta_1+\beta_1^2+h_3t(z-1)-2\beta_1(\beta_1+\beta_2-1)z\\&\quad+(\beta_1+\beta_2-1)(\beta_1+\beta_2)z^2+t(\beta_1-(h_2+h_1-h_0+3\beta_1+\beta_2)z\\&\quad+(h_2-h_0+2(\beta_1+\beta_2))z^2)\big)K(z)\\&\quad+(z-1)z\big((-t-2\beta_1+2(t+\beta_1+\beta_2)z)K'(z)+(z-1)zK''(z)\big)\Big)\Bigg]=0.\end{aligned}\tag{3.36}$$

Next, we divide equation (3.36) by $-\frac{1}{t}z^{\beta_1-2}(1-z)^{\beta_2-2}z(z-1)$, to force the coefficient of $K''(z)$ and $K'(z)$ to match those of (3.34). This gives us

$$\begin{aligned}&(1-z)zK''(z)+(t+2\beta_1-2(t+\beta_1+\beta_2)z)K'(z)\\&+\frac{1}{z(1-z)}\big(-\beta_1+\beta_1^2+h_3t(z-1)-2\beta_1(\beta_1+\beta_2-1)z+(\beta_1+\beta_2-1)(\beta_1+\beta_2)z^2\\&+t(\beta_1-(h_2+h_1-h_0+3\beta_1+\beta_2)z+(h_2-h_0+2(\beta_1+\beta_2))z^2)\big)K(z).\end{aligned}\tag{3.37}$$

Comparing (3.37) with (3.34), the coefficient of $K'(z)$ gives the constraints

$$\begin{aligned}C&=t+2\beta_1\\A+B&=t+\beta_1+\beta_2.\end{aligned}\tag{3.38}$$

Finally we turn to the term in front of $K(z)$. For this to reduce to a hypergeometric equation, we need this to be of the form $-AB\cdot K(z)$. This gives the equation

$$\begin{aligned}\big(-\beta_1+\beta_1^2&+h_3t(z-1)-2\beta_1(\beta_1+\beta_2-1)z+(\beta_1+\beta_2-1)(\beta_1+\beta_2)z^2\\&+t(\beta_1-(h_2+h_1-h_0+3\beta_1+\beta_2)z\\&\quad+(h_2-h_0+2(\beta_1+\beta_2))z^2)\big)=ABz^2-ABz.\end{aligned}\tag{3.39}$$

Equating coefficients of this quadratic in z gives

$$\begin{aligned}-h_3t-\beta_1+t\beta_1+\beta_1^2&=0\\h_3t-2\beta_1(-1+\beta_1+\beta_2)-t(h_2+h_1-h_0+3\beta_1+\beta_2)&=-AB\\((-1+\beta_1+\beta_2)(\beta_1+\beta_2)+t(h_2-h_0+2(\beta_1+\beta_2)))&=AB.\end{aligned}\tag{3.40}$$

Solving (3.38) and (3.40) simultaneously will give 5 equations with 5 unknowns that are quadratic in nature. This gives eight sets of solutions, which can be reduced to four by making use of the identity $F(A,B;C;z)=F(B,A;C;z)$. Before presenting these solutions, it will be useful to introduce

$$\begin{aligned} d_1 &= \sqrt{1-2t+4h_1t+t^2} \\ d_2 &= \sqrt{1-4t-4h_2t+4h_0t+4t^2} \\ d_3 &= \sqrt{1-2t+4h_3t+t^2}. \end{aligned} \tag{3.41}$$

The remaining four sets of solutions are then

$$\begin{aligned} K_1(z) &= F(\tfrac{1}{2}(1-d_3-d_1-d_2), \tfrac{1}{2}(1-d_3-d_1+d_2); 1-d_3; z) \\ K_2(z) &= F(\tfrac{1}{2}(1-d_3+d_1-d_2), \tfrac{1}{2}(1-d_3+d_1+d_2); 1-d_3; z) \\ K_3(z) &= F(\tfrac{1}{2}(1+d_3-d_1-d_2), \tfrac{1}{2}(1+d_3-d_1+d_2); 1+d_3; z) \\ K_4(z) &= F(\tfrac{1}{2}(1+d_3+d_1-d_2), \tfrac{1}{2}(1+d_3+d_1+d_2); 1+d_3; z). \end{aligned} \tag{3.42}$$

Remembering that $H_i(z)=z^{\beta_1}(1-z)^{\beta_2}K_i(z)$ we get the following solutions to the BPZ equation (3.32)

$$\begin{aligned} H_1(z) &= z^{\frac{1}{2}(1-t-d_3-2\mu_{23})}(1-z)^{\frac{1}{2}(1-t-d_1-2\mu_{21})} \\ &\qquad\qquad F(\tfrac{1}{2}(1-d_3-d_1-d_2), \tfrac{1}{2}(1-d_3-d_1+d_2); 1-d_3; z) \\ H_2(z) &= z^{\frac{1}{2}(1-t-d_3-2\mu_{23})}(1-z)^{\frac{1}{2}(1-t+d_1-2\mu_{21})} \\ &\qquad\qquad F(\tfrac{1}{2}(1-d_3+d_1-d_2), \tfrac{1}{2}(1-d_3+d_1+d_2); 1-d_3; z) \\ H_3(z) &= z^{\frac{1}{2}(1-t+d_3-2\mu_{23})}(1-z)^{\frac{1}{2}(1-t-d_1-2\mu_{21})} \\ &\qquad\qquad F(\tfrac{1}{2}(1+d_3-d_1-d_2), \tfrac{1}{2}(1+d_3-d_1+d_2); 1+d_3; z) \\ H_4(z) &= z^{\frac{1}{2}(1-t+d_3-2\mu_{23})}(1-z)^{\frac{1}{2}(1-t+d_1-2\mu_{21})} \\ &\qquad\qquad F(\tfrac{1}{2}(1+d_3+d_1-d_2), \tfrac{1}{2}(1+d_3+d_1+d_2); 1+d_3; z). \end{aligned} \tag{3.43}$$

Actually, these 4 solutions to the BPZ equation are actually only two solutions, as (3.32) is a second order ordinary differential equation. Using (B.13) we have

$$\begin{aligned} H_1(z) &= z^{\frac{1}{2}(1-t-d_3-2\mu_{23})}(1-z)^{\frac{1}{2}(1-t-d_1-2\mu_{21})} \\ &\qquad F(\tfrac{1}{2}(1-d_3-d_1-d_2), \tfrac{1}{2}(1-d_3-d_1+d_2); 1-d_3; z) \\ &= z^{\frac{1}{2}(1-t-d_3-2\mu_{23})}(1-z)^{\frac{1}{2}(1-t-d_1-2\mu_{21})} z^{1-d_3-\frac{1}{2}(1-d_3-d_1-d_2)-\frac{1}{2}(1-d_3-d_1+d_2)} \\ &\quad F(1-d_3-\tfrac{1}{2}(1-d_3-d_1-d_2), 1-d_3-\tfrac{1}{2}(1-d_3-d_1+d_2); 1-d_3; z) \\ &= z^{\frac{1}{2}(1-t+d_3-2\mu_{23})}(1-z)^{\frac{1}{2}(1-t-d_1-2\mu_{21})} \\ &\qquad F(\tfrac{1}{2}(1+d_3-d_1-d_2), \tfrac{1}{2}(1+d_3-d_1+d_2); 1+d_3; z) \\ &= H_3(z). \end{aligned}$$

A similar calculation shows $H_2(z)=H_4(z)$. This leaves us with just two linearly independent solutions to (3.32), as expected.

Now, using our 4-point function (2.51d) we can expand our correlator with $z_1 \to 0, z_2 \to 1$ and $z_0 \to \infty$ as

$$F(z) = G(z)z^{\mu_{23}}(1-z)^{\mu_{21}}.$$

Remembering that $G(z) = z^{-\mu_{23}}(1-z)^{-\mu_{21}}H(z)$, and taking $H_1(z)$ and $H_3(z)$ from (3.43) we finally have two linearly independent conformal blocks

$$\begin{aligned} F_1(z) &= z^{\frac{1}{2}(1-t-d_3)}(1-z)^{\frac{1}{2}(1-t-d_1)} \\ &\qquad F(\tfrac{1}{2}(1-d_3-d_1-d_2), \tfrac{1}{2}(1-d_3-d_1+d_2); 1-d_3; z) \\ F_2(z) &= z^{\frac{1}{2}(1-t+d_3)}(1-z)^{\frac{1}{2}(1-t-d_1)} \\ &\qquad F(\tfrac{1}{2}(1+d_3-d_1-d_2), \tfrac{1}{2}(1+d_3-d_1+d_2); 1+d_3; z). \end{aligned} \tag{3.44}$$

Now denoting $F^{(4)}(z,\bar{z})$ to be a 4 point correlation function with $z_1 \to \infty, z_2 \to 1, z_3 \to z$ and $z_4 \to 0$, this gives the final 4-point correlation function the decomposition:

$$F^{(4)}(z,\bar{z}) = \sum_{p,q=1}^{2} C_{p,q} F_p(z) F_q(\bar{z}), \tag{3.45}$$

for F_p in (3.44). We now have all the tools required to compute structure constants C_{ijk} in the minimal models.

Remark 20. While we take $H_1(z)$ and $H_3(z)$ in (3.44), and will be used in general, we will take different solutions from (3.43) to solve the Yang-Lee singularity and the Ising model. This is to match the analysis in [DMS97].

Chapter 4

Structure Constants of Minimal Models

In this chapter we will take all our operator formalism and tools of conformal field theory to solve some examples of the minimal models, meaning we will find the 3 point constants as determined by the 3-point correlator. We will begin by describing the formalism of the minimal models, then look at two examples in detail. Firstly the Yang-Lee singularity, then the Ising model, then more general cases. For a summary of the minimal models, see [DMS97]. They are also discussed in [Rib14] and [Run00, MS89, MS90, BPZ84].

4.1 Minimal Models

The minimal models are a collection of *rational conformal field theories* that are built up of finitely many irreducible Virasoro representations. A minimal model $\mathrm{M}(p,q)$ is parameterised by two co-prime integers p and q. In this thesis, we will adopt the convention that $p < q$. In the minimal models, a primary field $\phi_{r,s}$ will have conformal weight $h_{r,s}$. The values r and s have restrictions such that $1 \leq r < q$ and $1 \leq s < p$.

The spectrum of the quantum state space of the minimal models will be

$$\mathcal{S} \cong \bigoplus_{r,s} V_{h_{r,s}} \otimes \overline{V}_{h_{r,s}}. \tag{4.1}$$

Here, r and s will take values in the Kac table Fig. 4.1. This is a theorem of Wang [Wan93], but was used in [BPZ84].

These minimal models are *diagonal*, or A-type, since both factors of the tensor product in (4.1) are identical. So summing over all allowed values of r and s would count each representation twice.

Remark 21. In the Virasoro minimal models, a primary field $\phi(z)$ is equal to its conjugate, $\phi^*(z) = \phi(z)$.

In these conformal field theories, the central charge is parameterised by the two co-prime integers p, q by

$$c = 1 - 6\frac{(p-q)^2}{pq}. \tag{4.2}$$

The conformal weight of a field $\phi_{r,s}$ is

$$h_{r,s} = \frac{(pr - qs)^2 - (p-q)^2}{4pq}. \tag{4.3}$$

Consider a family of conformal fields $[\phi_{r,s}]$. The allowed values r, s can be plotted as in Fig. 4.1.

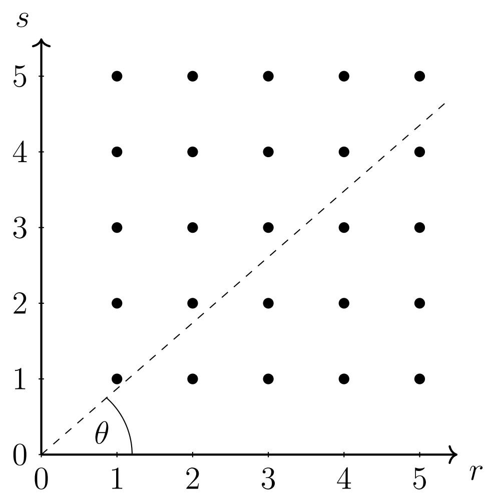

Figure 4.1: Kac table for a general value c. Each vertex (r, s) is associated with a conformal field $\phi_{r,s}$ of conformal dimension $h_{r,s}$. Each field above the line has a corresponding field below the line, which are identified in the minimal model.

