The Bosonic Ghost System in Conformal Field Theory

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Abstract

Bosonic ghost systems are perhaps the least well understood of the free field conformal field theories (CFTs). The system is generated by two bosonic fields with opposite charges and conformal dimensions summing to 1. This work starts by introducing some background knowledge related to CFT, then discusses the construction of the ghost system. One particular automorphism is the spectral flow automorphism, which helps twist a module or a field. The thesis then focuses on computing the correlation functions of fields with or without spectral flow by solving differential equations including the Ward identities and an analogue of the Knizhnik-Zamolodchikov equation. We finish by showing that the results match the known fusion rules for the ghost CFT.

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Chapter 1

Introduction

Conformal Field Theory (CFT) is one of the special cases of Quantum Field Theory (QFT), where strong symmetries apply, which means in this case that CFTs are of invariant under conformal transformations. They were formally introduced in 1984 by Belavin, Polyakov and Zamolodchikov in [BPZ84], which discussed the consequences of conformal invariance of two-dimensional quantum field theory. More applications were developed rapidly and, CFT found significant contributions to condensed matter physics, statistical mechanics [ISZ88, RS93, Gur93], and string theory [BP09, GSW12a, GSW12b]. The simplest example is the free boson, which is discussed as an introductory topic in multiple textbooks such as [DFMS97, BP09, Sch97], and lecture notes including [Rid13]. In 2014, the bosonic ghost system of central charge 2 was addressed by David Ridout and Simon Wood in [RW14], and shown that it is actually a logarithmic conformal field theory by using fusion to construct representations on which the hamiltonian acts non-diagonalisably.

Ghost systems have a long history in CFT, being first suggested in an application of superstring theory in 1986 by Friedan, Martinec and Shenker[FMS86]. They developed a way to quantize Faddeev-Popov ghosts [Fad09] in the conformal gauge [Pol81] and connect the conformal algebra with the BRST quantization method [BBH00]. We will not discuss this application to string theory here, but more information can be found in [BLT13, Erb21]. In addition, bosonic ghosts can be used to construct more complicated theories, such as quantum hamiltonian reduction, which is discussed in [KRW03] and [KW04].

In this thesis, we will introduce the same setup as in [RW14] about the bosonic ghost system, but with a different approach to fusion namely solving differential equations related to correlation functions. At the end, we will investigate if the solutions match with the fusion rules reported in [RW14]. These rules have since been confirmed in [AP19] and [AW22].

Outline

Throughout this thesis, we will explore the terminologies in the ghost CFT, and derive correlation functions for the fields we define. We will start by introducing some background knowledge in Chapter 2, including Lie algebras, representations, and conformal transformations, focusing on parts relevant to the ghost conformal field theory. Moreover, we will discuss fields, states and modes, as well as the fact that modes generate fields and each field has a corresponding state. We remark that the energy-momentum tensor generated by Virasoro modes are the fields that exist in all CFTs, so we will use it as an example to introduce the operator production expansion.

Next in Chapter 3, we start to consider the CFT called the bosonic ghost system based on [RW14]. There are two Virasoro primary fields generating the ghost CFT, and in addition, there is a current and an energy-momentum tensor. Commutation relations between the two modes can then be derived when we have the OPE between the two corresponding fields. Additionally, the spectral flow automorphism will be introduced, and it can act on modes or fields. Then we define different types of primary fields, corresponding to spectral flowed relaxed highest weight states, the relaxed highest weight states and the (conjugate) vacuum states. Acting with modes on each type of state generates different types of modules. One last step before computing the correlators is to compute the Ward identities for the ghost CFT, which are different from the general forms found in textbooks.

With primary fields defined and the Ward identities obtained, we can derive differential equations for n-point correlation functions in Chapter 4. Starting by using the ghost primaries, we substitute the Ward identities for 1-point, 2-point, 3-point and 4-point functions to obtain general solutions for the correlators. Then we derive a version of the Knizhnik-Zamolodchikov equation to further constrain the solutions. Lastly, we will twist one of the fields by acting with spectral flow, and compute the n-point correlators again up to $n = 3$. We then finish the thesis by showing that the solutions agree with the fusion rules in [RW14].

Chapter 2

Background Knowledge

This chapter aims to develop some basic principles and notations that are useful in CFT. We will only discuss the knowledge that will be used directly later on. A standard textbook about CFT is [DFMS97], while other details such as Lie algebras, representations can be found in references [FS03], [RW15] and [Kac98] for interest.

2.1 Lie Algebra

One of the basic math structures underlying CFT is that of a Lie algebra [FS03, page 47-60].[1] A Lie algebra $\mathfrak{g}$ is a non-commutative, non-associative algebra, where a Lie bracket is denoted by $[,]$. The axioms of a Lie algebra satisfy bilinearity, antisymmetry, and the Jacobi identity, respectively written as:

$$\begin{gathered}
[ax+by,z]=a[x,z]+b[y,z], \quad [x,ay+bz]=a[x,y]+b[x,z], \\
[x,y]=-[y,x], \\
[[x,y],z]+[[y,z],x]+[[z,x],y]=0,
\end{gathered} \tag{2.1}$$

with $x,y,z\in\mathfrak{g}$ and a,b being elements in a field $\mathbb{F}$.[2]

The standard example is the general linear Lie algebra, denoted by $\mathfrak{gl}(n;\mathbb{R})$ or $\mathfrak{gl}(n;\mathbb{C})$, where the Lie bracket is given by the commutator

$$[A,B]=AB-BA, \tag{2.2}$$

with A,B being any $n\times n$ matrices in a vector space. In addition, A and B commute with each other if $[A,B]=0$. The Lie algebra is abelian if $[A,B]=0$ for all A,B.

[1] This textbook is particularly designed for mathematics physicists, while more comprehensive textbooks include [Hum78, Hal15].

[2] The only fields we focus on here are $\mathbb{R}$ and $\mathbb{C}$.

A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h}$, which is closed under the Lie bracket: $x, y \in \mathfrak{h} \Rightarrow [x, y] \in \mathfrak{h}$. In CFT, operators and generators of the fields often form Lie algebras. For example, we have $\mathfrak{gl}(1)$, a one-dimensional Lie algebra, and it is spanned by a single element a. On the other hand, $\widehat{\mathfrak{gl}}(1)$ has the basis $\{a_n : n \in \mathbb{Z}\} \cup \{k\}$, and brackets given by

$$[a_n, a_m] = m\delta_{m+n}k, \quad [a_m, k] = 0. \tag{2.3}$$

In quantum theory, the a_n are called operators, and in CFT they will be modes of a holomorphic field.[3]

Additionally, the Virasoro algebra $\mathfrak{Vir}$ is an important algebra in CFT. It has an infinite basis given by $\{L_n : n \in \mathbb{Z}\} \cup \{c\}$ with Lie bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c, \quad [L_m, c] = 0, \tag{2.4}$$

where L_n are modes of the energy-momentum tensor which will be introduced in Section 2.8.

2.2 Representation Theory

A representation [FS03, page 64-77][4] is a linear map π from $\mathfrak{g}$ to $\mathfrak{gl}(V)$, with V being a vector space. The map satisfies

$$\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x), \tag{2.5}$$

for $x, y \in V$.

V is said to be a $\mathfrak{g}$-module, as it is a vector space on which $\mathfrak{g}$ acts. A submodule of a $\mathfrak{g}$-module V is then a subspace $W \subseteq V$, which is preserved by the $\mathfrak{g}$-action, i.e. $w \in W$ implies $x \cdot w \in W$ for all $x \in \mathfrak{g}$. Every $\mathfrak{g}$-module V has two submodules V and $\{0\}$. If there are other submodules, then V is reducible, if not then it is irreducible.

2.3 Lie Algebra and Representation Theory in CFT

In CFT, we decompose our Lie algebra $\mathfrak{g}$ into a direct sum of three Lie subalgebras, i.e.

$$\mathfrak{g} = \mathfrak{g}^{<} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{>}, \tag{2.6}$$

where each subalgebra contains creation modes, zero modes, and annihilation modes[5] respectively. A Cartan subalgebra is then an abelian Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}^0$. Its elements can also

[3]A holomorphic function in dimension n is a complex-valued function that is complex differentiable in a neighbourhood of each point in a domain in $\mathbb{C}^n$. Here, the function is in dimension 1.

[4]Again, the information about representations are quite brief, more details can be found in [FH91].

[5]More details in Section 2.5.

be seen as quantum observables[6] as most physicists do.

We get to choose these subalgebras. Our choice then defines two types of vectors as follows:

1. Relaxed highest weight vectors [RW15] are eigenvectors for $\mathfrak{h}$ and annihilated by $\mathfrak{g}^>$. This implies that they are vacuum vectors also known as ground states.

2. Highest weight vectors are relaxed highest weight vectors where we have chosen $\mathfrak{h} = \mathfrak{g}^0$

The word 'relaxed' means that we allow more possibilities for such vectors.

Hence, modules can be generated by those vectors by acting with creation modes in $\mathfrak{g}$. To say that, a (relaxed) highest weight vector v generates a $\mathfrak{g}$-module M means that every $w \in M$ can be written as a linear combination of vectors $x_1 \cdot x_2 \cdots x_n \cdot v$ for some modes $x_1, \cdots, x_n \in \mathfrak{g}^< \oplus \mathfrak{g}^0$.

We call such a module a Verma module [Maz99], because the creation modes act freely. It means that the only relations they obey are the Lie bracket.

2.4 Conformal Invariance

Conformal transformations are those which preserve angles. They include translations, rotations, dilations and special conformal transformations [DFMS97, page 95-113]. In coordinates, there is a finite dimensional space of conformal transformations to transform a vector from x to x'. On the other hand, infinitesimal conformal transformations means transforming the vector x to $x + \varepsilon(x)$, where ε is extremely small. In 2D, the space of infinitesimal conformal transformation is infinite dimensional. The textbook [GRAS96] can be a further reference for interest.

Consider a function $\phi(x)$, which is also called a field, that is infinitesimal conformal transformed to $\phi(x + \varepsilon)$. By applying the infinitesimal Taylor expansion, we can obtain

$$\phi(x + \varepsilon) = \phi(x) + \varepsilon^\mu \partial_\mu \phi(x). \tag{2.7}$$

Generators of such transformations, are differential operators on $\phi(x)$, and generate a Lie algebra called the conformal algebra. Here, the generators are simply $\varepsilon^\mu \partial_\mu$, written in Einstein summation form.[7] They give us elements of the conformal Lie algebra, and each type of transformation corresponds to different generators. As we care more about what happens in 2 dimensions, more information about conformal transformations in higher dimensions can be found in the textbook [Hen99] for interest.

[6] Quantum observables are the physical quantities that can be measured.

[7] This convention simplifies $y = \sum_{i=1}^n c_i x^i$ to $y = c_i x^i$, with n being the number of coordinates.

Now, let us focus on conformal transformations in 2 dimensions, which are on a complex plane. The generators will then reduce to a holomorphic or antiholomorphic function in order to preserve the angles. A basis for the corresponding Lie algebra is then

$$\{\varepsilon(z) = z^n,\ \bar{\varepsilon}(\bar{z}) = \bar{z}^n : n \in \mathbb{Z}\}, \tag{2.8}$$

and thus we can choose a basis of generators to be

$$\ell_n = -z^{n+1}\partial, \quad \bar{\ell_n} = -\bar{z}^{n+1}\bar{\partial}. \tag{2.9}$$

By computing the commutation relations between the generators, we can then prove they are the Virasoro modes in (2.4), but with $c = 0$. The reason for the constrained c value is that we are only considering classical fields (i.e. functions).

For a quantum field $\phi(z)$, it can be Fourier decomposed as

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h_\phi}, \tag{2.10}$$

c can then be non-zero. Here, ϕ_n are non-commuting operators, and h_ϕ is the conformal dimension[8] of the field. It is the eigenvalue of the Virasoro mode L_0 acting on the state corresponding to $\phi(z)$, which will be introduced in Section 2.6. In addition, the Virasoro modes in quantum physics, denoted by L_n, are the modes of an energy momentum tensor $T(z)$, which must exist for the theory to be conformal. We now suggest the conformal dimension of $T(z)$ to be 2, and will check if it agrees with the ghost system in Section 3.1.2. It can then be written as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{2.11}$$

2.5 Normal Ordering

Normal ordering [BP09, p37-41] is a simple but useful tool which helps reorder annihilation and creation operators to ensure the quantised equation is well defined. It also allows us to rewrite the infinite sum of a quantised field (2.10) into two sums, each with one bound.

Consider the modes a_n of a field $A(z)$, and a vacuum state $|\Omega\rangle$.[9] The a_n are called annihilation operators when $n > 0$, while they are creation operations when $n < 0$. Zero mode is when $n = 0$, which we also call it the quantum observable. However, acting with a_n on the vacuum, denoted by $a_n |\Omega\rangle$, gives zero when $n > -h_A$, whether it is an annihilation operator

[8] A conformal dimension is also called a conformal weight or conformal spin or the energy of a field.
[9] A vacuum state is the ground state with minimal energy and maximum symmetry.

or not. Hence, we define the normal-ordered product of a_n and b_m as

$$: a_n b_m : = \begin{cases} a_n b_m & \text{if } n < -h_A, \\ b_m a_n & \text{otherwise,} \end{cases} \tag{2.12}$$

where b_n are the modes of another field $B(z)$. We can then write the normal ordered product of $A(z)$ and $B(w)$ as

$$: A(z)B(w) := \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} : a_n b_m : z^{-n-h_A} w^{-n-h_B}. \tag{2.13}$$

2.6 State-Field Correspondence

The corresponding state of any field $\phi(z)$ defined in (2.10) is

$$|\phi\rangle = \lim_{z\to 0} \phi(z) |\Omega\rangle . \tag{2.14}$$

The relation shows that there is a bijection between states and fields, in other words, every state in CFT corresponds to a unique field. It will be very useful when we derive the OPE of two fields in Section 2.8.

Let us consider the energy momentum $T(z)$ defined in (2.11), we can compute the corresponding state of such a field.

$$\begin{aligned} |T\rangle = \lim_{z\to 0} T(z) |\Omega\rangle &= \lim_{z\to 0} \sum_{n\in\mathbb{Z}} L_n |\Omega\rangle z^{-n-2} \\ &= \lim_{z\to 0} \sum_{n\leq -2} L_n |\Omega\rangle z^{-n-2} = L_{-2} |\Omega\rangle , \end{aligned} \tag{2.15}$$

since the L_n annihilate the vacuum for $n > -2$ as stated in Section 2.5, and the terms for $n < -2$ go to zero as $z \to 0$. We can also compute $|\partial T\rangle$ using the same method

$$|\partial T\rangle = \lim_{z\to 0} \sum_{n\in\mathbb{Z}} (-n-2) L_n |\Omega\rangle z^{-n-3} = L_{-3} |0\rangle . \tag{2.16}$$

One can mention by experience that the Virasoro mode with index -1 always corresponds to ∂ for any field $\phi(z)$, namely

$$L_{-1} |\phi\rangle \quad \leftrightarrow \quad |\partial\phi\rangle . \tag{2.17}$$

A partial explanation in classical physics can be L_{-1} being an infinitesimal translation operator coincides with the ∂ in Section 2.4.

2.7 Primary fields

A primary field is a field which corresponds to a vacuum (highest weight state) $|\Omega\rangle$, while a secondary field is a field which corresponds to a descendant of $|\Omega\rangle$. However, in a more precise definition, the primary field depends on which algebra we are considering. We say that a Virasoro primary is a field corresponding to a highest weight state for the Virasoro algebra, with

$$L_0 |\phi\rangle = h_\phi |\phi\rangle \,, \; L_n |\phi\rangle = 0 \; \forall n > 0, \tag{2.18}$$

where h_ϕ is the conformal dimension appearing in (2.10). We will define another type of primary field when we study the bosonic ghosts CFT in Chapter 3.

2.8 Operator Production Expansion

We have discussed normal ordering in Section 2.5, which was the ordering between operators. Where here, radial ordering is the ordering between fields, which allows fields to act in time order. Radial ordering (time ordering) of two arbitrary fields is:

$$\mathcal{R}\{\phi(z)\psi(w)\} = \begin{cases} \phi(z)\psi(w) & \text{if } |z| > |w|, \\ \psi(w)\phi(z) & \text{if } |z| < |w|. \end{cases} \tag{2.19}$$

As we often do not care about the regular terms, the operator production expansion (OPE) [BP09, p23-27] of two fields is introduced as the singular terms of the radial ordering product.

We can then try to compute the OPE of two general fields $A(z)$ and $B(w)$. It is an expansion of the product $A(z)B(w)$ as a Laurent series in $z - w$, shown as

$$\begin{aligned} A(z)B(w) &= \sum_{m\in\mathbb{Z}} \psi_m(w)(z-w)^{-m-h_A} \\ \Rightarrow \lim_{w\to 0} A(z)B(w) |\Omega\rangle &= \lim_{w\to 0} \sum_{m\in\mathbb{Z}} \psi_m(w)(z-w)^{-m-h_A} |\Omega\rangle \\ \Rightarrow \; A(z) |B\rangle &= \sum_{m\in\mathbb{Z}} |\psi\rangle \, z^{-m-h_A} \\ \Rightarrow \sum_{m\in\mathbb{Z}} A_m z^{-m-h_A} |B\rangle &= \sum_{m\in\mathbb{Z}} |\psi\rangle \, z^{-m-h_A} \\ \Rightarrow \; A_m |B\rangle &= |\psi\rangle \,, \end{aligned} \tag{2.20}$$

where h_A is the conformal dimension of the field $A(z)$, and we assume it to be an integer. We identify the unknown fields $\psi_m(w)$ using the state-field correspondence (2.14). Hence, the

OPE is written as

$$A(z)B(w) = \sum_{m\in\mathbb{Z}} \psi_m(w)(z-w)^{-m-h_A}, \tag{2.21}$$

where some of the terms can be removed depending on the property of $B(w)$.

