Coset construction for the $N = 2$ and $\mathfrak{osp}(1|2)$ minimal models

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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Abstract

The thesis presents the study of the $N = 2$ and $\mathfrak{osp}(1|2)$ minimal models at admissible levels using the method of coset constructions. These sophisticated minimal models are rich in mathematical structure and come with various interesting features for us to investigate. First, some general principles of conformal field theory are reviewed, notations used throughout the thesis are established. The ideas are then illustrated with three examples of bosonic conformal field theories, namely, the free boson, the Virasoro minimal models, and the admissible-level Wess-Zumino-Witten models of affine $\mathfrak{sl}_2$. The concept of supersymmetry is then introduced, and examples of fermionic conformal field theories are discussed.

Of the two minimal models of interest, the $N = 2$ minimal model, tensored with a free boson, can be extended into an $\mathfrak{sl}_2$ minimal model tensored with a pair of fermionic ghosts, whereas an $\mathfrak{osp}(1|2)$ minimal model is an extension of the tensor product of certain Virasoro and $\mathfrak{sl}_2$ minimal models. We can therefore induce the known structures of the representations of the coset components and get a rather complete picture for the minimal models we want to investigate. In particular, the irreducible highest-weight modules (including the relaxed highest-weight modules, which result in a continuous spectrum) are classified, their characters and Grothendieck fusion rules are computed. The genuine fusion products and the projective covers of the irreducibles are conjectured.

The thesis concludes with a vision of how this method can be used for the study of other affine superalgebras. This provides a promising approach to solving superconformal field theories that are currently little known in the literature.

Contents

Chapter 1

Introduction

1.1 Background and Motivation

Other than the $\mathfrak{bc}$-ghost system, which falls into a special class of algebras known as the *Clifford algebras*, there are two general classes of conformal field theories we shall discuss in this thesis: the minimal models of Virasoro (super)algebras whose (super)conformal symmetry is defined by a (super) energy-momentum tensor and the Wess-Zumino-Witten models with an underlying $\widehat{\mathfrak{g}}_{\mathsf{k}}$ affine symmetry and a central charge

$$\mathsf{c} = \frac{\mathsf{k}\dim(\mathfrak{g})}{\mathsf{k}+g}, \tag{1.1.1}$$

where $\dim(\mathfrak{g})$ and g are respectively the dimension of the finite dimensional Lie algebra and its *dual Coxeter number*.

Superconformal field theories (SCFTs) in two dimensions have a long history in mathematical physics. The superalgebras are extended from the Virasoro algebra by the number N of fermionic partners of the energy-momentum tensor. The minimal models associated with such theories have caught much attention over the past years because their module structures are relevant to the development of many areas, such as string theory. As an integral part of the construction of the fermionic string theory, superconformal symmetry introduces spin operators, which are used for building space-time fermion vertices and the supersymmetry charge of the fermionic string [1]. Besides string theory, the logarithmic minimal models associated with SCFTs have been used for the modelling of statistical lattice models, such as the critical fused lattice model, in the continuum scaling limit. These theories are able to describe special critical points at which the macroscopic physics is supersymmetric.

A simple example of a superconformal algebra, after the Virasoro algebra itself, is the $N = 1$ superconformal algebra, which has been extensively studied in the literature. The structure of the theory in the Neveu-Schwarz sector was quickly settled as a consequence of its debut in superstring theory and statistical mechanics. The more intricate Ramond sector of the theory is also well-understood through studies such as in [2]. Motivated by lattice conjectures, the fusion rules for the rational minimal models in both sectors were discussed in [3, 4] and those of the logarithmic minimal models were reported in [5, 6]. A major application of the $N = 2$ theory is in the construction of string models corresponding to Calabi-Yau manifolds, which are seen to be exact vacua of string theory, retaining their topological and geometrical characteristics. String models based on $N = 2$ supersymmetry was discussed in [7], their relation with Calabi-Yau manifolds were shown in [8, 9]. Despite its rich and fascinating mathematical structures [10], the $N = 3$ theory has received less attention compared to its superconformal cousins in the physics world because no consistent superstring theories have so far been constructed with the $N = 3$ supersymmetry. However, this super-symmetry has been applied in the study of D3-branes in non-gravitational four-dimensional conformal field theories [11]. The interest in the $N = 4$ superconformal algebra comes from the fact that they possess non-trivial field realizations, namely non-linear σ-models taking values on hyper Kähler manifolds [12]. These theories fall into one of the unitary representations of the $N = 4$ theory. With index theory, the $N = 4$ superconformal algebra creates an interesting link between differential geometry and topological algebra [13]. Other applications of the $N = 4$ superconformal algebra include its use in constructing an $\mathfrak{su}(2)$ spinning string [14]. There are, of course, SCFTs with their numbers of supersymmetries larger than four, whose super energy-momentum tensors have negative conformal dimensions. Such theories, as well as the $N = 3, 4$ theories are beyond the scope of this thesis.

One of the goals of this thesis to describe in detail the minimal models associated with the $N = 2$ superconformal algebra. The $N = 2$ algebra first appeared as a gauge algebra in the study of fermionic strings and colour confinement [14]. After this debut in superstring theory, the structure theory of the algebra in the unitary case was settled [15, 16]. A determinant formula was conjectured [17] for the Neveu-Schwarz, the Ramond and the twisted sectors. The $N = 2$ modular invariance properties were studied in [18]. In the following years, some very interesting works [19–21] attempted to elucidate the module structure of the theory, including the construction of highest-weight modules of the unitary $N = 2$ theories in terms of conformal theory with $\mathbb{Z}_n$-symmetry and a free scalar field by Qiu [22]. A recent interest in the non-unitary case [23] introduced a framework including all weight modules and studied some of the characters in the context of vertex algebras. The more intricate non-unitary aspects of the $N = 2$ theory. It is therefore desirable to provide an exposition for the non-unitary minimal models of the $N = 2$ theory. An expediting tool, the coset construction, is expected to be crucial to this endeavour.

As mentioned at the start, the symmetry of the second group of conformal field theories, the Wess-Zumino-Witten models, is provided by an affine Lie algebra. Following the standard treatment of extending a finite Lie algebra $\mathfrak{g}$ to its affine version $\widehat{\mathfrak{g}}_{\mathsf{k}}$, we tensor it with a set of Laurent polynomials and adjoin it the level operator $\hat{k}$. The corresponding Wess-Zumino-Witten models are endowed with conformal symmetry through *the Sugawara construction*, which is the construction of the energy-momentum tensor $T(z)$ in terms of the currents of the affine algebra. The central charge associated with this construction is found to be given by (1.1.1).

A conformal field theory is said to be *rational* if it has a finite number of primary fields and its state space $\mathcal{H}$ can be written in

$$\mathcal{H} = \bigoplus_i \mathsf{M}_i \left(\mathcal{V}_i \otimes \overline{\mathcal{V}}_i\right), \tag{1.1.2}$$

where M_i are coefficients of non-negative integers and the $\mathcal{V}_i$ are the irreducibles. We denote the conjugate of $\mathcal{V}_i$ in the anti-holomorphic algebra by $\overline{\mathcal{V}}_i$. The unitarity of a Wess-Zumino-Witten model is determined by its level k, the eigenvalue with respect to $\hat{k}$. Taking $\widehat{\mathfrak{sl}}_2$ as an example, its minimal model vertex operator algebra (VOA) is rational and unitary (its modules contain no states with zero or negative norms) when k is a non-negative integer, and non-unitary when k is a fraction satisfying [24]

$$\mathsf{k}+2 = \frac{u}{v}, \quad u \in \mathbb{Z}_{\geq 2}, \quad v \in \mathbb{Z}_{\geq 2}, \quad \gcd\{u,v\} = 1.$$

In both cases, the level is referred to as *admissible*, and there are a finite number of highest-weight modules in the minimal models. On the other hand, at a non-admissible level, the VOA has infinitely many highest-weight modules, the models are no longer minimal. The unitary Wess-Zumino-Witten models have long been regarded as the fundamental building blocks of unitary conformal field theory. The fractional level theories [25–27] were first proposed as a speculative generalisation of the non-negative integer level theories. The purpose of conjecturing these models is to construct, in a similar way as in the unitary case [28], the non-unitary Virasoro minimal models as cosets.

In the unitary case, the fusion rules of unitary conformal field theories are readily computed from their modular properties and the Verlinde formula [29]. Unfortunately, it was realised that this method leads to negative fusion coefficients for fractional level Wess-Zumino-Witten models [30,31], whose highest-weight modules were first studied by Adamovič and Milas [32]. This motivated physicists to consider a new class of modules, now known as *the relaxed highest-weight modules*. It was pointed out that the problem of negative coefficients stems from attempting to build the theory with an insufficiently rich category of modules. Gaberdiel found [33] that the fusion of admissible modules may not be decomposed into direct sums of admissible modules as in the unitary case. By studying a simple case of $\widehat{\mathfrak{sl}}_2$ at level $-\frac{4}{3}$, he was able to show, using a purely algebraic algorithm, that the minimal model contains modules which are reducible but indecomposable. The image of the spectral flow automorphism of these modules appear in the fusion rules and have unbounded conformal dimensions. It was discovered that admissible level $\widehat{\mathfrak{sl}}_2$-theories naturally allowed for a continuous parametrised family of such relaxed highest-weight modules [34,35].

The root cause of the negative fusion coefficients was pointed out by Ridout [25] through a careful analysis of $\widehat{\mathfrak{sl}}_2$ at level $-\frac{1}{2}$. It was observed that the irreducible module characters are not linearly independent. More precisely, the modular transformations of the characters in the preceding analyses did not properly account for the non-trivial convergence properties. Characters, in this case, must be treated as distributions instead of meromorphic functions. In his subsequent works [26, 36, 37], Ridout generalised this special case to all admissible levels for $\widehat{\mathfrak{sl}}_2$ and arrived at Grothendieck fusion rules which are consistent with those from independent computations [27, 33].

Besides the $N = 2$ minimal models, the thesis also aims to describe the minimal models associated with the affine Kac-Moody superalgebra $\widehat{\mathfrak{osp}}(1|2)$ at admissible levels. This superalgebra is an extension of $\widehat{\mathfrak{sl}}_2$ by two fermionic generators. The conformal field theories associated with $\widehat{\mathfrak{osp}}(1|2)$ have been studied at both integer and fractional levels in the literature [38–40]. However, only irreducible modules in the Neveu-Schwarz sector were considered in these works, in which the issue of negative fusion coefficients appeared. The complete spectrum of irreducible and reducible but indecomposable modules in both the Neveu-Schwarz and the Ramond sector was first discussed in [41] for a particular level $\mathsf{k} = -\frac{5}{4}$. This paper also presented Grothendieck fusion rules of these modules computed from a modified Verlinde formula. The fusion coefficients are now indeed non-negative integers.

The methodology we present in this thesis, for the study of the $N = 2$ and the $\widehat{\mathfrak{osp}}(1|2)$ minimal models, is known as the *coset construction*. The coset construction for $N = 2$ minimal models was first proposed by Kazama and Suzuki in 1989 [42, 43] as the commutant of a free boson in the tensor product of $\widehat{\mathfrak{sl}}_2$ with another free boson. An obvious vulnerability of this construction is the lack of fermionic algebras, therefore preventing any possibility of supersymmetry. Progress was made by Eholzer and Hübel [44] in which they replaced one of the free bosons with two free fermions and investigated the unitary $N = 2$ minimal models. However, we found one actually needs to make a change of basis of the coordinates of the two free fermions and work with a $\mathfrak{bc}$-ghost system. With this modified coset, we decomposed modules of $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$ into a direct sum of Fock spaces with $N = 2$-modules, which were found to be irreducible highest-weight modules. Various aspects, such as characters and fusion rules were computed in both unitary and non-unitary cases. This work has appeared in [45].

The construction of conformal field theories from affine Kac-Moody algebras $\widehat{\mathfrak{g}}$ has a long history. The theory for this method was first proposed in [46], in which the construction of the unitary Virasoro minimal models in terms of affine $\widehat{\mathfrak{sl}}_2$ was presented as an example. Following from this, Kent, in his PhD thesis [47], proposed that the non-unitary case should exist.

The decomposition of admissible $\widehat{\mathfrak{osp}}(1|2)$ modules into $\widehat{\mathfrak{sl}}_2$ and Virasoro modules was first considered in [38]. The paper provided a classification of the $\widehat{\mathfrak{osp}}(1|2)$-modules, which did not include the aforementioned relaxed highest-weight modules. Characters over these modules were computed by combining characters of the $\widehat{\mathfrak{sl}}_2$ and the Virasoro modules. Fusion rules computed from the Verlinde formula carried negative coefficients. An immediate improvement we made upon these results is to present a more complete classification of the $\widehat{\mathfrak{osp}}(1|2)$ modules by filling in the missing part on the relaxed highest-weight modules. The completeness of our classification was proven [45] with the help of Zhu's algebra. The $\widehat{\mathfrak{osp}}(1|2)$ fusion rules suffer the same problem of negative coefficients as the $\widehat{\mathfrak{sl}}_2$. It was proposed that, for fermionic theories, one can derive variations of the Verlinde formula as in [6, 48]. And for certain non-rational theories, there is a generalisation called the standard Verlinde formula [49, 50] that is conjectured to give the Grothendieck fusion coefficients of the theory. A fermionic version of the standard Verlinde formula was recently tested successfully in [41] for the $\mathfrak{osp}(1|2)$ minimal model at level $\mathsf{k} = -\frac{5}{4}$.

For both target superalgebras ($\widehat{\mathfrak{osp}}(1|2)$ and the $N = 2$ theory), we adopted an alternative approach for computing their Grothendieck fusion rules, instead of the Verlinde formula. This method exploits the known fusion rules of the algebras in their coset components. The modules of the coset component algebras are combined appropriately and induced to modules of the larger algebras. The approach considerably simplifies the derivation of fusion rules compared to other methods involving Verlinde formula and keeps track of the parities of the resultant modules. From the obtained Grothendieck fusion rules, we further identify staggered modules, which are believed to be projective. Through the conjectured projective covers, the Grothendieck fusion rules are lifted to actual fusion rules of the target algebras.

1.2 Overview

The projects presented in this thesis are parts of a programme to understand the minimal models associated to a Lie superalgebra. While the theories with non-negative integer levels and simple Lie algebras lead to rational conformal field theories and, as such, are very well understood, the situation is much more complicated and rich for other levels or when superalgebras are involved. Indeed, the non-rational admissible-level minimal models are expected to be prime examples of logarithmic conformal field theories, these being models that admit representations on which the hamiltonian acts non-diagonalisably, leading to correlation functions with logarithmic singularities. Another interesting feature of these models is that they have a continuous spectrum of modules. The thesis consists of a detailed study of the minimal models associated to the $N = 2$ superconformal field theory and affine $\mathfrak{osp}(1|2)$ algebra at admissible levels.

The thesis starts with a brief account of the principles of conformal field theory which will be used in the following chapter. It serves to establish the notation and conventions, and most importantly, to motivate fusion (Section 2.3) and automorphisms (Section 2.4). We have attempted to concentrate the reader's attention on representations, while introducing key concepts such as the state-field correspondence, operator product expansions, null fields and the correlation function constraints they impose. For completeness, we introduce the conventional method of computing fusion rules using the Verlinde formula. This relies on performing S-transforms on the characters of the modules in order to obtain fusion coefficients. It is worth pointing out that this method is not suitable for the superalgebras we will study because of the difficulty in calculating the S-transforms of some characters as well as the problem of negative fusion coefficients. This motivates us to adopt an alternative and obviously simpler method for computing these fusion rules, which we shall present in Chapter 5 and Chapter 6. We finish the chapter by discussing the action of automorphisms on a state and then promote the action to the level of modules. These automorphisms help to establish a relation between known modules and new modules, and proves to be one of the most fruitful tools throughout the thesis.

In Chapter 3, we illustrate the general discussion in Chapter 2 with three examples of bosonic conformal field theories. These are the free boson (Section 3.1), the Virasoro (Section 3.2) and affine $\mathfrak{sl}_2$ (Section 3.3) minimal models. The first two theories are well known and discussed in the literature. The purpose of these reviews is to complement what follows rather than to reiterate the treatments in the literature. We also have a very good picture for affine $\mathfrak{sl}_2$. Our detailed discussion of the $\mathfrak{sl}_2$ minimal models is based on [32–34, 36, 51]. Our understanding of the $N = 2$ and $\mathfrak{osp}(1|2)$ minimal models are both extended from those of $\mathfrak{sl}_2$. The section presents the general theory of relaxed highest-weight modules of the $\mathfrak{sl}_2$ minimal models. These natural generalisations of the usual highest-weight modules were introduced in the conformal field theory literature in [52], though they had already appeared in mathematics classifications such as [32], but have only recently been formalised in a general setting [35]. Since then, the role played by irreducible relaxed highest-weight modules in facilitating the study of general admissible-level Wess-Zumino-Witten models has been widely appreciated and the field has been rapidly developing, see [53–56] for example.

Following the bosonic theories, we elucidate five examples of fermionic conformal field theories in Chapter 4. The chapter begins with a brief account of the free fermion (Section 4.1.1) and the $\mathfrak{bc}$-ghosts (Section 4.1.2) following the standard treatment in the literature. We then proceed to the $N = 1$ superconformal field theory, which is obtained by extending the Virasoro algebra with one degree of supersymmetry. After introducing the relevant theory through representation theory, we compute the fusion rules in both the Neveu-Schwarz and Ramond sectors by constructing PDEs from null fields of the theory. This method follows analogously to [57, Sec. 7.3] in which the fusion rules for the Virasoro minimal models are computed. Aside from the demand of the result, we get a taste of the complexity of this method when the theory is extended beyond the Virasoro case. Other than the complication by parities, the large numbers of correlation functions and PDEs make the problem considerably more difficult to solve. This motivates us to search for alternative methods of studying superconformal field theories with higher degree of supersymmetries such as the $N = 2$ and affine $\mathfrak{osp}(1|2)$ theories. The chapter concludes with algebraic preliminaries of the $N = 2$, affine $\mathfrak{osp}(1|2)$ theories and their minimal models.

Chapter 5 is dedicated to the understanding of the $N = 2$ minimal models which appear constantly in string theory [58]. Of course there is quite some literature on the subject, from physics [44, 48], mathematical physics [52, 59] and mathematics [15, 60, 61] perspectives. We exploit the known coset [38] of an $N = 2$ minimal model as the commutant of a free boson inside the tensor product of $\widehat{\mathfrak{sl}}_2$ with the $\mathfrak{bc}$-ghost, and establish an efficient procedure to extract representation theory of the $N = 2$ theory out of this coset.

The chapter begins by establishing the exact relation between $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$ and its subalgebras. We derive expressions for the generating fields of the $N = 2$ theory and the free boson in terms of those of $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$. This, along with the extremal state method, allows us to compute the branching rules of the coset, that is, how $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$-modules decompose into modules of $N = 2$ and Fock spaces. From branching rules, we identify the $N = 2$-modules by giving explicit formulae for their charges, conformal dimensions and parities which characterize an $N = 2$-module. Along with providing a complete classification of the irreducible $N = 2$-modules, we discuss a group of reducible but indecomposable modules which turn out to play an important role in the construction of the irreducibles by resolutions.

Section 5.2 is devoted to the computation of the characters of the $N = 2$-modules obtained in Section 5.1. We divide the discussion into two parts: the characters of the unitary minimal models, which are computed using the residue method [44], and those of the logarithmic models, which we compute by writing them as an infinite sum of the reducible characters using resolutions. Following a general discussion in Appendix A, Section 5.1 develops the basic strategy for the thesis, induction. This formalism has recently been developed in detail and rigour in [62–65]. Inducing a module of the subalgebra $N = 2$ tensored by a free boson yields an $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$-module. The main new outcome of this uniform and rather direct treatment of the minimal models is fusion rules. The technique of induction provides a straight-forward way of computing fusion without the Verlinde formula or PDEs. Since $N = 2$-modules tensored with Fock spaces induce to $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$-modules, whose fusion rules are known, we can simply extract Fock spaces from the result and are left with the $N = 2$ fusion rules. With the help of this method, we present the Grothendieck fusion rules for the non-unitary $N = 2$ minimal models in Section 5.3.1. The genuine fusion products and the projective covers of the irreducible modules are conjectured in Section 5.3.2.

Chapter 5 finishes with an application of the $N = 2$ fusion rules, where we examine a specific minimal model $\mathsf{M}^{\mathrm{N=2}}(4,3)$. One of the modules of the minimal module corresponds to an order 2 simple current and is found to satisfy the same algebra as the fermionic generating field in the $N = 1$ theory. We use the simple current to extend the $N = 2$ minimal model into a W-algebra, which has three degrees of supersymmetries.

The material presented in Chapter 5 includes content from [66], a paper in collaboration with Creutzig, Ridout and Wood. The calculations in this chapter are performed by the author with guidance from Ridout and Wood. The computation of fusion rules in Section 5.3 is based on theorems provided by Creutzig and Kanade.

In Chapter 6, we present our study of the minimal models associated to $\mathfrak{osp}(1|2)$. These minimal models are extensions of the tensor product of certain Virasoro and $\mathfrak{sl}_2$ minimal models, we can induce the known structures of the representations of the latter models to get a rather complete understanding of the minimal models of $\mathfrak{osp}(1|2)$. This method is referred to as the inverse coset method. The presentation of the chapter follows from similar ideas as in Chapter 5. Main results of this chapter include the classification of the irreducible relaxed highest-weight modules, their characters and Grothendieck fusion rules. We also discuss conjectures for the (genuine) fusion products and the projective covers of the irreducibles.

Part of Chapter 6 was presented in [45], a paper in collaboration with Creutzig, Ridout and Kanade. The calculations in this chapter are guided by Ridout and performed by the author. The computation of fusion rules is again based on theorems provided by Creutzig and Kanade. Section 6.4 on the completeness of the our classification of irreducible highest-weight $\mathsf{B}_{0|1}(u,v)$-modules was based on a discussion with Ridout and Creutzig, details on this topic can be found in [45].

The thesis concludes with an appendix on a few aspects of the coset theory which are used throughout the thesis. In the appendix, we summarise in general the idea of induction. Two important statements about induction are made: A module, under certain conditions, induced from an irreducible module is irreducible, and induction is preserved by fusion.

Chapter 2

Conformal Field Theory

In this chapter, we introduce some of the principles of conformal field theory. Rather than a comprehensive review, the chapter aims to refresh the reader's memory and establish notation. It begins with presenting conformal invariance in the classical picture, pointing out why conformal field theory is special in two-dimensions. We then proceed to the quantisation of the theory which is performed in the framework of radial ordering. One aim of the chapter is to introduce the fusion process. The main goal of the thesis is to compute such fusions for certain superalgebras. The chapter finishes with a general discussion on automorphisms, which will form a basic tool in what follows. The texts [57, 67, 68] are excellent sources for the subject.

2.1 Conformal Invariance

A quantum field theory is endowed with a set of fields $\Phi(x)$, whose dynamics is specified by an action functional $S[\Phi]$. Generally, any quantum field theory is expected to admit certain isometries. For example, in Minkowski space, the metric is usually required to be invariant under transformations in the Poincaré group. Transformations which preserve the metric up to a non-zero scaling factor are referred to as *conformal transformations*. It is not hard to check that, in a d-dimensional manifold, the angle between any two curves is invariant under conformal transformations.

The next step is to determine the *conformal algebra* from the infinitesimal conformal transformations, which are integrated into a finite-dimensional global conformal group. In a flat euclidean space, up to a first-order infinitesimal coordinate transformation $x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon^\mu(x)$, one can derive that the scaling factor of the metric Λ and the dimension of the manifold d are related by

$$1 - \Lambda = \frac{2}{d}(\partial \cdot \epsilon). \tag{2.1.1}$$

When $d > 2$, an infinitesimal transformation is strongly constrained by the conformal condition. There is only a finite number, $\frac{1}{2}(d+1)(d+2)$ to be exact, of linearly independent infinitesimal conformal transformations, so these can be integrated to a finite-dimensional

Lie group of global conformal transformations. It is the $d = 2$ case for which the conformal field theory becomes special. Its powerful symmetries enable the calculation of exact solutions without perturbation methods such as perturbation theory and Feynman diagrams.

Consider the length squared between two infinitesimally separated points. The quantity does not depend on the choice of coordinates since it is physically measurable:

$$\mathrm{d}s^2 = g'_{\mu\nu}\mathrm{d}x'^{\mu}\mathrm{d}x'^{\nu} = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}. \tag{2.1.2}$$

From this, we can deduce that the transformed metric $g'_{\mu\nu}$ is related to the original metric (in the case that the original metric is flat and constant) by

$$g_{\mu\nu} = g'_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \tag{2.1.3}$$

Therefore, for an infinitesimal transformation to be conformal, we require

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (1 - \Lambda) g_{\mu\nu}. \tag{2.1.4}$$

Comparing this with (2.1.1) with $d = 2$ yields

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = g_{\mu\nu}(\partial \cdot \epsilon) = g_{\mu\nu} g^{\rho\sigma} \partial_\rho \epsilon_\sigma. \tag{2.1.5}$$

For the euclidean space, this condition split into the following equations:

$$2\partial_1\epsilon_1 = \partial_1\epsilon_1 + \partial_2\epsilon_2 = 2\partial_2\epsilon_2, \quad \partial_1\epsilon_2 + \partial_2\epsilon_1 = 0, \tag{2.1.6}$$

which are recognised as the Cauchy-Riemann equations. With a change of basis from $\mathbb{R}^2$ to $\mathbb{C}$, the solutions of (2.1.6) can be found to be spanned by the holomorphic functions $\epsilon(z) = z^n$ and anti-holomorphic functions $\overline{\epsilon}(\overline{z}) = \overline{z}^n$.

A basis of generators with respect to the infinitesimal transformations is constructed as

$$\ell_n = -z^{n+1}\partial, \quad \overline{\ell}_n = -\overline{z}^{n+1}\overline{\partial}, \qquad n \in \mathbb{Z}. \tag{2.1.7}$$

These differential operators generate a infinite-dimensional Lie algebra called the *conformal algebra*. It follows that these basis elements give two commuting copies of the *Witt algebra*:

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \quad [\overline{\ell}_m, \overline{\ell}_n] = (m-n)\overline{\ell}_{m+n}, \quad [\overline{\ell}_m, \ell_n] = 0. \tag{2.1.8}$$

The Virasoro algebra is a central extension of the Witt algebra, denoted by $\mathfrak{Vir}$, and is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\mathsf{C}}{12}(m^3 - m)\delta_{m+n,0}, \quad [L_n, \mathsf{C}] = 0, \tag{2.1.9}$$

where C is known as the *central charge operator* and is an element of the algebra. The eigenvalue with respect to C, denoted by c, is the *central charge* of the algebra. The L_n (with $n \in \mathbb{Z}$) are known as Virasoro operators. A representation of the Virasoro algebra is therefore a projective representation of the Witt algebra. The Virasoro algebra is a Lie algebra, which means it satisfies bilinearity, antisymmetry and the Jacobi identity. The non-commutative and non-associative operation $[\,,\,]$ is known as a Lie bracket, or more commonly, a commutator. Equation (2.1.7) has a finite-dimensional subalgebra which is spanned by $\{\ell_{-1}, \overline{\ell}_{-1}, \ell_0, \overline{\ell}_0, \ell_1, \overline{\ell}_1\}$. This subalgebra integrates to give *global conformal transformations* including translations, dilations, rotations and special conformal transformations. Transformations described by modes other than these are referred to as *local conformal transformations*.

The holomorphic and the anti-holomorphic algebras are identical and commuting, each copy is referred to as a *chiral algebra*. For the rest of the thesis, we shall only discuss the holomorphic part of the theory. The reader should always keep in mind that, the genuine conformal field theory is really obtained by combining the two chiral halves together. This is of particular importance for calculations related to the physical world.

2.2 The Chiral Algebra

2.2.1 Representations of the Virasoro algebra

In a chiral conformal field theory, the quantum state space, which admits a representation of the Virasoro algebra, is a complex vector space. This space is acted on by the Virasoro operators L_n introduced in (2.1.9). Among the L_n, we define L_0 to be the energy operator, its eigenvalue on a state is referred to as the energy or *the conformal dimension*. In a rational conformal field theory, the central charge operator C acts as a multiple of the identity, and L_0 may be diagonalised on the representation space. All other L_n modes are partitioned into two groups:

- L_n with $n > 0$, *the annihilation (or lowering) operators*;
- L_n with $n < 0$, *the creation (or raising) operators*.

The quantum state space admits a scalar product $\langle\ ,\ \rangle$, which is defined by an adjoint operation, denoted by †, satisfying

$$L_n^\dagger = L_{-n}, \quad \mathsf{C}^\dagger = \mathsf{C}. \tag{2.2.1}$$

We declare that the scalar product of the vacuum state v with itself is unity:

$$\langle v, v \rangle = 1, \tag{2.2.2}$$

and the scalar product is bilinear and invariant with respect to the adjoint:

$$\left\langle x \cdot w_1, w_2 \right\rangle = \left\langle w_1, x^\dagger \cdot w_2 \right\rangle, \tag{2.2.3}$$

where x is an element of the algebra, and w_1, w_2 are in the state space.

A Verma module of $\mathfrak{Vir}$, denoted by $V(\mathsf{c}; h)$, is uniquely determined by its central charge and the conformal dimension of its highest-weight state. It is constructed by acting with creation operators on a *highest-weight state* $|h\rangle$ of conformal dimension (energy) h, satisfying

$$L_0|h\rangle = h|h\rangle, \quad \mathsf{C}|h\rangle = \mathsf{c}|h\rangle, \tag{2.2.4}$$

whereas $|h\rangle$ is required to be zero when being acted on by the annihilation operators:

$$L_n|h\rangle = 0 \quad (n > 0). \tag{2.2.5}$$

The Poincaré-Birkhoff-Witt (PBW) standard basis vector for a Verma module takes the form

$$L_{n_1} L_{n_2} \cdots L_{n_k} |h\rangle \quad (n_1 \leq n_2 \leq \ldots \leq n_k \leq -1), \tag{2.2.6}$$

where the order of the n_i is chosen to be increasing by convention. States of the form (2.2.6) are the so-called *descendant states*.

A Verma module may contain a highest-weight submodule. For the existence of a non-trivial submodule, its highest-weight state must be a linear combination of descendant states of the form (2.2.6). This state is known as a *singular vector*, and the module which the submodule is contained in is *reducible*. Following from (2.2.6) and the adjoint operation defined in (2.2.1), one can easily show that a singular vector is orthogonal to all states in the Verma module. It is therefore also referred to as a *null state*. A module becomes *irreducible* once we set all its singular vectors and their descendants to zero. The irreducible quotient module of a given Verma module $V(\mathsf{c}, h)$ is uniquely determined, we denote it by $M(\mathsf{c}, h)$. These modules are the building blocks of the minimal models which we shall introduce later in the thesis.

In physics, it is natural to expect the representations to be unitary, that is, they contain no states with negative-norms. This imposes constraints on the central charge and the conformal dimension of the module. In particular, a module cannot be unitary if $\mathsf{c} < 0$ or $h < 0$. This means the energy of the highest weight state must be at least zero. The highest-weight state annihilated by L_{-1} with energy zero is referred to as the *vacuum*, it is denoted by $|0\rangle$.

2.2.2 Conformal fields

In section 2.1, we have seen that conformal field theories are exceptionally powerful in two dimensions because of the existence of infinitely many generators. The system which we use to embed the infinite number of degrees of freedom is a scalar field $\phi(x,t)$ in terms of position x and time t with a lorentzian metric. In order to define a conformal field theory on a complex plane, which is euclidean, we perform Wick rotation by letting $t = i\tau$. The function is now defined on a complex plane with variable $z = \mathrm{e}^{2\pi(\tau+\mathrm{i}x)}$. The anti-holomorphic part of the theory is then described in terms of the complex conjugate of z.

In many examples of field theories, it is possible to write down the action of a theory and obtain equations of motion by extremising the action as a functional of fields. The equation of motion determines how the fields evolve over time. The fields, which are periodic in z, can be Fourier decomposed as

$$\phi(z) = \sum_n \phi_n z^{-n-h}, \tag{2.2.7}$$

where the Fourier modes ϕ_n are identified as quantum operators satisfying certain commutation relations endowed by the algebra of the theory. Here, h is the conformal dimension of the field as measured by the eigenvalue with respect to L_0. As an example, let us consider a special field which is the conserved current corresponding to the conformal symmetry, *the energy-momentum field*, denoted by $T(z)$. The field is associated with a rank 2 tensor $T^{\mu\nu}$ which measures how the μth component of the momentum and energy flux varies in the νth direction of the spacetime. Upon an arbitrary infinitesimal transformation in the coordinate $z' = z+\eta$, $\bar{z}' = \bar{z}+\bar{\eta}$, the change in action in terms of $T^{\mu\nu}$ is given by

$$\delta S = \int T^{\mu\nu}\partial_\mu\eta_\nu \mathrm{d}z\mathrm{d}\bar{z}. \tag{2.2.8}$$

The field has a conformal dimension of 2 and is expanded as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{2.2.9}$$

The expansion modes are identified as the Virasoro operators satisfying (2.1.9).

In the previous section, we have introduced how operators act on states to form a representation. One of the strengths of a conformal field theory is its ability to associate a given field $\phi(z)$ to a state through the '*state-field correspondence*' defined as

$$\lim_{z\to 0}\phi(z)|0\rangle = |\phi\rangle, \tag{2.2.10}$$

where $\phi(z)$ is then Fourier expanded according to (2.2.7) before acting on the vacuum $|0\rangle$.

The state $|\phi\rangle$ is known as an 'asymptotic in-state' in scattering theory, because in the Wick-rotated $\mathbb{R}^2$ plane, $z \to 0$ corresponds to $\tau \to -\infty$, which is the infinite past. To illustrate the correspondence, let us find the state which corresponds to the energy-momentum tensor $T(z)$:

$$\begin{aligned}\lim_{z\to 0} T(z)|0\rangle &= \lim_{z\to 0} \sum_{n\in\mathbb{Z}} L_n|0\rangle z^{-n-2} \\ &= \lim_{z\to 0}\left(\sum_{n\leq -3} L_n|0\rangle z^{n-2} + L_{-2}|0\rangle z^0 + L_{-1}|0\rangle z^{-1} + \sum_{n\geq 0} L_n|0\rangle z^{n-2}\right), \qquad (2.2.11\text{a})\end{aligned}$$

where in the first term, the polynomial in z vanishes as $z \to 0$. The last term also becomes zero since the vacuum is annihilated by non-negative modes of L_n. For the limit in z to exist, we force $L_{-1}|0\rangle$ in the second last term to be zero. This is also the reason that we required the vacuum to be annihilated by L_{-1} in its definition at the end of Section 2.2.1. Overall, we find that

$$\lim_{z\to 0} T(z)|0\rangle = L_{-2}|0\rangle, \qquad (2.2.12)$$

so the state corresponding to $T(z)$ is $L_{-2}|0\rangle$. It is also worth pointing out that, as one may expect, the state corresponding to the identity field is the vacuum state $|0\rangle$. In the previous section, we have defined a highest-weight state as a state which is annihilated by positive modes of L_n. The fields corresponding to such states are known as Virasoro *primary fields*. Fields which correspond to descendants of the Virasoro primary field in the same representation are referred to as *secondary fields*.

2.2.3 Chiral algebras

In a classical theory, the order of fields in the product form is usually irrelevant. The quantisation of a classical conformal field theory is performed in the framework of *radial quantisation*, in which space and time of a field $\phi(z)$ are defined to run in the angular and radial directions of z, respectively. The *radial ordering* of the bosonic fields $\phi(z)$ and $\psi(w)$ is defined as

$$\mathcal{R}\Big\{\phi(z)\psi(w)\Big\} = \begin{cases}\phi(z)\psi(w) & \text{if } |z| > |w| \\ \psi(w)\phi(z) & \text{if } |z| < |w|,\end{cases} \qquad (2.2.13)$$

where we assume $|z| \neq |w|$. Since the magnitude of the coordinates is related to its time component, radial ordering is also referred to as *time ordering*. The definition in (2.2.13) arranges the fields in such a way that when being applied to a state (we assume this state occurs at time zero), the field which occurs earlier in time (smaller coordinate magnitude) is applied first to the state.

The space of fields of a conformal field theory is endowed with a product called *the operator product expansion (OPE)*. As the name suggests, it expands the radially ordered product of two conformal fields as a Laurent series of their coordinates. The operator product expansion of $\phi(z)$ and $\psi(z)$ is given by

$$\mathcal{R}\Big\{\phi(z)\psi(w)\Big\} = \sum_{i=-\infty}^{\infty} \frac{A_i(w)}{(z-w)^i}, \tag{2.2.14}$$

where h_ϕ, h_ψ and h_i are the conformal dimensions of $\phi(z)$, $\psi(z)$ and $A_i(z)$, respectively. This expansion has poles at $z = w$ for $i+1 < 0$. The collection of all its regular terms helps to define the ordering of modes. This is denoted as $:\phi(z)\psi(w):$ and is called a *normal ordering* of the two fields. Since the normally ordered product has no singularities, we are allowed to Taylor expand the product at $z = w$ as

$$\lim_{z\to w} :\phi(z)\psi(w): = \lim_{z\to w}\left(\sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} :\partial^n\phi(z)\psi(w):\right) = :\phi(w)\psi(w):. \tag{2.2.15}$$

To see this indeed imposes an ordering on the operators, consider $:\phi(w)\psi(w):$ as the contour integral of the radially ordered product:

$$:\phi(w)\psi(w): = \oint_w \mathcal{R}\Big\{\phi(z)\psi(w)\Big\}(z-w)^{-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{2.2.16}$$

It is a common trick to split the z-contour around w into two opposite running contours around the origin which are infinitely close to each other. As shown in Figure 2.1, equation (2.2.16) can now be written as

$$:\phi(w)\psi(w): = \oint_{|z|>|w|} \frac{\phi(z)\psi(w)}{z-w}\frac{\mathrm{d}z}{2\pi\mathrm{i}} - \oint_{|z|<|w|} \frac{\psi(w)\phi(z)}{z-w}\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{2.2.17}$$

The $(z-w)^{-1}$ factor can be expanded as a convergent geometric series depending on its convergence region. In the first term where $|z| > |w|$, we have $(z-w)^{-1} = \sum_{n=0}^{\infty} w^n/z^{n+1}$, whereas the same factor in the second term is expanded as $-\sum_{n=0}^{\infty} z^n/w^{n+1}$. With the fields expanded in terms of modes, (2.2.17) becomes

$$:\phi(w)\psi(w): = \left(\sum_{n,m+h_\phi\le 0} \phi_m\psi_n + \sum_{n,m+h_\psi>0} \phi_n\psi_m\right) w^{-m-n-h_\phi-h_\psi}, \tag{2.2.18}$$

where h_ϕ and h_ψ are the conformal dimensions of fields $\phi(z)$ and $\psi(z)$ respectively. It is therefore natural to define the normal ordering of modes as

$$:\phi_m\psi_n: = \begin{cases} \phi_m\psi_n & \text{if } m+h_\phi \le 0, \\ \psi_n\phi_m & \text{if } m+h_\phi > 0. \end{cases} \tag{2.2.19}$$

In the next chapter, we shall see, with an example, how normal ordering helps to remove

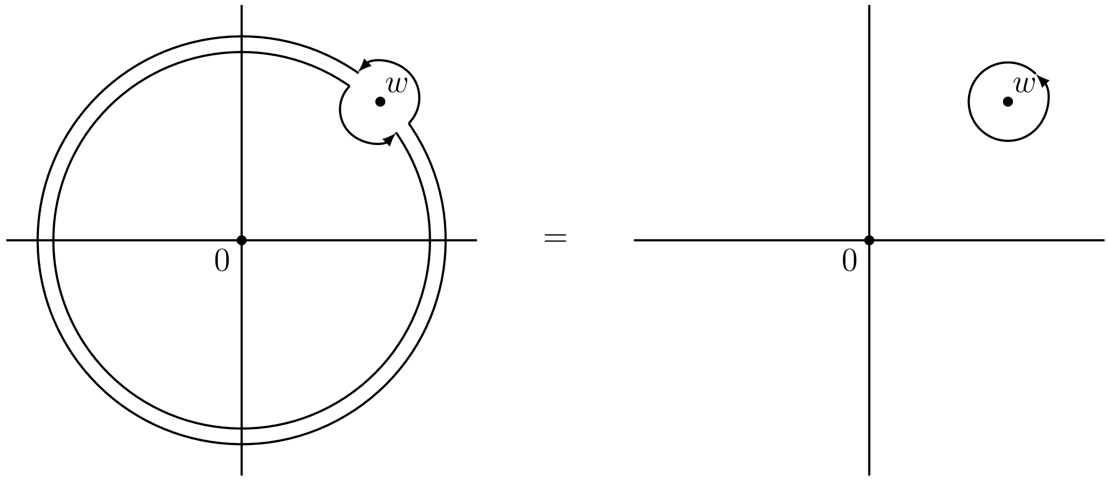

Figure 2.1: The equivalence between two infinitely close, opposite running contours around the origin and an infinitely small contour around w.

divergences and truncates the L_0-eigenvalue of a state to a reasonable finite answer.

Unless otherwise indicated, the product of fields with different arguments is always assumed to be radially ordered. We usually remove the '$\mathcal{R}\{\ \ \}$' symbol for simplicity. As we shall see later, the singular terms are the ones of importance in an OPE, the normal ordered terms can always be implicitly remembered. We shall omit the normal ordered terms and indicate the omission by replacing the equal sign by '$\sim$'. Mathematically, the OPE defines an algebraic operation over fields, whose Fourier expansion yields modes. It is natural to expect the modes to be endowed with a closely related algebraic operation, i.e., commutation relations. Recall (2.2.7) for the Fourier expansion of the field $\phi(z)$ of conformal dimension h_ϕ. Its n-th mode is expressed as a contour integral of z about the origin as

$$\phi_n = \oint_0 \phi(z) z^{n+h_\phi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{2.2.20}$$

The commutation relation between ϕ_n and the m-th mode of another field $\psi(w)$, in terms of the contour integrals, is given by

$$\begin{aligned}[\phi_n, \psi_m] = \oint_0 \oint_0 \phi(z)\psi(w) z^{n+h_\phi-1} w^{m+h_\psi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \\ - \oint_0 \oint_0 \psi(w)\phi(z) z^{n+h_\phi-1} w^{m+h_\psi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}}.\end{aligned} \tag{2.2.21}$$

The products of fields in both terms on the right-hand side are not radially ordered. If we assume $|z| > |w|$ in the first integral, and $|z| < |w|$ in the second, then both products can be radially-ordered as $\mathcal{R}\{\phi(z)\psi(w)\}$, and the two terms can be combined into a single term:

$$[\phi_n,\psi_m] = \left(\oint_0\oint_0\limits_{|z|>|w|} - \oint_0\oint_0\limits_{|z|>|w|}\right)\mathcal{R}\{\phi(z)\psi(w)\}z^{n+h_\phi-1}w^{m+h_\psi-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}. \tag{2.2.22}$$

By taking the limit $z\to w$, the two opposite-running z-contours around the origin can be deformed into a single contour around w. This is the reverse of Figure 2.1. With respect to the new contour, (2.2.22) can now be written as

$$[\phi_n,\psi_m] = \oint_0\oint_w \mathcal{R}\{\phi(z)\psi(w)\}z^{n+h_\phi-1}w^{m+h_\psi-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{d}w}{2\pi\mathrm{i}} \tag{2.2.23a}$$

$$= \oint_0\oint_w \sum_{i\in\mathbb{Z}}\frac{A_i(w)}{(z-w)^{i+1}}z^{n+h_\phi-1}w^{m+h_\psi-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}, \tag{2.2.23b}$$

where we have expanded the radially-ordered product as a formal Laurent series using (2.2.14). Regular terms in (2.2.23b) do not contribute to the integral, whereas the singular terms can be evaluated with Cauchy's integral theorem as

$$[\phi_n,\psi_m] = \sum_{i=0}^{n+h_\phi-1}\binom{n+h_\phi-1}{i}\oint_0 A_i(w)w^{m+n+h_\phi+h_\psi-2-i}\frac{\mathrm{d}w}{2\pi\mathrm{i}}. \tag{2.2.24}$$

As an example, let us try and compute the OPE between the energy-momentum tensor $T(z)$ with itself, using the commutator for the Virasoro algebra (2.1.9). It follows from (2.2.24) that

$$[L_n,L_m] = \sum_{i=0}^{n+1}\binom{n+1}{i}\oint_0 A_i(w)w^{m+n+2-i}\frac{\mathrm{d}w}{2\pi\mathrm{i}}. \tag{2.2.25}$$

In order to find the $A_i(w)$, we expand the commutator on the left-hand side and write the modes in terms of contour integrals according to (2.2.20):

$$\begin{aligned}&(m-n)L_{m+n}+\frac{\mathsf{c}}{12}(m^3-m)\delta_{m+n,0}\\ =&(m-n)\oint_0 T(w)w^{n+m+1}\frac{\mathrm{d}w}{2\pi\mathrm{i}}+\frac{m^3-m}{12}\oint_0\frac{\mathsf{c}}{2}w^{n+m-1}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\\ =&\binom{m+1}{0}\oint_0 w^{n+m+2}\partial T(w)\frac{\mathrm{d}w}{2\pi\mathrm{i}}+\binom{m+1}{1}\oint_0 2T(w)w^{n+m+1}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\\ &+\binom{m+1}{3}\oint_0\frac{\mathsf{c}}{2}w^{n+m-1}\frac{\mathrm{d}w}{2\pi\mathrm{i}},\end{aligned} \tag{2.2.26a}$$

where we have written $(m-n)$ on the right-hand side of (2.2.26a) as $2(m+1)-(m+n+2)$ and performed integration by parts on its first term. Comparing this with (3.3.12), we have

$$A_0(w)=\partial T(w),\quad A_1(w)=2T(w),\quad A_3(w)=\frac{\mathsf{c}}{2}, \tag{2.2.27}$$

and $A_i(w) = 0$ for $i = 2$ and $i \geq 4$. The OPE between two energy-momentum tensors is therefore forced by the Virasoro algebra to be

$$\mathcal{R}\Big\{T(z)T(w)\Big\} \sim \frac{\mathrm{c}/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{2.2.28}$$

Another interesting example is the OPE between $T(z)$ and a primary field $\phi(w)$. Instead of the commutation relation, one can derive this by acting with $T(z)$ on both sides of the state-field correspondence (2.2.10) of the primary field, and then perform an expansion of the field product as in the previous example. The OPE is found to be

$$T(z)\phi(w) \sim \frac{h_\phi \phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}, \tag{2.2.29}$$

where h_ϕ is the conformal dimension of $\phi(w)$. This OPE is often used for characterising the concept of a primary field with respect to the Virasoro algebra. We say a field is Virasoro-primary if and only if its OPE with $T(z)$ takes the form of (2.2.29).

2.3 Fusion

The motivation for constructing a *fusion* operation comes from our demand of computing *correlation functions*, which is the goal of a conformal field theory or any quantum field theory. Correlation functions are physically measurable quantities of the form

$$\langle 0|\mathcal{R}\Big\{\phi_1(z_1)\phi_2(z_2)\ldots\phi_n(z_n)\Big\}|0\rangle. \tag{2.3.1}$$

They contain products of fields which are expanded at short distance according to the OPE stated in (2.2.14). It would be useful to introduce the *family* of a primary field here, which is defined as the primary field itself and all its secondary fields. Since the correlation function of a secondary field is simply related to that of a primary field by a differential operator, all correlation functions can be obtained from those involving primary fields. It is not hard to verify, using the constraint of the conformal algebra, that a 1-point correlation function is zero unless the field is the identity field. The 2- and 3-point correlation functions of Virasoro primary fields take the form of

$$\langle 0|\phi_i(z_i)\phi_j(z_j)|0\rangle = \frac{C_{ij}}{(z_i - z_j)^{2h_i}}, \tag{2.3.2a}$$

$$\langle 0|\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)|0\rangle = \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_1-z_3)^{h_1-h_2+h_3}(z_2-z_3)^{-h_1+h_2+h_3}} \tag{2.3.2b}$$

where the field ϕ_i has conformal dimension h_i and the constant C_{123} depends on the primary fields in the correlator. In the 2-point correlation function, it is always possible to choose a basis of fields so that $C_{ij} = \delta_{ij^*}$, where j^* stands for the index of $\phi_j^*(z_j)$, the conjugate field of $\phi_j(z_j)$. Furthermore, we say the two primary fields from this basis are conjugate to each other if the corresponding 2-point correlation function is non-zero. In this case, we obtain

$$\langle 0|\phi_i(z_i)\phi_i^*(z_j)|0\rangle = \frac{1}{(z_i - z_j)^{2h_i}}. \tag{2.3.3}$$

A 3-point correlation function can be expanded as

$$\langle 0|\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)|0\rangle = \sum_{i=-\infty}^{\infty} \frac{1}{(z_1 - z_2)^i} \langle 0|A_i(z_2)\phi_3(z_3)|0\rangle, \tag{2.3.4}$$

where we have assumed the theory is rational and the OPE does not contain logarithmic terms. Again, it is possible to choose a basis for the primary fields, and write $A_i(z_2)$ as a linear combination of the basis primary fields and their descendants. If this linear combination contains a conjugate field of $\phi_3(z_3)$, then $\langle 0|A_i(z_2)\phi_3(z_3)|0\rangle$ is non-zero, and so is the 3-point correlation function. This leads to the idea of fusion, which is defined as an operation on the primary fields of a theory. In a rational conformal field theory, the fusion of two primary fields is the sum of primary fields which appear in their OPE. It is further observed that a descendant field cannot appear in the OPE unless its primary field does. Therefore, it is sufficient to only record the primary fields appearing. It is common to use $[\phi]$ to denote the family of fields of $\phi(z)$ in fusion rules. We shall omit the square brackets for simplicity but always keep in mind that it is the families of fields that are involved. As the number of primary fields gets larger, the task of computing an n-point correlation function becomes daunting. Knowing which fields appear in the OPE of these primary fields may significantly simplify the problem.