If the slope of the dashed line $\tan\theta$ is irrational, then the dashed line in (4.1) will never pass through a vertex (r, s), except for the origin of course. We let the gradient of the line be $\tan\theta = p/q$ and we set $t = p/q$. Our central charge (4.2) and conformal dimension (4.3) becomes (3.6).

From (4.3), we can see that the conformal dimension satisfies the identity $h_{r,s} = h_{q-r,p-s}$. Therefore $\phi_{r,s} = \phi_{q-r,p-s}$.

Remark 22. The collection of fields above the dashed line in Fig. 4.1 will each be unique, with a corresponding field below the line.

Consider the field $\phi_{2,1}(z)$, with z being a cross ratio. A 4-point correlation function containing this field will have its unknown function of the cross ratio obey the BPZ equation (3.32). Therefore, any correlator that we consider will be able to be expressed in terms of a linear combination of conformal blocks (3.33). We will compare this with (2.51d), and take operator product expansions, giving expansion of correlators, controlled by fusion.

Fusion between two general fields in the minimal models is given by

$$\phi_{(r_1,s_1)} \times \phi_{(r_2,s_2)} = \sum_{k\overset{2}{=}|r_1-r_2|+1}^{k_{\max}} \sum_{l\overset{2}{=}|s_1-s_2|+1}^{l_{\max}} \phi_{(k,l)}. \tag{4.4}$$

where the superscript 2 above an equals sign indicates that k and l increment by 2 instead of 1. We also have

$$\begin{aligned} k_{\max} &= \min(r_1 + r_2 - 1, 2q - r_1 - r_2 - 1) \\ l_{\max} &= \min(r_1 + r_2 - 1, 2q - s_1 - s_2 - 1). \end{aligned} \tag{4.5}$$

This is stated and proved in [DMS97].

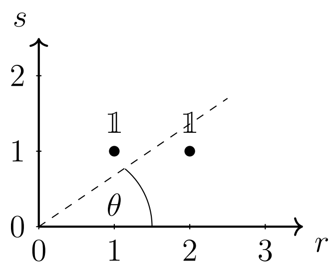

Figure 4.2: M(2,3) Kac Table

The simplest minimal model is the trivial minimal model $M(2,3)$ with central charge $c = 0$. The only fusion rule is $\mathbb{1} \times \mathbb{1} = \mathbb{1}$. The only structure constant is the trivial three point structure constant $C_{\mathbb{1}\mathbb{1}\mathbb{1}} = 1$.

We will also assume that our structure constants C_{ijk} are invariant under permutation of the fields. Note that any 3-point structure constant involving the identity field $C_{ij\mathbb{1}}$ will be 1 or 0, as fusion gives $\langle \phi_i \phi_j \mathbb{1} \rangle = \langle \phi_i \phi_j \rangle = \delta_{ij}$, which have all constants normalised to 1 by (2.58).

4.2 Structure Constants

4.2.1 Yang-Lee Singularity: M(2,5)

We will now look at our first example of calculating a three point structure constant. A study on the Yang-Lee singularity can be found in [Fis78], and its association to the minimal model $M(2,5)$ is due to [Car85]. This minimal model $M(2,5)$, has central charge

$$c = 1 - 6\frac{(5-2)^2}{10} = -\frac{22}{5}.$$

From the symmetry $\phi_{r,s} = \phi_{5-r,2-s}$, $\phi_{2,1} = \phi_{3,1}$ and $\phi_{1,1} = \phi_{4,1}$. This is indicated by the dashed line in Fig. 4.3, where the fields are labelled $\phi_{1,1} = \mathbb{1}$ and $\phi_{2,1} = \Phi$.

The conformal weight of $h_{\mathbb{1}} = 0$ as this is the identity field, and the conformal weight of Φ is $h_\Phi = -\frac{1}{5}$ using (3.6).

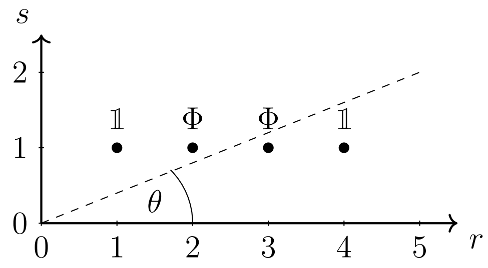

Figure 4.3: M(2,5) Kac Table

The fusion rules for these fields can be calculated using (3.26).

$$\begin{aligned} \mathbb{1} \times \mathbb{1} &= \mathbb{1} \\ \Phi \times \mathbb{1} &= \Phi \\ \Phi \times \Phi &= \mathbb{1} + \Phi. \end{aligned} \tag{4.6}$$

Any structure constant involving the identity will simply be 1, unless it is of the form ${C_{\Phi \mathbb{1}}}^{\mathbb{1}}$, in which case it will be zero (2.58). The aim of this calculation is to compute the only non-trivial structure constant of the Yang-Lee singularity: ${C_{\Phi\Phi}}^{\Phi} = C_{\Phi\Phi\Phi}$ (2.63). The only fusion rule which will give this constant is $\Phi \times \Phi = \mathbb{1} + \Phi$.

Now consider the 4 point correlation function, where every field is Φ:

$$F^{(4)}(z, \bar{z}) = \lim_{w \to \infty} |w|^{-\frac{4}{5}} \langle 0| \Phi(w, \bar{w}) \Phi(1,1) \Phi(z, \bar{z}) \Phi(0,0) |0\rangle \,. \tag{4.7}$$

Taking the operator product expansion with the last two fields

$$F^{(4)}(z, \bar{z}) = \lim_{w \to \infty} |w|^{-\frac{4}{5}} \langle 0| \Phi(w, \bar{w}) \Phi(1,1) \sum_k \frac{{C_{\Phi\Phi}}^{k} \phi_k(0,0)}{|z|^{2h_\Phi + 2h_\Phi - 2h_k}} |0\rangle + \cdots ,$$

where k is summed over the primary fields in the Kac table in Fig. 4.3. From the fusion rules for M(2, 5), we know that this sum gets contributions from two primary fields: $\mathbb{1}$ and Φ. As stated in Remark 16, we will not consider descendants, as they will not be important for this calculation. They will merely be represented as "$\ldots$".

This gives us

$$\begin{aligned} F^{(4)}(z, \bar{z}) &= \lim_{w \to \infty} |w|^{-\frac{4}{5}} \langle 0| \Phi(w, \bar{w}) \Phi(1,1) \frac{{C_{\Phi\Phi}}^{\mathbb{1}} \mathbb{1}(0,0)}{|z|^{2h_\Phi + 2h_\Phi - 2h_{\mathbb{1}}}} |0\rangle + \cdots \\ &\quad + \lim_{w \to \infty} |w|^{-\frac{4}{5}} \langle 0| \Phi(w, \bar{w}) \Phi(1,1) \frac{{C_{\Phi\Phi}}^{\Phi} \Phi(0,0)}{|z|^{2h_\Phi + 2h_\Phi - 2h_{\mathbb{1}}}} |0\rangle + \cdots \\ &= \lim_{w \to \infty} |w|^{-\frac{4}{5}} \langle 0| \Phi(w, \bar{w}) \Phi(1,1) |0\rangle |z|^{4/5} + \cdots \\ &\quad + \lim_{w \to \infty} |w|^{-\frac{4}{5}} {C_{\Phi\Phi}}^{\Phi} \langle 0| \Phi(w, \bar{w}) \Phi(1,1) \Phi(0,0) |0\rangle |z|^{2/5} + \cdots , \end{aligned}$$

where we have used ${C_{\Phi\Phi}}^{\mathbb{1}} = 1$.

Computing the two and three point functions from (2.51b) and (2.51c), we find

$$\begin{aligned} F^{(4)}(z,\bar{z}) &= \lim_{w\to\infty} |w|^{-\frac{4}{5}} |w|^{4/5}|z|^{4/5} + \cdots \\ &\quad + \lim_{w\to\infty} |w|^{-\frac{4}{5}} C_{\Phi\Phi}{}^{\Phi} C_{\Phi\Phi\Phi} |w|^{2/5}|w-1|^{2/5}|z|^{2/5} + \cdots \\ &= |z|^{4/5} + \cdots + C_{\Phi\Phi\Phi}^2 |z|^{2/5} + \cdots , \end{aligned} \tag{4.8}$$

where we used the result that $C_{\Phi\Phi\Phi} = C_{\Phi\Phi}{}^{\Phi}$ (2.63).

Now, as discussed in Chapter 3, the expansions we find from operator product expansions must be the same as the conformal block decomposition (3.33). Using our form of the 4 point function (2.51d) we see

$$\begin{aligned} F^{(4)}(z,\bar{z}) &= \lim_{w\to\infty} |w|^{-4/5} \langle 0| \, \Phi(w,\bar{w})\Phi(1,1)\Phi(z,\bar{z})\Phi(0,0) \, |0\rangle \\ &= \lim_{w\to\infty} G(z)|w|^{-4/5}|w-1|^{4/15}|w-z|^{4/15}|w|^{4/15}|1-z|^{4/15}|z|^{4/15} \\ &= G(z)|1-z|^{4/15}|z|^{4/15}, \end{aligned}$$

where $G(z)$ is a function of the cross ratio z. Since our correlator (4.7) contains a $\phi_{2,1}$, we can make use of our BPZ equation (3.32) to solve for $H(z)$. Taking (3.44), and identifying

$$d_1 = d_2 = d_3 = \frac{1}{5},$$

our correlator is built from two functions[1]

$$\begin{aligned} F_1(z) &= z^{2/5}(1-z)^{2/5} F\left(\tfrac{3}{5}, \tfrac{4}{5}; \tfrac{6}{5}; z\right) \\ F_2(z) &= z^{1/5}(1-z)^{1/5} F\left(\tfrac{2}{5}, \tfrac{3}{5}; \tfrac{4}{5}; z\right), \end{aligned} \tag{4.9}$$

giving our conformal block decomposition (3.33) to be

$$F^{(4)}(z,\bar{z}) = \sum_{i,j=1}^{2} C_{i,j} F_i(z) F_j(\bar{z}), \tag{4.10}$$

for the conformal blocks in (4.9). Expanding our (4.10), we find

$$\begin{aligned} F^{(4)}(z,\bar{z}) &= C_{1,1}|z|^{4/5}|1-z|^{4/5} \left| F\left(\tfrac{3}{5}, \tfrac{4}{5}; \tfrac{6}{5}; z\right)\right|^2 \\ &\quad + C_{1,2} z^{2/5}(1-z)^{2/5} F\left(\tfrac{3}{5}, \tfrac{4}{5}; \tfrac{6}{5}; z\right) \bar{z}^{1/5}(1-\bar{z})^{1/5} F\left(\tfrac{2}{5}, \tfrac{3}{5}; \tfrac{4}{5}; \bar{z}\right) \\ &\quad + C_{2,1} z^{1/5}(1-z)^{1/5} F\left(\tfrac{2}{5}, \tfrac{3}{5}; \tfrac{4}{5}; z\right) \bar{z}^{2/5}(1-\bar{z})^{2/5} F\left(\tfrac{3}{5}, \tfrac{4}{5}; \tfrac{6}{5}; \bar{z}\right) \\ &\quad + C_{2,2}|z|^{2/5}|1-z|^{2/5} \left| F\left(\tfrac{2}{5}, \tfrac{3}{5}; \tfrac{4}{5}; z\right)\right|^2 . \end{aligned}$$

Imposing the condition that this must be single valued, as per Remark 14, we set $C_{1,2} = 0$ and $C_{2,1} = 0$. This leaves us with the decomposition

$$\begin{aligned} F^{(4)}(z,\bar{z}) &= C_{1,1}|z|^{4/5}|1-z|^{4/5} \left| F\left(\tfrac{3}{5}, \tfrac{4}{5}; \tfrac{6}{5}; z\right)\right|^2 \\ &\quad + C_{2,2}|z|^{2/5}|1-z|^{2/5} \left| F\left(\tfrac{2}{5}, \tfrac{3}{5}; \tfrac{4}{5}; z\right)\right|^2 . \end{aligned}$$

[1] In this calculation we have taken our linearly independent solutions to be $H_2(z)$ and $H_4(z)$ in (3.43) as in [DMS97].