For example, we can derive the OPE of the energy-momentum tensor $T(z)$ with a Virasoro primary field $\phi(w)$, and use our definition from (2.18) that $L_n |\phi\rangle = 0$ for $n > 0$,

$$\begin{aligned} T(z)\phi(w) &= \sum_{m\leq 0} \psi_m(w)(z-w)^{-m-h_A} \\ &= \sum_{n\leq 2} \psi_{n-2}(w)(z-w)^{-n} \\ &\sim \frac{\psi_0(w)}{(z-w)^2} + \frac{\psi_{-1}(w)}{z-w} \\ &\sim \frac{h_\phi \phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}, \end{aligned} \tag{2.22}$$

where we were given that $|\psi_0\rangle = L_0 |\phi\rangle = h_\phi |\phi\rangle$, and thus we have $\psi_0(w) = h_\phi \phi(w)$. We have also applied (2.17) so that $\psi_{-1}(w) = \partial\phi(w)$. In other words, an arbitrary field $\phi(w)$ that has such OPE with $T(z)$ is called a Virasoro primary field, and the coefficient in the $\frac{1}{(z-w)^2}$ term is the conformal dimension of such a field.

The commutation relation between L_n and $\phi(w)$ can now be derived by applying the Cauchy integral theorem and this OPE

$$\begin{aligned} [L_n, \phi(w)] &= \oint_w T(z)\phi(w) z^{n+1} \frac{dz}{2\pi\mathfrak{i}} \\ &= \oint_w \left(\frac{h_\phi \phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} \right) z^{n+1} \frac{dz}{2\pi\mathfrak{i}} \\ &= (n+1) w^n h_\phi \phi(w) + w^{n+1} \partial\phi(w). \end{aligned} \tag{2.23}$$

Additionally, the OPE of $T(z)T(w)$ can be computed as well to check if the energy momentum tensor is a primary field. Since we have derived $|T\rangle = L_{-2} |\Omega\rangle$ in (2.15), we can use the commutator of Virasoro modes (2.4) to derive

$$\begin{aligned} T(z) |T\rangle &= \sum_{n\in\mathbb{Z}} L_n L_{-2} z^{-n-2} |\Omega\rangle \\ &= \left(\sum_{n\leq -2} L_n L_{-2} + \sum_{n>-2} ([L_n, L_{-2}] + L_{-2} L_n) \right) |\Omega\rangle z^{-n-2}, \end{aligned}$$

where L_n annihilates the vacuum for $n > -2$, we can then obtain

$$
\begin{aligned}
T(z)\,|T\rangle &= \left(\sum_{n \le -2} L_n L_{-2} + \sum_{n > 2} (n+2) L_{n-2} \right) |\Omega\rangle\, z^{-n-2} \\
&\quad + \left(\frac{L_{-3}}{z} + \frac{2L_{-2}}{z^2} + \frac{3L_{-1}}{z^3} + \frac{4L_0 + \frac{1}{2}c}{z^4} \right) |\Omega\rangle \\
&= \left(\frac{\frac{1}{2}c}{z^4} + \frac{2L_{-2}}{z^2} + \frac{L_{-3}}{z} \right) |\Omega\rangle + \sum_{n \le -2} L_n L_{-2} z^{-n-2}\,|\Omega\rangle\,.
\end{aligned} \tag{2.24}
$$

The terms for $n \le -2$ correspond to regular terms in the OPE, which we ignore. The OPE is thus

$$
T(z)T(w) \sim \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{2.25}
$$

by applying (2.15) and (2.16). Comparing with (2.22), $T(z)$ is not a primary field, but the coefficient of the OPE with $\frac{1}{(z-w)^4}$ is half of the central charge.

2.9 Wick's Theorem

Lastly, Wick's theorem [Kac98] tells how to decompose the radially-ordered product of an arbitrary number of free fields, given that we know the OPE for two. Radially-ordered product is obtained by summing all possible contractions of the fields in normally-ordered product. This theorem only works if we have OPE of two fields containing identity field only, i.e. $A(z)B(w) \sim \frac{k}{(z-w)^{h_A+h_B}}$ for $k \in \mathbb{C}$.

For example, if we have two fields written as $:A(z)B(z):C(w)$ by knowing the OPEs of any of the two, we can compute the OPE of these fields being

$$
:A(z)B(z):C(w) \sim \,:A(z)B(z)C(w): + :\wick{\c1 A(z)B(z)\c1 C(w)}: + :\wick{A(z)\c1 B(z)\c1 C(z)}:\,, \tag{2.26}
$$

where two fields being contracted are replaced by their OPE, and the contraction is defined as

$$
(\cdots)\wick{\c1 A(z)(\cdots)\c1 B(w)}(\cdots) = \frac{k}{(z-w)^{h_A+h_B}}(\cdots)(\cdots)(\cdots). \tag{2.27}
$$

Similarly, for the OPE of two fields $:A(z)B(z)::C(w)D(w):$ we have

$$
\begin{aligned}
:A(z)B(z)::C(w)D(w): \sim \,&:\wick{\c1 A(z)B(z)\c1 C(w)D(w)}: + :\wick{\c1 A(z)B(z)C(w)\c1 D(w)}: \\
&:\wick{A(z)\c1 B(z)\c1 C(w)D(w)}: + :\wick{A(z)\c1 B(z)C(w)\c1 D(w)}: \\
&+ \wick{\c1 A(z)\c2 B(z)\c2 C(w)\c1 D(w)} + \wick{\c1 A(z)\c2 B(z)\c1 C(w)\c2 D(w)}.
\end{aligned} \tag{2.28}
$$

We will only use Wick's theorem for deriving OPEs in Section 3.1.2, where other extended details can be found in [Kac98].

Chapter 3

Introduction to Bosonic Ghost

After gaining some background knowledge related to conformal field theory, we will start to consider CFT in the ghost system, which was first introduced in [FMS86], and discuss about the main concepts based on [RW14]. There are two Virasoro primary fields generating the ghost CFT. We will compute the OPEs and commutation relations related to the energy-momentum tensor from Section 2.4, and a new field called a current. After that, the Ward identities will be derived to compute correlators in Chapter 4.

3.1 Ghost Algebras

3.1.1 Fields introduction

The two Virasoro primary fields generating the bosonic ghost system [RW14] are $\beta(z)$ with charge 1 and conformal weight 1, and $\gamma(z)$ with charge -1 and conformal weight 0. The charges are the coefficients of (3.3) and (3.5) which will be mentioned later in Section 3.1.2. Their OPEs are defined by

$$\begin{aligned} &\beta(z)\beta(w) = \gamma(z)\gamma(w) \sim 0 \\ &\beta(z)\gamma(w) \sim -\frac{1}{z-w}. \end{aligned} \tag{3.1}$$

We define the current and energy-momentum tensor in the bosonic ghost CFT, with radial ordering defined in (2.8):

$$J(z) = :\beta(z)\gamma(w): \quad T(z) = -:\beta(z)\partial\gamma(z):. \tag{3.2}$$

3.1.2 OPEs and Commutation Relations

We can indeed show that the conformal charges and weights of the bosonic ghost fields agree with Section 3.1.1 by deriving the OPEs.

$$\begin{aligned} J(z)\beta(w) &= :\beta(z)\gamma(z):\beta(w) \\ &\sim :\beta(z)\gamma(z)\beta(w): + :\beta(z)\gamma(z)\beta(w): + :\beta(z)\gamma(z)\beta(w): \\ &\sim \frac{1}{z-w}\beta(z) \sim \frac{1}{z-w}\beta(w), \end{aligned} \tag{3.3}$$

where we applied Wick's theorem (2.26) on the second step and Taylor expanded around $z = w$ for the last step. We define a free boson primary field with the corresponding state satisfying

$$J_0 \left|\phi\right\rangle = j_\phi \left|\phi\right\rangle, \quad J_n \left|\phi\right\rangle = 0 \ \forall n > 0, \tag{3.4}$$

so that the field has such an OPE with the current from (3.3), and their coefficient of the field on the RHS of the OPE is the charge j. It means that $\beta(z)$ is a free boson primary of charge $j = 1$. The OPE of $J(z)\gamma(w)$ is similarly given by

$$J(z)\gamma(w) \sim -\frac{1}{z-w}\gamma(w), \tag{3.5}$$

thus $\gamma(z)$ is a free boson primary of charge -1. We similarly compute

$$T(z)\beta(w) \sim \frac{1}{(z-w)^2}\beta(w) + \frac{1}{z-w}\partial\beta(w), \tag{3.6}$$

so $\beta(z)$ is a Virasoro primary with conformal weight $h_\beta = 1$ based on (2.22), and

$$T(z)\gamma(w) \sim \frac{1}{z-w}\partial\gamma(z), \tag{3.7}$$

so $\gamma(z)$ is a Virasoro primary with conformal weight $h_\gamma = 0$. Finally

$$
\begin{aligned}
T(z)T(w) &= :\beta(z)\partial\gamma(z)::\beta(w)\partial\gamma(w): \\
&\sim :\beta(z)\partial\gamma(z)\beta(w)\partial\gamma(w): \; + \; :\beta(z)\partial\gamma(z)\beta(w)\partial\gamma(w): \; + \beta(z)\partial\gamma(z)\beta(w)\partial\gamma(w) \\
&\sim -\frac{1}{(z-w)^2}(:\beta(w)\partial\gamma(w): + \; :\partial\gamma(w)\beta(w):) \\
&\quad -\frac{1}{(z-w)}(:\partial^2\gamma(w)\beta(w): + \; :\partial\beta(w)\partial\gamma(w):) \; + \frac{1}{(z-w)^4} \\
&\sim \frac{-2}{(z-w)^2} :\beta(w)\partial\gamma(w): \; - \frac{1}{z-w}\frac{\partial}{\partial w}(:\beta(w)\partial\gamma(w):) + \frac{1}{(z-w)^4} \\
&\sim \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \frac{1}{(z-w)^4},
\end{aligned}
\tag{3.8}
$$

where we have used (2.28), and the contractions are done by applying the OPEs in (3.1). By comparing with (2.22), $T(z)$ is not a primary field but is an energy-momentum tensor with central charge of this ghost system being 2. In fact, $J(z)$ is not a Virasoro primary field either but it has conformal weight 1,

$$
T(z)J(w) \sim \frac{1}{z-w}\partial J(w) + \frac{1}{(z-w)^2}J(w) - \frac{1}{(z-w)^3}. \tag{3.9}
$$

By writing the ghost fields in terms of modes β_n and γ_n

$$
\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-1} \qquad \gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n}, \tag{3.10}
$$

commutation relations can be derived. For $[\beta_m, \gamma_n]$, we can use the Cauchy integral theorem, which is the same as in (2.23).

$$
\begin{aligned}
[\beta_m, \gamma_n] &= \beta_m\gamma_n - \gamma_n\beta_m \\
&= \oint_0\oint_0 \beta(z)\gamma(w) z^m w^{n-1}\frac{dz}{2\pi\mathrm{i}}\frac{dw}{2\pi\mathrm{i}} - \oint_0\oint_0 \gamma(w)\beta(z) z^m w^{n-1}\frac{dz}{2\pi\mathrm{i}}\frac{dw}{2\pi\mathrm{i}} \\
&= \left[\oint_0\oint_{0 |z|>|w|} - \oint_0\oint_{0 |z|<|w|}\right] \beta(z)\gamma(w) z^m w^{n-1}\frac{dz}{2\pi\mathrm{i}}\frac{dw}{2\pi\mathrm{i}} \\
&= \oint_0\oint_w \left[-\frac{1}{z-w} + \; :\beta(z)\gamma(w):\right] z^m w^{n-1}\frac{dz}{2\pi\mathrm{i}}\frac{dw}{2\pi\mathrm{i}} \\
&= -\oint_0 w^m w^{n-1}\frac{dw}{2\pi\mathrm{i}} \\
&= -\delta_{m+n=0}\mathbf{1},
\end{aligned}
\tag{3.11}
$$

where the difference of contour around 0, $\oint_{0|z|>|w|} - \oint_{0|z|<|w|}$ can be replaced by a contour around $\oint_w$, and we have substituted (3.1) into our derivation. Hence, β_m and γ_n commute except when $m+n=0$. With the same method, one can show that β_m and γ_m both commute with themselves.

$$[\beta_m, \gamma_n] = -\delta_{m+n=0}\mathbf{1}, \quad [\beta_m, \beta_n] = [\gamma_m, \gamma_n] = 0. \tag{3.12}$$

We define the Lie algebra with such commutation relations as our ghost Lie algebra $\mathfrak{g}$.

Then the current and energy-momentum tensor can be written in terms of modes as well:

$$\begin{aligned} J(z) &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-1}z^{-n} : \beta_m\gamma_n : \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} : \beta_m\gamma_n : z^{-m-n-1} = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} T(z) &= - : \beta(z)\partial\gamma(z) : \\ &= -\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} : \beta_m\gamma_n : z^{-m-1}(-n)z^{-n-1} = \sum_{m\in\mathbb{Z}}\sum_{r\in\mathbb{Z}}(r-m) : \beta_m\gamma_{r-m} : z^{-r-2} \\ &= \sum_{n\leq -2}\sum_{m\in\mathbb{Z}}(n-m) : \beta_m\gamma_{n-m} : z^{-n-2} = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \end{aligned} \tag{3.14}$$

By using normal ordering (2.5) and commutation relations (3.12) between β_m and γ_n:

$$J_n = \begin{cases} \sum_{r\in\mathbb{Z}} \beta_r\gamma_{n-r} & (n \neq 0), \\ \sum_{r\leq -1} \beta_r\gamma_{-r} + \sum_{r\geq 0} \gamma_{-r}\beta_r & (n = 0) \end{cases}, \tag{3.15}$$

$$L_n = \begin{cases} \sum_{m\in\mathbb{Z}}(n-m)\beta_m\gamma_{n-m} & (n \neq 0), \\ \sum_{m\leq -1}(-m)\beta_m\gamma_{-m} + \sum_{m\geq 0}(-m)\gamma_{-m}\beta_m & (n = 0). \end{cases} \tag{3.16}$$

OPE and commutation relations of $J(z)$ with a free boson primary field can be derived in the same way as $T(z)$ in (2.23) by using (2.20).

$$J(z)\phi(w) = \sum_{m\leq 0} \psi_m(w)(z-w)^{-m-1} \sim \frac{\psi_0(w)}{z-w} \sim \frac{j\phi(w)}{z-w}, \tag{3.17}$$

as we have $|\psi_0\rangle = J_0|\phi\rangle = j|\phi\rangle$ and $|\psi_m\rangle = 0$ if $m > 0$ recalling from (3.4). After deriving the OPE, we can easily find the commutation relation between J_n and $\phi(w)$

$$[J_n, \phi(w)] = \oint_w J(z)\phi(w)z^n \frac{dz}{2\pi\mathfrak{i}} = \oint_w \frac{j\phi(w)}{z-w} z^n \frac{dz}{2\pi\mathfrak{i}} = j\phi(w)w^n. \tag{3.18}$$

The commutation relations of $[L_n, \phi(w)]$ and $[J_n, \phi(w)]$ will help us to derive the Ward identities of n-point ghost fields in Section 3.4.1.

3.1.3 Spectral Flow Automorphism

A module can be twisted by applying spectral flow σ^ℓ, which will be useful later in Section 4.2. Here, we only focus on $\ell \in \mathbb{Z}$. The spectral flow automorphism is given by

$$\sigma^\ell(\beta_n) = \beta_{n-\ell}, \quad \sigma^\ell(\gamma_n) = \gamma_{n+\ell}, \quad \sigma^\ell(\mathbf{1}) = \mathbf{1}. \tag{3.19}$$

The spectral flow is invertible as shown below,

$$\begin{aligned} \beta_n = \sigma^{-\ell}(\beta_{n-\ell}) \quad &\Rightarrow \quad \sigma^{-\ell}(\beta_n) = \beta_{n+\ell}, \\ \gamma_n = \sigma^{-\ell}(\gamma_{n+\ell}) \quad &\Rightarrow \quad \sigma^{-\ell}(\gamma_n) = \gamma_{n-\ell}. \end{aligned} \tag{3.20}$$

Additionally,

$$\begin{aligned} \sigma^\ell([\beta_m, \gamma_n]) = \sigma^\ell(-\delta_{m+n=0}\mathbf{1}) = -\delta_{m+n=0}\mathbf{1}, \\ [\sigma^\ell(\beta_m), \sigma^\ell(\gamma_n)] = [\beta_{m-\ell}, \gamma_{n+\ell}] = -\delta_{m+n=0}\mathbf{1}, \end{aligned} \tag{3.21}$$

shows that the spectral flow preserves the Lie brackets. This means it is an automorphism of the ghost Lie algebra $\mathfrak{g}$.