A generic fusion rule takes the form

$$\phi_i \times \phi_j = \sum_k \mathsf{N}_{ij}{}^k \phi_k, \tag{2.3.5}$$

where the fusion coefficients $\mathsf{N}_{ij}{}^k$ are non-negative integers. The fusion rule is associative and commutative (therefore $\mathsf{N}_{ij}{}^k = \mathsf{N}_{ji}{}^k$). This comes from the associativity of operators on the state space and the commutativity and associativity of radial ordering of fields. The identity field provides the unit and is denoted as $\phi_0(z)$ here. Since $\phi_0 \times \phi_i = \phi_i$, we have $\mathsf{N}_{0j}{}^k = \delta_{jk}$. Fusion only helps to determine if the constant in a correlation function vanishes. To explicitly determine this value, one must employ other methods such as free field realisations or the conformal bootstrap method.

The concept of fusion can be naturally lifted to the level of modules. Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ be highest-weight modules of a rational conformal field theory generated by primary fields $\phi_1(z_1)$, $\phi_2(z_2)$ and $\phi_3(z_3)$. The fusion operation on modules $\mathcal{M}$ and $\mathcal{N}$ is a product which makes $\mathcal{M} \times \mathcal{N}$ a module. We know the module $\mathcal{P}$ is in $\mathcal{M} \times \mathcal{N}$ if the 3-point function

$\langle 0|\phi_1(z_1)\phi_2(z_2)\phi_3^*(z_3)|0\rangle$ is non-zero. Following from (2.3.5), the generic form of a fusion rule in terms of modules is

$$\mathcal{M} \times \mathcal{N} = \bigoplus_{\mathcal{P}} \mathsf{N}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \mathcal{P}, \tag{2.3.6}$$

again in a rational conformal field theory. The fusion coefficient $\mathsf{N}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ represents the multiplicity of the module $\mathcal{P}$.

It is most common to compute the fusion rules of a conformal field theory using the Verlinde formula. To see how this works, we have to introduce one of the most important tools in representation theory, the *characters*. A character is defined as the trace of the operator q^H over the entire state space. The variable q is defined in terms of τ, a variable in the upper half complex plane, as $\mathsf{q} = e^{2\pi i\tau}$, and it satisfies $|\mathsf{q}| < 1$ and $\text{Im}(\tau) > 0$. The Hamiltonian $H = L_0 - \frac{\mathsf{c}}{24}$ contains a correction factor $-\frac{\mathsf{c}}{24}$, which allows us to work on a cylinder rather than a complex plane. The generic form of the character of a L_0-graded module $\mathcal{M}$ is given by

$$\text{Ch}[\mathcal{M}](\mathsf{q}) = \text{Tr}_{\mathcal{M}}\mathsf{q}^{L_0-\frac{\mathsf{c}}{24}} = \sum_h \mathsf{q}^{h-\frac{\mathsf{c}}{24}}, \tag{2.3.7}$$

where the right-most term is being summed over independent eigenvectors (of conformal dimension h) with respect to L_0 in the module $\mathcal{M}$. The character of a Virasoro Verma module, or in fact, any generic Verma module can be calculated by first considering its PBW basis which was stated in (2.2.6), where L_n can be replaced by an appropriate choice of operators for other types of Verma modules. The L_0-eigenvalue of a state in the Verma module is simply the sum of all occupation numbers $\sum_n nj_n$. The number of states on a fixed level of L_0-eigenvalue m is therefore given by the number of partitions of m into positive integers. Following from (2.3.7), the character takes the form

$$\text{Ch}[V(\mathsf{c},h)](\mathsf{q}) = \mathsf{q}^{h-\mathsf{c}/24}\sum_{m=0}^{\infty}\mathsf{p}(m)\mathsf{q}^m, \tag{2.3.8}$$

where $\mathsf{p}(m)$ is the number of partitions of m. This sum can be shown to be equivalent to the inverse of the Euler function $\varphi(\mathsf{q}) = \prod_{i=1}^{\infty}(1-\mathsf{q}^i)$. The character is more commonly written as

$$\text{Ch}[V(\mathsf{c},h)](\mathsf{q}) = \mathsf{q}^{h-\frac{\mathsf{c}}{24}}\prod_{i=1}^{\infty}\frac{1}{(1-\mathsf{q}^i)} = \frac{\mathsf{q}^h}{\eta(\mathsf{q})}, \tag{2.3.9}$$

where the Dedekind function $\eta(\mathsf{q})$ is related to the Euler function by

$$\eta(\mathsf{q}) = \mathsf{q}^{1/24}\varphi(\mathsf{q}). \tag{2.3.10}$$

The concept of character is one of most important in conformal field theory. It establishes a powerful tool for calculating aspects of interest for the theory, including module classification, modularity and fusion rules, as we will discuss later in the thesis. Under the modular transformation $\tau \to -1/\tau$, the characters of these modules transform as

$$\mathrm{Ch}[\mathcal{M}](\tilde{\mathsf{q}}) = \sum_{\mathcal{N}} \mathcal{S}_{\mathcal{M}\mathcal{N}} \mathrm{Ch}[\mathcal{N}](\mathsf{q}), \tag{2.3.11}$$

where $\tilde{\mathsf{q}} = e^{-2\pi \mathrm{i}/\tau}$. This modular transform is called the *S-transform*. The Verlinde formula for a rational conformal field theory [29] states that the fusion coefficient in (2.3.5) can be computed as follows

$$\mathsf{N}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} = \sum_{\mathcal{Q}} \frac{\mathcal{S}_{\mathcal{M}\mathcal{Q}} \mathcal{S}_{\mathcal{N}\mathcal{Q}} \mathcal{S}^{*}_{\mathcal{P}\mathcal{Q}}}{\mathcal{S}_{0\mathcal{Q}}}, \tag{2.3.12}$$

where $\mathcal{S}_{\mathcal{M}\mathcal{N}}$ are the S-transform coefficients from (2.3.11), which form the entries of a matrix called *the S-matrix*.

Generalisations of the fusion rules described above exist for other chiral theories. It turns out that in many conformal field theories, especially the superconformal field theories that we will investigate in this thesis, it is daunting to compute the S-matrix of the modules. Another problem is that the denominator $\mathcal{S}_{0\mathcal{Q}}$ on the right-hand side of (2.3.12) can be zero for superconformal field theories. The Verlinde formula must be modified in order to make sense for theories with super-symmetries. There are of course many other methods of computing fusion rules, such as the conformal bootstrap method [69], the Coulomb gas method [70] or from correlation functions. In this thesis, we will adopt yet another method for computing the fusion rules. This will involve constructing a coset for the algebra of interest in terms of well-studied ones as we will describe in detail in Chapter 5 and Chapter 6.

2.4 Automorphisms

As we will see in later chapters, some modules of a conformal field theory are structurally identical though not being technically isomorphic. This can be naturally explained by the action of automorphisms on these modules. It is always useful to consider families of modules which are related by an automorphism ω. Let x be an operator and m be an element of module $\mathcal{M}$. In order to distinguish the elements of $\mathcal{M}$ from those of the resulting twisted module, which we shall denote by $\omega(\mathcal{M})$, we introduce a different notation for the states of the twisted modules:

$$\omega(\mathcal{M}) = \{\omega(m) : m \in \mathcal{M}\}. \tag{2.4.1}$$

Acting with x on the twisted element $\omega(m)$, we have

$$x \cdot \omega(m) = \omega(\omega^{-1}(x) \cdot m), \tag{2.4.2}$$

where we have demanded $\omega(a \cdot b) = \omega(a) \cdot \omega(b)$. This action promotes ω to an invertible action on modules which twists the operator before acting on m. It is the invertibility of ω which preserves the structure of the modules. There are two types of automorphisms which are of special interest to us — spectral flow (σ) [25] and conjugation (γ). The first one brings a module to a new module by changing its conformal dimension. The second one is important for modules which are graded by parameters, which we shall refer to as 'charges', other than the conformal dimension. It maps between modules by negating their charges. We shall illustrate the action of these automorphisms on modules with examples in Chapter 3 and Chapter 4, and see how they facilitate our computation in Chapter 5 and Chapter 6.

Automorphisms serve as an important tool in the study of conformal field theories. It saves us from the tedious task of examining each and all modules in a theory. Instead, one is able to examine just a few modules and create a link with the rest of the modules. This is particularly helpful in computing characters and fusion rules. In later chapters, illustrated with examples, we will see how characters of modules related by automorphisms are equivalent up to a change of variables. Fusion rules are preserved by the automorphisms in a sense that they satisfy

$$\sigma^m(\mathcal{M}) \times \sigma^n(\mathcal{N}) = \sigma^m \circ \sigma^n(\mathcal{M} \times \mathcal{N}) = \sigma^{m+n}(\mathcal{M} \times \mathcal{N}) \tag{2.4.3a}$$

$$\gamma(\mathcal{M}) \times \gamma(\mathcal{N}) = \gamma(\mathcal{M} \times \mathcal{N}). \tag{2.4.3b}$$

The conjugation is usually an order 2 operation on modules: $\gamma^2 = 1$. The problem of computing fusion for all modules in a theory therefore reduces into computing those for a minimum set and then applying automorphisms to the results.

The advantage of having automorphisms not only exists in establishing connections between modules. They also guide us in searching for new modules. This is particularly useful for modules with infinitely many ground states (these are states with the minimal conformal dimension in a module), on which the action of automorphisms preserves the modules on the level of vector spaces, but let the algebra to act differently. It is only when taking all these modules into account that a complete classification of modules for a theory can be provided.

Chapter 3

Bosonic Conformal Field Theories

This chapter illustrates the general features of conformal field theory described in Chapter 2 with examples of bosonic conformal field theories including the free boson, the Virasoro minimal models and the Wess-Zumino-Witten models associated with $\widehat{\mathfrak{sl}}_2$. These theories are the building blocks for the superalgebras we shall construct in Chapter 5 and Chapter 6. The three bosonic theories are presented as a review based on existing literature rather than new results.

3.1 The Free Boson

This section gives a detailed account of the free boson, which is one of the best and simplest illustrations of two-dimensional conformal field theory. The introduction starts in a classical picture, from which we see the necessity of quantising the theory by normal ordering so that the conformal symmetry can make sense in physics. The description for the free boson follows the standard treatments which can be found in [57, 71].

3.1.1 The conformal symmetry of a free boson

A free boson is represented by a free, massless, spinless closed string on a cylinder (or *world-sheet*). A bosonic string is described by the scalar field $\varphi(x,t)$, which has a time component t and a position component x of periodicity of L. The world-sheet is mapped by φ from the cylinder to a space-time, which is 26-dimensional in bosonic string theory. However, this is an unnecessary complication to our calculation, we will assume that the space-time is simply $\mathbb{R}$. The action associated with the free boson is

$$S[\varphi] = \frac{1}{2g} \int_{S^1 \times \mathbb{R}} \varphi \partial_\mu \partial^\mu \varphi \, \mathrm{d}x \, \mathrm{d}t, \tag{3.1.1}$$

where 'g' is a coupling constant. The classical equation of motion of a free boson is derived from the principle of stationary action as

$$\partial_t^2 \varphi = \partial_x^2 \varphi, \tag{3.1.2}$$

which is the familiar wave equation. The free boson can therefore be represented as waves propagating around the cylinder. To map the free boson from a cylinder to a complex plane, we apply Wick rotation by letting $\tau = \mathrm{i}t$, then make a change of variables by letting $z = \mathrm{e}^{2\pi(\tau+\mathrm{i}x)/L}$ and $\bar{z} = \mathrm{e}^{2\pi(\tau-\mathrm{i}x)/L}$. In terms of z and $\bar{z}$, the equation of motion (3.1.2) becomes

$$\partial_z \bar{\partial}_{\bar{z}} \varphi(z, \bar{z}) = 0, \tag{3.1.3}$$

from which we observe that $\partial\varphi(z)$ is holomorphic and $\bar{\partial}\varphi(\bar{z})$ is anti-holomorphic.

We shall only discuss the holomorphic part of the theory for the moment, the procedure for the anti-holomorphic part follows identically and commutes with the other part. The field $\partial\varphi(z)$ is referred to as the generating field of the theory. It has a conformal dimension of 1 and is Fourier expanded as

$$\partial\varphi(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \tag{3.1.4}$$

where the quantum operator a_n satisfies the following commutation relation

$$[a_m, a_n] = m\delta_{m+n,0}1. \tag{3.1.5}$$

The algebra defined by the free boson field and the above commutation relation is known as the *Heisenberg algebra* and is denoted by $\mathcal{H}$. The operators a_n are partitioned into three groups according to their behaviours when acting on the highest-weight state of momentum p, $|p\rangle$:

$$a_n = \begin{cases} a_n \,(n > 0) & \text{annihilation operators}, a_n|p\rangle = 0, \\ a_0 & \text{momentum operator}, a_0|p\rangle = p|p\rangle, \\ a_{-n} \,(n < 0) & \text{creation operators, act freely on } |p\rangle. \end{cases} \tag{3.1.6}$$

In a classical picture, if the reader imagines a free boson as a piece of closed string, its a_0-eigenvalue p is the momentum at which the string's centre of mass moves through the space-time. A creation operator a_{-n} introduces vibrational modes to the string, with the frequency of the vibration increasing with n. For each a_n, we propose an adjoint $a_n^\dagger$ and define it as $a_n^\dagger = a_{-n}$.

We now want to investigate how the conformal symmetry is embedded in the Heisenberg algebra by finding the energy-momentum tensor $T(z)$ in terms of the free boson field. Recall (3.1.1) for the action of a free boson. The change in action as a functional of the field φ under an infinitesimal transformation $\varphi \longrightarrow \varphi' = \varphi + \eta$ is given by

$$S' = S + \int \left(-\frac{1}{g} \partial\varphi\partial\varphi\bar{\partial}\eta - \frac{1}{g} \bar{\partial}\varphi\bar{\partial}\varphi\partial\bar{\eta} \right) \mathrm{d}z\mathrm{d}\bar{z}, \tag{3.1.7}$$

from which we identify the components of the energy-momentum tensor, by comparing with (2.2.8), as

$$T^{\bar{z}z} = -\frac{1}{g}\partial\varphi\partial\varphi, \quad T^{z\bar{z}} = -\frac{1}{g}\bar{\partial}\varphi\bar{\partial}\varphi, \quad T^{zz} = T^{\bar{z}\bar{z}} = 0. \tag{3.1.8}$$

We shall choose $g = \frac{1}{2}$ and define the renormalised energy-momentum fields for the free boson to be

$$T(z) = -\frac{g}{2}T^{\bar{z}z} = \frac{1}{2}\partial\varphi(z)\partial\varphi(z), \quad \overline{T}(\bar{z}) = -\frac{g}{2}T^{z\bar{z}} = \frac{1}{2}\bar{\partial}\varphi(\bar{z})\bar{\partial}\varphi(\bar{z}). \tag{3.1.9}$$

Let us consider the Fourier expansion of the holomorphic field $T(z)$, which can alternatively be obtained by Fourier expanding the two $\partial\varphi(z)$ fields

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \frac{1}{2}\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} a_r a_s z^{-r-s-2}. \tag{3.1.10}$$

Comparing the two expressions, the L_n mode is expressed in terms of the free boson operators a_n as

$$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{n-r}, \tag{3.1.11}$$

which turns out to be a divergent quantity as we will now illustrate by an example. The operator L_0, which measures the energy of a state (up to a scale factor), according to (3.1.11), is given by $L_0 = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{-r}$. We will apply L_0 to the highest-weight state with a momentum of p, and see if the energy eigenvalue is quadratic in terms of p:

$$L_0|p\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{-r}|p\rangle = \frac{1}{2}\left(\sum_{r<0} a_r a_{-r}|p\rangle + a_0^2|p\rangle + \sum_{r>0} a_r a_{-r}|p\rangle\right) \tag{3.1.12a}$$

$$= \frac{p^2}{2}|p\rangle + \sum_{r<0}\left([a_r, a_{-r}] - a_{-r}a_r\right)|p\rangle \tag{3.1.12b}$$

$$= \left(\frac{p^2}{2} + \sum_{r>0} r\right)|p\rangle. \tag{3.1.12c}$$

Unfortunately, the eigenvalue of $|p\rangle$ diverges to infinity as r is summed over all positive integers.

As we have introduced in the previous chapter, for the product of two fields to make sense in a quantised theory, they must be normally ordered. The quantised energy-momentum tensor is therefore given by

$$T(z) = \frac{1}{2}{:}\partial\varphi(z)\partial\varphi(z){:} = \sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\sum_{r\in\mathbb{Z}}{:}a_r a_{n-r}{:}\right) z^{-n-2}, \tag{3.1.13}$$

from which we identify the Virasoro operator as

$$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} {:}a_r a_{n-r}{:}, \tag{3.1.14}$$

where, according to (2.2.19), the ordering of free boson modes is

$${:}a_m a_n{:} = \begin{cases} a_m a_n & \text{if } m < 0, \\ a_n a_m & \text{otherwise.} \end{cases} \tag{3.1.15}$$

Let us now repeat (3.1.12) with this quantised version of L_0:

$$L_0|p\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} {:}a_r a_{-r}{:}|p\rangle = \frac{1}{2}\left(\sum_{r<0} a_r a_{-r}|p\rangle + a_0^2|p\rangle + \sum_{r>0} a_{-r}a_r|p\rangle\right) = \frac{p^2}{2}|p\rangle. \tag{3.1.16a}$$

The L_0-eigenvalue is indeed the energy of the ground state, half of its momentum squared, up to a scale factor. Actually, we can generalise the above argument further to show that the action of any Virasoro operator L_n on any state $\psi(z)$ of a Fock space $\mathcal{F}_p$, is well-defined and finite:

$$L_n|\psi\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} {:}a_r a_{n-r}{:}|\psi\rangle = \frac{1}{2}\left(\sum_{r<0} a_r a_{n-r}|\psi\rangle + \sum_{r\geq 0} a_{n-r}a_r|\psi\rangle\right), \tag{3.1.17}$$

where $|\psi\rangle$ can be a descendant state of a highest-weight state. In the first term of (3.1.17), for a small enough r, $|\psi\rangle$ is annihilated by a_{n-r}, whereas in the second term, $a_r|\psi\rangle$ is 0 given that r is large enough. Both sums therefore contain finitely many terms, given n is finite. Since the action of Virasoro modes on Fock spaces is well-defined, we propose that quantum state spaces should be constructed from Fock spaces. Note that if $L_0|\psi\rangle = h|\psi\rangle$, then both $a_{-n}|\psi\rangle$ and $L_{-n}|\psi\rangle$ are eigenvectors of L_0 with eigenvalue $h+n$. We also remark that the free boson modes satisfy ${:}a_m a_n{:} = {:}a_n a_m{:}$. According to (3.1.5), when $m \neq n$, a_m and a_n commute, so normal ordering becomes redundant. And it is straight-forward to check that ${:}a_n a_{-n}{:} = {:}a_{-n}a_n{:}$. Therefore, ${:}a_m a_n{:} = {:}a_n a_m{:}$ for all $m, n \in \mathbb{Z}$.

The OPE of two free boson fields can be derived following from the commutation relation stated in (3.1.5). Consider the expansion of the following OPE while assuming $|z| > |w|$:

$$\mathcal{R}\Big\{\partial\varphi(z)\partial\varphi(w)\Big\} = \sum_{r,s\in\mathbb{Z}} a_r a_s z^{-r-1}w^{-s-1} \tag{3.1.18}$$

The product of modes is normally ordered if $r < 0$ and equals to ${:}a_r a_s{:} + [a_r, a_s]$ if $r \geq 0$. We can therefore write (3.1.18) as

$$\mathcal{R}\Big\{\partial\varphi(z)\partial\varphi(w)\Big\} = \sum_{r,s\in\mathbb{Z}} {:}a_r a_s{:}z^{-r-1}w^{-s-1} + \frac{1}{z^2}\sum_{r=1}^{\infty} r\left(\frac{w}{z}\right)^{r-1}. \tag{3.1.19}$$

Since $|z| > |w|$, the sum in the last term is the derivative of a convergent geometric series of ratio w/z, whilst the first term is simply the normally-ordered product of the two free boson fields. The radially-ordered product is now simplified as

$$\mathcal{R}\Big\{\partial\varphi(z)\partial\varphi(w)\Big\} = {:}\partial\varphi(z)\partial\varphi(w){:} + \frac{1}{(z-w)^2}. \tag{3.1.20}$$

In the other case where we assume $|z| < |w|$, a similar calculation yields

$$\mathcal{R}\Big\{\partial\varphi(z)\partial\varphi(w)\Big\} = {:}\partial\varphi(w)\partial\varphi(z){:} + \frac{1}{(w-z)^2}. \tag{3.1.21}$$

As previously mentioned, ${:}a_m a_n{:} = {:}a_n a_m{:}$ which leads to ${:}\partial\varphi(z)\partial\varphi(w){:} = {:}\partial\varphi(w)\partial\varphi(z){:}$. Therefore, identical results follow from both cases:

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{1}{(z-w)^2}. \tag{3.1.22}$$

The OPE of $T(z)$ with $\partial\varphi(w)$ can be calculated from Wick's theorem as

$$T(z)\partial\varphi(w) \sim \frac{1}{2}{:}\partial\varphi(z)\partial\varphi(z){:}\partial\varphi(w) + \frac{1}{2}{:}\partial\varphi(z)\partial\varphi(z){:}\partial\varphi(w) \sim \frac{\partial\varphi(z)}{(z-w)^2}. \tag{3.1.23}$$

Taylor expanding $\partial\varphi(z)$ at $z = w$ as $\partial\varphi(z) = \partial\varphi(w) + (z-w)\partial^2\varphi(w) + \cdots$ gives

$$T(z)\partial\varphi(w) \sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}. \tag{3.1.24}$$

In the previous chapter, we stated that a field is Virasoro-primary if and only if its OPE with $T(z)$ satisfies (2.2.29). The OPE (3.1.24) shows that the free boson field is indeed primary with a conformal dimension of 1.

As a final task of the section, we have to prove the free boson is indeed a conformal field theory with its conformal symmetry defined by (3.1.12a). One way of showing this is to derive the OPE between two energy-momentum tensors, this can be calculated by Wick's theorem as

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{3.1.25}$$

which is consistent with (2.2.28) with a central charge 1. Alternatively, at the level of modes, one can show that the L_n defined in (3.1.14) satisfy the conformal algebra (2.1.9). The proof for this is tedious but straight-forward, we will present a brief outline of the procedure instead of giving a detailed account. We first compute the commutator between

L_m and a_n as

$$[L_m, a_n] = [\frac{1}{2}\sum_{k\in\mathbb{Z}} :a_k a_{m-k}:, a_m] = \frac{1}{2}\sum_{k\le -1}[a_k a_{m-k}, a_n] + \frac{1}{2}\sum_{k\ge 0}[a_{m-k}a_k, a_n] = -n a_{m+n}. \tag{3.1.26}$$

The commutator of two Virasoro modes follows from this and is given by

$$[L_m, L_n] = \frac{1}{2}\sum_{r\in\mathbb{Z}}[:a_r a_{m-r}:, L_n] = \frac{1}{2}\sum_{r\le -1}(m-r)a_r a_{m-r+n} + \frac{1}{2}\sum_{r\ge 0} a_{m-r+n}a_r \\ + \frac{1}{2}\sum_{r\le 1} r a_{n+r}a_{m-r} + \frac{1}{2}\sum_{r\ge 0} r a_{m-r}a_{n+r}. \tag{3.1.27}$$

When $m \neq n$, the modes in each term of (3.1.27) commute and the four terms combine into two as

$$[L_m, L_n] = \frac{1}{2}\sum_{r\in\mathbb{Z}}(m-r)a_{m-r+n}a_r + \frac{1}{2}\sum_{r\in\mathbb{Z}}(n-r)a_{m-(n-r)}a_{n+(r-n)} \\ = (m-n)\frac{1}{2}\sum_{r\in\mathbb{Z}} :a_{m+n-r}a_r: = (m-n)L_{m+n} \tag{3.1.28}$$

In the case where $m = n$, (3.1.27) reduces to

$$[L_m, L_{-m}] = 2mL_0 + \frac{1}{2}\sum_{r=0}^{\infty} r(m-n). \tag{3.1.29}$$

Combining (3.1.28) and (3.1.29) leads to

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}, \tag{3.1.30}$$

which is the Virasoro algebra with a central charge of 1.

The free boson theory remains one of the most useful models in conformal field theory. Its attractive features such as being exactly solvable serve as a fundamental tool in string theory. As we will see later in the thesis, it is the fundamental building block of many more complicated theories.

3.1.2 Characters and fusion rules

A Verma or highest-weight module of the Heisenberg algebra is known as a *Fock space*. It is induced by letting all creation operators a_{-n} ($n \ge 1$) act freely on a highest-weight state $|p\rangle$, where $p \in \mathbb{R}$ is the momentum (a_0-eigenvalue) of the state. A standard basis for the Fock space, denoted by $\mathcal{F}_p$, is given by

$$\left\{ \cdots a_{-3}^{j_3} a_{-2}^{j_2} a_{-1}^{j_1} |p\rangle \,\middle|\, j_n \in \mathbb{Z}_{\geq 0},\ \sum_n j_n < \infty \text{ and } p \in \mathbb{R} \right\}. \tag{3.1.31}$$

The number of creation operators in each basis state is required to be finite. A Fock space is an irreducible Verma module, which makes it automatically highest-weight. The complete character of a Fock space not only records the conformal dimensions of the states but also their momentum (a_0-eigenvalues). Following from (2.3.7), the character of $\mathcal{F}_p$ is given by

$$\mathrm{Ch}[\mathcal{F}_p](\mathsf{x};\mathsf{q}) = \mathrm{Tr}_{\mathcal{F}_p} \mathsf{x}^{a_0} \mathsf{q}^{L_0 - \frac{\mathsf{c}}{24}} = \mathsf{x}^p \mathsf{q}^{\frac{p^2}{2} - \frac{1}{24}} \prod_{i=1}^{\infty} \frac{1}{(1-\mathsf{q}^i)} = \frac{\mathsf{x}^p \mathsf{q}^{\frac{p^2}{2}}}{\eta(\mathsf{q})}. \tag{3.1.32}$$

Notice that all Fock spaces are structurally identical and their characters only differ by the factor $\mathsf{x}^p \mathsf{q}^{p^2/2}$. This can be explained by the action of automorphisms, spectral flow (σ_{fb}) and conjugation (γ_{fb}), which act on the Heisenberg algebra as

$$\begin{aligned} \gamma_{\mathrm{fb}}(a_n) &= -a_n, \quad \gamma_{\mathrm{fb}}(L_n^{\mathrm{fb}}) = L_n^{\mathrm{fb}} \\ \sigma_{\mathrm{fb}}^{\ell}(a_n) &= a_n - \ell\delta_{n,0}, \quad \sigma_{\mathrm{fb}}^{\ell}(L_n^{\mathrm{fb}}) = L_n^{\mathrm{fb}} - \ell a_n + \tfrac{1}{2}\ell^2 \delta_{n,0}. \end{aligned} \tag{3.1.33}$$

These automorphisms leave the identity operator invariant. Using the invertibility relation (2.3.12), one can induce these to the level of modules, they act on Fock spaces $\mathcal{F}_p$ as

$$\gamma_{\mathrm{fb}}(\mathcal{F}_p) \cong \mathcal{F}_{-p}, \quad \sigma_{\mathrm{fb}}^{\ell}(\mathcal{F}_p) \cong \mathcal{F}_{p+\ell}. \tag{3.1.34}$$

The fusion rules of the Fock spaces can be easily calculated from a generalisation of the Verlinde formula (2.3.12), with the discrete sum replaced by a continuous integral over p. The result is well known as

$$\mathcal{F}_p \times \mathcal{F}_{p'} = \mathcal{F}_{p+p'}. \tag{3.1.35}$$

We remark that fusion conserves the indices of the Fock spaces, which in terms of physics is simply the conservation of momentum.

3.2 Virasoro Minimal Models

In 1984, Belavin, Polyakov and Zamolodchikov studied a special class of rational conformal field theories (RCFTs) with central charge $\mathsf{c} < 1$. They called them the *'minimal models'* [72]. These are some of the most fundamental RCFTs based on the Virasoro algebra.

We have mentioned, in section 2.2.1, the existence of singular vectors in a Verma module of the Virasoro algebra. Such zero-norm vectors (and their descendents) are orthogonal to the whole Verma module and generate their own Verma submodules of the original Verma module. By quotienting out the submodules generated by the singular vectors, one may construct an irreducible representation of the Virasoro algebra. Such representations are the building blocks of minimal models.

Virasoro minimal models, denoted by $\mathsf{M}(p,q)$, are characterised by two positive integers p and q (p, $q \geq 2$) with no non-trivial common divisors. The Virasoro algebra associated with $\mathsf{M}(p,q)$ has central charge

$$\mathsf{c} = 1 - \frac{6(p-q)^2}{pq}. \tag{3.2.1}$$

The irreducible highest-weight modules of $\mathsf{M}(p,q)$ are denoted by $\mathcal{V}_{r,s}$, where r and s are positive integers with $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$. These modules are characterised by the conformal dimensions of the their highest-weight states, which are given by

$$\Delta^{\text{Vir}}_{r,s} = \frac{(qr-ps)^2 - (p-q)^2}{4pq}. \tag{3.2.2}$$

It is easy to check that the conformal dimension satisfies the symmetry

$$\Delta^{\text{Vir}}_{r,s} = \Delta^{\text{Vir}}_{p-r,q-s}, \tag{3.2.3}$$

which says that the total number of distinct modules is $\frac{1}{2}(p-1)(q-1)$.

An application of minimal models is the well-known *Ising model* which explains the statistical dynamics of ferromagnetism. The Ising model consists of a square lattice with a classical spin located at each lattice site. Each spin takes value of either 1 or -1. A spin at position i is denoted by σ_i, it interacts with its four nearest neighbours at j with an interaction strength J. The total energy of the lattice is given by summing over the interaction energies of all lattice points, this is given by

$$E = -J \sum_{<i,j>} \sigma_i \sigma_j, \tag{3.2.4}$$

where $< i,j >$ means summing over nearest neighbours. We choose J to be positive so that it is energetically favourable to have neighbouring spins aligned in the same direction ($\sigma_i = \sigma_j$). The Ising model undergoes a second-order phase transition at a critical temperature known as the Curie temperature (T_c), above which spontaneous magnetisation is lost. At T_c, the model displays a very interesting property — it is statistically invariant under rescaling. This encourages us to study the model with conformal field theory, which describes the scaling limit of lattice models.

The Ising model is mathematically described by the Virasoro minimal model $\mathsf{M}(3,4)$. Its three constituent observables, the unit (1), the spin (σ) and the energy (ϵ) are represented

$\Delta^{\mathrm{Vir}}_{r,s}$	s = 1	2	3
r = 1	0	$\frac{1}{16}$	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{1}{16}$	0

Table 3.1: Kac table for $\mathsf{M}(3,4)$ with the symmetry $\Delta^{\mathrm{Vir}}_{r,s} = \Delta^{\mathrm{Vir}}_{3-r,4-s}$.

by the three primary fields of conformal dimensions $\Delta^{\mathrm{Vir}}_{1,1} = 0$, $\Delta^{\mathrm{Vir}}_{1,2} = \frac{1}{16}$ and $\Delta^{\mathrm{Vir}}_{2,1} = \frac{1}{2}$, respectively. It is conventional to present modules of a minimal model in a Kac table, in which we arrange modules according to their r- and s-labels. This, for $\mathsf{M}(3,4)$, is displayed in Table 3.1. With this setup, one can compute the exact form of the singular vectors by requiring their corresponding fields and $T(z)$ to satisfy the OPE as given in (2.2.29). One can now construct solvable PDEs for correlations functions using these singular vectors. It is even possible to compare the correlation functions obtained through this pure algebraic method with experimental results [73–75]. For example, the 2-point correlation function for spins is computed to be

$$\langle 0|\sigma(z)\sigma(w)|0\rangle = \frac{1}{(z-w)^{1/8}}. \tag{3.2.5}$$

Combining this with a similar the 2-point correlation function from the anti-holomorphic sector gives

$$\langle 0|\sigma(z,\bar{z})\sigma(w,\bar{w})|0\rangle = \frac{1}{|z-w|^{1/4}}. \tag{3.2.6}$$

The exponent of the bulk correlation function predicts that the spin interaction between two lattice points decreases with a power of $\frac{1}{4}$ as they move away from each other.

A module of a minimal model is said to be *unitary* if it contains only positive-norm states. This imposes constraints on the conformal dimension of the highest-weight state Δ and the central charge c. These constraints are found by first considering the so-called *Gram matrix*, whose entries are inner products between all basis states. We shall refer to the negative sum of indices of the Virasoro modes in a state as its *grade*. At grade ℓ, the determinant of the Gram matrix $M^{(\ell)}$, known as the *Kac determinant* is given by [76–78]

$$\det M^{(\ell)} = \alpha_\ell \prod_{\substack{r,s\geq 1\\ rs\leq \ell}} \left[\Delta - \Delta_{r,s}(\mathsf{c})\right]^{\mathrm{p}(\ell-rs)}, \tag{3.2.7}$$

where $\mathrm{p}(\ell-rs)$ is the number of partitions of the integer $\ell-rs$, and α_ℓ is a positive constant given by

$$\alpha_\ell = \prod_{\substack{r,s\geq 1\\ rs\leq \ell}} [(2r)^s s!]^{m(r,s)}, \quad \text{with} \quad m(r,s) = \mathrm{p}(\ell-rs) - \mathrm{p}(\ell-r(s+1)). \tag{3.2.8}$$

A module is unitary if and only if its Kac determinant is positive at all grades. Following from this, one can show that

- All modules with negative central charge or conformal dimension are non-unitary.
- All irreducible highest-weight modules with $\mathsf{c} \geq 1$ and $\Delta \geq 0$ are unitary.
- For $0 < \mathsf{c} < 1$ and $\Delta > 0$, all modules are non-unitary except for the following discrete set [79]

$$\mathsf{c}(m) = 1 - \frac{6}{m(m+1)} \tag{3.2.9a}$$

$$\Delta_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, \tag{3.2.9b}$$

where $m \geq 2$ is an integer, $1 \leq r \leq m-1$ and $1 \leq s \leq m$. Formulae (3.2.9a) and (3.2.9b) give the central charge and the conformal dimensions of the highest-weight modules of the minimal model $\mathsf{M}(m, m+1)$, respectively.

We say a minimal model is unitary if all its modules are unitary. Recall (3.1.19) for conformal dimensions characterising the modules of $\mathsf{M}(p,q)$. Bézout's lemma states that there exists a pair of integers r_0 and s_0 satisfying $1 \leq r_0 \leq p-1$ and $1 \leq s_0 \leq q-1$ such that

$$pr_0 - qs_0 = 1. \tag{3.2.10}$$

There is therefore a module in the Kac table of $\mathsf{M}(p,q)$ with conformal dimension

$$\Delta_{r_0,s_0} = \frac{1-(p-q)^2}{4pq}, \tag{3.2.11}$$

which can only be non-negative if $|p-q| = 1$. This is the condition for a minimal model to be unitary. The integers r_0 and s_0, in this case, can be solved from (3.2.10) to both be 1, with $\Delta_{1,1} = 0$. This module is generated from the true vacuum $|0\rangle$, which corresponds to the identity field.

An irreducible module of a minimal model contains 'fewer' states than the generic Verma modules. Its character therefore does not take the simple form of (2.3.9). One needs to subtract the characters of the submodules generated by its singular vectors at all grades. It follows from the Kac determinant that singular vectors exist at grade $\ell = rs$ if $\Delta_{r,s}$ equals to the conformal dimension of the highest-weight state of the Verma module. The process of finding grades of singular vectors can be assisted by the Kac symmetries stated in (3.2.3). However, the task of quotienting out the Verma modules at these levels is not so straight-forward. The braiding structure of a Verma module can be quite complicated. Reading irreducible characters from it requires repetitively subtracting and adding submodules for all grades. Following this procedure, the character formula [80,81] for an irreducible Virasoro module of highest-weight $\Delta_{r,s}$ in $\mathsf{M}(p,q)$ in found to be

$$\begin{aligned}\chi^{(p,q)}_{r,s}(\mathsf{q}) = \mathrm{Ch}[\mathcal{V}_{r,s}](\mathsf{q}) = \mathrm{Tr}_{\mathcal{V}_{r,s}}\mathsf{q}^{L_0^{\mathrm{Vir}}-\mathsf{c}^{\mathrm{Vir}}/24} \\ = \frac{1}{\eta(\mathsf{q})}\sum_{n\in\mathbb{Z}}\left[\mathsf{q}^{(2pqn+qr-ps)^2/4pq} - \mathsf{q}^{(2pqn+qr+ps)^2/4pq}\right].\end{aligned} \tag{3.2.12}$$

The base case for the fusion rules can be computed by constructing PDEs for 3-point correlation functions using singular vectors at low levels. This provides a constraint on the conformal dimensions of the three primary fields, and therefore the fusion rules. The generating fusions are

$$\mathcal{V}_{1,2}\times\mathcal{V}_{r,s} = \mathcal{V}_{r,s-1}\oplus\mathcal{V}_{r,s+1}, \quad \mathcal{V}_{2,1}\times\mathcal{V}_{r,s} = \mathcal{V}_{r-1,s}\oplus\mathcal{V}_{r+1,s}. \tag{3.2.13}$$

One can induce these base cases to obtain fusion between any two modules. The result is further truncated by the commutativity of the algebra, which leads to the following fusion rule [72, 82] for $\mathsf{M}(p,q)$

$$\mathcal{V}_{r,s}\times\mathcal{V}_{r',s'} = \bigoplus_{r''=1}^{p-1}\bigoplus_{s''=1}^{q-1}\mathsf{N}^{[p,q]\,(r'',s'')}_{(r,s),(r',s')}\mathcal{V}_{r'',s''}, \tag{3.2.14}$$

where the Virasoro fusion coefficient $\mathsf{N}^{[p,q]\,(r'',s'')}_{(r,s),(r',s')} = \mathsf{N}^{[p]\,r''}_{r,r'}\mathsf{N}^{[q]\,s''}_{s,s'}$ and

$$\mathsf{N}^{[t]\,k}_{i,j} = \begin{cases} 1, & \text{if } |i-j|+1\le k\le \min\{i+j-1, 2t-i-j-1\} \text{ and } i+j+k \text{ is odd,} \\ 0, & \text{otherwise.}\end{cases} \tag{3.2.15}$$

In particular, the Virasoro coefficient satisfies the following identities

$$\mathsf{N}^{[t]\,k}_{i,j} = \mathsf{N}^{[t]\,k}_{j,i}, \quad \mathsf{N}^{[t]\,k}_{1,j} = \delta_{j,k}, \quad \mathsf{N}^{[t]\,k}_{i,t-1} = \delta_{k,t-i}. \tag{3.2.16}$$

We note that $\mathcal{V}_{1,1} = \mathcal{V}_{p-1,q-1}$ is the vacuum module satisfying $\mathcal{V}_{1,1}\times\mathcal{V}_{r,s} = \mathcal{V}_{p-1,q-1}\times\mathcal{V}_{r,s} = \mathcal{V}_{r,s}$. With p and q both greater than 2, $\mathcal{V}_{p-1,1} = \mathcal{V}_{1,q-1}$ is an example of *a simple current*. A simple current is a special type of primary field (or its corresponding highest-weight module) whose OPE with any other primary field gives just one (in the regular and singular terms) primary field. The notion of simple currents is another important tool in CFT. As we will see in later chapters, simple currents are often used for extending an algebra and helping with the study of a new algebra. In the case of $\mathsf{M}(p,q)$ with $p,q>2$, we have an order-2 simple current $\mathcal{V}_{p-1,1}$ satisfying $\mathcal{V}_{p-1,1}\times\mathcal{V}_{p-1,1} = \mathcal{V}_{1,1}$.

3.3 Affine $\widehat{\mathfrak{sl}}_2$ and Wess-Zumino-Witten Models

3.3.1 The affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$

An affine algebra $\widehat{\mathfrak{g}}$ is constructed from a finite dimensional reductive algebra $\mathfrak{g}$ by tensoring it with Laurent polynomials and adding a central element $\hat{\mathsf{k}}$ as follows:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \hat{\mathsf{k}}. \tag{3.3.1}$$

Besides the central element $\hat{\mathsf{k}}$, the generators of $\widehat{\mathfrak{g}}$ take the form $J^a \otimes t^n$, which are often written as J^a_n, where J^a are generators of the finite algebra $\mathfrak{g}$. These affine generators satisfy the commutation relations

$$[J^a_m, J^b_n] = [J_a, J_b] + m\kappa(J^a, J^b)\delta_{m+n,0}\hat{\mathsf{k}} \quad \text{and} \quad [J^a_n, \hat{\mathsf{k}}] = 0, \tag{3.3.2}$$

where $\kappa(J_a, J_b)$ is the Killing form of J_a with J_b, which is guaranteed by the reductive property of $\mathfrak{g}$ to be non-degenerate when $\mathfrak{g}$ is semisimple. The Heisenberg algebra, for example, is an affine algebra extended from $\mathfrak{u}(1)$, whose level is assigned as 1.

Following from this generic setup, the affine algebra $\widehat{\mathfrak{sl}}_2$ has generators $\{e_n, h_n, f_n \,|\, n \in \mathbb{Z}\}$ satisfying

$$\begin{gathered}[h_m, e_n] = 2e_{m+n}, \quad [h_m, h_n] = 2m\delta_{m+n,0}\hat{\mathsf{k}}, \quad [e_m, f_n] = h_{m+n} + m\delta_{m+n,0}\hat{\mathsf{k}}, \\ [h_m, f_n] = -2f_{m+n}, \quad [e_m, e_n] = [f_m, f_n] = 0,\end{gathered} \tag{3.3.3}$$

because the non-vanishing values of the Killing form are

$$\kappa(h,h) = 2, \quad \kappa(e,f) = \kappa(f,e) = 1. \tag{3.3.4}$$

With regard to conformal field theory, these generators are the Fourier modes of conformal fields $e(z)$, $f(z)$ and $h(z)$, which are expanded as

$$j(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1}, \tag{3.3.5}$$

where $j \in \{e, h, f\}$. All three fields are bosonic and have conformal dimension 1, they are related to the states $j_{-1}|0\rangle$ by the state-field correspondence. The OPEs of these fields can be computed from commutators (3.3.3) using contour integrals. The general form is found to be

$$J^a(z)J^b(w) \sim \frac{\kappa(J^a, J^b)\mathsf{k}}{(z-w)^2} + \frac{[J^a, J^b](w)}{z-w}, \tag{3.3.6}$$

where k is the eigenvalue of $\hat{\mathsf{k}}$ and is known as the *level* of the affine algebra. It follows from (3.3.6) that the OPEs of the generating fields in $\widehat{\mathfrak{sl}}_2$ are given by

$$\begin{gathered} h(z)e(w) \sim \frac{2e(w)}{z-w}, \qquad h(z)h(w) \sim \frac{2\mathsf{k}}{(z-w)^2}, \qquad h(z)f(w) \sim \frac{-2f(w)}{z-w}, \\ e(z)f(w) \sim \frac{\mathsf{k}}{(z-w)^2} + \frac{h(w)}{z-w}, \qquad e(z)e(w) \sim f(z)f(w) \sim 0. \end{gathered} \tag{3.3.7}$$

The process of endowing an affine algebra with conformal symmetry is known as the *Sugawara construction*. It provides an expression for the energy-momentum tensor $T(z)$ in terms of the generating fields of the algebra. This expression is found by writing $T(z)$ as a linear combination of all fields with conformal dimension 2, then constraining the coefficients so that (2.2.28) is satisfied. Note that all fields appear in this expression must be primary fields of conformal dimension 1, $\partial h(z)$ for example, is not allowed. In the case of $\widehat{\mathfrak{sl}}_2$, the Sugawara construction gives

$$T(z) = \sum_{n\in\mathbb{Z}} L_n^{\mathrm{sl}} z^{-n-2} = \frac{1}{2(\mathsf{k}+2)}\left[\frac{1}{2}{:}hh{:}(z) + {:}ef{:}(z) + {:}fe{:}(z)\right]. \tag{3.3.8}$$

The conformal dimension of a state can now be measured as the L_0-eigenvalue

$$L_0 w = \Delta w. \tag{3.3.9}$$

The commutators of the L_n can be verified to be those of $\mathfrak{Vir}$ with the central charge

$$\mathsf{c}^{\mathrm{sl}} = \frac{3\mathsf{k}}{\mathsf{k}+2}. \tag{3.3.10}$$

To construct a highest-weight module for $\widehat{\mathfrak{sl}}_2$, one performs a triangular decomposition of $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ into three subalgebras

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \oplus \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_-, \tag{3.3.11}$$

where $\widehat{\mathfrak{g}}_+$ and $\widehat{\mathfrak{g}}_-$ contain the positive (j_n with $n > 0$, $j \in \{e,h,f\}$) and negative modes (j_n with $n < 0$) of $\widehat{\mathfrak{sl}}_2$, respectively, whereas $\widehat{\mathfrak{g}}_0$ is spanned by $\{\hat{\mathsf{k}}, h_0, f_0, e_0\}$. A highest-weight state in $\widehat{\mathfrak{sl}}_2$ is defined as a state which is annihilated by the positive modes plus e_0. It is not hard to verify from (3.3.8) that such states are automatically annihilated by the positive modes L_n. An $\widehat{\mathfrak{sl}}_2$-primary is therefore also a Virasoro primary. We shall start constructing a highest-weight $\widehat{\mathfrak{g}}_0$-module by requiring that the highest-weight state $w = |\lambda^{\mathrm{sl}}, \Delta, \mathsf{k}\rangle$ be annihilated by e_0, and

$$h_0 w = \lambda^{\mathrm{sl}} w, \quad \hat{\mathsf{k}} w = \mathsf{k} w. \tag{3.3.12}$$

When f_0 is allowed to act freely on w, the $\widehat{\mathfrak{g}}_0$-module is a Verma module. If the $\widehat{\mathfrak{g}}_0$-module

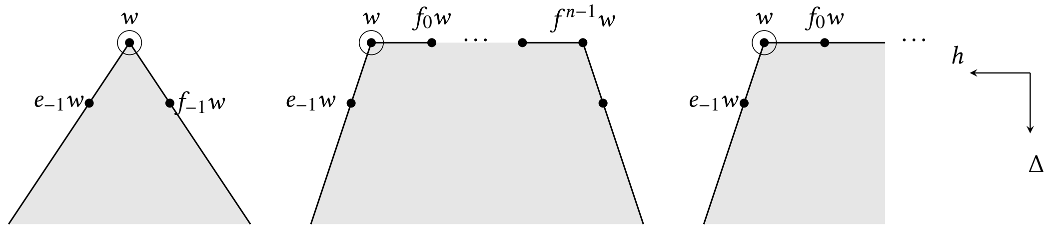

Figure 3.1: Generic level k highest-weight modules for $\widehat{\mathfrak{sl}}_2$. The charges and conformal dimensions of the states inside a module increases rightward and downward, respectively. The highest-weight state of each module is circled and labelled w. From left to right, the modules are induced from a one-dimensional, a finite-dimensional and a Verma $\widehat{\mathfrak{g}}_0$-module. The first module is known as the universal vacuum module, denoted by $\widehat{\mathfrak{sl}}(2)_{\mathsf{k}}$.

has a singular vector of the form $f_0^n w$, which generates a submodule, the quotient of the $\widehat{\mathfrak{g}}_0$-module obtained by setting the submodule to 0 is finite dimensional. The charges of the states of the quotient module are then bounded above by λ^{sl}. To induce this to an affine module, we let $\widehat{\mathfrak{g}}_+$ act as zero whereas $\widehat{\mathfrak{g}}_-$ acts freely. The resulting $\widehat{\mathfrak{g}}$-module is a highest-weight module characterised by the highest-weight state w.

In the special case where the $\widehat{\mathfrak{g}}_0$-module is one-dimensional, that is, if we require $f_0 w = 0$ with $w = |0, 0, \mathsf{k}\rangle$, the corresponding affine module is known as the *universal vacuum module*. This module carries the structure of an affine vertex algebra. The PBW basis is

$$\left\{ (f_{-n}^{\ell_n} h_{-n}^{j_n} e_{-n}^{k_n}) \cdots (f_{-2}^{\ell_2} h_{-2}^{j_2} e_{-2}^{k_2})(f_{-1}^{\ell_1} h_{-1}^{j_1} e_{-1}^{k_1}) w \;\middle|\; \ell_i, j_i, k_i \in \mathbb{Z}_{\geq 0}, \sum_i \ell_i + j_i + k_i < \infty \right\}. \quad (3.3.13)$$

We shall denote this Verma module at level k by $\widehat{\mathfrak{sl}}(2)_{\mathsf{k}}$. The concepts of highest-weight and universal vacuum modules are illustrated in Figure 3.1.