Now we expand this function around $z = 0$ using (B.5) to arrive at

$$F^{(4)}(z,\bar{z}) = C_{1,1}|z|^{4/5} + C_{2,2}|z|^{2/5} + \cdots .$$

Comparing this to (4.8), we find

$$C_{1,1} = 1, \quad C_{2,2} = C_{\Phi\Phi\Phi}^2. \tag{4.11}$$

This leaves us with the linear combination of conformal blocks

$$\begin{aligned} F^{(4)}(z,\bar{z}) = {} & |z(1-z)|^{4/5} \left|F\left(\tfrac{3}{5},\tfrac{4}{5};\tfrac{6}{5};z\right)\right|^2 \\ & + C_{\Phi\Phi\Phi}^2|z(1-z)|^{2/5}\left|F\left(\tfrac{2}{5},\tfrac{3}{5};\tfrac{4}{5};z\right)\right|^2. \end{aligned} \tag{4.12}$$

This expansion can also be taken around $z = 1$, as per Remark 19. We have the transformation property (B.15) for hypergeometric functions. Taking (4.12) and splitting the hypergeometric functions into holomorphic and antiholomorphic functions

$$\begin{aligned} F^{(4)}(z,\bar{z}) = {} & |z(1-z)|^{4/5} F\left(\tfrac{3}{5},\tfrac{4}{5};\tfrac{6}{5};z\right) F\left(\tfrac{3}{5},\tfrac{4}{5};\tfrac{6}{5};\bar{z}\right) \\ & + C_{\Phi\Phi\Phi}^2|z(1-z)|^{2/5} F\left(\tfrac{2}{5},\tfrac{3}{5};\tfrac{4}{5};z\right) F\left(\tfrac{2}{5},\tfrac{3}{5};\tfrac{4}{5};\bar{z}\right). \end{aligned} \tag{4.13}$$

We now apply (B.15) to expand the correlator around $z = 1$ instead of $z = 0$ to get

$$\begin{aligned} & F^{(4)}(z,\bar{z}) = |1-z|^{4/5} \\ & \cdot \left[\left(\frac{\Gamma(\frac{6}{5})\Gamma(-\frac{1}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{2}{5})} F\left(\tfrac{3}{5},\tfrac{4}{5};\tfrac{6}{5};1-z\right) + \frac{\Gamma(\frac{6}{5})\Gamma(\frac{1}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})}(1-z)^{-1/5} F\left(\tfrac{3}{5},\tfrac{2}{5};\tfrac{4}{5};1-z\right) \right) \right. \\ & \left. \left(\frac{\Gamma(\frac{6}{5})\Gamma(-\frac{1}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{2}{5})} F\left(\tfrac{3}{5},\tfrac{4}{5};\tfrac{6}{5};1-\bar{z}\right) + \frac{\Gamma(\frac{6}{5})\Gamma(\frac{1}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})}(1-\bar{z})^{-1/5} F\left(\tfrac{3}{5},\tfrac{2}{5};\tfrac{4}{5};1-\bar{z}\right) \right) \right] + \ldots \\ & + C_{\Phi\Phi\Phi}^2|1-z|^{2/5} \\ & \cdot \left[\left(\frac{\Gamma(\frac{4}{5})\Gamma(-\frac{1}{5})}{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})} F\left(\tfrac{3}{5},\tfrac{4}{5};\tfrac{6}{5};1-z\right) + \frac{\Gamma(\frac{4}{5})\Gamma(\frac{1}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{2}{5})}(1-z)^{-1/5} F\left(\tfrac{3}{5},\tfrac{2}{5};\tfrac{4}{5};1-z\right) \right) \right. \\ & \left. \left(\frac{\Gamma(\frac{4}{5})\Gamma(-\frac{1}{5})}{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})} F\left(\tfrac{3}{5},\tfrac{4}{5};\tfrac{6}{5};1-\bar{z}\right) + \frac{\Gamma(\frac{4}{5})\Gamma(\frac{1}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{2}{5})}(1-\bar{z})^{-1/5} F\left(\tfrac{3}{5},\tfrac{2}{5};\tfrac{4}{5};1-\bar{z}\right) \right) \right] + \ldots. \end{aligned}$$

Expanding out the brackets, and setting to zero any term that does not consist of absolute values of z only, as per Remark 14, we get the constraint

$$\frac{\Gamma(\frac{6}{5})^2\Gamma(-\frac{1}{5})\Gamma(\frac{1}{5})}{\Gamma(\frac{3}{5})^2\Gamma(\frac{2}{5})\Gamma(\frac{4}{5})} + C_{\Phi\Phi\Phi}^2 \frac{\Gamma(\frac{4}{5})^2\Gamma(-\frac{1}{5})\Gamma(\frac{1}{5})}{\Gamma(\frac{2}{5})^2\Gamma(\frac{3}{5})\Gamma(\frac{1}{5})} = 0.$$

Solving this, we find

$$C_{\Phi\Phi\Phi}^2 = -\frac{\Gamma(\frac{6}{5})^2\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})^3} \approx -3.65312. \tag{4.14}$$

This result has been reconciled numerically in [MW08], via transfer matrix and finite-size scaling techniques.

We are only able to ever solve the square of the three point constant, as no other equations exist to deduce whether the positive or negative square root result is correct. This is because we can multiply a primary field ϕ by -1, and it is still primary and the two point function (2.58) is normalised in the same way. However, under this multiplication, the operator product expansion constants change as ${C_{ij}}^k \mapsto -{C_{ij}}^k$.

Having found the only non-trivial three point constant, we have solved the minimal model M(2,5).

4.2.2 Critical Ising Model: M(3,4)

The simplest non-trivial unitary minimal model is the critical Ising model. The Ising model connection to the minimal model M(3,4) is due to [BPZ84], it models magnetic atomic spin.

This is perhaps one of the most famous statistical models, we aim to compute the structure constants in the field theory. The central charge of this theory is

$$c = 1 - 6\frac{(4-3)^2}{12} = \frac{1}{2}.$$

The Kac table is given below, where there are now 3 primary fields, $\phi_{(1,1)} = \mathbb{1}$, $\phi_{(2,1)} = \sigma$ and $\phi_{(2,2)} = \varepsilon$. We can set these fields

$$\begin{aligned} \phi_{1,1} &= \phi_{4-1,3-1} = \phi_{3,2} = \mathbb{1} \\ \phi_{1,2} &= \phi_{4-1,3-1} = \phi_{3,1} = \varepsilon \\ \phi_{2,2} &= \phi_{4-2,3-2} = \phi_{2,1} = \sigma. \end{aligned}$$

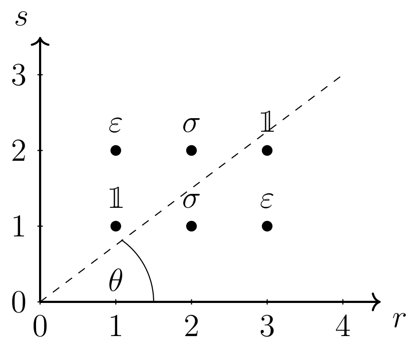

Figure 4.4: M(3,4) Kac Table

The fusion rules are

$$\begin{aligned} \sigma \times \sigma &= \mathbb{1} + \varepsilon \\ \sigma \times \varepsilon &= \sigma \\ \varepsilon \times \varepsilon &= \mathbb{1}. \end{aligned} \tag{4.15}$$

From the fusion rules, there is only one non-zero structure constant that is not simply 1 or 0: ${C_{\sigma\sigma}}^{\varepsilon}$. Consider the following correlation function,

$$F^{(4)}(z,\bar{z}) = \lim_{w\to\infty} |w|^{1/4} \langle 0| \sigma(w,\bar{w})\sigma(1,1)\sigma(z,\bar{z})\sigma(0,0) |0\rangle .$$

The method for obtaining an expansion will be identical to the Yang-Lee singularity (4.8). Taking the operator product expansion of $\sigma(z,\bar{z})\sigma(0,0)$ we obtain the expansion

$$
\begin{aligned}
F^{(4)}(z,\bar{z}) &= \lim_{w\to\infty} |w|^{1/4} \langle 0|\, \sigma(w,\bar{w})\sigma(1,1)\sigma(z,\bar{z})\sigma(0,0)\, |0\rangle \\
&= \lim_{w\to\infty} \left(\frac{|w|^{1/4}}{|z|^{1/4}} \langle 0|\, \sigma(w,\bar{w})\sigma(1,1)\mathbb{1}(0,0)\, |0\rangle + \cdots \right. \\
&\qquad \left. + \frac{C_{\sigma\sigma}{}^{\varepsilon}|w|^{1/4}}{|z|^{-3/4}} \langle 0|\, \sigma(w,\bar{w})\sigma(1,1)\varepsilon(0,0)\, |0\rangle + \cdots \right) \\
&= \lim_{w\to\infty} \left(\frac{|w|^{1/4}}{|z|^{1/4}|w-1|^{1/4}} + \cdots + \frac{C_{\sigma\sigma\varepsilon}^2 |w|^{1/4}|z|^{3/4}}{|w-1|^{1/8}|w|^{1/8}} + \cdots \right) \\
&= \frac{1}{|z|^{1/4}} + \cdots + C_{\sigma\sigma\varepsilon}^2 |z|^{3/4} + \cdots .
\end{aligned} \tag{4.16}
$$

The 4-point function has the form (3.33). Using (3.44) we get the decomposition[2]

$$
\begin{aligned}
F^{(4)}(z,\bar{z}) &= C_{1,1}|z(1-z)|^{-1/4} \left| F\left(\tfrac{3}{4},\tfrac{1}{4};\tfrac{1}{2};z\right)\right|^2 \\
&\quad + C_{2,2}|z(1-z)|^{3/4} \left| F\left(\tfrac{1}{4},\tfrac{3}{4};\tfrac{3}{2};z\right)\right|^2 .
\end{aligned} \tag{4.17}
$$

where we have set $C_{1,2} = 0$ and $C_{2,1} = 0$ to ensure the correlator is single valued. Expanding (4.17) and comparing with (4.16), we once again read off

$$
C_{1,1} = 1, \quad C_{2,2} = C_{\sigma\sigma\varepsilon}^2 .
$$

This gives our correlator

$$
\begin{aligned}
F^{(4)}(z,\bar{z}) &= |z(1-z)|^{-1/4} \left| F\left(\tfrac{3}{4},\tfrac{1}{4};\tfrac{1}{2};z\right)\right|^2 \\
&\quad + C_{\sigma\sigma\varepsilon}^2|z(1-z)|^{3/4} \left| F\left(\tfrac{1}{4},\tfrac{3}{4};\tfrac{3}{2};z\right)\right|^2 .
\end{aligned} \tag{4.18}
$$

Expanding (4.18) around $z = 1$ using (B.15) and setting the terms that are not entirely absolute values of z to zero, we get the equation

$$
\begin{aligned}
C_{\sigma\sigma\varepsilon}^2 \frac{\Gamma(\frac{3}{2})^2\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})^2\Gamma(\frac{3}{4})} &= -\frac{\Gamma(\frac{1}{2})^3\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})^2\Gamma(-\frac{1}{4})} \\
C_{\sigma\sigma\varepsilon}^2 &= -\frac{\Gamma(\frac{3}{4})}{\Gamma(-\frac{1}{4})}.
\end{aligned}
$$

where we have used the gamma function recursion relation $\Gamma(z+1) = z\Gamma(z)$. Evaluating the gamma functions, we are left with

$$
C_{\sigma\sigma\varepsilon}^2 = \frac{1}{4}. \tag{4.19}
$$

[2] In this calculation we have taken our linearly independent solutions to be $H_2(z)$ and $H_3(z)$ in (3.43) as in [DMS97].