We can then derive the action of spectral flow on J_n. First for the simplier case when $n \neq 0$, we have

$$\sigma^\ell(J_n) = \sum_{r\in\mathbb{Z}} \sigma^\ell(\beta_r\gamma_{n-r}) = \sum_{r\in\mathbb{Z}} \sigma^\ell(\beta_r)\sigma^\ell(\gamma_{n-r}) = \sum_{r\in\mathbb{Z}} \beta_{r-\ell}\gamma_{n-r+\ell} = \sum_{m\in\mathbb{Z}} \beta_m\gamma_{n-m} = J_n, \tag{3.22}$$

where we replaced m as $r-\ell$ for the second last step. Then for $n = 0$,

$$\begin{aligned} \sigma^\ell(J_0) &= \sum_{r\le -1} \sigma^\ell(\beta_r\gamma_{-r}) + \sum_{r\ge 0} \sigma^\ell(\gamma_{-r}\beta_r) = \sum_{r\le -1} \beta_{r-\ell}\gamma_{-r+\ell} + \sum_{r\ge -\ell} \gamma_{-r+\ell}\beta_{r-\ell} \\ &= \sum_{m\le -1-\ell} \beta_m\gamma_{-m} + \sum_{m\ge -\ell} \gamma_{-m}\beta_m \\ &= \sum_{m\le -1} \beta_m\gamma_{-m} + \sum_{m\ge 0} \gamma_{-m}\beta_m - \sum_{m=-\ell}^{-1} \beta_m\gamma_{-m} + \sum_{m=1}^{\ell} \gamma_{-m}\beta_m \\ &= J_0 + \sum_{m=1}^{\ell}(-\beta_{-m}\gamma_m + \gamma_{-m}\beta_m) = J_0 - \sum_{m=1}^{\ell}[\beta_m, \gamma_{-m}] \\ &= J_0 + \ell\mathbf{1}. \end{aligned} \tag{3.23}$$

Combining the two results gives us that

$$\sigma^\ell(J_n) = J_n + \ell\delta_{n=0}\mathbf{1}. \tag{3.24}$$

With the same method, we can compute the spectral flow on the Virasoro modes,

$$\sigma^{\ell}(L_n) = L_n - \ell J_n - \frac{1}{2}\ell\,(\ell-1)\,\delta_{n=0}\mathbf{1}. \tag{3.25}$$

J_0 and L_0 can also act on a vector v in a ghost module, which we will define later in Section 3.3. If they satisfy $J_0 v = jv,\ L_0 v = hv$, then

$$J_0\sigma^{\ell}(v) = (j-\ell)\sigma^{\ell}(v), \tag{3.26}$$

$$L_0\sigma^{\ell}(v) = \left[h + \ell j + \frac{1}{2}\ell\,(\ell-1)\right]\sigma^{\ell}(v). \tag{3.27}$$

To show this we first have

$$J_0\sigma^{\ell}(v) = \sigma^{\ell}(\sigma^{-\ell}(J_0)v) = \sigma^{\ell}((J_0-\ell\mathbf{1})v) = \sigma^{\ell}((j-\ell)v) = (j-\ell)\sigma^{\ell}(v). \tag{3.28}$$

and the derivation of $L_0\sigma^{\ell}(v)$ follows the same steps.

3.2 Primary fields

Following from the definition in Section 2.7, we get to choose our own primary fields for the ghost system. Firstly, a Virasoro primary exists in all CFT, with the corresponding state following same rules as in (2.18). We have also defined a free boson primary in Section 3.1.2 with the corresponding state being (3.4).

Then we define a primary field as $\phi^{\ell}(z)$ with the corresponding state $\left|\phi^{\ell}\right\rangle$, for $\ell \in \mathbb{Z}$. The state is called a spectrally flowed relaxed highest weight state, defined so that

$$\begin{aligned} \beta_{n-\ell}\left|\phi^{\ell}\right\rangle &= 0, & \gamma_{n+\ell}\left|\phi^{\ell}\right\rangle &= 0, \quad \forall n > 0 \\ J_0\left|\phi^{\ell}\right\rangle &= j_{\phi^{\ell}}\left|\phi^{\ell}\right\rangle, & L_0\left|\phi^{\ell}\right\rangle &= h_{\phi^{\ell}}\left|\phi^{\ell}\right\rangle, \end{aligned} \tag{3.29}$$

by recalling that $j_{\phi^{\ell}}$ being the eigenvalue of $J_0\left|\phi^{\ell}\right\rangle$, is the charge, and $h_{\phi^{\ell}}$ is the conformal dimension of the primary field. One can show that

$$L_n\left|\phi^{\ell}\right\rangle = J_n\left|\phi^{\ell}\right\rangle = 0 \quad \forall n > 0, \tag{3.30}$$

by substituting (3.13) and (3.16) separately. Also, notice that a primary field is a Virasoro primary and a free boson primary.

For a more specific case, we can define a ghost primary as $\phi(z)$, and it corresponds to a relaxed highest weight state $|\phi\rangle$. We remark that $\phi(z) = \phi^{0}(z)$, which means it is a primary field for $\ell = 0$. The field also follows the rules in (3.29), but with $\ell = 0$ strictly, writing as

$$\begin{aligned} \beta_n \left|\phi\right\rangle &= 0, & \gamma_n \left|\phi\right\rangle &= 0, \quad \forall n > 0 \\ J_0 \left|\phi\right\rangle &= j_\phi \left|\phi\right\rangle, & L_0 \left|\phi\right\rangle &= h_\phi \left|\phi\right\rangle, \end{aligned} \tag{3.31}$$

where $j_\phi = j_{\phi^0}$ and $h_\phi = h_{\phi^0}$. The word "relaxed" means the same as in Section 2.3. We can even compute the exact value of h_ϕ by expressing L_0 as summation of β and γ modes

$$\begin{aligned} L_0 \left|\phi\right\rangle &= \sum_{m\leq -1} (-m)\beta_m\gamma_{-m} \left|\phi\right\rangle + \sum_{m\geq 0} (-m)\gamma_{-m}\beta_m \left|\phi\right\rangle = 0 \\ \Rightarrow h_\phi \left|\phi\right\rangle &= 0, \end{aligned} \tag{3.32}$$

because positive modes annihilates the ghost primary. Hence, we can state that the conformal dimension of a ghost primary is always 0.

Lastly, there are two special cases, we denote them by ghost vacuum and conjugate ghost vacuum. We choose the ghost vacuum as being a ghost primary for which the corresponding state also satisfies

$$\beta_0 \left|\phi\right\rangle = 0. \tag{3.33}$$

The state $\left|\phi\right\rangle$ is a highest weight state, as we restricted the action of the zero mode β_0. On the other hand, a conjugate ghost vacuum is a ghost primary as well but with the additional constraint on the state being

$$\gamma_0 \left|\phi\right\rangle = 0. \tag{3.34}$$

The state $\left|\phi\right\rangle$ here can then be named as a conjugate highest weight state. In fact, the reason that we call the above primaries ghost vacuums and conjugate ghost vacuum is that the only states that follow such constraints are the vacuum and its conjugate, respectively.

With the primary fields introduced in (3.29), we can actually compute general OPEs of $\beta(z)$, $\gamma(z)$ with a primary field, denoted by $\phi^\ell(w)$, where the corresponding state follows (3.29). First we can notice that β_n annihilates $\phi^\ell(w)$ when $n > -\ell$, and γ_n annihilates the fields for $n > \ell$. Hence, the expansion of $\beta(z)\phi^\ell(w)$ is

$$\beta(z)\phi^\ell(w) = \sum_{n\leq -\ell} \psi_n(w)(z-w)^{-n-1}, \tag{3.35}$$

and by following the steps from (2.20), we get that $\left|\psi_n\right\rangle = \beta_n \left|\phi^\ell\right\rangle$. Since the terms are singular only when $n \geq 0$, we need to separate into two cases

$$\beta(z)\phi^\ell(w) \sim \begin{cases} \displaystyle\sum_{n=0}^{-\ell} \frac{(\beta_n\phi^n)(w)}{(z-w)^{n+1}} & \ell \leq 0 \\ 0 & \ell > 0 \end{cases}. \tag{3.36}$$

Similarly, we have

$$\gamma(z)\phi^{\ell}(w) \sim \begin{cases} \displaystyle\sum_{n=1}^{\ell} \frac{(\gamma_n \phi^n)(w)}{(z-w)^n} & \ell > 0 \\ 0 & \ell \leq 0 \end{cases} . \tag{3.37}$$

3.3 Representation Theory

With different constraints on states as shown in Section 3.2, we can choose to generate different modules. Starting from the simplest case, modes acting on the highest weight state with constraints (3.33) generate a Verma module as in Section 2.3. We can visualise the module as in Fig. 3.1, showing that $|\phi\rangle$ is the only highest weight state, and its corresponding field

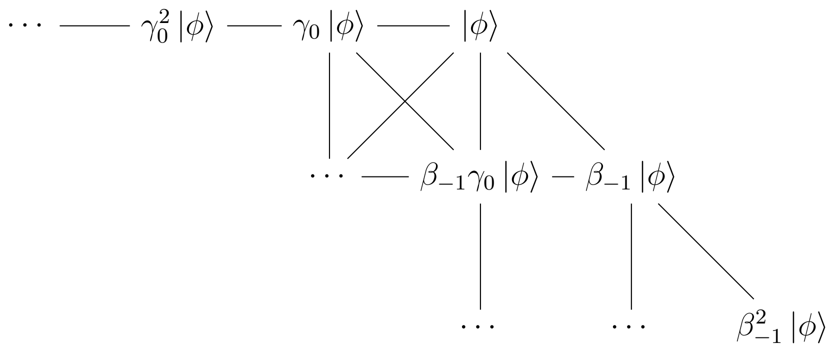

Figure 3.1: Verma module

is a ghost vacuum as we defined in Section 3.2. Meanwhile, all the other states are called descendant states. It turns out that the only possible highest weight state is the vacuum $|\Omega\rangle$ as shown in [RW14], and the module it generates is irreducible, where we have defined irreducibility in Section 2.2.

With the same idea, a conjugate Verma module can be generated by representations of a conjugate highest weight state, and it is also unique and irreducible.

On the other hand, relaxed Verma modules can be generated when we allow creators and zero modes β_0 and γ_0 to act freely on a relaxed highest weight state $|\phi\rangle$. It is visualised as in Fig. 3.2 . The states at the top of the module are all relaxed highest weight states, with corresponding fields as ghost primaries, as we defined in (3.31). In this case, there are infinitely many relaxed Verma modules, labelled by j_ϕ, which is the charge of the field from Section 3.2. We get to choose different weight vectors as described in proposition 1 in [RW14], and note from the proposition that the module is irreducible unless $j_\phi \in \mathbb{Z}$. In fact, the relaxed Verma modules that differ by an integer of j_ϕ are the same module as stated in [RW14].

Lastly, the representations of a spectral flowed relaxed highest weight state with a charge j_{ϕ^ℓ} can generate an even more general module, which we call a spectral flowed relaxed Verma

$$
\begin{tikzcd}[arrows=dash]
\cdots \arrow[r] & \gamma_0 \left|\phi\right\rangle \arrow[r] \arrow[d] \arrow[dr] & \left|\phi\right\rangle \arrow[r] \arrow[d] \arrow[dl] \arrow[dr] & \beta_0 \left|\phi\right\rangle \arrow[r] \arrow[d] \arrow[dl] & \cdots \\
\cdots \arrow[r] & \gamma_{-1} \left|\phi\right\rangle \arrow[r] \arrow[d] & \gamma_{-1}\beta_0 \left|\phi\right\rangle \arrow[r] \arrow[d] & \beta_{-1} \left|\phi\right\rangle \arrow[r] \arrow[d] & \cdots \\
& \cdots & \cdots & \cdots &
\end{tikzcd}
$$

Figure 3.2: Relaxed Verma module

module. The module can also be briefly visualised as in Fig. 3.3. By comparing with Fig. 3.2,

$$
\begin{tikzcd}[arrows=dash]
\ddots \arrow[dr] & & & & \\
& \gamma_\ell \left|\phi^\ell\right\rangle \arrow[dr] & & & \\
& \cdots & \left|\phi^\ell\right\rangle \arrow[dr] & & \\
& & & \beta_{-\ell} \left|\phi^\ell\right\rangle \arrow[dr] & \\
& & & & \ddots
\end{tikzcd}
$$

Figure 3.3: Spectral flowed Verma module

this is actually generated by twisting a relaxed Verma module by σ^ℓ. The corresponding fields of the states at the top are now the primary fields, which agree with (3.29).

We have mentioned the charges of the fields, and we can show how they change for different ℓ,

$$
J_0 \left|\phi^\ell\right\rangle = j_{\phi^\ell} \left|\phi^\ell\right\rangle = (j_\phi - \ell) \left|\phi^\ell\right\rangle , \tag{3.38}
$$

by using (3.26) directly. Hence, we have the charge for a spectral flowed relaxed highest weight state being $j_{\phi^\ell} = j_\phi - \ell$. With the same idea, we can use (3.27) to find the conformal dimension h_{ϕ^ℓ}, which is

$$
L_0 \left|\phi^\ell\right\rangle = L_0 \sigma^\ell(\left|\phi\right\rangle) = \left(\ell(j+\ell) - \frac{1}{2}\ell(\ell+1)\right) \left|\phi^\ell\right\rangle . \tag{3.39}
$$

The conformal dimension for a spectral flowed relaxed highest weight state is

$$
h_{\phi^\ell} = \ell(j+\ell) - \frac{1}{2}\ell(\ell+1). \tag{3.40}
$$

One can also notice that this agrees with (3.32): substituting $\ell = 0$ gives $h_\phi = 0$.

To summarise the representations, we have defined 4 types of modules:

- A Verma module generated by the modes β_{n-1}, γ_n for $n \leq 0$ acting on highest weight states.
- A conjugated Verma module generated by the modes β_n, γ_{n-1} for $n \leq 0$ acting on conjugate highest weight states.
- A relaxed Verma module generated by non-positive modes β_n, γ_n acting on relaxed highest weight states. The corresponding fields are with charges j_ϕ and conformal dimensions $h_\phi = 0$.
- A spectral flow relaxed Verma module generated by non-positive modes $\beta_{n-\ell}, \gamma_{n-\ell}$ acting on spectral flowed relaxed highest weight states, where the corresponding fields have charges $j_{\phi^\ell} = j_\phi - \ell$ and conformal dimensions $h_{\phi^\ell} = (j + \ell) - \frac{1}{2}\ell(\ell + 1)$.

As a Verma module is unique, and the only field corresponding to the highest weight state is the identity field, the correlation functions are hence less worthy to investigate. The same idea applies for the conjugate Verma module, so the first two modules are unique. Therefore, we are more interested in the third and fourth modules above, which will be the modules to compute the correlation functions in Section 4.1 and Section 4.2 respectively.

3.4 Ward Identity

After choosing the appropriate module, we can start to derive the constraints on correlation functions. Recall from Section 2.5 that the Virasoro modes L_n with indices $n > -2$ annihilate the vacuum. We can then derive three constraints by applying L_{-1}, L_0, L_1 on the correlation function of an n-point function, denoted by

$$\langle \Omega, \phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)\Omega \rangle \,, \tag{3.41}$$

where Ω is the vacuum we defined in Section 2.5. Or more conveniently, we can write this as

$$\langle \phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n) \rangle \,, \tag{3.42}$$

depending on whether we need to act with an operator on the vacuum or not. We call such constraints the Ward identities. Additionally, the constraint of acting with J_0 on the correlator will be mentioned later in Section 3.4.4.

3.4.1 Ward Identity Contradiction

Ward identities can be derived by acting with L_{-1}, L_0, L_1 on the correlation function. We start by assuming that the adjoint of L_n is simply L_{-n}, written as

$$L_n^\dagger = L_{-n}. \tag{3.43}$$

In fact, this is the case for most CFTs, for example in the free boson. Some articles even state this as a general result in CFT, such as [DFMS97, page 202]. First we start by considering a 2-point function $\langle \phi_1(z_1)\phi_2(z_2)\rangle$, of two Virasoro primary fields with h_1 and h_2 being the conformal dimensions respectively.[1] Acting with L_{-1} on the correlator gives us that

$$\begin{aligned}\langle \Omega, L_{-1}\phi_1(z_1)\phi_2(z_2)\Omega\rangle &= \left\langle L_{-1}^\dagger \Omega, \phi_1(z_1)\phi_2(z_2)\Omega\right\rangle \\ &= \langle L_1\Omega, \phi_1(z_1)\phi_2(z_2)\Omega\rangle = 0.\end{aligned} \tag{3.44}$$

In the meantime, by using (2.23) we can obtain

$$\begin{aligned}\langle \Omega, L_{-1}\phi_1(z_1)\phi_2(z_2)\Omega\rangle &= \langle \Omega, ([L_{-1}, \phi_1(z_1)]\phi_2(z_2) + \phi_1(z_1)[L_{-1}, \phi_2(z_2)] + \phi_1(z_1)\phi_2(z_2)L_{-1})\,\Omega\rangle \\ &= \langle \partial\phi_1(z_1)\phi_2(z_2) + \phi_1(z_1)\partial\phi_2(z_2)\rangle\,.\end{aligned} \tag{3.45}$$

With the same method by using the adjoints of L_0 and L_1 and applying (2.23), we come up with three constraints.

$$(\partial_1 + \partial_2)\langle \phi_1(z_1)\phi_2(z_2)\rangle = 0, \tag{3.46a}$$

$$(z_1\partial_1 + z_2\partial_2 + h_1 + h_2)\langle \phi_1(z_1)\phi_2(z_2)\rangle = 0, \tag{3.46b}$$

$$(z_1^2\partial_1 + z_2^2\partial_2 + 2hz_1 + 2h_2z_2)\langle \phi_1(z_1)\phi_2(z_2)\rangle = 0, \tag{3.46c}$$

where the ∂_i stands for $\frac{\partial}{\partial z_i}$. These are the Virasoro Ward identities for the 2-point function of Virasoro primaries, and we can then solve for the two point functions using those constraints.