The generators of $\widehat{\mathfrak{sl}}_2$ admit a number of automorphisms including spectral flow $\sigma_{\text{sl}}^{\ell}$, where $\ell \in \mathbb{Z}$, and conjugation γ_{sl}. Both automorphisms preserve the level $\hat{\mathsf{k}}$ and they act on the other generators by

$$\begin{aligned} \sigma_{\text{sl}}^{\ell}(e_n) = e_{n-\ell}, \quad \sigma_{\text{sl}}^{\ell}(f_n) = f_{n+\ell}, \quad \sigma_{\text{sl}}^{\ell}(h_n) = h_n + \delta_{n,0}\ell\hat{\mathsf{k}}, \\ \gamma_{\text{sl}}(e_n) = -f_n, \quad \gamma_{\text{sl}}(f_n) = -e_n, \quad \gamma_{\text{sl}}(h_n) = -h_n. \end{aligned} \quad (3.3.14)$$

The zero mode L_0^{sl} of the energy momentum tensor is preserved by the action of conjugation but changed by spectral flow according to

$$\sigma_{\text{sl}}^{\ell}\left(L_0^{\text{sl}}\right) = L_0^{\text{sl}} - \frac{1}{2}\ell h_0 + \frac{1}{4}\ell^2\hat{\mathsf{k}}. \quad (3.3.15)$$

3.3.2 Unitary Wess-Zumino-Witten models

One can deduce by studying its structure that the universal vacuum module $\widehat{\mathfrak{sl}}(2)_\mathsf{k}$ is reducible when the level k satisfies

$$\mathsf{k}+2=\frac{u}{v}, \quad u\in\mathbb{Z}_{\geq 2}, \quad v\in\mathbb{Z}_{\geq 1}, \quad \gcd\{u,v\}=1. \tag{3.3.16}$$

These levels are called *admissible* [83, 84], and the universal vacuum module $\widehat{\mathfrak{sl}}(2)_\mathsf{k}$ has maximal proper submodule generated by a singular vector. Analogous to Virasoro minimal models, we define the minimal models of $\widehat{\mathfrak{sl}}_2$, known as the Wess-Zumino-Witten models and denoted by $A_1(u,v)$, to be the irreducible quotients of $\widehat{\mathfrak{sl}}(2)_\mathsf{k}$ by these submodules at admissible levels. Now writing the central charge of the minimal model in terms of u and v, (3.3.10) becomes

$$\mathsf{c}^{\text{sl}}=3-\frac{6v}{u}. \tag{3.3.17}$$

A minimal model is unitary if all its modules are unitary. Such a minimal model has $v=1$, and its level $\mathsf{k}=u-2$ is therefore a non-negative integer. In this case, $A_1(u,1)$ is referred to as *rational*, which means it has a finite number ($u-1$ actually) of irreducible modules $\mathcal{L}_{r,0}$, where $1\leq r\leq u-1$. These modules are integrable [70] highest-weight modules, whose highest-weight states have h_0-charges and conformal dimensions

$$\lambda^{\text{sl}}_{r,0}=r-1 \quad \text{and} \quad \Delta^{\text{sl}}_{r,0}=\frac{r^2-1}{4u}, \tag{3.3.18}$$

respectively.

Recall (2.4.1) and (2.3.12) for the action of automorphisms on modules. The highest-weight modules of $A_1(u,v)$ are self-conjugate and are acted on by spectral flows as

$$\sigma_{\text{sl}}\left(\mathcal{L}_{r,0}\right)=\mathcal{L}_{u-r,0}. \tag{3.3.19}$$

The characters of these modules are given by

$$\text{Ch}[\mathcal{L}_{r,0}](\mathsf{z};\mathsf{q})=\text{Tr}_{\mathcal{L}_{r,0}}\mathsf{z}^{h_0}\mathsf{q}^{L^{\text{sl}}_0-\mathsf{c}^{\text{sl}}/24}=\frac{\mathsf{q}^{\Delta^{\text{sl}}_{r,0}-\mathsf{c}^{\text{sl}}/24+1/8}}{\mathrm{i}\vartheta_1(\mathsf{z}^2;\mathsf{q})}\sum_{j\in\mathbb{Z}}\left(\mathsf{z}^{2uj+r}-\mathsf{z}^{-2uj-r}\right)\mathsf{q}^{j(uj+r)}, \tag{3.3.20}$$

where we introduce alternative variables ζ and τ defined by $\mathsf{z}=e^{2\pi\mathrm{i}\zeta}$ and $\mathsf{q}=e^{2\pi\mathrm{i}\tau}$ for the study of modular properties. ϑ_1 denotes a Jacobi theta function. Here is our convention for Jacobi theta functions

$$\vartheta_1(\mathsf{z};\mathsf{q})=-\mathrm{i}\sum_{n\in\mathbb{Z}}(-1)^n\mathsf{z}^{n+1/2}\mathsf{q}^{(n+1/2)^2/2}=-\mathrm{i}\mathsf{z}^{1/2}\mathsf{q}^{1/8}\prod_{i=1}^{\infty}(1-\mathsf{z}\mathsf{q}^i)(1-\mathsf{q}^i)(1-\mathsf{z}^{-1}\mathsf{q}^{i-1})$$

$$\vartheta_2(\mathsf{z};\mathsf{q}) = \sum_{n\in\mathbb{Z}} \mathsf{z}^{n+1/2}\mathsf{q}^{(n+1/2)^2/2} = \mathsf{z}^{1/2}\mathsf{q}^{1/8}\prod_{i=1}^{\infty}(1+\mathsf{z}\mathsf{q}^i)(1-\mathsf{q}^i)(1+\mathsf{z}^{-1}\mathsf{q}^{i-1})$$
$$\vartheta_3(\mathsf{z};\mathsf{q}) = \sum_{n\in\mathbb{Z}} \mathsf{z}^{n}\mathsf{q}^{n^2/2} = \prod_{i=1}^{\infty}(1+\mathsf{z}\mathsf{q}^{i-1/2})(1-\mathsf{q}^i)(1+\mathsf{z}^{-1}\mathsf{q}^{i-1/2})$$
$$\vartheta_4(\mathsf{z};\mathsf{q}) = \sum_{n\in\mathbb{Z}} (-1)^n\mathsf{z}^{n}\mathsf{q}^{n^2/2} = \prod_{i=1}^{\infty}(1-\mathsf{z}\mathsf{q}^{i-1/2})(1-\mathsf{q}^i)(1-\mathsf{z}^{-1}\mathsf{q}^{i-1/2}).$$

With respect to the characters (3.3.20), the action of modular group element $\mathcal{S}$ is defined as the following coordinate transformation

$$\mathcal{S}(\zeta;\tau) = \left(\frac{\zeta}{\tau};\frac{-1}{\tau}\right). \tag{3.3.22}$$

The $\mathcal{S}$-modular property of the characters, with respect to this transformation, takes the form

$$\mathrm{Ch}[\mathcal{L}_{r,0}]\left(\frac{\zeta}{\tau};\frac{-1}{\tau}\right) = \mathsf{e}^{-\mathsf{i}\zeta^2/2\tau}\sum_{r'}\mathcal{S}_{rr'}\mathrm{Ch}[\mathcal{L}_{r',0}](\zeta;\tau), \tag{3.3.23}$$

where the entries of the $\mathcal{S}$-matrix $\mathcal{S}_{rr'}$ are given by

$$\mathcal{S}_{rr'} = \sqrt{\frac{2}{\mathsf{k}+2}}\sin\left(\frac{\pi(r-1)(r'-1)}{\mathsf{k}+2}\right), \tag{3.3.24}$$

where k is again the level of the minimal model. Note that from (3.3.20) to (3.3.23), we follow the standard abuse of notation and write the characters as functions of ζ and τ instead of z and q. Recall (2.3.12), we can now calculate the fusion coefficients using the Verlinde formula. The fusion rules for $A_1(u,1)$ are given by

$$\mathcal{L}_{r,0}\times\mathcal{L}_{r',0} = \bigoplus_{r''=1}^{u-1}\mathsf{N}_{r,r'}^{[u]\,r''}\mathcal{L}_{r'',0}, \tag{3.3.25}$$

where the Virasoro coefficient $\mathsf{N}_{r,r'}^{[u]\,r''}$ was defined in (3.2.15). The vacuum module of the minimal model is $\mathcal{L}_{1,0}$, and for $u>2$, $\mathcal{L}_{u-1,0}$ is an order 2 simple current.

3.3.3 Fractional level Wess-Zumino-Witten models

When $v\neq 1$ in $A_1(u,v)$, the level of the minimal model is a fraction. In this case, we generalise the parametrisation of the h_0-charge and conformal dimension from (3.3.18) to

$$\lambda_{r,s}^{\mathrm{sl}} = r-1-\frac{u}{v}s, \quad \Delta_{r,s}^{\mathrm{sl}} = \frac{(vr-us)^2-v^2}{4uv}. \tag{3.3.26}$$

With this parametrisation, the irreducible $A_1(u,v)$-modules come in several different classes, including those in the following list [32, 85]:

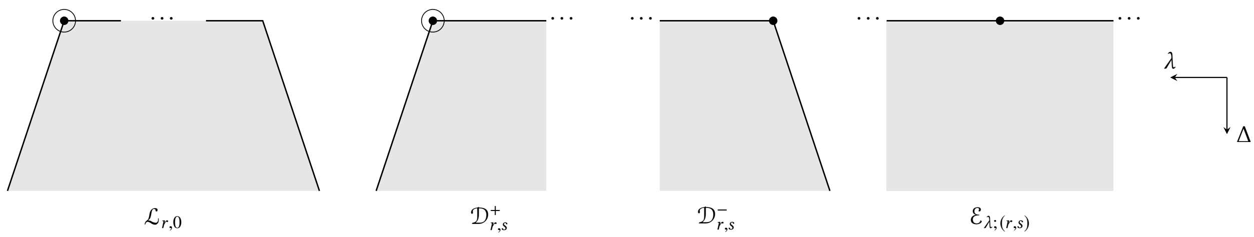

Figure 3.2: Different classes of irreducible $A_1(u,v)$-modules at a fractional level. Characterising states are indicated by black dots, among which highest-weight states are circled.

- The $\mathcal{L}_{r,0}$ with $1 \le r \le u-1$: Each is an irreducible highest-weight module whose space of ground states is finite-dimensional. The highest-weight vector of each module has h_0-charge $\lambda^{\mathrm{sl}}_{r,0}$ and conformal dimension $\Delta^{\mathrm{sl}}_{r,0}$.

- The $\mathcal{D}^{+}_{r,s}$ with $1 \le r \le u-1$ and $1 \le s \le v-1$: Each is an irreducible highest-weight module whose highest-weight vector has charge $\lambda^{\mathrm{sl}}_{r,s}$ and conformal dimension $\Delta^{\mathrm{sl}}_{r,s}$. The space of ground states forms an irreducible infinite-dimensional Verma module for the horizontal subalgebra $\mathfrak{sl}_2$.

- The $\mathcal{D}^{-}_{r,s}$ with $1 \le r \le u-1$ and $1 \le s \le v-1$: These are defined to be the conjugates of the $\mathcal{D}^{+}_{r,s}$, meaning that $\mathcal{D}^{-}_{r,s}$ is obtained from $\mathcal{D}^{+}_{r,s}$ by twisting the $A_1(u,v)$-action by the Weyl reflection of $\mathfrak{sl}_2$. It follows that the ground states of the $\mathcal{D}^{-}_{r,s}$ also have conformal dimension $\Delta^{\mathrm{sl}}_{r,s}$. However, these modules are not highest-weight modules since the h_0-charges of the ground states are not bounded above.

- The $\mathcal{E}_{\lambda;(r,s)}$ with $1 \le r \le u-1$, $1 \le s \le v-1$ and $\lambda \in \mathbb{C}/2\mathbb{Z}$ with $\lambda \ne \lambda^{\mathrm{sl}}_{r,s}, \lambda^{\mathrm{sl}}_{u-r,v-s} \pmod 2$: Each is an irreducible *relaxed highest-weight module* which is generated by a relaxed highest-weight state, this in turn being defined as an eigenstate of h_0 which is annihilated by the modes e_n, h_n and f_n, with $n > 0$. The ground states have h_0-charges in $\lambda + 2\mathbb{Z}$ and conformal dimension $\Delta^{\mathrm{sl}}_{r,s}$. These modules are neither highest-weight nor lowest-weight. The space of ground states is again infinite-dimensional. Such modules with the same r-, s-labels form a *family* of relaxed modules with the same conformal dimension but a continuous spectrum of h_0-charges.

- Spectral flows of all of the irreducible modules above. This generally gives new irreducibles, though there are some isomorphisms to note, in particular

$$\sigma^{\pm 1}_{\mathrm{sl}}\left(\mathcal{L}_{r,0}\right) \cong \mathcal{D}^{\pm}_{u-r,v-1}, \qquad \sigma^{\pm 1}_{\mathrm{sl}}\left(\mathcal{D}^{\pm}_{r,s}\right) \cong \mathcal{D}^{\pm}_{u-r,v-1-s} \quad (s \ne v-1). \tag{3.3.27}$$

The module diagrams for the $\mathcal{L}$-, $\mathcal{D}$- and $\mathcal{E}$-types of modules are illustrated in Figure 3.2.

There exist additional classes of irreducible $A_1(u,v)$-modules, for instance the Whittaker modules of [54]. However, these are not expected to be needed for the purpose of this thesis, that is, to construct certain (logarithmic) conformal field theories. One does, however, need certain reducible but indecomposable $A_1(u,v)$-modules, in particular the relaxed highest-weight modules $\mathcal{E}^{\pm}_{r,s}$. These have ground states whose h_0-charges are equal to $\lambda^{\text{sl}}_{r,s}$ (mod 2) and whose conformal dimension is $\Delta^{\text{sl}}_{r,s}$. We shall now employ a well-known trick [86] to construct $\mathcal{L}$- and $\mathcal{D}$-type modules in terms of these reducible $\mathcal{E}$-type modules.

The module $\mathcal{E}^{\pm}_{r,s}$ is reducible with a submodule isomorphic to $\mathcal{D}^{\pm}_{r,s}$ and its quotient by this submodule being isomorphic to $\mathcal{D}^{\mp}_{u-r,v-s}$. In other words, the following sequence is *exact*:

$$0 \longrightarrow \mathcal{D}^{\pm}_{r,s} \xrightarrow{\iota} \mathcal{E}^{\pm}_{r,s} \xrightarrow{\pi} \mathcal{D}^{\mp}_{u-r,v-s} \longrightarrow 0, \tag{3.3.28}$$

which means that the map ι is injective, while π is surjective. Recall (3.3.27), an $\mathcal{L}$-type module is related to a $\mathcal{D}$-type by the spectral flow. One can therefore derive a similar short sequence involving an $\mathcal{L}$-type module

$$0 \longrightarrow \sigma_{\text{sl}}\left(\mathcal{D}^{+}_{r,1}\right) \xrightarrow{\iota} \sigma_{\text{sl}}\left(\mathcal{E}^{+}_{r,1}\right) \xrightarrow{\pi} \mathcal{L}_{r,0} \longrightarrow 0. \tag{3.3.29}$$

The relations between the modules in (3.3.28) and (3.3.29) are depicted in Figure 3.3. These short sequences yield the following relations between the characters of the modules.

$$\text{Ch}[\mathcal{E}^{\pm}_{r,s}] = \text{Ch}[\mathcal{D}^{\pm}_{r,s}] + \text{Ch}[\mathcal{D}^{\mp}_{u-r,v-s}], \tag{3.3.30a}$$

$$\text{Ch}[\sigma_{\text{sl}}\left(\mathcal{E}^{+}_{r,1}\right)] = \text{Ch}[\sigma_{\text{sl}}\left(\mathcal{D}^{+}_{r,1}\right)] + \text{Ch}[\mathcal{L}_{r,0}]. \tag{3.3.30b}$$

Consider two short sequences of the generic forms

$$0 \longrightarrow \mathcal{D}_1 \xrightarrow{\iota} \mathcal{E}_1 \xrightarrow{\pi} \mathcal{D}_2 \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{D}_3 \xrightarrow{\iota'} \mathcal{E}_1 \xrightarrow{\pi'} \mathcal{D}_1 \longrightarrow 0, \tag{3.3.31}$$

in which the second module in the first sequence is the same as the fourth module of the second. This construction allows two sequences to be *spliced* together to form a longer sequence

$$0 \longrightarrow \mathcal{D}_3 \xrightarrow{\iota'} \mathcal{E}_2 \xrightarrow{\iota\circ\pi'} \mathcal{E}_1 \xrightarrow{\pi} \mathcal{D}_2 \longrightarrow 0, \tag{3.3.32}$$

which is again exact.

Let us now see how we can take advantage of this splicing to construct modules and compute their characters. We start from the following short sequences

$$0 \longrightarrow \sigma_{\text{sl}}\left(\mathcal{D}^{+}_{r,s+1}\right) \xrightarrow{\iota} \sigma_{\text{sl}}\left(\mathcal{E}^{+}_{r,s+1}\right) \xrightarrow{\pi} \mathcal{D}^{+}_{r,s} \longrightarrow 0, \quad (s \neq v-1) \tag{3.3.33a}$$

$$0 \longrightarrow \sigma^2_{\text{sl}}\left(\mathcal{D}^{+}_{u-r,1}\right) \xrightarrow{\iota} \sigma_{\text{sl}}\left(\mathcal{E}^{+}_{u-r,1}\right) \xrightarrow{\pi} \mathcal{D}^{+}_{r,v-1} \longrightarrow 0, \quad (s = v-1) \tag{3.3.33b}$$

which are obtained by replacing s in (3.3.29) by $s+1$ and applying spectral flow to each module. We shall take the $s \neq v-1$ case as an example and repeat the operation of replacing s by $s+1$ and spectral flow to (3.3.33a), which yields

$$0 \to \sigma_{\text{sl}}^2\left(\mathcal{D}^+_{r,s+2}\right) \xrightarrow{\iota'} \sigma_{\text{sl}}^2\left(\mathcal{E}^+_{r,s+2}\right) \xrightarrow{\pi'} \sigma_{\text{sl}}\left(\mathcal{D}^+_{r,s+1}\right) \to 0, \quad (s \neq v-2, v-1). \tag{3.3.34}$$

We can now splice (3.3.33a) and (3.3.34) together according to (3.3.32) to obtain a longer exact sequence

$$0 \to \sigma_{\text{sl}}^2\left(\mathcal{D}^+_{r,s+2}\right) \xrightarrow{\iota'} \sigma_{\text{sl}}^2\left(\mathcal{E}^+_{r,s+2}\right) \xrightarrow{\iota\circ\pi'} \sigma_{\text{sl}}\left(\mathcal{E}^+_{r,s+1}\right) \xrightarrow{\pi} \mathcal{D}^+_{r,s} \to 0. \tag{3.3.35}$$

It is actually possible to repeat this process infinitely, splicing further short sequences to (3.3.35) and arrive at the following infinitely long exact sequence, which is called a *resolution*:

$$\begin{aligned}\cdots \longrightarrow \sigma_{\text{sl}}^{3v-s-1}\left(\mathcal{E}^+_{r,v-1}\right) \longrightarrow \cdots \longrightarrow \sigma_{\text{sl}}^{2v-s+2}\left(\mathcal{E}^+_{r,2}\right) \longrightarrow \sigma_{\text{sl}}^{2v-s+1}\left(\mathcal{E}^+_{r,1}\right) \\ \longrightarrow \sigma_{\text{sl}}^{2v-s-1}\left(\mathcal{E}^+_{u-r,v-1}\right) \longrightarrow \cdots \longrightarrow \sigma_{\text{sl}}^{v-s+2}\left(\mathcal{E}^+_{u-r,2}\right) \longrightarrow \sigma_{\text{sl}}^{v-s+1}\left(\mathcal{E}^+_{u-r,1}\right) \\ \longrightarrow \sigma_{\text{sl}}^{v-s-1}\left(\mathcal{E}^+_{r,v-1}\right) \longrightarrow \cdots \longrightarrow \sigma_{\text{sl}}^{2}\left(\mathcal{E}^+_{r,s+2}\right) \longrightarrow \sigma_{\text{sl}}\left(\mathcal{E}^+_{r,s+1}\right) \longrightarrow \mathcal{D}^+_{r,s} \longrightarrow 0.\end{aligned} \tag{3.3.36}$$

The exact same resolution is obtained when considering the other case where $s = v-1$ in (3.3.33b).

Now starting from (3.3.29), the resolution involving an $\mathcal{L}$-type module obtained in a similar process is found to be

$$\begin{aligned}\cdots \longrightarrow \sigma_{\text{sl}}^{3v-1}\left(\mathcal{E}^+_{r,v-1}\right) \longrightarrow \cdots \longrightarrow \sigma_{\text{sl}}^{2v+2}\left(\mathcal{E}^+_{r,2}\right) \longrightarrow \sigma_{\text{sl}}^{2v+1}\left(\mathcal{E}^+_{r,1}\right) \\ \longrightarrow \sigma_{\text{sl}}^{2v-1}\left(\mathcal{E}^+_{u-r,v-1}\right) \longrightarrow \cdots \longrightarrow \sigma_{\text{sl}}^{v+2}\left(\mathcal{E}^+_{u-r,2}\right) \longrightarrow \sigma_{\text{sl}}^{v+1}\left(\mathcal{E}^+_{u-r,1}\right) \\ \longrightarrow \sigma_{\text{sl}}^{v-1}\left(\mathcal{E}^+_{r,v-1}\right) \longrightarrow \cdots \longrightarrow \sigma_{\text{sl}}^{2}\left(\mathcal{E}^+_{r,2}\right) \longrightarrow \sigma_{\text{sl}}\left(\mathcal{E}^+_{r,1}\right) \longrightarrow \mathcal{L}_{r,0} \longrightarrow 0.\end{aligned} \tag{3.3.37}$$

Following from (3.3.30), we can now write the characters of the $\mathcal{L}$- and $\mathcal{D}$-type modules as infinite linear combinations of the reducible $\mathcal{E}$-type characters and their spectral flows:

$$\text{Ch}[\mathcal{L}_{r,0}] = \sum_{s=1}^{v-1}(-1)^{s-1}\sum_{\ell=0}^{\infty}\left\{\text{Ch}[\sigma_{\text{sl}}^{2v\ell+s}\left(\mathcal{E}^+_{r,s}\right)] - \text{Ch}[\sigma_{\text{sl}}^{2v(\ell+1)-s}\left(\mathcal{E}^+_{u-r,v-s}\right)]\right\}, \tag{3.3.38a}$$

$$\text{Ch}[\mathcal{D}^\pm_{r,s}] = (-1)^{v-1-s}\text{Ch}[\sigma_{\text{sl}}^{v-s}\left(\mathcal{L}_{u-r,0}\right)] + \sum_{s'=s+1}^{v-1}(-1)^{s'-s-1}\text{Ch}[\sigma_{\text{sl}}^{s'-s}\left(\mathcal{E}^\pm_{r,s'}\right)]. \tag{3.3.38b}$$

The reducible and irreducible $\mathcal{E}$-type modules have reasonably well-understood structures and are chosen as the *standard modules* [49]. Such modules are reasonably well-understood and their characters have the most satisfactory modular transformation properties. They act as the building blocks for other types of modules as described by the resolutions. The characters of the $\mathcal{E}$-type modules take the simple form of

$$\mathrm{Ch}[\mathcal{E}^{\pm}_{r,s}](\mathsf{z};\mathsf{q}) = \mathsf{z}^{\pm\lambda^{\mathrm{sl}}_{r,s}} \frac{\mathrm{Ch}[\mathcal{V}_{r,s}](\mathsf{q})}{\eta(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} \mathsf{z}^{\pm 2n}, \tag{3.3.39a}$$

$$\mathrm{Ch}[\mathcal{E}_{\lambda;(r,s)}](\mathsf{z};\mathsf{q}) = \mathsf{z}^{\lambda} \frac{\mathrm{Ch}[\mathcal{V}_{r,s}](\mathsf{q})}{\eta(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} \mathsf{z}^{2n}, \tag{3.3.39b}$$

where $\mathcal{V}_{r,s}$ is a module of the Virasoro minimal model $\mathsf{M}(u,v)$. Since $\mathcal{E}^-_{r,s}$ and $\mathcal{E}^+_{r,s}$ are related by conjugation, which negates the h_0-charge of a state according to (3.3.14), the character of one module is obtained from the other by inverting z. The action of spectral flow upon the character of any $A_1(u,v)$-module $\mathcal{M}$ can also be derived from (3.3.14) and (3.3.15) as

$$\mathrm{Ch}[\sigma^{\ell}_{\mathrm{sl}}(\mathcal{M})](\mathsf{z};\mathsf{q}) = \mathsf{z}^{\ell}\mathsf{q}^{\ell^2/4}\mathrm{Ch}[\mathcal{M}]\left(\mathsf{z}\mathsf{q}^{\ell};\mathsf{q}\right). \tag{3.3.40}$$

This can be used to compute the characters of the spectral-flowed $\mathcal{E}^{\pm}_{r,s}$-modules which appear on the right-hand side of (3.3.38). Another way of computing the $\mathcal{L}$- and $\mathcal{D}$-type module characters is to study their braiding structures which may be obtained from the Kac-Kazhdan [87] formula. After removing the submodules generated by singular vectors at all levels, the characters are computed to be

$$\mathrm{Ch}[\mathcal{L}_{r,0}](\mathsf{z};\mathsf{q}) = \frac{\mathsf{q}^{\Delta^{\mathrm{sl}}_{r,0}-\mathsf{c}/24+1/8}}{\mathrm{i}\vartheta_1(\mathsf{z}^2;\mathsf{q})} \sum_{j\in\mathbb{Z}} \left(\mathsf{z}^{2uj+r} - \mathsf{z}^{-2uj-r}\right)\mathsf{q}^{vj(uj+r)}, \tag{3.3.41a}$$

$$\mathrm{Ch}[\mathcal{D}^{\pm}_{r,s}](\mathsf{z};\mathsf{q}) = \frac{\mathsf{z}^{\pm(\lambda^{\mathrm{sl}}_{r,s}+1)}\mathsf{q}^{\Delta^{\mathrm{sl}}_{r,s}-\mathsf{c}/24+1/8}}{\pm\mathrm{i}\vartheta_1(w^2;q)} \sum_{j\in\mathbb{Z}} \left[\mathsf{z}^{\pm 2uj}\mathsf{q}^{j(uvj+vr-us)} - \mathsf{z}^{\pm 2(uj-r)}\mathsf{q}^{(uj-r)(vj-s)}\right]. \tag{3.3.41b}$$

It was pointed out in [37] that such characters should be considered as algebraic distributions rather than meromorphic functions of z. It follows that the characters given in (3.3.38) are only valid when expanded in the following regions [83, 88]

$$|\mathsf{q}| < 1, \quad \begin{cases} 1 < |\mathsf{z}|^2 < |\mathsf{q}|^{-1} & (s \neq v-1) \\ 1 < |\mathsf{z}|^2 < |\mathsf{q}|^{-2} & (s = v-1), \end{cases} \tag{3.3.42}$$

assuming that $v > 1$.

The Grothendieck fusion rules of the irreducible relaxed highest-weight $A_1(u,v)$-modules were computed in [36] under the conjecture that *the standard Verlinde formula* [49, 50, 89] gives coefficients of the Grothendieck fusion. The fusion rules of type $\mathcal{L}_{r,0} \times \mathcal{L}_{r',0}$ were recently proven in [90] and confirm the Verlinde conjectures. The results, which were shown to be consistent with the irreducible fusion rules of [33], for $(u,v) = (2,3)$

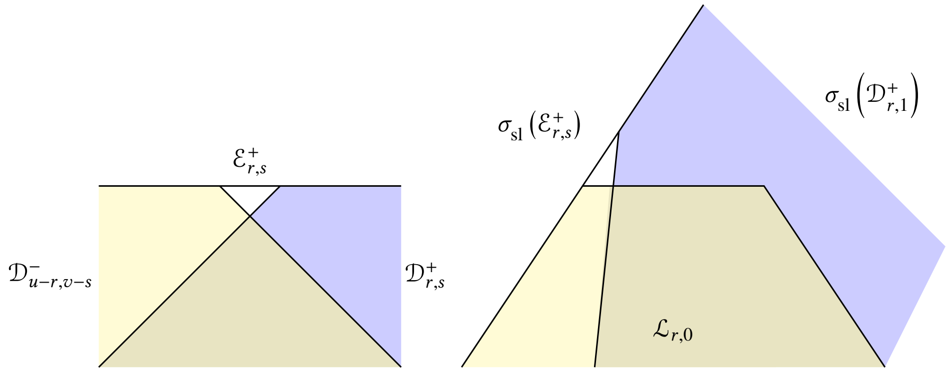

Figure 3.3: Module diagrams of short sequences (3.3.28) and (3.3.29). The reducible but indecomposable $\mathcal{E}^+_{r,s}$ and its spectral flow in each diagram are given by a submodule (blue) and a quotient module (yellow) 'glued' together.

(see [37] for some corrections), and [27], for $(u,v) = (3,2)$, are recorded as follows

$$\left[\mathcal{L}_{r,0}\right] \boxtimes \left[\mathcal{L}_{r',0}\right] = \sum_{r''} \mathsf{N}^{[u]\,r''}_{r,r'} \left[\mathcal{L}_{r'',0}\right], \tag{3.3.43a}$$

$$\left[\mathcal{L}_{r,0}\right] \boxtimes \left[\mathcal{D}^+_{r',s'}\right] = \sum_{r''} \mathsf{N}^{[u]\,r''}_{r,r'} \left[\mathcal{D}^+_{r'',s'}\right], \tag{3.3.43b}$$

$$\left[\mathcal{L}_{r,0}\right] \boxtimes \left[\mathcal{E}_{\lambda';(r',s')}\right] = \sum_{r''} \mathsf{N}^{[u]\,r''}_{r,r'} \left[\mathcal{E}_{\lambda'+r-1;(r'',s)}\right], \tag{3.3.43c}$$

$$\left[\mathcal{D}^+_{r,s}\right] \boxtimes \left[\mathcal{D}^+_{r',s'}\right] = \begin{cases} \displaystyle\sum_{r'',s''} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s')} \left[\sigma_{\mathrm{sl}}\left(\mathcal{E}_{\lambda_{r'',s+s'+1};(r'',s'')}\right)\right] \\ \displaystyle\qquad + \sum_{r''} \mathsf{N}^{[u]\,r''}_{r,r'} \left[\mathcal{D}^+_{r'',s+s'}\right], & \text{if } s+s' < v, \\ \displaystyle\sum_{r'',s''} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s+1),(r',s'+1)} \left[\sigma_{\mathrm{sl}}\left(\mathcal{E}_{\lambda^{\mathrm{sl}}_{r'',s+s'+1};(r'',s'')}\right)\right] \\ \displaystyle\qquad + \sum_{r''} \mathsf{N}^{[u]\,r''}_{r,r'} \left[\sigma_{\mathrm{sl}}\left(\mathcal{D}^+_{u-r'',s+s'-v+1}\right)\right], & \text{if } s+s' \geq v \end{cases} \tag{3.3.43d}$$

$$\begin{aligned} \left[\mathcal{D}^+_{r,s}\right] \boxtimes \left[\mathcal{E}_{\lambda';(r',s')}\right] = &\sum_{r'',s''} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s+1),(r',s')} \left[\mathcal{E}_{\lambda'+\lambda^{\mathrm{sl}}_{r,s};(r'',s'')}\right] \\ &+ \sum_{r'',s''} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s')} \left[\sigma_{\mathrm{sl}}\left(\mathcal{E}_{\lambda'+\lambda^{\mathrm{sl}}_{r,s+1};(r'',s'')}\right)\right], \end{aligned} \tag{3.3.43e}$$

$$\begin{aligned} \left[\mathcal{E}_{\lambda;(r,s)}\right] \boxtimes \left[\mathcal{E}_{\lambda';(r',s')}\right] = &\sum_{r'',s''} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s')} \left\{\left[\sigma_{\mathrm{sl}}\left(\mathcal{E}_{\lambda+\lambda'-k;(r'',s'')}\right)\right] + \left[\sigma_{\mathrm{sl}}^{-1}\left(\mathcal{E}_{\lambda+\lambda'+k;(r'',s'')}\right)\right]\right\} \\ &+ \sum_{r'',s''} \left(\mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s'-1)} + \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s'+1)}\right) \left[\mathcal{E}_{\lambda+\lambda';(r'',s'')}\right], \end{aligned} \tag{3.3.43f}$$

where r'' and s'' are summed from 1 to $u-1$, and from 1 to $v-1$, respectively. We refer to (3.2.15) for the definition of the (Virasoro) fusion coefficients that appear.

It was shown in [36, 91] that the Grothendieck fusion rules of the spectral flows of these $A_1(u,v)$-modules satisfy

$$\left[\sigma_{\mathrm{sl}}^{m}(\mathcal{M})\right] \boxtimes \left[\sigma_{\mathrm{sl}}^{n}(\mathcal{N})\right] = \sigma_{\mathrm{sl}}^{m+n}\left(\left[\mathcal{M}\right] \boxtimes \left[\mathcal{N}\right]\right) \qquad m,n \in \mathbb{Z} \tag{3.3.44}$$

3.3.4 Staggered modules of $A_1(u,v)$

The known fusion rules for $(u,v) = (2,3)$ and $(3,2)$ involve additional reducible, but indecomposable, $A_1(u,v)$-modules with four composition factors each. They are examples of staggered modules, in the sense of [36,92], possessing a non-diagonalisable action of L_0^{sl}. As such, they are responsible for the logarithmic nature of the corresponding conformal field theories. We believe that these staggered modules are projective and are therefore the projective covers of their irreducible heads (in an appropriate category of $A_1(u,v)$-modules). We elucidate this belief below and extend it to all admissible levels.

For convenience, we shall enforce the following notation for (and only for) the rest of the section:

$$\mathcal{D}_{r,-1}^{\pm} = \mathcal{D}_{r,1}^{\mp}, \qquad \mathcal{D}_{r,0}^{+} \equiv \mathcal{L}_{r,0} \equiv \mathcal{D}_{r,0}^{-} \qquad \text{and} \qquad \mathcal{D}_{r,v}^{\pm} = \sigma_{\mathrm{sl}}^{\pm 1}\left(\mathcal{D}_{u-r,1}^{\pm}\right). \tag{3.3.45}$$

The projective covers of the $\mathcal{D}_{r,s}^{\pm}$, for $s = 0,1,\ldots,v-1$, shall be denoted by $\mathcal{S}_{r,s}$. We shall sometimes drop the label $\pm$ when $s = 0$ in accordance with the second identification of (3.3.45).

The structures of the (conjectured) projective covers will be described in terms of their *Loewy diagrams*. This is a picture in which the composition factors of the module are arranged in horizontal layers. The bottom layer contains the composition factors of the module's socle (the sum of the irreducible submodules of the module). The next layer up contains the composition factors of the socle of the quotient of the module by its socle. This continues up until we reach the top layer which contains the composition factors of the module's head. We refer to [36, App. A.4] for an elementary introduction to Loewy diagrams that describes the idea in detail.

With this background in place, we can now conjecture the following for the projective covers of the irreducible $A_1(u,v)$-modules:

- The irreducible $\sigma_{\mathrm{sl}}^{\ell}\left(\mathcal{E}_{\lambda;(r,s)}\right)$, with $\ell \in \mathbb{Z}$, $1 \geq r \geq u-1$, $1 \geq s \geq v-1$ and $\lambda \neq \lambda_{r,s}^{\mathrm{sl}}, \lambda_{u-r,v-s}^{\mathrm{sl}}$ (mod 2), are projective and are hence their own projective covers.

- The Loewy diagram of the projective cover $\mathcal{S}_{r,s}$ of $\mathcal{D}^{\pm}_{r,s}$ is

$$\begin{array}{ccccc} & & \mathcal{D}^{\pm}_{r,s} & & \\ & \swarrow & & \searrow & \\ \sigma_{\mathrm{sl}}^{-1}\left(\mathcal{D}^{\pm}_{r,s-1}\right) & & \mathcal{S}_{r,s} & & \sigma_{\mathrm{sl}}\left(\mathcal{D}^{\pm}_{r,s+1}\right) \\ & \searrow & & \swarrow & \\ & & \mathcal{D}^{\pm}_{r,s} & & \end{array} \qquad (s = 0, 1, \dots, v-1). \tag{3.3.46}$$

The projective cover of $\sigma_{\mathrm{sl}}^{\ell}\left(\mathcal{D}^{\pm}_{r,s}\right)$ is then $\sigma_{\mathrm{sl}}^{\ell}\left(\mathcal{S}^{\pm}_{r,s}\right)$ and its Loewy diagram is obtained from that of $\mathcal{D}^{\pm}_{r,s}$ by applying $\sigma_{\mathrm{sl}}^{\ell}$ to each composition factor. (Indeed, that of $\mathcal{S}_{r,v-1}$ is the image under σ_{sl} of that of $\mathcal{S}_{r,0}$.)

Evidence for the conjectured Loewy diagrams (3.3.46) comes from trying to lift the Grothendieck fusion rules of (3.3.41) to actual fusion rules. We expect that the physically consistent category of $A_1(u,v)$-modules should be, among other things, rigid and tensor. The associative tensor product is, of course, fusion and rigidity ensures that fusing with any fixed module defines an exact functor on the category [93, Prop. 4.2.1]. This means that the Grothendieck group of the category inherits a well-defined product $\boxtimes$ between two modules $\mathcal{M}$ and $\mathcal{N}$ from the fusion product $\times$ as follows

$$\left[\mathcal{M} \times \mathcal{N}\right] = \left[\mathcal{M}\right] \boxtimes \left[\mathcal{N}\right]. \tag{3.3.47}$$

Another consequence of rigidity is that the projectives of the category form a tensor ideal: the fusion product of a projective, in particular one of the irreducible $\mathcal{E}_{\lambda;(r,s)}$, with any module is again projective [93, Prop. 4.2.12].

As the $\mathcal{L}_{r,0}$, $\mathcal{D}^{\pm}_{r,s}$ and $\mathcal{E}^{\pm}_{r,s}$, along with their spectral flows, cannot be projective, there are not many ways to arrange the composition factors, obtained from (3.3.41), of a fusion product involving an irreducible $\mathcal{E}_{\lambda;(r,s)}$ so that the result could be projective. Indeed, if we also insist on projectives being self-dual, a desirable property in view of the non-degeneracy of two-point correlation functions [94], then the arrangement is often essentially unique. This is reflected in the following statement for a particular subset of the $A_1(u,v)$ fusion rules:

Let $\lambda \neq \lambda^{\mathrm{sl}}_{1,1}, \lambda^{\mathrm{sl}}_{u-1,v-1} \pmod 2$ and $\mu \neq \lambda^{\mathrm{sl}}_{r,s}, \lambda^{\mathrm{sl}}_{u-r,v-s} \pmod 2$. Then, for all $1 \le r \le u-1$

and $2 \le s \le v-2$ (which requires that $v \ge 4$), we have the fusion rules

$$\mathcal{E}_{\lambda;(1,1)} \times \mathcal{E}_{\mu;(r,s)} = \begin{cases} \mathcal{S}^+_{r,s-1} \oplus \sigma_{\mathrm{sl}}^{-1}\left(\mathcal{E}_{\lambda+\mu+\mathsf{k};(r,s)}\right) \oplus \mathcal{E}_{\lambda+\mu;(r,s+1)}, & \text{if } \lambda+\mu = \lambda^{\mathrm{sl}}_{r,s-1}, \\ \mathcal{S}^+_{u-r,v-s-1} \oplus \sigma_{\mathrm{sl}}^{-1}\left(\mathcal{E}_{\lambda+\mu+\mathsf{k};(r,s)}\right) \oplus \mathcal{E}_{\lambda+\mu;(r,s-1)}, & \text{if } \lambda+\mu = \lambda^{\mathrm{sl}}_{u-r,v-s-1}, \\ \mathcal{S}^-_{u-r,v-s-1} \oplus \sigma_{\mathrm{sl}}\left(\mathcal{E}_{\lambda+\mu-\mathsf{k};(r,s)}\right) \oplus \mathcal{E}_{\lambda+\mu;(r,s-1)}, & \text{if } \lambda+\mu = \lambda^{\mathrm{sl}}_{r,s+1}, \\ \mathcal{S}^-_{r,s-1} \oplus \sigma_{\mathrm{sl}}\left(\mathcal{E}_{\lambda+\mu-\mathsf{k};(r,s)}\right) \oplus \mathcal{E}_{\lambda+\mu;(r,s+1)}, & \text{if } \lambda+\mu = \lambda^{\mathrm{sl}}_{u-r,v-s+1}, \\ \sigma_{\mathrm{sl}}\left(\mathcal{E}_{\lambda+\mu-\mathsf{k};(r,s)}\right) \oplus \sigma_{\mathrm{sl}}^{-1}\left(\mathcal{E}_{\lambda+\mu+\mathsf{k};(r,s)}\right) & \\ \qquad \oplus \mathcal{E}_{\lambda+\mu;(r,s-1)} \oplus \mathcal{E}_{\lambda+\mu;(r,s+1)}, & \text{otherwise,} \end{cases} \tag{3.3.48}$$

where $\lambda+\mu$ is always understood mod 2. When $s=1$ or $s=v-1$, these conjectured fusion rules are modified to remove any $\mathcal{E}_{\nu;(r,s')}$ with $s'=0$ or v, and remove any direct summands that do not appear in all expressions corresponding to the same value of $\lambda+\mu$ (mod 2). For example, the fusion rule for $s=1$, $v \ge 3$ and $\lambda+\mu = \lambda^{\mathrm{sl}}_{r,0}$ (mod 2) becomes

$$\mathcal{E}_{\lambda;(1,1)} \times \mathcal{E}_{\mu;(r,1)} = \mathcal{S}_{r,0} \oplus \mathcal{E}_{\lambda+\mu;(r,2)}, \tag{3.3.49}$$

because $\lambda^{\mathrm{sl}}_{r,0} = \lambda^{\mathrm{sl}}_{u-r,v}$ and the spectrally flowed summands in the first and fourth cases of (3.3.48) are different. When $v=2$, we would also have to remove the $\mathcal{E}_{\lambda+\mu;(r,2)}$ from the right-hand side. In fact, the Loewy diagrams (3.3.46) were deduced by analysing the possible arrangements for the composition factors appearing in the Grothendieck counterpart (3.3.41) (with $r,s=1$). It is, of course, possible to similarly conjecture the remaining fusion rules involving staggered $A_1(u,v)$-modules, coming from fusing the $\mathcal{D}$- and the irreducible $\mathcal{E}$-type modules:

$$\mathcal{E}_{\lambda;(1,1)} \times \mathcal{D}^+_{1,s} = \begin{cases} \mathcal{S}^+_{u-1,v-s-1}, & \text{if } \lambda+\lambda^{\mathrm{sl}}_{1,s} = \lambda^{\mathrm{sl}}_{u-1,v-s-1} \\ \mathcal{E}_{\lambda+\lambda^{\mathrm{sl}}_{1,s};(1,s+1)} \oplus \sigma_{\mathrm{sl}}\left(\mathcal{E}_{\lambda+\lambda^{\mathrm{sl}}_{1,s+1};(1,s)}\right), & \text{otherwise,} \end{cases} \tag{3.3.50a}$$

$$\mathcal{E}_{\lambda;(1,s)} \times \mathcal{D}^+_{1,1} = \begin{cases} \mathcal{S}^+_{1,s-1} \oplus \mathcal{E}_{\lambda+\lambda^{\mathrm{sl}}_{1,1};(1,s+1)}, & \text{if } \lambda+\lambda^{\mathrm{sl}}_{1,1} = \lambda^{\mathrm{sl}}_{1,s-1} \\ \mathcal{S}^+_{u-1,v-s-1} \oplus \mathcal{E}_{\lambda+\lambda^{\mathrm{sl}}_{1,1};(1,s-1)}, & \text{if } \lambda+\lambda^{\mathrm{sl}}_{1,1} = \lambda^{\mathrm{sl}}_{u-1,v-s-1} \\ \mathcal{E}_{\lambda+\lambda^{\mathrm{sl}}_{1,1};(1,s-1)} \oplus \mathcal{E}_{\lambda+\lambda^{\mathrm{sl}}_{1,1};(1,s+1)} & \\ \qquad \oplus \sigma_{\mathrm{sl}}\left(\mathcal{E}_{\lambda+\lambda^{\mathrm{sl}}_{1,2};(1,s)}\right), & \text{otherwise.} \end{cases} \tag{3.3.50b}$$

As in the case of fusing two $\mathcal{E}$-type modules, for special values of $s=1$ or $v-1$, we truncate terms in (3.3.50) whose s' is outside the range of 1 to $v-1$.

The fusion rules (3.3.48) and (3.3.50) provide the minimal generating set for mathematically inducing to the complete set of fusion rules involving staggered modules.

One might expect the fusion between two $\mathcal{D}$-type modules to give rise to staggered modules as well, which turns out to be false. For example, the generating fusion rules are

$$\left[\mathcal{D}^+_{1,1}\right] \boxtimes \left[\mathcal{D}^+_{1,s}\right] = \begin{cases} \left[\sigma_{\text{sl}}\left(\mathcal{E}_{\lambda^{\text{sl}}_{1,s+2};(1,s)}\right)\right] + \left[\mathcal{D}^+_{1,1+s}\right], & s < v-1, \\ \left[\sigma_{\text{sl}}\left(\mathcal{D}^+_{u-1,1}\right)\right], & s = v-1. \end{cases} \tag{3.3.51}$$

The module $\sigma_{\text{sl}}\left(\mathcal{E}_{\lambda^{\text{sl}}_{1,s+2};(1,s)}\right)$ is irreducible at all admissible levels. No staggered modules can be formed from the modules on the right-hand side of (3.3.51). The proper fusion rule therefore is simply given by a direct sum of all the modules involved

$$\mathcal{D}^+_{1,1} \times \mathcal{D}^+_{1,s} = \begin{cases} \sigma_{\text{sl}}\left(\mathcal{E}_{\lambda^{\text{sl}}_{1,s+2};(1,s)}\right) \oplus \mathcal{D}^+_{1,1+s}, & s < v-1, \\ \sigma_{\text{sl}}\left(\mathcal{D}^+_{u-1,1}\right), & s = v-1. \end{cases} \tag{3.3.52}$$

Mathematically inducing this to a general fusion rule between two arbitrary $\mathcal{D}$-type modules leads to a fusion of the same form as (3.3.43a):

$$\mathcal{D}^+_{r,s} \times \mathcal{D}^+_{r',s'} = \begin{cases} \bigoplus_{r'',s''} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s')}\, \sigma_{\text{sl}}\left(\mathcal{E}_{\lambda^{\text{sl}}_{r'',s+s'+1};(r'',s'')}\right) \oplus \bigoplus_{r''} \mathsf{N}^{[u]\,r''}_{r,r'}\, \mathcal{D}^+_{r'',s+s'}, & \text{if } s+s' < v, \\ \bigoplus_{r'',s''} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s+1),(r',s'+1)}\, \sigma_{\text{sl}}\left(\mathcal{E}_{\lambda^{\text{sl}}_{r'',s+s'+1};(r'',s'')}\right) & \\ \qquad \oplus \bigoplus_{r''} \mathsf{N}^{[u]\,r''}_{r,r'}\, \sigma_{\text{sl}}\left(\mathcal{D}^+_{u-r'',s+s'-v+1}\right), & \text{if } s+s' \geq v. \end{cases} \tag{3.3.53}$$

Chapter 4

Fermionic Conformal Field Theories

Fermionic fields are characterised by anti-symmetry under field exchange in a many-field system, which follows from anti-commutation relations between the two fields. This chapter provides accounts of conformal field theories with such fermionic symmetries, including the free fermion [57], the $\mathfrak{bc}$*-ghosts, the* $N = 1$ *[95–97] and* $N = 2$ *[98, 99] superconformal field theories and the affine Lie Kac-Moody algebra* $\widehat{\mathfrak{osp}}(1|2)$ *[38–40].*

4.1 Elementary Fermionic Conformal Field Theories

The generating fields of the $\mathfrak{bc}$-ghost system are fermionic, which means their coordinates are periodic under a 4π rotation about the origin. Such fields therefore satisfy the following boundary condition under a rotation of 2π

$$x(\mathsf{e}^{2\pi\mathsf{i}}z) = \begin{cases} x(z), & \text{Neveu-Schwarz sector,} \\ -x(z), & \text{Ramond sector.} \end{cases} \tag{4.1.1}$$

Fields from different sectors are expanded differently, the Fourier modes can be either integers or half-integers depending on the sector:

$$x(z) = \sum_n x_n z^{-n-\Delta}, \tag{4.1.2}$$

where Δ is the conformal dimension of the field $x(z)$. The powers of z in this expansion are integers in the Neveu-Schwarz sector, and half-integers in the Ramond sector. This means for $x(z)$ with a half-integer conformal dimension, its Fourier mode indices $n \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz sector and $n \in \mathbb{Z}$ in the Ramond sector.