4.3 More Structure Constants

4.3.1 A General Structure Constant: M(p,q)

We will now generalise the method used for our previous two examples. Suppose we want to solve a minimal model $\mathrm{M}(p,q)$ with a correlator

$$F^{(4)}(z,\bar{z}) = \lim_{w\to\infty} |w|^{4h_{2,1}} \langle 0|\, \phi_{2,1}(w,\bar{w})\phi_{2,1}(1,1)\phi_{2,1}(z,\bar{z})\phi_{2,1}(0,0)\,|0\rangle\,.$$

As before, we can fuse the last two fields to give

$$\begin{aligned}
F^{(4)}(z,\bar{z}) &= \lim_{w\to\infty}\Bigg(\frac{1}{|z|^{4h_{21}}}|w|^{4h_{2,1}}\langle 0|\,\phi_{2,1}(w,\bar{w})\phi_{2,1}(1,1)\phi_{(1,1)}(0,0)\,|0\rangle+\cdots\\
&\quad+\frac{C_{(2,1)(2,1)}{}^{(3,1)}}{|z|^{4h_{2,1}-2h_{3,1}}}|w|^{4h_{2,1}}\langle 0|\,\phi_{2,1}(w,\bar{w})\phi_{2,1}(1,1)\phi_{(3,1)}(0,0)\,|0\rangle+\cdots\Bigg)\\
&= \lim_{w\to\infty}\Bigg(\frac{1}{|z|^{4h_{21}}}|w|^{4h_{2,1}}\langle 0|\,\phi_{2,1}(w,\bar{w})\phi_{2,1}(1,1)\,|0\rangle+\cdots\\
&\quad+\frac{C_{(2,1)(2,1)}{}^{(3,1)}}{|z|^{4h_{2,1}-2h_{3,1}}}|w|^{4h_{2,1}}\langle 0|\,\phi_{2,1}(w,\bar{w})\phi_{2,1}(1,1)\phi_{(3,1)}(0,0)\,|0\rangle+\cdots\Bigg)\\
&= \frac{1}{|z|^{4h_{21}}}+\cdots+\frac{\left(C_{(2,1)(2,1)}{}^{(3,1)}\right)^2}{|z|^{4h_{2,1}-2h_{3,1}}}+\cdots\,.
\end{aligned}$$

Using (3.6) to compute $h_{2,1}$ and $h_{3,1}$ in terms of t, we are left with the expansion

$$F^{(4)}(z,\bar{z}) = |z|^{2-3t}+\cdots+\left(C_{(2,1)(2,1)}{}^{(3,1)}\right)^2|z|^t+\cdots\,. \tag{4.20}$$

Now using (3.44) we know the correlator takes the form[3]

$$\begin{aligned}
F^{(4)}(z,\bar{z}) = C_{1,1}&|z|^{1-t-d_3}|1-z|^{1-t-d_1}\\
&\cdot|F(\tfrac{1}{2}(1-d_3-d_1-d_2),\tfrac{1}{2}(1-d_3-d_1+d_2);1-d_3;z)|^2\\
+C_{2,2}&|z|^{1-t+d_3}|1-z|^{1-t-d_1}\\
&\cdot|F(\tfrac{1}{2}(1+d_3-d_1-d_2),\tfrac{1}{2}(1+d_3-d_1+d_2);1+d_3;z)|^2,
\end{aligned}$$

where we have set $C_{1,2}=0$ and $C_{2,1}=0$. Computing

$$d_1=d_2=d_3=\sqrt{(1-2t)^2}=1-2t,$$

where we will take the positive square root, as the negative case will just switch the powers of the leading order of the $|z|$ terms, this gives us the following correlator

$$\begin{aligned}
F^{(4)}(z,\bar{z}) = C_{1,1}|z|^t|1-z|^t|F(3t-1,t;2t;z)|^2&\\
+C_{2,2}|z|^{2-3t}|1-z|^t|F(t,1-t;2-2t;z)|^2&.
\end{aligned}$$

Expanding around $z=0$ we have

$$F^{(4)}(z,\bar{z}) = C_{1,1}|z|^t+C_{2,2}|z|^{2-3t}+\cdots\,. \tag{4.21}$$

[3] We will use $H_1(z)$ and $H_3(z)$ from (3.43) as in (3.44) for the remainder of the thesis.

As usual we compare our expansions (4.20) and (4.21) to conclude

$$C_{1,1} = \left(C_{(2,1)(2,1)}{}^{(3,1)}\right)^2, \quad C_{2,2} = 1. \tag{4.22}$$

Now taking the correlator

$$F^{(4)}(z,\bar{z}) = \left(C_{(2,1)(2,1)}{}^{(3,1)}\right)^2 |z|^t |1-z|^t |F(3t-1,t;2t;z)|^2 + |z|^{2-3t}|1-z|^t|F(t,1-t;2-2t;z)|^2, \tag{4.23}$$

expanding around $z = 1$ using (B.15) and setting the terms that are not single valued to zero gives the equation

$$\left(C_{(2,1)(2,1)}{}^{(3,1)}\right)^2 \frac{\Gamma(2t)^2\Gamma(1-2t)\Gamma(2t-1)}{\Gamma(3t-1)\Gamma(t)^2\Gamma(1-t)} = -\frac{\Gamma(2-2t)^2\Gamma(1-2t)\Gamma(2t-1)}{\Gamma(t)\Gamma(1-t)^2\Gamma(2-3t)}. \tag{4.24}$$

Solving (4.24) we have

$$\left(C_{(2,1)(2,1)}{}^{(3,1)}\right)^2 = -\frac{\Gamma(2-2t)^2\Gamma(t)\Gamma(3t-1)}{\Gamma(2t)^2\Gamma(1-t)\Gamma(2-3t)}. \tag{4.25}$$

Substituting $t = 2/5$ and $t = 3/4$ into (4.25) recovers (4.14) and (4.19) respectively. We now know the structure constants $(C_{(2,1)(2,1)}{}^{(3,1)})^2$ for any minimal model $\mathrm{M}(p,q)$.

4.3.2 Coupled Constants: M(3,5)

So far, the minimal models we have considered can have all non-trivial structure constants obtained by simply inserting $t = p/q$ into (4.25). However, in the minimal model $\mathrm{M}(3,5)$, we will see how we will obtain quadratic equations for the structure constants.

The central charge of this minimal model is $c = -\frac{3}{5}$. We define the primary fields $\phi_{1,1} = \mathbb{1}$, $\phi_{2,1} = \Phi$, $\phi_{3,1} = \Psi$ and $\phi_{4,1} = \Theta$. They have conformal weights

$$h_\Phi = \tfrac{3}{4}, \quad h_\Psi = \tfrac{7}{3}, \quad h_\Theta = \tfrac{19}{4}.$$

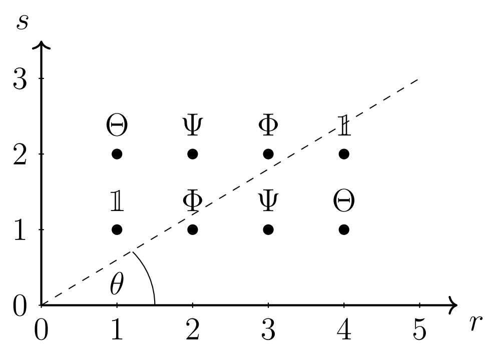

Figure 4.5: M(5,3) Kac Table

Using the property $\phi_{r,s} = \phi_{5-r,3-s}$, we can see

$$\begin{aligned}
\mathbb{1} &= \phi_{1,1} = \phi_{5-1,3-1} = \phi_{4,2}\\
\Phi &= \phi_{2,1} = \phi_{5-2,3-1} = \phi_{3,2}\\
\Psi &= \phi_{3,1} = \phi_{5-3,3-1} = \phi_{2,2}\\
\Theta &= \phi_{4,1} = \phi_{5-4,3-1} = \phi_{1,2}.
\end{aligned}$$

These fields have the following fusion rules:

$$\begin{aligned}
\Phi \times \Phi &= \mathbb{1} + \Psi \\
\Phi \times \Psi &= \Phi + \Theta \\
\Phi \times \Theta &= \Psi \\
\Psi \times \Theta &= \Phi \\
\Psi \times \Psi &= \mathbb{1} + \Psi \\
\Theta \times \Theta &= \mathbb{1}.
\end{aligned} \tag{4.26}$$

Now, we need to check which structure constants may be non-zero. We know from conformal invariance that a 3-point correlation function (2.51c) will be zero if and only if $C_{123} = 0$. We also recall that a 2-point correlator will only be non-zero if both fields are the same. Using our fusion rules (4.26), always fusing the last two fields, we compute the only non-zero 3-point functions are

$$\begin{aligned}
\langle \Psi\Phi\Phi \rangle &\approx \langle \mathbb{1}\Psi \rangle + \langle \Psi\Psi \rangle \approx \langle \Psi\Psi \rangle, \\
\langle \Psi\Psi\Psi \rangle &\approx \langle \mathbb{1}\Psi \rangle + \langle \Psi\Psi \rangle \approx \langle \Psi\Psi \rangle, \\
\langle \Phi\Psi\Theta \rangle &\approx \langle \Phi\Phi \rangle,
\end{aligned}$$

up to permutations. The constants of the two point functions $C_{\Psi\Psi} = 1$ and $C_{\Phi\Phi} = 1$, and any other 3-point correlator is 0 from fusion and 2-point function normalisation (2.58).

To compute these non-zero constants, we consider the following 4-point functions, incdicating which structure constants they give

$$\begin{aligned}
\langle \Phi\Phi\Phi\Phi \rangle &\leftrightarrow C^2_{\Phi\Phi\Psi} \\
\langle \Psi\Psi\Phi\Phi \rangle &\leftrightarrow C_{\Psi\Psi\Psi} C_{\Phi\Phi\Psi} \\
\langle \Phi\Psi\Phi\Psi \rangle &\leftrightarrow C^2_{\Phi\Psi\Theta}.
\end{aligned}$$

Using our formula (4.25), we compute the first constant to be

$$C^2_{\Phi\Phi\Psi} = -\frac{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})^3}{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{6}{5})^2}. \tag{4.27}$$

Now consider the 4-point correlation function

$$F(z,\bar{z}) = \lim_{w\to\infty} |w|^{28/3} \langle 0| \, \Psi(w,\bar{w})\Psi(1,1)\Phi(z,\bar{z})\Phi(0,0) \, |0\rangle \,.$$

As we have outlined in the previous section, we take operator product expansions, compare with the expansion from (3.33) to obtain the operator product expansion constants. Finally, we use (B.15), and set the non-single valued terms to zero to arrive at the equation

$$C_{\Phi\Phi\Psi} C_{\Psi\Psi\Psi} = -\frac{\Gamma(-\frac{1}{5})\Gamma(\frac{3}{5})^2\Gamma(\frac{4}{5})^2\Gamma(\frac{7}{5})}{\Gamma(-\frac{2}{5})\Gamma(\frac{2}{5})^2\Gamma(\frac{6}{5})^3}.$$

Squaring the equation,

$$C^2_{\Phi\Phi\Psi} C^2_{\Psi\Psi\Psi} = \frac{\Gamma(-\frac{1}{5})^2\Gamma(\frac{3}{5})^4\Gamma(\frac{4}{5})^4\Gamma(\frac{7}{5})^2}{\Gamma(-\frac{2}{5})^2\Gamma(\frac{2}{5})^4\Gamma(\frac{6}{5})^6}. \tag{4.28}$$

Substituting (4.27) into (4.28) gives

$$-\frac{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})^3}{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{6}{5})^2}C^2_{\Psi\Psi\Psi} = \frac{\Gamma(-\frac{1}{5})^2\Gamma(\frac{3}{5})^4\Gamma(\frac{4}{5})^4\Gamma(\frac{7}{5})^2}{\Gamma(-\frac{2}{5})^2\Gamma(\frac{2}{5})^4\Gamma(\frac{6}{5})^6}. \tag{4.29}$$

