Conformal field theory and vertex operator algebras

by

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Abstract

We first cover some background on Lie algebras and their representations that is required for the main sections. We then introduce conformal field theory via a prototypical example: the free boson. As a part of the analysis we introduce key structural elements of the vertex operator algebra: states and fields, the state-field correspondence, and the operator product expansion. We then define the vertex operator algebra structure, and discuss the Zhu algebras that can be associated to them. We then define several key examples of vertex operator algebras and for each we compute the associated Zhu algebra and discuss the consequences for the representation theory. Finally we discuss level one Zhu algebras and identify the vector space underlying the level one Zhu algebra associated to the free boson. A level one version of the Zhu product is computed, but we do not identify the full algebra structure; further work is required.

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Chapter 1

Introduction

Quantum field theories are incredibly successful in describing the most fundamental building blocks of nature, but they are highly mathematically complex. We therefore frequently utilise the symmetries inherent in quantum field theories in order to solve problems. By a *symmetry* we mean any transformation that can be applied to a physical theory without altering the underlying physics. Symmetries of a physical theory can often be used to simplify problems that might be otherwise intractable, and continuous symmetries give rise to conservation laws that are fundamental in the description of a system's behaviour. Typical symmetries of a relativistic quantum field theory are spacetime translation invariance and Lorentz transformation invariance. A *conformal field theory* is a quantum field theory that in addition to the standard symmetries is also invariant under *conformal transformations*.

A conformal transformation is one that preserves angles, such as scaling, translations and rotations. The algebra of 2-dimensional local conformal transformations is generated by two copies of the Witt algebra, which implies that the symmetry algebra of a 2-dimensional conformal field theory is the unique central extension of the Witt algebra, which is called the *Virasoro algebra*.

In [1], Witten developed string theories in which spacetime was replaced by compact Lie groups. These models correspond to conformal field theories constructed from representations of affine Kac-Moody algebras. Contemporaneously, Belavin, Polyakov and Zamolodchikov published their seminal paper [2], in which they construct an important class of conformal field theories called the Virasoro minimal models, showing that they can be constructed from representations of the Virasoro algebra.

In [3] Borcherds defined the structure of a vertex algebra and was eventually successful in using it to prove *monstrous moonshine*, a long-standing conjecture of Conway and Norton [4]. Borcherds' vertex algebras were modified and extended into vertex operator algebras in works such as [5], [6], and [7], and these were recognised to be the algebraic structures axiomatising the symmetries of conformal field theories.

A new tool for studying the representation theory of rational vertex operator algebras came in the form of Zhu's algebra, which he constructed in his Ph.D thesis [8]. This is an associative algebra that can be associated to a vertex operator algebra, and Zhu demonstrated a bijective correspondence between the irreducible representations of the Zhu algebra and those of the original vertex operator algebra. In [9] Zhu and Frenkel compute the Zhu algebra of Witten's affine Kac-Moody vertex operator algebras, and of the Virasoro vertex operator algebra, and in [10] Wang computes the Zhu algebra of the Virasoro minimal models.

Another development came from [11] in which Dong, Li and Mason generalise the Zhu algebra $A(V)$ for a vertex operator algebra V to a sequence of associative algebras $A_n(V)$, $n \geq 0$. These higher Zhu algebras can be used to probe the indecomposable representations that occur for non-rational vertex operator algebras. There is some recent development of this higher level Zhu algebra theory in the works [12] and [13] in which Barron et al. refurbish some of the results of Dong, Li and Mason, prove some of their conjectures, and compute some key examples.

In this thesis we begin by presenting some elementary theory of Lie algebras that will be necessary background for the later material. We also define the Kac-Moody and Virasoro Lie algebras that form symmetry algebras for the conformal field theories of interest.

In Chapter 3 we introduce conformal field theory by defining and discussing a key example: the free boson. A key motive in this will be to introduce the essential elements of the vertex operator algebra, namely states and the state space, fields, the state-field correspondence, normal ordering, and the operator product expansion.

We begin Chapter 4 by defining the vertex operator algebra structure, as well as the Zhu algebra that can be associated to a vertex operator algebra. We use the results of [8], which give a bijective correspondence between the irreducible modules of a vertex operator algebra and those of its associated Zhu algebra. Since the Zhu algebra is always simpler than the original vertex operator algebra, this permits an easier analysis of the irreducible modules of a vertex operator algebra.

We then define the key objects of interest: the free boson vertex operator algebra, the Virasoro vertex operator algebra, and the Virasoro minimal models. For each example we compute the associated Zhu algebra and use it to identify the irreducible modules of the vertex operator algebra. For this we use the results of [14] which permit a more concrete method of computing the Zhu algebra than is typical in the literature.

We then conclude the thesis by beginning an analysis of the level one Zhu algebra for the free boson vertex operator algebra. We identify the underlying vector space, and define a level one Zhu product, but we do not complete the identification of the full associative algebra structure. We mention the possibility of future work to complete this analysis.

Chapter 2

Lie algebras

We will show in Chapter 3 that the vertex operator algebras associated to conformal field theories are closely related to certain Lie algebras, and the quantum state spaces form modules over these Lie algebras. Therefore in this chapter we briefly discuss the general theory of Lie algebras and their representations in order to have a framework for describing these objects. A *Lie algebra* consists of a vector space $\mathfrak{g}$ (over either $\mathbb{R}$ or $\mathbb{C}$) together with the *Lie bracket*, a bilinear antisymmetric map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{2.1}$$

that satisfies the *Jacobi identity*

$$\big[x,[y,z]\big] + \big[y,[z,x]\big] + \big[z,[x,y]\big] = 0 \qquad\qquad \forall x,y,z \in \mathfrak{g}. \tag{2.2}$$

This definition is motivated by the prototypical example of a Lie bracket, the *commutator*

$$[x,y] = xy - yx \tag{2.3}$$

which satisfies all the properties required of a Lie bracket.

We also define a *Lie algebra homomorphism* $\phi : \mathfrak{g} \to \mathfrak{h}$ between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ to be a linear map that preserves the bracket, that is, for all $x, y \in \mathfrak{g}$

$$\phi\big([x,y]\big) = \big[\phi(x), \phi(y)\big]. \tag{2.4}$$

If a Lie algebra homomorphism is also a bijection we call it a *Lie algebra isomorphism.*

Any associative algebra A can be made into a Lie algebra by defining the bracket according to (2.3). In particular, we have an important class of Lie algebras called *matrix Lie algebras.* The set of all $n \times n$ matrices with entries from a field of scalars $\mathbb{F}$, with bracket given by the commutator, forms the *general linear Lie algebra*, denoted by $\mathfrak{gl}(n;\mathbb{F})$. We may take for the field of scalars in the underlying vector space either $\mathbb{R}$ or $\mathbb{C}$, making $\mathfrak{gl}(n;\mathbb{F})$ either a *real* or *complex* Lie algebra, respectively.

A *Lie subalgebra* is a vector subspace of a Lie algebra that itself forms a Lie algebra under the same Lie bracket. $\mathfrak{gl}(n;\mathbb{F})$ has many interesting Lie subalgebras that are often useful in physical applications. For example, define the *special unitary Lie algebra* $\mathfrak{su}(n)$ to be the vector space of traceless and skew-hermitian $n \times n$ complex matrices:

$$\mathfrak{su}(n) = \left\{ A \in M_n(\mathbb{C}) \mid \mathrm{tr} A = 0 \text{ and } A^\dagger = -A \right\} \subset \mathfrak{gl}(n;\mathbb{C}) \tag{2.5}$$

Even though $\mathfrak{su}(n)$ is defined in terms of complex matrices, this is a real Lie algebra. The particular Lie algebra $\mathfrak{su}(2)$ has basis $\{i\sigma_x, i\sigma_y, i\sigma_z\}$ where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6}$$

are the *Pauli matrices*, which are used in the description of quantum spin.

A Lie algebra $\mathfrak{g}$ is called *abelian* when $[x,y] = [y,x]$ for all $x, y \in \mathfrak{g}$. If this is the case then antisymmetry implies that $[x,y] = 0$ for all $x, y \in \mathfrak{g}$. Conversely, if $[x,y] = 0$ for all $x, y \in \mathfrak{g}$ it is immediate that $\mathfrak{g}$ is abelian. So the abelian Lie algebras are exactly those whose bracket is identically zero.

The *dimension* of a Lie algebra is just the dimension of the underlying vector space. Consider a 1-dimensional Lie algebra $\mathfrak{g}$, generated by a single element $x \in \mathfrak{g}$. Bilinearity means that the bracket is entirely determined by the value of $[x,x]$, but antisymmetry implies that $[x,x] = 0$, which in turn means $\mathfrak{g}$ is abelian by the above.

We now come to those Lie algebras that will appear in the conformal field theories we are considering. First we define the infinite-dimensional complex Lie algebra $\widehat{\mathfrak{gl}}(1)$ having basis $\{a_n \mid n \in \mathbb{Z}\} \cup \{k\}$ and with bracket

$$[a_m, a_n] = m\delta_{m+n=0}k, \qquad [a_m, k] = 0. \tag{2.7}$$

This type of Lie algebra is usually called a *Kac-Moody algebra*, and more details about such Lie algebras can be found in Kac' book [15].

Next we define the *Virasoro algebra*, denoted by $\mathfrak{Vir}$, having basis $\{L_n \mid n \in \mathbb{Z}\} \cup \{c\}$ and bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n=0}c, \qquad [L_m, c] = 0. \tag{2.8}$$

The Virasoro algebra first appeared in [16], and its role in conformal field theory was identified in [2]. We will find both of these Lie algebras occurring as symmetry algebras in conformal field theories.

In conformal field theory, the elements of the Lie algebras are considered to be operators which act upon quantum state spaces. We describe this mathematically by representations. A *representation* of a Lie algebra $\mathfrak{g}$ is a linear map

$$\rho : \mathfrak{g} \to \mathrm{End}V, \tag{2.9}$$

where V is a vector space, that satisfies for all $x, y \in \mathfrak{g}$

$$\rho\big([x,y]\big) = \rho(x)\rho(y) - \rho(y)\rho(x). \tag{2.10}$$

Note that for any vector space V, the endomorphism space $\mathrm{End}V$ forms an associative algebra, and therefore a Lie algebra with bracket given by the commutator (2.3). With this structure a representation $\rho : \mathfrak{g} \to \mathrm{End}V$ becomes a Lie algebra homomorphism. It is common to associate an element $x \in \mathfrak{g}$ to its image $\rho(x) \in \mathrm{End}V$, and say that x *acts upon* $v \in V$, writing

$$xv \equiv \rho(x)(v). \tag{2.11}$$

In this framework V becomes a $\mathfrak{g}$-*module*, and the condition (2.10) becomes

$$[x,y]v = x(yv) - y(xv) \tag{2.12}$$

for all $x, y \in \mathfrak{g}$ and $v \in V$.

A *submodule* of a $\mathfrak{g}$-module V is a vector subspace $W \subseteq V$ that is closed under action by elements of $\mathfrak{g}$, that is, if $xw \in W$ for all $x \in \mathfrak{g}$ and $w \in W$. Every module V contains the trivial submodules V and $\{0\}$. If V has any other submodules, we call V a *reducible* module. Conversely, if V has only the trivial submodules, it is called *irreducible*.

Given a $\mathfrak{g}$-module V and a submodule $W \subseteq V$ we can take the quotient vector space V/W since a submodule is also a vector subspace. We can make V/W into a *quotient module* by defining the action of $x \in \mathfrak{g}$ on an equivalence class $\overline{v}$ by

$$x\overline{v} = \overline{xv} \tag{2.13}$$

where v may be any representative of its equivalence class. Since W is a submodule, we have

$$v \sim v' \implies v' - v \in W \implies x(v' - v) = xv' - xv \in W \implies xv \sim xv' \tag{2.14}$$

so this action is well-defined.

Chapter 3

Conformal Field Theory

3.1 The free boson

In this section we present an introduction to conformal field theory and vertex algebras via a prototypical example: the free boson. The content of this section is largely adapted from the textbook [17] and from the notes [18].

The *classical free boson* is described by a scalar field φ defined on a cylinder of circumference L:

$$\varphi(x,t) = \varphi(x+L,t). \tag{3.1}$$

We use a *Lorentzian metric* on the cylinder, i.e, we use signature $(-,+)$ for (t,x). The *action* of the free boson is

$$S[\varphi] = \frac{1}{2g}\int_{S^1\times\mathbb{R}} \partial_\mu\varphi\partial^\mu\varphi dxdt \tag{3.2}$$

where μ is summed over the set $\{x,t\}$. From this we can obtain the equations of motion in the style of Lagrangian mechanics, by extremising the action as a functional of φ.