4.1.1 The free fermion

The simplest fermionic theory, the free fermion, appears with the action

$$S\left[\psi, \overline{\psi}\right] = \frac{1}{2\pi} \int \left(\psi \bar{\partial} \psi + \overline{\psi} \partial \overline{\psi}\right) \mathrm{d}z \, \mathrm{d}\bar{z}, \tag{4.1.3}$$

from which one can derive the equations of motion

$$\partial\overline{\psi} = 0, \quad \overline{\partial}\psi = 0 \tag{4.1.4}$$

using the stationary action principle. The field $\psi(z)$ (of conformal dimension $\frac{1}{2}$) is therefore holomorphic and is Fourier expanded as

$$\psi(z) = \begin{cases} \sum_{n\in\mathbb{Z}+1/2} b_n z^{-n-1/2} & \text{Neveu-Schwarz sector,} \\ \sum_{n\in\mathbb{Z}} b_n z^{-n-1/2} & \text{Ramond sector.} \end{cases} \tag{4.1.5}$$

The expansion of the anti-holomorphic field $\overline{\psi}(\overline{z})$ is analogous. Concentrating on the holomorphic sector, the operators b_n along with the identity operator span an infinite dimensional Lie superalgebra. From canonical quantisation, one can derive the following anti-commutation relation between the b_n

$$\{b_m, b_n\} = b_m b_n + b_n b_m = \delta_{m+n,0}. \tag{4.1.6}$$

In the Neveu-Schwarz sector, the algebra is spanned by $\{b_n \mid n \in \mathbb{Z} + \frac{1}{2}\} \cup \{1\}$. We partition the $\{b_n\}$ into two sets: the ones with positive indices as annihilators and those with negative indices as creators. The adjoint for each b_n is proposed to be

$$b_n^\dagger = b_{-n}. \tag{4.1.7}$$

It follows from (4.1.6) that

$$\{b_n, b_n\} = 2b_n^2 = 0 \quad \text{for all } n \in \mathbb{Z} + \frac{1}{2}. \tag{4.1.8}$$

This is consistent with the Pauli exclusion principle, which states that no two identical fermions can occupy the same quantum state simultaneously, that is, $b_n^2 = 0$ for all $n \in \mathbb{Z} + \frac{1}{2}$. The only vacuum state in the Neveu-Schwarz sector is the true vacuum of zero conformal dimension, we shall denote this by $|0_{\text{NS}}\rangle$. Unlike the free boson, there is no notion of momentum in a free fermion theory. Neveu-Schwarz vacua of conformal dimensions other than 0 therefore do not exist. The state-field correspondence is defined over the Neveu-Schwarz true vacuum module, for example

$$\lim_{z\to 0} \psi(z)|0_{\text{NS}}\rangle = b_{-1/2}|0_{\text{NS}}\rangle, \tag{4.1.9}$$

with the vacuum state $|0_{\text{NS}}\rangle$ itself corresponding to the identity field. Using the state-field correspondence, one can Fourier expand the fields and derive the OPE between two free fermions as

$$\psi(z)\psi(w) \sim \frac{1}{z-w}. \tag{4.1.10}$$

The classical energy-momentum tensor $T(z)$ of the free fermion theory can be derived using Noether's theorem. To quantise the theory, we define the normal ordering of two b_n modes in the Neveu-Schwarz sector as

$$:b_m b_n: = \begin{cases} b_m b_n, & \text{if } m \le -\dfrac{1}{2}, \\ -b_n b_m, & \text{if } m \ge \dfrac{1}{2}. \end{cases} \tag{4.1.11}$$

With this definition, the quantised energy-momentum tensor is given by

$$T(z) = -\frac{1}{2} :\psi(z)\partial\psi(z):. \tag{4.1.12}$$

From the OPE of $T(z)$ with itself, one can observe the central charge of the free fermion to be $\frac{1}{2}$. It can be calculated from (4.1.12) that the OPE between $T(z)$ and $\psi(z)$ is given by

$$T(z)\psi(w) \sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}, \tag{4.1.13}$$

which shows that $\psi(z)$ is a Virasoro primary field with conformal dimension $\frac{1}{2}$.

In comparison, the behaviour of the free fermion is more complicated in the Ramond sector, operators b_n now have integer indices. In order to study its modules, we introduce a vacuum state, denoted by $|0_{\mathrm{R}}\rangle$, which is required to be annihilated by all positive modes of b_n. It follows from (4.1.6) that b_0 in the Ramond sector satisfies

$$b_n^2 = \frac{1}{2}\delta_{n,0} \quad \Rightarrow \quad b_0^2 = \frac{1}{2}. \tag{4.1.14}$$

The operators b_{-n}, $n \ge 0$ are creators, among which b_0 does not raise the conformal dimension of a state but reverses its parity, the Ramond Verma module thus has two independent ground states.

Note that for parity to make sense, we declare that $|0_{\mathrm{R}}\rangle$ and $b_0|0_{\mathrm{R}}\rangle$ are independent states by insisting that the representations are always direct sums of a bosonic and a fermionic subspace. The resulting representation is reducible with respect to the fermion algebra because it splits as the direct sum of two modules with one ground state each. These ground states are eigenvectors of b_0 but cannot be consistently assigned a parity. However, the Ramond Verma module is irreducible with respect to to the extended algebra generated by the b_n and an operator $(-1)^F$ which anti-commutes with the b_n, that is F equals 0 and 1 when acting on bosonic and fermionic subspaces, respectively [97].

We will exploit correlation functions defined over this state to calculate the conformal dimension of $|0_{\mathrm{R}}\rangle$. Expand the following 2-point correlation function

$$\langle 0_{\mathrm{R}} \mid \partial\psi(z)\psi(w) \mid 0_{\mathrm{R}}\rangle = \partial_z \langle 0_{\mathrm{R}} \mid \psi(z)\psi(w) \mid 0_{\mathrm{R}}\rangle$$

$$
\begin{aligned}
&= \partial_z \langle 0_\mathrm{R} | \sum_{m,n\in\mathbb{Z}} b_n b_m z^{-n-1/2} w^{-m-1/2} |0_\mathrm{R}\rangle \\
&= \partial_z \left(\frac{1/2}{\sqrt{zw}} + \frac{1}{\sqrt{zw}} \sum_{n=1}^{\infty} \left(\frac{w}{z}\right)^n \right) \\
&= \frac{-\sqrt{z/w}-\sqrt{w/z}}{2(z-w)^2} + \frac{1}{4(z-w)} \left(\frac{1}{\sqrt{zw}} - \sqrt{\frac{w}{z^3}} \right).
\end{aligned} \tag{4.1.15}
$$

Now consider the OPE of the two fields in the above correlation function with the first regular term written out:

$$
\begin{aligned}
\partial\psi(z)\psi(w) &= -\frac{1}{(z-w)^2} + :\partial\psi(w)\psi(w): + \mathcal{O}(z-w) \\
&= -\frac{1}{(z-w)^2} + 2T(w) + \mathcal{O}(z-w)
\end{aligned} \tag{4.1.16}
$$

Inserting this into the correlator function yields

$$
\begin{aligned}
\langle 0_\mathrm{R} \mid \partial\psi(z)\psi(w) \mid 0_\mathrm{R}\rangle &= \langle 0_\mathrm{R} \mid \frac{-1}{(w-z)^2} \mid 0_\mathrm{R}\rangle + 2\sum_{n\in\mathbb{Z}} \langle 0_\mathrm{R} \mid L_n z^{-n-2} \mid 0_\mathrm{R}\rangle + \mathcal{O}(z-w) \\
&= \frac{-1}{(w-z)^2} + 2\langle 0_\mathrm{R} \mid L_0 z^{-2} \mid 0_\mathrm{R}\rangle + \mathcal{O}(z-w),
\end{aligned} \tag{4.1.17}
$$

where we have expanded $T(z)$ in term of Virasoro modes L_n. When $n > 1$, the L_n annihilate $|0_\mathrm{R}\rangle$, and when $n < 1$, they annihilate $|0_\mathrm{R}\rangle$. Hence the only term left is the L_0 mode. Comparing the limits of (4.1.17) with (4.1.15) as z approaches w, we arrive at

$$
2\langle 0_\mathrm{R} \mid L_0 \mid 0_\mathrm{R}\rangle = \frac{1}{8} \quad \Rightarrow \quad L_0|0_\mathrm{R}\rangle = \frac{1}{16}|0_\mathrm{R}\rangle. \tag{4.1.18}
$$

The Ramond vacuum therefore is not the true vacuum, since its conformal dimension is not 0 but $\frac{1}{16}$.

Because $|0_\mathrm{R}\rangle$ is not a true vacuum, we cannot define state-field correspondence with it in the Ramond sector. Any attempt would quickly lead to a divergence problem when taking the limit of the coordinate of a field approaching 0.

4.1.2 The $\mathfrak{bc}$-ghost system

The ghost superalgebra $\mathfrak{bc}$ is generated by linear combinations of two free fermionic fields $\psi_1(z_1)$ and $\psi_2(z_2)$, which we discussed in the previous section. We shall call them $b(z)$ and $c(z)$, which are given by

$$
b(z) = \frac{1}{\sqrt{2}}\psi_1(z) + \frac{\mathfrak{i}}{\sqrt{2}}\psi_2(z), \quad c(z) = \frac{1}{\sqrt{2}}\psi_1(z) - \frac{\mathfrak{i}}{\sqrt{2}}\psi_2(z). \tag{4.1.19}
$$

The OPEs of these fields are defined as

$$b(z)c(w) \sim c(z)b(w) \sim \frac{1}{z-w}, \quad b(z)b(w) \sim c(z)c(w) \sim 0. \tag{4.1.20}$$

Using the method of contour integrals, one can derive the anti-commutation relations between the modes of $b(z)$ and $c(z)$ from the above OPEs:

$$\{b_n, c_m\} = \delta_{m+n}, \quad \{b_n, b_m\} = \{c_n, c_m\} = 0. \tag{4.1.21}$$

The energy-momentum tensor is chosen as

$$T^{\mathrm{bc}}(z) = \frac{1}{2}\Big(- :b\,\partial c:(z) + :\partial b\, c:(z)\Big), \tag{4.1.22}$$

where the normal ordering of fermionic modes given in general is

$$:\phi_m \psi_n: = \begin{cases} \phi_m \psi_n, & \text{if } m + h_\phi \leq 0, \\ -\psi_n \phi_m, & \text{if } m + h_\phi > 0. \end{cases} \tag{4.1.23}$$

with h_ϕ being the conformal dimension of $\phi(z)$. Notice that this is almost identical to the bosonic case (2.2.19), except for the gaining of a minus sign when swapping the order of the two modes. The central charge associated with the energy-momentum tensor (4.1.22) is calculated to be 1. And the conformal dimensions of $b(z)$ and $c(z)$, as measured by the L_0-eigenvalue, is $\frac{1}{2}$ for both fields.

As a useful tool for visualising the structure of the $\mathfrak{bc}$-modules, a Heisenberg field of conformal dimension 1 is constructed from the two generating fields as $Q(z) = :bc:(z)$, whose OPEs with $b(z)$ and $c(z)$ are given by

$$Q(z)b(w) \sim \frac{b(w)}{z-w}, \quad Q(z)c(w) \sim -\frac{c(w)}{z-w}. \tag{4.1.24}$$

The eigenvalue of the zero mode Q_0 therefore measures the difference in numbers of b_n and c_n operators of a state, with 1 corresponding to a b_n mode and -1 to a c_n mode. This number is referred to as the $\mathfrak{bc}$-charge (or ghost number) of a state.

Since the $\mathfrak{bc}$-system is generated by (the linear combinations of) two free fermions, they therefore can be both in the Neveu-Schwarz sector, both in the Ramond or each in one sector. For the purpose of this thesis, we shall disregard the last case where the algebra is mixed. Because, in this case, $Q(z)$ and $T(z)$ have half-integer moding and therefore do not incorporate conformal symmetry. This is irrelevant to the superalgebras that we will construct in later chapters. The highest-weight state of a Neveu-Schwarz module of $\mathfrak{bc}$ is given by the tensor product of the two Neveu-Schwarz vacua of two free fermions, we will simply denote it by $|0\rangle$. The *parity* of a state refers to if it is bosonic or fermionic. Of course, a state can be a linear combination of two states of opposite parities. We define a parity reversal functor Π to change the parity of a state. States $|0\rangle$ (defined as bosonic) and $\Pi|0\rangle$ therefore have opposite parities, and generate two non-isomorphic irreducible modules, which are denoted by $\mathcal{N}_0$ and $\mathcal{N}_2$, respectively. To build a Neveu-Schwarz module, let all positive modes (annihilation operators) act as zero on the highest-weight state, and all negative modes (creation operators) act at most once. The PBW-basis for such a module is

$$\left\{(c^{k_n}_{-n+\frac{1}{2}}b^{j_n}_{-n+\frac{1}{2}})\cdots(c^{k_2}_{-\frac{3}{2}}b^{j_2}_{-\frac{3}{2}})(c^{k_1}_{-\frac{1}{2}}b^{j_1}_{-\frac{1}{2}})w \,\middle|\, j_i,k_i\in\{0,1\} \text{ and } \sum_{i=1}(j_i+k_i)<\infty\right\}, \tag{4.1.25}$$

where $w=|0\rangle$ or $\Pi|0\rangle$.

The operators of $\mathfrak{bc}$ in the Ramond sector have integer mode indices. We shall define b_0 and all the positive modes to be annihilators while c_0 and the negative modes as creators. As discussed in the free fermion case, one can introduce the operator $(-1)^F$ into the Ramond sector as to ensure the linear independence of a highest-weight state w and its parity reversal, which generate two irreducible modules of opposite parities. Alternatively, one can insist that irreducible will always mean $\mathbb{Z}_2$-graded irreducible, that is the module has no non-zero proper $\mathbb{Z}_2$-graded submodules. The $\mathbb{Z}_2$-grading here means the decomposition of a module as a direct sum of the bosonic and fermionic subspaces.

We shall denote the two irreducible Ramond $\mathfrak{bc}$-modules of bosonic and fermionic parities by $\mathcal{N}_1$ and $\mathcal{N}_3$, respectively. These two modules are not isomorphic only when considered as modules of the Ramond algebra extended by $(-1)^F$. The highest-weight state of a Ramond module has conformal dimension $\frac{1}{8}$, which comes from the two Ramond vacua of the free fermions. To see how $Q_0 = {:}bc{:}_0$ acts on a highest-weight state, consider the generalised commutation relation [57]

$$\sum_{i\geq 0}\left[A_{-j}B_j+(-1)^{\bar{A}\bar{B}}B_{-j-1}A_{j+1}\right]=\sum_{j\geq-\Delta_A}\binom{\Delta_A}{i+\Delta_A}{:}A_jB{:}_0, \tag{4.1.26}$$

where $\bar{A}$ is the parity of A, with 0 being bosonic and 1 fermionic, and Δ_A is the conformal dimension of A.

We replace A and B in (4.1.26) by b and c, respectively, and act both sides on a Ramond highest-weight state w. Equation (4.1.26) then becomes

$$b_0c_0w=\left[\binom{\frac{1}{2}}{0}{:}b_{-\frac{1}{2}}c{:}_0+\binom{\frac{1}{2}}{1}{:}b_{\frac{1}{2}}c{:}_0\right]w=({:}bc{:}_0+\frac{1}{2})w. \tag{4.1.27}$$

Swapping b_0 and c_0 on the left-hand side using the first anti-commutator in (4.1.21) gives

$$Q_0w={:}bc{:}_0w=\frac{1}{2}w. \tag{4.1.28}$$

The highest-weight state of the Ramond $\mathfrak{bc}$-module therefore has charge $\frac{1}{2}$. Unlike the free fermion, the Ramond module for the $\mathfrak{bc}$-system is no longer doubly-degenerate, the state $c_0 w$ is now a descendant state with charge $-\frac{1}{2}$. The PBW basis for such a module is

$$\left\{(c_{-n}^{k_n} b_{-n}^{j_n}) \cdots (c_{-2}^{k_2} b_{-2}^{j_2})(c_{-1}^{k_1} b_{-1}^{j_1}) c_0^{j_0} w \,\middle|\, j_i, k_i \in \{0,1\} \text{ and } \sum_{i=1} (k_i + j_i) < \infty \right\}, \tag{4.1.29}$$

where w is the highest-weight of the module.

As in any fermionic theory, it is appropriate to consider the character and the supercharacter of a module $\mathcal{N}$. For fermionic ghosts, these are defined as

$$\mathrm{Ch}[\mathcal{N}](\mathsf{z};\mathsf{q}) = \mathrm{Tr}_{\mathcal{N}} \mathsf{z}^{Q_0} \mathsf{q}^{L_0^{\mathrm{bc}} - 1/24}, \quad \mathrm{Sch}[\mathcal{N}](\mathsf{z};\mathsf{q}) = \mathrm{Tr}_{\mathcal{N}} (-1)^F \mathsf{z}^{Q_0} \mathsf{q}^{L_0^{\mathrm{bc}} - 1/24}, \tag{4.1.30}$$

where F equals 0 on the bosonic subspace and as 1 on the fermionic subspace. The $\mathfrak{bc}$-characters and supercharacters are easily calculated from their PBW-bases to be

$$\mathrm{Ch}[\mathcal{N}_0](\mathsf{z};\mathsf{q}) = \mathrm{Ch}[\mathcal{N}_2](\mathsf{z};\mathsf{q}) = \frac{\vartheta_3(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})}, \tag{4.1.31a}$$

$$\mathrm{Ch}[\mathcal{N}_1](\mathsf{z};\mathsf{q}) = \mathrm{Ch}[\mathcal{N}_3](\mathsf{z};\mathsf{q}) = \frac{\vartheta_2(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})}, \tag{4.1.31b}$$

$$\mathrm{Sch}[\mathcal{N}_0](\mathsf{z};\mathsf{q}) = -\mathrm{Sch}[\mathcal{N}_2](\mathsf{z};\mathsf{q}) = \frac{\vartheta_4(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})}, \tag{4.1.31c}$$

$$\mathrm{Sch}[\mathcal{N}_1](\mathsf{z};\mathsf{q}) = -\mathrm{Sch}[\mathcal{N}_3](\mathsf{z};\mathsf{q}) = \frac{\mathrm{i}\vartheta_1(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})}, \tag{4.1.31d}$$

where ϑ_i denotes the Jacobi theta functions, which are defined as

$$\vartheta_1(\mathsf{z};\mathsf{q}) = -\mathrm{i} \sum_{n\in\mathbb{Z}} (-1)^n z^{n+1/2} q^{(n+1/2)^2/2}, \quad \vartheta_2(\mathsf{z};\mathsf{q}) = \sum_{n\in\mathbb{Z}} z^{n+1/2} q^{(n+1/2)^2/2}, \tag{4.1.32a}$$

$$\vartheta_3(\mathsf{z};\mathsf{q}) = \sum_{n\in\mathbb{Z}} z^n q^{n^2/2}, \qquad \vartheta_4(\mathsf{z};\mathsf{q}) = \sum_{n\in\mathbb{Z}} (-1)^n z^n q^{n^2/2}. \tag{4.1.32b}$$

The conjugation automorphism γ_{bc} and the spectral flow isomorphisms $\sigma_{\mathrm{bc}}^{\ell}$, $\ell \in \mathbb{Z}/2$, act on the superalgebra $\mathfrak{bc}$ as

$$\begin{aligned} &\gamma_{\mathrm{bc}}(b_n) = c_n, \quad \gamma_{\mathrm{bc}}(c_n) = b_n, \quad \gamma_{\mathrm{bc}}(Q_n) = -Q_n, \quad \gamma_{\mathrm{bc}}(L_n^{\mathrm{bc}}) = L_n^{\mathrm{bc}}, \\ &\sigma_{\mathrm{bc}}^{\ell}(b_n) = b_{n-\ell}, \quad \sigma_{\mathrm{bc}}^{\ell}(c_n) = c_{n+\ell}, \quad \sigma_{\mathrm{bc}}^{\ell}(Q_n) = Q_n - \ell\delta_{n,0}, \quad \sigma_{\mathrm{bc}}^{\ell}\left(L_n^{\mathrm{bc}}\right) = L_n^{\mathrm{bc}} - \ell Q_n + \tfrac{1}{2}\ell^2\delta_{n,0}. \end{aligned} \tag{4.1.33}$$

It is now easily verified that twisting the modules introduced above by these automorphisms leads to

$$\gamma_{\mathrm{bc}}(\mathcal{N}_i) = \mathcal{N}_{-i}, \quad \sigma_{\mathrm{bc}}^{\ell}(\mathcal{N}_i) = \mathcal{N}_{i+2\ell}, \tag{4.1.34}$$

where we understand that the ghost modules indices are taken mod 4. Note that conjugation preserves a module in the Neveu-Schwarz sector but reverses the parity of a Ramond module. The spectral flow index is allowed to be a half integer in a fermionic theory, and when $\ell \in \mathbb{Z}+\frac{1}{2}$, the sector of a module is swapped. Up to an isomorphism, σ_{bc} may be identified with the parity reversal operator Π, whilst σ_{bc}^2 is a non-trivial automorphism of each $\mathcal{N}_i$.

Finally, the fusion rules of the $\mathfrak{bc}$-ghost system can be easily deduced from the fermionic Verlinde formula of [6]. Here we exploit the method of induction, which is elucidated in Appendix A, to compute these fusion rules from those of the Heisenberg algebra as stated in (3.1.35). We know from the celebrated boson-fermion correspondence [100–102], the $\mathfrak{bc}$-fields $b(z)$ and $c(z)$ are respectively identified with the vertex operators $V_1(z)$ and $V_{-1}(z)$ (both of conformal dimension $\frac{1}{2}$) of a free boson. Recall that a vertex operator $V_p(z)$ is related to the ground states $|p\rangle$ of a free boson by the state-field correspondence and is expanded as

$$V_p(z) = :e^{p\varphi(z)}: = \sum_{n=0}^{\infty} \frac{p^n}{n!} :\overbrace{\varphi(z)\cdots\varphi(z)}^{n \text{ times}}:. \tag{4.1.35}$$

The fields $V_1(z)$ and $V_{-1}(z)$ correspond to ground states of momenta 1 and -1, respectively, which in turn generate Fock spaces $\mathcal{F}_1$ and $\mathcal{F}_{-1}$. The two fields therefore act as simple currents allowing us to extend the Heisenberg vertex algebra $\mathcal{F}_0$. This means that the Heisenberg algebra is embedded in the $\mathfrak{bc}$-ghosts, which can be realised as an infinite-order simple current extension of the Heisenberg algebra. The vacuum module of $\mathfrak{bc}$, $\mathcal{N}_0$, is decomposed into Fock spaces as

$$\mathcal{N}_0 \downarrow = \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_p. \tag{4.1.36}$$

This is the branching rule of the $\mathfrak{bc}$-vacuum module. On the other hand, the induction of a Fock space $\mathcal{F}_i$ is defined by fusing it with the $\mathfrak{bc}$-vacuum

$$\mathcal{F}_i \uparrow = \mathcal{F}_i \times \mathcal{N}_0, \tag{4.1.37}$$

To obtain a $\mathfrak{bc}$-module $\mathcal{N}_i$, consider the induction of the Fock space $\mathcal{F}_{2n+i/2}$. Restricting the induced module using the branching rule (4.1.36) gives

$$\mathcal{F}_{2n+i/2} \uparrow\downarrow = \mathcal{F}_{2n+i/2} \times \left(\bigoplus_{p\in\mathbb{Z}} \mathcal{F}_p \right) = \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_{p+i/2} = \mathcal{N}_i \downarrow, \tag{4.1.38}$$

that is,

$$\mathcal{N}_i = \mathcal{F}_{2n+i/2} \uparrow, \tag{4.1.39}$$

where $n \in \mathbb{Z}$ and i is taken to be mod 4.

It follows from (4.1.38) that $\mathcal{N}_0$ and $\mathcal{N}_2$ have the same decomposition as Fock spaces. It is through parities that the two modules are distinguished. Using (4.1.39), we shall define the parity of $\mathcal{N}_i$ as the parity of its highest-weight state, while letting the highest-weight state of $\mathcal{F}_{2n+i/2}$ be bosonic. For example, the module $\mathcal{N}_0$ is induced from $\mathcal{F}_0$, the highest-weight state of $\mathcal{N}_0$ is given by the highest-weight state of $\mathcal{F}_0$, which is bosonic, and so is the parity of $\mathcal{N}_0$. On the other hand, $\mathcal{N}_2$ is induced from $\mathcal{F}_1$, whose highest-weight state is assigned with a bosonic parity. With this assignment, the highest-weight state of $\mathcal{N}_2$, which is the highest-weight state of $\mathcal{F}_0$, is fermionic. The parity of $\mathcal{N}_2$ is therefore fermionic. In the same fashion, $\mathcal{N}_1$ and $\mathcal{N}_3$ are defined to have bosonic and fermionic parities, respectively.

A main result of [65] is that induction is preserved by fusion. We can therefore compute the fusion between two $\mathfrak{bc}$-modules as

$$\mathcal{N}_i \times \mathcal{N}_j = \mathcal{F}_{2n+i/2}\uparrow \times \mathcal{F}_{2m+j/2}\uparrow = \mathcal{F}_{2(m+n)+(i+j)/2}\uparrow = \mathcal{N}_{i+j}, \tag{4.1.40}$$

where $m, n \in \mathbb{Z}$ and the addition in the index of the final $\mathfrak{bc}$-module is again understood to be mod 4.

4.2 The N=1 superconformal field theory

4.2.1 The chiral theory

The $N = 1$ superconformal field theory is motivated from its formulation in a superspace, in which the ordinary coordinates, z and $\bar{z}$, are twinned with the fermionic coordinates θ and $\bar{\theta}$ (known as Grassmann variables), respectively. The supersymmetric version of the energy-momentum tensor is decomposed as

$$\hat{T}(\zeta) = \theta T(z) + \frac{1}{2}G(z), \tag{4.2.1}$$

where $\zeta = (z;\theta)$. The fermionic field $G(z)$ has conformal dimension $\frac{3}{2}$ and is referred to as the superpartner of the usual energy-momentum tensor $T(z)$.

The rest of the section is formulated in the z-space rather than superspace. The $N = 1$ superalgebra is an extended algebra of the Virasoro algebra by the fermionic field $G(z)$. The OPEs between the generating fields are

$$T(z)T(w) \sim \frac{\mathsf{c}^{\mathrm{N=1}}/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{4.2.2a}$$

$$T(z)G(w) \sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}, \tag{4.2.2b}$$

$$G(z)G(w) \sim \frac{2\mathsf{c}^{\mathrm{N=1}}/3}{(z-w)^3} + \frac{2T(w)}{z-w}. \tag{4.2.2c}$$

These OPEs lead to the (anti-)commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\mathsf{c}^{\mathrm{N=1}}}{12}(m^3 - m)\delta_{m+n,0}, \tag{4.2.3a}$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}, \tag{4.2.3b}$$

$$\{G_r\,, G_s\} = 2L_{r+s} + \frac{\mathsf{c}^{\mathrm{N=1}}}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, \tag{4.2.3c}$$

where $m, n \in \mathbb{Z}$ and G_r are the Fourier expansion modes in $G(z) = \sum_r G_r z^{-r-\frac{3}{2}}$. Since $G(z)$ is a fermionic field, it can act on either the Neveu-Schwarz or the Ramond sector. The mode indices are half-integers in the Neveu-Schwarz sector and integers in the Ramond sector.

The first step in the analysis of the representations of the $N = 1$ superalgebra is again to define the highest-weight states and to specify their relations to the generating fields $T(z)$ and $G(z)$. For both sectors, we shall define annihilation operators as those whose mode indices are greater than zero. Highest-weight states $|\phi^{\pm}\rangle$ are eigenstates of L_0 satisfying

$$L_n|\phi^{\pm}\rangle = G_r|\phi^{\pm}\rangle = 0, \quad \text{for all } n, r > 0, \tag{4.2.4}$$

where we use $\pm$ to label the parity of a state, with $+$ $(-)$ being bosonic (fermionic). The conformal dimension of the highest-weight state is the eigenvalue with respect to L_0. The complete Verma module is generated by the action of the creation operators L_{-n} and G_{-r}, where $n > 0$ and $r \geq 0$. The zeroth mode of $G(z)$ in the Ramond sector is also regarded as a creation operator.

The vacuum states $|0^{\pm}\rangle$, which correspond to the identity fields $1^{\pm}(z)$, live in the Neveu-Schwarz sector. The vacuum states have the minimum energy (conformal dimension) 0 and are invariant under all global superconformal transformations. This requires them to satisfy

$$L_n|0^{\pm}\rangle = G_m|0^{\pm}\rangle = 0, \quad \text{where } n \geq -1,\, m \geq -\frac{1}{2}. \tag{4.2.5}$$

The bosonic Neveu-Schwarz vacuum state $|0^{+}\rangle$ allows the state-field correspondence to be defined. The highest-weight states $|\phi^{\pm}\rangle$ are related to the primary fields $\phi^{\pm}(z)$ by

$$|\phi^{\pm}\rangle = \lim_{z\to 0} \phi^{\pm}(z)|0^{+}\rangle. \tag{4.2.6}$$

We shall denote the field corresponding to the descendent state $G_{-\frac{1}{2}}|\phi^{\pm}\rangle$ by $\xi^{\mp}(z)$

$$G_{-\frac{1}{2}}|\phi^{\pm}\rangle = \lim_{z\to 0} \xi^{\mp}(z)|0^{+}\rangle. \tag{4.2.7}$$

Notice that the fields $\xi^{\pm}(z)$, even though being descendent fields in the $N = 1$ theory, are actually primary with respect to the Virasoro subalgebra. One can easily verify, using (4.2.3b), that the corresponding states of $\xi^{\pm}(z)$, $G_{-\frac{1}{2}}|\phi^{\mp}\rangle$, are annihilated by all positive modes of L_n.

Let us now proceed to the Ramond sector. Following the definition of (4.2.4), the states $|\phi^{\pm}\rangle$ and $G_0|\phi^{\pm}\rangle$ are the highest-weight states of a reducible Ramond module. As explained in [97], this space is always a reducible representation of the Ramond algebra but is an irreducible representation of the extension of the Ramond algebra by the operator $(-1)^F$. We also note that the space with highest-weight space given by these two states has no *graded* invariant subspaces and so is irreducible if we only consider graded representations. As noted above, this is the sense of irreducibility we will use in the rest of the thesis. This allows us to declare that $|\phi^{\pm}\rangle$ and $G_0|\phi^{\pm}\rangle$ are independent states in an $N = 1$ Verma module.

Consider the anti-commutation relation (4.2.3c). By letting $r = s = 0$, we arrive at

$$G_0^2|\phi^{\pm}\rangle = \left(L_0 - \frac{\mathsf{C}^{\mathrm{N=1}}}{24}\right)|\phi^{\pm}\rangle = \left(\Delta_\phi - \frac{\mathsf{c}^{\mathrm{N=1}}}{24}\right)|\phi^{\pm}\rangle. \tag{4.2.8}$$

This means, if normalised appropriately, the states $|\phi^{+}\rangle$ and $|\phi^{-}\rangle$ are related under the action of G_0 by

$$G_0|\phi^{+}\rangle = \sqrt{\Delta_\phi - \frac{\mathsf{c}^{\mathrm{N=1}}}{24}}|\phi^{-}\rangle \quad \text{and} \quad G_0|\phi^{-}\rangle = \sqrt{\Delta_\phi - \frac{\mathsf{c}^{\mathrm{N=1}}}{24}}|\phi^{+}\rangle, \tag{4.2.9}$$

where Δ_ϕ is the conformal dimension of $|\phi^{\pm}\rangle$. When $\Delta_\phi \neq \mathsf{c}^{\mathrm{N=1}}/24$, the states $G_0|\phi^{\pm}\rangle$ are thus indistinguishable from $|\phi^{\mp}\rangle$ apart from a scalar multiple. When $\Delta_\phi \neq \mathsf{c}^{\mathrm{N=1}}/24$, an irreducible $N = 1$ Ramond module may therefore be identified with its parity-reversed version and so its parity need not be specified. However, in the special case where $\Delta_\phi = \mathsf{c}^{\mathrm{N=1}}/24$, the action of G_0 annihilates the highest-weight state of the irreducible module, and the ground state is no longer doubly degenerate. In this case, the parity of the irreducible Ramond module needs to be specified.

A super-minimal model, denoted by $\mathsf{M}^{\mathrm{N=1}}(p,q)$, is parametrised by two positive integers $p, q \geq 2$, where $p - q \in 2\mathbb{Z}$, and $(p-q)/2$ and p are coprime to each other. Without loss of generality, let us take $p < q$. An analysis analogous to the Virasoro minimal models [46] shows that $\mathsf{M}^{\mathrm{N=1}}(p,q)$ is non-unitary unless $q = p + 2$. The central charge associated with $\mathsf{M}^{\mathrm{N=1}}(p,q)$ is given by

$$\mathsf{c}^{\mathrm{N=1}} = \frac{3}{2}\left(1 - \frac{2(p-q)^2}{pq}\right). \tag{4.2.10}$$

The bosonic $N = 1$ irreducible highest-weight modules, denoted by $\mathcal{W}_{r,s}$, are characterised by their bosonic parity, sectors and conformal dimensions of their highest-weight

$\Delta^{\text{N=1}}_{\alpha,\beta}$	s: 1	2	3	4	5
r: 1	0	$\frac{1}{16}$	$\frac{1}{6}$	$\frac{9}{16}$	1
2	$\frac{3}{8}$	$\frac{1}{16}$	$\frac{1}{24}$	$\frac{1}{16}$	$\frac{3}{8}$
3	1	$\frac{9}{16}$	$\frac{1}{6}$	$\frac{1}{16}$	0

Table 4.1: The Kac table for the $N = 1$ super-minimal model $\mathsf{M}^{\text{N=1}}(4,6)$ of central charge 1. Neveu-Schwarz and Ramond (shaded) modules alternate throughout the table. The Kac table possesses the symmetry given in (4.2.12).

states. The formula for the conformal dimension of the module $\mathcal{W}_{r,s}$ is given by

$$\Delta^{\text{N=1}}_{r,s} = \frac{(rq-sp)^2-(p-q)^2}{8pq} + \frac{1}{32}\left(1-(-1)^{r+s}\right), \tag{4.2.11}$$

where $1 \le r \le p-1$ and $1 \le s \le q-1$. The action of the parity reversal functor Π on $\mathcal{W}_{r,s}$ gives their fermionic partners $\Pi\mathcal{W}_{r,s}$.

Notice that the second factor in this formula is non-vanishing only when $r+s$ is odd. The modules, in this case, belong to the Ramond sector. When arranged according to their r-, s-labels, the modules from the two sectors are found to be alternatively distributed along the table. An example of an $N = 1$ Kac table is illustrated in Table 4.1. Not all pairs of (r,s) yield distinct modules because of the symmetry

$$\Delta^{\text{N=1}}_{r,s} = \Delta^{\text{N=1}}_{p-r,q-s}. \tag{4.2.12}$$

Unlike the Virasoro minimal model $\mathsf{M}(p,q)$, where p and q are required to be coprime, the two parameters of $\mathsf{M}^{\text{N=1}}(p,q)$ are allowed to be both even. In this case, there can be a fixed point of the relation (4.2.12) which is located at the centre of the Kac table:

$$r = \frac{p}{2} \quad \text{and} \quad s = \frac{q}{2}. \tag{4.2.13}$$

One can calculate from (4.2.10) and (4.2.11) that this module (which is always in the Ramond sector) has a conformal dimension of $\Delta^{\text{N=1}}_{p/2,q/2} = \mathsf{c}^{\text{N=1}}/24$. The module is referred to as *central*.

We shall denote a Verma module generated by the bosonic highest-weight state of conformal dimension $\Delta^{\text{N=1}}_{r,s}$ by ${}^{\text{N=1}}\mathcal{V}^{\text{NS/R}}_{r,s}$. In the Neveu-Schwarz sector, the Verma module has PBW-basis

$$\left\{ \left(G^{k_n}_{-n+1/2}L^{j_n}_{-n}\right)\cdots\left(G^{k_2}_{-3/2}L^{j_2}_{-2}\right)\left(G^{k_1}_{-1/2}L^{j_1}_{-1}\right)|\phi^+_{r,s}\rangle \;\middle|\; j_i \in \mathbb{Z}_{\ge 0},\ k_i \in \{0,1\} \text{ and } \sum_{i=1}^{\infty}(j_i+k_i) < \infty \right\}$$

and in the Ramond sector, the PBW-basis of the Verma module is

$$\left\{\left(G_{-n}^{k_n}L_{-n}^{j_n}\right)\cdots\left(G_{-2}^{k_2}L_{-2}^{j_2}\right)\left(G_{-1}^{k_1}L_{-1}^{j_1}\right)G_0^{k_0}|\phi_{r,s}^+\rangle \,\middle|\, j_i\in\mathbb{Z}_{\geq 0},\ k_i\in\{0,1\} \text{ and } \sum_{i=1}^{\infty}(j_i+k_i)<\infty\right\}.$$

The characters of the Verma modules in the two sectors computed from the PBW-bases are given by

$$\mathrm{Ch}[{}^{\mathrm{N=1}}\mathcal{V}_{r,s}^{\mathrm{NS}}](\mathsf{q}) = \mathsf{q}^{\Delta_{r,s}^{\mathrm{N=1}}-\mathrm{c}^{\mathrm{N=1}}/24}\prod_{n=1}^{\infty}\frac{1+\mathsf{q}^{n-1/2}}{1-\mathsf{q}^n}, \qquad r+s\in 2\mathbb{Z}, \tag{4.2.14a}$$

$$\mathrm{Ch}[{}^{\mathrm{N=1}}\mathcal{V}_{r,s}^{\mathrm{R}}](\mathsf{q}) = 2\mathsf{q}^{\Delta_{r,s}^{\mathrm{N=1}}-\mathrm{c}^{\mathrm{N=1}}/24}\prod_{n=1}^{\infty}\frac{1+\mathsf{q}^{n}}{1-\mathsf{q}^n}, \qquad r+s\in 2\mathbb{Z}+1. \tag{4.2.14b}$$

The irreducible modules $\mathcal{W}_{r,s}$ are the simple quotients of the Verma modules ${}^{\mathrm{N=1}}\mathcal{V}_{r,s}^{\mathrm{NS/R}}$. In the Neveu-Schwarz sector, the embedding diagram for the $N=1$ Neveu-Schwarz Verma modules is identical to that of the Virasoro case, not only in its structure but also in the labelling of the singular vectors. The only difference is that the levels at which the singular vectors occur in the Neveu-Schwarz $N=1$ case is half that of the Virasoro Verma module. The (super)characters of these irreducible modules are computed [103], with an assist of the $N=1$ Kac determinant formulae [104] to be

$$\mathrm{Ch}\left[\mathcal{W}_{r,s}\right](\mathsf{q}) = \begin{cases} \dfrac{1}{\eta(\mathsf{q})}\sqrt{\dfrac{\vartheta_3(1;\mathsf{q})}{\eta(\mathsf{q})}}\displaystyle\sum_{n\in\mathbb{Z}}\Big[\mathsf{q}^{(2npq+qr-us)^2/8pq} \\ \qquad -\mathsf{q}^{(2npq+qr+us)^2/8pq}\Big], & \text{if } r+s\in 2\mathbb{Z}, \\ \dfrac{1}{\eta(\mathsf{q})}\sqrt{\dfrac{\vartheta_2(1;\mathsf{q})}{2\eta(\mathsf{q})}}\displaystyle\sum_{n\in\mathbb{Z}}\Big[\mathsf{q}^{(2npq+qr-us)^2/8pq} \\ \qquad -\mathsf{q}^{(2npq+qr+us)^2/8pq}\Big], & \text{if } (r,s)=(\frac{p}{2},\frac{q}{2}). \\ \dfrac{2}{\eta(\mathsf{q})}\sqrt{\dfrac{\vartheta_2(1;\mathsf{q})}{2\eta(\mathsf{q})}}\displaystyle\sum_{n\in\mathbb{Z}}\Big[\mathsf{q}^{(2npq+qr-us)^2/8pq} \\ \qquad -\mathsf{q}^{(2npq+qr+us)^2/8pq}\Big], & \text{otherwise.} \end{cases} \tag{4.2.15}$$

The (super)characters of the irreducible modules were detailed in [6, 105], in which a fermionic version of the standard Verlinde formula is introduced and used to compute the (super)characters of the Ramond modules. These are given by

$$\mathrm{Sch}\left[\mathcal{W}_{r,s}\right](\mathsf{q}) = \begin{cases} \dfrac{1}{\eta(\mathsf{q})}\sqrt{\dfrac{\vartheta_4(1;\mathsf{q})}{\eta(\mathsf{q})}}\displaystyle\sum_{n\in\mathbb{Z}}(-1)^{np}\Big[\mathsf{q}^{(2npq+qr-us)^2/8pq} \\ \qquad -(-1)^r\mathsf{q}^{(2npq+qr+us)^2/8pq}\Big], & \text{if } r+s\in 2\mathbb{Z}, \\ 0, & \text{if } r+s\in 2\mathbb{Z}+1 \text{ and } (r,s)\neq(\frac{p}{2},\frac{q}{2}). \end{cases} \tag{4.2.16}$$

A subtlety exists in the supercharacter for the central Ramond module. Since the result is not used for in this thesis, We shall avoid this complication by ignoring the formula in this special case.

Note that the (super)characters of the $N = 1$ modules generated by a fermionic highest-weight state satisfy

$$\mathrm{Ch}\left[\Pi\mathcal{W}_{r,s}\right](\mathsf{q}) = \mathrm{Ch}\left[\mathcal{W}_{r,s}\right](\mathsf{q}), \quad \mathrm{Sch}\left[\Pi\mathcal{W}_{r,s}\right](\mathsf{q}) = -\mathrm{Sch}\left[\mathcal{W}_{r,s}\right](\mathsf{q}). \tag{4.2.17}$$

4.2.2 Fusion rules in the Neveu-Schwarz sector

Modifications of the Verlinde formula [48] are required in order for it to work in a fermionic theory. We will avoid this complication by computing the fusion by the alternative method of constructing PDEs for correlation functions. Other than the additional concept of parity, the process is analogous to the Virasoro fusion rules though significantly more complicated. We shall therefore be brief here and only state the differences from the Virasoro case.

Recall that in the Neveu-Schwarz sector, there are four Virasoro primary fields $\phi^{\pm}(z)$ and $\xi^{\mp}(z) = (G_{-1/2}\phi^{\pm})(z)$ (in two representations), corresponding to highest-weight states $|\phi^{\pm}\rangle$ and $N = 1$-descendent states $|\xi^{\mp}\rangle = G_{-1/2}|\phi^{\pm}\rangle$. The fields $\xi^{\pm}(z)$, according to the mode expansion (2.2.20), can be written in terms of a contour integral as

$$\xi^{\pm}(z) = \oint_z G(w)\phi^{\mp}(z)\frac{\mathrm{d}w}{2\pi\mathrm{i}}. \tag{4.2.18}$$

The first the non-trivial singular states [105, 106] $|\chi^{\pm}_{\alpha,\beta}\rangle$ of an irreducible $N = 1$ module in the Neveu-Schwarz sector occur at level $\frac{3}{2}$, where $(\alpha,\beta) = (1,3)$ or $(3,1)$. The singular vector is found by the constraint $G_{1/2}|\chi^{\pm}_{\alpha,\beta}\rangle = G_{3/2}|\chi^{\pm}_{\alpha,\beta}\rangle = 0$ to be

$$|\chi^{\pm}_{\alpha,\beta}\rangle = \left[G_{-\frac{1}{2}}L_{-1} - \left(\Delta^{\mathrm{N=1}}_{\alpha,\beta} + \frac{1}{2}\right)G_{-\frac{3}{2}}\right]|\phi^{\mp}_{\alpha,\beta}\rangle, \tag{4.2.19}$$

whose corresponding field takes the form

$$\chi^{\pm}_{\alpha,\beta}(z) = \oint_z G(w)\partial\phi^{\mp}_{\alpha,\beta}(z)\frac{\mathrm{d}w}{2\pi\mathrm{i}} - \left(\Delta^{\mathrm{N=1}}_{\alpha,\beta} + \frac{1}{2}\right)\oint_z G(w)\phi^{\mp}_{\alpha,\beta}(z)(z-w)^{-1}\frac{\mathrm{d}w}{2\pi\mathrm{i}}. \qquad (4.2.20)$$

A 3-point correlation function of the form

$$\langle 0|\chi^{\pm}_{\alpha,\beta}(z)\varphi_1(z_1)\varphi_2(z_2)|0\rangle, \qquad (4.2.21)$$

where $\varphi_i = \phi^{\pm}$ or $\xi^{\pm}$ is a Virasoro primary, must vanish, because of the singular field.

The aim of this section is to compute the fusion between two Neveu-Schwarz $N = 1$-primary fields with positive parities using the vanishing correlation function (4.2.21). We shall insert the expression for $\xi^{\pm}_{\alpha,\beta}(z)$ by substituting (4.2.20) into (4.2.21) in order to create a PDE of non-vanishing correlation functions. This provides us with a constraint on the conformal dimensions of the primary fields in a non-vanishing correlator, which will be interpreted as a constraint on fusion.

First of all, we would like the overall parity of the fields in (4.2.20) to be bosonic, so that the PDE we will construct has non-vanishing terms. For example, we can take the singular field to be fermionic, and the two Virasoro primary fields as bosonic and fermionic by choosing $\varphi_1 = \phi^{+}_1$ and $\varphi_2 = \xi^{-}_2$. Inserting (4.2.20) into this correlator, we have

$$\begin{aligned}\langle 0|\chi^{-}_{\alpha,\beta}(z)\phi^{+}_1(z_1)\xi^{-}_2(z_2)|0\rangle &= \langle 0|\oint_z G(w)\partial\phi^{+}_{\alpha,\beta}(z)\phi^{+}_1(z_1)\xi^{-}_2(z_2)\frac{\mathrm{d}w}{2\pi\mathrm{i}}|0\rangle \\ &- \left(\Delta^{\mathrm{N=1}}_{\alpha,\beta} + \frac{1}{2}\right)\langle 0|\oint_z G(w)\phi^{+}_{\alpha,\beta}(z)(w-z)^{-1}\phi^{+}_1(z_1)\xi^{-}_2(z_2)\frac{\mathrm{d}w}{2\pi\mathrm{i}}|0\rangle.\end{aligned} \qquad (4.2.22)$$

The integrands have potential poles at $w = z$, z_1 and z_2. We change the contours around these poles into a contour at the infinity subtracted by one at the origin as depicted in Figure 4.1. These contours are therefore related by

$$\oint_z + \oint_{z_1} + \oint_{z_2} = \oint_0 - \oint_\infty. \qquad (4.2.23)$$

Fourier expanding the fields inside each integral shows that both terms on the right-hand side of (4.2.22) vanish when evaluated around infinity and zero. The contour along z in (4.2.22) can then be written as minus the contour around z_1 and z_2:

$$\begin{aligned}\langle 0|\chi^{-}_{\alpha,\beta}(z)\phi^{+}_1(z_1)\xi^{-}_2(z_2)|0\rangle &= \left(-\oint_{z_1} - \oint_{z_2}\right)\langle 0|G(w)\partial\phi^{+}_{\alpha,\beta}(z)\phi^{+}_1(z_1)\xi^{-}_2(z_2)|0\rangle\frac{\mathrm{d}w}{2\pi\mathrm{i}} \\ &+ \left(\Delta^{\mathrm{N=1}}_{\alpha,\beta} + \frac{1}{2}\right)\left(\oint_{z_1} + \oint_{z_2}\right)\langle 0|G(w)\phi^{+}_{\alpha,\beta}(z)(w-z)^{-1}\phi^{+}_1(z_1)\xi^{-}_2(z_2)|0\rangle\frac{\mathrm{d}w}{2\pi\mathrm{i}}.\end{aligned}$$

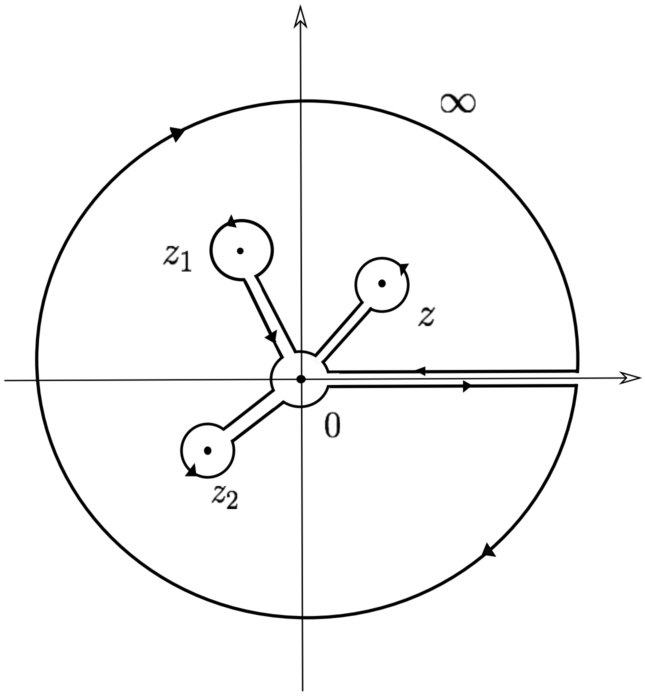

Figure 4.1: The contours around the poles at $w = z$, z_1 and z_2 can be deformed into a contour at infinity subtracted by a contour around zero.