(3.46a) shows that the 2-point correlator does not depend on $z_1 + z_2$. To see this, define $z = z_1 + z_2,\ z_{12} = z_1 - z_2$, and thus

$$\partial = \partial_1 + \partial_2, \quad \partial_{12} = \partial_1 - \partial_2. \tag{3.47}$$

Then (3.65a) becomes

$$(\partial_1 + \partial_2)\langle \phi_1(z_1)\phi_2(z_2)\rangle = \partial\langle \phi_1, \phi_2\rangle = 0, \tag{3.48}$$

[1] Here h means the same as Δ_{ϕ_i}, being the conformal dimension of the field $\phi_i(z)$.

so the correlator is a function depending only on z_{12}:

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = f(z_{12}). \tag{3.49}$$

Substituting into (3.46b), we can compute

$$\begin{aligned}
0 &= (z_1\partial_1 + z_2\partial_2 + h_1 + h_2)f(z_{12}) \\
&= \left(\frac{1}{2}(z+z_{12})(\partial+\partial_{12}) + \frac{1}{2}(z-z_{12})(\partial-\partial_{12}) + h_1 + h_2\right) f(z_{12}) \\
&= \left(\frac{1}{2}(z\partial + z\partial_{12} + z_{12}\partial + z_{12}\partial_{12} + z\partial - z\partial_{12} - z_{12}\partial + z_{12}\partial_{12}) + h_1 + h_2\right) f(z_{12}) \\
&= (z_{12}\partial_{12} + h_1 + h_2)f(z_{12}),
\end{aligned} \tag{3.50}$$

where we have used (3.48) to remove the terms with ∂. This is then easily solved:

$$\begin{aligned}
&\partial_{12}f(z_{12}) + \frac{h_1+h_2}{z_{12}}f(z_{12}) = 0 \\
\Rightarrow\ & z_{12}^{h_1+h_2}\partial_{12}f(z_{12}) + z_{12}^{h_1+h_2-1}(h_1+h_2)f(z_{12}) = 0 \\
\Rightarrow\ & \partial_{12}(z^{h_1+h_2}f(z_{12})) = 0 \\
\Rightarrow\ & f(z_{12}) = \frac{C_{12}}{z_{12}^{h_1+h_2}},
\end{aligned} \tag{3.51}$$

where C_{12} is a constant. Lastly, we apply the third constraint (3.46c):

$$\begin{aligned}
0 &= (z_1^2\partial_1 + z_2^2\partial_2 + 2h_1z_1 + 2h_2z_2)f(z_{12}) \\
&= ((\frac{z+z_{12}}{2})^2(\partial+\partial_{12}) + (\frac{z-z_{12}}{2})^2(\partial-\partial_{12}) + 2h_1z_1 + 2h_2z_2)f(z_{12}) \\
&= (\frac{1}{4}(z+z_{12})^2\partial_{12} - \frac{1}{4}(z-z_{12})^2\partial_{12} + h_1(z+z_{12}) + h_2(z-z_{12}))f(z_{12}) \\
&= (z(z_{12}\partial_{12} + (h_1+h_2)) + (h_1-h_2)z_{12})f(z_{12}) \\
&= (h_1-h_2)z_{12}f(z_{12}) \\
&= (h_1-h_2)\frac{C_{12}}{z_{12}^{h_1+h_2-1}} = 0,
\end{aligned} \tag{3.52}$$

where the terms including z are removed by using (3.51). The solution for the 2-point correlator is then

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \frac{C}{z_{12}^{h_1+h_2}}\delta_{h_1=h_2}. \tag{3.53}$$

Note that the solution is 0 unless $h_1 = h_2$. In fact, this is a standard result for a 2-point correlator with Virasoro primaries, which may be found in the textbook [DFMS97, eq(5.25)].

However, in the ghost CFT, we have $\beta(z)$ and $\gamma(w)$ as Virasoro primaries, and were given from (3.1) that

$$\langle \beta(z)\gamma(w)\rangle = \frac{1}{z-w}, \tag{3.54}$$

where $h_\beta = 1,\ h_\gamma = 0$. This contradicts (3.53) because $h_\beta \neq h_\gamma$, and thus the assumption (3.43) must not be true here. We thus aim to derive the correct Ward identities for the ghost CFT, and use them to solve for the ghost correlation functions.

3.4.2 Ward Identities for Primaries Fields

To derive the adjoints of the Virasoro modes, we need to find the adjoints for the ghost modes β_n and γ_n first, by guessing that

$$\beta_n^\dagger = \varepsilon_1 \gamma_{-n}, \quad \gamma_n^\dagger = \varepsilon_2 \beta_{-n}, \tag{3.55}$$

and using the axiom $(AB)^\dagger = B^\dagger A^\dagger$.

$$\begin{aligned} [\gamma_m, \beta_n]^\dagger &= -[\gamma_m^\dagger, \beta_m^\dagger] = -[\varepsilon_2\beta_{-m}, \varepsilon_1\gamma_{-n}] = \varepsilon_1\varepsilon_2\delta_{-m-n,0}\mathbf{1} \\ [\gamma_m, \beta_n] &= \delta_{m+n,0}\mathbf{1}, \end{aligned} \tag{3.56}$$

with help from (3.12). As the identity is self-adjoint, the commutator of γ_m and β_n is then self-adjoint. Thus we can set $\varepsilon_1 = \varepsilon_2 = 1$ and obtain the adjoints as

$$\beta_n^\dagger = \gamma_{-n}, \quad \gamma_n^\dagger = \beta_{-n}. \tag{3.57}$$

The adjoints for J_n and L_n can then be computed by using (3.15) and (3.16).

Firstly consider the case of J_n with $n = 0$,

$$\begin{aligned} J_0^\dagger &= [\sum_{r\leq -1} \beta_r\gamma_{-r} + \sum_{r\geq 0}\gamma_{-r}\beta_r]^\dagger = \sum_{r\leq -1}\gamma_{-r}^\dagger\beta_r^\dagger + \sum_{r\geq 0}\beta_r^\dagger\gamma_{-r}^\dagger \\ &= \sum_{r\leq -1}\beta_r\gamma_{-r} + \sum_{r\geq 0}\gamma_{-r}\beta_r = J_0. \end{aligned} \tag{3.58}$$

This zero mode is then self-adjoint. When $n \neq 0$, the ghost modes commute with each other, we then have

$$\begin{aligned} J_n^\dagger &= \sum_{r\in\mathbb{Z}} (\beta_r\gamma_{n-r})^\dagger = \sum_{r\in\mathbb{Z}} \gamma_{-r}\beta_{-n+r} \\ &= \sum_{m\in\mathbb{Z}} \gamma_{-n-m}\beta_m = J_{-n}, \end{aligned} \tag{3.59}$$

by letting $m = -n + r$ and thus $r = m + n$. The adjoints for L_n follow from the same method, such that

$$\begin{aligned}
L_0^\dagger &= \sum_{m\leq -1}(-m)(\beta_m\gamma_{-m})^\dagger + \sum_{m\geq 0}(-m)(\gamma_{-m}\beta_m)^\dagger \\
&= \sum_{m\leq -1}(-m)\gamma_{-m}\beta_m + \sum_{m\geq 0}(-m)\beta_m\gamma_{-m} \\
&= L_0,
\end{aligned} \tag{3.60}$$

and for $n \neq 0$, we have

$$\begin{aligned}
L_n^\dagger &= [\sum_{m\in\mathbb{Z}}(n-m)\beta_m\gamma_{n-m}]^\dagger = \sum_{m\in\mathbb{Z}}(n-m)\beta_{-n+m}\gamma_{-m} \\
&= \sum_{m\in\mathbb{Z}}(-m)\beta_m\gamma_{-n-m} = \sum_{m\in\mathbb{Z}}(-n-m)\beta_m\gamma_{-n-m} + \sum_{m\in\mathbb{Z}} n\beta_m\gamma_{-n-m} \\
&= L_{-n} + nJ_{-n}.
\end{aligned} \tag{3.61}$$

This is in fact different from what we assumed in (3.43).

Now, we need to use the correct adjoints for L_n to rederive the Ward identities. Recall from (3.13), J_n annihilates the vacuum for all $n > -1$. We can then apply the Virasoro modes on an n-point correlator of primary fields. More specifically, the fields only need to be both Virasoro primaries and free boson primaries, which means that the corresponding states satisfy (2.18) and (3.4) at the same time. First for L_{-1}, we have

$$\begin{aligned}
\langle \Omega, L_{-1}\phi_1(z_1)\cdots\phi_n(z_2)\Omega\rangle &= \left\langle L_{-1}^\dagger\Omega, \phi_1(z_1)\cdots\phi_n(z_2)\Omega\right\rangle \\
&= \langle (L_1 - J_1)\Omega, \phi_1(z_1)\cdots\phi_n(z_2)\Omega\rangle \\
&= 0,
\end{aligned} \tag{3.62}$$

as L_1 and J_1 both annihilates the vacuum. Additionally, L_0 is self adjoint as shown in (3.60) and gives zero when acting on the correlator. Therefore, the derivations of the first two Ward identities are the same as in Section 3.4.1. The only difference is now we are dealing with an n-point function of primary fields as defined in Section 3.2. We are then more interested in what happens with the L_1 Ward identity:

$$\begin{aligned}
\langle \Omega, L_1\phi_1(z_1)\cdots\phi_n(z_n)\Omega\rangle &= \left\langle L_1^\dagger, \phi_1(z_1)\cdots\phi_n(z_n)\Omega\right\rangle \\
&= \langle (L_{-1} + J_{-1})\Omega, \phi_1(z_1)\cdots\phi_n(z_n)\Omega\rangle \\
&= \langle J_{-1}\Omega, \phi_1(z_1)\cdots\phi_n(z_n)\Omega\rangle \,.
\end{aligned} \tag{3.63}$$

The result is nonzero as J_{-1} does not annihilate the vacuum. Hence, we need to compute the right hand side of (3.63) by using (3.18):

$$\begin{aligned}\langle J_{-1}\Omega, \phi_1(z_1)\cdots\phi_n(z_n)\Omega\rangle &= \langle\Omega, J_1\phi_1(z_1)\cdots\phi_n(z_n)\Omega\rangle \\ &= \langle\Omega, [J_1,\phi_1(z_1)]\phi_2(z_2)\cdots\phi_n(z_n)+\cdots+ \\ &\quad \phi_1(z_1)\cdots\phi_{n-1}(z_{n-1})[J_1,\phi_n(z_n)]+\phi_1(z_1)\cdots\phi_n(z_n)J_1\Omega\rangle \\ &= \langle(j_1z_1+j_2z_2+\cdots+j_3z_3)\phi_1(z_1)\cdots\phi_n(z_n)\rangle\,, \end{aligned} \tag{3.64}$$

where $j_1, j_2, \ldots, j_n$ are the charges of $\phi_1(z_1), \phi_2(z_2), \ldots, \phi_n(z_n)$ respectively and we have used the condition that $\phi_i(z_i)$ are free boson primaries. We can then substitute the result back in (3.63), and apply (2.23) to compute the left hand side of (3.63) by noticing that $\phi_i(z_i)$ are Virasoro primaries as well.

Combining all the above results, we then obtain the Ward identities for an n-point correlator in the ghost CFT:

$$\sum_{i=1}^{n}\partial_i\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = 0, \tag{3.65a}$$

$$\sum_{i=1}^{n}(h_i+z_i\partial_i)\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = 0, \tag{3.65b}$$

$$\sum_{i=1}^{n}(2h_iz_i+z_i^2\partial_i-j_iz_i)\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = 0. \tag{3.65c}$$

These identities will be used throughout all of our calculations in Chapter 4.

3.4.3 Checking Ward Identities

One thing before we start to compute the solutions of the correlators, is to check if our new Ward identities are consistent with the 2-point correlator of Virasoro primaries $\beta(z)$ and $\gamma(z)$ (3.54). Substitute this into the Ward identities (3.65), and (3.65a) gives

$$(\partial_z+\partial_w)\langle\beta(z)\gamma(w)\rangle = (\partial_z+\partial_w)\frac{-1}{z-w} = 0, \tag{3.66}$$

while (3.65b) gives

$$(1+z\partial_z+w\partial_w)\left(\frac{-1}{z-w}\right) = 0. \tag{3.67}$$

Lastly, (3.65c) tells us that

$$(2z+z^2\partial_z+w^2\partial_w-(z-w))\frac{-1}{z-w} = 0. \tag{3.68}$$

It is easy to verify that all three are true. Therefore, the new Ward identities agree with (3.1), and we use these to compute other correlators with more general primary fields.

3.4.4 Additional constraint

In fact, there is an additional Ward identity obtained by acting with J_0, given by

$$\begin{aligned}\langle\Omega, J_0\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)\Omega\rangle &= \sum_{i=1}^{n}\langle\Omega, j_i\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)\Omega\rangle + \langle\Omega, \phi_1\cdots\phi_n J_0\Omega\rangle \\ &= \sum_{i=1}^{n}\langle\Omega, j_i\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)\Omega\rangle\,, \end{aligned} \tag{3.69}$$

by recalling that J_0 annihilates the vacuum and using (3.18) again. Then we use the fact that J_0 is self-adjoint from (3.58), and the LHS of the above equation is

$$\langle\Omega, J_0\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)\Omega\rangle = \langle J_0\Omega, \phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)\Omega\rangle = 0. \tag{3.70}$$

Hence we have the fourth constraint on the correlator, written as

$$\sum_{i=1}^{n} j_i\,\langle\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)\rangle = 0. \tag{3.71}$$

The constraint implies that the n-point correlation function must be proportional to the delta function $\delta_{j_1+\cdots+j_n=0}$, so it can be imposed directly into the solutions in Chapter 4.

Chapter 4

Correlators of Bosonic Ghosts

Now we have defined the modules we care about, and found the correct Ward identites for the ghost CFT. We can start to solve our correlation relations of ghost (3.31) or primary fields (3.29) by using the Ward identities and deriving analogues of the KZ equations. In this chapter, we will sometimes write $\phi_i(z_i)$ as ϕ_i for simplicity.

4.1 Fields without Spectral Flow

In this section, we choose the relaxed Verma module defined in Section 3.3, and try to solve for the n-point correlators up to $n = 4$. The fields are then primaries with $\ell = 0$. In particular, they are Virasoro and free boson primaries.

4.1.1 1-Point Function

Starting with the most straightfoward case, a 1-point correlator is $\langle \phi_1(z_1) \rangle$ with conformal dimension h_1 and charge j_1. Then (3.65a) tells us that

$$\partial_1 \langle \phi_1(z_1) \rangle = 0, \tag{4.1}$$

which means $\langle \phi_1(z_1) \rangle$ is a constant function $\langle \phi_1(z_1) \rangle = C_1$. By applying (3.65b), we have

$$\begin{aligned}(h_1 + z_1\partial_1) \langle \phi_1(z_1) \rangle &= 0 \\ \Rightarrow \; h_1 \langle \phi_1(z_1) \rangle &= 0 \\ \Rightarrow \; \langle \phi_1(z_1) \rangle &= C_1 \delta_{h=0}. \end{aligned} \tag{4.2}$$

Lastly, (3.65c) gives that

$$\begin{aligned} \big((2h_1 - j_1)z_1 + z_1^2\partial_1\big)\langle\phi_1(z_1)\rangle &= 0 \\ \Rightarrow\ -j_1 z_1 \langle\phi_1(z_1)\rangle &= 0 \\ \Rightarrow\ \langle\phi_1(z_1)\rangle &= C_1\delta_{h_1=0}\delta_{j_1=0}. \end{aligned} \tag{4.3}$$

Recall that the fourth Ward identity (3.71) gives $j_1 = 0$ for a 1-point function, which is automatically satisfied from the solution. We then have the solution as

$$\langle\phi_1(z_1)\rangle = C_1\delta_{h_1=0}\delta_{j_1=0}. \tag{4.4}$$

This implies there are non-zero solutions only when $\phi_1(z_1)$ is a vacuum with the conformal dimension $h_1 = 0$ and charge $j_1 = 0$. We can also remark that correlator of 1-point function is not affected by the new version of Ward identities (3.65), so it gives the same answer as being a vacuum as in the free boson in [Rid13, eq(6.23)].

4.1.2 2-Point Function

As we have computed the 2-point function of Virasoro primaries in Section 3.4.1, and (3.46a), (3.46b) agree with the Ward identities in ghost CFT (3.65a) and (3.65b). We can use the result (3.50) directly, which is

$$f(z_{12}) = \frac{C_{12}}{z_{12}^{h_1+h_2}}, \tag{4.5}$$

where C_{12} is a constant. Then, use (3.65c) to further constrain the correlation function and have

$$(z_1^2\partial_1 + z_2^2\partial_2 + 2h_1z_1 + 2h_2z_2 - j_1z_1 - j_2z_2)f(z_{12}) = 0. \tag{4.6}$$

By applying $z = z_1 + z_2, z_{12} = z_1 - z_2, \partial = \partial_1 + \partial_2, \partial_{12} = \partial_1 - \partial_2$ again to obtain

$$\begin{aligned} 0 &= ((\frac{z+z_{12}}{2})^2(\partial + \partial_{12}) + (\frac{z-z_{12}}{2})^2(\partial - \partial_{12}) + (2h_1 - j_1)(\frac{z+z_{12}}{2}) \\ &\quad + (2h_2 - j_2)(\frac{z-z_{12}}{2}))f(z_{12}) \\ &= (z(z_{12}\partial_{12} + (h_1+h_2)) + (h_1-h_2)z_{12} - \frac{1}{2}j_1(z+z_{12}) - \frac{1}{2}j_2(z-z_{12}))f(z_{12}) \\ &= (z_{12}(h_1 - h_2 - \frac{1}{2}j_1 + \frac{1}{2}j_2) - \frac{1}{2}z(j_1+j_2))\frac{C_{12}}{z_{12}^{h_1+h_2}}, \end{aligned} \tag{4.7}$$

where we have used (3.51) again to remove $z(z_{12}\partial_{12} + h_1 + h_2)$. Then if C_{12} is non zero, we will have

$$z_{12}(h_1 - h_2 - \frac{1}{2}j_1 + \frac{1}{2}j_2) - \frac{1}{2}z(j_1 + j_2) = 0. \tag{4.8}$$

The coefficients in z_{12} and z must be zero for non-zero solutions. Hence we have

$$h_1 - h_2 = \frac{1}{2}(j_1 - j_2), \quad j_1 + j_2 = 0, \tag{4.9}$$

where the fourth Ward identity (3.71) is automatically satisfied. The Ward identities give the general solution

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \frac{C_{12}\delta_{h_1-h_2=j_1}\delta_{j_1+j_2=0}}{(z_1 - z_2)^{h_1+h_2}}, \tag{4.10}$$

for a constant C_{12}. Comparing to (3.53), we can notice the correlators for ghost primaries differs from the textbook result by shifting $h_1 - h_2 = 0$ to $h_1 - h_2 = j_1$. Additionally, we can further solve for the solution by applying (3.32), such that $h_1 = h_2 = 0$. To satisfy the delta functions, we need to have $j_1 = j_2 = 0$. Therefore, the solution for the 2-point ghost primaries can be written as

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = C_{12}\delta_{h_1=h_2=j_1=j_2=0}, \tag{4.11}$$

implying that solutions can be nonzero only when $\phi_1(z_1)$ and $\phi_2(z_2)$ are vacuums.