Consider an infinitesimal transformation $\varphi \mapsto \varphi' = \varphi + \eta$, and the corresponding change in the action

$$S[\varphi'] - S[\varphi] = \frac{1}{2g}\int (\partial_\mu\varphi'\partial^\mu\varphi' - \partial_\mu\partial^\mu\varphi)dxdt. \tag{3.3}$$

After integrating by parts, noting that boundary terms vanish by periodicity in x, and assuming that φ and its derivatives vanish as $t \to \pm\infty$, we obtain the expression

$$S[\varphi'] - S[\varphi] = -\frac{1}{g}\int \eta\partial_\mu\partial^\mu\varphi dxdt. \tag{3.4}$$

To obtain an extremum of φ, this difference should be zero for an arbitrary infinitesimal η, which leads to the equations of motion

$$\partial_\mu \partial^\mu \varphi = 0 \qquad \Longrightarrow \qquad \partial_t^2 \varphi = \partial_x^2 \varphi. \tag{3.5}$$

We now want to confirm that our theory satisfies conformal invariance. An arbitrary infinitesimal conformal transformation on the complex plane can be written

$$z \mapsto z' = z + \epsilon(z), \qquad \bar{z} \mapsto \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) \tag{3.6}$$

where $\epsilon, \bar{\epsilon}$ are holomorphic. This fact can be found in [18] Chapter 2. In order to use this, we will map from the cylinder onto the complex plane using a transformation called a *Wick rotation*, in which we set $\tau = it$ so that the Lorentzian metric on (t, x) becomes a Euclidean metric (+,+) on (τ, x). We now map to the complex plane by setting

$$z = e^{2\pi(\tau + ix)/L} = e^{2\pi i(x-t)/L}, \qquad \bar{z} = e^{2\pi(\tau - ix)/L} = e^{-2\pi i(x+t)/L}. \tag{3.7}$$

which transforms the derivatives as

$$\partial_t = -\frac{2\pi i}{L} z\partial - \frac{2\pi i}{L}\bar{z}\bar{\partial}, \qquad \partial_x = \frac{2\pi i}{L} z\partial - \frac{2\pi i}{L}\bar{z}\bar{\partial}. \tag{3.8}$$

The equations of motion become

$$\partial\bar{\partial}\varphi = 0, \tag{3.9}$$

i.e, $\partial\varphi(z)$ is holomorphic, while $\bar{\partial}\varphi(\bar{z})$ is antiholomorphic.

We can now ask whether the theory is conformally invariant. More specifically, we want to know whether the action is invariant under a conformal transformation. The key idea here is that a conformal transformation is just a change of coordinates, and since we want the free boson field to be physically measurable, it cannot depend on the choice of coordinates, i.e, we stipulate

$$\varphi'(z', \bar{z}') = \varphi(z, \bar{z}). \tag{3.10}$$

Now, under the transformation (3.6) we have

$$\partial \equiv \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}\frac{\partial z'}{\partial z} = (1 + \partial\epsilon)\partial', \qquad \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}'}\frac{\partial \bar{z}'}{\partial \bar{z}} = (1 + \bar{\partial}\bar{\epsilon})\bar{\partial}' \tag{3.11}$$

and

$$dz' = \frac{\partial z'}{\partial z} dz = (1 + \partial\epsilon)dz, \qquad d\bar{z}' = \frac{\partial \bar{z}'}{\partial \bar{z}} d\bar{z} = (1 + \bar{\partial}\bar{\epsilon})d\bar{z}. \tag{3.12}$$

Writing

$$\frac{1}{1 + \partial\epsilon} = \frac{1 - \partial\epsilon}{(1 + \partial\epsilon)(1 - \partial\epsilon)} = \frac{1 - \partial\epsilon}{1 - (\partial\epsilon)^2}, \tag{3.13}$$

and similarly for $1/(1 + \bar{\partial}\bar{\epsilon})$, we ignore the squared ϵ terms to write

$$\partial' = (1 - \partial\epsilon)\partial, \qquad \bar{\partial}' = (1 - \bar{\partial}\bar{\epsilon})\bar{\partial} \tag{3.14}$$

and we can now observe

$$\begin{aligned}S'[\varphi'] &= \frac{1}{g}\int_{\mathbb{C}^2}\partial'\varphi'(z',\overline{z}')\overline{\partial}'\varphi'(z',\overline{z}')dz'd\overline{z}' \\ &= \frac{1}{g}\int_{\mathbb{C}^2}(1-\partial\epsilon)\partial\varphi(z,\overline{z})(1-\overline{\partial}\overline{\epsilon})\overline{\partial}\varphi(z,\overline{z})(1+\partial\epsilon)(1+\overline{\partial}\overline{\epsilon})dzd\overline{z} \\ &= \frac{1}{g}\int_{\mathbb{C}^2}\partial\varphi(z,\overline{z})\overline{\partial}\varphi(z,\overline{z})dzd\overline{z} = S[\varphi],\end{aligned} \tag{3.15}$$

again ignoring the terms in ϵ squared. This shows that the free boson is conformally invariant.

We now want to quantise the theory, which we accomplish by canonical quantisation. We Fourier expand the holomorphic field

$$\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \tag{3.16}$$

considering the modes a_n to be operators, which can be shown to satisfy the commutation relations

$$[a_m, a_n] = m\delta_{m+n=0}\frac{g}{4\pi^2}. \tag{3.17}$$

These are the $\widehat{\mathfrak{gl}}(1)$ commutation relations (2.7), with the central element k replaced by the scalar $g/4\pi^2$. We can rescale the modes by $\sqrt{g}/2\pi$ to obtain the more convenient commutation relation

$$[a_m, a_n] = m\delta_{m+n=0}. \tag{3.18}$$

We will derive these relations explicitly later in this Chapter.

The operators a_n act on a vector space called the *state space*. We construct the state space from a special element called a *vacuum*, denoted by $|p\rangle$, where p is a scalar called the *momentum* of the vacuum. The a_n are then characterised by their action upon the vacua. The a_n with $n > 0$ are called *annihilation operators*, satisfying

$$a_n |p\rangle = 0. \tag{3.19}$$

For $n = 0$, the *zero mode* a_0 satisfies

$$a_0 |p\rangle = p\,|p\rangle\,. \tag{3.20}$$

and is therefore known as the *momentum operator*. If $n < 0$ then a_n is a *creation operator*, which creates a new state when applied to a vacuum. Multiple creation operators can be applied to create further states, for example,

$$a_{-n_1}|p\rangle\,, \qquad a_{-n_1}a_{-n_2}|p\rangle\,, \qquad a_{-n_1}a_{-n_2}a_{-n_3}|p\rangle\,, \qquad n_i > 0 \tag{3.21}$$

are all distinct non-zero states. Note that since the modes a_n obey the $\widehat{\mathfrak{gl}}(1)$ commutation relations (3.18), any state $a_{-n_1}\cdots a_{-n_k}|p\rangle$ will be annihilated by any annihilation operator a_m with $m > \max\{n_1,\ldots,n_k\}$, since if $m \neq n_i$ for all i, then a_m freely commutes all the way to the right, where it annihilates the vacuum.

The vector space generated from $|p\rangle$ by applying creation operators, often called the *Fock space* $\mathcal{F}_p$, is spanned by the set

$$\{a_{-n_1} \cdots a_{-n_k} |0\rangle \mid k \geq 0;\, n_i > 0\}. \tag{3.22}$$

Now notice that the commutation relations (3.18) imply that all the creation operators commute amongst themselves, and therefore we can choose a canonical ordering for the indices, leading to the smaller spanning set

$$\{a_{-n_1} \cdots a_{-n_k} |0\rangle \mid k \geq 0;\, n_1 \geq n_2 \geq \cdots \geq n_k > 0\}. \tag{3.23}$$

That this set is in fact a basis is a result known as the Poincaré -Birkhoff-Witt theorem. This result is found in many textbooks on the subject of Lie Algebras, and in particular, a proof can be found in Section 9.4 of [19].

Now, we expect to find the Virasoro algebra as a symmetry algebra for a conformal field theory, and it will appear upon consideration of the *energy-momentum tensor*, which is defined classically by

$$T(z) = \frac{1}{2}\partial\varphi(z)\partial\varphi(z). \tag{3.24}$$

Canonical quantisation gives

$$T(z) = \frac{1}{2}\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} a_r a_s z^{-r-s-2} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[\sum_{r\in\mathbb{Z}} a_r a_{n-r}\right] z^{-n-2} \tag{3.25}$$

suggesting the modes of the EM tensor should be

$$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{n-r}. \tag{3.26}$$

To make this definition, we must ensure that each L_n acts in a well-defined way upon the states of the theory, which we accomplish by introducing normal ordering. The *normally-ordered product* of two modes is defined by

$$:a_m a_n: = \begin{cases} a_m a_n & \text{if } m \leq -1 \\ a_n a_m & \text{otherwise.} \end{cases} \tag{3.27}$$

We also speak of a normally-ordered product of fields, which in this case is defined as one might expect, in terms of the normally-ordered product of modes:

$$:\partial\varphi(z)\partial\varphi(z): = \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} :a_r a_s:\, z^{-r-s-2}. \tag{3.28}$$

We extend these normally-ordered products to arbitrarily many modes (or fields) inductively:

$$:a_{k_1} a_{k_2} \cdots a_{k_m}: = :a_{k_1}\, :a_{k_2} \cdots a_{k_m}::\,. \tag{3.29}$$

We now define the energy-momentum tensor as the normally-ordered product

$$T(z) = \frac{1}{2} : \partial\varphi(z)\partial\varphi(z) : = \frac{1}{2}\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} : a_r a_s : z^{-r-s-2} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[\sum_{r\in\mathbb{Z}} : a_r a_{n-r} :\right] z^{-n-2} \tag{3.30}$$

which suggests the appropriate definition of the modes is

$$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} : a_r a_{n-r} : = \frac{1}{2}\left[\sum_{r\leq -1} a_r a_{n-r} + \sum_{r\geq 0} a_{n-r} a_r\right]. \tag{3.31}$$

Defined in this way the L_n always act in a well-defined way on any state. To see why, let $|\psi\rangle$ be an arbitrary state, and consider

$$L_n |\psi\rangle = \frac{1}{2}\sum_{r\leq -1} a_r a_{n-r} |\psi\rangle + \frac{1}{2}\sum_{r\geq 0} a_{n-r} a_r |\psi\rangle . \tag{3.32}$$

In this expression, for fixed n the left sum terms are zero for sufficiently negative r, and the right sum terms are zero for sufficiently large r. So the whole expression is finite, and gives a well-defined action for L_n.

With the L_n now well-defined we can compute their commutation relations. First consider L_m with $m \neq 0$, which can be rewritten as

$$L_m = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{m-r} \tag{3.33}$$

without normal ordering as the modes all commute in this case. From this we compute

$$\begin{aligned} [L_m, a_n] &= \frac{1}{2}\sum_{r\in\mathbb{Z}} [a_r a_{m-r}, a_n] \\ &= \frac{1}{2}\sum_{r\in\mathbb{Z}} \big(a_r[a_{m-r}, a_n] + [a_r, a_n]a_{m-r}\big) \\ &= \frac{1}{2}\sum_{r\in\mathbb{Z}} \big((m-r)\delta_{m-r+n=0} a_r + r\delta_{r+n=0} a_{m-r}\big) \\ &= \frac{1}{2}\left(-n a_{m+n} - n a_{m+n}\right) = -n a_{m+n}. \end{aligned} \tag{3.34}$$

For $m = 0$, notice we can write

$$L_0 = \frac{1}{2} a_0^2 + \sum_{r>0} a_{-r} a_r \tag{3.35}$$

and therefore

$$\begin{aligned} [L_0, a_n] &= \frac{1}{2}[a_0^2, a_n] + \sum_{r>0} [a_{-r} a_r, a_n] \\ &= \sum_{r>0} \big(a_{-r}[a_r, a_n] + [a_{-r}, a_n]a_r\big) \\ &= \sum_{r>0} \left(r\delta_{r+n=0} a_{-r} - r\delta_{n-r=0} a_r\right). \end{aligned} \tag{3.36}$$

Now since $r > 0$, when $n > 0$ the left term is always zero, while only $r = n$ remains on the right, and similarly for $n < 0$, the right term is always zero, while $r = -n$ remains on the left. When $n = 0$, since a_0 is central in $\widehat{\mathfrak{gl}}(1)$, we have

$$[L_0, a_0] = 0$$

and therefore

$$[L_0, a_n] = -na_n \tag{3.37}$$

for all $n \in \mathbb{Z}$, which combined with (3.34) gives

$$[L_m, a_n] = -na_{m+n} \tag{3.38}$$

for all $m, n \in \mathbb{Z}$.

Now consider $m, n \in \mathbb{Z}$ satisfying $m + n \neq 0$. Then (3.38) implies

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{r\leq -1}[L_m, a_r a_{n-r}] + \frac{1}{2}\sum_{r\geq 0}[L_m, a_{n-r}a_r] \\
&= \frac{1}{2}\sum_{r\leq -1}\big(a_r[L_m, a_{n-r}] + [L_m, a_r]a_{n-r}\big) + \frac{1}{2}\sum_{r\geq 0}\big(a_{n-r}[L_m, a_r] + [L_m, a_{n-r}]a_r\big) \\
&= \frac{1}{2}\sum_{r\leq -1}\big((r-n)a_r a_{m+n-r} - r a_{m+r}a_{n-r}\big) + \frac{1}{2}\sum_{r\geq 0}\big(-r a_{n-r}a_{m+r} + (r-n)a_{m+n-r}a_r\big) \\
&= \frac{1}{2}\left(\sum_{r\leq -1}(r-n)a_r a_{m+n-r} + \sum_{r\geq 0}(r-n)a_{m+n-r}a_r\right) - \frac{1}{2}\left(\sum_{r\leq -1} r a_{m+r}a_{n-r} + \sum_{r\geq 0} r a_{n-r}a_{m+r}\right) \\
&= \frac{1}{2}\sum_{r\in\mathbb{Z}}(r-n)a_r a_{m+n-r} - \frac{1}{2}\sum_{r\in\mathbb{Z}} r a_{m+r}a_{n-r} \\
&= \frac{1}{2}\sum_{r\in\mathbb{Z}}(r-n)a_r a_{m+n-r} - \frac{1}{2}\sum_{r\in\mathbb{Z}}(r-m)a_r a_{m+n-r} \\
&= (m-n)\frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{m+n-r} = (m-n)L_{m+n}
\end{aligned} \tag{3.39}$$

where we have used that all the modes commute since $m+n \neq 0$, and performing the change of variables $r \mapsto r - m$ to obtain the second to last line.