To evaluate the integrals, first perform OPEs for $G(w)$ with $\phi_1^+(z_1)$ and $\xi_2^-(z_2)$. Notice that

$$G_{\frac{1}{2}}|\xi^\pm\rangle = G_{\frac{1}{2}}G_{-\frac{1}{2}}|\phi^\mp\rangle = \{G_{\frac{1}{2}}, G_{-\frac{1}{2}}\}|\phi^\mp\rangle = 2L_0|\phi^\mp\rangle = 2h|\phi^\mp\rangle, \tag{4.2.24}$$

where h is the conformal dimension of the highest-weight states $|\phi^\pm\rangle$. This leads to the OPEs

$$G(z)\xi^\pm(w) \sim \frac{2h\phi^\pm(w)}{(z-w)^2} + \frac{\partial\phi^\pm(w)}{z-w}, \tag{4.2.25a}$$

$$G(z)\phi^\pm(w) \sim \frac{\xi^\mp(w)}{z-w}. \tag{4.2.25b}$$

Substituting the OPEs of $G(z)$ into (4.2.24), we arrive at a PDE for 3-point correlation functions of the $N = 1$ fields

$$\left[-\partial_2\partial + \left(h_{\alpha,\beta} + \frac{1}{2}\right)\frac{1}{z_2 - z}\partial_2 - \frac{(2h_{\alpha,\beta}+1)h_2}{(z_2-z)^2}\right]\langle 0|\phi^+_{\alpha,\beta}(z)\phi_1^+(z_1)\phi_2^+(z_2)|0\rangle$$
$$- \left[\partial - \left(h_{\alpha,\beta} + \frac{1}{2}\right)\frac{1}{z_1 - z}\right]\langle 0|\phi^+_{\alpha,\beta}(z)\xi_1^-(z_1)\xi_2^-(z_2)|0\rangle = 0, \tag{4.2.26}$$

where $h_{\alpha,\beta}$, h_1 and h_2 are the conformal dimensions of $\phi_{\alpha,\beta}(z)$, $\phi_1(z_1)$ and $\phi_2(z_2)$, respectively. The partial derivatives with respect to to z, z_1 and z_2 are respectively denoted by ∂, ∂_1 and ∂_2. Using the same procedure, we examine another vanishing correlator

$\langle 0|\chi^-_{\alpha,\beta}(z)\xi^-_1(z_1)\phi^+_2(z_2)|0\rangle$ from which we construct the following PDE

$$\begin{aligned}\left[-\partial_1\partial+\left(h_{\alpha,\beta}+\frac{1}{2}\right)\frac{1}{z_1-z}\partial_1-\frac{(2h_{\alpha,\beta}+1)h_1}{(z_1-z)^2}\right]\langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle\\ +\left[\partial-\left(h_{\alpha,\beta}+\frac{1}{2}\right)\frac{1}{z_2-z}\right]\langle 0|\phi^+_{\alpha,\beta}(z)\xi^-_1(z_1)\xi^-_2(z_2)|0\rangle=0.\end{aligned} \tag{4.2.27}$$

Observe that (4.2.26) and (4.2.27) are coupled equations for the correlators $\langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle$ and $\langle 0|\phi^+_{\alpha,\beta}(z)\xi^-_1(z_1)\xi^-_2(z_2)|0\rangle$. Since the fields involved in the two PDEs are all Virasoro primaries, the correlators must take the general form given in (2.3.2b). We wish to determine how the constants, $C_{\phi\phi\phi}$ and $C_{\phi\xi\xi}$, associated with these correlators are related. To achieve this, consider now another correlation function of the form $\langle 0|G_{-1/2}\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\xi^+_2(z_2)|0\rangle$, which vanishes because $\langle 0|$ is annihilated by $G_{-1/2}$. We then commute the $G_{-1/2}$-mode through to the right using the following (anti-)commutation relations derived from (4.2.25)

$$[G_m,\phi^{\pm}(w)]_{\pm}=\oint_w\frac{\xi^{\pm}(w)}{z-w}z^{m+1/2}\frac{\mathrm{d}z}{2\pi\mathrm{i}}=w^{m+1/2}\xi^{\pm}(w), \tag{4.2.28a}$$

$$\begin{aligned}[G_m,\xi^{\pm}(w)]_{\pm}&=\oint_w\left(\frac{2h\phi^{\pm}(w)}{(z-w)^2}+\frac{\partial\phi^{\pm}(w)}{z-w}\right)z^{m+\frac{1}{2}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\\ &=(2m+1)hw^{m-1/2}\phi^{\pm}(w)+w^{m+\frac{1}{2}}\partial\phi^{\pm}(w),\end{aligned} \tag{4.2.28b}$$

where h is the conformal dimension of $\phi(z)$, and '$[\ ,\ ]_+$' stands for the commutation relation denoted by $[\ ,\ ]$ previously, and '$[\ ,\ ]_-$' the anti-commutator $\{\ ,\ \}$.

Back to the vanishing correlator $\langle 0|G_{-1/2}\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\xi^+_2(z_2)|0\rangle$, as we commute $G_{-1/2}$ to the right, we arrive at the relation

$$\begin{aligned}\langle 0|G_{-1/2}\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\xi^+_2(z_2)|0\rangle=\langle 0|\xi^-_{\alpha,\beta}(z)\phi^+_1(z_1)\xi^+_2(z_2)|0\rangle+\\ \langle 0|\phi^+_{\alpha,\beta}(z)\xi^-_1(z_1)\xi^-_2(z_2)|0\rangle+\partial_2\langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle=0.\end{aligned} \tag{4.2.29}$$

Analogously, by considering other vanishing correlators, we come up with expressions similar to (4.2.29):

$$\begin{aligned}&\langle 0|\xi^-_{\alpha,\beta}(z)\phi^+_1(z_1)\xi^-_2(z_2)|0\rangle+\langle 0|\xi^-_{\alpha,\beta}(z)\xi^-_1(z_1)\phi^-_2(z_2)|0\rangle\\ &\qquad+\partial\langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle=0,\end{aligned} \tag{4.2.30a}$$

$$\begin{aligned}&\langle 0|\xi^-_{\alpha,\beta}(z)\xi^-_1(z_1)\phi^+_2(z_2)|0\rangle+\langle 0|\phi^+_{\alpha,\beta}(z)\xi^-_1(z_1)\xi^-_2(z_2)|0\rangle\\ &\qquad+\partial_1\langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle=0.\end{aligned} \tag{4.2.30b}$$

The relations (4.2.29) and (4.2.30) provide three constraints on the four correlators in them. We can write the constants of three correlators in terms of the constant of the fourth. Assuming none of the correlators vanishes, we normalise them such that the constant associated with $\langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle$, denoted by $C_{\phi\phi\phi}$, is 1. We thus obtain the following four correlators:

$$\begin{aligned}
\langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle &= \frac{1}{(z-z_1)^{h_{\alpha,\beta}+h_1-h_2}(z-z_2)^{h_{\alpha,\beta}-h_1+h_2}(z_1-z_2)^{-h_{\alpha,\beta}+h_1+h_2}},\\
\langle 0|\phi^+_{\alpha,\beta}(z)\xi^-_1(z_1)\xi^-_2(z_2)|0\rangle &= \frac{h_{\alpha,\beta}-h_1-h_2}{(z-z_1)^{h_{\alpha,\beta}+h_1-h_2}(z-z_2)^{h_{\alpha,\beta}-h_1+h_2}(z_1-z_2)^{-h_{\alpha,\beta}+h_1+h_2+1}},\\
\langle 0|\xi^-_{\alpha,\beta}(z)\phi^+_1(z_1)\xi^-_2(z_2)|0\rangle &= \frac{-h_{\alpha,\beta}+h_1-h_2}{(z-z_1)^{h_{\alpha,\beta}+h_1-h_2}(z-z_2)^{h_{\alpha,\beta}-h_1+h_2+1}(z_1-z_2)^{-h_{\alpha,\beta}+h_1+h_2}},\\
\langle 0|\xi^-_{\alpha,\beta}(z)\xi^-_1(z_1)\phi^+_2(z_2)|0\rangle &= \frac{-h_{\alpha,\beta}-h_1+h_2}{(z-z_1)^{h_{\alpha,\beta}+h_1-h_2+1}(z-z_2)^{h_{\alpha,\beta}-h_1+h_2}(z_1-z_2)^{h_{\alpha,\beta}+h_1-h_2+1}}.
\end{aligned} \tag{4.2.31}$$

We can now substitute (4.2.31) into the PDEs (4.2.26) and (4.2.27) obtained previously. Recall that we have assumed that none of the correlators appearing in the PDEs are zero. For the PDEs to hold, the conformal dimensions of the three primary fields $\phi_{\alpha,\beta}$, ϕ_1 and ϕ_2 must satisfy

$$h_2 = \frac{1}{4}\left(1+2h_{\alpha,\beta}+4h_1 \pm \sqrt{1-4h_{\alpha,\beta}+4h^2_{\alpha,\beta}+16h_1+32h_{\alpha,\beta}h_1}\right). \tag{4.2.32}$$

This is the condition for a 3-point correlation function in the Neveu-Schwarz $N=1$ theory to be nonzero. Now, let $h_{\alpha,\beta}=\Delta^{\mathrm{N=1}}_{1,3}$ and $h_1=\Delta^{\mathrm{N=1}}_{r,s}$. Then h_2 is solved to be equal to $\Delta^{\mathrm{N=1}}_{r,s\pm 2}$, which yields the fusion rule

$$\mathcal{W}_{1,3}\times\mathcal{W}_{r,s} = \mathcal{W}_{r,s-2}\oplus\mathcal{W}_{r,s+2}\oplus\cdots. \tag{4.2.33}$$

And when taking $h_{\alpha,\beta}=\Delta^{\mathrm{N=1}}_{3,1}$, the fusion rule becomes

$$\mathcal{W}_{3,1}\times\mathcal{W}_{r,s} = \mathcal{W}_{r-2,s}\oplus\mathcal{W}_{r+2,s}\oplus\cdots. \tag{4.2.34}$$

Note that these fusions are not yet complete, since the fusion of two bosonic module may also give fermionic modules as a result. This is what we will investigate next.

In order to obtain the fermionic modules appearing in the fusion, we shall let $\phi_2(z_2)$ have negative parity while fixing the parities of $\chi_{\alpha,\beta}$ and $\phi_1(z_1)$ as before. Again, we construct PDEs by using OPEs between the fields in vanishing 3-point correlation functions. The constants associated with the non-vanishing correlators involved in these PDEs are computed as in (4.2.29) and (4.2.30). Among these, two of the correlators take the form

$$\begin{aligned}
&\langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\xi^+_2(z_2)|0\rangle\\
&\quad= \frac{1}{(z-z_1)^{h_{\alpha,\beta}+h_1-h_2-1/2}(z-z_2)^{h_{\alpha,\beta}-h_1+h_2+1/2}(z_1-z_2)^{-h_{\alpha,\beta}+h_1+h_2+1/2}},
\end{aligned} \tag{4.2.35a}$$

$$\langle 0|\phi^+_{\alpha,\beta}(z)\xi^-_1(z_1)\phi^-_2(z_2)|0\rangle$$
$$= \frac{-1}{(z-z_1)^{h_{\alpha,\beta}+h_1-h_2+1/2}(z-z_2)^{h_{\alpha,\beta}-h_1+h_2-1/2}(z_1-z_2)^{-h_{\alpha,\beta}+h_1+h_2+1/2}}, \quad (4.2.35b)$$

where the first correlator is normalised so that its associated constant $C_{\phi\phi\xi}$ is 1, and the constant of the second correlator found to be related to $C_{\phi\phi\xi}$ by a negative sign. The constraint in the conformal dimensions of $\phi_{\alpha,\beta}(z)$, $\phi_1(z_1)$ and $\phi_2(z_2)$ leads to

$$\mathcal{W}_{3,1} \times \mathcal{W}_{r,s} = \mathcal{W}_{1,3} \times \mathcal{W}_{r,s} = \Pi\mathcal{W}_{r,s} \oplus \cdots . \quad (4.2.36)$$

Combining this with (4.2.33) and (4.2.34), the complete fusion of $\mathcal{W}_{1,3}$ (and $\mathcal{W}_{3,1}$) with $\mathcal{W}_{r,s}$ is

$$\mathcal{W}_{1,3} \times \mathcal{W}_{r,s} = \mathcal{W}_{r,s-2} \oplus \Pi\mathcal{W}_{r,s} \oplus \mathcal{W}_{r,s+2}, \quad (4.2.37a)$$
$$\mathcal{W}_{3,1} \times \mathcal{W}_{r,s} = \mathcal{W}_{r-2,s} \oplus \Pi\mathcal{W}_{r,s} \oplus \mathcal{W}_{r+2,s}. \quad (4.2.37b)$$

However, (4.2.37) is not quite enough, as the base cases, for deducing the general fusion rules by induction. We also need

$$\mathcal{W}_{2,2} \times \mathcal{W}_{r,s} = \mathcal{W}_{r-1,s-1} \oplus \Pi\mathcal{W}_{r-1,s+1} \oplus \mathcal{W}_{r+1,s+1} \oplus \Pi\mathcal{W}_{r+1,s-1} \quad (4.2.38)$$

which may be calculated from the level-1 singular vector

$$|\chi^\pm_{2,2}\rangle = L^2_{-1}|\phi^\pm_{2,2}\rangle + \frac{2h_{2,2}(10h_{2,2}-1)}{1-7h_{2,2}}L_{-2}|\phi^\pm_{2,2}\rangle - \frac{(1-4h_{2,2})^2}{1-7h_{2,2}}G_{-\frac{3}{2}}G_{-\frac{1}{2}}|\phi^\pm_{2,2}\rangle, \quad (4.2.39)$$

where $|\phi^\pm_{2,2}\rangle$ has conformal dimension $h_{2,2}$ and corresponds to $\phi^\pm_{2,2}(z)$ by the state-field correspondence.

Equations (4.2.37) and (4.2.39) form the base cases from which we can induce to the general Neveu-Schwarz fusion rule. The induction and truncation processes follow exactly as in the Virasoro case, which we shall not repeat here. It turns out that, other than the concept of parities, the $N=1$ fusion in the Neveu-Schwarz sector is exactly the same as the Virasoro fusion:

$$\mathcal{W}_{r,s} \times \mathcal{W}_{r',s'} = \bigoplus_{r''=1}^{p-1}\bigoplus_{s''=1}^{q-1} \mathsf{N}^{[p,q]\,(r'',s'')}_{(r,s),(r',s')}\Pi^{(t-t_{\max})/2}\mathcal{W}_{r'',s''}, \quad (4.2.40)$$

where the fusion coefficients are identical to the Virasoro coefficients (3.2.15). Notice that the fusion coefficient requires that, if given $r+s$, $r'+s' \in 2\mathbb{Z}$, the modules appear on the right-hand side of the fusion must have $r''+s'' \in 2\mathbb{Z}$. This means the fusion of two Neveu-Schwarz modules yields Neveu-Schwarz modules only. The parity of the resulting modules is observed to follow a pattern described as follows. First denote the sum of r''

and s'' of a module by t. The module with the maximal value of t has bosonic parity; denote this maximal t by $t_{\max}$. The fusion coefficient requires the module with $t = t_{\max}$ to be unique. Now for any module on the right-hand side of (4.2.54), its parity is bosonic if $t - t_{\max} = 0 \bmod 4$, and fermionic if $t - t_{\max} = 2 \bmod 4$.

Lastly, parity is preserved with respect to fusion. Fusions between modules with parities other than bosonic can be computed using the identity

$$\Pi^i(\mathcal{M}) \times \Pi^j(\mathcal{N}) = \Pi^{i+j}(\mathcal{M} \times \mathcal{N}), \tag{4.2.41}$$

where $\mathcal{M}$ and $\mathcal{N}$ are modules of the super-minimal model $\mathsf{M}^{\mathrm{N=1}}(p,q)$.

4.2.3 Fusion rules involving Ramond fields

As discussed earlier in the chapter, there are two types of Ramond modules, distinguished by their conformal dimensions — the doubly degenerate modules which do not carry an overall parity and the central modules which are generated by a single highest-weight state and can be either bosonic or fermionic. We shall only concern ourselves with the doubly degenerate Ramond modules in this section and avoid the complications such as parity brought by the Ramond central modules. Fusion rules involving central Ramond modules can be derived following an analogous but more careful procedure as that presented in this section.

In a 3-point correlation function, for the overall sector of fields to be Neveu-Schwarz, there can only be two Ramond fields. We will consider two cases: The fusion of a Ramond field with a Neveu-Schwarz field, which yields a Ramond field; and the fusion between two Ramond fields, which gives a Neveu-Schwarz field in return. In either case, we need a non-trivial singular vector at the lowest level possible. This singular is found at level-1 to be [107, 108]

$$|\chi^{\pm}_{\alpha,\beta}\rangle = \left(L_{-1} - \frac{6}{8\Delta^{\mathrm{N=1}}_{\alpha,\beta} + \mathsf{c}} G_{-1}G_0\right)|\phi^{\pm}_{\alpha,\beta}\rangle, \tag{4.2.42}$$

where $(\alpha,\beta) = (1,2)$ or $(2,1)$. Notice that the levels of a Ramond module are integers, because of the integer modes G_n.

Let us start on the first case by considering a vanishing correlation function of the form $\langle 0|\chi^{+}_{\alpha,\beta}(z)\phi^{+}_{1}(z_1)\phi^{+}_{2}(z_2)|0\rangle = 0$, where $\chi^{\pm}_{\alpha,\beta}(z)$ is the unphysical Ramond field corresponding to the state (4.2.42), and ϕ^{+}_{1} and ϕ^{+}_{2} are Neveu-Schwarz and Ramond fields, respectively. To compute the fusion between $\phi_{\alpha,\beta}(z)$ and $\phi_1(z)$, we need to construct a PDE from the vanishing correlator. As in the previous section, we shall start by writing the modes in

(4.2.42) in terms of contour integrals:

$$\begin{aligned}\langle 0|\chi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle &= \partial\langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle \\ &\quad - \frac{6}{8\Delta^{N=1}_{r,s}+c}\langle 0|\oint_z G(w)(w-z)^{-\frac{1}{2}}G_0\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)\frac{dw}{2\pi i}|0\rangle,\end{aligned} \tag{4.2.43}$$

where ∂ denotes the partial derivative with respect to z. Note that even though the integral term contains the factor $(w-z)^{-1/2}$, there is no branch cut in the integral. The OPE between $G(w)$ and $\phi^+_{\alpha,\beta}(z)$ introduces half-integer poles that combine with the factor to make the integral analytic. To evaluate this integral, we change the contour around z to contours around z_1 and z_2 according to (4.2.23) and contract $G(w)$ with $\phi^+_1(z_1)$ and $\phi^+_2(z_2)$, respectively. The OPE between $G(w)$ and the Neveu-Schwarz field $\phi^+_1(z_1)$ follows from (4.2.25b), whereas ϕ^+_2 is a Ramond field satisfying

$$G(w)\phi^\pm_2(z_2) \sim \frac{G_0\phi^\pm_2(z_2)}{(w-z_2)^{3/2}} + \frac{G_{-1}\phi^\pm_2(z_2)}{(w-z_2)^{1/2}}. \tag{4.2.44}$$

The integral in (4.2.43) then becomes

$$I = \oint_z G(w)(w-z)^{-\frac{1}{2}}G_0\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)\frac{dw}{2\pi i} \tag{4.2.45a}$$

$$\begin{aligned}&= -\oint_{z_1}(w-z)^{-\frac{1}{2}}G_0\phi^+_{\alpha,\beta}(z)\left(\frac{\xi^+_1(z_1)}{w-z_1}\right)\phi^+_2(z_2)\frac{dw}{2\pi i} \\ &\qquad -\oint_{z_2}(w-z)^{-\frac{1}{2}}G_0\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\left(\frac{G_0\phi^+_2(z_2)}{(w-z_2)^{3/2}}+\frac{G_{-1}\phi^+_2(z_2)}{(w-z_2)^{1/2}}\right)\frac{dw}{2\pi i},\end{aligned} \tag{4.2.45b}$$

where the integrand in the second term contains half-integer poles at $w = z_2$.

One of the difficulties with computing fusion rules for Ramond fields is the existence of half-integer poles such as those in (4.2.45), which cannot be evaluated by Cauchy's theorem. This problem can be solved by inserting a 'fudge factor' into the integrand and binomially expanding the factor. The process leads to a relation between an integral with integer poles and one with half-integer poles, which we wish to calculate. Instead of computing (4.2.45) directly, we insert into it a factor of $(w-z_2)^{1/2}$. The new integral has integer poles and can be evaluated:

$$I' = \oint_z G(w)(w-z)^{-\frac{1}{2}}(w-z_2)^{\frac{1}{2}}G_0\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)\frac{dw}{2\pi i} \tag{4.2.46a}$$

$$\begin{aligned}&= -\oint_{z_1}(w-z)^{-\frac{1}{2}}(w-z_2)^{\frac{1}{2}}G_0\phi^+_{\alpha,\beta}(z)\left(\frac{\xi^+_1(z_1)}{w-z_1}\right)\phi^+_2(z_2)\frac{dw}{2\pi i} \\ &\qquad -\oint_{z_2}(w-z)^{-\frac{1}{2}}G_0\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\left(\frac{G_0\phi^+_2(z_2)}{w-z_2}+G_{-1}\phi^+_2(z_2)\right)\frac{dw}{2\pi i},\end{aligned}$$

$$= -(w-z)^{-\frac{1}{2}}(w-z_2)^{\frac{1}{2}} G_0 \phi^+_{\alpha,\beta}(z) \xi^+_1(z_1) \phi^+_2(z_2) - (w-z)^{-\frac{1}{2}} G_0 \phi^+_{\alpha,\beta}(z) \phi^+_1(z_1) G_0 \phi^+_2(z_2). \tag{4.2.46b}$$

The trick is to come up with a relation between (4.2.45a) and (4.2.46a), such that (4.2.45a) can be written in terms of evaluable quantities. This is achieved by binomially expanding the inserted factor as

$$(w-z_2)^{\frac{1}{2}} = (w-z+z-z_2)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (w-z)^k (z-z_2)^{\frac{1}{2}-k}, \tag{4.2.47}$$

The series is checked to be convergent for w close to z. With this expansion, (4.2.46a) becomes

$$I' = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \oint_z G(w)(w-z)^{k-\frac{1}{2}} (z-z_2)^{\frac{1}{2}-k} G_0 \phi^+_{\alpha,\beta}(z) \phi^+_1(z_1) \phi^+_2(z_2) \frac{\mathrm{d}w}{2\pi \mathrm{i}}. \tag{4.2.48}$$

Notice that, in this infinite series, the terms with $k > 1$ have no poles in w, the integral therefore vanishes for these terms. We only need to consider the contribution from the $k = 1$ and 0 terms, whose sum is evaluated as

$$\begin{aligned} I' = &\frac{1}{2} G_0^2 \phi^+_{\alpha,\beta}(z) \phi^+_1(z_1) \phi^+_2(z_2)(z-z_2)^{-\frac{1}{2}} \\ &+ (z-z_2)^{\frac{1}{2}} \oint_z G(w)(w-z)^{-\frac{1}{2}} G_0 \phi^+_{\alpha,\beta}(z) \phi^+_1(z_1) \phi^+_2(z_2) \frac{\mathrm{d}w}{2\pi \mathrm{i}}. \end{aligned} \tag{4.2.49}$$

The integral we want to compute is contained in the second term of (4.2.49). We now know that integrals I and I' are related by

$$I' = (z-z_2)^{\frac{1}{2}} I + \frac{1}{2} G_0^2 \phi^+_{\alpha,\beta}(z) \phi^+_1(z_1) \phi^+_2(z_2)(z-z_2)^{-\frac{1}{2}}. \tag{4.2.50}$$

Substituting the result for I' given by (4.2.46b), we arrive at

$$\begin{aligned} &\oint_z G(w)(w-z)^{-\frac{1}{2}} G_0 \phi^+_{\alpha,\beta}(z) \phi^+_1(z_1) \phi^+_2(z_2) \frac{\mathrm{d}w}{2\pi \mathrm{i}} \\ =&\frac{(z_1-z_2)^{1/2}}{(z-z_2)^{1/2}(z_1-z)^{1/2}} G_0 \phi^+_{\alpha,\beta}(z) \xi^-_1(z_1) \phi^+_2(z_2) \\ &+ \frac{1}{(z_2-z)^{1/2}(z-z_2)^{1/2}} G_0 \phi^+_{\alpha,\beta}(z) \phi^+_1(z_1) G_0 \phi^+_2(z_2) - \frac{1}{2(z-z_2)} G_0^2 \phi^+_{\alpha,\beta}(z) \phi^+_1(z_1) \phi^+_2(z_2). \end{aligned} \tag{4.2.51}$$

With this result, one can now construct a PDE using the vanishing correlator (4.2.43):

$$\begin{aligned}\left(\partial + \frac{3(h_{\alpha,\beta} - \mathsf{c}/24)}{8h_{\alpha,\beta} + \mathsf{c}} \frac{1}{z - z_2}\right) \langle 0|\phi^+_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^+_2(z_2)|0\rangle \\ + \frac{6\sqrt{(h_{\alpha,\beta} - \mathsf{c}/24)(h_2 - \mathsf{c}/24)}}{8h_{\alpha,\beta} + \mathsf{c}} \frac{\langle 0|\phi^-_{\alpha,\beta}(z)\phi^+_1(z_1)\phi^-_2(z_2)|0\rangle}{(z - z_2)^{1/2}(z_2 - z)^{\frac{1}{2}}} \\ - \frac{6\sqrt{h_{\alpha,\beta} - \mathsf{c}/24}}{8h_{\alpha,\beta} + \mathsf{c}} \frac{(z_1 - z_2)^{1/2}}{(z_1 - z)^{1/2}(z - z_2)^{1/2}} \langle 0|\phi^-_{\alpha,\beta}(z)\xi^-_1(z_1)\phi^+_2(z_2)|0\rangle = 0,\end{aligned} \tag{4.2.52}$$

where the action of G_0 on a field is further simplified according to (4.2.9).

Following a similar process as in the Neveu-Schwarz case, the PDE gives constraints on the conformal dimensions, $h_{\alpha,\beta}$, h_1 and h_2, of the three primary fields, which leads to the following fusions

$$\begin{aligned}\mathcal{W}_{1,2} \times \mathcal{W}_{r,s} &= \mathcal{W}_{r,s-1} \oplus \mathcal{W}_{r,s+1}, \\ \mathcal{W}_{2,1} \times \mathcal{W}_{r,s} &= \mathcal{W}_{r-1,s} \oplus \mathcal{W}_{r+1,s},\end{aligned} \tag{4.2.53}$$

where $r + s \in 2\mathbb{Z}$. Parities in these fusions are redundant because ϕ_1 is the only module which carries a parity, this parity does not affect the result of the fusion.

The general fusion rule between a non-central Ramond and a Neveu-Schwarz module induced from the basic cases given in (4.2.53) turns out to take the same form as the fusion between two Neveu-Schwarz fields

$$\mathcal{W}_{r,s} \times \mathcal{W}_{r',s'} = \bigoplus_{r''=1}^{p-1} \bigoplus_{s''=1}^{q-1} \mathsf{N}^{[p,q]\,(r'',s'')}_{(r,s),(r',s')} \mathcal{W}_{r'',s''}, \tag{4.2.54}$$

with $r + s \in 2\mathbb{Z} + 1$, $(r, s) \neq (\frac{p}{2}, \frac{q}{2})$, $r' + s' \in 2\mathbb{Z}$, $r'' + s'' \in 2\mathbb{Z} + 1$ and $(r'', s'') \neq (\frac{p}{2}, \frac{q}{2})$. Here we have assumed that the Ramond modules appearing on the right-hand side are non-central. In the case that the right-hand side indeed contains a central module $\mathcal{W}_{p/2,q/2}$, its super-partner $\Pi\mathcal{W}_{p/2,q/2}$ must appear as well.

Since fields in a correlation function are commutative up to a scalar, one can swap $\phi_1(z_1)$ and $\phi_2(z_2)$ in the above calculation and obtain the fusion rule between two Ramond modules. We shall not detail this calculation here, since it follows analogously from the previous. The fusion rules between two non-central Ramond modules are given by

$$\mathcal{W}_{r,s} \times \mathcal{W}_{r',s'} = \bigoplus_{r''=1}^{p-1} \bigoplus_{s''=1}^{q-1} \mathsf{N}^{[p,q]\,(r'',s'')}_{(r,s),(r',s')} \left(\mathcal{W}_{r'',s''} + \Pi\mathcal{W}_{r'',s''}\right), \tag{4.2.55}$$

where $r + s$ and $r' + s' \in 2\mathbb{Z} + 1$, (r, s) and $(r', s') \neq (\frac{p}{2}, \frac{q}{2})$, $r'' + s'' \in 2\mathbb{Z}$.

4.3 The $N = 2$ superconformal field theory

4.3.1 Algebraic preliminaries

A further extension of the Virasoro algebra by two fermionic fields $G^+(z)$ and $G^-(z)$ of conformal dimension $\frac{3}{2}$ leads to the $N = 2$ superconformal algebras. The super-energy-momentum tensor has a conformal dimension of 1 and takes the form

$$\hat{T}(\zeta) = \theta_1\theta_2 T(z) + \alpha_1\theta_1 G^+(z) + \alpha_2\theta_1 G^+(z) + \alpha_3 J(z), \tag{4.3.1}$$

where α_i are normalising constants, and the coordinates of the superspace is given by z twinned with two Grassmann variables of conformal dimension $-\frac{1}{2}$, θ_1 and θ_2. Other than the Virasoro energy-momentum tensor $T(z)$ and the two fermionic fields $G^\pm(z)$, which encode the supersymmetries, the set of generating fields also includes a bosonic field $J(z)$ of conformal dimension 1 which generates the Heisenberg algebra. The operator product expansions between these fields are

$$\begin{gathered}
T(z)T(w) \sim \frac{\mathsf{c}^{\mathrm{N=2}}/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\
T(z)G^\pm(w) \sim \frac{\frac{3}{2}G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(z)}{z-w}, \qquad T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \\
J(z)J(w) \sim \frac{\mathsf{c}^{\mathrm{N=2}}/3}{(z-w)^2}, \qquad J(z)G^\pm(w) \sim \frac{\pm G^\pm(w)}{z-w}, \qquad G^\pm(z)G^\pm(w) \sim 0, \\
G^\pm(z)G^\mp(w) \sim \frac{(2\mathsf{c}^{\mathrm{N=2}}/3)}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2T(w) \pm J(w)}{z-w}.
\end{gathered} \tag{4.3.2}$$

The OPEs on the second line show that $G^\pm(z)$ and $J(z)$ are Virasoro-primary fields of conformal dimensions $\frac{3}{2}$ and 1, respectively. From (4.3.2), one can derive the (anti-)commutation relations between the Fourier modes of the fields:

$$\begin{gathered}
[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}(m^3 - m)\delta_{m+n,0}\mathsf{C}^{\mathrm{N=2}}, \\
[L_m, J_n] = -nJ_{m+n}, \qquad [J_m, J_n] = \tfrac{1}{3}m\delta_{m+n,0}\mathsf{C}^{\mathrm{N=2}}, \\
[L_m, G^\pm_s] = (\tfrac{1}{2}m - s)G^\pm_{m+s}, \qquad [J_m, G^\pm_s] = \pm G^\pm_{m+s}, \\
\{G^\pm_r, G^\mp_s\} = 2L_{r+s} \pm (r-s)J_{r+s} + \tfrac{1}{12}(4r^2 - 1)\delta_{r+s,0}\mathsf{C}^{\mathrm{N=2}}.
\end{gathered} \tag{4.3.3}$$

Similar to the algebras we introduced previously, the $N = 2$ algebra is parametrised by the central charge $\mathsf{c}^{\mathrm{N=2}} \in \mathbb{C}$, the eigenvalue with respect to the operator $\mathsf{C}^{\mathrm{N=2}}$.

As in any fermionic theory, the mode indices are sector-dependent. This is determined by the boundary conditions imposed on the fermionic fields. The moding in the three sectors of the $N = 2$ algebra is summarised by

	L_m	J_n	G^+_r	G^-_s
Neveu-Schwarz	$m \in \mathbb{Z}$	$n \in \mathbb{Z}$	$r \in \mathbb{Z}+\frac{1}{2}$	$s \in \mathbb{Z}+\frac{1}{2}$
Ramond	$m \in \mathbb{Z}$	$n \in \mathbb{Z}$	$r \in \mathbb{Z}$	$s \in \mathbb{Z}$

We will not concern ourselves with the twisted sector, in which $G^{\pm}(z)$ have different boundary conditions, because the moding, in this case, is not well adapted to the $G^{\pm}_n$ basis elements. This sector is not usually considered to be part of the $N = 2$ minimal models.

Let $\mathfrak{g}_{\text{NS}}$ denote the $N = 2$ algebra with Neveu-Schwarz moding. We decompose it as $\mathfrak{g}_{\text{NS}} = \mathfrak{g}^+_{\text{NS}} \oplus \mathfrak{h} \oplus \mathfrak{g}^-_{\text{NS}}$, where $\mathfrak{g}^+_{\text{NS}}$ and $\mathfrak{g}^-_{\text{NS}}$ are spanned by the positive and the negative modes of $\mathfrak{g}_{\text{NS}}$, respectively, while $\mathfrak{h} = \text{span}\{J_0, L_0, \mathsf{C}^{\text{N=2}}\}$. A state in the Neveu-Schwarz $N = 2$ sector is characterised by its conformal weight (L_0-eigenvalue), its charge (J_0-eigenvalue) and its parity. The parity of a state is indicated by $\pm$, with $+$ ($-$) denoting bosonic (fermionic). A highest-weight state in $\mathfrak{g}_{\text{NS}}$, denoted by $|j, \Delta, \pm\rangle$, is simultaneously an eigenstate of J_0, L_0 and $\mathsf{C}^{\text{N=2}}$ satisfying

$$J_0|j,\Delta,\pm\rangle = j|j,\Delta,\pm\rangle, \quad L_0|j,\Delta,\pm\rangle = \Delta|j,\Delta,\pm\rangle, \quad \mathsf{C}^{\text{N=2}}|j,\Delta,\pm\rangle = \mathsf{c}^{\text{N=2}}|j,\Delta,\pm\rangle \tag{4.3.4}$$

and is annihilated by $\mathfrak{g}^+_{\text{NS}}$. We shall denote the one-dimensional $(\mathfrak{g}^+_{\text{NS}} \oplus \mathfrak{h})$-module spanned by the highest-weight vector $|j, \Delta, \pm\rangle$ by $\mathbb{C}^{\text{NS},\pm}_{j,\Delta}$. Inducing $\mathbb{C}^{\text{NS},\pm}_{j,\Delta}$ to a full $\mathfrak{g}_{\text{NS}}$-module by letting $\mathfrak{g}^-_{\text{NS}}$ act freely gives a Verma module, which we shall denote by $\mathcal{V}^{\text{NS},\pm}_{j,\Delta}$. Now take the Verma module $\mathcal{V}^{\text{NS},+}_{0,0}$ and quotient by the submodules generated by the singular vectors $G^{\pm}_{-1/2}|0,0,+\rangle$. This quotient module is known as the universal vacuum module of the $N = 2$ theory, and is denoted by $\mathcal{U}^{\text{N=2}}_{\mathsf{c}}$. We shall likewise denote the irreducible quotient of $\mathcal{V}^{\text{NS},\pm}_{j,\Delta}$ by $\mathcal{L}^{\text{NS},\pm}_{j,\Delta}$.

In the Ramond sector, the $G^{\pm}_n$ modes in the $N = 2$ algebra $\mathfrak{g}_{\text{R}}$ have integer indices. The algebra is similarly decomposed into $\mathfrak{g}^{\pm}_{\text{R}}$ and $\mathfrak{h}$ except that we now let $G^+_0 \in \mathfrak{g}^+_{\text{R}}$ and $G^-_0 \in \mathfrak{g}^-_{\text{R}}$. A Ramond highest-weight vector is then a simultaneous eigenvector of J_0, L_0 and $\mathsf{C}^{\text{N=2}}$ as in (4.3.4), and is annihilated by $\mathfrak{g}^+_{\text{R}}$. The Ramond Verma module $\mathcal{V}^{\text{R},\pm}_{j,\Delta}$ is again constructed by letting the one-dimensional $(\mathfrak{g}^+_{\text{R}} \oplus \mathfrak{h})$-module $\mathbb{C}^{\text{R},\pm}_{j,\Delta}$ be induced to a full Ramond module, while letting $\mathfrak{g}^+_{\text{R}}$ act freely. The irreducible quotients of $\mathcal{V}^{\text{R},\pm}_{j,\Delta}$ are likewise denoted by $\mathcal{L}^{\text{R},\pm}_{j,\Delta}$.

4.3.2 The $N = 2$ super-minimal models and automorphisms

The universal vacuum module is not simple when the central charge is [109]

$$\mathsf{c}^{\text{N=2}} = 3 - \frac{6v}{u}, \quad u \in \mathbb{Z}_{\geq 2},\ v \in \mathbb{Z}_{\geq 1},\ \gcd\{u,v\} = 1. \tag{4.3.5}$$

For these central charges, the unique simple quotient of the universal $N = 2$ algebra $\mathcal{L}^{\text{NS/R},\pm}_{j,\Delta}$ gives the super-minimal models of the $N = 2$ theory, which are denoted by $\mathsf{M}^{\text{N=2}}(u, v)$. One of the goals of this thesis is to examine such super-minimal models with the method of coset constructions, which we will detail in Chapter 5.

The $N = 2$ Lie superalgebras admit many automorphisms including the conjugation automorphism $\gamma^{\text{N=2}}$, given by

$$\gamma_{\text{N=2}}(L_n) = L_n, \quad \gamma_{\text{N=2}}(J_n) = -J_n, \quad \gamma_{\text{N=2}}(G_s^{\pm}) = G_s^{\mp}, \quad \gamma_{\text{N=2}}\left(\mathsf{C}^{\text{N=2}}\right) = \mathsf{C}^{\text{N=2}}, \tag{4.3.6}$$

and the spectral flow automorphisms $\sigma^{\ell}_{\text{N=2}}$, $\ell \in \mathbb{Z}/2$, given by

$$\sigma^{\ell}_{\text{N=2}}(L_n) = L_n - \ell J_n + \frac{1}{6}\ell^2 \delta_{n,0}\mathsf{C}^{\text{N=2}}, \quad \sigma^{\ell}_{\text{N=2}}(J_n) = J_n - \frac{1}{3}\ell \delta_{n,0}\mathsf{C}^{\text{N=2}}, \tag{4.3.7}$$

$$\sigma^{\ell}_{\text{N=2}}(G_s^{\pm}) = G^{\pm}_{s+\ell}, \quad \sigma^{\ell}_{\text{N=2}}\left(\mathsf{C}^{\text{N=2}}\right) = \mathsf{C}^{\text{N=2}}. \tag{4.3.8}$$

Taking $\ell \in \mathbb{Z} + \frac{1}{2}$ changes the moding of the fermions, meaning that half-integer spectral flows define isomorphisms between the Neveu-Schwarz and the Ramond $N = 2$ Lie superalgebras. Both conjugation and spectral flow lift to automorphisms of the universal $N = 2$ vertex superalgebras as well as their minimal model quotients. Following from (2.4.2), one can easily check that the result of acting with an automorphism on a state v satisfies

$$\begin{aligned} &L_0\gamma_{\text{N=2}}(v) = \Delta\gamma_{\text{N=2}}(v), && J_0\gamma_{\text{N=2}}(v) = -j\gamma_{\text{N=2}}(v), \\ &L_0\sigma^{\ell}_{\text{N=2}}(v) = (\Delta + \ell j + \frac{1}{6}\ell^2\mathsf{c}^{\text{N=2}})\sigma^{\ell}_{\text{N=2}}(v), && J_0\sigma^{\ell}_{\text{N=2}}(v) = (j + \frac{1}{3}\ell\mathsf{c}^{\text{N=2}})\sigma^{\ell}_{\text{N=2}}(v). \end{aligned} \tag{4.3.9}$$

The states $\gamma_{\text{N=2}}(v)$ and $\sigma^{\ell}_{\text{N=2}}(v)$ are therefore weight vectors, meaning that they have a fixed weight and charge. One can then deduce from this the following isomorphisms among irreducible $N = 2$ modules

$$\begin{aligned} &\gamma_{\text{N=2}}\left(\mathcal{L}^{\text{NS},\pm}_{j,\Delta}\right) \cong \mathcal{L}^{\text{NS},\pm}_{-j,\Delta}, \quad \gamma_{\text{N=2}}\left(\mathcal{L}^{\text{R},\pm}_{j,\Delta}\right) \cong \begin{cases} \mathcal{L}^{\text{R},\pm}_{-j,\Delta}, & \text{if } \Delta = \frac{\mathsf{c}^{\text{N=2}}}{24}, \\ \mathcal{L}^{\text{R},\mp}_{-j+1,\Delta}, & \text{otherwise}, \end{cases} \\ &\sigma^{1/2}_{\text{N=2}}\left(\mathcal{L}^{\text{NS},\pm}_{j,\Delta}\right) \cong \mathcal{L}^{\text{R},\pm}_{j+\mathsf{c}^{\text{N=2}}/6,\Delta+j/2+\mathsf{c}^{\text{N=2}}/24}, \\ &\sigma^{1/2}_{\text{N=2}}\left(\mathcal{L}^{\text{R},\pm}_{j,\Delta}\right) \cong \begin{cases} \mathcal{L}^{\text{NS},\pm}_{j+\mathsf{c}^{\text{N=2}}/6,(j+\mathsf{c}^{\text{N=2}}/6)/2}, & \text{if } \Delta = \frac{\mathsf{c}^{\text{N=2}}}{24}, \\ \mathcal{L}^{\text{NS},\mp}_{j-1+\mathsf{c}/6,\Delta+(j-1)/2+\mathsf{c}/24}, & \text{otherwise}. \end{cases} \end{aligned} \tag{4.3.10}$$

4.4 $\widehat{\mathfrak{osp}}(1|2)$ and its minimal models

4.4.1 The finite algebra

The Lie superalgebra $\mathfrak{osp}(1|2)$ has basis $\{e, h, f, x, y\}$, in which the bosonic subalgebra $\mathfrak{g}_0 = \text{span}\{e, h, f\}$ is isomorphic to $\mathfrak{sl}_2$, satisfying

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{4.4.1}$$

The elements x and y are fermionic, they span the fermionic subalgebra $\mathfrak{g}_1$. The remaining (anti-)commutation relations are

$$\begin{aligned} [h,x] &= x, & [e,x] &= 0, & [f,x] &= -y, \\ [h,y] &= -y, & [e,y] &= -x, & [f,y] &= 0, \\ \{x,y\} &= h, & \{x,x\} &= 2e, & \{y,y\} &= -2f. \end{aligned} \tag{4.4.2}$$

A module $\mathcal{M}$ of $\mathfrak{osp}(1|2)$, as any module of a superalgebra, can be decomposed into the direct sum of a bosonic and a fermionic subspace, $\mathcal{M}_0$ and $\mathcal{M}_1$. The parity of $\mathcal{M}$ is $\mathbb{Z}_2$-graded as usual with $\mathfrak{g}_i\mathcal{M}_j \subseteq \mathcal{M}_{i+j}$, for all $i,j \in \mathbb{Z}_2$.

A state in an $\mathfrak{osp}(1|2)$-module is characterised by its $\mathfrak{osp}$-weight (h-eigenvalue) λ and its parity. The bosonic irreducible modules include the following types:

- The $\overline{\mathcal{A}}_\lambda$, where $\lambda \in \mathbb{Z}_{\geq 0}$. Each is a $(2\lambda+1)$-dimensional highest-weight and lowest-weight module, with a unique highest-weight λ and a unique lowest-weight $-\lambda$. The operators e, f, x, y are nilpotent. The highest-weight and the lowest-weight states are bosonic.

- The $\overline{\mathcal{B}}^+_\lambda$, where $\lambda \notin \mathbb{Z}_{\geq 0}$. Each is an infinite-dimensional Verma module of highest-weight λ. The operators e and x are nilpotent, while f and y act on the highest-weight state freely. The highest-weight state is bosonic.

- The $\overline{\mathcal{B}}^-_\lambda$, where $\lambda \notin \mathbb{Z}_{\geq 0}$. Each is an infinite-dimensional lowest-weight module with the lowest-weight $-\lambda$. The operators f and y are nilpotent, while e and x act on the lowest-weight state freely. These are the conjugation of $\overline{\mathcal{B}}^-_\lambda$.

The modules listed above, and their parity reversals, exhaust the irreducible highest- and lowest-weight modules of $\mathfrak{osp}(1|2)$. We included a bar in the notation for all above finite modules in order to distinguish them from the affine modules we will introduce later on. However, there is an additional group of irreducible modules that need to be included. These are the infinite-dimensional modules with no highest nor lowest weights. All operators act freely on a characterising state of weight $\lambda \in \mathbb{C}/2\mathbb{Z}$, and other states in the module have weights in $\lambda + \mathbb{Z}$. Such modules are referred to as *dense modules*. It turns out that dense modules of $\mathfrak{osp}(1|2)$ are not uniquely determined by the weights of their characterising states. This motivates us to introduce an extra operator, the *super-Casimir*, denoted by ς, which is given by

$$\varsigma = xy - yx + \frac{1}{2} \tag{4.4.3}$$

and satisfies

$$[\varsigma, \mathfrak{g}_0] = \{\varsigma, \mathfrak{g}_1\} = 0. \tag{4.4.4}$$

The name 'super-Casimir' is chosen in reference with the word 'Casimir', which plays a similar role as the energy-momentum tensor in an affine algebra. Let the ς-eigenvalue on a bosonic state of an $\mathfrak{osp}(1|2)$ dense module be Σ. The fermionic subspace of the dense module then has ς-eigenvalue $-\Sigma$. We shall denote such dense modules by $\overline{\mathcal{C}}_{\lambda,\Sigma}$, where $\lambda^2 \neq (\Sigma - \frac{1}{2})^2$ is required by irreducibility [41]. The parity reversals of the $\overline{\mathcal{C}}_{\lambda,\Sigma}$ of course also contribute to the list of irreducible $\mathfrak{osp}(1|2)$-modules.

The modules $\overline{\mathcal{C}}^{+}_{\lambda,\Sigma}$ with $\lambda^2 = (\Sigma - \frac{1}{2})^2$ are reducible but indecomposable. They are notationally distinguished from their conjugate modules $\overline{\mathcal{C}}^{-}_{\lambda,\Sigma}$ by the sign on the top right corner. These modules are characterised by the following short sequence

$$0 \longrightarrow \Pi\overline{\mathcal{B}}^{\pm}_{\lambda+1} \longrightarrow \overline{\mathcal{C}}^{\pm}_{\lambda,\Sigma} \longrightarrow \overline{\mathcal{B}}^{\mp}_{\lambda} \longrightarrow 0, \tag{4.4.5}$$

where $\Sigma = \mp\lambda + \frac{1}{2}$ for $\overline{\mathcal{C}}^{\pm}_{\lambda,\Sigma}$.

4.4.2 The affine algebra and the minimal models

The affinisation from $\mathfrak{osp}(1|2)$ to $\widehat{\mathfrak{osp}}(1|2)$ follows from (3.3.1), where we tensored the finite algebra with Laurent polynomials and added a level operator $\hat{\mathsf{k}}$. The affine algebra $\widehat{\mathfrak{osp}}(1|2)$ is generated by bosonic modes e_n, h_n and f_n, as well as fermionic modes x_n and y_n. Their non-zero (anti-)commutation relations are given by (3.3.3) (the bosonic subalgebra of $\widehat{\mathfrak{osp}}(1|2)$ is isomorphic to $\widehat{\mathfrak{sl}}_2$) along with

$$\begin{gathered}[e_m, y_s] = -x_{m+s}, \quad [h_m, x_s] = x_{m+s}, \quad [h_m, y_s] = -y_{m+s}, \quad [f_m, x_s] = -y_{m+s}, \\ \{x_r, x_s\} = 2e_{r+s}, \quad \{x_r, y_s\} = h_{r+s} + 2r\delta_{r+s,0}\mathsf{k}, \quad \{y_r, y_s\} = -2f_{r+s}.\end{gathered} \tag{4.4.6}$$

The fermionic modes have integer indices in the Neveu-Schwarz sector and half-integer indices in the Ramond sector.

The conformal symmetry is provided by the Sugawara construction with the following choice of energy-momentum tensor:

$$T^{\text{osp}}(z) = \frac{1}{2\mathsf{k}+3}\left[\frac{1}{2}{:}hh{:}(z) + {:}ef{:}(z) + {:}fe{:}(z) - \frac{1}{2}{:}xy{:}(z) + \frac{1}{2}{:}yx{:}(z)\right]. \tag{4.4.7}$$

The central charge associated with this energy-momentum tensor is found to be

$$\mathsf{c}^{\text{osp}} = \frac{2\mathsf{k}}{2\mathsf{k}+3}. \tag{4.4.8}$$

A state in an $\widehat{\mathfrak{osp}}(1|2)$-module is characterised by its charge (h_0-eigenvalue), conformal dimension and parity. Consider the subalgebra $\mathfrak{g}_v$ in the Neveu-Schwarz sector generated by all positive modes, y_0 and f_0. A highest-weight state is defined to be an eigenstate of h_0 and $\hat{\mathsf{k}}$ (with eigenvalues λ and k, respectively) which is annihilated by all operators in $\mathfrak{g}_v$. We shall denote the highest-weight state of charge h and conformal dimension Δ by $|h, \Delta, \pm\rangle$, where the bosonic and fermionic parities of the states are denoted by + and

−, respectively. A Verma module of $\widehat{\mathfrak{osp}}(1|2)$ is given by inducing the one-dimensional module spanned by $|h,\Delta,\pm\rangle$ to a full $\widehat{\mathfrak{osp}}(1|2)$-module while requiring all modes in $\mathfrak{g}_v$ to act as zero, while $\hat{\mathsf{k}}$ acts as the multiple k.

For the universal vacuum module $\mathcal{U}^{\text{osp}}_{\mathsf{k}}$, which lives in the Neveu-Schwarz sector, first consider another subalgebra $\mathfrak{g}_u$, generated by $\{\hat{\mathsf{k}}, e_n, f_n, h_n, x_n, y_n \,|\, n \in \mathbb{Z} \text{ and } n \geq 0\}$. Inducing the one-dimensional submodule spanned by $|0,0,+\rangle$ to a full $\widehat{\mathfrak{osp}}(1|2)$-module while letting $\hat{\mathsf{k}}$ act as the scalar k and the rest of $\mathfrak{g}_u$ act as zero yields $\mathcal{U}^{\text{osp}}_{\mathsf{k}}$.