Solving this we find

$$C^2_{\Psi\Psi\Psi} = -\frac{\Gamma(-\frac{1}{5})^2\Gamma(\frac{3}{5})^3\Gamma(\frac{4}{5})\Gamma(\frac{7}{5})^2\Gamma(\frac{1}{5})}{\Gamma(-\frac{2}{5})^2\Gamma(\frac{2}{5})^3\Gamma(\frac{6}{5})^4}. \tag{4.30}$$

The last constant to determine is $C_{\Phi\Psi\Theta}$. Taking the correlation function $\langle\Phi\Psi\Phi\Psi\rangle$ and following the previous method gives

$$C^2_{\Phi\Psi\Theta} = -\frac{\Gamma(\frac{1}{5})^2\Gamma(\frac{6}{5})\Gamma(\frac{7}{5})}{\Gamma(-\frac{2}{5})\Gamma(-\frac{1}{5})\Gamma(\frac{9}{5})^2}C^2_{\Phi\Phi\Psi}.$$

Using (4.27) we evaluate

$$\begin{aligned} C^2_{\Phi\Psi\Theta} &= \frac{\Gamma(\frac{1}{5})^2\Gamma(\frac{6}{5})\Gamma(\frac{7}{5})}{\Gamma(-\frac{2}{5})\Gamma(-\frac{1}{5})\Gamma(\frac{9}{5})^2}\cdot\frac{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})^3}{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{6}{5})^2} \\ &= \frac{\Gamma(\frac{1}{5})\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})^3\Gamma(\frac{7}{5})}{\Gamma(-\frac{2}{5})\Gamma(-\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{6}{5})\Gamma(\frac{9}{5})^2} \\ &= \frac{1}{4}, \end{aligned} \tag{4.31}$$

where once again we made very good use of the gamma function property $\Gamma(z+1) = z\Gamma(z)$. We now have all the structure constants for the minimal model $\mathrm{M}(3,5)$.

4.3.3 General Coupled Constants: M(p,q)

To conclude, we will now derive constraints for structure constants for a general correlation function in the minimal models. Consider the 4-point correlator

$$F^{(4)}(z,\bar{z}) = \lim_{w\to\infty}|w|^{4h_{r_1,s_1}}\langle 0|\,\phi_{r_1,s_1}(w,\bar{w})\phi_{r_2,s_2}(1,1)\phi_{2,1}(z,\bar{z})\phi_{r_3,s_3}(0,0)\,|0\rangle\,.$$

Taking the operator product expansion in the last two fields

$$\begin{aligned} F^{(4)}(z,\bar{z}) &= \lim_{w\to\infty}|w|^{4h_{r_1,s_1}}\langle 0|\,\phi_{r_1,s_1}(w,\bar{w})\phi_{r_2,s_2}(1,1)\sum_k\frac{C_{(2,1)(r_3,s_3)}{}^k\phi_k(0,0)}{|z|^{2h_{21}+2h_{r_3,s_3}-2h_k}}\,|0\rangle+\cdots \\ &= \lim_{w\to\infty}\Big(C_{(2,1)(r_3,s_3)}{}^{(r_3-1,s_3)}|w|^{4h_{r_1,s_1}}\langle 0|\,\phi_{r_1,s_1}(w,\bar{w})\phi_{r_2,s_2}(1,1)\phi_{r_3-1,s_3}(0,0)\,|0\rangle \\ &\qquad\cdot|z|^{2h_{r_3-1,s_3}-2h_{2,1}-2h_{r_3,s_3}}+\cdots \\ &\quad+C_{(2,1)(r_3,s_3)}{}^{(r_3+1,s_3)}|w|^{4h_{r_1,s_1}}\langle 0|\,\phi_{r_1,s_1}(w,\bar{w})\phi_{r_2,s_2}(1,1)\phi_{r_3+1,s_3}(0,0)\,|0\rangle \\ &\qquad\cdot|z|^{2h_{r_3+1,s_3}-2h_{2,1}-2h_{r_3,s_3}}+\cdots\Big) \\ &= C_{(2,1)(r_3,s_3)}{}^{(r_3-1,s_3)}C_{(r_1,s_1)(r_2,s_2)(r_3-1,s_3)}|z|^{2h_{r_3-1,s_3}-2h_{2,1}-2h_{r_3,s_3}} \\ &+C_{(2,1)(r_3,s_3)}{}^{(r_3+1,s_3)}C_{(r_1,s_1)(r_2,s_2)(r_3+1,s_3)}|z|^{2h_{r_3+1,s_3}-2h_{2,1}-2h_{r_3,s_3}}+\cdots\,. \end{aligned} \tag{4.32}$$

Using (3.6) we can express the powers in terms of r_3, s_3 and t to leave us with

$$F^{(4)}(z,\bar{z}) = C_{(2,1)(r_3,s_3)}{}^{(r_3-1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3-1,s_3)} |z|^{1-t(r_3+1)+s_3} + \cdots$$
$$+ C_{(2,1)(r_3,s_3)}{}^{(r_3+1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3+1,s_3)} |z|^{1+t(r_3-1)-s_3} + \cdots . \quad (4.33)$$

We still need to use fusion to determine precisely which fields will give non-zero 3-point structure constants, but we will leave this for the moment, and move onto our other expansion. Using (3.44) and (3.33) as before we have

$$\begin{aligned} F^{(4)}(z,\bar{z}) = {}& C_{1,1}|z|^{1-t-d_3}|1-z|^{1-t-d_1} \\ & |F(\tfrac{1}{2}(1-d_3-d_1-d_2), \tfrac{1}{2}(1-d_3-d_1+d_2); 1-d_3; z)|^2 \\ & + C_{2,2}|z|^{1-t+d_3}|1-z|^{1-t-d_1} \\ & |F(\tfrac{1}{2}(1+d_3-d_1-d_2), \tfrac{1}{2}(1+d_3-d_1+d_2); 1+d_3; z)|^2. \end{aligned}$$

Expanding around $z = 0$ gives

$$F^{(4)}(z,\bar{z}) = C_{1,1}|z|^{1-t-d_3} + C_{2,2}|z|^{1-t+d_3} + \ldots . \quad (4.34)$$

Now since d_1, d_2 and d_3 depend on the conformal weights, we compute

$$\begin{aligned} d_1 &= \sqrt{1-2t+4h_{r_2,s_2}t+t^2} \\ &= s_2 - r_2 t, \\ d_2 &= \sqrt{1-4t-4h_{2,1}t+4h_{r_1,s_1}t+4t^2} \\ &= s_1 - r_1 t, \\ d_3 &= \sqrt{1-2t+4h_{r_3,s_3}t+t^2} \\ &= s_3 - r_3 t. \end{aligned} \quad (4.35)$$

Substituting in (4.34) we now have

$$F^{(4)}(z,\bar{z}) = C_{1,1}|z|^{1-t(r_3-1)-s_3} + C_{2,2}|z|^{1-t(r_3+1)+s_3} + \ldots . \quad (4.36)$$

So comparing (4.33) and (4.36), we have

$$\begin{aligned} C_{1,1} &= C_{(2,1)(r_3,s_3)}{}^{(r_3+1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3+1,s_3)} \\ C_{2,2} &= C_{(2,1)(r_3,s_3)}{}^{(r_3-1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3-1,s_3)}. \end{aligned}$$

Now we will return to (4.32) and discuss which fields ϕ_{r_1,s_1},ϕ_{r_2,s_2},ϕ_{r_3,s_3} can occur. From (4.32) We have two 3-point correlation functions to consider:

$$\langle 0|\, \phi_{r_1,s_1}\phi_{r_2,s_2}\phi_{r_3-1,s_3}\, |0\rangle\,, \quad \langle 0|\, \phi_{r_1,s_1}\phi_{r_2,s_2}\phi_{r_3+1,s_3}\, |0\rangle\,.$$

Using (4.4), we can fuse the second and third fields to give

$$\begin{aligned} \langle 0|\, \phi_{r_1,s_1}\phi_{r_2,s_2}\phi_{r_3-1,s_3}\, |0\rangle &= \sum_{k\overset{2}{=}|r_2-r_3+1|+1}^{k^-_{\max}} \sum_{l\overset{2}{=}|s_2-s_3|+1}^{l_{\max}} \langle 0|\, \phi_{r_1,s_1}\phi_{k,l}\, |0\rangle\,, \\ \langle 0|\, \phi_{r_1,s_1}\phi_{r_2,s_2}\phi_{r_3+1,s_3}\, |0\rangle &= \sum_{k\overset{2}{=}|r_2-r_3-1|+1}^{k^+_{\max}} \sum_{l\overset{2}{=}|s_2-s_3|+1}^{l_{\max}} \langle 0|\, \phi_{r_1,s_1}\phi_{k,l}\, |0\rangle\,, \end{aligned} \quad (4.37)$$

where

$$\begin{aligned} k^-_{\max} &= \min(r_2 + r_3 - 2, 2q - r_2 - r_3) \\ k^+_{\max} &= \min(r_2 + r_3, 2q - r_2 - r_3 - 2) \\ l_{\max} &= \min(s_2 + s_3 - 1, 2p - s_2 - s_3 - 1). \end{aligned} \tag{4.38}$$

Now, we need both 2-point correlation functions to be non-zero if we wish to compute structure constants that are not trivially zero.

The sums over l and k are independent, so we can consider the cases separately. Firstly, consider the values that s_1 can take to match l in both correlators. Since the fusion in s_2 and s_3 is identical, the only constraint we have is that s_1 must be in the set of values l takes:

$$s_1 \in \{|s_2 - s_3| + 1, |s_2 - s_3| + 3, \cdots, l_{\max}\} := S.$$

That was not too painful. Unfortunately, the values that r_1 can take is a bit more complicated due to the k values not obviously matching in both sums.

To calculate the upper bound on the values of r_1, we will take the lowest upper bound on k in both sums (4.37), this involves taking a combination of both $k^+_{\max}$ and $k^-_{\max}$ (4.38). The upper bound will therefore be

$$r_{\max} = \min(r_2 + r_3 - 2, 2q - r_2 - r_3 - 2).$$

Now to make the derivation of the lower bound on the set of values that r_1 takes, we will consider the cases $r_2 > r_3$, $r_2 < r_3$, and $r_2 = r_3$ separately.

First, assume that $r_2 > r_3$. The sums start from

$$\begin{aligned} k &= |r_2 - r_3 + 1| + 1 = r_2 - r_3 + 2 \\ k &= |r_2 - r_3 - 1| + 1 = r_2 - r_3. \end{aligned}$$

So we can see that the sum starting from $|r_2 - r_3 + 1| + 1$ will always be 2 ahead of the sum starting from $|r_2 - r_3 - 1| + 1$. Since the sum runs over every second term, taking

$$r_1 \in \{|r_2 - r_3 + 1| + 1, \cdots, r_{\max}\} := R_1,$$

will give non-zero 2-point functions in both of (4.37).

Now consider the case $r_2 < r_3$. Then the sums will start from

$$\begin{aligned} k &= |r_2 - r_3 + 1| + 1 = r_3 - r_2 \\ k &= |r_2 - r_3 - 1| + 1 = r_3 - r_2 + 2. \end{aligned}$$

So in this case r_1 will take values in the set

$$r_1 \in \{|r_2 - r_3 - 1| + 1, \cdots, r_{\max}\} := R_2.$$

Finally, when $r_2 = r_3$, both the sums in (4.37) r_1 will need to start from 2, increment by 2 and need to finish at $r_{\max}$ for both correlators to be non-zero. This method is illustrated in Fig. 4.6 for $r_2 > r_3$.

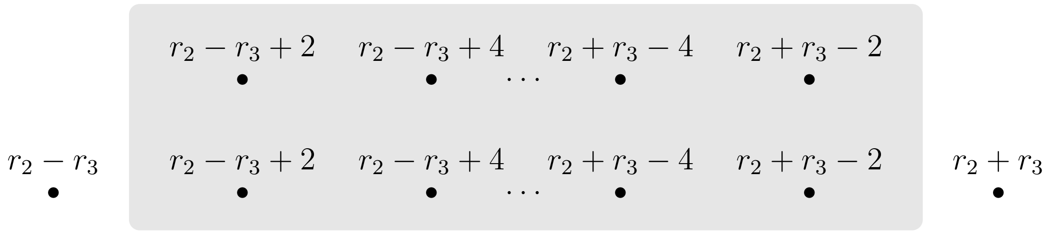

Figure 4.6: Fusion restrictions for r_1 when $r_2 > r_3$ and r_2, r_3 are sufficiently small. The first row is $\phi_{r_2,s_2} \times \phi_{r_3-1,s_3}$ and the second is $\phi_{r_2,s_2} \times \phi_{r_3+1,s_3}$. Values for r_1 that give both non-zero correlators are in the shaded grey region.