Now, if $m \neq 0$ then we have

$$\begin{aligned}
[L_m, L_{-m}] &= [L_m, \frac{1}{2}\sum_{r\leq -1} a_r a_{-m-r} + \frac{1}{2}\sum_{r\geq 0} a_{-m-r} a_r] \\
&= \frac{1}{2}\sum_{r\leq -1}[L_m, a_r a_{-m-r}] + \frac{1}{2}\sum_{r\geq 0}[L_m, a_{-m-r} a_r] \\
&= \frac{1}{2}\sum_{r\leq -1}\big(a_r[L_m, a_{-m-r}] + [L_m, a_r]a_{-m-r}\big) + \frac{1}{2}\sum_{r\geq 0}\big(a_{-m-r}[L_m, a_r] + [L_m, a_{-m-r}]a_r\big) \\
&= \frac{1}{2}\sum_{r\leq -1}\big((m+r)a_r a_{-r} - r a_{m+r} a_{-m-r}\big) + \frac{1}{2}\sum_{r\geq 0}\big(-r a_{-m-r} a_{m+r} + (m+r) a_{-r} a_r\big) \\
&= \frac{1}{2}\sum_{r\in\mathbb{Z}}(m+r) : a_r a_{-r} : -\frac{1}{2}\left(\sum_{r\leq -1} r a_{m+r} a_{-m-r} + \sum_{r\geq 0} r a_{-m-r} a_{m+r}\right) \\
&= \frac{1}{2}\sum_{r\in\mathbb{Z}}(m+r) : a_r a_{-r} : -\frac{1}{2}\left(\sum_{r\leq m-1}(r-m) a_r a_{-r} + \sum_{r\geq m}(r-m) a_{-r} a_r\right)
\end{aligned} \tag{3.40}$$

If $m > 0$ then we can write

$$\begin{aligned}
&\sum_{r\leq m-1}(r-m)a_r a_{-r} + \sum_{r\geq m}(r-m)a_{-r}a_r \\
&\quad= \sum_{r\leq -1}(r-m)a_r a_{-r} + \sum_{0\leq r<m}(r-m)a_r a_{-r} + \sum_{r\geq 0}(r-m)a_{-r}a_r - \sum_{0\leq r<m}(r-m)a_{-r}a_r \\
&\quad= \sum_{r\in\mathbb{Z}}(r-m) : a_r a_{-r} : + \sum_{0\leq r<m}(r-m)[a_r, a_{-r}] \\
&\quad= \sum_{r\in\mathbb{Z}}(r-m) : a_r a_{-r} : + \sum_{0\leq r<m} r(r-m) \\
&\quad= \sum_{r\in\mathbb{Z}}(r-m) : a_r a_{-r} : + \frac{1}{6}(m - m^3)
\end{aligned} \tag{3.41}$$

Inserting this back into (3.40) gives

$$\begin{aligned}
[L_m, L_{-m}] &= \frac{1}{2}\sum_{r\in\mathbb{Z}}(m+r) : a_r a_{-r} : -\frac{1}{2}\left(\sum_{r\in\mathbb{Z}}(r-m) : a_r a_{-r} : + \sum_{0\leq r<m} r(r-m)\right) \\
&= m\sum_{r\in\mathbb{Z}} : a_r a_{-r} : -\frac{1}{2}\left(\frac{1}{6}(m-m^3)\right) \\
&= 2mL_0 + \frac{m^3 - m}{12} \qquad (m > 0)
\end{aligned} \tag{3.42}$$

Now for $m < 0$ antisymmetry implies

$$\begin{aligned}
[L_m, L_{-m}] = -[L_{-m}, L_m] &= -\left(2(-m)L_0 + \frac{(-m)^3 - (-m)}{12}\right) \\
&= 2mL_0 + \frac{m^3 - m}{12} \qquad (m < 0)
\end{aligned} \tag{3.43}$$

and therefore

$$[L_m, L_{-m}] = 2mL_0 + \frac{m^3 - m}{12} \qquad (m \neq 0) \tag{3.44}$$

The combination of the cases (3.39) and (3.44), as well as the immediate fact that $[L_m, L_m] = 0$ $m \in \mathbb{Z}$, gives the full result for $m, n \in \mathbb{Z}$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n=0}, \tag{3.45}$$

which are the commutation relations for the Virasoro algebra as defined in (2.8) with $c = 1$.

We now introduce a crucial component of conformal field theory, the *state-field correspondence*. This is a bijective correspondence between quantum states and quantum fields. The map in one direction can be written simply: the state corresponding to a field $\psi(z)$ is given by

$$|\psi\rangle = \lim_{z\to 0} \psi(z) |0\rangle \tag{3.46}$$

where $|0\rangle$ is the zero-momentum vacuum state. The reason for taking the limit $z \to 0$ is that this limit corresponds under (3.7) to taking $t \to -\infty$ on the cylinder on which we originally defined the free boson, thus taking the state to the infinite past. By way of example, and for later use, we compute some states of particular interest:

$$\begin{aligned}
|\partial^k \varphi\rangle = \lim_{z\to 0} \partial^k \varphi(z) |0\rangle &= \lim_{z\to 0} \sum_{n\in\mathbb{Z}} (-n-1)(-n-2)\cdots(-n-k+1) a_n |0\rangle z^{-n-k} \\
&= \lim_{z\to 0} \sum_{n\le -k} (-n-1)(-n-2)\cdots(-n-k+1) a_n |0\rangle z^{-n-k} \\
&= (k-1)! a_{-k} |0\rangle
\end{aligned} \tag{3.47}$$

and writing $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$,

$$\begin{aligned}
|T\rangle = \lim_{z\to 0} T(z) |0\rangle &= \lim_{z\to 0} \sum_{n\in\mathbb{Z}} L_n |0\rangle z^{-n-2} \\
&= \lim_{z\to 0} \sum_{n<-2} L_n |0\rangle z^{-n-2} \\
&= L_{-2} |0\rangle,
\end{aligned} \tag{3.48}$$

where we have used that

$$L_{-1} |0\rangle = \frac{1}{2} \sum_{r\le -1} a_r a_{-1-r} |0\rangle + \frac{1}{2} \sum_{r\ge 0} a_{-1-r} a_r |0\rangle = 0 \tag{3.49}$$

We now introduce the *operator product expansion*, which will allow us to separate the singularities of a product of fields from the regular terms, simplifying computation of many integrals and facilitating calculations in the vertex operator algebras that we will discuss in Chapter 4. In a similar fashion to how normal ordering ensures a well-defined product of two fields with the same argument, we introduce radial ordering to define a product of fields with different arguments. The *radially-ordered product* of two fields is defined by

$$\mathcal{R}\{A(z)B(w)\} = \begin{cases} A(z)B(w) & \text{if } |z| > |w|, \\ B(w)A(z) & \text{if } |z| < |w| \end{cases} \tag{3.50}$$

with the product of three or more fields defined by extending inductively. We note that that the radial direction in $\mathbb{C}$ corresponds under the map (3.7) to the time direction on the cylinder, thus radial ordering of the fields on $\mathbb{C}$ is equivalent to time ordering on the cylinder.

The radially-ordered product can now be expanded into an operator product expansion. We present the method for $\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\}$ as an example. First suppose $|z| > |w|$, then we have

$$\begin{aligned} \mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} = \partial\varphi(z)\partial\varphi(w) &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} a_r a_s z^{-r-1} w^{-s-1} \\ &= \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} a_r a_{n-r} z^{-r-1} w^{-n-1+r} \end{aligned} \tag{3.51}$$

using a change of variables $s \mapsto n - r$. Now, when $n \neq 0$, a_r commutes with a_{n-r}, so we can add normal ordering to those terms, and extract the $n = 0$ term:

$$\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} = \sum_{n\neq 0}\sum_{r\in\mathbb{Z}} : a_r a_{n-r} : z^{-r-1} w^{-n-1+r} + \sum_{r\in\mathbb{Z}} a_r a_{-r} z^{-r-1} w^{r-1} \tag{3.52}$$

Consider for a moment only $n = 0$. When $r \leq -1$, the product $a_r a_{-r}$ is already normally ordered. When $r \geq 0$ we can obtain normal ordering by using the commutator (3.18) to write $a_r a_{-r} = a_{-r} a_r + r$ which gives for the second term in (3.52)

$$\sum_{r\in\mathbb{Z}} a_r a_{-r} z^{-r-1} w^{r-1} = \sum_{r\in\mathbb{Z}} : a_r a_{-r} : z^{-r-1} w^{r-1} + \sum_{r\geq 0} r z^{-r-1} w^{r-1} \tag{3.53}$$

where now the trailing term can be rearranged into the derivative of a geometric series:

$$\sum_{r\geq 0} r z^{-r-1} w^{r-1} = \frac{1}{z^2}\sum_{r\geq 0} r \left(\frac{w}{z}\right)^{r-1} = \frac{1}{z^2}\frac{1}{\left(1 - w/z\right)^2} = \frac{1}{(z-w)^2}, \tag{3.54}$$

where convergence is ensured since $|z| > |w|$. We now have

$$\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} = \frac{1}{(z-w)^2} + \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} : a_r a_{n-r} : z^{-r-1} w^{-n-1+r} \tag{3.55}$$

and we can change variables back in the first term with $n \mapsto s + r$ giving

$$\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} = \frac{1}{(z-w)^2} + \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} : a_r a_s : z^{-r-1} w^{-s-1} = \frac{1}{(z-w)^2} + : \partial\varphi(z)\partial\varphi(w) : . \tag{3.56}$$

If on the other hand $|z| < |w|$ an almost identical computation gives the expansion

$$\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} = \frac{1}{(w-z)^2} + : \partial\varphi(w)\partial\varphi(z) : . \tag{3.57}$$

Finally noting that $:\partial\varphi(w)\partial\varphi(z): = :\partial\varphi(z)\partial\varphi(w):$ and Taylor expanding the normally ordered product we obtain the full operator product expansion

$$\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} = \frac{1}{(z-w)^2} + :\partial\varphi(w)\partial\varphi(w): + :\partial^2\varphi(w)\partial\varphi(w):(z-w) + \cdots \tag{3.58}$$

Since a product of fields with different arguments is only well-defined when it is radially ordered, in future we will assume all such products are radially ordered unless otherwise specified, and omit the notation, that is,

$$\mathcal{R}\{A(z)B(w)\} \equiv A(z)B(w). \tag{3.59}$$

Note that the expansion (3.58) provides an explicit formula for the normally ordered product of fields:

$$:\partial\varphi(w)\partial\varphi(w): = \oint_w \frac{\partial\varphi(z)\partial\varphi(w)}{z-w}\frac{dz}{2\pi i}. \tag{3.60}$$

We will use this formula as our definition of the normally ordered product of general fields:

$$:A(z)B(z): = \oint_w \frac{A(z)B(w)}{z-w}\frac{dz}{2\pi i} \tag{3.61}$$

and extend to three or more fields inductively as in (3.29):

$$:A_1(z_1)A_2(z_2)\cdots A_m(z_m): = :A_1(z_1):A_2(z_2)\cdots A_m(z_m):: \tag{3.62}$$

Now, having obtained the OPE, we can use it to compute many quantities of interest. For example, we can calculate the commutation relations for the modes of the fields, now justifying what we earlier assumed in (3.18). Noting that

$$\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \qquad \Longleftrightarrow \qquad a_n = \oint_0 \partial\varphi(z) z^n \frac{dz}{2\pi i} \tag{3.63}$$

we can write

$$[a_m, a_n] = \oint_0\oint_0 \partial\varphi(z)\partial\varphi(w) z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i} - \oint_0\oint_0 \partial\varphi(w)\partial\varphi(z) z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i} \tag{3.64}$$

where the field products are not yet radially ordered. To simplify this we fix w and deform the z contours so that $|z| > |w|$ in the first term and $|z| < |w|$ in the second term. In this way the field products become radially ordered, so the integrands can be identified, and the integrals combined

$$[a_m, a_n] = \left[\underset{|z|>|w|}{\oint_0\oint_0} - \underset{|z|<|w|}{\oint_0\oint_0}\right] \partial\varphi(z)\partial\varphi(w) z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}, \tag{3.65}$$

wrapping up much of the complexity into the contours. Now we recognise that this difference of contours is equivalent to first integrating z on a small circle around w, then integrating w around a circle about the origin

$$\left[\oint_0 \oint_{\substack{0\\|z|>|w|}} - \oint_0 \oint_{\substack{0\\|z|<|w|}}\right] = \oint_0 \oint_w . \tag{3.66}$$

This now lets us insert the OPE and complete the calculation using residue calculus

$$\begin{aligned}
[a_m, a_n] &= \oint_0 \oint_w \partial\varphi(z)\partial\varphi(w) z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \oint_0 \oint_w \left[\frac{1}{(z-w)^2} + :\partial\varphi(z)\partial\varphi(w):\right] z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \oint_0 \left[\frac{z^m}{(z-w)^2}\frac{dz}{2\pi i}\right] w^n \frac{dw}{2\pi i} = \int_0 m w^{m-1} w^n \frac{dw}{2\pi i} = m\delta_{m+n=0}.
\end{aligned} \tag{3.67}$$

We note that in this calculation, only the singular terms of the OPE contributed to the result. This is usually the case, and so we often omit the regular terms when writing an OPE:

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{1}{(z-w)^2}, \tag{3.68}$$

using a '~' to indicate that the regular terms have been dropped.