The level-k $\widehat{\mathfrak{osp}}(1|2)$ minimal model $\mathsf{B}_{0|1}(u,v)$ is defined to be the simple quotient of the universal supervertex operator algebra associated to $\widehat{\mathfrak{osp}}(1|2)$ with [110]

$$\mathsf{k} = -\frac{3}{2} + \frac{u}{2v}, \qquad u \in \mathbb{Z}_{\geq 2},\ v \in \mathbb{Z}_{\geq 1},\ \frac{u+v}{2} \in \mathbb{Z},\ \gcd\left\{u, \frac{u+v}{2}\right\} = 1. \tag{4.4.9}$$

The central charge associated with the energy-momentum tensor in (4.4.7) in terms of u and v is given by

$$\mathsf{c}^{\text{osp}} = 1 - \frac{3v}{u}. \tag{4.4.10}$$

Spectral flow acts on the generators of $\widehat{\mathfrak{osp}}(1|2)$ and the Virasoro zero mode L_0^{osp} obtained from (4.4.7) as follows:

$$\begin{gathered}
\sigma^{\ell}_{\text{osp}}(e_n) = e_{n-\ell}, \qquad \sigma^{\ell}_{\text{osp}}(h_n) = h_n - \delta_{n,0}\ell\mathsf{k}, \qquad \sigma^{\ell}_{\text{osp}}(f_n) = f_{n+\ell}, \\
\sigma^{\ell}_{\text{osp}}(x_n) = x_{n-\ell/2}, \qquad \sigma^{\ell}_{\text{osp}}(y_n) = y_{n+\ell/2}, \\
\sigma^{\ell}_{\text{osp}}(L_0^{\text{osp}}) = L_0^{\text{osp}} - \frac{1}{2}\ell h_0 + \frac{1}{4}\ell^2\mathsf{k}.
\end{gathered} \tag{4.4.11}$$

Note that restricting σ^{osp} to the bosonic subalgebra $\widehat{\mathfrak{sl}}_2$ recovers σ^{sl}. As with $A_1(u,v)$-modules, the spectral flow $\sigma^{\ell}_{\text{sl}}(\mathcal{M})$ of a $\mathsf{B}_{0|1}(u,v)$-module $\mathcal{M}$ is another $\mathsf{B}_{0|1}(u,v)$-module. If $\ell \in 2\mathbb{Z}$, then spectral flow preserves the sector (Neveu-Schwarz or Ramond) of the module while these sectors are exchanged if $\ell \in 2\mathbb{Z}+1$.

The conjugation automorphism acts on the $\widehat{\mathfrak{osp}}(1|2)$-modes as

$$\begin{aligned}
\gamma_{\text{osp}}(e_n) &= -f_n, & \gamma_{\text{osp}}(h_n) &= -h_n, & \gamma_{\text{osp}}(f_n) &= -e_n, \\
\gamma_{\text{osp}}(x_n) &= -y_n, & \gamma_{\text{osp}}(y_n) &= x_n, & \gamma_{\text{osp}}(L_0^{\text{osp}}) &= L_0^{\text{osp}}.
\end{aligned} \tag{4.4.12}$$

The classification of irreducible relaxed highest-weight $\mathsf{B}_{0|1}(u,v)$-modules has only recently been completed in [111], see also [56]. Our aim here is to provide an alternative classification that relies on a coset construction. This has the advantage that it will also allow us to easily deduce the characters, which were also only recently calculated [55], as well as the Grothendieck fusion rules, which were previously unknown. To prepare for this classification and to fix notation, we introduce the irreducible relaxed highest-weight $\widehat{\mathfrak{osp}}$)(1|2)-modules following [41]:

- The ${}^{\mathrm{NS}}\mathcal{A}_\lambda$ (${}^{\mathrm{R}}\mathcal{A}_\lambda$), where $\lambda \in \mathbb{Z}_{\geq 0}$. Each is an irreducible highest-weight module in the Neveu-Schwarz (Ramond) sector whose space of ground states forms an irreducible finite-dimensional module $\overline{\mathcal{A}}_\lambda$ for $\mathfrak{osp}(1|2)$ ($\mathcal{L}_{\lambda,0}$ for $\mathfrak{sl}_2$). The highest-weight vector of each module is bosonic with h_0-charge λ and conformal dimension $\frac{\lambda(\lambda+1)}{2(2\mathsf{k}+3)}$.

- The ${}^{\mathrm{NS}}\mathcal{B}^+_\lambda$ (${}^{\mathrm{R}}\mathcal{B}^+_\lambda$), where $\lambda \notin \mathbb{Z}_{\geq 0}$. Each is an irreducible highest-weight module in the Neveu-Schwarz (Ramond) sector whose space of ground states forms an irreducible infinite-dimensional Verma module for $\mathfrak{osp}(1|2)$ ($\mathfrak{sl}_2$). The highest-weight state of each module is likewise bosonic with h_0-charge λ and conformal dimension $\frac{\lambda(\lambda+1)}{2(2\mathsf{k}+3)}$.

- The ${}^{\mathrm{NS}}\mathcal{B}^-_\lambda$ (${}^{\mathrm{R}}\mathcal{B}^-_\lambda$), where $\lambda \notin \mathbb{Z}_{\geq 0}$, that are the conjugates of the ${}^{\mathrm{NS}}\mathcal{B}^+_\lambda$ (${}^{\mathrm{R}}\mathcal{B}^+_\lambda$).

- The ${}^{\mathrm{NS}}\mathcal{C}_{\lambda,\Sigma}$ (${}^{\mathrm{R}}\mathcal{C}_{\lambda,q}$), where $\lambda \in \mathbb{C}$ and $\Sigma \in \mathbb{C}$ ($q \in \mathbb{C}$) satisfy $\lambda \neq \pm(\Sigma - \frac{1}{2}) \pmod 2$ ($\lambda \neq -1 \pm \sqrt{1+2q} \pmod 2$). Each is an irreducible relaxed highest-weight module whose ground states have h_0-charges equal to $\lambda \pmod 2$ and conformal dimension given by

$$ {}^{\mathrm{NS}}\mathcal{C}_{\lambda,\Sigma} : \quad \frac{\Sigma^2 - 1/4}{2(2\mathsf{k}+3)}, \qquad {}^{\mathrm{R}}\mathcal{C}_{\lambda,q} : \quad \frac{q - \mathsf{k}/4}{2\mathsf{k}+3}. \tag{4.4.13} $$

  The ground state of h_0-charge λ is bosonic. Here, Σ denotes the eigenvalue of the $\mathfrak{osp}$ super-Casimir [112]

$$ \varsigma = x_0 y_0 - y_0 x_0 + \frac{1}{2} \tag{4.4.14} $$

  on the bosonic ground states, while q denotes the ground state eigenvalue of the $\mathfrak{sl}_2$ quadratic Casimir

$$ \Omega = \frac{1}{2} h_0^2 + e_0 f_0 + f_0 e_0. \tag{4.4.15} $$

- The parity reversals of the above irreducibles obtained by declaring in each case that the ground state of h_0-charge λ is fermionic rather than bosonic. Parity reversal will be denoted by Π.

Of course, the spectral flows of these irreducible relaxed highest-weight modules will again be irreducible, though they are usually not relaxed nor highest-weight.

Coset Construction for $N = 2$ minimal models

So far, we have discussed two general classes of CFTs, the Virasoro minimal models and their supersymmetric extensions, and the minimal models corresponding to the affine Kac-Moody (super)algebras. For the second class, we have assigned a Sugawara energy-momentum tensor to an algebra so as to provide it with a conformal symmetry. The coset construction essentially builds a commutant from the CFTs we introduced in Chapters 3 and 4, which helps with constructing and studying new algebras. As we will soon find out, the method provides the framework for a complete classification for the new algebras, and allows automorphisms, characters, fusion rules, etc. to be calculated in a simpler way than using a Verlinde formula.

5.1 The embedding

The coset construction, often referred to as the *Goddard-Kent-Olive (GKO) construction*, was first proposed in [46, 113]. The papers provided a method of constructing representations of the Virasoro algebra out of those of affine Kac-Moody algebras, yielding the full discrete series of irreducible representations of the Virasoro algebra. They also proposed the coset for the $N = 1$ superconformal algebra in which the irreducible highest-weight modules in the Neveu-Schwarz and the Ramond sectors are described. Following a theorem of Frenkel-Zhu [114] which generalises the GKO construction to certain vertex operator algebras, Li and Lepowsky [115] presented an even more general discussion that works for all vertex operator algebras.

We shall rephrase the theorem in the context of CFT as follows. Let $\mathfrak{g}_v$ be a vertex operator algebra with its conformal symmetry provided by the energy-momentum tensor $T^v(z)$. Assume $\mathfrak{g}_v$ has a subalgebra $\mathfrak{g}_s$ with an energy-momentum tensor $T^s(z)$. Denote the central charges of $\mathfrak{g}_v$ and $\mathfrak{g}_s$ be c_v and c_s, respective. We define the (coset) commutant of $\mathfrak{g}_s$ in $\mathfrak{g}_v$, denoted by

$$\mathfrak{g}_{v/s} = \mathrm{Com}(\mathfrak{g}_s, \mathfrak{g}_v) = \frac{\mathfrak{g}_s}{\mathfrak{g}_v}, \tag{5.1.1}$$

as the algebra generated by the set of fields in $\mathfrak{g}_v$ which commute with $\mathfrak{g}_s$. The coset algebra $\mathfrak{g}_{v/s}$ also carries a conformal symmetry with its energy-momentum tensor given by

$$T^{v/s}(z) = T^v(z) - T^s(z) \tag{5.1.2}$$

associated with the central charge $\mathsf{c}_v - \mathsf{c}_s$.

As is well known, see [38] for an early instance and [116, Thm. 8.4] for a proof, the $N = 2$ minimal model $\mathsf{M}^{\mathrm{N=2}}(u, v)$ can be represented as the following coset (commutant)

$$\mathsf{M}^{\mathrm{N=2}}(u, v) = \mathrm{Com}(\mathcal{H}, A_1(u, v) \otimes \mathfrak{bc}) = \frac{A_1(u, v) \otimes \mathfrak{bc}}{\mathcal{H}}, \tag{5.1.3}$$

where $A_1(u, v)$, $\mathfrak{bc}$ and $\mathcal{H}$ denote the simple vertex superalgebras associated to the affine $\widehat{\mathfrak{sl}}(2)$ at level $\mathsf{k} = -2 + \frac{u}{v}$, the fermionic ghost algebra and the Heisenberg (free boson) algebra, respectively.

Note that for the coset (5.1.3) to be valid, we adopt a different normalisation of the free boson field by choosing g in (3.1.8) to be $\frac{1}{2(\mathsf{k}+2)}$. The OPE between this normalised free boson and itself is given by

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{2\mathsf{k} + 4}{(z - w)^2} \tag{5.1.4}$$

rather than (3.1.22), where k is the level of $A_1(u, v)$. The normalised energy-momentum tensor, denoted by $T^{\mathrm{fb}}(z)$, with respect to this change becomes

$$T^{\mathrm{fb}}(z) = \frac{1}{4(\mathsf{k} + 2)} :\partial\varphi(z)\partial\varphi(z):. \tag{5.1.5}$$

It follows from (5.1.3) that $\mathsf{M}^{\mathrm{N=2}}(u, v)$ commutes with $\mathcal{H}$, and they are both subalgebras of $A_1(u, v) \otimes \mathfrak{bc}$. The tensor product of the two commuting subalgebras is therefore embedded in $A_1(u, v) \otimes \mathfrak{bc}$:

$$\mathcal{H} \otimes \mathsf{M}^{\mathrm{N=2}}(u, v) \hookrightarrow A_1(u, v) \otimes \mathfrak{bc}. \tag{5.1.6}$$

At the level of generating fields, there is an embedding which allows us to express $\partial\varphi(z)$ of $\mathcal{H}$, as well as $T^{\mathrm{N=2}}(z)$, $G^{\pm}(z)$ and $J(z)$ of $\mathsf{M}^{\mathrm{N=2}}(u, v)$ in terms of $b(z)$, $c(z)$ of $\mathfrak{bc}$, and $e(z)$, $f(z)$ and $h(z)$ of $A_1(u, v)$. These are given by

$$\partial\varphi(z) = h(z) + 2Q(z), \tag{5.1.7a}$$

$$J(z) = \frac{1}{t}h(z) - \frac{\mathsf{k}}{t}Q(z), \qquad G^+(z) = \sqrt{\frac{2}{t}}e(z)c(z), \qquad G^-(z) = \sqrt{\frac{2}{t}}f(z)b(z), \tag{5.1.7b}$$

$$\begin{aligned} T^{\mathrm{N=2}}(z) &= T^{\mathrm{sl}}(z) + T^{\mathrm{bc}}(z) - T^{\mathrm{fb}}(z) \\ &= \frac{1}{2t}\left[:ef:(z) + :fe:(z) - \frac{1}{t}h(z)Q(z) + \frac{\mathsf{k}}{2t}:QQ:(z)\right], \end{aligned} \tag{5.1.7c}$$

where $t = \mathsf{k}+2$. These expressions are found by writing the generating fields of $\mathcal{H}$ and $\mathsf{M}^{\mathsf{N=2}}(u,v)$ as a linear combination of fields in $A_1(u,v)\otimes \mathfrak{bc}$ with the same conformal dimension. The coefficients of the linear combinations can be computed by requiring the expressions to satisfy the $\mathcal{H}$ and the $N=2$ algebras, and $\partial\varphi(z)$ to have regular OPEs with the $\mathsf{M}^{\mathsf{N=2}}(u,v)$ generating fields. It is easy to check that these criteria are indeed satisfied by (5.1.7). For example, with (5.1.7a), the OPE of $\partial\varphi(z)$ with itself is given by

$$\begin{aligned}\partial\varphi(z)\partial\varphi(w) &= [h(z)+2Q(z)]\,[h(w)+2Q(w)]\\ &= h(z)h(w)+4Q(z)Q(w)\\ &\sim \frac{2\mathsf{k}}{(z-w)^2}+\frac{4}{(z-w^2)}=\frac{2\mathsf{k}+4}{(z-w)^2},\end{aligned} \tag{5.1.8}$$

which is consistent with (5.1.4).

Of course, the identifications (5.1.7b) and (5.1.7c) only define a non-zero homomorphism of vertex operator superalgebras from the tensor product of $\mathcal{H}$ with the universal $N=2$ algebra $\mathcal{U}^{\mathsf{N=2}}_{\mathsf{c}}$ to $A_1(u,v)\otimes\mathfrak{bc}$. We therefore have an embedding of $\mathcal{H}\otimes\mathsf{V}$ into $A_1(u,v)\otimes\mathfrak{bc}$, where V is some (indecomposable) quotient of $\mathcal{U}^{\mathsf{N=2}}_{\mathsf{c}}$ by the kernel of the embedding. As the zero modes h_0 and Q_0 act diagonalisably on $A_1(u,v)$ and $\mathfrak{bc}$, respectively, $a_0 = h_0+2Q_0$ acts diagonalisably on their tensor product. Since the h_0- and $2Q_0$-eigenvalues of the states in the respective $A_1(u,v)$- and $\mathfrak{bc}$-vertex algebras are even, the a_0-eigenvalue for the states in the tensored module $A_1(u,v)\otimes\mathfrak{bc}$ must be even. The tensored module must then be decomposed into a direct sum of even-indexed Fock spaces tensored with appropriate coset modules as follows:

$$(A_1(u,v)\otimes\mathfrak{bc})\downarrow\cong\bigoplus_{p\in2\mathbb{Z}}\mathcal{F}_p\otimes\mathcal{C}_p. \tag{5.1.9}$$

Here, the $\mathcal{C}_p$ are V-modules and, as $\mathcal{F}_0=\mathcal{H}$, the discussion above forces $\mathcal{C}_0=\mathsf{V}$. However, $A_1(u,v)\otimes\mathfrak{bc}$ is simple as a vertex operator superalgebra, since both $A_1(u,v)$ and $\mathfrak{bc}$ are, hence V is simple by a result of Kac and Radul [117, Thm. 1.1] (see [64, Sec. 3.1] for a detailed discussion). In other words, $\mathsf{V}=\mathsf{M}^{\mathsf{N=2}}(u,v)$ and we have proven the desired embedding (5.1.7).

This simple proof stands in contrast to many of the arguments found in the literature. One of the first arguments to address the simplicity of the coset (5.1.3) is found in [44], where it is established using explicit character computations. However, this relied upon the Verma module embedding diagrams of [16, 118] which are not universally acknowledged. A different proof appears in [15], based on the coset-inspired categorical equivalences sketched in [52] but only recently proven in [23]. Another proof, based on invariant theory, has recently appeared in [116].

5.1.1 Branching rules for unitary $N = 2$ minimal models

Recall that the $N = 2$ minimal model $\mathsf{M}^{\mathrm{N=2}}(u, v)$ is parametrised by two positive coprime integers $u \neq 1$ and v, which also describe the $\widehat{\mathfrak{sl}}_2$ minimal model $A_1(u, v)$ in the coset construction (5.1.3). The minimal model $\mathsf{M}^{\mathrm{N=2}}(u, v)$ is unitary and rational when $v = 1$ and is non-unitary and logarithmic otherwise.

The explicit decomposition of a module of an algebra into the direct sum of modules of its subalgebra is known as a *branching rule*. An example of a branching rule was given in (5.1.9), where the restriction operation ($\downarrow$) was used to indicate the action of decomposing an $A_1(u, v) \otimes \mathfrak{bc}$-module. The aim of this section is to employ *the method of extremal states* to find such branching rules for all modules of $A_1(u, 1) \otimes \mathfrak{bc}$, which take the generic form

$$(\mathcal{L}_{r,0} \otimes \mathcal{N}_i) \downarrow = \bigoplus_p \mathcal{F}_p \otimes {}^{[i]}\mathsf{C}_{p,r}. \tag{5.1.10}$$

Recall that the $\mathsf{M}^{\mathrm{N=2}}(u, 1)$-modules are denoted by $\mathcal{L}^{\mathrm{NS/R},\pm}_{j,\Delta}$. We will identify these modules with the coset modules ${}^{[i]}\mathsf{C}_{p,r}$ by giving explicit formulae for the charge and conformal dimension of ${}^{[i]}\mathsf{C}_{p,r}$. We would like to remind the reader that, as discussed after (5.1.9), decomposing irreducible modules of $A_1(u, 1)$ with $\mathfrak{bc}$-modules gives rise to irreducible modules. The modules ${}^{[i]}\mathsf{C}_{p,r}$ are therefore irreducible. We determine which Fock spaces and $\mathsf{M}^{\mathrm{N=2}}(u, 1)$-modules are obtained from the decomposition of a particular $A_1(u, 1) \otimes \mathfrak{bc}$-module by computing the exact values of p.

The *extremal states* of a module are defined to be those states which, for a given fixed charge, have the minimal possible conformal dimension. In the case at hand, the extremal states are the minimal conformal dimension states of $\mathcal{L}_{r,0} \otimes \mathcal{N}_i$ in each subspace of constant a_0-charge, where we recall that $a_0 = h_0 + 2Q_0$. The minimality condition ensures that such a state is necessarily annihilated by the positive modes of the Heisenberg algebra $\mathcal{H}$ and $\mathsf{M}^{\mathrm{N=2}}(u, 1)$. As both $\mathcal{F}_p$ and the ${}^{[i]}\mathsf{C}_{p,r}$ are irreducible, they may be identified by computing the a_0-, J_0- and L_0-eigenvalues of the highest-weight extremal states.

To illustrate, we consider $\mathcal{L}_{r,0} \otimes \mathcal{N}_0$. Its extremal states may be readily found as a subset of the states obtained by tensoring an extremal state of $\mathcal{L}_{r,0}$ with one of $\mathcal{N}_0$. Let $|r\rangle$ and $|0^+\rangle$ denote the highest-weight states of $\mathcal{L}_{r,0}$ and $\mathcal{N}_0$, respectively, recalling that the h_0-charge of $|r\rangle$ is $r - 1$ and the Q_0-charge of $|0^+\rangle$ is 0. The extremal states of $\mathcal{L}_{r,0}$ and $\mathcal{N}_0$ include

$$\begin{aligned} &f_0^m |r\rangle \quad (m = 0, 1, \dots, r-1), \\ &e_{-1}^n |r\rangle \quad (n = 0, 1, \dots, u-r-1), \end{aligned} \qquad b_{-1/2}|0^+\rangle, \quad |0^+\rangle, \quad c_{-1/2}|0^+\rangle \tag{5.1.11}$$

(there are many others, but these will suffice for our analysis). In $\mathcal{L}_{r,0} \otimes \mathcal{N}_0$, minimising conformal dimensions now easily verifies that the extremal state of a_0-charge $r - 1 - 2m$, $m = 0, 1, \dots, r-1$ has the form $f_0^m |r\rangle \otimes |0^+\rangle$ and that of a_0-charge $r + 1 + 2n$, $n = 0, 1, \dots, u -$

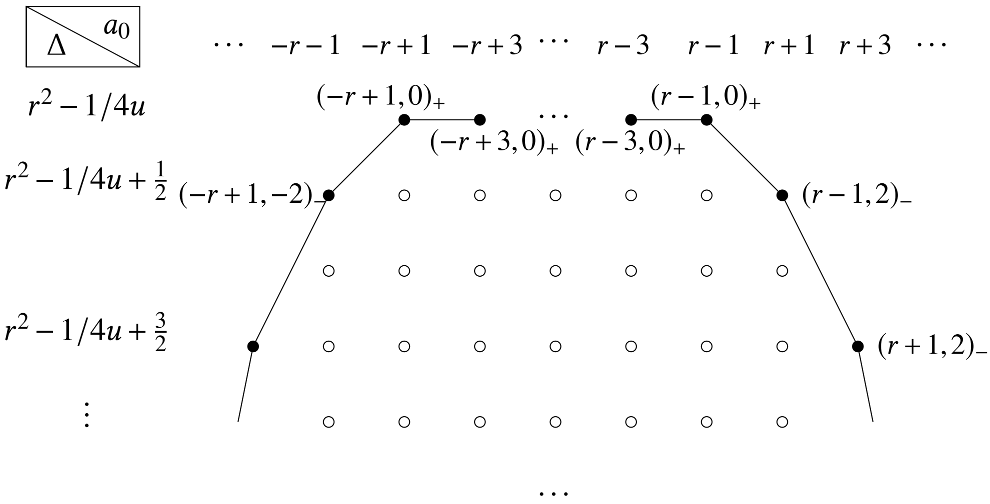

Figure 5.1: Module diagram for $\mathcal{L}_{r,0} \otimes \mathcal{N}_0$. The states of the module are represented by dots with extremal states indicated by black solid dots, they are located at the top of each column. The tuple associated with each extremal state gives its h_0- and $2Q_0$-charges, these two parameters combine to give the a_0-charge according to (5.1.7a). The parity of a state is indicated by the $\pm$-subscript of its tuple.

$r-1$, has the form $e_{-1}^n |r\rangle \otimes b_{-1/2}|0^+\rangle$. The former are therefore bosonic with conformal weight Δ_r whilst the latter are fermionic with conformal weight $\Delta_r + n + \frac{1}{2}$, where Δ_r is the conformal dimension of $|r\rangle$.

Identifying these extremal states as highest-weight states of $\mathcal{F}_p \otimes {}^{[0]}\mathsf{C}_{p,r}$, with $p = r-1-2m$ or $p = r+1+2n$, we use $J_0 = \frac{1}{u}h_0 - \frac{u-2}{u}Q_0$ and $L_0 = L_0^{\text{sl}} + L_0^{\text{bc}} - L_0^{\text{fb}}$ to identify the irreducible $\mathsf{M}^{\text{N=2}}(u,1)$-modules that they generate. In this way, we find that the dictionary between the coset and $N=2$ notations for these modules is given by

$$
{}^{[0]}\mathsf{C}_{p,r} \cong \mathcal{L}_{j,\Delta}^{\text{NS},\bullet}, \text{ where}
$$
$$
\begin{cases} \bullet = +, & j = \dfrac{p}{u}, \qquad \Delta = \dfrac{r^2-p^2-1}{4u}, \qquad (p = -r+1,\dots,r-1), \\ \bullet = -, & j = \dfrac{p}{u}-1, \quad \Delta = \dfrac{r^2-p^2-1}{4u} + \dfrac{p-r}{2}, \quad (p = r+1,\dots,2u-r-1). \end{cases} \tag{5.1.12}
$$

We also find that the charges and conformal dimensions of the coset modules are $2u$-periodic in p. Therefore, for those $p \in r-1+2\mathbb{Z}$ which do not fall in the range $-r+1,\dots,2u-r-1$, we can identify the module as one of those in (5.1.12) by

$$
{}^{[0]}\mathsf{C}_{p,r} \cong {}^{[0]}\mathsf{C}_{p\pm 2u,r}. \tag{5.1.13}
$$

The dictionary for Ramond modules is similarly found to be

$^{[1]}\mathrm{C}_{p,r} \cong \mathcal{L}^{\mathrm{R},\bullet}_{j,\Delta}$, where

$$\begin{cases} \bullet = -, \quad j = \dfrac{p}{u} + \dfrac{1}{2}, \quad \Delta = \dfrac{r^2 - p^2 - 1}{4u} + \dfrac{1}{8}, & (p = -r, \dots, r-2), \\ \bullet = +, \quad j = \dfrac{p}{u} - \dfrac{1}{2}, \quad \Delta = \dfrac{r^2 - p^2 - 1}{4u} + \dfrac{1}{8} + \dfrac{p-r}{2}, & (p = r, \dots, 2u - r - 2), \end{cases} \tag{5.1.14}$$

with $^{[1]}\mathrm{C}_{p,r} \cong {}^{[1]}\mathrm{C}_{p\pm 2u,r}$, if $p \in r + 2\mathbb{Z}$ does not fall in the range $-r, \dots 2u - r - 2$. Note that when comparing the dictionary for the Ramond with that of the Neveu-Schwarz modules, similar to the unitary case, the J_0-charges and conformal dimensions are shifted by $\frac{1}{2}$ and $\frac{1}{8}$, respectively, while parities are reversed. The dictionaries for $i = 2$ and 3 are obtained from those for $i = 0$ and 1, respectively, by reversing parities.

We remark that if p and r satisfy $-r \le p \le r - 1$, then $p + u$ and $u - r$ satisfy $u - r \le u + p \le 2u - (u - r) - 1$. In other words, the two branches of each dictionary are exchangeable under an isomorphism. It follows that we may restrict to a single branch, say that for $-r \le p \le r - 1$, remembering that the other just corresponds to its parity-reversal. We therefore have a uniform parametrisation for the irreducible $\mathsf{M}^{\mathrm{N=2}}(u,1)$-modules obtained through the coset construction:

$$\mathcal{L}^{\mathrm{NS/R},\pm}_{j(p),\Delta(p,r)}, \qquad \begin{aligned} &r = 1, 2, \dots, u-1, \\ &p = -r, -r+1, \dots r-1, \end{aligned} \qquad \begin{aligned} j(p) &= \frac{p}{u} + \frac{1 + (-1)^{p+r}}{4}, \\ \Delta(p,r) &= \frac{r^2 - 1 - p^2}{4u} + \frac{1 + (-1)^{p+r}}{16}. \end{aligned} \tag{5.1.15}$$

The module is Neveu-Schwarz for $p + r$ odd and Ramond for $p + r$ even. The branching rule following from this is given by

$$(\mathcal{L}_{r,0} \otimes \mathcal{N}_i) \downarrow \cong \bigoplus_{p \in i + r - 1 + 2\mathbb{Z}} \mathcal{F}_p \otimes {}^{[i]}\mathrm{C}_{p,r}, \tag{5.1.16}$$

again with the periodicity stated in (5.1.13).

Observe from the formula for $j(p)$ that modules with different parameters p in (5.1.15) are not isomorphic. Comparing $\Delta(p,r)$ and $\Delta(p,r')$ now shows that the modules in (5.1.15) are all distinct, hence that the coset construction produces precisely $2u(u-1)$ inequivalent irreducible $\mathsf{M}^{\mathrm{N=2}}(u,1)$-modules (including parity). In fact, it is easy to show that there can be no more than $2u(u-1)$. This relies on the result [64, Thm. 4.3] that given any irreducible $\mathsf{M}^{\mathrm{N=2}}(u,1)$-module $\mathcal{C}$, one can find a Fock space $\mathcal{F}_p$ such that $\mathcal{F}_p \otimes \mathcal{C}$ may be induced to an $A_1(u,1) \otimes \mathfrak{bc}$-module using the embedding (5.1.7). The induced module will then decompose as a direct sum of irreducibles $\mathcal{M} \otimes \mathcal{N}_i$, meaning that each $\mathcal{M}$ is an irreducible $A_1(u,1)$-module, and thus $\mathcal{F}_p \otimes \mathcal{C}$ will appear in the branching rule of at least one of the $\mathcal{M} \otimes \mathcal{N}_i$. However, we have determined the branching rules for a complete set of irreducible $A_1(u,1)$-modules, so $\mathcal{C}$ must be one of the $2u(u-1)$ irreducible $\mathsf{M}^{\mathrm{N=2}}(u,1)$-modules identified above.

		p					
	$j;\Delta$	-3	-2	-1	0	1	2
	1			$\frac{1}{4};\frac{1}{16}$	$0;0$		
r	2		$0;\frac{1}{16}$	$-\frac{1}{4};\frac{1}{8}$	$\frac{1}{2};\frac{5}{16}$	$\frac{1}{4};\frac{1}{8}$	
	3	$-\frac{1}{4};\frac{1}{16}$	$-\frac{1}{2};\frac{1}{4}$	$\frac{1}{4};\frac{9}{16}$	$0;\frac{1}{2}$	$\frac{3}{4};\frac{9}{16}$	$\frac{1}{2};\frac{1}{4}$

Table 5.1: The reduced Kac table of $\mathsf{M}^{\mathrm{N=2}}(4,1)$ ($c = \frac{3}{2}$) in which parity is ignored and the 'Kac symmetry' (5.1.43) is used to remove half of the modules. Each irreducible module is labelled by its charge j and its conformal dimension Δ. The sector is indicated by shading Ramond cells. The charges and conformal dimensions in this pyramidal table are computed using (5.1.12).

		p									
	$\pm;j;\Delta$	-3	-2	-1	0	1	2	3	4	5	6
	1			$-;\frac{1}{4};\frac{1}{16}$	$+;0;0$	$+;-\frac{1}{4};\frac{1}{16}$	$-;-\frac{1}{2};\frac{1}{4}$	$+;\frac{1}{4};\frac{9}{16}$	$-;0;\frac{1}{2}$	$+;\frac{3}{4};\frac{9}{16}$	$-;\frac{1}{2};\frac{1}{4}$
r	2		$-;0;\frac{1}{16}$	$+;-\frac{1}{4};\frac{1}{8}$	$-;\frac{1}{2};\frac{5}{16}$	$+;\frac{1}{4};\frac{1}{8}$	$+;0;\frac{1}{16}$	$-;-\frac{1}{4};\frac{1}{8}$	$+;\frac{1}{2};\frac{5}{16}$	$-;\frac{1}{4};\frac{1}{8}$	
	3	$-;-\frac{1}{4};\frac{1}{16}$	$+;-\frac{1}{2};\frac{1}{4}$	$-;\frac{1}{4};\frac{9}{16}$	$+;0;\frac{1}{2}$	$-;\frac{3}{4};\frac{9}{16}$	$+;\frac{1}{2};\frac{1}{4}$	$+;\frac{1}{4};\frac{1}{16}$	$-;0;0$		

Table 5.2: The Kac table of $\mathsf{M}^{\mathrm{N=2}}(4,1)$ ($\mathsf{c} = \frac{3}{2}$). Each irreducible module is labelled by its parity $\pm$, its charge j and its conformal dimension Δ. The sector is indicated by shading Ramond cells.

One can arrange the identifying data of these irreducibles ${}^{[i]}\mathsf{C}_{p,r}$ into a table reminiscent of the Kac table of the Virasoro minimal models. We label the rows of this Kac table by $r = 1,\dots,u-1$ and the columns by $p = -r,\dots,2u-r-1$, illustrating it for $\mathsf{M}^{\mathrm{N=2}}(4,1)$ in Table 5.2. At the cost of ignoring the parity information, we constrain the value of p from $-r$ to $r-1$. This reduced Kac table is more helpful in the realisation of the symmetry of the modules, which we will describe in Section 5.1.4. The reduced table for $\mathsf{M}^{\mathrm{N=2}}(4,1)$ is illustrated in Table 5.1.

5.1.2 Branching rules for non-unitary $N = 2$ minimal models

The method of extremal states described above can be readily generalised to the study of the non-unitary $N = 2$ minimal models $\mathsf{M}^{\mathrm{N=2}}(u,v)$, where $u,v \in \mathbb{Z}_{\geq 2}$ and $\gcd\{u,v\} = 1$.

Recall from Section 3.3 that the modules of a non-unitary Wess-Zumino-Witten model $A_1(u,v)$ come in various types as listed at the beginning of Section 3.3.3. Following (5.1.3), the coset modules we construct must also come in different types. The procedure for constructing these coset modules follows exactly from the unitary case, in which we identify the extremal states of the $A_1(u,v)$- and $\mathfrak{bc}$-modules, tensoring the extremal states such that the conformal dimension is minimised. The tensored module is then decomposed into a direct sum of Fock spaces tensored with $\mathsf{M}^{\mathrm{N=2}}(u,v)$-modules.

The resulting branching rules have the form

$$(\mathcal{L}_{r,0}\otimes\mathcal{N}_i)\downarrow\cong\bigoplus_{p\in i+\lambda^{\mathrm{sl}}_{r,0}+2\mathbb{Z}}\mathcal{F}_p\otimes{}^{[i]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)},\qquad(\mathcal{D}^+_{r,s}\otimes\mathcal{N}_i)\downarrow\cong\bigoplus_{p\in i+\lambda^{\mathrm{sl}}_{r,s}+2\mathbb{Z}}\mathcal{F}_p\otimes{}^{[i]}\mathsf{C}^{\mathsf{D}}_{p;(r,s)},\tag{5.1.17a}$$

$$(\mathcal{E}_{\lambda;(r,s)}\otimes\mathcal{N}_i)\downarrow\cong\bigoplus_{p\in i+\lambda+2\mathbb{Z}}\mathcal{F}_p\otimes{}^{[i]}\mathsf{C}^{\mathsf{E}}_{p;(r,s)},\tag{5.1.17b}$$

where $1\le r\le u-1$, $1\le s\le v-1$, $i\in\{0,1,2,3\}$ and $\lambda\ne\lambda^{\mathrm{sl}}_{r,s},\lambda^{\mathrm{sl}}_{u-r,v-s}$ mod 2. The ${}^{[i]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}$, ${}^{[i]}\mathsf{C}^{\mathsf{D}}_{p;(r,s)}$ and ${}^{[i]}\mathsf{C}^{\mathsf{E}}_{p;(r,s)}$ are then irreducible $\mathsf{M}^{\mathrm{N=2}}(u,v)$-modules, by [64, Thm. 3.8]. Note that it is not necessary to consider the branching rules involving the $\mathcal{D}^-_{r,s}$ because, as shown in (3.3.27), they are spectral flow images of the $\mathcal{D}^+_{r,s}$, hence will not produce new $\mathsf{M}^{\mathrm{N=2}}(u,v)$-modules. (Actually, this also applies to the $\mathcal{L}_{r,0}$ unless we remove the branching rules corresponding to $\mathcal{D}^+_{r,v-1}$.) In contrast to the unitary case, the periodicity condition (5.1.13) is no longer valid in the non-unitary case.

By identifying the extremal states of $A_1(u,v)\otimes\mathfrak{bc}$-modules as highest-weight vectors in $\mathcal{H}\otimes\mathsf{M}^{\mathrm{N=2}}(u,v)$-modules, we are able to identify infinitely many inequivalent irreducible $\mathsf{M}^{\mathrm{N=2}}(u,v)$-modules in the branching rules of the $\mathcal{L}_{r,0}$, $\mathcal{D}^+_{r,s}$ and $\mathcal{E}_{\lambda;(r,s)}$ (leaving out those of the $\mathcal{D}^+_{r,v-1}$). The dictionary for identifying $\mathcal{L}$-type $\mathsf{M}^{\mathrm{N=2}}(u,v)$-modules is

$${}^{[0]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}\cong\mathcal{L}^{\mathrm{NS},\bullet}_{j,\Delta},\ \text{where } p\in\lambda^{\mathrm{sl}}_{r,0}+2\mathbb{Z},\ \text{and}$$
$$\begin{cases}\bullet=-,\ j=\dfrac{p}{t}+1,\ \Delta=\Delta^{\mathrm{N=2}}_{p;(r,0)}-\dfrac{p+r}{2}, & (p\le -r-1)\\ \bullet=+,\ j=\dfrac{p}{t},\quad \Delta=\Delta^{\mathrm{N=2}}_{p;(r,0)}, & (1-r\le p\le r-1),\\ \bullet=-,\ j=\dfrac{p}{t}-1,\ \Delta=\Delta^{\mathrm{N=2}}_{p;(r,0)}+\dfrac{p-r}{2}, & (p\ge r+1),\end{cases}\tag{5.1.18a}$$

and for the Ramond sector

$${}^{[1]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}\cong\mathcal{L}^{\mathrm{R},\bullet}_{j,\Delta},\ \text{where } p\in\lambda^{\mathrm{sl}}_{r,0}+1+2\mathbb{Z},\ \text{and}$$
$$\begin{cases}\bullet=+,\ j=\dfrac{p}{t}+\dfrac{3}{2},\ \Delta=\Delta^{\mathrm{N=2}}_{p;(r,0)}+\dfrac{1}{8}-\dfrac{p+r}{2}, & (p\le -r-2),\\ \bullet=-,\ j=\dfrac{p}{t}+\dfrac{1}{2},\ \Delta=\Delta^{\mathrm{N=2}}_{p;(r,0)}+\dfrac{1}{8}, & (-r\le p\le r-2),\\ \bullet=+,\ j=\dfrac{p}{t}-\dfrac{1}{2},\ \Delta=\Delta^{\mathrm{N=2}}_{p;(r,0)}+\dfrac{1}{8}+\dfrac{p-r}{2}, & (p\ge r),\end{cases}\tag{5.1.18b}$$

where $t=\frac{u}{v}$ and

$$\Delta^{\mathrm{N=2}}_{p;(r,s)}=\Delta^{\mathrm{sl}}_{r,s}-\frac{p^2}{4t}.\tag{5.1.19}$$

The dictionaries for the $\mathcal{D}$- and $\mathcal{E}$-type irreducibles are as follows:

$${}^{[0]}\mathsf{C}^{\mathsf{D}}_{p;(r,s)}\cong\mathcal{L}^{\mathrm{NS},\bullet}_{j,\Delta},\ \text{where } p\in\lambda^{\mathrm{sl}}_{r,s}+2\mathbb{Z},\ \text{and}$$

$$\begin{cases} \bullet = +,\ j = \dfrac{p}{t}, & \Delta = \Delta^{\text{N=2}}_{p;(r,s)}, & (p \le \lambda^{\text{sl}}_{r,s}), \\ \bullet = -,\ j = \dfrac{p}{t} - 1, & \Delta = \Delta^{\text{N=2}}_{p;(r,s)} + \dfrac{p - \lambda^{\text{sl}}_{r,s} - 1}{2}, & (p \ge \lambda^{\text{sl}}_{r,s} + 2), \end{cases} \tag{5.1.20a}$$

$${}^{[1]}\mathsf{C}^{\mathsf{D}}_{p;(r,s)} \cong \mathcal{L}^{\text{R},\bullet}_{j,\Delta}, \text{ where } p \in \lambda^{\text{sl}}_{r,s} + 1 + 2\mathbb{Z}, \text{ and}$$

$$\begin{cases} \bullet = -,\ j = \dfrac{p}{t} + \dfrac{1}{2}, & \Delta = \Delta^{\text{N=2}}_{p;(r,s)} + \dfrac{1}{8}, & (p \le \lambda^{\text{sl}}_{r,s} - 1), \\ \bullet = +,\ j = \dfrac{p}{t} - \dfrac{1}{2}, & \Delta = \Delta^{\text{N=2}}_{p;(r,s)} + \dfrac{1}{8} + \dfrac{p - \lambda^{\text{sl}}_{r,s} - 1}{2}, & (p \ge \lambda^{\text{sl}}_{r,s} + 1), \end{cases} \tag{5.1.20b}$$

$$\begin{aligned} {}^{[0]}\mathsf{C}^{\mathsf{E}}_{p;(r,s)} &\cong \mathcal{L}^{\text{NS},+}_{p/t,\Delta^{\text{N=2}}_{p;(r,s)}}, \quad (p \in \lambda + 2\mathbb{Z}), \\ {}^{[1]}\mathsf{C}^{\mathsf{E}}_{p;(r,s)} &\cong \mathcal{L}^{\text{R},-}_{p/t+1/2,\Delta^{\text{N=2}}_{p;(r,s)}+1/8}, \quad (p \in \lambda + 1 + 2\mathbb{Z}). \end{aligned} \tag{5.1.21}$$

In addition to the branching rules of the irreducible $A_1(u,v)$-modules described above, we can similarly deduce the branching rules of the reducible indecomposable modules $\mathcal{E}^{\pm}_{r,s}$. Since the set of possible charges of $\mathcal{E}^{\pm}_{r,s}$ is $\lambda^{\text{sl}}_{r,s} + 2\mathbb{Z}$, the branching rules have the form

$$(\mathcal{E}^{\pm}_{r,s} \otimes \mathcal{N}_i) \downarrow \cong \bigoplus_{p \in \lambda^{\text{sl}}_{r,s} + i + 2\mathbb{Z}} \mathcal{F}_p \otimes {}^{[i]}\mathsf{C}^{\pm}_{p;(r,s)}. \tag{5.1.22}$$

Given the $A_1(u,v)$-exact sequences (3.3.28), we can tensor each module by the $\mathfrak{bc}$-module $\mathcal{N}_i$, then decompose according to the branching rules stated above. Factoring out the common Fock spaces leaves us with the following $N = 2$ exact sequences for ${}^{[i]}\mathsf{C}^{\pm}_{p;(r,s)}$:

$$\begin{gathered} 0 \longrightarrow {}^{[i]}\mathsf{C}^{\mathsf{D}}_{p;(r,s)} \longrightarrow {}^{[i]}\mathsf{C}^{+}_{p;(r,s)} \longrightarrow {}^{[i+2]}\mathsf{C}^{\mathsf{D}}_{p+t;(r,s-1)} \longrightarrow 0, \\ 0 \longrightarrow {}^{[i+2]}\mathsf{C}^{\mathsf{D}}_{p+t;(u-r,v-s-1)} \longrightarrow {}^{[i]}\mathsf{C}^{-}_{p;(r,s)} \longrightarrow {}^{[i]}\mathsf{C}^{\mathsf{D}}_{p;(u-r,v-s)} \longrightarrow 0. \end{gathered} \tag{5.1.23}$$

In these sequences, any occurrence of ${}^{[i]}\mathsf{C}^{\mathsf{D}}_{p;(r,0)}$ should be replaced by ${}^{[i]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}$. In analogy to the nomenclature of $\widehat{\mathfrak{sl}}_2$, the modules ${}^{[i]}\mathsf{C}^{\mathsf{E}}_{p;(r,s)}$ and ${}^{[i]}\mathsf{C}^{\pm}_{p;(r,s)}$ will be referred to as *standard modules*. Further, the modules ${}^{[i]}\mathsf{C}^{\mathsf{E}}_{p;(r,s)}$ will be referred to as *typical modules* while all other indecomposable modules will be referred to as *atypical* modules.

5.1.3 Induction

The theory of the opposite operation to restriction, known as *induction* and denoted by $\downarrow$, is elucidated in Appendix A. We shall briefly illustrate the discussion with the $N = 2$ coset in this section. Recall the branching rule for the vacuum module of $A_1(u,v) \otimes \mathfrak{bc}$

$$(\mathcal{L}_{1,0} \otimes \mathcal{N}_0) \downarrow = \bigoplus_{p \in 2\mathbb{Z}} \mathcal{F}_p \otimes {}^{[0]}\mathsf{C}^{\mathsf{L}}_{p;(1,0)}. \tag{5.1.24}$$

We define the induction of a $\mathcal{H} \otimes \mathsf{M}^{\mathsf{N=2}}(u,v)$-module $\mathcal{M}$ to be an $A_1(u,v) \otimes \mathfrak{bc}$-module obtained by fusing $\mathcal{M}$ with the vacuum module of $A_1(u,v) \otimes \mathfrak{bc}$:

$$\mathcal{M}\uparrow = \mathcal{M} \times (\mathcal{L}_{1,0} \otimes \mathcal{N}_0) \quad \Rightarrow \quad \mathcal{M}\uparrow\downarrow = \bigoplus_{p \in 2\mathbb{Z}} \left(\mathcal{F}_p \otimes {}^{[0]}\mathsf{C}^{\mathsf{L}}_{p;(1,0)}\right) \times \mathcal{M}. \tag{5.1.25}$$

It is therefore possible to construct $A_1(u,v) \otimes \mathfrak{bc}$-modules by inducing certain $\mathcal{H} \otimes \mathsf{M}^{\mathsf{N=2}}(u,v)$-modules. For example, let us consider the induction of the following module with $p' \in \lambda^{\mathrm{sl}}_{r,0} + i + 2\mathbb{Z}$:

$$\begin{aligned}
\left(\mathcal{F}_{p'} \otimes {}^{[i]}\mathsf{C}^{\mathsf{L}}_{p';(r,0)}\right)\uparrow\downarrow &= \left(\mathcal{F}_{p'} \otimes {}^{[i]}\mathsf{C}^{\mathsf{L}}_{p';(r,0)}\right) \times \left(\bigoplus_{p \in 2\mathbb{Z}} \mathcal{F}_p \otimes {}^{[0]}\mathsf{C}^{\mathsf{L}}_{p;(1,0)}\right) && (5.1.26a)\\
&= \bigoplus_{p \in 2\mathbb{Z}} \; \bigoplus_{p' \in \lambda^{\mathrm{sl}}_{r,0}+i+2\mathbb{Z}} \mathcal{F}_{p+p'} \otimes {}^{[i]}\mathsf{C}^{\mathsf{L}}_{p+p';(r,0)} && (5.1.26b)\\
&= \bigoplus_{m \in \lambda^{\mathrm{sl}}_{r,0}+i+2\mathbb{Z}} \mathcal{F}_m \otimes {}^{[i]}\mathsf{C}^{\mathsf{L}}_{m;(r,0)}\\
&= \mathcal{L}_{r,0} \otimes \mathcal{N}_i \downarrow,
\end{aligned}$$

from which we conclude

$$\left(\mathcal{F}_{p'} \otimes {}^{[i]}\mathsf{C}^{\mathsf{L}}_{p';(r,0)}\right)\uparrow = \mathcal{L}_{r,0} \otimes \mathcal{N}_i, \tag{5.1.27}$$

for any $p' \in \lambda^{\mathrm{sl}}_{r,0} + i + 2\mathbb{Z}$. Notice that it only makes sense to induce a $\mathcal{H} \otimes \mathsf{M}^{\mathsf{N=2}}(u,v)$-module whose Fock space label is equal (mod 2) to the Fock space part of the $N = 2$-labels, that is, the module $\mathcal{F}_p \otimes {}^{[i]}\mathsf{C}^{\mathsf{L}}_{p';(r,0)}$ can only be induced if $p = p' \pmod 2$. The induction of such modules are identified with $A_1(u,v) \otimes \mathfrak{bc}$-modules according to the branching rules (5.1.17).

The result (5.1.27) is identical to the induction of modules in the unitary minimal models, while the procedure works analogously for all other types of irreducible modules in the non-unitary case. We summarise the results as follows:

$$\left(\mathcal{F}_p \otimes {}^{[i]}\mathsf{C}^{\mathsf{D}}_{p;(r,s)}\right)\uparrow = \mathcal{D}^{+}_{r,s} \otimes \mathcal{N}_i, \quad \left(\mathcal{F}_{p'} \otimes {}^{[i]}\mathsf{C}^{\mathsf{E}}_{p';(r,s)}\right)\uparrow = \mathcal{E}_{\lambda;(r,s)} \otimes \mathcal{N}_i, \tag{5.1.28}$$

where $p \in \lambda^{\mathrm{sl}}_{r,s} + i + 2\mathbb{Z}$ and $p' \in \lambda + i + 2\mathbb{Z}$ with $\lambda \neq \lambda^{\mathrm{sl}}_{r,s}, \lambda^{\mathrm{sl}}_{u-r,v-s} \pmod 2$.

5.1.4 Automorphisms

In Section 4.3.2, we introduced two types of automorphisms, conjugation and spectral flow, and their actions on the $N = 2$ generators, which were then promoted to actions on the $N = 2$-modules. These results were displayed in (4.3.6) – (4.3.10). Similar formulae were studied for the $\widehat{\mathfrak{sl}}_2$-, $\mathfrak{bc}$- and Heisenberg algebras. Following from the coset construction (5.1.3), we want to study the relations between these automorphisms, In particular, we want to find the powers of the automorphisms, in terms of ℓ and m, on the right-hand side of the following relations:

$$\gamma_{\mathrm{sl}}^{\ell} \otimes \gamma_{\mathrm{bc}}^{m} \cong \gamma_{\mathrm{fb}}^{?} \otimes \gamma_{\mathrm{N=2}}^{?}, \quad \sigma_{\mathrm{sl}}^{j} \otimes \sigma_{\mathrm{bc}}^{p} \cong \sigma_{\mathrm{fb}}^{?} \otimes \sigma_{\mathrm{N=2}}^{?}. \tag{5.1.29}$$

These numbers can be found by looking at how the automorphisms act with respect to the embedding (5.1.7).