4.1.3 3-Point Function

Now, for a 3-point correlator, we can start by introducing the fourth Ward identity (3.71) such that

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \delta_{j_1+j_2+j_3=0}F(z_1, z_2, z_3), \tag{4.12}$$

where $F(z_1, z_2, z_3)$ is the function we want to solve for later. We then let $z = z_1+z_2+z_3$, $z_{12} = z_1 - z_2$, $z_{23} = z_2 - z_3$, and $\partial = \partial_1 + \partial_2 + \partial_3$, $\partial_{12} = \partial_1 - \partial_2$, $\partial_{23} = \partial_2 - \partial_3$. Expressing them in terms of z_1, z_2 and z_3 gives

$$\begin{aligned}
3z_1 &= z + 2z_{12} + z_{23}, & \partial_1 &= \partial + \partial_{12}, \\
3z_2 &= z - z_{12} + z_{23}, & \partial_2 &= \partial - \partial_{12} + \partial_{23}, \\
3z_3 &= z - z_{12} - 2z_{23}, & \partial_3 &= \partial - \partial_{23}.
\end{aligned} \tag{4.13}$$

Again, (3.65a) tells us the correlator does not depend on $z_1 + z_2 + z_3$,

$$\begin{aligned}
(\partial_1 + \partial_2 + \partial_3)\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle &= \partial\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = 0 \\
\Rightarrow \langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle &= \delta_{j_1+j_2+j_3=0}f(z_{12}, z_{23}).
\end{aligned} \tag{4.14}$$

Now express (3.65b) in terms of the expressions in (4.13).

$$0 = (z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + h_1 + h_2 + h_3)f(z_{12}, z_{23})\delta_{j_1+j_2+j_3=0}. \tag{4.15}$$

Then substitute (4.13) into the first three terms to have

$$\begin{aligned}
z_1\partial_1 + z_2\partial_2 + z_3\partial_3 &= \frac{1}{3}[(z + 2z_{12} + z_{23})(\partial + \partial_{12}) \\
&\quad + (z - z_{12} + z_{23})(\partial - \partial_{12} + \partial_{23}) + (z - z_{12} - 2z_{23})(\partial - \partial_{23})] \\
&= \frac{1}{3}(z\partial_{12} + 2z_{12}\partial_{12} + z_{23}\partial_{12} - z\partial_{12} + z\partial_{23} + z_{12}\partial_{12} - z_{12}\partial_{23} \\
&\quad - z_{23}\partial_{12} + z_{23}\partial_{23} - z\partial_{23} + z_{12}\partial_{23} + 2z_{23}\partial_{23}) \\
&= z_{12}\partial_{12} + z_{23}\partial_{23}.
\end{aligned} \tag{4.16}$$

We have changed equation (4.15) into

$$0 = (z_{12}\partial_{12} + z_{23}\partial_{23} + h_1 + h_2 + h_3)f(z_{12}, z_{23})\delta_{j_1+j_2+j_3=0}. \tag{4.17}$$

This is a first order PDE, so we can apply the method of characteristics by parametrising z_i to be dependent on t. Then by chain rule $\frac{df}{dt} = \frac{\partial f}{\partial z_{12}}\frac{dz_{12}}{dt} + \frac{\partial f}{\partial z_{23}}\frac{dz_{23}}{dt}$, and we have

$$\begin{aligned}
\frac{dz_{12}}{dt} &= z_{12}, \quad \frac{dz_{23}}{dt} = z_{23}, \\
\frac{df}{dt} &= (z_{12}\partial_{12} + z_{23}\partial_{23})f(z_{12}, z_{23}) = -(h_1 + h_2 + h_3)f(z_{12}, z_{23}),
\end{aligned} \tag{4.18}$$

by rearranging (4.17) to obtain $\frac{df}{dt}$. Then we solve for the three ODEs, giving us the following solutions

$$z_{12} = z_{12}(0)e^t, \quad z_{23} = z_{23}(0)e^t, \quad f = f(0)e^{-(h_1+h_2+h_3)t}. \tag{4.19}$$

Notice that $\frac{z_{23}}{z_{12}} = \frac{z_{23}(0)}{z_{12}(0)}$ is a constant with respect to t, so we can write $f(0)$ in terms of $\frac{z_{23}}{z_{12}}$. The solution for f will be

$$f(z_{12}, z_{23}) = g(\frac{z_{23}}{z_{12}})z_{12}^{-(h_1+h_2+h_3)}. \tag{4.20}$$

Now we want to solve for the function $g(\frac{z_{23}}{z_{12}})$ by first changing the variables as $v = \frac{z_{23}}{z_{12}}$ and $u = z_{12}$ so that we have

$$\frac{\partial g}{\partial z_1} = \frac{dg}{dv}\frac{\partial v}{\partial z_1} = -\frac{v}{u}\partial_v g, \quad \frac{\partial g}{\partial z_2} = \frac{dg}{dv}\frac{\partial v}{\partial z_2} = \frac{1+v}{u}\partial_v g, \quad \frac{\partial g}{\partial z_3} = \frac{dg}{dv}\frac{\partial v}{\partial z_3} = -\frac{1}{u}\partial_v g, \tag{4.21}$$

and

$$f(u, v) = g(v)u^{-h_1-h_2-h_3}. \tag{4.22}$$

Then use (3.65c) to obtain the equation

$$0 = (z_1^2\partial_1 + z_2^2\partial_2 + z_3^2\partial_3 + (2h_1 - j_1)z_1 + (2h_2 - j_2)z_2 + (2h_3 - j_3)z_3)f(z_{12}, z_{23}). \tag{4.23}$$

Substitute (4.22) and compute the partial derivatives, and as the equation is long, we break it into two expressions. We first compute the terms $z_1^2\partial_1 + z_2^2\partial_2 + z_3^2\partial_3 + 2h_1z_1 + 2h_2z_2 + 2h_3z_3$, denoted by $(*)$, as some terms may cancel out

$$\begin{aligned}(*) = {} & z_1^2(-h_1 - h_2 - h_3)u^{-h_1-h_2-h_3-1}g(v) + z_1^2u^{-h_1-h_2-h_3}\left(\frac{-v}{u}\right)\partial_v g\\ & + z_2^2(h_1 + h_2 + h_3)u^{-h_1-h_2-h_3-1}g(v) + z_2^2u^{-h_1-h_2-h_3}\left(\frac{1+v}{u}\right)\partial_v g\\ & + z_3^2u^{-h_1-h_2-h_3}\left(\frac{-1}{u}\right)\partial_v g + 2(h_1z_1 + h_2z_2 + h_3z_3)u^{-h_1-h_2-h_3}g(v).\end{aligned}$$

Extract the common factors and write as

$$\begin{aligned}& = (-z_1^2v + z_2^2(1+v) - z_3^2)u^{-h_1-h_2-h_3-1}\partial_v g\\ & \quad - (z_1^2 - z_2^2)(h_1 + h_2 + h_3)u^{-h_1-h_2-h_3-1}g(v) + 2(h_1z_1 + h_2z_2 + h_3z_3)u^{-h_1-h_2-h_3}g(v)\\ & = (-(z_1^2 - z_2^2)v + (z_2^2 - z_3^2))u^{-h_1-h_2-h_3-1}\partial_v g\\ & \quad + (h_1z_1 - h_1z_2 + h_2z_2 - h_2z_1 + 2h_3z_3 - h_3z_1 - h_3z_2)u^{-h_1-h_2-h_3}g(v)\end{aligned}$$

Then write v back to $\frac{z_{23}}{z_{12}}$, and extract common terms so that we have

$$\begin{aligned}(*) & = (z_2 - z_3)(z_2 + z_3 - z_1 - z_2)u^{-h_1-h_2-h_3-1}\partial_v g + h_1u - h_2u - h_3(2uv + u)u^{-h_1-h_2-h_3}g(v)\\ & = -uv(u + uv)u^{-h_1-h_2-h_3-1}\partial_v g + u(h_1 - h_2 - 2h_3v - h_3)u^{-h_1-h_2-h_3}g(v)\\ & = -v(u + uv)u^{-h_1-h_2-h_3}\partial_v g + u(h_1 - h_2 - 2h_3v - h_3)u^{-h_1-h_2-h_3}g(v),\end{aligned} \tag{4.24}$$

where the second last step is obtained by noticing $z_{23} = uv$. Then we focus back on the terms with j's, we will need to substitute (4.13) and write it in terms of u and v as well.

$$\begin{aligned}j_1z_1 + j_2z_2 + j_3z_3 & = \frac{1}{3}j_1(z + 2z_{12} + z_{23}) + \frac{1}{3}j_2(z - z_{12} + z_{23}) + \frac{1}{3}j_3(z - z_{12} - 2z_{23})\\ & = \frac{1}{3}\left(z(j_1 + j_2 + j_3) + z_{12}(2j_1 - j_2 - j_3) + z_{23}(j_1 + j_2 - 2j_3)\right).\end{aligned} \tag{4.25}$$

We then use (3.71) to have $j_1 + j_2 + j_3 = 0$, so that the expression becomes

$$j_1z_1 + j_2z_2 + j_3z_3 = j_1z_{12} - j_3z_{23} = j_1u - j_3uv. \tag{4.26}$$

After simplifying all the terms, we can substitute them back into (4.23) to obtain

$$\begin{aligned}
&0 = -uv(1+v)u^{-h_1-h_2-h_3}\partial_v g + u[h_1 - h_2 - 2h_3 v - h_3 - (j_1 - j_3 v)]u^{-h_1-h_2-h_3} g(v) \\
\Rightarrow\ &0 = -v(1+v)\partial_v g + (h_1 - h_2 - 2h_3 v - h_3 - j_1 + j_3 v) g(v),
\end{aligned} \tag{4.27}$$

where we divided the equation by $u^{-h_1-h_2-h_3+1}$. Now the equation becomes an ODE that depends on v only, so we can solve for it to obtain:

$$\begin{aligned}
&\partial_v g = \frac{h_1 - h_2 - 2h_3 v - h_3 - j_1 + j_3 v}{v(1+v)} g(v) = \left[\frac{h_1 - h_2 - h_3 - j_1}{v(1+v)} + \frac{j_3 - 2j_3}{1+v}\right] g(v) \\
\Rightarrow\ &g(v) = \frac{C_{123}}{v^{-h_1+h_2+h_3+j_1}(1+v)^{h_1-h_2+h_3-j_3-j_1}}.
\end{aligned} \tag{4.28}$$

Finally, we can substitute $g(v)$ back into (4.22) and write everything in z_j's again by reminding that $z_{ij} = z_i - z_j$.

$$\begin{aligned}
f(z_{12}, z_{23}) &= \delta_{j_1+j_2+j_3=0} z_{12}^{-h_1-h_2-h_3} g(\frac{z_{23}}{z_{12}}) \\
&= \frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{h_1+h_2+h_3}(\frac{z_{23}}{z_{12}})^{-h_1+h_2+h_3+j_1}(1+\frac{z_{23}}{z_{12}})^{h_1-h_2+h_3-j_3-j_1}} \\
&= \frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{h_1+h_2-h_3+j_3} z_{23}^{-h_1+h_2+h_3+j_1} z_{13}^{h_1-h_2+h_3+j_2}}.
\end{aligned} \tag{4.29}$$

Therefore, the general solution for a 3-point correlation function is

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{h_1+h_2-h_3+j_3} z_{23}^{-h_1+h_2+h_3+j_1} z_{13}^{h_1-h_2+h_3+j_2}}, \tag{4.30}$$

where C_{123} is a constant. Now we can use (3.32) again to simplify the solution for the correlator of ghost primaries

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{j_3} z_{23}^{j_1} z_{13}^{j_2}}, \tag{4.31}$$

as $h_1 = h_2 = h_3 = 0$. Note that unlike in 1-point and 2-point function, the solution does not tell us about the exact values of j_1, j_2 and j_3.

4.1.4 4-Point Function

Remark that 1-point, 2-point and 3-point functions can be solved up to constant factors directly from the Ward identities. However, a 4-point function is a little more tricky to solve. To start, again, we apply (3.71) to write the correlator as

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \delta_{j_1+j_2+j_3+j_4=0} F(z_1, z_2, z_3, z_4), \tag{4.32}$$

we make a change of variables by letting

$$\begin{aligned} z &= z_1 + z_2 + z_3 + z_4, \qquad & \partial &= \partial_1 + \partial_2 + \partial_3 + \partial_4, \\ z_{12} &= z_1 - z_2, & \partial_{12} &= \partial_1 - \partial_2, \\ z_{23} &= z_2 - z_3, & \partial_{23} &= \partial_2 - \partial_3, \\ z_{34} &= z_3 - z_4, & \partial_{34} &= \partial_3 - \partial_4, \end{aligned} \tag{4.33}$$

and rearrange the expressions to write z_1, z_2, z_3, z_4 in terms of $z, z_{12}, z_{23}, z_{34}$:

$$\begin{aligned} 4z_1 &= z + 3z_{12} + 2z_{23} + z_{34}, \qquad & \partial_1 &= \partial + \partial_{12}, \\ 4z_2 &= z - z_{12} + 2z_{23} + z_{34}, & \partial_2 &= \partial - \partial_{12} + \partial_{23}, \\ 4z_3 &= z - z_{12} - 2z_{23} + z_{34}, & \partial_3 &= \partial - \partial_{23} + \partial_{34}, \\ 4z_4 &= z - z_{12} - 2z_{23} - 3z_{34}, & \partial_4 &= \partial - \partial_{34}. \end{aligned} \tag{4.34}$$

Then following the same steps, we apply (3.65a) and find out that the correlator does not depend on z.

$$\begin{aligned} 0 &= (\partial_1 + \partial_2 + \partial_3 + \partial_4) \langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle \\ &= \partial \langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle , \end{aligned} \tag{4.35}$$

so that we have

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \delta_{j_1+j_2+j_3+j_4=0} f(z_{12}, z_{23}, z_{34}). \tag{4.36}$$

Apply the second Ward identity (3.65b), and substitute (4.34) to obtain

$$\begin{aligned} 0 &= (z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + z_4\partial_4 + h_1 + h_2 + h_3 + h_4) \langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle \\ &= (z_{12}\partial_{12} + z_{23}\partial_{23} + z_{34}\partial_{34} + h_1 + h_2 + h_3 + h_4) f(z_{12}, z_{23}, z_{34}) \delta_{j_1+j_2+j_3+j_4=0}. \end{aligned} \tag{4.37}$$

We skipped the steps for these substitutions as there are not any new techniques involved. This is again a PDE but with 3 variables, so we can use method of characteristics, the same way as in the 3-point function Section 4.1.3.

$$\frac{df}{dt} = \frac{\partial f}{\partial z_{12}}\frac{dz_{12}}{dt} + \frac{\partial f}{\partial z_{23}}\frac{dz_{23}}{dt} + \frac{\partial f}{\partial z_{34}}\frac{dz_{34}}{dt}, \tag{4.38}$$

so that

$$\begin{aligned}
\frac{dz_{12}}{dt} &= z_{12} \quad \Rightarrow \quad z_{12} = z_{12}(0)e^t, \\
\frac{dz_{23}}{dt} &= z_{23} \quad \Rightarrow \quad z_{23} = z_{23}(0)e^t, \\
\frac{dz_{34}}{dt} &= z_{34} \quad \Rightarrow \quad z_{34} = z_{34}(0)e^t.
\end{aligned} \tag{4.39}$$

Then use (4.37) to obtain

$$\begin{aligned}
\frac{df}{dt} &= (z_{12}\partial_{12} + z_{23}\partial_{23} + z_{34}\partial_{34})f \\
&= -(h_1 + h_2 + h_3 + h_4)f(z_{12}, z_{23}, z_{34}) \\
\Rightarrow \ f(z_{12}, z_{23}, z_{34}) &= z_{12}^{-h} g(\frac{z_{23}}{z_{12}}, \frac{z_{34}}{z_{12}}),
\end{aligned} \tag{4.40}$$

by noticing that $\frac{z_{23}}{z_{12}}$ and $\frac{z_{34}}{z_{12}}$ are constants independent of t.