In summary, the fusion rules (4.37) dictate that if we wish to solve for structure constants that are not trivially zero, r_1 and s_1 must be selected from the sets

$$\begin{cases} r_1 \in R_1 \text{ if } r_2 \geq r_3 \text{ and } r_2 > 1, \\ r_1 \in R_2 \text{ if } r_2 < r_3 \text{ and } r_2 > 1, \\ r_1 \in R_2 \text{ if } r_2 \geq r_3 \text{ and } r_2 = 1, \\ r_1 \in R_1 \text{ if } r_2 < r_3 \text{ and } r_2 = 1, \\ s_1 \in S. \end{cases} \tag{4.39}$$

The reason for swapping the R sets for $r_2 < 2$ is that the upper bound on the fusion rule $r_{\max}$ being too low when $r_2 = 1$. In the cases where the upper bound in R_1 and R_2 is lower than the given lower bound, these structure constants will be zero, as fusion dictates.

We can also take the values of r_3, s_3 and r_2, s_2 to be such that they will give unique fields on the Kac table. The value r_3 is slightly more constrained, as we need $r_3 - 1$ and $r_3 + 1$ to both be valid values on the Kac table. We also will take r_2 and r_3 to begin from 2.

Now we have a clear idea of which conformal fields will be in our correlation functions. We are left with the conformal block decomposition for our correlator

$$\begin{aligned} F^{(4)}(z,\bar{z}) = {} & C_{(2,1)(r_3,s_3)}^{\quad (r_3+1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3+1,s_3)} |z|^{1-t-d_3} |1-z|^{1-t-d_1} \\ & |F(\tfrac{1}{2}(1-d_3-d_1-d_2), \tfrac{1}{2}(1-d_3-d_1+d_2); 1-d_3; z)|^2 \\ & + C_{(2,1)(r_3,s_3)}^{\quad (r_3-1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3-1,s_3)} |z|^{1-t+d_3} |1-z|^{1-t-d_1} \\ & |F(\tfrac{1}{2}(1+d_3-d_1-d_2), \tfrac{1}{2}(1+d_3-d_1+d_2); 1+d_3; z)|^2. \end{aligned} \tag{4.40}$$

Expanding (4.40) about $z = 1$, setting the non single valued terms to zero, cancelling terms and rearranging leaves us with

$$\begin{aligned} & \frac{C_{(2,1)(r_3,s_3)}^{\quad (r_3+1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3+1,s_3)}}{C_{(2,1)(r_3,s_3)}^{\quad (r_3-1,s_3)} C_{(r_1,s_1)(r_2,s_2)(r_3-1,s_3)}} \\ & = -\frac{\Gamma[1+d_3]^2 \Gamma[\frac{1}{2}(1-d_3-d_1-d_2)] \Gamma[\frac{1}{2}(1-d_3+d_1-d_2)]}{\Gamma[1-d_3]^2 \Gamma[\frac{1}{2}(1+d_3-d_1-d_2)] \Gamma[\frac{1}{2}(1+d_3+d_1-d_2)]} \\ & \quad \cdot \frac{\Gamma[\frac{1}{2}(1-d_3-d_1+d_2)] \Gamma[\frac{1}{2}(1-d_3+d_1+d_2)]}{\Gamma[\frac{1}{2}(1+d_3-d_1+d_2)] \Gamma[\frac{1}{2}(1+d_3+d_1+d_2)]}. \end{aligned} \tag{4.41}$$

There we have it. We are able now to solve for a wide array of minimal model structure constants by iterating through unique (r, s) labels in (4.41).

Now, we have probably led the reader to believe that any minimal model can have all structure constants solved by (4.41). [4] While (4.41) will be able to solve all the structure constants in the minimal models we have considered so far, it is not necessarily true that it can solve every structure constant for any minimal model.

This is due to the constraint that we had to include a $\phi_{2,1}$ field in our correlator. This decision was made back in Chapter 3, when we chose to take the singular vector $|\chi_{2,1}\rangle$ (3.13) for all the differential equation and conformal block decomposition. In order to derive other structure constants, one would need to derive other BPZ equations using different singular vectors, such as in Remark 17, or higher order singular vectors.

Now having a general formula (4.41) for computing structure constants, the method of solving will look like the following algorithm:

1. Select r_3, s_3, r_2, s_2 to give fields ϕ_{r_3,s_3}, ϕ_{r_2,s_2} on the Kac table, where $2 \leq r_3 < q - 1$.
2. For the fields chosen ϕ_{r_3,s_3}, ϕ_{r_2,s_2}, calculate all possible fields ϕ_{r_1,s_1}, where r_1 and s_1 are found using (4.39).
3. Output the valid quadratic equations for the chosen fields ϕ_{r_3,s_3}, ϕ_{r_2,s_2} and ϕ_{r_1,s_1}, by substituting in their r, s labels into (4.41).
4. Repeat for all valid fields ϕ_{r_3,s_3}, ϕ_{r_2,s_2}.
5. Attempt to solve the quadratic equations simultaneously.

Having a general method to calculate a subset of structure constants for any minimal model $\mathrm{M}(p, q)$, we can solve for a vast array of structure constants.

4.3.4 General Coupled Constants Example: M(4,5)

To conclude this thesis, we will illustrate the power of this algorithm by solving a subset of non-zero structure constants in the minimal model $\mathrm{M}(4, 5)$.

The minimal model $\mathrm{M}(4, 5)$ has been identified with the tricritical Ising model [DMS97]. There is quite a lot that could be said about this minimal model, including supersymmetry, but for the purposes of this example we will show how the algorithm produces quadratic equations.

The Kac table is given in Fig. 4.7. Since the fields above and below the dashed line in Fig. 4.7, we only need to consider the ones above the line. We will make the

[4] This may be true for a minimal model $\mathrm{M}(p, q)$ with $p < 4$, but not in general.

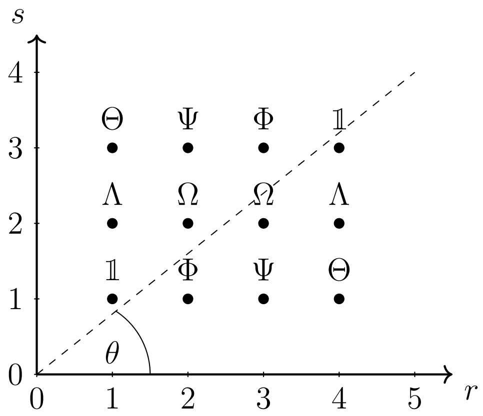

Figure 4.7: M(4,5) Kac Table

following identifications to simplify the notation [5]

$$
\begin{aligned}
\phi_{1,1} &= \phi_{4,3} = \mathbb{1}\\
\phi_{2,1} &= \phi_{3,3} = \Phi\\
\phi_{3,1} &= \phi_{2,3} = \Psi\\
\phi_{4,1} &= \phi_{1,3} = \Theta\\
\phi_{3,2} &= \phi_{2,2} = \Omega\\
\phi_{4,2} &= \phi_{1,2} = \Lambda.
\end{aligned}
$$

Applying the algorithm to the equation (4.41), using (4.39) gives

1. $(r_3, s_3) = (2,1), (r_2, s_2) = (2,1) \implies (r_1, s_1) = (2,1)$.

$$C^2_{\Phi\Phi\Psi} = -\frac{\Gamma[\frac{2}{5}]^2\Gamma[\frac{4}{5}]\Gamma[\frac{7}{5}]}{\Gamma[-\frac{2}{5}]\Gamma[\frac{1}{5}]\Gamma[\frac{8}{5}]^2}.$$

2. $(r_3, s_3) = (2,1), (r_2, s_2) = (3,1) \implies (r_1, s_1) = (3,1)$.

$$\Gamma[-\tfrac{3}{5}]\Gamma[\tfrac{2}{5}]^2\Gamma[\tfrac{4}{5}]^2\Gamma[\tfrac{11}{5}] = -\Gamma[-\tfrac{6}{5}]\Gamma[\tfrac{1}{5}]^2\Gamma[\tfrac{8}{5}]^3 C_{\Phi\Phi\Psi}C_{\Psi\Psi\Psi}.$$

3. $(r_3, s_3) = (2,1), (r_2, s_2) = (3,2) \implies (r_1, s_1) = (3,2)$.

$$\Gamma[\tfrac{2}{5}]^3\Gamma[\tfrac{4}{5}]^2\Gamma[\tfrac{6}{5}] = -\Gamma[-\tfrac{1}{5}]\Gamma[\tfrac{1}{5}]^2\Gamma[\tfrac{3}{5}]\Gamma[\tfrac{8}{5}]^2 C_{\Phi\Phi\Psi}C_{\Omega\Omega\Psi}.$$

4. $(r_3, s_3) = (3,1), (r_2, s_2) = (2,1) \implies (r_1, s_1) = (3,1)$.

$$\Gamma[-\tfrac{2}{5}]^2\Gamma[\tfrac{8}{5}]\Gamma[\tfrac{11}{5}]C^2_{\Phi\Phi\Psi} = -\Gamma[-\tfrac{6}{5}]\Gamma[-\tfrac{3}{5}]\Gamma[\tfrac{12}{5}]^2 C^2_{\Phi\Psi\Theta}.$$

5. $(r_3, s_3) = (3,1), (r_2, s_2) = (3,1) \implies (r_1, s_1) = (2,1)$.

$$\Gamma[-\tfrac{2}{5}]^2\Gamma[\tfrac{8}{5}]\Gamma[\tfrac{11}{5}]C^2_{\Phi\Phi\Psi} = -\Gamma[-\tfrac{6}{5}]\Gamma[-\tfrac{3}{5}]\Gamma[\tfrac{12}{5}]^2 C^2_{\Phi\Psi\Theta}.$$

[5]This is not the notation found in [DMS97], there is a good reason for this: the author finds it unpalatable.