Let us start from conjugation. Recall (4.1.33) for the action of conjugation on the following modes:

$$\gamma_{\mathrm{sl}}(e_n) = f_n, \quad \gamma_{\mathrm{sl}}(f_n) = e_n, \quad \gamma_{\mathrm{bc}}(b_n) = c_n, \quad \gamma_{\mathrm{bc}}(c_n) = b_n, \tag{5.1.30}$$

from which we can observe that the conjugation on G_s^+ must correspond to the action of γ_{sl} on e_n tensored with γ_{bc} on c_{s-n}. This constricts both values of ℓ and m in (5.1.29) to 1, in which case $G_s^+ = \sqrt{2/t}\sum_{n\in\mathbb{Z}} e_n c_{s-n}$ is sent to $G_s^- = \sqrt{2/t}\sum_{n\in\mathbb{Z}} f_n b_{s-n}$ by $\gamma_{\mathrm{sl}} \otimes \gamma_{\mathrm{bc}}$. But we know from (4.3.6) that G_s^+ is mapped to G_s^- by $\gamma_{\mathrm{N=2}}$. The first identity in (5.1.29) then becomes

$$\gamma_{\mathrm{sl}} \otimes \gamma_{\mathrm{bc}} = \gamma_{\mathrm{fb}}^{?} \otimes \gamma_{\mathrm{N=2}}. \tag{5.1.31}$$

It is obvious that any other combinations of $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$-conjugations do not translate into meaningful conjugation in the $N = 2$ theory.

Let us now look at how $\gamma_{\mathrm{sl}} \otimes \gamma_{\mathrm{bc}}$ is related to γ_{fb}. Recall that the Heisenberg mode a_n is embedded in $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$ as $h_n + 2Q_n$, which under conjugation becomes

$$(\gamma_{\mathrm{sl}} \otimes \gamma_{\mathrm{bc}})(h_n + 2Q_n) = \gamma_{\mathrm{sl}}(h_n) + \gamma_{\mathrm{bc}}(2Q_n) = -h_n - 2Q_n \equiv \gamma_{\mathrm{fb}}(a_n) = -a_n, \tag{5.1.32}$$

where we have used $\gamma_{\mathrm{bc}}(Q_n) = -Q_n$ as stated in (4.1.33). The action of $\gamma_{\mathrm{sl}} \otimes \gamma_{\mathrm{bc}}$ therefore leads to γ_{fb} in $\mathcal{H}$ with the embedding (5.1.7). This, along with (5.1.31), yields the conclusion that conjugation is conserved by the coset:

$$\gamma_{\mathrm{sl}} \otimes \gamma_{\mathrm{bc}} = \gamma_{\mathrm{fb}} \otimes \gamma_{\mathrm{N=2}}. \tag{5.1.33}$$

The formula for spectral flow can be found following a similar procedure. For example, the spectral flow $\sigma_{\mathrm{sl}}^{j} \otimes \sigma_{\mathrm{bc}}^{p}$ acts on the embedding (5.1.7b) of G_s^- in $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$ as

$$(\sigma_{\mathrm{sl}}^{j} \otimes \sigma_{\mathrm{bc}}^{p})\sqrt{\frac{2}{t}}\sum_{n\in\mathbb{Z}} f_n b_{s-n} = \sqrt{\frac{2}{t}}\sum_{n\in\mathbb{Z}} f_{n+j} b_{s-n-p}, \tag{5.1.34}$$

which is the embedding of mode G^-_{s+j-p} in $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$, and can be obtained through $\sigma_{\mathrm{N=2}}^{j-p}(G_s^-)$.

The action of $\sigma_{\text{sl}}^{j} \otimes \sigma_{\text{bc}}^{p}$ therefore corresponds to $\sigma_{\text{N=2}}^{j-p}$ in $\mathsf{M}^{\text{N=2}}(u,v)$:

$$\sigma_{\text{sl}}^{j} \otimes \sigma_{\text{bc}}^{p} = \sigma_{\text{fb}}^{?} \otimes \sigma_{\text{N=2}}^{j-p}. \tag{5.1.35}$$

To determine the question mark in (5.1.35) associated with the spectral flow of the Heisenberg algebra, recall the embedding of the free boson mode in $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$: $a_n = h_n + 2Q_n$, which satisfies

$$(\sigma_{\text{sl}}^{j} \otimes \sigma_{\text{bc}}^{p})(h_n + 2Q_n) = h_n - j\mathsf{k}\delta_{n,0} + 2(Q_n - p\delta_{n,0}) \equiv a_n - (j\mathsf{k} + 2p)\delta_{n,0}, \tag{5.1.36}$$

whose rightmost equality can also be obtained by

$$\sigma_{\text{fb}}^{j\mathsf{k}+2p}(a_n) = a_n - (j\mathsf{k} + 2p)\delta_{n,0}. \tag{5.1.37}$$

Combining this result with (5.1.35), we arrive at

$$\sigma_{\text{sl}}^{j} \otimes \sigma_{\text{bc}}^{p} = \sigma_{\text{fb}}^{j\mathsf{k}+2p} \otimes \sigma_{\text{N=2}}^{j-p}, \tag{5.1.38}$$

where $j \in \mathbb{Z}$ and $p \in \mathbb{Z}/2$. We remark that this relation works for all generators other than the ones we illustrated in the previous examples. It is a very powerful relation in many aspects. For example, it provides an alternative method for computing the branching rules other than the extremal state method.

Let $\mathcal{M}_\lambda$ be an indecomposable module of $A_1(u,v)$ with h_0-eigenvalues in $\lambda + \mathbb{C}/2\mathbb{Z}$. The field identifications (5.1.7) then imply that the eigenvalues of the Heisenberg zero mode a_0 on $\mathcal{M}_\lambda \otimes \mathcal{N}_i$, where $i \in \{0,1,2,3\}$, lie in $\lambda + i + 2\mathbb{Z}$. This means that the branching rule has the form

$$(\mathcal{M}_\lambda \otimes \mathcal{N}_i) \downarrow \cong \bigoplus_{p \in \lambda+i+2\mathbb{Z}} \mathcal{F}_p \otimes {}^{[i]}\mathcal{C}_p^{\mathcal{M}}, \tag{5.1.39}$$

for some $\mathsf{M}^{\text{N=2}}(u,v)$-modules ${}^{[i]}\mathcal{C}_p^{\mathcal{M}}$. If $\mathcal{M}_\lambda$ is irreducible, then these ${}^{[i]}\mathcal{C}_p^{\mathcal{M}}$-modules will be as well by [64, Thm. 3.8]. Noting that, from (3.3.14), $\gamma_{\text{sl}}(\mathcal{M}_\lambda)$ and $\sigma_{\text{sl}}^{\ell}(\mathcal{M}_\lambda)$ have h_0-charges $-\lambda + \mathbb{C}/2\mathbb{Z}$ and $\lambda + \ell\mathsf{k} + \mathbb{C}/2\mathbb{Z}$, respectively, we can now derive many identifications among the $N = 2$ modules appearing in the branching rules of $\mathcal{M}_\lambda$ and its twists.

For example, following from the generalised branching rule (5.1.39), the decomposition of the module $\sigma_{\text{sl}}(\mathcal{M}_\lambda) \otimes \sigma_{\text{bc}}(\mathcal{N}_i)$ takes the form

$$(\sigma_{\text{sl}}(\mathcal{M}_\lambda) \otimes \sigma_{\text{bc}}(\mathcal{N}_i)) \downarrow \cong \bigoplus_{p \in \lambda+\mathsf{k}+i+2\mathbb{Z}} \mathcal{F}_p \otimes {}^{[i+2]}\mathcal{C}_p^{\sigma(\mathcal{M})}, \tag{5.1.40}$$

where $\sigma_{\text{sl}}(\mathcal{M}_\lambda)$ has h_0-charge $\lambda + \mathsf{k} + \mathbb{C}/2\mathbb{Z}$ and we recall from (4.1.34) that $\sigma_{\text{bc}}(\mathcal{N}_i) = \mathcal{N}_{i+2}$. Alternatively, we can use (5.1.38) with $\ell = m = 1$, which gives $\sigma_{\text{sl}} \otimes \sigma_{\text{bc}} = \sigma_{\text{fb}}^{\mathsf{k}+2} \otimes 1_{\text{N=2}}$.

Applying this identity to the generalised branching rule (5.1.39) yields

$$\left(\sigma_{\mathrm{sl}}(\mathcal{M}_\lambda)\otimes\sigma_{\mathrm{bc}}(\mathcal{N}_i)\right)\downarrow\cong\bigoplus_{p\in\lambda+i+2\mathbb{Z}}\sigma_{\mathrm{fb}}^{k+2}(\mathcal{F}_p)\otimes{}^{[i]}\mathcal{C}_p^{\mathcal{M}}\cong\bigoplus_{p\in\lambda+i+2\mathbb{Z}}\mathcal{F}_{p+k+2}\otimes{}^{[i]}\mathcal{C}_p^{\mathcal{M}}. \tag{5.1.41}$$

Comparing this with (5.1.40), with an appropriate relabeling of the index p, we have

$${}^{[i]}\mathcal{C}_p^{\sigma(\mathcal{M})}\cong{}^{[i+2]}\mathcal{C}_{p-t}^{\mathcal{M}}, \tag{5.1.42}$$

where $t=k+2=u/v$ for $\mathsf{M}^{\mathrm{N=2}}(u,v)$. Similar identifications follow from applying $1_{\mathrm{sl}}\otimes\sigma_{\mathrm{bc}}^{-\ell}=\sigma_{\mathrm{fb}}^{-2\ell}\otimes\sigma_{\mathrm{N=2}}^{\ell}$ and $\gamma_{\mathrm{sl}}\otimes\gamma_{\mathrm{bc}}=\gamma_{\mathrm{fb}}\otimes\gamma_{\mathrm{N=2}}$ to (5.1.39), which we summarise as follows:

$${}^{[i]}\mathcal{C}_p^{\sigma^\ell(\mathcal{M})}\cong{}^{[i+2\ell]}\mathcal{C}_{p-\ell t}^{\mathcal{M}},\quad \sigma_{\mathrm{N=2}}^{\ell}\left({}^{[i]}\mathcal{C}_p^{\mathcal{M}}\right)\cong{}^{[i-2\ell]}\mathcal{C}_{p-2\ell}^{\mathcal{M}},\quad {}^{[i]}\mathcal{C}_p^{\gamma(\mathcal{M})}\cong\gamma_{\mathrm{N=2}}\left({}^{[-i]}\mathcal{C}_{-p}^{\mathcal{M}}\right). \tag{5.1.43}$$

Note also that the coset preserves parity, so ${}^{[i+2]}\mathcal{C}_p^{\mathcal{M}}\cong\Pi\,{}^{[i]}\mathcal{C}_p^{\mathcal{M}}$. We note, in particular, that the branching rules of the spectral flows of $\mathcal{M}_\lambda$ produce no $\mathsf{M}^{\mathrm{N=2}}(u,v)$-modules that have not already appeared in the branching rules of $\mathcal{M}_\lambda$.

Let us now apply (5.1.43) to the unitary minimal models $A_1(u,1)$, which have precisely $u-1$ inequivalent irreducibles $\mathcal{L}_{r,0}$, $r=1,\dots,u-1$, as we described in Section 3.3.2. The h_0-charges of these modules take values in $r-1+2\mathbb{Z}$. Using the extremal state method, we arrived at the branching rule (5.1.16). Due to the fact that $\sigma_{\mathrm{sl}}\left(\mathcal{L}_{r,0}\right)\cong\mathcal{L}_{u-r,0}$, the $\mathsf{M}^{\mathrm{N=2}}(u,1)$-modules ${}^{[i]}\mathsf{C}_{p,r}$ are not all inequivalent. Indeed, (5.1.42) implies that

$${}^{[i]}\mathsf{C}_{p,r}\cong{}^{[i+2]}\mathsf{C}_{p\pm u,u-r}\cong{}^{[i]}\mathsf{C}_{p\pm 2u,r},\qquad i\in\mathbb{Z}_4,\ r=1,\dots,u-1,\ p\in i+r-1+2\mathbb{Z}, \tag{5.1.44}$$

where $t=u$ for unitary minimal models. The isomorphisms ${}^{[i]}\mathsf{C}_{p,r}\cong{}^{[i]}\mathsf{C}_{p+2u,r}$ coincide with the periodicity condition we observed in (5.1.13) from the extremal state method. One can also deduce from these isomorphisms that the commutant of $\mathsf{M}^{\mathrm{N=2}}(u,1)\cong{}^{[0]}\mathsf{C}_{0,1}$ in $A_1(u,1)\otimes\mathfrak{bc}$ is not $\mathcal{H}\cong\mathcal{F}_0$, but is rather the compactified free boson (lattice vertex operator algebra) $\mathbb{F}_0\cong\bigoplus_{n\in 2u\mathbb{Z}}\mathcal{F}_n$. Accordingly, the branching rules (5.1.16) may be rewritten as direct sums over tensor products of $\mathbb{F}_0$- and $\mathsf{M}^{\mathrm{N=2}}(u,1)$-modules:

$$(\mathcal{M}\otimes\mathcal{N}_i)\downarrow\cong{\bigoplus_{p=1-r}^{2u-r-1}}'\,\mathbb{F}_p\otimes{}^{[i]}\mathcal{C}_p^{\mathcal{M}}, \tag{5.1.45}$$

where $\mathbb{F}_p\cong\bigoplus_{n\in p+2u\mathbb{Z}}\mathcal{F}_n$ and the prime next to the summation sign above indicates p is summed in increments of 2. These facts can be useful in many ways, in particular as $\mathbb{F}_p$ is rational, but will not be required in what follows.

It is not hard to verify, by inspection, that spectral flow and conjugation partition the $2u(u-1)$ simple modules into orbits in which the action of the automorphisms are closed. The number of orbits and the orbit lengths depend on the u-parameter of the minimal

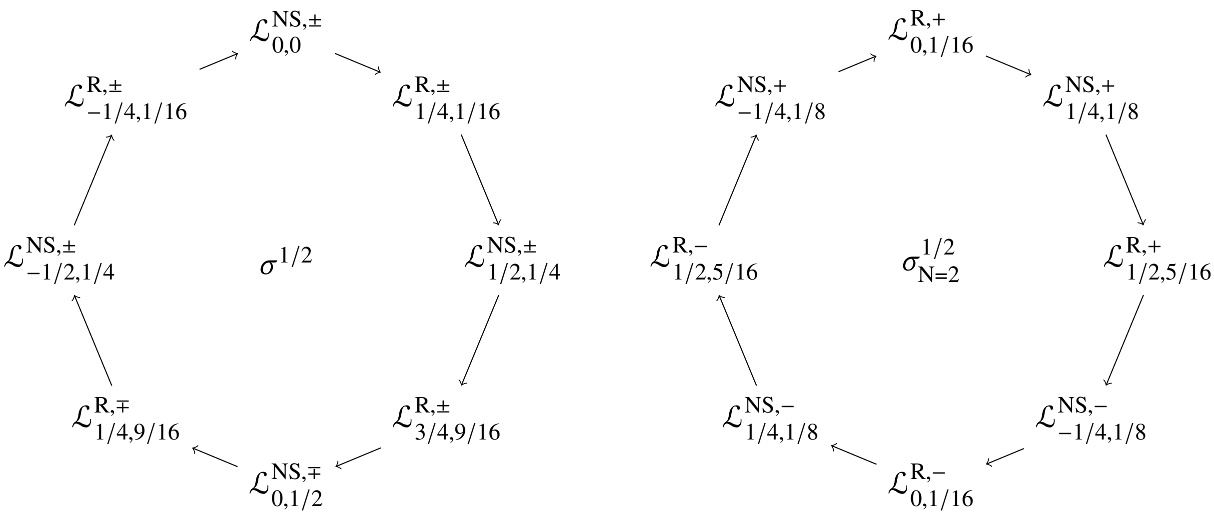

Figure 5.2: The action of the spectral flow automorphism $\sigma^{1/2}_{\mathrm{N=2}}$ on the $\mathsf{M}^{\mathrm{N=2}}(4,1)$ ($c = \frac{3}{2}$) Neveu-Schwarz and Ramond highest-weight modules. The orbit on the left is, in fact, two orbits, one being the parity reversal of the other. Conjugation is realised as reflection through the vertical diameters and parity reversal is implemented on the right orbit through rotation by π.

model. This can be summarised as the following list:

- u orbits when $u \in 4\mathbb{Z}+2$, two for each $r = 1, \dots, \frac{u}{2}$ (one being the parity reversal of the other). For $r < \frac{u}{2}$, the orbit length is $2u$, but for $r = \frac{u}{2}$, the orbit length is only u.

- $u-1$ orbits when $u \in 4\mathbb{Z}$, with two for each $r = 1, \dots, \frac{u}{2}-1$ (one being the parity reversal of the other) but only one for $r = \frac{u}{2}$ (closed under parity reversal). All orbits have length $2u$.

- $\frac{u-1}{2}$ orbits when $u \in 2\mathbb{Z}+1$, one for each $r = 1, \dots, \frac{u-1}{2}$, all of length $4u$ and all closed under parity reversal.

As an example, we illustrate the orbits for $\mathsf{M}^{\mathrm{N=2}}(4,1)$ in Figure 5.2.

We may choose representatives for each of these orbits as follows:

$$\begin{aligned} &{}^{[u/2\pm1]}\mathrm{C}_{0,u/2},\ {}^{[r\pm1]}\mathrm{C}_{0,r}, \quad 1 \le r < \frac{u}{2}, && \text{if } u \in 4\mathbb{Z}+2, \\ &{}^{[u/2-1]}\mathrm{C}_{0,u/2},\ {}^{[r\pm1]}\mathrm{C}_{0,r}, \quad 1 \le r < \frac{u}{2}, && \text{if } u \in 4\mathbb{Z}, \\ &{}^{[r-1]}\mathrm{C}_{0,r}, \quad 1 \le r \le \frac{u-1}{2}, && \text{if } u \text{ is odd.} \end{aligned} \tag{5.1.46}$$

This is easily deduced from the fact that spectral flow will only change the global parity of an irreducible if it is Ramond and doesn't have $\Delta = \frac{c}{24}$.

Recall the end of Section 5.1.1, we came up with a reduced Kac table and commented that the symmetries between modules are more readily realised from this Kac table. Let us now illustrate the actions of the conjugation and the spectral flow automorphisms on

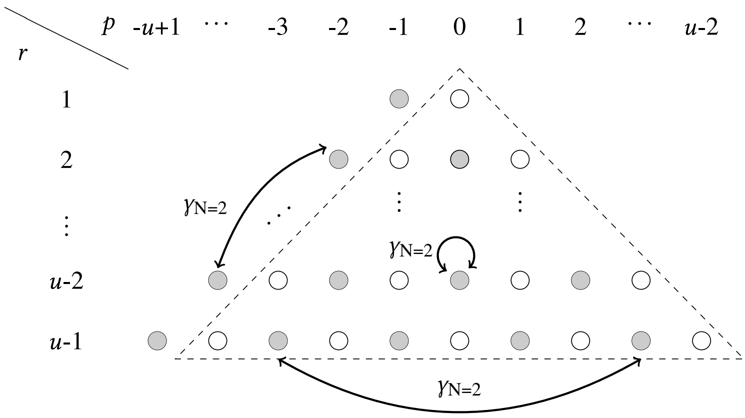

Figure 5.3: The action of conjugation on the Kac table of the minimal model $\mathsf{M}^{\mathrm{N=2}}(u,1)$. The white/grey circles represent Neveu-Schwarz/Ramond modules, respectively. In the dashed triangle, conjugation is effected by reflection about the central column ($p=0$). The modules outside this triangle form a strip on which conjugation is effected by reflection about the strip's middle point.

the $\mathsf{M}^{\mathrm{N=2}}(u,1)$-modules using the reduced Kac table. These are shown in Figure 5.3 and Figure 5.4, respectively.

In the non-unitary case, the automorphism relations (5.1.43) apply in exactly the same way, except that spectral flow is now of infinite order. We shall now look at how spectral flow provides information on the classification of modules of $\mathsf{M}^{\mathrm{N=2}}(u,v)$.

A collection of $\mathsf{M}^{\mathrm{N=2}}(u,v)$-modules that have the same $[i]$- and (r,s)-labels shall be referred to as being of the same family. Recall from (5.1.19) that the conformal dimension of the ground states of a standard module is a quadratic function with charge $j=p/t$. The parabolae describing different families of standard modules do not intersect, see Figure 5.5. It follows that the set of (isomorphism classes of the) standard modules are partitioned into families that are uniquely specified by the labels $[i]$ and (r,s). According to (5.1.43), applying spectral flow to these modules results in a shift in their Fock space labels, which again yields standard modules, and therefore does not yield new modules.

Recall from Section 5.1.2, the atypical standard modules ${}^{[i]}\mathsf{C}^{+}_{p;(r,s)}$ and ${}^{[i]}\mathsf{C}^{+}_{p;(u-r,v-s)}$ with $p\in\lambda_{r,s}+2\mathbb{Z}$ and $p\in\lambda_{u-r,v-s}+2\mathbb{Z}$ respectively, are reducible but indecomposable and correspond to certain points of the parabola describing the conformal dimensions of its family. One may therefore ask whether a given atypical standard is highest-weight or not. This is easily answered, for ${}^{[i]}\mathsf{C}^{+}_{p;(r,s)}$, by comparing the conformal dimensions of its submodule ${}^{[i]}\mathsf{C}^{\mathsf{D}}_{p;(r,s)}$ and its quotient ${}^{[i+2]}\mathsf{C}^{\mathsf{D}}_{p+t;(r,s-1)}$. Using (5.1.20), it turns out that ${}^{[i]}\mathsf{C}^{+}_{p;(r,s)}$ is a highest-weight module if and only if $p\geq\lambda_{r,s}+1$. Otherwise, ${}^{[i]}\mathsf{C}^{+}_{p;(r,s)}$ is the contragredient dual of a highest-weight module.

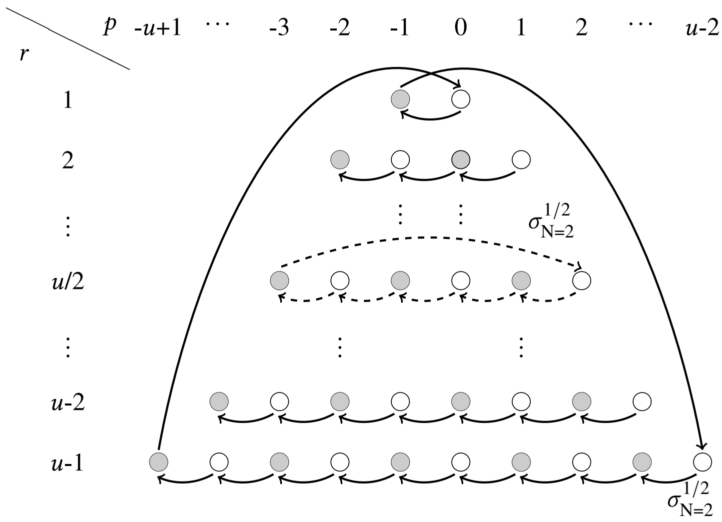

Figure 5.4: Spectral flow acting on the Kac table of the minimal model $\mathsf{M}^{\mathrm{N=2}}(u, 1)$. The nth and the nth-last rows together form a single orbit. The arrows pictured in each orbit illustrate the action of $\sigma^{1/2}_{\mathrm{N=2}}$. When u is even, there is a middle row in the Kac table which forms a closed orbit on its own.

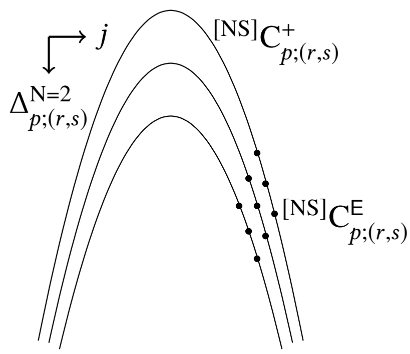

Figure 5.5: Families of standard $\mathsf{M}^{\mathrm{N=2}}(u, v)$-modules in the Neveu-Schwarz sector. Curves are parabolae parametrised in terms of J_0-charges and conformal dimensions. Each parabola is uniquely parametrised by r and s, with each point specifying a module. Some reducible E-type modules are indicated by the black solid dots.

5.2 Characters

In this section, we compute the $N = 2$ minimal model (super)characters in both the unitary and the non-unitary cases. The most important tool for these computations is the *residue method*, introduced by Gaberdiel and Eholzer [44] and outlined below. This method allows one to express (super)characters of $N = 2$-modules as residues of (super)characters of $A_1(u, v) \otimes \mathfrak{bc}$-modules. In the unitary case, only a certain subset of (super)characters will be computed in this way, with the remainder being deduced from spectral flow.

5.2.1 The residue method

We define the character and supercharacter of a $\mathsf{M}^{\mathsf{N=2}}(u, v)$-module $\mathcal{C}$ by

$$\mathsf{Ch}[\mathcal{C}](\mathsf{z};\mathsf{q}) = \mathrm{Tr}_{\mathcal{C}} \mathsf{z}^{J_0} q^{L_0^{\mathsf{N=2}} - \mathsf{C}/24} \quad \text{and} \quad \mathsf{Sch}[\mathcal{C}](\mathsf{z};\mathsf{q}) = \mathrm{Tr}_{\mathcal{C}} (-1)^F \mathsf{z}^{J_0} q^{L_0^{\mathsf{N=2}} - \mathsf{C}/24}, \tag{5.2.1}$$

respectively, where F acts as 0 on the bosonic subspace and as 1 on the fermionic subspace. Consider the generalised branching rule (5.1.16) for an $A_1(u, v)$-module $\mathcal{M}_\lambda$ of charge λ:

$$(\mathcal{M}_\lambda \otimes \mathcal{N}_i) \downarrow \cong \bigoplus_{p \in \lambda + i + 2\mathbb{Z}} \mathcal{F}_p \otimes {}^{[i]}\mathcal{C}_p^{\mathcal{M}}. \tag{5.2.2}$$

The (super)character of $\mathcal{M}_\lambda \otimes \mathcal{N}_i$ may then be computed as either an $A_1(u, v) \otimes \mathfrak{bc}$-module or as an $\mathcal{H} \otimes \mathsf{M}^{\mathsf{N=2}}(u, v)$-module, and the result must be the same. We shall parametrise the character decomposition following from (5.2.2) as

$$\mathsf{Ch}[\mathcal{M}_\lambda](\mathsf{w};\mathsf{q})\, \mathsf{Ch}[\mathcal{N}_i](\mathsf{x};\mathsf{q}) = \sum_{p \in \lambda + i + 2\mathbb{Z}} \mathsf{Ch}[\mathcal{F}_p](\mathsf{y};\mathsf{q})\, \mathsf{Ch}[{}^{[i]}\mathcal{C}_p^{\mathcal{M}}](\mathsf{z};\mathsf{q}), \tag{5.2.3}$$

which in terms of traces is

$$\mathrm{Tr}_{\mathcal{M}_L} \mathsf{w}^{h_0} \mathsf{x}^{Q_0} \mathsf{q}^{L_0^{\mathrm{sl}} + L_0^{\mathrm{bc}}} = \mathrm{Tr}_{\mathcal{M}_R} \mathsf{y}^{a_0} \mathsf{z}^{J_0} \mathsf{q}^{L_0^{\mathrm{fb}} + L_0^{\mathsf{N=2}}}, \tag{5.2.4}$$

where $\mathcal{M}_L$ and $\mathcal{M}_R$ are the modules on the left-hand side and right-hand side of (5.2.2). Since the modules are equivalent, the terms being summed over must also be equivalent, that is,

$$\mathsf{w}^{h_0} \mathsf{x}^{Q_0} \mathsf{q}^{L_0^{\mathrm{sl}} + L_0^{\mathrm{bc}}} = \mathsf{y}^{a_0} \mathsf{z}^{J_0} \mathsf{q}^{L_0^{\mathrm{fb}} + L_0^{\mathsf{N=2}}}. \tag{5.2.5}$$

The modes on right-hand side of (5.2.5) can be replaced in terms of the $A_1(u, v) \otimes \mathfrak{bc}$ modes using the embedding (5.1.7), which leads to

$$\mathsf{y}^{a_0} \mathsf{z}^{J_0} \mathsf{q}^{L_0^{\mathrm{fb}} + L_0^{\mathsf{N=2}}} = \mathsf{y}^{h_0 + 2Q_0} \mathsf{z}^{h_0/t - \mathsf{k} Q_0/t} \mathsf{q}^{L_0^{\mathrm{sl}} + L_0^{\mathrm{bc}}} = (\mathsf{y}\mathsf{z}^{1/t})^{h_0} (\mathsf{y}^2 \mathsf{z}^{-\mathsf{k}/t})^{Q_0} \mathsf{q}^{L_0^{\mathrm{sl}} + L_0^{\mathrm{bc}}}. \tag{5.2.6}$$

Identifying this with the left-hand side of (5.2.5), we arrive at

$$\mathsf{w} = \mathsf{y}\mathsf{z}^{1/t}, \quad \mathsf{x} = \mathsf{y}^2\mathsf{z}^{-\mathsf{k}/t}, \tag{5.2.7}$$

which allows us to convert (super)characters of $\mathcal{H} \otimes \mathsf{M}^{\mathsf{N}=2}(u,v)$- and $A_1(u,v) \otimes \mathfrak{bc}$-modules into one another

$$\begin{aligned} \mathrm{Ch}[\mathcal{M}_\lambda]\left(\mathsf{y}\mathsf{z}^{1/t};\mathsf{q}\right)\mathrm{Ch}[\mathcal{N}_i]\left(\mathsf{y}^2\mathsf{z}^{-\mathsf{k}/t};\mathsf{q}\right) &= \sum_{p\in\lambda+i+2\mathbb{Z}} \mathrm{Ch}[\mathcal{F}_p](y;\mathsf{q})\,\mathrm{Ch}[{}^{[i]}\mathcal{C}_p^{\mathcal{M}}](\mathsf{z};\mathsf{q}), \\ \mathrm{Ch}[\mathcal{M}_\lambda]\left(\mathsf{y}\mathsf{z}^{1/t};\mathsf{q}\right)\mathrm{Sch}[\mathcal{N}_i](\mathsf{y}^2\mathsf{z}^{-\mathsf{k}/t};\mathsf{q}) &= \sum_{p\in\lambda+i+2\mathbb{Z}} \mathrm{Ch}[\mathcal{F}_p](\mathsf{y};\mathsf{q})\,\mathrm{Sch}[{}^{[i]}\mathcal{C}_p^{\mathcal{M}}](\mathsf{z};\mathsf{q}). \end{aligned} \tag{5.2.8}$$

The simple form (3.1.32) of the free boson characters, the fact that they are proportional to a power of y in particular, implies the following residue formulae for all $p = \lambda + i \bmod 2$:

$$\begin{aligned} \mathrm{Ch}[{}^{[i]}\mathcal{C}_p^{\mathcal{M}}](\mathsf{z};\mathsf{q}) &= \mathrm{Res}_{\mathsf{y}=0}\left[\mathsf{y}^{-p-1}\eta(\mathsf{q})\mathsf{q}^{-p^2/4t}\mathrm{Ch}[\mathcal{M}_\lambda]\left(\mathsf{y}\mathsf{z}^{1/t};\mathsf{q}\right)\mathrm{Ch}[\mathcal{N}_i]\left(\mathsf{y}^2\mathsf{z}^{-\mathsf{k}/t};\mathsf{q}\right)\right], \\ \mathrm{Sch}[{}^{[i]}\mathcal{C}_p^{\mathcal{M}}](\mathsf{z};\mathsf{q}) &= \mathrm{Res}_{\mathsf{y}=0}\left[\mathsf{y}^{-p-1}\eta(\mathsf{q})\mathsf{q}^{-p^2/4t}\mathrm{Ch}[\mathcal{M}_\lambda]\left(\mathsf{y}\mathsf{z}^{1/t};\mathsf{q}\right)\mathrm{Sch}[\mathcal{N}_i](\mathsf{y}^2\mathsf{z}^{-\mathsf{k}/t};\mathsf{q})\right]. \end{aligned} \tag{5.2.9}$$

5.2.2 Unitary $N = 2$ minimal model characters

In [44], the residue formulae (5.2.9) was used to compute the characters of the vacuum $\mathsf{M}^{\mathsf{N}=2}(u,1)$-modules ${}^{[0]}\mathrm{C}_{0,1}$, specialised to $z = 1$. In this section, we extend their method to calculate unspecialised (super)character formulae for certain $\mathsf{M}^{\mathsf{N}=2}(u,1)$-modules, namely the ${}^{[i]}\mathrm{C}_{0,r}$, $r = 1,\ldots,u-1$. These are precisely the modules that are tensored with the vacuum Fock space $\mathcal{F}_0$ in the branching rules (5.1.10). From (5.1.46), we know that each spectral flow orbit contains at least one of these modules and so the (super)characters of the remaining modules may be obtained from these (super)characters by spectral flow.

In the course of calculating the residue formulae (5.2.9) for ${}^{[i]}\mathrm{C}_{0,r}$, we shall need the identity

$$\frac{1}{\prod_{n=1}^{\infty}(1-\mathsf{w}^2\mathsf{q}^{n-1})(1-\mathsf{w}^{-2}\mathsf{q}^n)} = \frac{\mathsf{q}^{1/12}}{\eta(\mathsf{q})^2}\sum_{\ell\in\mathbb{Z}}\phi_\ell(\mathsf{q})\mathsf{w}^{2\ell} \qquad (|\mathsf{q}| < |\mathsf{w}|^2 < 1), \tag{5.2.10}$$

where

$$\phi_\ell(\mathsf{q}) = \sum_{s=0}^{\infty}(-1)^s\mathsf{q}^{\ell s+s(s+1)/2}. \tag{5.2.11}$$

This was derived[1] in [44] from an identity given in [119]. The proof requires some delicacy with convergence regions and we shall take care to respect these in what follows.

[1]The formula in [44] contains a small typo, which we have fixed here, in the exponent of q in the first factor.

We shall first manipulate the $A_1(u,1)$ character formulae (3.3.20) by writing its $\vartheta_1(\mathsf{w}^2;\mathsf{q})$ factor in product form, which is then converted into an infinite sum by identity (5.2.10):

$$\begin{aligned}
\mathrm{Ch}[\mathcal{L}_{r,0}](\mathsf{w};\mathsf{q}) &= \frac{\mathsf{q}^{r^2/4u-1/8}}{\mathsf{w}} \frac{\sum_{j\in\mathbb{Z}} \mathsf{q}^{j(uj+r)}\left(\mathsf{w}^{2uj+r}-\mathsf{w}^{-2uj-r}\right)}{\prod_{i=1}^{\infty}\left(1-\mathsf{w}^2\mathsf{q}^i\right)\left(1-\mathsf{q}^i\right)\left(1-\mathsf{w}^{-2}\mathsf{q}^{i-1}\right)} \\
&= -\frac{\mathsf{w}}{\eta(\mathsf{q})} \frac{\sum_{j\in\mathbb{Z}} \mathsf{q}^{(2uj+r)^2/4u}\left(\mathsf{w}^{2uj+r}-\mathsf{w}^{-2uj-r}\right)}{\prod_{i=1}^{\infty}\left(1-\mathsf{w}^2\mathsf{q}^{i-1}\right)\left(1-\mathsf{w}^{-2}\mathsf{q}^i\right)} \\
&= \frac{\mathsf{w}}{\eta(\mathsf{q})^3} \sum_{j\in\mathbb{Z}} \mathsf{q}^{(2uj+r)^2/4u}\left(\mathsf{w}^{-2uj-r}-\mathsf{w}^{2uj+r}\right) \sum_{\ell\in\mathbb{Z}} \phi_\ell(\mathsf{q})\mathsf{w}^{2\ell}.
\end{aligned} \tag{5.2.12}$$

Combining this with the Neveu-Schwarz ghost characters (4.1.31), we find that the residue formula (5.2.9) for ${}^{[0]}\mathrm{C}_{0,r}$, with r odd, now yields

$$\begin{aligned}
\mathrm{Ch}[{}^{[0]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) = \frac{\mathsf{z}^{1/u}}{\eta(\mathsf{q})^3} &\sum_{j,\ell,n\in\mathbb{Z}} \mathsf{z}^{-n}\mathsf{q}^{n^2/2+(2uj+r)^2/4u} \\
&\mathrm{Res}_{\mathsf{y}=0}\left[\left(\mathsf{y}\mathsf{z}^{1/u}\right)^{-2uj-r+2n+2\ell} - \left(\mathsf{y}\mathsf{z}^{1/u}\right)^{2uj+r+2n+2\ell}\right]\phi_\ell(\mathsf{q}),
\end{aligned} \tag{5.2.13}$$

where $t=u$ for $\mathsf{M}^{\mathrm{N=2}}(u,1)$. The residue terms are given by the coefficients of the y^{-1} terms. We therefore choose $n=\frac{r+1}{2}-\ell+uj$ in the first term and $n=-\frac{r+1}{2}-\ell-uj$ in the second. The character (5.2.13) is then evaluated to be

$$\begin{aligned}
\mathrm{Ch}[{}^{[0]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) = \frac{1}{\eta(\mathsf{q})^3} \sum_{j,\ell\in\mathbb{Z}} \mathsf{q}^{(2uj+r)^2/4u} &\Bigg[\sum_{s=0}^{\infty}(-1)^s \mathsf{z}^{\ell-uj+(1-r)/2}\mathsf{q}^{\ell s+s(s+1)/2+(\ell-uj+(1-r)/2)^2/2} \\
&- \sum_{s=0}^{\infty}(-1)^s \mathsf{z}^{\ell+uj+(1+r)/2}\mathsf{q}^{\ell s+s(s+1)/2+(\ell+uj+(1+r)/2)^2/2}\Bigg],
\end{aligned} \tag{5.2.14}$$

where we have also substituted the series expansion (5.2.11). The exponents of z and q in the brackets simplify greatly upon replacing ℓ by $\ell-s+uj-\frac{1}{2}(1-r)$ in the first summand and by $\ell-s-uj-\frac{1}{2}(1+r)$ in the second:

$$\begin{aligned}
\mathrm{Ch}[{}^{[0]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) &= \frac{1}{\eta(\mathsf{q})^3} \sum_{j,l\in\mathbb{Z}} \mathsf{q}^{(2uj+r)^2/4u} \Bigg[\sum_{s=0}^{\infty}(-1)^s \mathsf{z}^{\ell-s}\mathsf{q}^{\ell^2/2+s(2uj+r)/2} \\
&\qquad - \sum_{s=0}^{\infty}(-1)^s \mathsf{z}^{\ell-s}\mathsf{q}^{\ell^2/2-s(2uj+r)/2}\Bigg] \\
&= \frac{\vartheta_3(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})^3} \sum_{j\in\mathbb{Z}} \mathsf{q}^{(2uj+r)^2/4u} \left[\sum_{s=0}^{\infty}\left(-\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2}\right)^s - \sum_{s=0}^{\infty}\left(-\mathsf{z}^{-1}\mathsf{q}^{-(2uj+r)/2}\right)^s\right].
\end{aligned} \tag{5.2.15a}$$

We have not combined the two sums over s into one, nor have we explicitly summed these geometric series. This is because their regions of convergence are j-dependent and, in fact, there is no region in which all these geometric series converge simultaneously. We instead proceed by recalling the product form

$$\vartheta_3(\mathsf{z};\mathsf{q}) = \prod_{i=1}^{\infty}(1+\mathsf{z}\mathsf{q}^{i-1/2})(1-\mathsf{q}^i)(1+\mathsf{z}^{-1}\mathsf{q}^{i-1/2}) \tag{5.2.16}$$

and noting the following formal power series identities:

$$(1-x)\sum_{n=0}^{\infty}x^n = 1, \qquad (1-x)\sum_{n=0}^{\infty}x^{-n} = -x. \tag{5.2.17}$$

Indeed, $\vartheta_3(\mathsf{z};\mathsf{q})$ will have a factor $(1+\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2})$ if $uj+\frac{r+1}{2}\in\mathbb{Z}_{>0}$, that is if $j\in\mathbb{Z}_{\geq 0}$, so for these j, we may take $x=-\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2}$ to obtain

$$(1+\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2})\sum_{s=0}^{\infty}\left(-\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2}\right)^s = 1. \tag{5.2.18}$$

Similarly, when $j\in\mathbb{Z}_{\geq 0}$, $(1+\mathsf{z}\mathsf{q}^{(2uj+r)/2})$ is a factor of $\vartheta_3(\mathsf{z};\mathsf{q})$ so putting $x=-\mathsf{z}\mathsf{q}^{(2uj+r)/2}$ results in

$$(1+\mathsf{z}\mathsf{q}^{(2uj+r)/2})\sum_{s=0}^{\infty}\left(-\mathsf{z}^{-1}\mathsf{q}^{-(2uj+r)/2}\right)^s = \mathsf{z}\mathsf{q}^{(2uj+r)/2}. \tag{5.2.19}$$

Similarly analysing the $j\in\mathbb{Z}_{<0}$ terms leads to the following character formula:

$$\begin{aligned}
\mathrm{Ch}[{}^{[0]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) &= \mathrm{Ch}[{}^{[2]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) \\
&= \frac{\vartheta_3(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})^3}\left[\sum_{j\geq 0}\mathsf{q}^{(2uj+r)^2/4u}\left(\frac{1}{1+\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2}} - \frac{\mathsf{z}\mathsf{q}^{(2uj+r)/2}}{1+\mathsf{z}\mathsf{q}^{(2uj+r)/2}}\right)\right. \\
&\qquad\left. + \sum_{j<0}\mathsf{q}^{(2uj+r)^2/4u}\left(\frac{\mathsf{z}\mathsf{q}^{-(2uj+r)/2}}{1+\mathsf{z}\mathsf{q}^{-(2uj+r)/2}} - \frac{1}{1+\mathsf{z}^{-1}\mathsf{q}^{-(2uj+r)/2}}\right)\right] \\
&= \frac{\vartheta_3(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})^3}\sum_{j\in\mathbb{Z}}\left(\frac{\mathsf{q}^{(2uj+r)^2/4u}}{1+\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2}} - \frac{\mathsf{q}^{(2uj+r)^2/4u}}{1+\mathsf{z}^{-1}\mathsf{q}^{-(2uj+r)/2}}\right).
\end{aligned} \tag{5.2.20}$$

This may of course be simplified further. Here, we content ourselves with remarking that this expression is non-singular for all non-zero z because the denominators that appear are all factors of $\vartheta_3(\mathsf{z};\mathsf{q})$. This character formula is therefore valid for $|\mathsf{q}|<1$ and $|\mathsf{z}|\neq 0$.

To compute the corresponding supercharacter, it suffices to note that taking the supertrace is equivalent to factorising out z to the power of the J_0-charge of the highest-weight vector and replacing z by $-\mathsf{z}$ in what remains. Since the J_0-charge of the highest-weight vector of ${}^{[0]}\mathrm{C}_{0,r}$ is 0, we obtain

$$\begin{aligned}\mathrm{Sch}[{}^{[0]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) &= -\mathrm{Sch}[{}^{[2]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) = \mathrm{Ch}[{}^{[0]}\mathrm{C}_{0,r}](-\mathsf{z};\mathsf{q}) \\ &= \frac{\vartheta_4(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})^3}\sum_{j\in\mathbb{Z}}\left(\frac{\mathsf{q}^{(2uj+r)^2/4u}}{1-\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2}}-\frac{\mathsf{q}^{(2uj+r)^2/4u}}{1-\mathsf{z}^{-1}\mathsf{q}^{-(2uj+r)/2}}\right),\end{aligned} \tag{5.2.21}$$

where we noticed the identity $\vartheta_3(-\mathsf{z};\mathsf{q}) = \vartheta_4(\mathsf{z};\mathsf{q})$. And as in (5.2.20), the denominators in the supercharacter formula are factors of $\vartheta_4(\mathsf{z};\mathsf{q})$.

Similarly, we can repeat the same calculations in the Ramond sector for ${}^{[1]}\mathrm{C}_{0,r}$ with $1 \le r \le u-1$ and r even, to conclude

$$\begin{aligned}\mathrm{Ch}[{}^{[1]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) &= \mathrm{Ch}[{}^{[3]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) \\ &= \frac{\vartheta_2(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})^3}\sum_{j\in\mathbb{Z}}\left(\frac{\mathsf{q}^{(2uj+r)^2/4u}}{1+\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2}}-\frac{\mathsf{q}^{(2uj+r)^2/4u}}{1+\mathsf{z}^{-1}\mathsf{q}^{-(2uj+r)/2}}\right),\end{aligned} \tag{5.2.22}$$

$$\begin{aligned}\mathrm{Sch}[{}^{[1]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) &= -\mathrm{Sch}[{}^{[3]}\mathrm{C}_{0,r}](\mathsf{z};\mathsf{q}) \\ &= \frac{\mathrm{i}\vartheta_1(\mathsf{z};\mathsf{q})}{\eta(\mathsf{q})^3}\sum_{j\in\mathbb{Z}}\left(\frac{\mathsf{q}^{(2uj+r)^2/4u}}{1-\mathsf{z}^{-1}\mathsf{q}^{(2uj+r)/2}}-\frac{\mathsf{q}^{(2uj+r)^2/4u}}{1-\mathsf{z}^{-1}\mathsf{q}^{-(2uj+r)/2}}\right),\end{aligned} \tag{5.2.23}$$

noticing the identity $\vartheta_2(-\mathsf{z};\mathsf{q}) = \mathrm{i}\vartheta_1(\mathsf{z};\mathsf{q})$.

Equations (5.2.20) and (5.2.22) provide the character formulae for $N = 2$ modules ${}^{[i]}\mathrm{C}_{n,r}$ with $n = 0$. These modules lie on the same column as the vacuum module within the Kac tables described in Section 5.1.2. We shall refer this column as the vacuum column of the Kac table. In Section 5.1.2, we discussed the orbits of simple $\mathsf{M}^{\mathrm{N=2}}(u,1)$ modules under spectral flow and conjugation and in (5.1.46) we gave a vacuum column representative for each orbit. Next we can use spectral flow to compute character formulae for all simple modules outside of the vacuum column.

Let $\mathcal{L}$ be an $\mathsf{M}^{\mathrm{N=2}}(u,v)$ module, and consider its ℓ-fold spectral flow $\sigma^{\ell}_{\mathrm{N=2}}(\mathcal{L})$. The character of $\sigma^{\ell}_{\mathrm{N=2}}(\mathcal{L})$ is then given by

$$\begin{aligned}\mathrm{Ch}[\sigma^{\ell}_{\mathrm{N=2}}(\mathcal{L})](\mathsf{z};\mathsf{q}) &= \mathrm{Tr}_{\sigma^{\ell}_{\mathrm{N=2}}(\mathcal{L})}\left[\mathsf{z}^{J_0}\mathsf{q}^{L_0-\mathsf{C}/24}\right] = \mathrm{Tr}_{\mathcal{L}}\left[\mathsf{z}^{\sigma^{-\ell}_{\mathrm{N=2}}(J_0)}\mathsf{q}^{\sigma^{-\ell}_{\mathrm{N=2}}(L_0)-\mathsf{C}/24}\right] \\ &= \mathrm{Tr}_{\mathcal{L}}\left[\mathsf{z}^{J_0+\mathsf{C}\ell/3}\mathsf{q}^{L_0+\ell J_0+\mathsf{C}\ell^2/6-\mathsf{C}/24}\right] \\ &= \mathsf{z}^{\mathsf{c}\ell/3}\mathsf{q}^{\mathsf{c}\ell^2/6}\mathrm{Ch}[\mathcal{L}]\left(\mathsf{z}\mathsf{q}^{\ell};\mathsf{q}\right).\end{aligned} \tag{5.2.24}$$

It follows from (5.1.43) that spectral flow acts on a coset module as

$$\sigma^{\ell}_{\mathrm{N=2}}\left({}^{[i]}\mathrm{C}_{p,r}\right) \cong {}^{[i-2\ell]}\mathrm{C}_{p-2\ell,r}. \tag{5.2.25}$$

Thus, ${}^{[i]}\mathrm{C}_{n,r}$ is simply the $-n/2$-times spectral flow on the vacuum column module ${}^{[i+n]}\mathrm{C}_{0,r}$, whose character can be computed as

$$\begin{aligned} \mathrm{Ch}[{}^{[i]}\mathrm{C}_{n,r}](\mathsf{z};\mathsf{q}) &= \mathrm{Ch}[\sigma_{\mathrm{N=2}}^{-n/2}({}^{[i+n]}\mathrm{C}_{0,r})](\mathsf{z};\mathsf{q}) \\ &= \mathsf{z}^{-n\mathsf{c}/6}\mathsf{q}^{n^2\mathsf{c}/24}\mathrm{Ch}[{}^{[i+n]}\mathrm{C}_{0,r}]\left(\mathsf{z}\mathsf{q}^{-n/2};\mathsf{q}\right) \end{aligned} \tag{5.2.26}$$

using (5.2.24). Similarly, for supercharacters we get

$$\mathrm{Sch}[{}^{[i]}\mathrm{C}_{n,r}](\mathsf{z};\mathsf{q}) = \mathsf{z}^{-n\mathsf{c}/6}\mathsf{q}^{n^2\mathsf{c}/24}\mathrm{Sch}[{}^{[i+n]}\mathrm{C}_{0,r}](\mathsf{z}\mathsf{q}^{-n/2};\mathsf{q}). \tag{5.2.27a}$$

5.2.3 Non-unitary minimal model characters

We now turn to the computation of the (super)characters of the standard modules of the non-unitary $N=2$ minimal models $\mathsf{M}^{\mathrm{N=2}}(u,v)$, $v>1$, again by taking residues of $A_1(u,v)\otimes\mathfrak{bc}$ characters. The characters of the irreducible atypical modules then follow implicitly by resolving them in terms of (atypical) standard modules. This approach to the irreducible atypical characters is rooted in the subtle question of how to correctly account for their non-trivial regions of convergence. The reader can refer to [49] for a review, including several examples in which this approach has succeeded.