So OPEs let us calculate commutation relations, but the converse is true as well: the commutation relations can be used to compute OPEs. To see how, assume that the product has a standard form

$$\partial\varphi(z)\partial\varphi(w) = \sum_{n\in\mathbb{Z}} \psi_n(w)(z-w)^{-n-1} \tag{3.69}$$

where the ψ_n are unknown fields. We use the state-field correspondence to map both sides to their states giving

$$\partial\varphi(z) |\partial\varphi\rangle = \sum_{n\in\mathbb{Z}} |\psi_n\rangle z^{-n-1} \tag{3.70}$$

and expanding $\partial\varphi(z)$ we obtain

$$\sum_{n\in\mathbb{Z}} a_n |\partial\varphi\rangle z^{-n-1} = \sum_{n\in\mathbb{Z}} |\psi_n\rangle z^{-n-1}. \tag{3.71}$$

Here we can match coefficients to conclude

$$|\psi_n\rangle = a_n |\partial\varphi\rangle = a_n a_{-1} |0\rangle . \tag{3.72}$$

The singular terms in the expansion (3.69) correspond to $n \geq 0$. When $n \geq 2$ the mode a_n commutes with a_{-1} and annihilates the vacuum, so these $|\psi_n\rangle$ are equal to zero. For $n = 0$ we have

$$|\psi_0\rangle = a_0 a_{-1} |0\rangle = a_{-1} a_0 |0\rangle = 0 \cdot |0\rangle = 0. \tag{3.73}$$

When $n = 1$ we have

$$|\psi_1\rangle = a_1 a_{-1} |0\rangle = a_{-1} a_1 |0\rangle + 1 \cdot \mathrm{Id} |0\rangle = |0\rangle , \tag{3.74}$$

so we conclude that $\psi_1(w) = \mathrm{Id}$, and therefore

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{1}{(z-w)^2}. \tag{3.75}$$

The regular terms can be computed similarly, if one were interested.

Chapter 4

Vertex operator algebras

Note: In this chapter we use the convention $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

The symmetries of a conformal field theory have the mathematical structure of a *vertex operator algebra* (VOA). We adapt the definition from that provided in [20]. A *vertex operator algebra* consists of an $\mathbb{N}$-graded vector space

$$\mathcal{F} = \bigoplus_{n \in \mathbb{N}} \mathcal{F}_n \tag{4.1}$$

called the *state space*, equipped with the *state-field correspondence*, a linear map

$$\begin{aligned} \mathcal{F} &\to (\operatorname{End} \mathcal{F})[[z, z^{-1}]] \\ v &\mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, \end{aligned} \tag{4.2}$$

where $Y(v, z)$ is known as the *vertex operator* or *field* associated to v. $\mathcal{F}$ contains two special elements, the *vacuum* $|0\rangle \in \mathcal{F}_0$, and the *conformal element* $\omega \in \mathcal{F}_2$. We say $v \in \mathcal{F}_n$ has *weight* n. These data are subject to the following axioms:

- **vacuum**:
$$Y\big(|0\rangle, z\big) = \mathrm{Id}_{\mathcal{F}} \qquad \text{and} \qquad Y(v, z)\,|0\rangle\big|_{z=0} = v. \tag{4.3}$$

- **Virasoro symmetry**: the modes of $T = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfy the Virasoro commutation relations
$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n=0}c \tag{4.4}$$
where $c \in \mathbb{C}$ is called the *central charge*.

- **compatability of L_0 with the grading**: for $v \in \mathcal{F}_n$,
$$L_0 v = nv. \tag{4.5}$$

- **derivative property of L_{-1}:**

$$[L_{-1}, Y(v,z)] = \partial Y(v,z). \tag{4.6}$$

- **Jacobi identity**: letting $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$ be the *formal delta function*,

$$z_1^{-1}\delta\left(\frac{z_2 - z_3}{z_1}\right)Y(u,z_2)Y(v,z_3) - z_1^{-1}\delta\left(\frac{z_2 - z_3}{z_1}\right)Y(v,z_3)Y(u,z_2)$$
$$= z_3^{-1}\delta\left(\frac{z_2 - z_1}{z_3}\right)Y\left(Y(u,z_1)v, z_3\right). \tag{4.7}$$

  This statement looks very different from the standard Jacobi identity for a Lie algebra bracket. We can illustrate the idea of the connection between the two identities with an analogy. We consider (4.7) acting on a third vector w, and then the ordering of the actions is analagous to

$$\left[u,[v,w]\right] - \left[v,[u,w]\right] = \left[[u,v],w\right]. \tag{4.8}$$

  Now by applying antisymmetry this becomes exactly the standard Jacobi identity for a Lie algebra.

In order to analyse the representation theory of a VOA V, we apply the simplifying results in [21]. There Zhu describes an associative algebra $A(V)$ that can be associated to a VOA V, and with the key result Theorem 2.2.2 demonstrates up to isomorphism a bijective correspondence between the irreducible modules of this *Zhu algebra* or *zero mode algebra* and those of the original VOA. The Zhu algebra typically has a more tractable representation theory, so the correspondence permits a simpler analysis of the VOA-modules.

We define the *Zhu product* on fields $A, B \in V$ by

$$A * B = \sum_{j\geq -h_A} \binom{h_A}{j+h_A} A_j B = \oint_0 A(z)B\frac{(1+z)^{h_A}}{z}\frac{dz}{2\pi i}. \tag{4.9}$$

as well as the *circle product*

$$A \circ B = \sum_{j\geq -h_A-1} \binom{h_A}{j+h_A+1} A_j B = \oint_0 A(z)B\frac{(1+z)^{h_A}}{z^2}\frac{dz}{2\pi i}. \tag{4.10}$$

Let $O(V)$ denote the span of all elements of the form $A \circ B$. Zhu's algebra is then defined to be the quotient

$$A(V) = V/O(V). \tag{4.11}$$

Theorem 2.1.1 of [21] states that $A(V)$ a unital associative algebra under (4.9).

In an appendix to [14], Ridout and Wood explain how the Zhu algebra of a VOA admits a concrete description as an algebra of the zero modes of the fields as they act upon ground states, where a *ground* *state* means a state $|v\rangle$ satisfying

$$B_n |v\rangle = 0, \qquad \forall n > 0, \;\; B \in V. \tag{4.12}$$

In this context we adapt the product (4.9) for use with the zero modes of fields A and B by defining

$$A_0 * B_0 = \sum_{0 \geq j \geq -h_A} \binom{h_A}{j + h_A} (A_j B)_0 \tag{4.13}$$

where $(A_j B)_0$ denotes the zero mode of the field corresponding to the state $A_j B$. We use the notation $A \sim B$ to indicate that operators A and B have the same action upon ground states.

To see how the form of the Zhu product is derived, consider the contour integral

$$\oint_0 \oint_w \frac{A(z)B(w)z^{h_A} w^{h_B - 1}}{z - w} \frac{dz}{2\pi i} \frac{dw}{2\pi i} \tag{4.14}$$

where $A(z), B(w)$ are arbitrary fields of conformal dimension h_A, h_B respectively. On one hand, by utilising the contour manipulation trick (3.66) and using geometric series we compute

$$\begin{aligned}
&\oint_0 \oint_w \frac{A(z)B(w)z^{h_A} w^{h_B - 1}}{z - w} \frac{dz}{2\pi i} \frac{dw}{2\pi i} \\
&\quad = \oint_0 \underset{|z|>|w|}{\oint_0} \frac{A(z)B(w)z^{h_A} w^{h_B - 1}}{z - w} \frac{dz}{2\pi i} \frac{dw}{2\pi i} - \oint_0 \underset{|z|<|w|}{\oint_0} \frac{B(w)A(z)z^{h_A} w^{h_B - 1}}{z - w} \frac{dz}{2\pi i} \frac{dw}{2\pi i} \\
&\quad = \oint_0 \underset{|z|>|w|}{\oint_0} \frac{A(z)B(w)z^{h_A - 1} w^{h_B - 1}}{1 - \frac{w}{z}} \frac{dz}{2\pi i} \frac{dw}{2\pi i} + \oint_0 \underset{|z|<|w|}{\oint_0} \frac{B(w)A(z)z^{h_A} w^{h_B - 2}}{1 - \frac{z}{w}} \frac{dz}{2\pi i} \frac{dw}{2\pi i} \\
&\quad = \oint_0 \underset{|z|>|w|}{\oint_0} \sum_{j \geq 0} A(z)B(w)z^{h_A - j - 1} w^{h_B + j - 1} \frac{dz}{2\pi i} \frac{dw}{2\pi i} + \oint_0 \underset{|z|<|w|}{\oint_0} \sum_{j \geq 0} B(w)A(z)z^{h_A + j} w^{h_B - j - 2} \frac{dz}{2\pi i} \frac{dw}{2\pi i} \\
&\quad = \sum_{j \geq 0} \left[\oint_0 A(z) z^{h_A - j - 1} \frac{dz}{2\pi i} \oint_0 B(w) w^{h_B + j - 1} \frac{dw}{2\pi i} + \oint_0 B(w) w^{h_B - j - 2} \frac{dw}{2\pi i} \oint_0 A(z) z^{h_A + j} \frac{dz}{2\pi i} \right] \\
&\quad = \sum_{j \geq 0} \left[A_{-j} B_j + B_{-j-1} A_{j+1} \right] \\
&\quad \sim A_0 B_0.
\end{aligned} \tag{4.15}$$

On the other hand, if we instead insert the general OPE

$$A(z)B(w) = \sum_{n \in \mathbb{Z}} (A_n B)(w)(z - w)^{-n - h_A} \tag{4.16}$$

then we have

$$\begin{aligned}
&\oint_0 \oint_w \frac{A(z)B(w)z^{h_A}w^{h_B-1}}{z-w}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&\quad = \oint_0 \oint_w \left[\sum_{j\in\mathbb{Z}}(A_jB)(w)(z-w)^{-j-h_A-1}\right] z^{h_A}w^{h_B-1}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&\quad = \oint_0 \sum_{j\in\mathbb{Z}}(A_jB)(w)w^{h_B-1}\oint_w \frac{z^{h_A}}{(z-w)^{j+h_A+1}}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&\quad = \oint_0 \sum_{0\geq j\geq -h_A}(A_jB)(w)w^{h_B-1}\left[\frac{h_A!}{(-j)!(h_A+j)!}w^{-j}\right]\frac{dw}{2\pi i} \\
&\quad = \sum_{0\geq j\geq -h_A}\binom{h_A}{-j}\oint_0 (A_jB)(w)w^{h_B-j-1}\frac{dw}{2\pi i} \\
&\quad = \sum_{0\geq j\geq -h_A}\binom{h_A}{j+h_A}(A_jB)_0,
\end{aligned} \tag{4.17}$$

exactly the form of the Zhu product in (4.13). Comparing (4.15) and (4.17) now justifies the Zhu product as a product of zero modes of fields acting on ground states. The circle product can also be derived using the same technique, but instead beginning from the expression

$$\oint_0 \oint_w \frac{A(z)B(w)z^{h_A+1}w^{h_B-1}}{(z-w)^2}\frac{dz}{2\pi i}\frac{dw}{2\pi i}. \tag{4.18}$$

In the computations of this chapter it will frequently be useful to have an explicit expression for the modes of a normally-ordered product of general fields. We compute this now from the definition

(3.61):

$$
\begin{aligned}
:AB: &= \oint_w \frac{\mathcal{R}\{A(z)B(w)\}}{z-w}\frac{dz}{2\pi i} \\
&= \oint_{|w|<|z|} \frac{A(z)B(w)}{z-w}\frac{dz}{2\pi i} - \oint_{|z|<|w|} \frac{B(w)A(z)}{z-w}\frac{dz}{2\pi i} \\
&= \oint_{|w|<|z|} A(z)B(w)z^{-1}\sum_{n\geq 0}\left(\frac{w}{z}\right)^n \frac{dz}{2\pi i} + \oint_{|z|<|w|} B(w)A(z)w^{-1}\sum_{n\geq 0}\left(\frac{z}{w}\right)^n \frac{dz}{2\pi i} \\
&= \sum_{n\geq 0} w^n \oint_{|w|<|z|} A(z)z^{-n-1}\frac{dz}{2\pi i}B(w) + \sum_{n\geq 0} w^{-n-1}B(w)\oint_{|z|<|w|} A(z)z^n\frac{dz}{2\pi i} \\
&= \sum_{n\geq 0} w^n A_{-n-h_A}B(w) + \sum_{n\geq 0} w^{-n-1}B(w)A_{n-h_A+1} \\
&= \sum_{n\geq 0} w^n A_{-n-h_A}\left(\sum_{m\in\mathbb{Z}} B_m w^{-m-h_B}\right) + \sum_{n\geq 0} w^{-n-1}\left(\sum_{m\in\mathbb{Z}} B_m w^{-m-h_B}\right)A_{n-h_A+1} \\
&= \sum_{m\in\mathbb{Z}}\left(\sum_{n\geq 0} w^{n-m-h_B}A_{-n-h_A}B_m + \sum_{n\geq 0} w^{-n-m-h_B-1}B_m A_{n-h_A+1}\right) \\
&= \sum_{m\in\mathbb{Z}}\left(\sum_{n\leq -1} w^{-n-m-h_B-1}A_{n-h_A+1}B_m + \sum_{n\geq 0} w^{-n-m-h_B-1}B_m A_{n-h_A+1}\right) \\
&= \sum_{m\in\mathbb{Z}}\left(\sum_{r\leq -h_A} w^{-r-m-h_A-h_B}A_r B_m + \sum_{r>-h_A} w^{-r-m-h_A-h_B}B_m A_r\right) \\
&= \sum_{n\in\mathbb{Z}}\left(\sum_{r\leq -h_A} A_r B_{n-r} + \sum_{r>-h_A} B_{n-r}A_r\right)w^{-n-h_A-h_B}.
\end{aligned}
\tag{4.19}
$$

This gives the n^{th} mode

$$
:AB:_n = \sum_{r\leq -h_A} A_r B_{n-r} + \sum_{r>-h_A} B_{n-r}A_r \tag{4.20}
$$

and in particular the zero mode

$$
:AB:_0 = \sum_{r\leq -h_A} A_r B_{-r} + \sum_{r>-h_A} B_{-r}A_r. \tag{4.21}
$$

4.1 The free boson

For this section let V denote the free boson VOA, also known in the literature as the Heisenberg VOA. For the state-field correspondence, we define the field $\psi(z) \equiv Y\left(|\psi\rangle, z\right)$ corresponding to the state $|\psi\rangle$ by the relation

$$
|\psi\rangle = \lim_{z\to 0}\psi(z)|0\rangle. \tag{4.22}
$$

The state space of V has basis

$$
\{a_{-n_1}\cdots a_{-n_k}|0\rangle \,\big|\, k\geq 0;\, n_1\geq n_2\geq\cdots\geq n_k>0\} \tag{4.23}
$$

where $|0\rangle$ is the vacuum, and the a_n are the modes of

$$Y(a_{-1}|0\rangle, z) \equiv \partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \tag{4.24}$$

satisfying the $\widehat{\mathfrak{gl}}(1)$ commutation relations (3.18), as well as

$$a_n|0\rangle = 0, \qquad \forall n \geq 0. \tag{4.25}$$

Section 6.3 of [20] covers many details of this and related structures, including a proof that this is indeed a vertex operator algebra.