Lastly, apply (3.65c) so that we have

$$\begin{aligned}
0 = (z_1^2\partial_1 + z_2^2\partial_2 + z_3^2\partial_3 + z_4^2\partial_4 + (2h_1 - j_1)z_1 + (2h_2 - j_2)z_2 \\
+ (2h_3 - j_3)z_3 + (2h_4 - j_4)z_4)z_{12}^{-h} g(\frac{z_{23}}{z_{12}}, \frac{z_{34}}{z_{12}}).
\end{aligned} \tag{4.41}$$

Now we can take $u = \frac{z_{23}}{z_{12}}$, $v = \frac{z_{34}}{z_{12}}$ so that $f = z_{12}^{-h} g(u, v)$. By using chain rule, we can find derivatives of g with respect to z_j's.

$$\begin{aligned}
\frac{\partial g}{\partial z_1} &= -\frac{z_{23}}{z_{12}^2}\partial_u g - \frac{z_{34}}{z_{12}^2}\partial_v g, & \frac{\partial g}{\partial z_2} &= \frac{z_{12} + z_{23}}{z_{12}^2}\partial_u g + \frac{z_{34}}{z_{12}^2}\partial_v g, \\
\frac{\partial g}{\partial z_3} &= -\frac{1}{z_{12}}\partial_u g + \frac{1}{z_{12}}\partial_v g, & \frac{\partial g}{\partial z_4} &= -\frac{1}{z_{12}}\partial_v g.
\end{aligned} \tag{4.42}$$

Then use (3.71), which tells us $j_1 + j_2 + j_3 + j_4 = 0$ to simplify $j_1 z_1 + j_2 z_2 + j_3 z_3 + j_4 z_4$

$$\begin{aligned}
j_1 z_1 + j_2 z_2 + j_3 z_3 + j_4 z_4 &= \frac{1}{4}[j_1(z + 3z_{12} + 2z_{23} + z_{34}) + j_2(z - z_{12} + 2z_{23} + z_{34}) \\
&\quad + j_3(z - z_{12} - 2z_{23} + z_{34}) + j_4(z - z_{12} - 2z_{23} - 3z_{34})] \\
&= \frac{1}{4}[z_{12}(3j_1 - j_2 - j_3 - j_4) + z_{23}(2j_1 + 2j_2 - 2j_3 - 2j_4) \\
&\quad + z_{34}(j_1 + j_2 + j_3 - 3j_4)] \\
&= j_1 z_{12} + (j_1 + j_2)z_{23} - j_4 z_{34}.
\end{aligned} \tag{4.43}$$

Now, substitute into the equation (4.41) and simplify the terms, giving us

$$\begin{aligned}
0 &= -u(1+u)z_{12}\partial_u g - v(1+2u+v)z_{12}\partial_v g \\
&\quad + \left[(h_1-h_2)z_{12} + h_3(-z_{12}-2z_{23}) + h_4(-z_{12}-2z_{23}-2z_{34})\right] g \\
&\quad - \left[(j_1 z_{12} + (j_1+j_2)z_{23} - j_4 z_{34})\right] g \\
\Rightarrow\ 0 &= -u(1+u)\partial_u g - v(1+2u+v)\partial_v g + (h_1-h_2-h_3-h_4-j_1)g \\
&\quad + [u(-2h_3-2h_4-(j_1+j_2)) + v(-2h_4+j_4)]g \\
&= u(1+u)\partial_u g + v(1+2u+v)\partial_v g + (-h_1+h_2+h_3+h_4+j_1)g \\
&\quad + [u(2h_3+2h_4+(j_1+j_2)) + v(2h_4-j_4)]g.
\end{aligned} \tag{4.44}$$

Apply the method of characteristics once more to obtain the below solution for g,

$$g = u^{h_1-h_2-h_3-h_4+j_1}(1+u)^{-h_1+h_2-h_3-h_4+j_2}\left(-\frac{u(1+u)}{v}\right)^{2h_4+j_4} G\left(\frac{u(1+u+v)}{v}\right). \tag{4.45}$$

Now we have an expression containing an unknown function of ratio G. By simplifying $\frac{u(1+u+v)}{v}$, we can find out that G depends on a ratio of z_j's. Hence, set $G = G(\eta)$, where $\eta = \frac{z_{14}z_{23}}{z_{12}z_{34}}$. We can substitute $u = \frac{z_{23}}{z_{12}}$, $v = \frac{z_{34}}{z_{12}}$ into g.

$$\begin{aligned}
&\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle \\
&= G(\eta) z_{12}^{-h_1-h_2-h_3-h_4}\left(\frac{z_{23}}{z_{12}}\right)^{h_1-h_2-h_3-h_4+j_1}\left(\frac{z_{13}}{z_{12}}\right)^{-h_1+h_2-h_3-h_4+j_2}\left(\frac{z_{23}z_{13}}{z_{12}z_{34}}\right)^{2h_4+j_4} \\
&= G(\eta) z_{12}^{-h_1-h_2+h_3-h_4+j_3} z_{23}^{h_1-h_2-h_3+h_4+j_1+j_4} z_{13}^{-h_1+h_2-h_3+h_4+j_2+j_4} z_{34}^{-2h_4-j_4}.
\end{aligned} \tag{4.46}$$

As we want to write it in terms of a products of z_{ij}'s. We guess that

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = H(\eta)\prod_{a<b} z_{ab}^{h/3-h_a-h_b-\alpha j_a-\beta j_b}. \tag{4.47}$$

Then divide the solution by (4.47), and find α and β by equating the powers of z_{12}, z_{23}, z_{34} and z_{14} separately,

$$\frac{G(\eta) z_{12}^{-h_1-h_2+h_3-h_4+j_3} z_{23}^{h_1-h_2-h_3+h_4+j_1+j_4} z_{13}^{-h_1+h_2-h_3+h_4+j_2+j_4} z_{34}^{-2h_4-j_4}}{H(\eta)\prod_{a<b} z_{ab}^{h/3-h_a-h_b-\alpha j_a-\beta j_b}}, \tag{4.48}$$

which gives $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$. The solution for 4-point correlation function that we obtain from the Ward identities is then

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \delta_{j_1+j_2+j_3+j_4=0} H(\eta)\prod_{a<b} z_{ab}^{h/3-h_a-h_b-j_a/2-j_b/2}, \tag{4.49}$$

by including the constraint in (3.4.4) where $H(\eta)$ is an undetermined function. Then substitute $h_1 = h_2 = h_3 = h_4 = 0$ so that

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \delta_{j_1+j_2+j_3+j_4=0} H(\eta) \prod_{a<b} z_{ab}^{-j_a/2-j_b/2}, \tag{4.50}$$

which is the solution for the 4-point correlator with ghost primaries.

4.1.5 Knizhnik-Zamolodchikov Equations

The Knizhnik-Zamolodchikov (KZ) equation for an n-point correlator are derived by acting with L_{-1} on $\phi_1(z_1)$ in the correlator$\langle\phi_1\phi_2\cdots\phi_n\rangle$. Recall from (2.17) that the correlator becomes

$$\langle L_{-1}\phi_1\phi_2\cdots\phi_n\rangle = \langle \partial\phi_1\phi_2\cdots\phi_n\rangle\,, \tag{4.51}$$

which forms the LHS of the KZ equation. For the other side of the KZ, one can substitute the mode expansion of L_{-1} from (3.16) such that

$$\begin{aligned} L_{-1} &= \sum_{m\in\mathbb{Z}}(-1-m)\beta_m\gamma_{1-m} \\ \Rightarrow\ L_{-1}\left|\phi_1\right\rangle &= \sum_{m\in\mathbb{Z}}(-1-m)\beta_m\gamma_{-1-m}\left|\phi\right\rangle = -\gamma_{-1}\beta_0\left|\phi_1\right\rangle \\ \Rightarrow\ \langle L_{-1}\phi_1\phi_2\cdots\phi_n\rangle &= -\left\langle(\gamma_{-1}\beta_0\phi_1)\phi_2\cdots\phi_n\right\rangle, \end{aligned} \tag{4.52}$$

as other modes annihilate the ghost primaries. β_0 acting on $\phi_1(z_1)$ is a primary field as well, but the additional γ_{-1} makes $\gamma_{-1}\beta_0\phi_1(z_1)$ a secondary field following from Fig. 3.2. This forms the RHS of the KZ equation. Combining (4.51) and (4.52) gives us the KZ equation as

$$\langle \partial\phi_1\phi_2\cdots\phi_n\rangle = -\left\langle(\gamma_{-1}\beta_0\phi_1)\phi_2\cdots\phi_n\right\rangle. \tag{4.53}$$

Substituting the solutions derived from the Ward identities to the LHS, and apply Cauchy integral theorem on the RHS may provide us with some new information about the constants in 2-point and 3-point correlators, and help us solve for the constant function $g(\eta)$ in (4.50). Remark that this is not the only form of the KZ equation as we can also act with L_{-1} on other fields to obtain the KZ as other forms. However, the equations appear to have the same solutions since all of the fields are ghost primaries here, while in Section 4.1 with primary fields involved, different KZ equations may provide us with different information.

Now with the KZ equation of ghost primaries introduced, we can start to simplify the RHS of (4.53). Apply Cauchy integral theorem and then use the fact that a contour around z_1 is the same as a very large contour around all z_i minus contours around the other z_i, for

$i = 2, 3, \dots, n$, written as

$$\oint_{z_1} = \oint_{\infty} - \sum_{i=2}^{n} \oint_{z_i} . \tag{4.54}$$

Then RHS of (4.53) becomes

$$\begin{aligned} - \langle (\gamma_{-1}\beta_0\phi_1)\phi_2 \cdots \phi_n \rangle &= - \oint_{z_1} \left\langle \frac{\gamma(z)(\beta_0\phi_1)(z_1)}{(z-z_1)^2}\phi_2 \cdots \phi_n \right\rangle \frac{dz}{2\pi i} \\ &= (-\oint_{\infty} + \sum_{i=2}^{n} \oint_{z_i}) \left\langle \frac{(\beta_0\phi_1)(z_1)}{(z-z_1)^2}\phi_2 \cdots \gamma(z)\phi_i \cdots \phi_n \right\rangle \frac{dz}{2\pi i}. \end{aligned} \tag{4.55}$$

Recall that we had $\gamma(z)\phi^{\ell}(w) \sim 0$ for $\ell \leq 0$, meaning that $\gamma(z)\phi_i(z_i)$ always has a regular OPE. Additionally, by intuition the infinite integral does not contribute here, but we will prove this formally in Section 4.2.1. Hence the RHS of the KZ equation gives us that

$$\Rightarrow - \langle (\gamma_{-1}\beta_0\phi_1)\phi_2 \cdots \phi_n \rangle = 0. \tag{4.56}$$

This is a general result for any n-point correlator. Then, compute the LHS of the KZ equation for 1-point, 2-point, 3-point and 4-point functions by substituting the results derived from the Ward identities.

Firstly, the KZ equation for 1-point correlator does not give us any new information as the first Ward identity (3.65a) already tells us that $\partial \langle \phi_1 \rangle = 0$. Then LHS is 0, agreeing with the RHS.

For the 2-point correlator, differentiate (4.10) with respect to z_1 gives zero as the solution is a constant.

$$\langle \partial\phi_1\phi_2 \rangle = \partial_1 (C_{12}\delta_{h_1=h_2=j_1=j_2=0}) = 0. \tag{4.57}$$

Hence, once again, the KZ equation does not tell us anything new as we already knew that the solution of the 2-point correlation function is a constant.

Now, differentiate the 3-point correlator with respect to z_1, we can obtain

$$\begin{aligned} \langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \rangle &= \partial_1 \left[\frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{j_3} z_{23}^{j_1} z_{13}^{j_2}} \right] \\ &= \left(\frac{-j_3}{z_{12}} + \frac{-j_2}{z_{13}} \right) \frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{j_3} z_{23}^{j_1} z_{13}^{j_2}}, \end{aligned} \tag{4.58}$$

which is the LHS of the KZ equation, and equating with the RHS gives us

$$\left(\frac{-j_3}{z_{12}} + \frac{-j_2}{z_{13}} \right) \frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{j_3} z_{23}^{j_1} z_{13}^{j_2}} = 0. \tag{4.59}$$

To satisfy this equation, we must have $j_1 = j_2 = j_3 = 0$ or $C_{123} = 0$. Hence, KZ tells us that the solution of 3-point correlator can only be nonzero when $\phi_1(z_1), \phi_2(z_2), \phi_3(z_3)$ are all vacuums.

Lastly, substituting the 4-point correlator into the LHS of the KZ equation tells us

$$\begin{aligned}\langle \partial\phi_1\phi_2\phi_3\phi_4\rangle &= \partial_1 \left[\delta_{j_1+j_2+j_3+j_4=0}\, g(\eta) \prod_{a<b} z_{ab}^{-j_a/2-j_b/2}\right] \\ &= \left(\frac{-j_1-j_2}{2z_{12}} + \frac{-j_1-j_3}{2z_{13}} + \frac{-j_1-j_4}{2z_{14}}\right) \delta_{j_1+j_2+j_3+j_4=0}\, g(\eta) \prod_{a<b} z_{ab}^{-j_a/2-j_b/2}. \end{aligned} \tag{4.60}$$

Again, to satisfy with the RHS of (4.53) and obtain nonzero solutions, we need

$$j_1 + j_2 = 0, \quad j_1 + j_3 = 0, \quad j_1 + j_4 = 0, \quad j_1 + j_2 + j_3 + j_4 = 0. \tag{4.61}$$

By combining the equations, we will have $j_1 = j_2 = j_3 = j_4 = 0$. Same as previous results, all of the ghost primaries in the correlator are vacuums for nonzero solutions, and the solution is a constant.

Therefore, we can conclude that the n-point correlation functions of ghost primaries are zero unless the fields are all vacuums, which is not an exciting result. Hence, we want to twist at least one field by acting with σ^ℓ on the field to have a primary field, and investigate what the solutions can be.

4.2 Fields with Spectral Flow

Now we start to compute the solutions of the n-point correlators involving one primary field $\phi_n^\ell(z)$, while the others remain as ghost primaries $\phi_i(z_i)$, for $i = 1, 2, \ldots, n-1$, written as

$$\left\langle \phi_1(z_1)\phi_2(z_2)\cdots\phi_{n-1}(z_{n-1})\phi_n^\ell(z_n)\right\rangle. \tag{4.62}$$

One thing makes life easier is that the Ward identities are the same for correlators of any primary fields with $\ell \in \mathbb{Z}$, as the ghost primaries are special form of the primaries with $\ell = 0$. We can use the results from Section 4.1 directly and only rederive the KZ equations.

4.2.1 Essential Steps Towards KZ Equations

Techniques from Section 3.1.3 can be used to derive the following results. Recall from (3.26) and (3.27) that the charge and conformal dimension of a primary field are

$$j_{\phi^\ell} = j_\phi - \ell, \quad h_{\phi^\ell} = \ell(j+\ell) - \frac{1}{2}\ell(\ell+1). \tag{4.63}$$

We can then act with L_{-1} on the primary field define in (3.29) to obtain

$$\begin{aligned} L_{-1}\left|\phi^{\ell}\right\rangle &= \sum_{m\in\mathbb{Z}}(-1-m)\beta_m\gamma_{-1-m}\left|\phi^{\ell}\right\rangle \\ &= \ell\beta_{-\ell-1}\gamma_{\ell}\left|\phi^{\ell}\right\rangle + (-1+\ell)\gamma_{\ell-1}\beta_{-\ell}\left|\phi^{\ell}\right\rangle , \end{aligned} \tag{4.64}$$

where remind from (2.17) that $L_{-1}\left|\phi^{\ell}\right\rangle$ corresponds to $\left|\partial\phi^{\ell}\right\rangle$. Then for J_{-1} acting on the primary field, we have

$$\begin{aligned} J_{-1}\left|\phi^{\ell}\right\rangle &= \sum_{m\in\mathbb{Z}}\beta_m\gamma_{-1-m}\left|\phi^{\ell}\right\rangle \\ &= \beta_{-\ell-1}\gamma_{\ell}\left|\phi^{\ell}\right\rangle + \gamma_{\ell-1}\beta_{-\ell}\left|\phi^{\ell}\right\rangle . \end{aligned} \tag{4.65}$$

This will be applied in our later derivations for the KZ equation.