6. $(r_3, s_3) = (3,1), (r_2, s_2) = (3,2) \implies (r_1, s_1) = (2,2)$.

$$\begin{aligned}\Gamma[-\tfrac{2}{5}]^2\Gamma[\tfrac{4}{5}]\Gamma[\tfrac{6}{5}]^2\Gamma[\tfrac{8}{5}]C_{\Omega\Omega\Phi}C_{\Phi\Phi\Psi} &= \\ &- \Gamma[-\tfrac{3}{5}]\Gamma[-\tfrac{1}{5}]^2\Gamma[\tfrac{1}{5}]\Gamma[\tfrac{12}{5}]^2 C_{\Omega\Omega\Theta}C_{\Phi\Psi\Theta}.\end{aligned}$$

7. $(r_3, s_3) = (3,2), (r_2, s_2) = (2,1) \implies (r_1, s_1) = (3,2)$.

$$\Gamma[\tfrac{3}{5}]^3\Gamma[\tfrac{6}{5}]C^2_{\Omega\Omega\Phi} = -\Gamma[-\tfrac{1}{5}]\Gamma[\tfrac{2}{5}]\Gamma[\tfrac{7}{5}]^2C^2_{\Phi\Omega\Lambda}.$$

8. $(r_3, s_3) = (3,2), (r_2, s_2) = (3,1) \implies (r_1, s_1) = (2,2)$.

$$\Gamma[\tfrac{1}{5}]\Gamma[\tfrac{3}{5}]^2\Gamma[\tfrac{8}{5}]C_{\Omega\Omega\Phi}C_{\Omega\Omega\Psi} = -\Gamma[-\tfrac{3}{5}]\Gamma[\tfrac{4}{5}]\Gamma[\tfrac{7}{5}]^2C_{\Phi\Omega\Lambda}C_{\Psi\Omega\Lambda}.$$

9. $(r_3, s_3) = (3,2), (r_2, s_2) = (3,2) \implies (r_1, s_1) = (2,1)$.

$$\Gamma[\tfrac{3}{5}]^3\Gamma[\tfrac{6}{5}]C^2_{\Omega\Omega\Phi} = -\Gamma[-\frac{1}{5}]\Gamma[\tfrac{2}{5}]\Gamma[\tfrac{7}{5}]^2C^2_{\Phi\Omega\Lambda}.$$

10. $(r_3, s_3) = (3,2), (r_2, s_2) = (3,2) \implies (r_1, s_1) = (2,3)$.

$$\Gamma[\tfrac{1}{5}]\Gamma[\tfrac{3}{5}]^2\Gamma[\tfrac{8}{5}]C_{\Omega\Omega\Psi}C_{\Omega\Omega\Phi} = -\Gamma[-\tfrac{3}{5}]\Gamma[\tfrac{4}{5}]\Gamma[\tfrac{7}{5}]^2C_{\Psi\Omega\Lambda}C_{\Phi\Omega\Lambda}.$$

The non-zero structure constants that are found here are

$$\begin{aligned}C^2_{\Phi\Phi\Psi} &= -\frac{\Gamma[\frac{2}{5}]^2\Gamma[\frac{4}{5}]\Gamma[\frac{7}{5}]}{\Gamma[-\frac{2}{5}]\Gamma[\frac{1}{5}]\Gamma[\frac{8}{5}]^2} \\ C^2_{\Psi\Psi\Psi} &= -\frac{\Gamma[-\frac{3}{5}]^2\Gamma[-\frac{2}{5}]\Gamma[\frac{2}{5}]^2\Gamma[\frac{4}{5}]^3\Gamma[\frac{11}{5}]^2}{\Gamma[-\frac{6}{5}]^2\Gamma[\frac{1}{5}]^3\Gamma[\frac{7}{5}]\Gamma[\frac{8}{5}]^4} \\ C^2_{\Omega\Omega\Psi} &= \frac{\Gamma[-\frac{2}{5}]\Gamma[\frac{2}{5}]^4\Gamma[\frac{4}{5}]^3\Gamma[\frac{6}{5}]^2}{\Gamma[-\frac{1}{5}]^2\Gamma[\frac{1}{5}]^3\Gamma[\frac{3}{5}]^2\Gamma[\frac{7}{5}]\Gamma[\frac{8}{5}]^2} \\ C^2_{\Phi\Psi\Theta} &= \frac{\Gamma[-\frac{2}{5}]\Gamma[\frac{2}{5}]^2\Gamma[\frac{4}{5}]\Gamma[\frac{7}{5}]\Gamma[\frac{11}{5}]}{\Gamma[-\frac{6}{5}]\Gamma[-\frac{3}{5}]\Gamma[\frac{1}{5}]\Gamma[\frac{8}{5}]\Gamma[\frac{12}{5}]^2} \\ C^2_{\Psi\Omega\Lambda} &= -\frac{\Gamma[-\frac{2}{5}]\Gamma[\frac{2}{5}]^5\Gamma[\frac{4}{5}]\Gamma[\frac{6}{5}]}{\Gamma[-\frac{3}{5}]^2\Gamma[-\frac{1}{5}]\Gamma[\frac{1}{5}]\Gamma[\frac{7}{5}]^3}.\end{aligned} \tag{4.42}$$

However, there remains 2 equations and 3 unknown constants

$$\begin{aligned}\Gamma[-\tfrac{2}{5}]^2\Gamma[\tfrac{4}{5}]\Gamma[\tfrac{6}{5}]^2\Gamma[\tfrac{8}{5}]&C_{\Omega\Omega\Phi}C_{\Phi\Phi\Psi} \\ &= -\Gamma[-\tfrac{3}{5}]\Gamma[-\tfrac{1}{5}]^2\Gamma[\tfrac{1}{5}]\Gamma[\tfrac{12}{5}]^2C_{\Omega\Omega\Theta}C_{\Phi\Psi\Theta} \\ \Gamma[\tfrac{3}{5}]^3\Gamma[\tfrac{6}{5}]C^2_{\Omega\Omega\Phi} &= -\Gamma[-\tfrac{1}{5}]\Gamma[\tfrac{2}{5}]\Gamma[\tfrac{7}{5}]^2C^2_{\Phi\Omega\Lambda}.\end{aligned} \tag{4.43}$$

Looking at (4.43), the constants still to be determined are $C_{\Phi\Omega\Lambda}$, $C_{\Omega\Omega\Theta}$ and $C_{\Omega\Omega\Phi}$.

We can see that this managed to give us 5 structure constants out of 8 that will be non-zero. If time permitted, we would need to derive a second differential equation involving Remark 17, leading to two coupled equations for the structure constants, allowing us to solve for the remaining 3 undetermined constants.

Chapter 5

Conclusion

We began this thesis with a general introduction to conformal field theory, with examples from the Virasoro algebra. Chapter 2 introduced the operator formalism of conformal field theory, including conformal transformations, the operator product expansion and correlation functions. This chapter primarily demonstrated how correlation functions are restricted by the Ward identities and that the structure constants appear in the OPE and the 3-point functions.

In Chapter 3, we further explored the symmetry of the Virasoro algebra. We found that by setting singular vectors to zero, we could obtain a second set of differential equations, known as BPZ equations. These equations were solved by converting them into hypergeometric equations, which have known solutions. Using these solutions, we found the BPZ solutions called conformal blocks, which were products of hypergeometric functions and powers of polynomials.

Finally, we considered the minimal models in Chapter 4. Using the correlator form from both the Ward identities and the conformal block decomposition, we computed the structure constants for many examples of minimal models, including $\mathrm{M}(2,5)$, $\mathrm{M}(3,4)$ and $\mathrm{M}(3,5)$. We concluded this discussion by demonstrating an algorithm one can use to derive a portion of the structure constants for any minimal model $\mathrm{M}(p,q)$, and found these equations to be quadratic in nature.

The largest restriction for computing all the structure constants in $\mathrm{M}(4,5)$ was the necessity of containing a $\phi_{2,1}$ field to make use of our BPZ equation (3.32). If time permitted, it would be interesting to try and find another second order ordinary differential equation using Remark 17, and perhaps derive another coupled set of equations similar to (4.41) to solve for the missing constants. Together, these two equations might not be the whole story for minimal models with very large values of p and q, and one might need to consider higher order BPZ equations.

As stated in the introduction, the minimal model structure constants have been found completely generally using a different method, by Dotsenko and Fateev [DF84, DF85]. It would be interesting to see if the algorithm obtained in Section 4.3.3 is consistent with these methods, and if they do give the same structure constants in general.

Furthermore, it would be worthwhile investigating how this method can be generalised beyond the Virasoro minimal models to the $N = 1$ super Virasoro minimal models. These families of conformal field theories are similar, so it is possible the same algorithm would apply with minimal modifications.

A further direction one could investigate is Liouville theory for central charges that are complex and continuous. Liouville theory is less well understood than rational conformal field theories such as the minimal models. An introduction to Liouville theory can be found in [Rib14].

Another direction of study could be logarithmic conformal field theory. These are field theories where the correlation functions may contain logarithmic singularities. These are much more difficult to solve, due to the complex logarithm being multi-valued and the constraint that correlators must be single valued. An introduction to these conformal field theories can be found in [CR13].

There are also numerous applications in representation theory and vertex operator algebras that one could explore, such as [RRR21]. Conformal field theory is a pathway to many fascinating areas of mathematics.

Appendix A

Correlation Functions

In this appendix, we will see how the Ward identities (2.50) constrain our correlators (2.45).

Firstly, we consider the 1-point function (2.51a). The identity (2.50a) gives $\partial_1 \langle 0| \phi_1(z_1) |0\rangle = 0$, so the correlator is a constant. Both (2.50b) and (2.50c) both give $h_1 \langle 0| \phi_1(z_1) |0\rangle = 0$, so our 1-point function is zero unless the conformal dimension $h_1 = 0$. Typically, the only primary field that has conformal dimension zero is the identity field $\mathbb{1}$. So we arrive at

$$\langle 0| \phi_1(z_1) |0\rangle = \delta_{\phi_1 = \mathbb{1}}. \tag{A.1}$$

Next, we look at the consequence (2.50) has on the 2-point correlation function (2.51b). We begin by making a change of variables $z = z_1 + z_2$ and $z_{12} = z_1 - z_2$. The Ward identity (2.50a) now tells us that the correlation function is a function of z_{12} only, $\langle 0| \phi_1(z_1)\phi_2(z_2) |0\rangle = f(z_{12})$. Now (2.50b) takes the form

$$\left(z_{12}\partial_{12} + h_1 + h_2\right) f(z_{12}) = 0,$$

which is a first order ordinary differential equation. The general solution is

$$f(z_{12}) = \frac{C_{12}}{z_{12}^{h_1+h_2}}.$$

Now substituting this into (2.50c), we obtain

$$\left(h_1 - h_2\right) \frac{C_{12}}{z_{12}^{h_1+h_2-1}} = 0,$$

therefore $h_1 = h_2$ or $C_{12} = 0$. Hence, we conclude

$$\langle 0| \phi_1(z_1)\phi_2(z_2) |0\rangle = \frac{C_{12}\delta_{h_1=h_2}}{\left(z_1 - z_2\right)^{h_1+h_2}}. \tag{A.2}$$

This is usually written as

$$\langle 0| \phi_1(z_1)\phi_2(z_2) |0\rangle = \frac{C_{12}}{\left(z_1 - z_2\right)^{2h}}, \tag{A.3}$$

where $h = h_1 = h_2$.

Now we consider the 3-point function (2.51c). We will once again make a change of coordinates, this time defining $z = z_1 + z_2 + z_3, z_{12} = z_1 - z_2$ and $z_{23} = z_2 - z_3$. The Ward identity (2.50a) tells us the correlator does not depend on z, so $\langle 0| \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) |0\rangle = f(z_{12}, z_{23})$. Substituting our change of coordinates into (2.50b), we see

$$(z_{12}\partial_{12} + z_{23}\partial_{23} + (h_1 + h_2 + h_3))\, f(z_{12}, z_{23}).$$

Solving this partial differential equation using method of characteristics, we acquire the solution

$$f(z_{12}, z_{23}) = z_{12}^{-h_1-h_2-h_3} g\left(\frac{z_{12}}{z_{23}}\right), \tag{A.4}$$

where g is an undetermined function of z_{12}/z_{23}. Once again, we make a change of coordinates $x = z_{12}/z_{23}$ and $y = z_{12}z_{23}$, this allows us to express the correlation function in terms of one variable x. After the dust has settled, we get the first order ordinary differential equation

$$\frac{\partial g}{\partial x} + \left(\frac{h_1 - h_2 + h_3}{x+1} - \frac{2h_3}{x}\right) g(x) = 0,$$

which has the general solution

$$g(x) = \frac{C_{123}}{(x+1)^{h_1-h_2-h_3} x^{-2h_3}}.$$

Substituting $x = z_{12}/z_{23}$ back in, and remembering (A.4), we are left with

$$\begin{aligned}&\langle 0| \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) |0\rangle \\ &\quad = \frac{C_{123}}{(z_1 - z_2)^{h_1+h_2-h_3}(z_1 - z_3)^{h_1-h_2+h_3}(z_2 - z_3)^{-h_1+h_2+h_3}}.\end{aligned} \tag{A.5}$$

Finally, we will see how the Ward identities constrain the 4-point correlator (2.51d). Since we only have 3 partial differential equations, we will not be able to solve this up to an unknown constant. We will however be able to solve it up to an unknown function of the cross ratio

$$\eta = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)}. \tag{A.6}$$

Once again to ensure the Ward identity (2.50a) removes a variable, we will make the change of coordinates $z = z_1 + z_2 + z_3 + z_4$, $z_{12} = z_1 - z_2$, $z_{23} = z_2 - z_3$ and $z_{34} = z_3 - z_4$. Now $\langle 0| \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4) |0\rangle = f(z_{12}, z_{23}, z_{34})$ is a function of only three variables. Substituting this into (2.50b), remembering that the correlator does not depend on z, we arrive at

$$(z_{12}\partial_{12} + z_{23}\partial_{23} + z_{34}\partial_{34} + (h_1 + h_2 + h_3 + h_4))\, f(z_{12}, z_{23}, z_{34}).$$