The space of fields of V has basis

$$\left\{ :\partial^{k_1}\varphi(z)\cdots\partial^{k_m}\varphi(z): \middle| m \geq 0;\, k_1,\dots,k_m \geq 1 \right\} \tag{4.26}$$

where normal ordering is defined as in (3.27) and (3.29). If we define

$$T(z) := \frac{1}{2} :\partial\varphi(z)\partial\varphi(z): = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[\sum_{r\leq -1} a_r a_{n-r} + \sum_{r\geq 0} a_{n-r}a_r\right] z^{-n-2} \tag{4.27}$$

then the modes

$$L_n := \frac{1}{2}\left[\sum_{r\leq -1} a_r a_{n-r} + \sum_{r\geq 0} a_{n-r}a_r\right] \tag{4.28}$$

satisfy the Virasoro commutation relations (3.45) with central charge $c = 1$, and L_{-1} satisfies the derivative property (4.6). By applying (4.22) to $T(z)$ we can identify the conformal element of the state space to be

$$\omega = \frac{1}{2}a_{-1}^2|0\rangle. \tag{4.29}$$

To compute the Zhu algebra for the free boson we need to identify how the zero modes of the fields act upon ground states, and we note that it is sufficient to compute how the zero modes of the basis fields (4.26) act on ground states. The $m = 1$ case is easy to compute directly:

$$\left(\partial^k\varphi\right)_0 = (-1)^{k-1}(k-1)!a_0. \tag{4.30}$$

Utilising (3.29) and (4.21) we can compute the general case

$$\begin{aligned} :\partial^{k_1}\varphi\partial^{k_2}\varphi\cdots\partial^{k_m}\varphi:_0 &= :\partial^{k_1}\varphi\left(:\partial^{k_2}\varphi\cdots\partial^{k_m}\varphi:\right):_0 \\ &= \sum_{r\leq -k_1}(\partial^{k_1}\varphi)_r(:\partial^{k_2}\varphi\cdots\partial^{k_m}\varphi:)_{-r} + \sum_{r>-k_1}(:\partial^{k_2}\varphi\cdots\partial^{k_m}\varphi:)_{-r}(\partial^{k_1}\varphi)_r \\ &\sim \sum_{0\geq r>-k_1}(:\partial^{k_2}\varphi\cdots\partial^{k_m}\varphi:)_{-r}(\partial^{k_1}\varphi)_r \end{aligned} \tag{4.31}$$

where we have discarded each summand having a positive mode acting first, as these always annihilate ground states. Now examining the mode

$$(\partial^{k_1}\varphi)_r = (-1)^{k_1-1}(r+1)(r+2)\cdots(r+k_1-1)a_r, \tag{4.32}$$

we see that it is zero for $r = -1, -2, \ldots, -k_1+1$, so these can be discarded as well, and we have

$$:\partial^{k_1}\varphi\partial^{k_2}\varphi\cdots\partial^{k_m}\varphi:_0 \sim (:\partial^{k_2}\varphi\cdots\partial^{k_m}\varphi:)_0(\partial^{k_1}\varphi)_0, \tag{4.33}$$

and by induction we conclude

$$:\partial^{k_1}\varphi\partial^{k_2}\varphi\cdots\partial^{k_m}\varphi:_0 \sim (\partial^{k_m}\varphi)_0\cdots(\partial^{k_2}\varphi)_0(\partial^{k_1}\varphi)_0. \tag{4.34}$$

This, together with the result (4.30), shows that when acting on ground states, the zero mode of every basis field is proportional to a_0^m where $m \in \mathbb{Z}_{\geq 0}$ is the number of fields in the normally-ordered product, taking the identity to be a product of zero fields.

The mapping of fields to zero modes acting on ground states is a projection, so the set

$$\{a_0^n \mid n \in \mathbb{Z}_{\geq 0}\} \tag{4.35}$$

forms a basis for the vector space of zero modes acting on ground states, thus we have the identification

$$A(V) \cong \mathbb{C}[a_0] \tag{4.36}$$

as vector spaces. Now to complete the identification of the algebra, it remains to show that the Zhu product (4.9) corresponds to standard multiplication in the polynomial algebra $\mathbb{C}[a_0]$, that is, we need

$$a_0^m * a_0^n = a_0^{m+n}. \tag{4.37}$$

We first note that a_0^n corresponds to the zero mode of $:\partial\varphi^n:$ acting on ground states, where we use the shorthand

$$:\partial\varphi^n: \;\equiv\; \underbrace{:\partial\varphi\cdots\partial\varphi:}_{n \text{ copies of } \partial\varphi}. \tag{4.38}$$

This means we have

$$\begin{aligned}
a_0 * a_0^n &= \sum_{j\geq -1}\binom{1}{j+1}(a_j :\partial\varphi^n:)_0 \\
&= (a_{-1} :\partial\varphi^n:)_0 + (a_0 :\partial\varphi^n:)_0 \\
&\mapsto \left(a_{-1}a_{-1}^n\,|0\rangle\right)_0 + \left(a_0 a_{-1}^n\,|0\rangle\right)_0 \\
&= \left(a_{-1}^{n+1}\,|0\rangle\right)_0 \\
&\mapsto :\partial\varphi^{n+1}:_0 \\
&\sim a_0^{n+1}.
\end{aligned} \tag{4.39}$$

where the symbol '$\mapsto$' indicates use of the state-field correspondence. Now writing

$$a_0^m * a_0^n = (\underbrace{a_0 * \cdots * a_0}_{m \text{ times}}) * a_0^n, \tag{4.40}$$

the result follows by associativity, showing that

$$A(V) \cong \mathbb{C}[a_0] \tag{4.41}$$

as associative algebras.

The representation theory of the associative algebra $\mathbb{C}[a_0]$ is very straightforward. Since it is abelian, all the irreducible modules are 1-dimensional, spanned by a single element $|p\rangle$. Theorem 2.2.2 of [21] gives a bijection up to isomorphism between irreducible V-modules and irreducible $A(V)$-modules. Therefore for each $p \in \mathbb{C}$ there is an irreducible module of V, which we denote by $\mathcal{F}_p$. Each $\mathcal{F}_p$ contains a *highest weight state* $|p\rangle$ satisfying

$$a_0 |p\rangle = p |p\rangle . \tag{4.42}$$

A basis for $\mathcal{F}_p$ is generated by applying creation operators a_n, $n > 0$, to $|p\rangle$, so the irreducible modules of V are precisely

$$\mathcal{F}_p = \operatorname{span}\left\{a_{-n_1} \cdots a_{-n_k} |p\rangle \,\middle|\, k \geq 0;\, n_1 \geq n_2 \geq \cdots \geq n_k \geq 0\right\} \tag{4.43}$$

for $p \in \mathbb{C}$.

4.2 Virasoro

For this section let V denote the Virasoro vertex operator algebra. The state space of V has basis

$$\{L_{-n_1} \cdots L_{-n_k} |0\rangle \,|\, k \geq 0;\, n_1 \geq n_2 \geq \cdots \geq n_k > 1\} \tag{4.44}$$

where $|0\rangle$ is the vacuum, and the L_n are the modes of

$$T(z) \equiv Y\left(L_{-2} |0\rangle, z\right) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \tag{4.45}$$

satisfying the Virasoro commutation relations (3.45), as well as

$$L_n |0\rangle = 0, \qquad n \geq -1 \tag{4.46}$$

The state-field correspondence and normal ordering are defined just as in the case of the free boson, according to (3.46), (3.27) and (3.29). Section 6.1 of [20] contains many further details, and a verification that this structure indeed forms a vertex operator algebra.

The space of fields of V has basis

$$\{:\partial^{k_1} T \cdots \partial^{k_m} T: \,|\, m \geq 0;\, k_1, \ldots, k_m \geq 0\}. \tag{4.47}$$

To identify the Zhu algebra $A(V)$, we must calculate how the zero modes of these basis fields act upon ground states. The $m = 1$ case is easy to compute directly:

$$\partial^k T = \sum_{n\in\mathbb{Z}} (-1)^k (n+2)\cdots(n+k+1) L_n z^{-n-k-2} \tag{4.48}$$

therefore

$$(\partial^k T)_0 = (-1)^k (k+1)! L_0. \tag{4.49}$$

Using the formula (4.20) the nth mode of a normally-ordered product of $\partial^k T$ with $B \in V$ can be written

$$:\partial^k T B:_n = \sum_{r\leq -k-2} (\partial^k T)_r B_{n-r} + \sum_{r>-k-2} B_{n-r} (\partial^k T)_r. \tag{4.50}$$

When $n \geq 0$ and $r \leq -k-2$ we have $n - r \geq k - 2 > 0$, so the first sum vanishes when acting on ground states, and similarly for $r > 0$ in the second. Now noting that

$$\left(\partial^{k_1} T\right)_r = (-1)^{k_1} (r+2)(r+3)\cdots(r+k_1+1) L_r = 0 \text{ when } r = -2, -3, \ldots, -k_1 - 1 \tag{4.51}$$

we obtain

$$:\partial^k T B:_n \sim B_n (\partial^k T)_0 + B_{n+1} (\partial^k T)_{-1} \tag{4.52}$$

on ground states. From this we see that when expanding a mode in this fashion, the sum of the indices on each term is conserved. In particular, in the case of expansion of the zero mode

$$:\partial^{k_1} T \cdots \partial^{k_m} T:_0, \tag{4.53}$$

each term in the complete expansion consists of some product of L_i with sum of indices equal to zero, so each term is either proportional to L_0^m, or otherwise must contain some L_n with $n > 0$. Consider such a term

$$L_n L_{m_1} \cdots L_{m_p} \tag{4.54}$$

with $n > 0$, $m_1, \ldots, m_p$ arbitrary and $n + \sum_i m_i = 0$. Applying the commutation relations gives

$$L_n L_{m_1} \cdots L_{m_p} = L_{m_1} L_n L_{m_2} \cdots L_{m_p} + (n - m_1) L_{n+m_1} L_{m_2} \cdots L_{m_p} + \frac{1}{12}(n^3 - n) L_{m_2} \cdots L_{m_p} \delta_{n+m_1=0}. \tag{4.55}$$

In the first term L_n is just commuted once to the right, while the second and third terms contain fewer modes, but still have indices summing to zero. By induction we can conclude that after completely commuting L_n to the right so that we have annihilation on ground states, we will only be left with terms consisting of powers of L_0, with exponents bounded above by the initial number of modes.

This means that the zero mode of every basis field $:\partial^{k_1} T \cdots \partial^{k_m} T:$, when acting on ground states, is equivalent to a polynomial in L_0 of degree m. Taking one polynomial of each degree gives a basis for the vector space of polynomials $\mathbb{C}[L_0]$, demonstrating that

$$A(V) \cong \mathbb{C}[L_0] \tag{4.56}$$

as vector spaces. We also note that under this mapping, the only data that is relevant about a basis field is the number of fields in the normally-ordered product. This means that it will be sufficient in what follows to show results for fields of the form $:T^n:$.

To complete the algebra identification, we need to verify that the Zhu product on the zero modes corresponds to multiplication in the polynomial algebra $\mathbb{C}[L_0]$. First we show that

$$:T^n:_0 \sim L_0(L_0+2)(L_0+4)\cdots(L_0+2n-2). \tag{4.57}$$

Let A be a field from V of conformal dimension h_A. The derivative property (4.6) implies that

$$\begin{aligned}
[L_{-1}, A_1] &= \oint_0 [L_{-1}, A(w)] w^{h_A} \frac{dw}{2\pi i} \\
&= \oint_0 \partial A(w) w^{h_A} \frac{dw}{2\pi i} \\
&= \big(\partial A(w)\big)_0 = -h_A A_0.
\end{aligned} \tag{4.58}$$

We now compute using (4.58) as well as (4.21),

$$\begin{aligned}
:T^n:_0 &= :T\ :T^{n-1}::_0 \\
&= \sum_{r\leq -2} L_r\ :T^{n-1}:_{-r}\ + \sum_{r>-2}\ :T^{n-1}:_{-r}\ L_r \\
&\sim :T^{n-1}:_0\ L_0 +\ :T^{n-1}:_1\ L_{-1} \\
&\sim :T^{n-1}:_0\ L_0\ + (2n-2)\ :T^{n-1}:_0
\end{aligned} \tag{4.59}$$

and (4.57) follows by induction.