Consider back to our large contour in (4.54), we need to check when it does not contribute. We are now choosing a mode A_m of a general field $A(z)$, that is acting on a primary field $\phi^{\ell}_n(z_n)$, the Cauchy integral theorem then gives

$$\left\langle\Omega, \phi_1(z_1)\phi_2(z_2)\cdots A_m\phi^{\ell}_n(z_n)\Omega\right\rangle = \oint_{z_n}\left\langle\Omega, \phi_1(z_1)\phi_2(z_2)\cdots A(z)\phi^{\ell}_n(z_n)\Omega\right\rangle(z-z_n)^{m+h_A-1}\frac{dz}{2\pi\mathfrak{i}}, \tag{4.66}$$

where h_A is the conformal weight of $A(z)$. We can then replace the contour as follows:

$$\oint_{z_n} = \oint_{\infty} - \sum_{i=1}^{n-1}\oint_{z_i} . \tag{4.67}$$

To check if the infinity contour contributes, we expand $A(z)$ as $A(z) = \sum_k A_k z^{-k-h_A}$ and set

$w = \frac{1}{z}$ to evaluate the contour around 0

$$
\begin{aligned}
&\oint_{\infty} \langle \Omega, A(z)\phi_1(z_1)\phi_2(z_2)\cdots\phi_n^{\ell}(z_n)\Omega\rangle (z-z_n)^{m+h_A-1}\frac{dz}{2\pi\mathfrak{i}} \\
&= \oint_{\infty} \sum_k \langle \Omega, A_k z^{-k-h_A}\phi_1(z_1)\phi_2(z_2)\cdots\phi_n^{\ell}(z_n)\Omega\rangle (z-z_n)^{m+h_A-1}\frac{dz}{2\pi\mathfrak{i}} \\
&= \oint_{\infty} \sum_k \left\langle A_k^{\dagger}\Omega, \phi_1(z_1)\phi_2(z_2)\cdots\phi_n^{\ell}(z_n)\Omega\right\rangle (z-z_n)^{m+h_A-1} z^{-k-h_A}\frac{dz}{2\pi\mathfrak{i}} \\
&= \oint_{\infty} \sum_{-k\le -h_{A^\dagger}} \langle (A^{\dagger})_{-k}\Omega, \phi_1(z_1)\phi_2(z_2)\cdots\phi_n^{\ell}(z_n)\Omega\rangle z^{-k-h_A}(z-z_n)^{m+h_A-1}\frac{dz}{2\pi\mathfrak{i}} \\
&= \oint_{0} \sum_{-k\le -h_{A^\dagger}} \langle (A^{\dagger})_{-k}\Omega, \phi_1(z_1)\phi_2(z_2)\cdots\phi_n^{\ell}(z_n)\rangle w^{k+h_A-2}(\frac{1}{w}-z_n)^{m+h_A-1}\frac{dw}{2\pi\mathfrak{i}} \\
&= \oint_{0} \sum_{k\ge h_{A^\dagger}} \langle (A^{\dagger})_{-k}\Omega, \phi_1(z_1)\phi_2(z_2)\cdots\phi_n^{\ell}(z_n)\rangle (1-z_n w)^{m+h_A-1} w^{k-m-1}\frac{dw}{2\pi\mathfrak{i}}. \qquad (4.68)
\end{aligned}
$$

In the ghost system, A_k can be γ_k, β_k or J_k, which annihilates the vacuum for any $k < -h_A$. Thus the nonzero terms remaining for $(A^{\dagger})_{-k}$ are $-k \le -h_{A^\dagger}$, explaining the second last step. We can also remark that changing the integral from infinity to 0 flips the contour from anticlockwise to clockwise. The last step is then positive. After adjusting the expression, we obtain the power of w to be $k-m-1$. As we do not want the integral to contribute, we need the term to be regular at $w=0$, which means $k-m-1 \ge 0$ is required, and from the summation that $k \ge h_{A^\dagger}$ also needs to be satisfied.

Therefore, a constraint for m can be obtained by substituting k in:

$$
\begin{aligned}
&m \le k-1 \ \forall k \ge h_{A^\dagger} \\
\Rightarrow\ &m \le h_{A^\dagger}-1, \qquad (4.69)
\end{aligned}
$$

by taking the minimum of k. We can say that for these values of m, (4.68) vanishes.

Now let A_m be γ_m or β_m to see which modes for each operator are allowed to use such trick. When $A_m = \beta_m$:

$$A_m^{\dagger} = \beta_m^{\dagger} = \gamma_{-m} \ \Rightarrow\ m \le -1. \qquad (4.70)$$

As $h_{A^\dagger} = 0$ being γ has conformal weight 0. When $A_m = \gamma_m$:

$$A_m^{\dagger} = \gamma_m^{\dagger} = \beta_{-m} \ \Rightarrow\ m \le 0. \qquad (4.71)$$

As $h_{A^\dagger} = 1$ being β has conformal weight 1.

Hence, the contour around infinity does not contribute except when mode index is negative for β and non-negative for γ, when the mode acts on a primary field. This has to be checked every time we apply such trick.

4.2.2 Knizhnik-Zamolodchikov Equations

Now we can derive a general form of the KZ equation for the n-point correlator of primary fields in (4.62). There are two methods to be used in deriving the KZ equation. Method 1 is the same as in Section 4.1.5, with L_{-1} acting on $\phi_1(z_1)$, so we have the KZ equation written as

$$\left\langle \partial\phi_1(z_1)\phi_2(z_2)\cdots\phi_n^\ell\right\rangle = -\left\langle \gamma_{-1}(\beta_0\phi_1)\phi_2\cdots\phi_n^\ell(z_n)\right\rangle . \tag{4.72}$$

Then we compute the RHS again using the trick from (4.54)

$$\begin{aligned} RHS &= -\oint_{z_1}\left\langle \frac{\gamma(z)(\beta_0\phi_1)}{(z-z_1)^2}\phi_2\cdots\phi_n^\ell\right\rangle dz \\ &= \sum_{i=2}^{n}\oint_{z_i}\left\langle \frac{(\beta_0\phi_1)}{(z-z_1)^2}\phi_2\cdots\gamma(z)\phi_i\cdots\phi_n^\ell\right\rangle dz \\ &= \oint_{z_n}\frac{1}{(z-z_1)^2}\left\langle (\beta_0\phi_1)\phi_2\cdots\phi_{n-1}\left[\frac{\gamma_\ell\phi_n^\ell}{(z-z_n)^\ell}+\cdots+\frac{\gamma_1\phi_n^\ell}{z-z_n}\right]\right\rangle . \end{aligned} \tag{4.73}$$

As γ is with mode -1 and based on (4.71), the contour around infinity does not contribute. Remind from (3.37) that $\gamma(z)\phi_i(z)$ are regular for $i = 2, 3, \ldots, n-1$, so the only term that contributes is when $\gamma(z)$ acts on ϕ_n^ℓ. Then Fourier expand $\gamma(z)$ and apply the definition of (3.29) that γ_n annihilates ϕ_n^ℓ for $n > \ell$. Hence we obtain the expression of the KZ in method 1.

Additionally, recall from (3.37) that $\gamma(z)\phi^\ell(w)$ is regular for $\ell \le 0$, so that RHS of the KZ equation gives zero as well for $\ell < 0$, and the correlators will give constant solutions for the non contributing infinite contour. We will then focus on $\ell > 0$ when we solve for the 2-point and 3-point correlators.

Method 2 is to apply L_{-1} on the field with spectral flow, which is always the last field $\phi_n^\ell(z_n)$ for an n-point correlator here. Hence the KZ equation becomes

$$\left\langle \phi_1(z_1)\phi_2(z_2)\cdots\partial\phi_n^\ell(z_n)\right\rangle = \left\langle \phi_1(z_1)\phi_2(z_2)\cdots(\ell\beta_{-\ell-1}\gamma_\ell+(-1+\ell)\gamma_{\ell-1}\beta_{-\ell})\,\phi_n^\ell(z_n)\right\rangle , \tag{4.74}$$

where we have applied (4.64) for the RHS. Then the RHS can be further written as

$$RHS = \oint_{z_n}\left(\frac{\ell}{(z-z_n)^{\ell+1}}\left\langle \phi_1\phi_2\cdots\beta(z)(\gamma_\ell\phi_n^\ell)\right\rangle + \frac{(-1+\ell)}{(z-z_n)^{-\ell+2}}\left\langle \phi_1\phi_2\cdots\gamma(z)(\beta_{-\ell}\phi_n^\ell)\right\rangle\right)\frac{dz}{2\pi\mathrm{i}}. \tag{4.75}$$

However, $\gamma_{\ell-1}$ will have a positive index when $\ell > 1$, meaning that the infinity contour can not be cancelled based on our result in (4.71). As we want this method to be workable for all $\ell > 0$, we will modify the KZ equation in (4.74). By observing (4.65), the expression

$$L_{-1} - (\ell - 1)J_{-1}, \tag{4.76}$$

happens to cancel out the term that contains $\gamma_{-\ell-1}$ in (4.64). We can then obtain a modified method 2 with the KZ equation, with the RHS being

$$\begin{aligned} RHS &= \left\langle \phi_1 \cdots \phi_{n-1} L_{-1} \phi_n^\ell \right\rangle - (\ell - 1) \left\langle \phi_1 \cdots \phi_{n-1} J_{-1} \phi_n^\ell \right\rangle \\ &= \left\langle \phi_1 \cdots \phi_{n-1} (\ell \beta_{-\ell-1} \gamma_\ell + (-1 + \ell) \gamma_{\ell-1} \beta_{-\ell} - (\ell - 1) \beta_{-\ell-1} \gamma_\ell - (\ell - 1) \beta_{-\ell} \gamma_{\ell-1}) \phi_n^\ell \right\rangle \\ &= \left\langle \phi_1 \cdots \phi_{n-1} \beta_{-\ell-1} (\gamma_\ell \phi_n^\ell) \right\rangle . \end{aligned} \tag{4.77}$$

The KZ equation is then

$$\left\langle \phi_1 \phi_2 \cdots \partial \phi_n^\ell \right\rangle - (\ell - 1) \left\langle \phi_1 \phi_2 \cdots J_{-1} \phi_n^\ell \right\rangle = \left\langle \phi_1 \cdots \phi_{n-1} \beta_{-\ell-1} (\gamma_\ell \phi_n^\ell) \right\rangle . \tag{4.78}$$

However, we do not have the corresponding state for $J_{-1}\phi_2^\ell$. We need to compute the term by using the Cauchy integral theorem

$$\begin{aligned} \left\langle \phi_1 \cdots J_{-1} \phi_n^\ell \right\rangle &= \oint_{z_n} \frac{\left\langle \phi_1 J(z) \phi_n^\ell \right\rangle}{z - z_n} \frac{dz}{2\pi \mathfrak{i}} = -\sum_{i=1}^{n-1} \oint_{z_i} \frac{\left\langle \phi_1 \cdots J(z) \phi_i \cdots \phi_n^\ell \right\rangle}{(z - z_n)} \frac{dz}{2\pi \mathfrak{i}} \\ &= -\sum_{i=1}^{n-1} \oint_{z_i} \frac{j_i \left\langle \phi_1 \cdots \phi_n^\ell \right\rangle}{(z - z_n)(z - z_i)} \frac{dz}{2\pi \mathfrak{i}}, \end{aligned} \tag{4.79}$$

by recalling from (3.17) that $J(z)\phi(w) \sim \frac{j\phi(w)}{z-w}$. Hence, the LHS becomes

$$LHS = \left\langle \phi_1 \phi_2 \cdots \partial \phi_n^\ell \right\rangle + (\ell - 1) \sum_{i=1}^{n-1} \oint_{z_i} \frac{j_i \left\langle \phi_1 \cdots \phi_n^\ell \right\rangle}{(z - z_n)(z - z_i)} \frac{dz}{2\pi \mathfrak{i}}. \tag{4.80}$$

Then apply the trick (4.54) on RHS again, where now the large contour vanishes according to (4.70), as the modes of β is $-\ell - 1$, which is negative for $\ell > 0$:

$$\begin{aligned} RHS &= \oint_{z_n} \left\langle \phi_1 \cdots \beta(z) (\gamma_\ell \phi_n^\ell) \right\rangle (z - z_n)^{-\ell-1} \frac{dz}{2\pi \mathfrak{i}} \\ &= -\sum_{i=1}^{n-1} \oint_{z_i} \left\langle \phi_1 \cdots \beta(z) \phi_i \cdots (\gamma_\ell \phi_n^\ell) \right\rangle (z - z_n)^{-\ell-1} \frac{dz}{2\pi \mathfrak{i}} \\ &= -\sum_{i=1}^{n-1} \oint_{z_i} \left\langle \phi_1 \cdots \beta_0 \phi_i \cdots (\gamma_\ell \phi_n^\ell) \right\rangle (z - z_n)^{-\ell-1} \frac{dz}{2\pi \mathfrak{i}}, \end{aligned} \tag{4.81}$$

where we have used the OPE for $\beta(z)\phi^\ell(w)$ from (3.36).

We will then use method 1 and modified method 2 to solve for the 2-point and 3-point correlators.

4.2.3 2-Point Function

The 2-point correlator with the second field $\phi_2(z_2)$ being spectral flowed is written as

$$\langle \phi_1(z_1)\phi_2^\ell(z_2)\rangle . \tag{4.82}$$

Define $\phi_1(z_1)$ to have the charge and conformal dimension $(j_1, h_1 = 0)$, where the conformal dimension of $\phi_1(z_1)$ is derived from (3.32) for $\ell = 0$. Also, we define $\phi_2^\ell(z_2)$ to have the charge and conformal dimension (j_2, h_2), where h_2 is given in (4.63). Apply method 1, we can first use (4.10) to derive the LHS of the KZ equation

$$LHS = \partial_1 \left[\frac{C_{12}\delta_{j_1+j_2=0}\delta_{h_1-h_2=j_1}}{(z_1-z_2)^{h_1+h_2}}\right] = \frac{-h_2 C_{12}\delta_{j_1+j_2=0}\delta_{-h_2=j_1}}{z_{12}^{h_2+1}}, \tag{4.83}$$

by recalling that $z_{12} = z_1 - z_2$.

Then for the RHS, we have

$$RHS = \oint_{z_n} \frac{1}{(z-z_1)^2}\left\langle (\beta_0\phi_1)\left[\frac{\gamma_\ell\phi_2}{(z-z_n)^\ell} + \cdots + \frac{\gamma_1\phi_2}{z-z_n}\right]\right\rangle . \tag{4.84}$$

As the integral depends on ℓ, we will try with different values of ℓ. Additionally, as we stated in Section 4.2.2, the RHS is zero for $\ell \leq 0$, so we will discuss the cases for $\ell > 0$ only.

First, consider $\ell = 1$, and apply (4.63) to obtain $h_2 = j_2$. Then we can express h_2 in terms of j_2 and (4.83) becomes

$$LHS = \frac{-j_2 C_{12}\delta_{j_1+j_2=0}\delta_{-h_2=j_1}}{z_{12}^{j_2+1}}. \tag{4.85}$$

Then RHS of the KZ equation is

$$\begin{aligned} RHS &= \oint_{z_2} \frac{1}{(z-z_1)^2(z-z_2)}\left\langle (\beta_0\phi_1)(\gamma_1\phi_2^1)\right\rangle \frac{dz}{2\pi\mathfrak{i}} \\ &= \frac{1}{(z_2-z_1)^2}\left\langle (\beta_0\phi_1)(\gamma_1\phi_2^1)\right\rangle . \end{aligned} \tag{4.86}$$

Now, we have two new primary fields in RHS, $(\beta_0\phi_1)(z_1)$ and $(\gamma_1\phi_2^1)(z_2)$, and denote the charges and conformal dimensions by $(j_1', 0), (j_2', h_2')$ respectively. To find the charges for these fields, we can use the fact in Section 3.1.1 that $\beta(z)$ has a charge of 1, and $\gamma(z)$ has the charge -1, so $j_1' = j_1 + 1$, $j_2' = j_2 - 1$. Additionally, the conformal dimension of $(\gamma_1\phi_2^1)(z_2)$ becomes $h_2' = j_2' = j_2 - 1$. Then we obtain the charges and conformal dimensions of the two fields as being $(j_1+1,0)$ and (j_2-1,j_2-1). Hence, the RHS follows by substituting this data into (3.50)

$$\begin{aligned} RHS &= \frac{1}{(z_1-z_2)^2}\frac{C'_{12}\delta_{j'_1+j'_2=0}}{(z_1-z_2)^{j_2-1}} \\ &= \frac{C'_{12}\delta_{j_1+j_2=0}}{(z_{12})^{j_2+1}}, \end{aligned} \tag{4.87}$$

where C'_{12} is another constant that is different from C_{12} in (4.85). Lastly, equating LHS and RHS gives us a KZ equation for the 2-point function:

$$\frac{-(j_2-1)C_{12}}{z_{12}^{j_2+1}} = \frac{C'_{12}}{(z_{12})^{j_2+1}}. \tag{4.88}$$

Therefore, the 2-point correlation function with $\ell=1$ has only the zero solution unless $C'_{12} = -(j_2-1)C_{12}$, so the 2-point correlator function $\langle\phi_1\phi_2^1\rangle$ need not to be constants.

Next, we can try to apply method 1 to compute the KZ equation for $\ell=2$. From (4.63) we have $h_2 = 2j_2+1$ for $\ell=2$, and method 1 gives:

$$\langle\partial\phi_1\phi_2^2\rangle = \oint_{z_2}\frac{1}{(z-z_1)^2}\left\langle(\beta_0\phi_1)\left(\frac{\gamma_2\phi_2^2}{(z-z_2)^2}+\frac{\gamma_1\phi_2^2}{z-z_2}\right)\right\rangle\frac{dz}{2\pi\mathfrak{i}}. \tag{4.89}$$

Notice that the second term contains a non-primary field with $(\gamma_1\phi_2^2)(z_1)$, the infinite integral is then non-zero as γ has index 1, and we do not have the technique to solve it for now.