Performing method of characteristics on this equation, we discover that

$$f(z_{12}, z_{23}, z_{34}) = z_{12}^{-h} g\left(\frac{z_{23}}{z_{12}}, \frac{z_{34}}{z_{12}}\right), \tag{A.7}$$

where $h = \sum_{i=1}^{4} h_i$. Now we will introduce two new variables $x = z_{23}/z_{12}$ and $y = z_{34}/z_{12}$, so now g is a function of just x and y. Substitute (A.7) into (2.50c) and after a non-trivial amount of simplification, we obtain the partial differential equation

$$\Big(x(x+1)\partial_x + ((1+2x+y)y)\partial_y \\ + (-h_1+h_2+h_3+h_4) + (2h_3+2h_4)x + 2h_4 y\Big) g(x,y) = 0.$$

Once more performing methods of characteristics, we find the result

$$g(x,y) = x^{h_1-h_2-h_3-h_4}(x+1)^{-h_1+h_2-h_3-h_4} \\ \cdot \left(\frac{x(x+1)}{y}\right)^{2h_4} h\left(\frac{x(1+x+y)}{y}\right).$$

substituting $x = z_{23}/z_{12}$ and $y = z_{34}/z_{12}$ back in, we are left with our 4 point correlation function

$$\langle 0|\,\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\,|0\rangle \\ = h\left(\eta\right) z_{12}^{-h_1-h_2+h_3-h_4} z_{13}^{-h_1+h_2-h_3+h_4} z_{23}^{h_1-h_2-h_3+h_4} z_{34}^{-2h_4}, \quad \text{(A.8)}$$

where $\eta = z_{12}z_{34}/z_{13}z_{24}$ is the cross ratio (A.6). Since the function $h(\eta)$ can be any function of the cross ratio, the form of the 4-point function is quite flexible. The most compact form is given by

$$\langle 0|\,\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\,|0\rangle = F(\eta)\prod_{i<j} z_{ij}^{\frac{h}{3}-h_i-h_j}. \quad \text{(A.9)}$$

To show how to get from (A.8) to (A.9), we take the ratio of both solutions and show that it is a function of the cross ratio

$$\frac{h(\eta)z_{12}^{-h_1-h_2+h_3-h_4} z_{13}^{-h_1+h_2-h_3+h_4} z_{23}^{h_1-h_2-h_3+h_4} z_{34}^{-2h_4}}{F(\eta)z_{12}^{\frac{h}{3}-h_1-h_2} z_{13}^{\frac{h}{3}-h_1-h_3} z_{14}^{\frac{h}{3}-h_1-h_4} z_{23}^{\frac{h}{3}-h_2-h_3} z_{24}^{\frac{h}{3}-h_2-h_4} z_{34}^{\frac{h}{3}-h_3-h_4}} \\ = \tilde{h}(\eta)(z_{12}z_{34})^{-\frac{h}{3}+h_3-h_4}(z_{13}z_{24})^{-\frac{h}{3}+h_2+h_4}(z_{14}z_{23})^{-\frac{h}{3}+h_1+h_4} \\ = \tilde{h}(\eta)\eta^{-\frac{h}{3}+h_3-h_4}(1-\eta)^{-\frac{h}{3}+h_1+h_4} \\ = H(\eta). \quad \text{(A.10)}$$

Therefore, rearranging (A.10) and redefining $F(\eta) := F(\eta)H(\eta)$ as just a function of the cross ratio and substituting into (A.8), we arrive at the factorised 4-point function (A.9).

Appendix B

Hypergeometric Series

This appendix will outline how to derive the hypergeometric equation, and many hypergeometric function identities that have been used throughout this thesis. The work discussed here may be found in [GR90, Sea91, Bai35, Bar08, WW90, Han13].

B.1 Hypergeometric Equation

The hypergeometric equation is

$$\left\{ z(1-z)\frac{\partial^2}{\partial z^2} + [C-(A+B+1)z]\frac{\partial}{\partial z} - AB \right\} K(z) = 0. \tag{B.1}$$

This equation has singularities at $z = 0, 1, \infty$. We make the series substitution

$$K(z) = \sum_{n=0}^{\infty} a_n z^{n+s},$$

where we assume $a_0 \neq 0$. The indicial equation is

$$s(s+C-1) = 0, \tag{B.2}$$

and the Frobenius method results in the recursion relation

$$a_{n+1} = \frac{(n+s)(n+s+A+B)+AB}{(n+s+1)(n+s+C)} a_n. \tag{B.3}$$

Substituting $s = 0$ into (B.3) gives

$$\begin{aligned} a_n &= \frac{(n+A)(n+B)}{(n+1)(n+C)} a_n \\ &= \frac{(A)_n (B)_n}{n!(C)_n} a_0, \end{aligned} \tag{B.4}$$

and choosing $a_0 = 1$ gives the first solution

$$K(z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(C)_n} \frac{z^n}{n!}. \tag{B.5}$$

Now the second solution near $z = 0$ for (B.1) will be when $s = 1 - C$ in (B.2). Substituting $s = 1 - C$ into (B.3) gives

$$\begin{aligned} a_{n+1} &= \frac{(n+s)(n+s+A+B)+AB}{(n+s+1)(n+s+C)} a_n \\ &= \frac{[n+(2-C)-1][n+(2-C)-1+A+B]+AB}{(n+1)(n+(2-c))} a_n. \end{aligned}$$

Redefining $A' = 1 + A - C$, $B' = 1 + B - C$ and $C' = 2 - C$, we arrive at

$$a_{n+1} = \frac{(n+A')(n+B')}{(n+1)(n+C')} a_n.$$

Comparing with (B.4), our second solution is thus

$$K(z) = z^{1-C} F(A', B'; C', z) = z^{1-C} F(1 + A - C, 1 + B - C; 2 - C; z). \tag{B.6}$$

B.2 Hypergeometric Function Identities

The hypergeometric function has the simple identity

$$F(A, B; C; z) = F(B, A; C; z). \tag{B.7}$$

Euler's integral formula for $|z| < 1$ and $\mathrm{Re}(C) > \mathrm{Re}(B) > 0$ is

$$F(A, B; C; z) = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 t^{B-1}(1-t)^{C-B-1}(1-zt)^{-A} \mathrm{d}t. \tag{B.8}$$

To prove this, we start with

$$\frac{(B)_n}{(C)_n} = \frac{\Gamma(B+n)\Gamma(C)}{\Gamma(C+n)\Gamma(B)} = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \cdot \frac{\Gamma(B+n)\Gamma(C-B)}{\Gamma(C+n)}. \tag{B.9}$$

Recall that the beta function is defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t, \tag{B.10}$$

for $x, y \in \mathbb{C}$ with $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$. The beta function can be expressed as

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{B.11}$$

under analytic continuation. Using this we can express the last factor in (B.9) as

$$\frac{\Gamma(B+n)\Gamma(C-B)}{\Gamma(C+n)} = \beta(B+n, C-B) = \int_0^1 t^{B+n-1}(1-t)^{C-B-1} \mathrm{d}t.$$

Substituting this into $F(A,B;C;z)$ we find

$$\begin{aligned}
F(A,B;C;z) &= \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \sum_{n=0}^{\infty} \frac{(A)_n}{n!} \int_0^1 t^{B+n-1}(1-t)^{C-B-1} z^n \mathrm{d}t \\
&= \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 t^{B-1}(1-t)^{C-B-1} \sum_{n=0}^{\infty} \frac{(A)_n (zt)^n \mathrm{d}t}{n!} \\
&= \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 t^{b-1}(1-t)^{C-B-1} \sum_{n=0}^{\infty} \frac{(A)_n (zt)^n \mathrm{d}t}{n!} \\
&= \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 t^{B-1}(1-t)^{C-B-1}(1-zt)^{-A} \mathrm{d}t,
\end{aligned}$$

where we substituted the binomial series $(1-zt)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n (tz)^n}{n!}$ for $|zt| < 1$ to arrive at the identity.

We can also use the Euler integral formula to derive two new identities, known as the *Pfaff Transformations*

$$F(A,B;C;z) = (1-z)^{-B} F\left(B, C-A; C; \frac{z}{z-1}\right)$$
$$F(A,B;C;z) = (1-z)^{-A} F\left(A, C-B; C; \frac{z}{z-1}\right)$$

Taking Euler's integral (B.8) and letting $t = 1-s$, we calculate

$$\begin{aligned}
F(A,B;C;z) &= \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 (1-s)^{B-1} s^{C-B-1}((1-z)+sz)^{-A} \mathrm{d}s \\
&= (1-z)^{-A} \int_0^1 s^{C-B-1}(1-s)^{B-1}\left(1-\frac{sz}{z-1}\right)^{-A} \mathrm{d}s \\
&= (1-z)^{-A} F(A, C-B; C; -\tfrac{z}{1-z}).
\end{aligned} \tag{B.12}$$

Applying (B.7) to (B.12) gives

$$F(A,B;C;z) = (1-z)^{-A} F\left(C-B, A; C; -\frac{z}{1-z}\right).$$

Now if we let $w = -\frac{z}{1-z}$, so $1-w = \frac{1}{1-z}$ and therefore $z = -\frac{w}{1-w}$, applying the Pfaff transformation again results in Euler's transformation

$$\begin{aligned}
F(A,B;C;z) &= (1-z)^{-A}(1-w)^{B-C} F\left(C-B, C-A; C; \frac{-w}{1-w}\right) \\
&= (1-z)^{C-A-B} F(C-A, C-B; C; z).
\end{aligned} \tag{B.13}$$

Using Euler's integral formula (B.8) we can calculate the hypergeometric function at $z = 1$

$$\begin{aligned}
F(A,B;C;1) &= \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 t^{B-1}(1-t)^{C-B-A-1} \mathrm{d}t \\
&= \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \cdot \frac{\Gamma(B)\Gamma(C-A-B)}{\Gamma(C-A)} \\
&= \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}
\end{aligned} \tag{B.14}$$

where we have used (B.11).

The final transformation considered in this appendix is a translation to an expansion around $z = 1$.

$$\begin{aligned} F(A,B;C,z) = {} & \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} F(A,B;A+B+1-C;1-z) \\ & + \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} (1-z)^{C-A-B} \\ & \quad \cdot F(C-A,C-B;1+C-A-B;1-z). \end{aligned} \tag{B.15}$$

Proof of this can be found in [Bar08, Bai35, WW90, Leb65].

First, the solution to (B.1) can near $z = 0$ can be expressed as

$$K(z) = D_1 F(A,B;C,z) + D_2 z^{1-C} F(1+A-C,1+B-C;2-C;z). \tag{B.16}$$

Under the transformation $z' = 1 - z$, the parameters in (B.16) change as

$$A' = A, \quad B' = B, \quad C' = 1 + A + B - C.$$

Therefore (B.16) becomes

$$\begin{aligned} F(A,B;C;z) = {} & D_1' F(A,B;1+A+B-C,1-z) \\ & + D_2'(1-z)^{C-A-B} F(C-A,C-B;1-A-B+C;1-z). \end{aligned} \tag{B.17}$$

We will now assume that $\mathrm{Re}(A + B) < \mathrm{Re}(C) < 1$. Taking $z \to 1$ and $z \to 0$ in (B.17) and using (B.14), we obtain

$$\begin{aligned} D_1' &= \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} \\ 1 &= D_1' \frac{\Gamma(1+A+B-C)\Gamma(1-C)}{\Gamma(1+A-C)\Gamma(1+B-C)} + D_2' \frac{\Gamma(1-A-B+C)\Gamma(1-C)}{\Gamma(1-A)\Gamma(1-B)} \end{aligned} \tag{B.18}$$

Solving (B.18) for D_2', and substituting both D_1' and D_2' into (B.17), we arrive at the identity (B.15).

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