Now, under the Zhu product we have

$$\begin{aligned}
L_0 * :T^n:_0 &= \sum_{0\geq j\geq -2} \binom{2}{j+2} \big(L_j :T^n:\big)_0 \\
&= \left(2n :T^n:\right)_0 + 2\left(\partial :T^n:\right)_0 + :T^{n+1}:_0 \\
&= 2n :T^n:_0 - 4n :T^n:_0 + :T^{n+1}:_0 \\
&= :T^{n+1}:_0\ -2n :T^n:_0 \\
&= L_0(L_0+2)\cdots(L_0+2n-2)(L_0+2n) - 2nL_0(L_0+2)\cdots(L_0+2n-2)
\end{aligned} \tag{4.60}$$

and if we instead compute using standard polynomial multiplication:

$$\begin{aligned}
L_0 \times :T^n:_0 &= L_0 \times L_0(L_0+2)\cdots(L_0+2n-2) \\
&= (L_0+2n-2n)L_0(L_0+2)\cdots(L_0+2n-2) \\
&= L_0(L_0+2)\cdots(L_0+2n-2)(L_0+2n) - 2nL_0(L_0+2)\cdots(L_0+2n-2)
\end{aligned} \tag{4.61}$$

we find that multiplying by L_0 under the Zhu product is the same as standard polynomial multiplication. Now writing

$$:T^m:_0 * :T^n:_0 \sim L_0(L_0+2)(L_0+4)\cdots(L_0+2n-2) * :T^n:_0, \tag{4.62}$$

associativity of $*$ and the fact that $L_0 * :T^n:_0$ is just polynomial multiplication gives the equivalence

$$:T^m:_0 * :T^n:_0 \sim :T^m:_0 \times :T^n:_0 \tag{4.63}$$

and therefore the Zhu product corresponds to polynomial multiplication for the whole of $A(V)$. This gives the result

$$A(V) \cong \mathbb{C}[L_0] \tag{4.64}$$

as associative algebras.

This is the same agreeable result as section 4.1, and we have that the irreducible modules for Virasoro VOA are generated from highest weight vectors $|p\rangle$ for $p \in \mathbb{C}$ which satisfy

$$L_0\,|p\rangle = p\,|p\rangle\,. \tag{4.65}$$

We denote them by

$$V_p = \operatorname{span}\left\{L_{-n_1}\cdots L_{-n_k}\,|p\rangle \,\middle|\, k \geq 0;\, n_1 \geq n_2 \geq \cdots \geq n_k \geq 0\right\}. \tag{4.66}$$

4.3 Virasoro minimal models

The Virasoro central charge c can theoretically take any value in $\mathbb{C}$. To this point we have either left it arbitrary, or considered the case of $c = 1$, which one might call the "generic" Virasoro vertex operator algebra. We now consider a class of vertex operator algebras known as the *Virasoro minimal models*.

These VOAs are characterised by central charge value

$$c = 1 - \frac{6(p-p')^2}{pp'} \tag{4.67}$$

where $p, p' \in \mathbb{Z}_{\geq 2}$ are coprime, and are typically denoted by $M(p,p')$. The minimal models are very useful, and many examples have been identified with well-known physical models. In particular, $M(2,5)$ was identified with the Yang-Lee edge singularity in [22], and $M(3,4)$ was identified with the Ising model in [2].

The construction of minimal models is concerned with removing so-called *singular vectors*, which are a consequence of the physical interpretation of the theory. Singular vectors are defined in Chapter 7 of [17] to be vectors that are annihilated by all L_n with $n > 0$. The motivation for identifying such vectors comes by considering the physical observables of a quantum theory, which are written as inner products of states.

We can define a bilinear form on the state space as in Chapter 7 of [17] using the *Hermitian conjugate*

$$L_{-n}^{\dagger} = L_n, \tag{4.68}$$

and so that the form of two states

$$L_{-k_1} \cdots L_{-k_n} |0\rangle \qquad \text{and} \qquad L_{-l_1} \cdots L_{-l_m} |0\rangle \tag{4.69}$$

is written

$$\langle 0| L_{k_m} \cdots L_{k_1} L_{-l_1} \cdots L_{-l_m} |0\rangle \tag{4.70}$$

where $\langle 0|$ is a *dual state* to $|0\rangle$ satisfying

$$\langle 0| L_j = 0, \qquad j < 0. \tag{4.71}$$

The inner product (4.70) is evaluated by passing the L_{k_i} over the L_{-l_j} until they hit $|0\rangle$, and we also specify

$$\langle 0 \mid 0 \rangle = 0. \tag{4.72}$$

Having defined this, we can state another characteristic property of a singular vector $|\chi\rangle$, that it is orthogonal to the entire state space with respect to (4.70), and in particular,

$$\langle \chi \mid \chi \rangle = 0. \tag{4.73}$$

This means a singular vector has no non-zero measurable quantity associated to it: it is *unphysical*, and is in some sense an artefact that ought to be removed from the mathematical theory if one wants a sensible physical model.

The descendants of the field corresponding to a singular vector $|\chi\rangle$ comprise a non-zero proper ideal. This then corresponds to a non-zero proper submodule of the state space, meaning that it is reducible. We remove all these vectors by taking a quotient by the submodule generated by $|\chi\rangle$. We refer to [23] for the useful fact that a Virasoro module contains a maximum of one such singular submodule. This means that after taking the quotient, we obtain an irreducible module: a Virasoro minimal model.

To analyse the representation theory of a minimal model $M(p,p')$ we would like to again use the Zhu algebra and [21] Theorem 2.2.2. To do this we refer to Proposition 1.4.2 in [9] which states that if I is an ideal of a vertex operator algebra V such that $\mathrm{Id} \notin I$ and the conformal element $\omega \notin I$ then we have the isomorphism of associative algebras

$$A(V/I) \cong A(V)/A(I), \tag{4.74}$$

where $A(I)$ is the image of the ideal I in the Zhu algebra $A(V)$. For a minimal model we have V being the generic Virasoro VOA, and I the ideal generated by a singular vector $\chi \in V$. By computing the image $A(I)$ and taking the quotient, we will obtain the desired Zhu algebra.

A singular vector $|\chi\rangle$ satisfies

$$L_n |\chi\rangle = 0, \qquad n > 0. \tag{4.75}$$

Notice however that if $n \geq 3$ the commutation relations give

$$[L_{n-1}, L_1] = (n-2)L_n \implies L_n = \frac{1}{n-2}\left(L_{n-1}L_1 - L_1 L_{n-1}\right) \tag{4.76}$$

so for $n \geq 3$,

$$L_n |\chi\rangle = \frac{1}{n-2}\left(L_{n-1}L_1 |\chi\rangle - L_1 L_{n-1} |\chi\rangle\right), \tag{4.77}$$

therefore by induction as long as

$$L_1 |\chi\rangle = L_2 |\chi\rangle = 0, \tag{4.78}$$

the condition (4.75) is satisfied. The non-trivial generating singular vector in the minimal model $M(p, p')$ has conformal dimension $(p-1)(p'-1)$; this result can be found in Chapter 7 of [17]. For this section we will write $A \equiv B$ to indicate that operators A, B satisfy

$$A |0\rangle = B |0\rangle . \tag{4.79}$$

The first, most obvious example to consider is $M(2, 3)$. In this case we have

$$c_{2,3} = 1 - \frac{6(3-2)^2}{3 \cdot 2} = 0 \tag{4.80}$$

giving commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{4.81}$$

Observe that for this case we have

$$L_1 L_{-2} |0\rangle = (L_{-2}L_1 + 3L_{-1}) |0\rangle = 0 \tag{4.82}$$

and

$$L_2 L_{-2} |0\rangle = (L_{-2}L_2 + 4L_0) |0\rangle = 0, \tag{4.83}$$

so $L_{-2} |0\rangle$ is a singular vector. Now notice that if we set $L_{-2} |0\rangle = 0$, then

$$0 = L_{-1}L_{-2} |0\rangle = (L_{-2}L_{-1} + L_{-3}) |0\rangle = L_{-3} |0\rangle \tag{4.84}$$

so $L_{-3} |0\rangle$ is generated by $L_{-2} |0\rangle$. If we iterate application of L_{-1}, we find that $L_{-m} |0\rangle$ are all generated by $L_{-2} |0\rangle$ for $m \geq 2$, and therefore the ideal generated by $L_{-2} |0\rangle$ contains all vectors of higher conformal dimension. Thus, $M(2, 3)$ is the trivial VOA, containing only $|0\rangle$ and its scalar multiples. It is always handy to have a zero object, but we now move on to more interesting examples.

4.3.1 $M(2,5)$

In this case we have

$$c_{2,5} = 1 - \frac{6(5-2)^2}{5 \cdot 2} = -\frac{22}{5} \tag{4.85}$$

giving commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} - \frac{11(m^3-m)}{30}\delta_{m+n=0} \cdot \mathrm{Id}. \tag{4.86}$$

We would like to find $|\chi\rangle$ of conformal dimension $h_\chi = (5-1)(2-1) = 4$ such that $L_1 |\chi\rangle = L_2 |\chi\rangle = 0$. There are five basis vectors of conformal dimension 4, but utilising $L_{-1}|0\rangle = 0$ leaves only two:

$$L_{-4} \qquad \text{and} \qquad L_{-2}^2. \tag{4.87}$$

We can therefore write an arbitrary vector of conformal dimension 4 as

$$|\chi\rangle = \left(aL_{-2}^2 + bL_{-4}\right)|0\rangle. \tag{4.88}$$

The condition (4.78) will allow us to solve for a and b only up to a constant scaling factor. This is expected, since the multiples of a singular vector should also be singular.

Quick calculations give

$$\begin{aligned} L_1 L_{-2}^2 &\equiv 3L_{-3} \\ L_1 L_{-4} &\equiv 5L_{-3} \end{aligned} \tag{4.89}$$

therefore

$$L_1|\chi\rangle = (3a+5b)L_{-3}|0\rangle. \tag{4.90}$$

More quick calculations give

$$\begin{aligned} L_2 L_{-2}^2 &\equiv \frac{18}{5}L_{-2}, \\ L_2 L_{-4} &\equiv 6L_{-2} \end{aligned} \tag{4.91}$$

therefore

$$L_2|\chi\rangle = \left(\frac{18a}{5} + 6b\right)L_{-2}|0\rangle. \tag{4.92}$$

Imposing the condition (4.78), linear independence gives the linear system

$$\begin{aligned} 3a + 5b &= 0 \\ \frac{18a}{5} + 6b &= 0. \end{aligned} \tag{4.93}$$

with solutions

$$a = 5z, \quad b = -3z, \quad z \in \mathbb{C}. \tag{4.94}$$

This is sufficient to fix a non-trivial singular vector for $M(2,5)$

$$|\chi\rangle = \left(5L_{-2}^2 - 3L_{-4}\right)|0\rangle\,. \tag{4.95}$$

This corresponds under the state-field correspondence to the field

$$\chi(z) = 5 : TT : -\frac{3}{2}\partial^2 T. \tag{4.96}$$

Let $I = \langle\chi\rangle$ be the ideal generated by χ. The minimal model $M(2,5)$ is the quotient V/I. By (4.74) to compute the Zhu algebra $A(V/I)$ we need to compute the image of I in $A(V)$, which we will denote by $A(I)$. We claim that χ_0 generates the whole of $A(I)$,

$$A(I) = \langle\chi_0\rangle, \tag{4.97}$$

i.e, any field generated by χ has zero mode generated by χ_0.

To see this, first note that every state generated from χ can be written

$$L_{-r_1}\cdots L_{-r_k}|\chi\rangle \tag{4.98}$$

for some $k \geq 0, r_i \geq 0$. Under the state field correspondence these correspond to the fields

$$\frac{1}{(r_1-2)!\cdots(r_k-2)!} : \partial^{r_1-2}T\cdots\partial^{r_k-2}T\chi : . \tag{4.99}$$

We note that since in the following argument we will reduce each field to its modes, we do not need to explicitly consider those fields where χ is replaced by some $\partial^j\chi$, as $(\partial^j\chi)_n$ is proportional to χ_n for all j.

We will need the commutation relations $[L_m, \chi_n]$. To compute them, note that $L_0|\chi\rangle = 4|\chi\rangle$ and $L_n|\chi\rangle = 0$ for $n > 0$ imply the OPE

$$T(z)\chi(w) = \frac{4\chi(w)}{(z-w)^2} + \frac{\partial\chi(w)}{z-w} + \text{regular terms}. \tag{4.100}$$

This now lets us compute

$$\begin{aligned}
[L_m, \chi_n] &= \oint_0\oint_w \left[\frac{4\chi(w)}{(z-w)^2} + \frac{\partial\chi(w)}{z-w} + \text{regular terms}\right] z^{m+1}w^{n+3}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \oint_0 \left[4(m+1)\chi(w)w^m + \partial\chi(w)w^{m+1}\right] w^{n+3}\frac{dw}{2\pi i} \\
&= 4(m+1)\oint_0 \chi(w)w^{m+n+3}\frac{dw}{2\pi i} + \oint_0 \partial\chi(w)w^{m+n+4}\frac{dw}{2\pi i} \\
&= 4(m+1)\chi_{m+n} + (\partial\chi)_{m+n} = (3m-n)\chi_{m+n}.
\end{aligned} \tag{4.101}$$

Now consider

$$
\begin{aligned}
(L_{-r_1}\cdots L_{-r_k}\chi)_0 \propto &: \partial^{r_1-2}T\cdots\partial^{r_k-2}T\chi :_0 \\
= &\sum_{s\le -r_1}(\partial^{r_1-2}T)_s \;:\partial^{r_2-2}T\cdots\partial^{r_k-2}T\chi:_{-s} \\
&\qquad + \sum_{s>-r_1} :\partial^{r_2-2}T\cdots\partial^{r_k-2}T\chi:_{-s}\,(\partial^{r_1-2}T)_s \\
\sim &\sum_{0\ge s>-r_1} :\partial^{r_2-2}T\cdots\partial^{r_k-2}T\chi:_{-s}\,(\partial^{r_1-2}T)_s \\
= &\sum_{0\ge s>-r_1}(-1)^{r_1-2}(s+2)\cdots(s+r_1-1):\partial^{r_2-2}T\cdots\partial^{r_k-2}T\chi:_{-s}L_s \\
= &\sum_{s=0,-1}(-1)^{r_1-2}(s+2)\cdots(s+r_1-1):\partial^{r_2-2}T\cdots\partial^{r_k-2}T\chi:_{-s}L_s
\end{aligned}
\tag{4.102}
$$