Alternatively, we can try to apply the modified method 2 in Section 4.2.2, starting by checking if $\ell=1$ agrees with the result from method 1. From (4.80), the second term of the LHS of the KZ equation vanishes, so we have

$$LHS = \partial_2\left[\frac{C_{12}\delta_{j_1+j_2=0}\delta_{-h_2=j_1}}{(z_1-z_2)^{h_2}}\right] = \frac{j_2C_{12}\delta_{j_1+j_2=0}}{z_{12}^{j_2+1}}. \tag{4.90}$$

From what we derived in (4.78),

$$RHS = -\oint_{z_1}\beta(z)\frac{\langle\phi_1(\gamma_1\phi_2^1)\rangle}{(z-z_2)^2}\frac{dz}{2\pi\mathfrak{i}} = -\frac{1}{(z_1-z_2)^2}\left\langle(\beta_0\phi_1)(\gamma_1\phi_2^1)\right\rangle = -\frac{C'_{12}\delta_{j_1+j_2=0}}{z_{12}^{j_2+1}}. \tag{4.91}$$

We can see that power of z_{12} in the RHS in the modified method 2 is the same as in method 1. Equating LHS and RHS gives us

$$\frac{j_2C_{12}\delta_{j_1+j_2=0}}{z_{12}^{j_2+1}} = -\frac{C'_{12}\delta_{j_1+j_2=0}}{z_{12}^{j_2+1}}, \tag{4.92}$$

which agrees with (4.88) for having nonzero solutions when $C'_{12} = j_2C_{12}$ is satisfied. Hence, both methods give us the same result, and then we can use the modified method 2 to compute for $\ell = 2$. From (4.80) we have

$$
\begin{aligned}
LHS &= \partial_2 \left\langle \phi_1 \phi_2^2 \right\rangle + (2-1) \oint_{z_1} \frac{j_1 \left\langle \phi_1 \phi_2^2 \right\rangle}{(z-z_2)(z-z_1)} \frac{dz}{2\pi \mathfrak{i}} \\
&= \frac{-h_2 C_{12} \delta_{j_1+j_2=0}}{z_{12}^{h_2+1}} + \frac{j_1 C_{12} \delta_{j_1+j_2=0}}{z_{12}^{h_2+1}}.
\end{aligned} \tag{4.93}
$$

Then use (4.81) so that

$$
RHS = -\oint_{z_1} \frac{\left\langle (\beta_0 \phi_1)(\gamma_\ell \phi_3^\ell) \right\rangle}{(z-z_2)^{\ell+1}(z-z_1)} \frac{dz}{2\pi \mathfrak{i}} = \frac{-C'_{12} \delta_{j_1+j_2=0}}{z_{12}^{\ell+h'_2+1}}, \tag{4.94}
$$

by substituting (4.10) with conformal charges and weights $(j'_1 = j_1+1, h_1 = 0), (j'_2 = j_2-1, h'_2)$. As we have $h_2 = 2j_2 + 1$ from (4.63), and the LHS and RHS of the KZ equation are

$$
LHS = \frac{(-2j_2 - 1 + j_1) C_{12} \delta_{j_1+j_2=0}}{z_{12}^{2j_2+2}}, \quad RHS = \frac{-C'_{12} \delta_{j_1+j_2=0}}{z_{12}^{2j_2+4}}. \tag{4.95}
$$

Equating LHS and RHS does not give a non-zero solution as the powers of z_{12} are different. The only possibility is when $C_{12} = C_{12'} = 0$. With the same idea, the powers of z_{12} do not match in the KZ equation for $\ell > 2$, all of them give solutions as zeros. As a matter of fact, if we combine the deltas $\delta_{j_1+j_2=0}$ and $\delta_{h_1-h_2=j_1}$ in the result from the Ward identities (4.10), we can find out that $h_2 = j_2$ must be true for LHS of the KZ equation (4.83) to be non-zero, given that $h_1 = 0$. Hence, the only 2-point correlator that has non trivial solutions is when $\ell = 1$.

4.2.4 3-Point Function

Based on our previous experience, we can use method 1 for the $\ell = 1$ case to find the solutions for 3-point correlator $\left\langle \phi_1(z_1)\phi_2(z_2)\phi_3^\ell(z_3) \right\rangle$ as well, and the modified method 2 to check for $\ell = 1$ and derive for $\ell > 1$.

First with method 1, consider for $\ell = 1$, we have $h_2 = j_2$, apply (4.30) and the LHS of the KZ equation is

$$
LHS = \partial_1 \left[\frac{C_{123} \delta_{j_1+j_2+j_3=0}}{z_{23}^{j_3+j_1} z_{13}^{j_3+j_2}} \right] = \frac{-(j_3+j_2) C_{123} \delta_{j_1+j_2+j_3=0}}{z_{23}^{j_3+j_1} z_{13}^{j_3+j_2+1}}. \tag{4.96}
$$

Then, we compute the RHS as being

$$\begin{aligned}RHS &= \left\langle \partial\phi_1(z_1)\phi_2(z_2)\phi_3^1(z_3)\right\rangle = \oint_{z_3} \frac{1}{(z-z_1)^2(z-z_3)} \left\langle (\beta_0\phi_1)\phi_2(\gamma_1\phi_3^1)\right\rangle \\ &= \frac{1}{z_{13}^2}\left\langle (\beta_0\phi_1)\phi_2(\gamma_1\phi_3^1)\right\rangle = \frac{1}{z_{13}^2}\frac{C'_{123}\delta_{j_1+j_2+j_3=0}}{z_{23}^{j_3+j_1}\, z_{13}^{j_3+j_2-1}},\end{aligned} \tag{4.97}$$

with charges and conformal dimensions of the three primary fields $(\beta_0\phi_1)(z_1), \phi_2(z_2), (\gamma_1\phi_3^1)(z_3)$ being $(j_1' = j_1+1, 0), (j_2, 0), (j_3' = j_3-1, h_3' = j_3-1)$ respectively. Equating the RHS and LHS to find the relations between the constants:

$$\frac{-(j_3+j_2)C_{123}\delta_{j_1+j_2+j_3=0}}{z_{23}^{j_3+j_1}\, z_{13}^{j_3+j_2+1}} = \frac{C'_{123}\delta_{j_1+j_2+j_3=0}}{z_{23}^{j_3+j_1}\, z_{13}^{j_3+j_2+1}}. \tag{4.98}$$

The powers of the variables are the same, so the correlator has non-zero solutions when $C'_{123} = -(j_3+j_2)C_{123}$.

After that, we will apply modified method 2 as shown in (4.78) and compute the LHS and RHS of the KZ equation for $\ell > 0$. The LHS can be computed by using (4.80) again

$$\begin{aligned}LHS &= \left\langle \phi_1\phi_2\partial\phi_3^\ell\right\rangle + (\ell-1)\left(\oint_{z_1}\frac{j_1\left\langle\phi_1\phi_2\phi_3^\ell\right\rangle}{(z-z_3)(z-z_1)}\frac{dz}{2\pi \mathrm{i}} - \oint_{z_2}\frac{j_2\left\langle\phi_1\phi_2\phi_3^\ell\right\rangle}{(z-z_3)(z-z_2)}\frac{dz}{2\pi \mathrm{i}}\right) \\ &= \left\langle \phi_1\phi_2\partial\phi_3^\ell\right\rangle + \frac{j_1\left\langle\phi_1\phi_2\phi_3^\ell\right\rangle}{z_{13}} + \frac{j_2\left\langle\phi_1\phi_2\phi_3^\ell\right\rangle}{z_{23}}.\end{aligned} \tag{4.99}$$

Hence, we can obtain the LHS by substituting the result derived from the Ward identities (4.30):

$$LHS = \left(\frac{h_3+j_2+(\ell-1)j_1}{z_{13}} + \frac{h_3+j_1+(\ell-1)j_2}{z_{23}}\right)\frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{-h_3+j_3}\, z_{13}^{h_3+j_2}\, z_{23}^{h_3+j_1}}, \tag{4.100}$$

with conformal charges and dimensions of ϕ_1, ϕ_2, ϕ_3 being $(j_1, 0), (j_2, 0), (j_3, h_3)$ respectively. While the RHS is obtained from (4.81) such that

$$\begin{aligned}RHS &= -\oint_{z_1}\frac{\left\langle\beta_0\phi_1\phi_2(\gamma_\ell\phi_3^\ell)\right\rangle}{(z-z_3)^{\ell+1}(z-z_1)}\frac{dz}{2\pi \mathrm{i}} - \oint_{z_2}\frac{\left\langle\phi_1(\beta_0\phi_2)(\gamma_\ell\phi_3^\ell)\right\rangle}{(z_2-z_3)^{\ell+1}(z-z_2)}\frac{dz}{2\pi \mathrm{i}} \\ &= \frac{-C'_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{-h_3'+j_3-1}\, z_{13}^{h_3'+j_2+\ell+1}\, z_{23}^{h_3'+j_1+1}} - \frac{C''_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{-h_3'+j_3-1}\, z_{13}^{h_3'+j_2+1}\, z_{23}^{h_3'+j_1+\ell+1}},\end{aligned} \tag{4.101}$$

by applying (4.30) again with conformal charges and dimensions $(j_1+1, 0), (j_2, 0), (j_3-1, h_3')$ respectively for the first term and $(j_1, 0), (j_2+1, 0), (j_3-1, h_3')$ respectively for the second term. According to our previous result (4.63), which is $h_3 = \ell(j_3+\ell) - \frac{1}{2}\ell(\ell+1)$, we can further express h_3' in terms of h_3 as $h_3' = \ell(j_3-1+\ell) - \frac{1}{2}\ell(\ell+1) = h_3 - \ell$. Then the RHS can be written in terms of h_3:

$$RHS = \frac{-C'_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{-h_3+j_3-1+\ell}z_{13}^{h_3+j_2+1}z_{23}^{h_3+j_1+1-\ell}} - \frac{C''_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{-h_3+j_3-1+\ell}z_{13}^{h_3+j_2+1-\ell}z_{23}^{h_3+j_1+1}}$$
$$= \frac{\delta_{j_1+j_2+j_3=0}}{z_{12}^{-h_3+j_3}z_{13}^{h_3+j_2}z_{23}^{h_3+j_1}}\left(\frac{-C'_{123}}{z_{12}^{-1+\ell}z_{13}z_{23}^{1-\ell}} + \frac{-C''_{123}}{z_{12}^{-1+\ell}z_{13}^{1-\ell}z_{23}}\right). \tag{4.102}$$

To check if this method works in the 3-point correlator, we can check again with $\ell = 1$ by substituting $h_3 = j_3$ in the equations,

$$LHS = \left(\frac{j_3 + j_2}{z_{13}} + \frac{j_3 + j_1}{z_{23}}\right)\frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{0}z_{13}^{j_3+j_2}z_{23}^{j_3+j_1}}$$
$$RHS = \frac{\delta_{j_1+j_2+j_3=0}}{z_{12}^{-j_3+j_3}z_{13}^{j_3+j_2}z_{23}^{h_3+j_1}}\left(\frac{-C'_{123}}{z_{12}^{0}z_{13}z_{23}^{0}} + \frac{-C''_{123}}{z_{12}^{0}z_{13}^{0}z_{23}}\right). \tag{4.103}$$

When equating LHS and RHS, we obtain the two relations between the constants:

$$C'_{123} = -C_{123}(j_3 + j_2) = C_{123}j_1$$
$$C''_{123} = -C_{123}(j_3 + j_1) = C_{123}j_2. \tag{4.104}$$

Hence, the 3-point correlator may have non-zero solutions for $\ell = 1$, which agrees with method 1.

Now can work on the cases of larger ℓ, for instance when $\ell = 2$, by substituting $h_3 = 2j_3+1$ into LHS and RHS to obtain

$$LHS = \left(\frac{2j_3 - 1 + j_2 + j_1}{z_{13}} + \frac{2j_3 - 1 + j_1 + j_2}{z_{23}}\right)\frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{-j_3-1}z_{13}^{2j_3+j_2+1}z_{23}^{2j_3+j_1+1}}$$
$$= \left(\frac{j_3 - 1}{z_{13}} + \frac{j_3 - 1}{z_{23}}\right)\frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{-j_3-1}z_{13}^{2j_3+j_2+1}z_{23}^{2j_3+j_1+1}}$$
$$RHS = -\frac{\delta_{j_1+j_2+j_3=0}}{z_{12}^{-j_3-1}z_{13}^{2j_3+j_2+1}z_{23}^{2j_3+j_1+1}}\left(C'_{123}\frac{z_{23}}{z_{13}z_{12}} + C''_{123}\frac{z_{13}}{z_{23}z_{12}}\right)$$
$$= -\frac{\delta_{j_1+j_2+j_3=0}}{z_{12}^{-j_3-1}z_{13}^{2j_3+j_2+1}z_{23}^{2j_3+j_1+1}}\left(C'_{123}\frac{z_{12} - z_{13}}{z_{13}z_{12}} + C''_{123}\frac{z_{12} + z_{23}}{z_{23}z_{12}}\right)$$
$$= -\frac{\delta_{j_1+j_2+j_3=0}}{z_{12}^{-j_3-1}z_{13}^{2j_3+j_2+1}z_{23}^{2j_3+j_1+1}}\left(C'_{123}\left(\frac{1}{z_{13}} - \frac{1}{z_{12}}\right) + C''_{123}\left(\frac{1}{z_{23}} + \frac{1}{z_{12}}\right)\right). \tag{4.105}$$

Matching the common powers will give us the relations between the constants as

$$C'_{123} = -C_{123}(j_3 - 1)$$
$$C''_{123} = C_{123}(j_3 - 1)$$
$$C'_{123} + C''_{123} = 0. \tag{4.106}$$

Therefore, a 3-point correlator with $\ell = 2$ may have non-zero solutions if the constants follow the above constraints.

Furthermore, we can consider the case for $\ell = 3$, where $h_3 = 3j_3$. Then LHS is written as

$$LHS = (\frac{3j_3 + j_2 + 2j_1}{z_{13}} + \frac{3j_3 + j_1 + 2j_2}{z_{23}})\frac{C_{123}\delta_{j_1+j_2+j_3=0}}{z_{12}^{-2j_3} z_{13}^{3j_3+j_2} z_{23}^{3j_3+j_1}}, \tag{4.107}$$

$$\begin{aligned} RHS &= \frac{\delta_{j_1+j_2+j_3=0}}{z_{12}^{-2j_3} z_{13}^{3j_3+j_2} z_{23}^{3j_3+j_1}}(\frac{-C'_{123}}{z_{12}^2 z_{13} z_{23}^{-2}} + \frac{-C''_{123}}{z_{12}^2 z_{13}^{-2} z_{23}}) \\ &= \frac{\delta_{j_1+j_2+j_3=0}}{z_{12}^{-h_3+j_3} z_{13}^{h_3+j_2} z_{23}^{h_3+j_1}}[-C'_{123}(\frac{z_{13}}{z_{12}^2} - \frac{2}{z_{12}} + \frac{z_{12}^2}{z_{13}}) - C''_{123}(\frac{z_{23}}{z_{12}^2} + \frac{2}{z_{12}} + \frac{z_{12}^2}{z_{23}})]. \end{aligned} \tag{4.108}$$

The powers of the terms on RHS do not match with LHS, which means the only solution is when $C'_{123} = C''_{123} = 0$. The same idea applies for larger ℓ, that means for $\ell > 2$ in 3-point function, there are no terms in RHS that can be expressed in terms of $\frac{1}{z_{13}}$ and $\frac{1}{z_{23}}$, thus all of them would give zero solutions.

To conclude, we can make connections with the fusion rules derived in [RW14]. Following from Corollary 10 in [RW14], we can decompose primary fields and descendants of primary fields, defined in Section 3.2, into a sum of a generic OPE. Ignoring the descendants, this gives the fusion rule

$$\phi_1 \times \phi_2^\ell = \phi_3^\ell + \phi_3^{\ell-1}, \tag{4.109}$$

where $j_3 = -j_1 - j_2$. Hence here, an OPE for the 3-point correlator can be obtained as being

$$\left\langle \phi_{j_1}\phi_{j_2}\phi_{j_3}^\ell \right\rangle \sim \left\langle \phi_{j_1}\phi_{j_2+j_3}^\ell \right\rangle + \left\langle \phi_{j_1}\phi_{j_2+j_3}^{\ell-1} \right\rangle. \tag{4.110}$$

Recall from Section 4.1.2 that $\langle \phi_{j_1}\phi_{j_2} \rangle$ only has non-zero solutions for $\ell = 1$. Substituting into (4.110) gives us that the 3-point correlator is non-zero when $\ell = 1$, where the first term does not vanish, or $\ell = 2$, where the second term does not vanish. Therefore, the results we derived in this chapter agree with the fusion rules.

Chapter 5

Conclusion

In this thesis, we started by introducing some fundamental information in Chapter 2 about general conformal field theory, including Lie algebras, representations and conformal transformations. Then we investigated how modes act on a state, which corresponds to a field. Primary fields are also introduced with an example of the energy-momentum tensor.

After gaining the necessary knowledge, we established the ghost CFT and discussed the ghost algebras in Chapter 3. With the spectral flow automorphisms introduced and the primary fields defined, we were able to generate modules, including a Verma module, where the only highest weight state was the vacuum, relaxed Verma modules, where the states correspond to ghost primaries, and spectral flowed relaxed Verma modules, with states corresponding to primary fields. We were interested in the last two modules. In addition, we found out that the usual Ward identities do not apply in ghost CFT because $L_n^\dagger \neq L_{-n}$ in general, so that we derived new constraints for the correlators, which were checked to be applicable.

In Chapter 4, we first tried to solve for the correlators with ghost primaries. After substituting the new Ward identities and solving for the differential equations, we were able to obtain some general solutions containing unknown constants for 1-point, 2-point and 3-point functions, while an unknown constant function for 4-point function. However, a version of the Knizhnik-Zamolodchikov (KZ) equation told us the solutions are all constants, and are only non-zero when the fields are vacuums. This forced us to twist one of the field by acting with the spectral flow σ^ℓ and rederive the correlations. Luckily, the solutions from the Ward identities are general, which means they are true for correlators of primary fields. After applying for the KZ equation, we discovered that the 2-point correlator is non-zero only when the module is twisted by 1, i.e. $\ell = 1$. The 3-point correlator is non-zero for $\ell = 1$ and $\ell = 2$. Then we concluded the thesis by confirming that these results agree with the fusion rules in [RW14]

Further research can be investigating the 4-point correlator $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4^\ell(z_4) \rangle$ in (4.49). By applying the KZ equation, the unknown function in the correlator is expected to be solved. In fact, [RW14] shows that 4-point correlator can give a logarithmic solution, which results in the ghost CFT being a logarithmic CFT. In addition, the solution can be substituted back into the 3-point correlator to deduce the values of the constant C_{123}.

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