By expanding we have extracted an L-mode while the sum of the indices remains equal to zero. If we repeat the process until termination, every term will be proportional to

$$\chi_m L_{m_1}\cdots L_{m_k} \tag{4.103}$$

for $m\ge 0$, $m_1,\ldots,m_k\le 0$ satisfying $m+\sum_i m_i = 0$. Applying the commutation relations (4.101) to a term of the form (4.103)

$$\chi_m L_{m_1}\cdots L_{m_k} = L_{m_1}\chi_m L_{m_2}\cdots L_{m_k} - (3m-m_1)\chi_{m+m_1}L_{m_2}\cdots L_{m_k} \tag{4.104}$$

we see that if χ_m is commuted all the way to the right, then if $m>0$ we have annihilation on ground states, and so only terms generated by χ_0 will remain. Furthermore, the extra terms introduced by the commutation will be proportional to

$$\chi_{m+\sum_i m_i} = \chi_0, \tag{4.105}$$

so we conclude that every field of the form (4.99) has zero mode generated by χ_0. This proves that

$$A(I) = \langle\chi_0\rangle \tag{4.106}$$

as desired. The Zhu algebra for $M(2,5)$ is therefore

$$A(V/I)\cong A(V)/A(I)\cong \mathbb{C}[L_0]/\langle\chi_0\rangle. \tag{4.107}$$

Having identified the Zhu algebra for $M(2,5)$ we can discuss the representations. It is a basic property of the quotient that if $\rho:\mathbb{C}[L_0]\to \mathrm{End}V$ is a representation that satisfies

$$\rho\,|_{\langle\chi_0\rangle} = 0 \tag{4.108}$$

then there exists a unique representation $\pi:\mathbb{C}[L_0]/\langle\chi_0\rangle\to\mathrm{End}V$ such that $\pi\circ q=\rho$ where q is the canonical quotient map. Since $A(I) = \langle \chi_0 \rangle$, for a representation ρ to satisfy (4.108) it is sufficient that it satisfies

$$\rho(\chi_0) = 0. \tag{4.109}$$

So the representations of $\mathbb{C}[L_0]$ satisfying (4.109) provide corresponding representations of the Zhu algebra $A(V/I)$, which in turn by Theorem 2.2.2 of [21] provide corresponding representations of the minimal model $M(2,5)$.

We now compute the modules explicitly. We have

$$\chi = 5 : TT : -\frac{3}{2}\partial^2 T \tag{4.110}$$

giving the zero mode

$$\begin{aligned} \chi_0 &= 5 \sum_{r \leq -2} L_r L_{-r} + 5 \sum_{r > -2} L_{-r} L_r - 9L_0 \\ &\sim 5L_1 L_{-1} + 5L_0^2 - 9L_0 \\ &= 5\left[L_{-1}L_1 + 2L_0\right] + 5L_0^2 - 9L_0 \\ &\sim 5L_0^2 + L_0. \end{aligned} \tag{4.111}$$

Consider now a Virasoro module V_p. If χ_0 acts as zero then we must have

$$\chi_0 \left|p\right\rangle = \left(5p^2 + p\right)\left|p\right\rangle = 0 \tag{4.112}$$

so

$$5p^2 + p = 0 \implies p = 0, -\frac{1}{5}. \tag{4.113}$$

Therefore the irreducible modules for the minimal model $M(2,5)$ are the Virasoro modules V_0 and $V_{-1/5}$.

4.3.2 $M(3,4)$

In this case we have

$$c_{3,4} = 1 - \frac{6(4-3)^2}{4\cdot 3} = \frac{1}{2} \tag{4.114}$$

therefore the Virasoro commutation relations are

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{24}\delta_{m+n=0} \cdot \mathrm{Id} \tag{4.115}$$

and the non-trivial singular vector $\left|\chi\right\rangle$ has conformal dimension $h_\chi = (4-1)(3-1) = 6$. There are 11 generating vectors of conformal dimension 6, but utilising $L_{-1}\left|0\right\rangle = 0$ shortens the list to just 4:

$$L_{-6}\left|0\right\rangle, \quad L_{-4}L_{-2}\left|0\right\rangle, \quad L_{-3}^2\left|0\right\rangle, \quad L_{-2}^3\left|0\right\rangle. \tag{4.116}$$

Let

$$|\chi\rangle = \left(aL_{-6} + bL_{-4}L_{-2} + cL_{-3}^2 + dL_{-2}^3\right)|0\rangle . \tag{4.117}$$

Some straightforward calculations give that

$$\begin{aligned} L_1L_{-6} &\equiv 7L_{-5}, \\ L_1L_{-4}L_{-2} &\equiv 5L_{-3}L_{-2}, \\ L_1L_{-3}^2 &\equiv 8L_{-3}L_{-2} + 4L_{-5}, \\ L_1L_{-2}^3 &\equiv 9L_{-3}L_{-2} + 6L_{-5}. \end{aligned} \tag{4.118}$$

We therefore have

$$L_1|\chi\rangle = \left[(7a + 4c + 6d)\,L_{-5} + (5b + 8c + 9d)\,L_{-3}L_{-2}\right]|0\rangle . \tag{4.119}$$

It is also easy to show

$$\begin{aligned} L_2L_{-6} &\equiv 8L_{-4}, \\ L_2L_{-4}L_{-2} &\equiv \frac{1}{4}L_{-4} + 6L_{-2}^2, \\ L_2L_{-3}^2 &\equiv 10L_{-4}, \\ L_2L_{-2}^3 &\equiv \frac{99}{4}L_{-2}^2, \end{aligned} \tag{4.120}$$

which means

$$L_2|\chi\rangle = \left[\left(8a + \frac{1}{4}b + 10c\right)L_{-4} + \left(6b + \frac{99}{4}d\right)L_{-2}^2\right]|0\rangle . \tag{4.121}$$

Imposing the condition (4.78) gives the linear system

$$\begin{aligned} 7a + 4c + 6d &= 0 \\ 5b + 8c + 9d &= 0 \\ 8a + \frac{1}{4}b + 10c &= 0 \\ 6b + \frac{99}{4}d &= 0 \end{aligned} \tag{4.122}$$

with solutions

$$a = 108n, \qquad b = 264n, \qquad c = -93n, \qquad d = -64n, \qquad n \in \mathbb{Z}, \tag{4.123}$$

allowing us to fix the minimal conformal dimension singular vector

$$|\chi\rangle = \left[108L_{-6} + 264L_{-4}L_{-2} - 93L_{-3}^2 - 64L_{-2}^3\right]|0\rangle . \tag{4.124}$$

which corresponds under the state field correspondence to

$$\frac{9}{2}\partial^4 T + 132 :\partial^2 TT: - 93 :\partial T\partial T: - 64 :TTT: \tag{4.125}$$

Now, upon a careful reading of the previous section on $M(2,5)$, one can observe that the only property of the singular vector used there other than its singularity was its conformal dimension, and when this appeared it only affected constant multipliers, which didn't affect the substance of the proportionality arguments. This means that the analysis used there is also valid here, and the results will be the same up to constant multipliers.

We can therefore say immediately that the Zhu algebra of $M(3,4) = V/I$ where V is the generic Virasoro VOA and $I = \langle\chi\rangle$ where χ is now defined by (4.125) is given by

$$A(V/I) \cong A(V)/A(I) \cong \mathbb{C}[L_0]/\langle\chi_0\rangle. \tag{4.126}$$

The analysis for the representations also carries over, and for each representation $\rho : \mathbb{C}[L_0] \to \mathrm{End}V$ that satisfies

$$\rho\,|_{\langle\chi_0\rangle} = 0 \tag{4.127}$$

there is a unique representation $\pi : \mathbb{C}[L_0]/\langle\chi_0\rangle \to \mathrm{End}V$ such that $\pi \circ q = \rho$ where q is the canonical quotient map, and by Theorem 2.2.2 of [21] for each of these we have a corresponding representation of the minimal model $M(3,4)$.

To compute the modules explicitly, we need the zero mode of χ. The calculation is quite lengthy and we omit it here, simply stating the result

$$\chi_0 \sim -64L_0^3 + 36L_0^2 - 2L_0. \tag{4.128}$$

Letting this act upon $|p\rangle$ for a Virasoro module V_p and setting the result to zero gives the equation

$$-32p^3 + 18p^2 - p = 0 \tag{4.129}$$

with solutions $p = 0, \frac{1}{2}, \frac{1}{16}$. Therefore the irreducible modules of $M(3,4)$ are the Virasoro modules $V_0, V_{1/2}, V_{1/16}$.

4.4 Level one Zhu algebras

In [11], Dong, Li and Mason generalise Zhu's associative algebra $A(V)$ for a vertex operator algebra V to a sequence of associative algebras $A_n(V)$ for $n = 0, 1, 2, \ldots$, often called *level n Zhu algebras*. The Zhu algebra we defined at the start of this chapter is the level zero algebra $A_0(V)$. In this section we consider the level one algebra $A_1(V)$ where V will be the free boson vertex operator algebra.

Recall that we treated $A_0(V)$ as the space of zero modes of fields acting on ground states. We will treat $A_1(V)$ as the space of zero modes acting on *near-ground states*, meaning states $|v\rangle$ satisfying

$$B_n\,|v\rangle = 0, \qquad \forall n > 1 \tag{4.130}$$

for any field $B \in V$. We note that an immediate consequence of this definition is that if $|v\rangle$ satisfies

(4.130) then

$$B^{m_1}_{n_1} \cdots B^{m_k}_{n_k} |v\rangle = 0 \tag{4.131}$$

whenever

$$\sum_{i=1}^{k} m_i n_i > 1. \tag{4.132}$$

We use the notation $A \sim_1 B$ to indicate that A and B have the same action on near-ground states.

As before, we first identify the underlying vector space of $A_1(V)$. The calculation is similar to (4.31), but now we consider the action on near-ground states. Since we are reducing each field to its modes, we need only consider fields of the form

$$: \partial\varphi \cdots \partial\varphi : \equiv : \partial\varphi^n :, \tag{4.133}$$

since the modes of $\partial^j \varphi$ are proportional to the modes of $\partial\varphi$. We now compute the zero mode of a field (4.133) as it acts under $\sim_1$:

$$\begin{aligned} : \partial\varphi^n :_0 &= \sum_{r \le -1} a_r \, : \partial\varphi^{n-1} :_{-r} + \sum_{r \ge 0} : \partial\varphi^{n-1} :_{-r} \, (\partial\varphi)_r \\ &\sim_1 a_{-1} \, : \partial\varphi^{n-1} :_1 + \, : \partial\varphi^{n-1} :_0 \, a_0 + \, : \partial\varphi^{n-1} :_{-1} \, a_1. \end{aligned} \tag{4.134}$$

In this process of breaking down the normally-ordered product, the sum over the indices is preserved in each term, therefore when the process terminates, each term will be proportional to a product of modes of the form

$$a_{r_1} \cdots a_{r_m}. \tag{4.135}$$

where $\sum_{i=1}^{m} r_i = 0$. This means that each term is either proportional to a_0^m, or else it contains some mode a_r with $r > 0$.

Consider a product of modes of the form (4.135) that contains a_r with $r > 1$. If this product does not also contain the mode a_{-r} to the right of a_r, then a_r commutes all the way to the right and annihilates on near-ground states. If it does contain a_{-r} to the right, then when they commute we must add a new term that is the same except a_r and a_{-r} are both eliminated. By induction we then have that there will be no terms remaining that contain a_r with $r > 1$.

What remains are products of a_0, a_1 and a_{-1}, with the number of a_1 factors equalling the number of a_{-1} factors. Using the commutation relations we can commute the a_1 to the right past each a_{-1} at the cost of introducing some extra terms each proportional to a_0^i for $i < m$. So we can reduce to terms of the form

$$a_0^i a_{-1}^k a_1^k \tag{4.136}$$

and by (4.131) and (4.132) this annihilates level one states when $k \ge 2$, so we also must have $k = 0, 1$.

So we have a basis for $A_1(V)$

$$\{a_0^i a_{-1}^k a_1^k\}_{i \ge 0; k=0,1}. \tag{4.137}$$

We can write this in terms of zero modes since

$$: \partial\varphi\partial\varphi :_0 = a_0^2 + 2a_{-1}a_1 \tag{4.138}$$

meaning that the set

$$\{a_0^i : \partial\varphi\partial\varphi :_0^k\}_{i\geq 0; k=0,1} \tag{4.139}$$

also forms a basis for $A_1(V)$. We therefore have the identification

$$A_1(V) \cong \mathbb{C}[a_0] \oplus \, : \partial\varphi\partial\varphi :_0 \mathbb{C}[a_0] \cong \mathbb{C}[a_0] \oplus \mathbb{C}[a_0] \tag{4.140}$$

as vector spaces.

We now begin an investigation of the associative algebra structure of $A_1(V)$. The level zero Zhu product was defined at the beginning of the chapter by an action upon ground states; we will need to modify this for action upon level one states. We compute another generalised commutation relation using contour manipulation:

$$\begin{aligned}
&\oint_0 \oint_w \frac{A(z)B(w)z^{h_A+1}w^{h_B-1}}{(z-w)^2}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \oint_{\substack{0 \\ |z|>|w|}} \oint_0 \frac{A(z)B(w)z^{h_A-1}w^{h_B-1}}{\left(1-\frac{w}{z}\right)^2}\frac{dz}{2\pi i}\frac{dw}{2\pi i} - \oint_{\substack{0 \\ |z|<|w|}} \oint_0 \frac{B(w)A(z)z^{h_A+1}w^{h_B-3}}{\left(1-\frac{z}{w}\right)^2}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \sum_{j\geq 0}(j+1)\left[\oint_{\substack{0 \\ |z|>|w|}} \oint_0 A(z)B(w)z^{h_A-j-1}w^{h_B+j-1}\frac{dz}{2\pi i}\frac{dw}{2\pi i} - \oint_{\substack{0 \\ |z|<|w|}} \oint_0 B(w)A(z)z^{h_A+j+1}w^{h_B-j-3}\frac{dz}{2\pi i}\frac{dw}{2\pi i}\right] \\
&= \sum_{j\geq 0}(j+1)\left[\oint_0 A(z)z^{h_A-j-1}\frac{dz}{2\pi i}\oint_0 B(w)w^{h_B+j-1}\frac{dw}{2\pi i} - \oint_0 B(w)w^{h_B-j-3}\frac{dw}{2\pi i}\oint_0 A(z)z^{h_A+j+1}\frac{dz}{2\pi i}\right] \\
&= \sum_{j\geq 0}(j+1)\left[A_{-j}B_j - B_{-j-2}A_{j+2}\right] \sim_1 A_0B_0 + 2A_{-1}B_1
\end{aligned} \tag{4.141}$$

where we have used the identity

$$\frac{1}{(1-x)^2} = \sum_{j\geq 0}(j+1)x^j, \qquad |x|<1. \tag{4.142}$$

Alternatively, inserting the OPE gives

$$
\begin{aligned}
&\oint_0 \oint_w \frac{A(z)B(w)z^{h_A+1}w^{h_B-1}}{(z-w)^2}\frac{dz}{2\pi i}\frac{dw}{2\pi i}\\
&\quad = \oint_0 \oint_w \left[\sum_{k\in\mathbb{Z}} (A_kB)(w)(z-w)^{-k-h_A-2}\right] z^{h_A+1}w^{h_B-1}\frac{dz}{2\pi i}\frac{dw}{2\pi i}\\
&\quad = \oint_0 \sum_{k\in\mathbb{Z}} (A_kB)(w)w^{h_B-1}\oint_w \frac{z^{h_A+1}}{(z-w)^{k+h_A+2}}\frac{dz}{2\pi i}\frac{dw}{2\pi i}\\
&\quad = \oint_0 \sum_{0\geq k\geq -h_A-1} (A_kB)(w)w^{h_B-1}\left[\binom{h_A+1}{h_A+k+1}w^{-k}\right]\frac{dw}{2\pi i}\\
&\quad = \sum_{0\geq k\geq -h_A-1}\binom{h_A+1}{h_A+k+1}\oint_0 (A_kB)(w)w^{h_B-k-1}\frac{dw}{2\pi i}\\
&\quad = \sum_{0\geq k\geq -h_A-1}\binom{h_A+1}{h_A+k+1}(A_kB)_0.
\end{aligned}
\tag{4.143}
$$

This is not yet a product of zero modes. To obtain what we want, we also compute

$$
\begin{aligned}
&\oint_0 \oint_w \frac{A(z)B(w)z^{h_A+2}w^{h_B-1}}{(z-w)^3}\frac{dz}{2\pi i}\frac{dw}{2\pi i}\\
&\quad = \oint_{\substack{0\\|z|>|w|}} \oint_0 \frac{A(z)B(w)z^{h_A-1}w^{h_B-1}}{\left(1-\frac{w}{z}\right)^3}\frac{dz}{2\pi i}\frac{dw}{2\pi i} + \oint_{\substack{0\\|z|<|w|}} \oint_0 \frac{B(w)A(z)z^{h_A+2}w^{h_B-4}}{\left(1-\frac{z}{w}\right)^3}\frac{dz}{2\pi i}\frac{dw}{2\pi i}\\
&\quad = \sum_{j\geq 0}\frac{(j+1)(j+2)}{2}\Bigg[\oint_{\substack{0\\|z|>|w|}} \oint_0 A(z)B(w)z^{h_A-j-1}w^{h_B+j-1}\frac{dz}{2\pi i}\frac{dw}{2\pi i}\\
&\qquad\qquad + \oint_{\substack{0\\|z|<|w|}} \oint_0 B(w)A(z)z^{h_A+j+2}w^{h_B-j-4}\frac{dz}{2\pi i}\frac{dw}{2\pi i}\Bigg]\\
&\quad = \sum_{j\geq 0}\frac{(j+1)(j+2)}{2}\Bigg[\oint_0 A(z)z^{h_A-j-1}\frac{dz}{2\pi i}\oint_0 B(w)w^{h_B+j-1}\frac{dw}{2\pi i}\\
&\qquad\qquad + \oint_0 B(w)w^{h_B-j-4}\frac{dw}{2\pi i}\oint_0 A(z)z^{h_A+j+2}\frac{dz}{2\pi i}\Bigg]\\
&\quad = \sum_{j\geq 0}\frac{(j+1)(j+2)}{2}\left[A_{-j}B_j - B_{-j-3}A_{j+3}\right] \sim_1 A_0B_0 + 3A_{-1}B_1
\end{aligned}
\tag{4.144}
$$

using

$$
\frac{1}{(1-x)^3} = \sum_{j\geq 0}\frac{(j+1)(j+2)}{2}x^j, \qquad |x|<1. \tag{4.145}
$$

We can now take a linear combination of (4.141) and (4.144):

$$
3\left(A_0B_0 + 2A_{-1}B_1\right) - 2\left(A_0B_0 + 3A_{-1}B_1\right) = A_0B_0. \tag{4.146}
$$

We will take the same linear combination on the other side to obtain the final definition. To do this we also need

$$
\begin{aligned}
&\oint_0 \oint_w \frac{A(z)B(w)z^{h_A+2}w^{h_B-1}}{(z-w)^3}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&\quad = \oint_0 \oint_w \left[\sum_{k\in\mathbb{Z}} (A_k B)(w)(z-w)^{-k-h_A-3}\right] z^{h_A+2}w^{h_B-1}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&\quad = \oint_0 \sum_{k\in\mathbb{Z}} (A_k B)(w)w^{h_B-1}\oint_w \frac{z^{h_A+2}}{(z-w)^{k+h_A+3}}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&\quad = \oint_0 \sum_{0\geq k\geq -h_A-2} (A_k B)(w)w^{h_B-1}\left[\binom{h_A+2}{h_A+k+2}w^{-k}\right]\frac{dw}{2\pi i} \\
&\quad = \sum_{0\geq k\geq -h_A-2}\binom{h_A+2}{h_A+k+2}\oint_0 (A_k B)(w)w^{h_B-k-1}\frac{dw}{2\pi i} \\
&\quad = \sum_{0\geq k\geq -h_A-2}\binom{h_A+2}{h_A+k+2}(A_k B)_0.
\end{aligned}
\tag{4.147}
$$

We therefore define the *level one Zhu product* on fields A and B to be

$$
\begin{aligned}
A *_1 B &= 3\sum_{0\geq k\geq -h_A-1}\binom{h_A+1}{h_A+k+1}(A_k B)_0 - 2\sum_{0\geq k\geq -h_A-2}\binom{h_A+2}{h_A+k+2}(A_k B)_0 \\
&= \sum_{0\geq k\geq -h_A-2}\left[3\binom{h_A+1}{h_A+k+1} - 2\binom{h_A+2}{h_A+k+2}\right](A_k B)_0
\end{aligned}
\tag{4.148}
$$

using the convention that $\binom{n}{r} = 0$ when $r < 0$. We then have

$$A *_1 B \sim_1 A_0 B_0. \tag{4.149}$$

We have now identified the underlying vector space and defined a level one Zhu product, but we will not proceed to identify the full associative algebra structure in this thesis. The result is known, and in Section 6.1 of [13] one finds Theorem 6.2:

$$A_1(V) \cong \mathbb{C}[x] \oplus \mathbb{C}[x] \tag{4.150}$$

as associative algebras. In order to prove this result in our framework, it only remains to check that the level one Zhu product we have defined corresponds to multiplication in the algebra $\mathbb{C}[a_0] \oplus \mathbb{C}[a_0]$. This is certainly possible, so the prospect for future work and completion of the identification exists.

Chapter 5

Conclusion

We began by covering the Lie algebra background necessary to fully understand the rest of the content, also defining the Lie algebras that occur as symmetry algebras for conformal field theories, namely the Kac-Moody algebra $\widehat{\mathfrak{gl}}(1)$ and the Virasoro algebra $\mathfrak{Vir}$.

In Chapter 3 we introduced conformal field theory via discussion of a key example, the free boson. In this process we focused on introducing the main structural elements of the vertex operator algebra structure that would be required for the analysis in Chapter 4. In particular we defined states and the state space, fields, the state-field correspondence, normal ordering, and the operator product expansion. We also used the introduced methods to identify the $\widehat{\mathfrak{gl}}(1)$ and $\mathfrak{Vir}$ symmetries in the free boson theory.

We began Chapter 4 by defining the structure of a vertex operator algebra. We then discussed the Zhu algebra that can be associated to a vertex operator algebra, and wrote down a definition. We also performed some computations on generalised commutation relations to derive the form of the associative multiplication in the Zhu algebra. We then referred to the results of Ridout and Wood in [14], which explain how the Zhu algebra can be described concretely as an algebra of zero modes of fields as they act upon ground states, and this description was used throughout the rest of the chapter.

We then moved on to our main examples of vertex operator algebras: the free boson, the Virasoro vertex operator algebra, and the Virasoro minimal models $M(2,5)$ and $M(3,4)$. For each of these examples we defined the vertex operator algebra structure in detail, then computed the Zhu algebra using Ridout and Wood's zero mode description.

Having obtained each Zhu algebra, we referred to the results of Zhu in [8] that show a bijective correspondence between the irreducible modules of a Zhu algebra and those of its corresponding vertex operator algebra. This allowed us to find the irreducible modules of each vertex operator algebra by identifying those of the Zhu algebra, and since the Zhu algebra is structurally simpler than the vertex operator algebra, the task was made much more tractable.

We concluded by beginning an analysis of the level one Zhu algebra $A_1(V)$ for V being the free boson vertex operator algebra, originally defined by Dong, Li and Mason in [11]. We successfully identified the vector space underlying $A_1(V)$, and defined a modified level one Zhu product upon it, but we did not fully identify the associative algebra structure of $A_1(V)$. We mentioned the known result and suggested the route that might have been taken to prove this in our framework.

Bibliography

[1] E. Witten. Non-abelian bosonization in two dimensions. *Communications in Mathematical Physics*, 92(4):455–472, 1984.

[2] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241:330–380, 1984.

[3] R. E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proceedings of the National Academy of Sciences*, 83(10):3068–3071, 1986.

[4] J. H. Conway and S. P. Norton. Monstrous moonshine. *Bulletin of the London Mathematical Society*, 11(3):308–339, 1979.

[5] I. B. Frenkel, J. Lepowsky, and A. Meurman. A natural representation of the Fischer-Griess Monster with the modular function J as character. *Proceedings of the National Academy of Sciences*, 81(10):3256–3260, 1984.

[6] I. B. Frenkel, Y-Z. Huang, and J. Lepowsky. *On axiomatic approaches to vertex operator algebras and modules*. American Mathematical Society, 1993.

[7] I. B. Frenkel, J. Lepowsky, and A. Meurman. *Vertex Operator Algebras and the Monster*. Elsevier, 1989.

[8] Y. Zhu. *Vertex operator algebras, elliptic functions and modular forms*. PhD thesis, Yale University, New Haven, CT, May 1990.

[9] B. I. Frenkel and Y. Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Mathematical Journal*, 66(1):123–168, 1992.

[10] W. Wang. Rationality of Virasoro vertex operator algebras. *International Mathematics Research Notices*, 1993(7):197–211, 04 1993.

[11] C. Dong, H. Li, and G. Mason. Vertex operator algebras and associative algebras. *Journal of Algebra*, 206:67–96, 1998.

[12] K. Barron, N. V. Werf, and J. Yang. Higher level zhu algebras and modules for vertex operator algebras. *Journal of Pure and Applied Algebra*, 223(8):3295–3317, 2019.

[13] D. Addabbo and K. Barron. On generators and relations for higher level Zhu algebras and applications. arXiv:2110.07671 [math.QA], 2021.

[14] D. Ridout and S. Wood. Relaxed singular vectors, Jack symmetric functions and fractional level $\hat{\mathfrak{sl}}(2)$ models. *Nuclear Physics B*, 894:621–664, 2015.

[15] V. Kac. *Infinite Dimensional Lie Algebras*. Progress in Mathematics. Birkháuser, Boston, 1983.

[16] M. A. Virasoro. Subsidiary conditions and ghosts in dual-resonance models. *Physical Review D*, 1:2933–2936, 1970.

[17] P. D. Francesco, P. Mathieu, and D. Sénéchal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer, 1997.

[18] D. Ridout. Topics in Mathematical Physics Part 1: An Introduction to Conformal Field Theory. Course notes for MATH3351, Australian National University, 2013.

[19] B. C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Springer, 2015.

[20] J. Lepowsky and H. Li. *Introduction to Vertex Operator Algebras and Their Representations*. Progress in Mathematics. Springer, 2004.

[21] Y. Zhu. Modular invariance of characters of vertex operator algebras. *Journal of the American Mathematical Society*, 9:237–302, 1996.

[22] J. L. Cardy. Conformal invariance and the Yang-Lee edge singularity in two dimensions. *Physical review letters*, 54(13):1354–1356, 1985.

[23] A. Astashkevich. On the Structure of Verma modules over Virasoro and Neveu-Schwarz Algebras. *Communications in Mathematical Physics*, 186(3):531–562, 1997.