As in the unitary case, plugging the character formulae for standard $A_1(u,v)$ modules into the residue formula (5.2.9) yields the $\mathsf{M}^{\mathrm{N=2}}(u,v)$ (super)character formulae and these residue formulae are arguably easier to evaluate than in the unitary case. We give the details for $\mathrm{Ch}[{}^{[0]}\mathrm{C}^{\mathrm{E}}_{p;(r,s)}](\mathsf{z};\mathsf{q})$. Recall that the $A_1(u,v)$ $\mathcal{E}$-type character formulae $\mathrm{Ch}[\mathcal{E}_{h;(r,s)}]$ (3.3.39) contain the algebraic delta function $\delta(\mathsf{w}^2)=\sum_{n\in\mathbb{Z}}\mathsf{w}^{2n}$ as a factor, which satisfies the identity $\mathsf{w}^{2m}\delta(\mathsf{w}^2)=\delta(\mathsf{w}^2)$, $m\in\mathbb{Z}$. Thus,

$$\begin{aligned} \mathrm{Ch}[{}^{[0]}\mathrm{C}^{\mathrm{E}}_{p;(r,s)}](\mathsf{z};\mathsf{q}) &= \mathrm{Res}_{\mathsf{y}=0}\left[\mathsf{y}^{-p-1}\eta(\mathsf{q})\mathsf{q}^{-p^2/4t}\mathrm{Ch}[\mathcal{E}_{p;(r,s)}]\left(\mathsf{y}\mathsf{z}^{1/t};\mathsf{q}\right)\mathrm{Ch}[\mathcal{N}_0]\left(\mathsf{y}^2\mathsf{z}^{-\mathsf{k}/t};\mathsf{q}\right)\right] \\ &= \mathrm{Res}_{\mathsf{y}=0}\left[\mathsf{y}^{-p-1}\mathsf{q}^{-p^2/4t}\left(\mathsf{y}\mathsf{z}^{1/t}\right)^p\frac{\chi^{(u,v)}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2}\delta\left(\mathsf{y}^2\mathsf{z}^{2/t}\right)\vartheta_3(\mathsf{y}^2\mathsf{z}^{-\mathsf{k}/t};\mathsf{q})\right] \\ &= \mathrm{Res}_{\mathsf{y}=0}\left[\mathsf{y}^{-1}\mathsf{q}^{-p^2/4t}\mathsf{z}^{p/t}\frac{\chi^{(u,v)}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2}\delta\left(\mathsf{y}^2\mathsf{z}^{2/t}\right)\vartheta_3(\mathsf{z}^{-1};\mathsf{q})\right] \\ &= \frac{\mathsf{z}^{p/t}}{\mathsf{q}^{p^2/4t}}\frac{\vartheta_3(\mathsf{z};\mathsf{q})\chi^{(u,v)}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2}, \end{aligned} \tag{5.2.28}$$

where we recall that $\chi^{(u,v)}_{r,s}(\mathsf{q})=\chi^{(u,v)}_{u-r,v-s}(\mathsf{q})$ is the character of the Virasoro module of highest weight $\Delta^{\mathrm{Vir}}_{r,s}$ in $\mathsf{M}(u,v)$. The third equality follows from the identity

$$\mathsf{y}^2\mathsf{z}^{-\mathsf{k}/t}\delta\left(\mathsf{y}^2\mathsf{z}^{2/t}\right)=\mathsf{z}^{-(2+\mathsf{k})/t}\delta\left(\mathsf{y}^2\mathsf{z}^{2/t}\right)=\mathsf{z}^{-1}\delta\left(\mathsf{y}^2\mathsf{z}^{2/t}\right), \tag{5.2.29}$$

with $t=\mathsf{k}+2$ and the fourth from $\vartheta_3(\mathsf{z}^{-1};\mathsf{q})=\vartheta_3(\mathsf{z};\mathsf{q})$. The remaining (super)character formulae follow similarly and are

$$\mathrm{Sch}[{}^{[0]}\mathsf{C}^{\mathsf{E}}_{p;(r,s)}](\mathsf{z};\mathsf{q}) = \frac{\mathsf{z}^{p/t}}{\mathsf{q}^{p^2/4t}} \frac{\vartheta_4(\mathsf{z};\mathsf{q})\chi^{(u,v)}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2}, \tag{5.2.30a}$$

$$\mathrm{Ch}[{}^{[1]}\mathsf{C}^{\mathsf{E}}_{p;(r,s)}](\mathsf{z};\mathsf{q}) = \frac{\mathsf{z}^{p/t}}{\mathsf{q}^{p^2/4t}} \frac{\vartheta_2(\mathsf{z};\mathsf{q})\chi^{(u,v)}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2}, \tag{5.2.30b}$$

$$\mathrm{Sch}[{}^{[1]}\mathsf{C}^{\mathsf{E}}_{p;(r,s)}](\mathsf{z};\mathsf{q}) = \frac{\mathsf{z}^{p/t}}{\mathsf{q}^{p^2/4t}} \frac{\mathsf{i}\vartheta_1(\mathsf{z};\mathsf{q})\chi^{(u,v)}_{r,s}(\mathsf{q})}{\eta(\mathsf{q})^2}, \tag{5.2.30c}$$

The exact sequences such as (5.1.23), which reducible standard modules satisfy, can be spliced together to form resolutions for atypical simple modules in terms of standard modules. Such resolutions are very helpful because the (super)characters of atypical simple modules can then be computed as alternating sums of standard module (super)characters through the Euler-Poincaré principle. Further, since restriction is exact we can compute the branching rules of $A_1(u,v)$ resolutions tensored with $\mathfrak{bc}$ modules.

The resolutions for $\mathcal{L}_{r,0}$ and $\mathcal{D}^+_{r,s}$ in $A_1(u,v)$ are given in (6.1.11) and (3.3.36) [36]. To obtain the corresponding resolution for $\mathsf{M}^{\mathsf{N=2}}(u,v)$, we tensor each $A_1(u,v)$-module by a $\mathfrak{bc}$-module $\mathcal{N}_i$, and then decompose each tensored module using the branching rules stated in (5.1.17). By isolating the coefficient of each Fock space of integral weight p and making use of the identities (5.1.43) to determine the branching rules of spectral flows of E-type modules, we get the $N=2$ resolution for ${}^{[i]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}$.

$$\begin{aligned} \cdots &\longrightarrow {}^{[i]}\mathsf{C}^{+}_{p-(3v-1)t;(r,v-1)} \longrightarrow \cdots \longrightarrow {}^{[i]}\mathsf{C}^{+}_{p-(2v+2)t;(r,2)} \longrightarrow {}^{[i]}\mathsf{C}^{+}_{p-(2v+1)t;(r,1)} \\ &\longrightarrow {}^{[i]}\mathsf{C}^{+}_{p-(2v-1)t;(u-r,v-1)} \longrightarrow \cdots \longrightarrow {}^{[i]}\mathsf{C}^{+}_{p-(v+2)t;(u-r,2)} \longrightarrow {}^{[i]}\mathsf{C}^{+}_{p-(v+1)t;(u-r,1)} \\ &\longrightarrow {}^{[i]}\mathsf{C}^{+}_{p-(v-1)t;(r,v-1)} \longrightarrow \cdots \longrightarrow {}^{[i]}\mathsf{C}^{+}_{p-2t;(r,2)} \longrightarrow {}^{[i]}\mathsf{C}^{+}_{p-t;(r,1)} \longrightarrow {}^{[i]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)} \longrightarrow 0, \end{aligned} \tag{5.2.31}$$

where $t = u/v$ and $p \in r-1+i \in 2\mathbb{Z}$. The Euler Poincaré principle now gives the (super)character of ${}^{[i]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}$ as the alternating sum of (super)characters of the preceding terms:

$$\begin{aligned} \mathrm{Ch}[{}^{[i]}\mathsf{C}^{\mathsf{E}}_{p;(r,0)}] = \sum_{s=1}^{v-1}(-1)^{s-1}\sum_{\ell=0}^{\infty}\Big(&\mathrm{Ch}[{}^{[i]}\mathsf{C}^{+}_{p-ts-2u\ell;(r,s)}] \\ &-\mathrm{Ch}[{}^{[i]}\mathsf{C}^{+}_{p+ts-2u(\ell+1);(u-r,v-s)}]\Big) \end{aligned} \tag{5.2.32}$$

$$\mathrm{Sch}[{}^{[i]}\mathsf{C}^{\mathsf{E}}_{p;(r,0)}] = \sum_{s=1}^{v-1}\sum_{\ell=0}^{\infty}\Big(\mathrm{Ch}[{}^{[i]}\mathsf{C}^{+}_{p+ts-2u(\ell+1);(u-r,v-s)}] - \mathrm{Ch}[{}^{[i]}\mathsf{C}^{+}_{p-ts-2u\ell;(r,s)}]\Big). \tag{5.2.33}$$

Substituting in the standard module character formulae which were given in (5.2.28), we arrive at the character formulae

$$\begin{aligned}
\mathrm{Ch}[{}^{[0]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}](\mathsf{z};\mathsf{q}) &= \frac{\vartheta_3(\mathsf{z};\mathsf{q})}{\eta^2(\mathsf{q})}\sum_{s=1}^{v-1}(-1)^{s-1}\chi^{(u,v)}_{r,s}(\mathsf{q}) \\
&\qquad\times\sum_{\ell=0}^{\infty}\left(\frac{\mathsf{z}^{p/t-(2v\ell+s)}}{\mathsf{q}^{(p-(2v\ell+s)t)^2/4t}}-\frac{\mathsf{z}^{p/t-(2v(\ell+1)-s)}}{\mathsf{q}^{(p-(2v(\ell+1)-s)t)^2/4t}}\right) \\
\mathrm{Sch}[{}^{[0]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}](\mathsf{z};\mathsf{q}) &= \frac{\vartheta_4(\mathsf{z};\mathsf{q})}{\eta^2(\mathsf{q})}\sum_{s=1}^{v-1}\chi^{(u,v)}_{r,s}(\mathsf{q}) \\
&\qquad\times\sum_{\ell=0}^{\infty}\left(\frac{\mathsf{z}^{p/t-(2v(\ell+1)-s)}}{\mathsf{q}^{(p-(2v(\ell+1)-s)t)^2/4t}}-\frac{\mathsf{z}^{p/t-(2v\ell+s)}}{\mathsf{q}^{(p-(2v\ell+s)t)^2/4t}}\right) \\
\mathrm{Ch}[{}^{[1]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}](\mathsf{z};\mathsf{q}) &= \frac{\vartheta_2(\mathsf{z};\mathsf{q})}{\eta^2(\mathsf{q})}\sum_{s=1}^{v-1}(-1)^{s-1}\chi^{(u,v)}_{r,s}(\mathsf{q}) \\
&\qquad\times\sum_{\ell=0}^{\infty}\left(\frac{\mathsf{z}^{p/t-(2v\ell+s)}}{\mathsf{q}^{(p-(2v\ell+s)t)^2/4t}}-\frac{\mathsf{z}^{p/t-(2v(\ell+1)-s)}}{\mathsf{q}^{(p-(2v(\ell+1)-s)t)^2/4t}}\right) \\
\mathrm{Sch}[{}^{[1]}\mathsf{C}^{\mathsf{L}}_{p;(r,0)}](\mathsf{z};\mathsf{q}) &= \frac{\mathrm{i}\vartheta_1(\mathsf{z};\mathsf{q})}{\eta^2(\mathsf{q})}\sum_{s=1}^{v-1}\chi^{(u,v)}_{r,s}(\mathsf{q}) \\
&\qquad\times\sum_{\ell=0}^{\infty}\left(\frac{\mathsf{z}^{p/t-(2v(\ell+1)-s)}}{\mathsf{q}^{(p-(2v(\ell+1)-s)t)^2/4t}}-\frac{\mathsf{z}^{p/t-(2v\ell+s)}}{\mathsf{q}^{(p-(2v\ell+s)t)^2/4t}}\right).
\end{aligned} \tag{5.2.34}$$

The (super)characters for branching rules of the remaining highest weight $A_1(u,v)$-modules, $\mathcal{D}^+_{r,s}$, can be computed in the same way. Alternatively, we can also use the relation (3.3.38b)

$$\begin{aligned}
\mathrm{Ch}[\mathcal{D}^+_{r,s}] &= (-1)^{v-1-s}\mathrm{Ch}[\sigma^{v-s}_{\mathrm{sl}}\left(\mathcal{L}_{u-r,0}\right)] \\
&\qquad+\sum_{j=1}^{v-s-1}(-1)^{j-1}\mathrm{Ch}[\sigma^{j}_{\mathrm{sl}}\left(\mathcal{E}^+_{r,s+j}\right)], \qquad (s<v-1),
\end{aligned} \tag{5.2.35}$$

and character decompositions to conclude

$$\begin{aligned}
\mathrm{Ch}[{}^{[i]}\mathsf{C}^{\mathsf{D}}_{p;(r,s)}] &= (-1)^{v-1-s}\mathrm{Ch}[{}^{[i]}\mathsf{C}^{\mathsf{L}}_{p-(v-s)t;(u-r,s)}]+\sum_{j=1}^{v-s-1}(-1)^{j-1}\mathrm{Ch}[{}^{[i]}\mathsf{C}^{\mathsf{E}}_{p-tj;(r,s')}], \\
\mathrm{Sch}[{}^{[i]}\mathsf{C}^{\mathsf{D}}_{p;(r,s)}] &= -\mathrm{Sch}[{}^{[i]}\mathsf{C}^{\mathsf{L}}_{p-(v-s)t;(u-r,s)}]-\sum_{j=1}^{v-s-1}\mathrm{Sch}[{}^{[i]}\mathsf{C}^{\mathsf{E}}_{p-tj;(r,s')}].
\end{aligned}$$

Again, (5.2.35) and (5.2.36) can be written into a form similar to (5.2.34) by substituting in the standard module character (5.2.30) and the $\mathcal{L}$-type character given in (5.2.32).

5.3 Fusion Rules

5.3.1 Grothendieck fusion rules

Let us start the section with a brief explanation of the concept of the Grothendieck group. In the context of superconformal field theory, we say two modules are isomorphic as elements of the Grothendieck group if they have the same characters and supercharacters. The set of isomorphic classes of modules form a group, which means we can add and subtract these classes since they are effectively identified with (super)characters. This group is known as the Grothendieck group, and it is the $\mathbb{Z}$-span of the isomorphism classes of the irreducibles. The image of a module in the Grothendieck group is the sum of the isomorphism classes of its composition factors. It is conjectured in [36] that fusion also respects these classes, this makes the Grothendieck group a ring with fusion as its product operation.

One of the most convenient ways to compute the fusion rules of a rational bosonic conformal field theory involves substituting its S-matrix entries into the Verlinde formula for fusion coefficients. For fermionic theories, one can derive variations of the Verlinde formula as in [6, 48]. For certain non-rational theories, there is a generalisation called the standard Verlinde formula [49, 50] that is conjectured to give the Grothendieck fusion coefficients of the theory, these being the structure constants of the Grothendieck ring of the fusion ring.

We shall, however, present an alternative approach to compute the (Grothendieck) fusion rules using the coset (5.1.3) and the known (Grothendieck) fusion rules of the $\widehat{\mathfrak{sl}}_2$ minimal models $A_1(p,u)$, the $\mathfrak{bc}$-ghost and the Heisenberg algebra. We shall illustrate the idea by computing the fusion of ${}^{[i]}\mathsf{C}^{\mathsf{L}}_{m;(r,0)}$ and ${}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(r',s')}$. These two modules tensored with $\mathcal{F}_m$ ($m \in \lambda^{\mathrm{sl}}_{r,0} + i + 2\mathbb{Z}$) and $\mathcal{F}_n$ ($n \in \lambda^{\mathrm{sl}}_{r',s'} + j + 2\mathbb{Z}$), respectively, induce to $A_1(u,v) \otimes \mathfrak{bc}$-modules according to (5.1.27) and (5.1.28). Thus,

$$\begin{aligned}
\left(\mathcal{L}_{r,0} \otimes \mathcal{N}_i\right) \times \left(\mathcal{D}^+_{r',s'} \otimes \mathcal{N}_j\right) &= \left(\mathcal{F}_m \otimes {}^{[i]}\mathsf{C}^{\mathsf{L}}_{m;(r,0)}\right) \uparrow \times \left(\mathcal{F}_n \otimes {}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(r',s')}\right) \uparrow \\
&\cong \left[\left(\mathcal{F}_m \otimes {}^{[i]}\mathsf{C}^{\mathsf{L}}_{m;(r,0)}\right) \times \left(\mathcal{F}_n {}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(r',s')}\right)\right] \uparrow \\
&\cong \left[\mathcal{F}_{m+n} \otimes \left({}^{[i]}\mathsf{C}^{\mathsf{L}}_{m;(r,0)} \times {}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(r',s')}\right)\right] \uparrow
\end{aligned} \tag{5.3.1}$$

as induction is preserved by fusion (A.1). We now use the $A_1(u,v)$ and $\mathfrak{bc}$ fusion rules to compute the left-hand side of (5.3.1) and identify the result as a direct sum of induced $\mathsf{M}^{\mathsf{N}=2}(u,v) \otimes \mathcal{H}$-modules

$$\begin{aligned}
\left(\mathcal{L}_{r,0} \otimes \mathcal{N}_i\right) \times \left(\mathcal{D}^+_{r',s'} \otimes \mathcal{N}_j\right) &= \left(\bigoplus_{r''} \mathsf{N}^{[u]\,r''}_{r,r'}\, \mathcal{D}^+_{r'',s'} \otimes \mathcal{N}_{i+j}\right) \\
&= \bigoplus_{r''} \mathsf{N}^{[u]\,r''}_{r,r'} \left(\mathcal{F}_p \otimes {}^{[i+j]}\mathsf{C}^{\mathsf{D}}_{p;(r'',s')}\right) \uparrow,
\end{aligned} \tag{5.3.2}$$

where $p \in \lambda^{\mathrm{sl}}_{r'',s'} + i + j + 2\mathbb{Z}$. We shall choose $p = m+n$, which can be easily checked to fall in the allowed range. Comparing the left-hand sides of (5.3.1) and (5.3.2) with this choice gives

$$\Big[\mathcal{F}_{m+n} \otimes \Big({}^{[i]}\mathsf{C}^{\mathsf{L}}_{m;(r,0)} \times {}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(r',s')}\Big)\Big] \uparrow = \Big(\bigoplus_{r''} \mathsf{N}^{[u]\, r''}_{r,r'} \mathcal{F}_{m+n} \otimes {}^{[i+j]}\mathsf{C}^{\mathsf{D}}_{m+n;(r'',s')}\Big) \uparrow . \tag{5.3.3}$$

Splitting out the common Fock space yields the following $\mathsf{M}^{\mathrm{N=2}}(u,v)$ fusion rules:

$${}^{[i]}\mathsf{C}^{\mathsf{L}}_{m;(r,0)} \times {}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(r',s')} = \bigoplus_{r''} \mathsf{N}^{[u]\, r''}_{r,r'}\, {}^{[i+j]}\mathsf{C}^{\mathsf{D}}_{m+n;(r'',s')}. \tag{5.3.4}$$

In an identical fashion, induction gives the following $\mathsf{B}_{0|1}(p,v)$ fusion rules:

$${}^{[i]}\mathsf{C}^{\mathsf{L}}_{m;(r,0)} \times {}^{[j]}\mathsf{C}^{\mathsf{L}}_{n;(r',0)} = \bigoplus_{r''=1}^{u-1} \mathsf{N}^{[u]\, r''}_{r,r'}\, {}^{[i+j]}\mathsf{C}^{\mathsf{L}}_{m+n;(r'',0)}, \tag{5.3.5a}$$

$${}^{[i]}\mathsf{C}^{\mathsf{L}}_{m;(r,0)} \times {}^{[j]}\mathsf{C}^{\mathsf{E}}_{n;(r',s')} = \bigoplus_{r''=1}^{u-1} \mathsf{N}^{[u]\, r''}_{r,r'}\, {}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(r'',s')}. \tag{5.3.5b}$$

Because fusion respects parity reversal and should respect spectral flow [120, Prop. 2.11 and Eq. (3.6)],

$$\begin{aligned} \mathcal{M} \times \Pi\mathcal{N} &\cong \Pi(\mathcal{M}\times\mathcal{N}) \cong \Pi\mathcal{M}\times\mathcal{N}, \\ \mathcal{M} \times \sigma_{\mathrm{N=2}}(\mathcal{N}) &\cong \sigma_{\mathrm{N=2}}(\mathcal{M}\times\mathcal{N}) \cong \sigma_{\mathrm{N=2}}(\mathcal{M})\times\mathcal{N}, \end{aligned} \tag{5.3.6}$$

these fusion rules imply many others. We remark that the fusion rules of the unitary minimal models $\mathsf{M}^{\mathrm{N=2}}(u,1)$ are given by (5.3.5a) alone.

Unfortunately, a complete set of irreducible $\mathsf{M}^{\mathrm{N=2}}(u,v)$ fusion rules cannot be obtained in this way because the required $A_1(u,v)$ fusion rules are not known. Instead, we have their Grothendieck versions [36] which were reproduced in (3.3.41). Recall that we denote the Grothendieck fusion operation by $\boxtimes$ and the image of a module $\mathcal{M}$ in the Grothendieck fusion ring by $[\mathcal{M}]$.

The fact that $\boxtimes$ is well defined is not at all obvious. A sufficient condition for this is that fusing with any fixed module from our category is exact, meaning that it respects the exactness of sequences. For rational theories, such as the $\mathsf{M}^{\mathrm{N=2}}(u,1)$, this is a theorem in the formalism of Huang, Lepowsky and Zhang [121]. However, for the $\mathsf{M}^{\mathrm{N=2}}(u,v)$ with $v \neq 1$, we have to assume that fusion is exact on a suitable module category. Granting this, it follows that the fusion and Grothendieck fusion products of two modules $\mathcal{M}$ and $\mathcal{N}$ are related by

$$\big[\mathcal{M}\times\mathcal{N}\big] = \big[\mathcal{M}\big] \boxtimes \big[\mathcal{N}\big]. \tag{5.3.7}$$

(This is, in fact, how $\boxtimes$ is defined.) The exactness assumption being made is strong, but is not expected to be problematic. Unfortunately, tools to verify it seem to be out of reach at present.

In any case, taking Grothendieck images respects tensor products and induction, the latter because it is defined in terms of fusion, hence the methods that led to the fusion rules (5.3.4) and (5.3.5) apply equally well to Grothendieck fusion rules. This procedure thus determines the Grothendieck fusion rules involving all types of the irreducible $\mathsf{M}^{\mathsf{N=2}}(u,v)$-modules. Those that are not just the Grothendieck images of (5.3.4) and (5.3.5) (or their parity-reversed and spectral-flowed versions) are:

$$\left[{}^{[i]}\mathsf{C}^{\mathsf{E}}_{m;(r,s)}\right] \times \left[{}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(r',s')}\right] = \sum_{r'',s''\in\mathbb{Z}} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s'+1)} \left[{}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(r'',s'')}\right] + \sum_{r'',s''\in\mathbb{Z}} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s')} \left[{}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(r'',s'')}\right], \tag{5.3.8a}$$

$$\left[{}^{[i]}\mathsf{C}^{\mathsf{E}}_{m;(r,s)}\right] \times \left[{}^{[j]}\mathsf{C}^{\mathsf{E}}_{n;(r',s')}\right] = \sum_{r'',s''\in\mathbb{Z}} \left(\mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s'-1)} + \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s'+1)}\right) \left[{}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(r'',s'')}\right] + \sum_{r'',s''\in\mathbb{Z}} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s')} \left(\left[{}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(r'',s'')}\right] + \left[{}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n+t;(r'',s'')}\right]\right) \tag{5.3.8b}$$

$$\left[{}^{[i]}\mathsf{C}^{\mathsf{D}}_{m;(r,s)}\right] \times \left[{}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(r',s')}\right] = \begin{cases} \displaystyle\sum_{r'',s''\in\mathbb{Z}} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s')} \left[{}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(r'',s'')}\right] + \sum_{r''\in\mathbb{Z}} \mathsf{N}^{[u]\,r''}_{r,r'} \left[{}^{[i+j]}\mathsf{C}^{\mathsf{D}}_{m+n;(r'',s+s')}\right], & \text{if } s+s'<v \\ \displaystyle\sum_{r'',s''\in\mathbb{Z}} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s+1),(r',s'+1)} \left[{}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(r'',s'')}\right] + \sum_{r''\in\mathbb{Z}} \mathsf{N}^{[u]\,r''}_{r,r'} \left[{}^{[i+j+2]}\mathsf{C}^{\mathsf{D}}_{m+n-t;(u-r'',s+s'-v+1)}\right], & \text{if } s+s'\geq v. \end{cases} \tag{5.3.8c}$$

Here, the sums over r'' always run from 1 to $p-1$ while the sums over s'' always run from 1 to $v-1$. These fusion rules can be extended to include parity reversals and spectral flows using the Grothendieck versions of (5.3.6).

5.3.2 Projective modules and fusion rules

Identifying genuine fusion rules from their Grothendieck version (5.3.8) can be challenging because they are expected to involve reducible but indecomposable modules in general. Recall that in Section 3.3.4, we conjectured the existence of indecomposable $A_1(u,v)$-modules on which L_0^{sl} acts non-semisimply [49,92]), these were referred to as the staggered modules and were conjectured to be projective. In this section, we shall examine the staggered modules of the $N=2$ minimal models $\mathsf{M}^{\mathsf{N=2}}(u,v)$ and report the fusion rules related to these modules.

Recall the staggered modules $\mathcal{S}_{r,s}$ of $A_1(u,v)$ whose Loewy diagram was given in

(3.3.46). Since projective modules induce to projective modules, we may lift the conjectures to the $N = 2$ theory by proposing

$$\left(\mathcal{F}_n \otimes {}^{[i]}\mathscr{P}_{n;(r,s)}\right) \uparrow \cong \mathcal{S}_{r,s} \otimes \mathcal{N}_i, \qquad 1 \le r \le u-1 \text{ and } 0 \le s \le v-1, \tag{5.3.9}$$

where we denote the $N = 2$ staggered module by ${}^{[i]}\mathscr{P}_{n;(r,s)}$. The relation is true for all $n \in i + \lambda^{\mathsf{sl}}_{r,s} + 2\mathbb{Z}$. Conversely, decomposing an $A_1(u,v)$ staggered module with a $\mathfrak{bc}$-module yields the branching rule

$$\left(\mathcal{S}_{r,s} \otimes \mathcal{N}_i\right) \downarrow = \bigoplus_{n \in i + \lambda^{\mathsf{sl}}_{r,s} + 2\mathbb{Z}} \mathcal{F}_n \otimes {}^{[i]}\mathscr{P}_{n;(r,s)}. \tag{5.3.10}$$

The Loewy diagram corresponding to the $\mathsf{M}^{\mathsf{N=2}}(u,v)$ projective cover ${}^{[i]}\mathscr{P}_{n;(r,s)}$ of ${}^{[i]}\mathsf{C}^{\mathsf{D}}_{n;(r,s)}$ is

$$\begin{array}{ccccc} & & {}^{[i]}\mathsf{C}^{\mathsf{D}}_{n;(r,s)} & & \\ & \swarrow & & \searrow & \\ {}^{[i+2]}\mathsf{C}^{\mathsf{D}}_{n+t;(r,s-1)} & & {}^{[i]}\mathscr{P}_{n;(r,s)} & & {}^{[i+2]}\mathsf{C}^{\mathsf{D}}_{n-t;(r,s+1)} \\ & \searrow & & \swarrow & \\ & & {}^{[i]}\mathsf{C}^{\mathsf{D}}_{n;(r,s)} & & \end{array} \qquad (s = 0,1,\dots,v-1). \tag{5.3.11}$$

For convenience, we force the following notation, which only applies to this section and nowhere else:

$${}^{[i]}\mathsf{C}^{\mathsf{D}}_{n;(r,-1)} \cong {}^{[i]}\mathsf{C}^{\mathsf{D}}_{n;(r,1)}, \quad {}^{[i]}\mathsf{C}^{\mathsf{D}}_{n;(r,0)} \cong {}^{[i]}\mathsf{C}^{\mathsf{L}}_{n;(r,0)} \quad \text{and} \quad {}^{[i]}\mathsf{C}^{\mathsf{D}}_{n;(r,v)} \cong {}^{[i+2]}\mathsf{C}^{\mathsf{D}}_{n-t;(u-r,1)}. \tag{5.3.12}$$

Analogous statements can be obtained by applying parity reversal or spectral flow to (5.3.11).

Using the usual induction and restriction trick, we are able to lift the conjectured $A_1(u,v)$ fusion rules (3.3.48) to $\mathsf{M}^{\mathsf{N=2}}(u,v)$ in order to show how the ${}^{[i]}\mathscr{P}_{n;(r,s-1)}$ arise. In fusing the two irreducible standard modules ${}^{[i]}\mathsf{C}^{\mathsf{E}}_{m;(1,1)}$ and ${}^{[j]}\mathsf{C}^{\mathsf{E}}_{n;(r,s)}$, we require $n \notin \lambda^{\mathsf{sl}}_{1,1} + i + 2\mathbb{Z}$ and $m \notin \lambda^{\mathsf{sl}}_{r,s} + j + 2\mathbb{Z}$. Then, for all $1 \le r \le u-1$ and $2 \le s \le v-2$ (which requires that $v \geq 4$), we have the fusion rules

$$
{}^{[i]}\mathsf{C}^{\mathsf{E}}_{m;(1,1)} \times {}^{[j]}\mathsf{C}^{\mathsf{E}}_{n;(r,s)} =
\begin{cases}
{}^{[i+j]}\mathscr{P}_{m+n;(r,s-1)} \oplus {}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n+t;(r,s)} \oplus {}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(r,s+1)}, & \text{if } m+n \in -\lambda^{\mathrm{sl}}_{r,s-1} + i + j + 2\mathbb{Z}, \\
{}^{[i+j]}\mathscr{P}_{m+n;(u-r,v-s-1)} \oplus {}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n+t;(r,s)} \oplus {}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(r,s-1)}, & \text{if } m+n \in \lambda^{\mathrm{sl}}_{u-r,v-s-1} + i + j + 2\mathbb{Z}, \\
{}^{[i+j+2]}\mathscr{P}_{m+n+t;(r,s)} \oplus {}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(r,s)} \oplus {}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(r,s-1)}, & \text{if } m+n \in \lambda^{\mathrm{sl}}_{r,s+1} + i + j + 2\mathbb{Z}, \\
{}^{[i+j+2]}\mathscr{P}_{m+n-t;(r,s)} \oplus {}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(r,s)} \oplus {}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(r,s+1)}, & \text{if } m+n \in \lambda^{\mathrm{sl}}_{u-r,v-s+1} + i + j + 2\mathbb{Z}, \\
{}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(r,s)} \oplus {}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n+t;(r,s)} & \\
\quad \oplus {}^{[i]}\mathsf{C}^{\mathsf{E}}_{m+n;(r,s-1)} \oplus {}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(1,s+1)}, & \text{otherwise,}
\end{cases}
\tag{5.3.13}
$$

where $m+n$ is always understood mod 2. Since these fusion rules are induced from the $A_1(u,v)$ $\mathcal{E}$-type modules, as in (3.3.48) when $s=1$ or $s=v-1$, we modify the right-hand side of (5.3.13) and remove any ${}^{[i]}\mathsf{C}^{\mathsf{E}}_{p;(r',s')}$ with $s'=0$ or v. And again, we would have to remove any summands that do not appear in all cases corresponding to the same value of $m+n$. The reader can visualise this through an analogous example we gave in (3.3.49) for the $\mathfrak{sl}_2$ case.

In a similar fashion, one can induce (3.3.50) to $\mathsf{M}^{\mathsf{N=2}}(u,v)$ and arrive at the following fusion rules

$$
{}^{[i]}\mathsf{C}^{\mathsf{E}}_{m;(1,1)} \times {}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(1,s)} =
\begin{cases}
{}^{[i+j]}\mathscr{P}_{m+n;(u-1,v-s-1)}, & \text{if } m+n \in \lambda^{\mathrm{sl}}_{1,s+1} + i + j + 2\mathbb{Z}, \\
{}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(1,s+1)} \oplus {}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(1,s)}, & \text{otherwise.}
\end{cases}
\tag{5.3.14a}
$$

$$
{}^{[i]}\mathsf{C}^{\mathsf{E}}_{m;(1,s)} \times {}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(1,1)} =
\begin{cases}
{}^{[i+j]}\mathscr{P}_{m+n;(1,s-1)} \oplus {}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(1,s+1)}, & \text{if } m+n \in \lambda^{\mathrm{sl}}_{1,s-1} + i + j + 2\mathbb{Z}, \\
{}^{[i+j]}\mathscr{P}_{m+n;(u-1,v-s-1)} \oplus {}^{[i+j]}\mathsf{C}^{\mathsf{D}}_{m+n;(1,s-1)}, & \text{if } m+n \in \lambda^{\mathrm{sl}}_{u-1,v-s-1} + i + j + 2\mathbb{Z}, \\
{}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(1,s-1)} \oplus {}^{[i+j]}\mathsf{C}^{\mathsf{E}}_{m+n;(1,s+1)} & \\
\quad \oplus {}^{[i+j+2]}\mathsf{C}^{\mathsf{D}}_{m+n-t;(1,s)}, & \text{otherwise,}
\end{cases}
\tag{5.3.14b}
$$

with a similar truncation rule as in (5.3.13) for the special case of $s=1$ or $s=v-1$.

The fusion rules (5.3.13) and (6.5.11) provide the minimal generating set from which general fusion rules can be determined. And these are the only fusion rules involving staggered modules. The fusion rule between ${}^{[i]}\mathsf{C}^{\mathsf{D}}_{n;(r,s)}$ and ${}^{[i]}\mathsf{C}^{\mathsf{D}}_{m;(r',s')}$ gives no staggered modules just like the fusion between two $\mathcal{D}$-type modules in $A_1(u,v)$. The proper fusion rule takes the same form as the Grothendieck fusion and is given by

$$
{}^{[i]}\mathsf{C}^{\mathsf{D}}_{m;(r,s)} \times {}^{[j]}\mathsf{C}^{\mathsf{D}}_{n;(r',s')} =
\begin{cases}
\bigoplus_{r'',s''} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s),(r',s')} {}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(r'',s'')} \\
\qquad \oplus \bigoplus_{r''} \mathsf{N}^{[u]\,r''}_{r,r'} {}^{[i+j]}\mathsf{C}^{\mathsf{D}}_{m+n;(r'',s+s')}, & \text{if } s+s' < v, \\
\bigoplus_{r'',s''} \mathsf{N}^{[u,v]\,(r'',s'')}_{(r,s+1),(r',s'+1)} {}^{[i+j+2]}\mathsf{C}^{\mathsf{E}}_{m+n-t;(r'',s'')} \\
\qquad \oplus \bigoplus_{r''} \mathsf{N}^{[u]\,r''}_{r,r'} {}^{[i+j+2]}\mathsf{C}^{\mathsf{D}}_{m+n-t;(u-r'',s+s'-v+1)}, & \text{if } s+s' \geq v.
\end{cases}
\tag{5.3.15}
$$

Again, we can extend these fusion rules in order to include parity reversals and spectral flows.

5.4 An application: A Simple Current Extension of the $\mathsf{M}^{\mathbf{N=2}}(4,3)$ Minimal Model

As an application of our results on the $N = 2$ fusion rules, we consider the minimal model $\mathsf{M}^{\mathsf{N=2}}(4,3)$ of central charge $-\frac{3}{2}$. A simple current is identified and used to extend the minimal model into one with an extra degree of supersymmetry.

Consider the branching rule given in (5.1.17), the decomposition of the order 2 simple currents $\mathcal{L}_{1,0}$ and $\mathcal{L}_{3,0}$ of $A_1(4,3)$, tensored with the $\mathfrak{bc}$-module $\mathcal{N}_0$ yields infinitely many $\mathsf{M}^{\mathsf{N=2}}(4,3)$ modules, which are all simple currents. Among these simple currents, ${}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(3,0)}$ is of order 2, whereas the rest of the simple currents are of infinite order, and ${}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(1,0)}$ is the vacuum module. We assign the field which corresponds to the highest-weight vector of ${}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(3,0)}$ to be $\phi(z)$. Using (5.3.5a), we find ${}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(3,0)}$ satisfies the following fusion rules:

$$
{}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(3,0)} \times {}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(1,0)} = {}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(3,0)}, \quad {}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(3,0)} \times {}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(3,0)} = {}^{[0]}\mathsf{C}^{\mathsf{L}}_{0;(1,0)}. \tag{5.4.1}
$$

$\phi(z)$ has 0 charge and a conformal dimension of $\frac{3}{2}$. Its OPE with the energy momentum tensor $T(z)$ from the Sugawara construction is easy, since it is a primary field:

$$
T(z)\phi(w) \sim \frac{\frac{3}{2}\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}. \tag{5.4.2}
$$

We know from the second fusion of (5.4.1), that the general form of the OPE of $\phi(z)$ with itself contains only the identity field or its bosonic descendants:

$$
\phi(z)\phi(w) \sim \frac{\alpha}{(z-w)^3} + \frac{\beta J(w)}{(z-w)^2} + \frac{\gamma T(w) + \delta\partial J(w) + \epsilon{:}JJ{:}(w)}{z-w}, \tag{5.4.3}
$$

with unknown constants α, β, γ, δ and ϵ.

To compute these constants, let us consider the OPE of $\phi(z)$ with itself which takes the generic form

$$\phi(z)\phi(w) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} (\phi_n\phi)(w)(z-w)^{-n-\frac{3}{2}}, \tag{5.4.4}$$

where the field $(\phi_n\phi)(w)$ is related to the state $\phi_n|\phi\rangle$ by the state-field correspondence. We now take certain values of n on the right-hand side of (5.4.4) and compare the terms with those of the same powers of $(z-w)$ in (5.4.3). When $n=\frac{3}{2}$, the term $(\phi_{3/2}\phi)(w)(z-w)^{-3}$ must equal to $\alpha(z-w)^{-3}$ in (5.4.3). This relation in terms of states is $\phi_{3/2}|\phi\rangle \equiv \alpha|0\rangle$. Applying the ket $\langle 0|$ to both sides of this relation gives

$$\langle 0|\phi_{3/2}|\phi\rangle = \langle\phi|\phi\rangle \equiv \alpha. \tag{5.4.5}$$

Taking $n=\frac{1}{2}$, we find $\phi_{1/2}|\phi\rangle \equiv \beta J_{-1}|0\rangle$. Applying $\langle J|$ to both sides gives

$$\langle J|\phi_{1/2}|\phi\rangle = \beta\frac{\mathsf{c}}{3}. \tag{5.4.6}$$

The right-hand side of the equation can be written as $\langle 0|J_1\phi_{1/2}|\phi\rangle$ which equals to $\langle 0|\phi_{1/2}J_1|\phi\rangle$ since $J(z)$ commutes with $\phi(z)$. This correlation function, however, is zero because $\langle 0|$ is annihilated by $\phi_{1/2}$. This shows that $\beta=0$. Taking $n=-\frac{1}{2}$ and equating similar terms from (5.4.4) and (5.4.3) leads to

$$\phi_{-\frac{1}{2}}|\phi\rangle = \gamma L_{-2}|0\rangle + \delta L_{-1}J_{-1}|0\rangle + \epsilon J_{-1}J_{-1}|0\rangle. \tag{5.4.7}$$

We then apply $\langle 0|L_2$ to both sides of (5.4.7). A simplification of the correlators using the commutation relation

$$[L_m,\phi_n] = (\frac{m}{2}-n)\phi_{m+n} \tag{5.4.8}$$

yields the relation

$$\frac{\gamma\mathsf{c}}{2}+\frac{\epsilon\mathsf{c}}{3} = \frac{3\alpha}{2}. \tag{5.4.9}$$

Back to (5.4.7), applying $\langle 0|J_2$ instead of $\langle 0|L_2$ to both sides gives $\delta=0$, while applying $\langle 0|J_1J_1$ to (5.4.7) leads to $\gamma=\epsilon$. Along with (5.4.9) and $\mathsf{c}=-\frac{3}{2}$ we are able to solve that $\gamma=\epsilon=-6\alpha/5$. That is, all non-zero variables depend on α. Note that $\phi(z)$ is the only primary field with conformal dimension $\frac{3}{2}$ in this theory. For the corresponding fusion to be non-zero, $\phi(z)$ must be self-conjugate, which means the OPE of $\phi(z)$ with itself must contain singular terms. We therefore require α to be non-zero and choose it to be $\alpha=-5/3$.

The OPE (5.4.3) then becomes

$$\phi(z)\phi(w) \sim \frac{-5/3}{(z-w)^3} + \frac{2T(w)+2{:}JJ{:}(w)}{z-w}, \tag{5.4.10}$$

which contains a nonlinear field in the last term. To simplify the algebra, we define a new energy-momentum tensor $\widetilde{T}(z)$ as

$$\widetilde{T}(z) = T(z) + {:}JJ{:}(z), \tag{5.4.11}$$

then (5.4.10) becomes

$$\phi(z)\phi(w) \sim \frac{-5/3}{(z-w)^3} + \frac{2\widetilde{T}(w)}{z-w}. \tag{5.4.12}$$

The field $\phi(z)$ is primary with respect to $\widetilde{T}(z)$, since $\phi(z)$ has no J_0-charge:

$$\widetilde{T}(z)\phi(w) = (T(z) + {:}JJ{:}(z))\,\phi(w) = T(z)\phi(w) \sim \frac{\frac{3}{2}\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} \tag{5.4.13}$$

$\widetilde{T}(z)$ satisfies

$$\widetilde{T}(z)\widetilde{T}(w) \sim \frac{-5/4}{(z-w)^4} + \frac{2\widetilde{T}(w)}{(z-w)^2} + \frac{\partial\widetilde{T}(w)}{z-w}. \tag{5.4.14}$$

This fixes the central charge $\widetilde{\mathsf{c}}$ of the energy-momentum tensor $\widetilde{T}(z)$ to be $-\frac{5}{2}$, so that the coefficient of the first term in (5.4.14) is $\widetilde{\mathsf{c}}/2$ and hence gives the Virasoro algebra. The coefficient in the first term of (5.4.10) is now $2\widetilde{\mathsf{c}}/3$. One may have noticed that equations (5.4.10), (5.4.13) and (5.4.14) give the $N = 1$ superconformal algebra with $\phi(z)$ being the fermionic generating field $G(z)$. The central charge $-\frac{5}{2}$ coincides that of the $N = 1$ logarithmic minimal model $\mathcal{M}^{N=1}(3,1)$, which has a chain-type singular vector structure [5].

By studying the $N = 2$ minimal models and their fusion rules, we found an order 2 simple current in $\mathsf{M}^{\mathrm{N=2}}(4,3)$ and identified it as the $G(z)$ field in an $N = 1$ sub-theory. This simple current extends the $N = 2$ algebra into a W-algebra with three fermionic generating fields of conformal dimension $\frac{3}{2}$, $G(z)$, $G^+(z)$, $G^-(z)$ and two bosonic generating fields $J(z)$ and $T(z)$ of conformal dimensions 1 and 2, respectively.

Chapter 6

The Inverse Coset Construction for $\mathfrak{osp}(1|2)$ Minimal Models

In this chapter, we study the minimal models associated to $\mathfrak{osp}(1|2)$. Our strategy is to invert the coset construction introduced in Chapter 5 and write the minimal models as extensions of the tensor product of certain Virasoro and $\mathfrak{sl}_2$ minimal models. We shall induce the known structures of the modules of the later models to get a rather complete understanding of the $\mathfrak{osp}(1|2)$ minimal models. In particular, we classify the irreducible relaxed highest-weight modules, determine their (super)characters and compute their Grothendieck fusion rules. We also conjecture their (genuine) fusion products and the projective covers of the irreducibles.

6.1 The Embedding and Inverting the Coset

Recall that the minimal model of $\widehat{\mathfrak{osp}}(1|2)$ at admissible levels

$$\mathsf{k} = -\frac{3}{2} + \frac{p}{2v}, \qquad p \in \mathbb{Z}_{\geq 2},\ v \in \mathbb{Z}_{\geq 1},\ \frac{p+v}{2} \in \mathbb{Z},\ \gcd\left\{p, \frac{p+v}{2}\right\} = 1 \tag{6.1.1}$$

is denoted by $\mathsf{B}_{0|1}(p,v)$. It is well known, see [38] for an early instance and [116, Thm. 8.4] for a proof, that the coset of this minimal model by that of the $\widehat{\mathfrak{sl}}(2)$ subalgebra $A_1(u,v)$ is the rational Virasoro minimal model $\mathsf{M}(p,u)$:

$$\mathsf{M}(p,u) \cong \mathrm{Com}(A_1(u,v), \mathsf{B}_{0|1}(p,v)) \equiv \frac{\mathsf{B}_{0|1}(p,v)}{A_1(u,v)} \qquad (p+v=2u). \tag{6.1.2}$$

This means that every module of $\mathsf{B}_{0|1}(p,v)$ is also a module of the tensor product of the two subalgebras $A_1(u,v)$ and $\mathsf{M}(p,u)$. We thus want to construct the representations of $\mathsf{B}_{0|1}(p,v)$ from the known ones of these subalgebras. The mathematical tool that accomplishes this is again *induction*. In the language of commutants, the Virasoro minimal model $\mathsf{M}(p,u)$ commutes with $A_1(u,v)$, and their tensor product is a subalgebra of $\mathsf{B}_{0|1}(p,v)$:

$$\mathsf{M}(p,u) \otimes A_1(u,v) \hookrightarrow \mathsf{B}_{0|1}(p,v). \tag{6.1.3}$$

The relation between the three parameters u, v and p is derived by comparing (6.1.1) and (3.3.16) while requiring the minimal models $\mathsf{B}_{0|1}(p,v)$ and $A_1(u,v)$ to have the same level k. Note that if $\mathsf{B}_{0|1}(p,v)$ is unitary, then both $\mathsf{M}(p,u)$ and $A_1(u,v)$ must be unitary. Thus, we must have $p-u=\pm 1$ and $v=1$. The only solution is $p=3$, $u=2$ and $v=1$, hence the only unitary $\widehat{\mathfrak{osp}}(1|2)$ minimal model is $\mathsf{B}_{0|1}(3,1)$ corresponding to $\mathsf{k}=0$. This is the trivial one-dimensional vertex operator superalgebra.

We shall now discuss a proof of the coset identification (6.1.3). At the level of the generating fields, the $\widehat{\mathfrak{sl}}_2$ fields $e(z)$, $h(z)$ and $f(z)$ are identified with their namesakes in $\mathsf{B}_{0|1}(p,v)$, while the Virasoro energy-momentum tensor, following from the coset, is identified with the difference of the energy-momentum tensors of $\widehat{\mathfrak{osp}}(1|2)$ and $\widehat{\mathfrak{sl}}_2$:

$$T^{\text{Vir}}(z) = T^{\text{osp}}(z) - T^{\text{sl}}(z). \tag{6.1.4}$$

This construction guarantees that T^{Vir} has regular operator product expansions with e, h and f [46]. Let V_{k} denote the tensor product of the Virasoro universal vacuum module of central charge $1-\frac{6(p-u)^2}{pu}$ and the $\widehat{\mathfrak{sl}}_2$ universal vacuum module of level k. The field identifications above then define a homomorphism of V_{k} into $\mathsf{B}_{0|1}(p,v)$.

To show that this descends to the embedding (6.1.3) and prove (6.1.2), we claim that it suffices to prove the following branching rule

$$\mathsf{B}_{0|1}(p,v)\downarrow\cong\bigoplus_{i=1}^{u-1}\mathcal{V}_{1,i}\otimes\mathcal{L}_{i,0}, \tag{6.1.5}$$

where we decompose $\mathsf{B}_{0|1}(p,v)$ as a V_{k}-module and note that the direct summands which appear are in fact $\mathsf{M}(p,u)\otimes A_1(u,v)$-modules. With this, the embedding (6.1.3) is now clear and the commutant of $A_1(u,v)$, here identified with its vacuum module $\mathcal{L}_{1,0}$, is obviously $\mathcal{V}_{1,1}$, the vacuum module of $\mathsf{M}(p,u)$, as claimed.

To show the modules appearing in the decomposition (6.1.5) are indeed irreducible, we refer to a straightforward, though somewhat lengthy, computation of the following character decomposition [122, Lem. 2.1]:

$$\text{Ch}[\mathsf{B}_{0|1}(p,v)](\mathsf{z};\mathsf{q}) = \text{Tr}_{\mathsf{B}_{0|1}(p,v)}\mathsf{z}^{h_0}\mathsf{q}^{L_0^{\text{osp}}-\mathsf{c}^{\text{osp}}/24} = \sum_{i=1}^{u-1}\chi_{1,i}^{(p,u)}(\mathsf{q})\,\text{Ch}[\mathcal{L}_{i,0}](\mathsf{z};\mathsf{q}), \tag{6.1.6}$$

where $\chi_{1,i}^{(p,u)}(\mathsf{q})$ and $\text{Ch}[\mathcal{L}_{i,0}](\mathsf{z};\mathsf{q})$ are the characters of irreducible modules of $\mathsf{M}(p,u)$ and $A_1(u,v)$, respectively. This calculation is performed at the level of meromorphic continuations of characters in $\mathsf{z}\in\mathbb{C}$ and $|\mathsf{q}|<1$, rather than as formal power series, hence its validity also requires the linear independence of these continuations (or careful attention to convergence regions). It is known that $\mathsf{M}(p,u)$ is rational [123, 124] and $A_1(u,v)$ is rational in category $\mathscr{O}$ [32, 34]. Unfortunately, the continuations of the irreducible

$A_1(u,v)$-characters in category $\mathscr{O}$ are not linearly independent if $v > 1$ [25]. We can rectify this by replacing category $\mathscr{O}$ by its Kazhdan-Lusztig (or ordinary) subcategory $\mathscr{KL}$ whose objects are the $A_1(u,v)$-modules in $\mathscr{O}$ with finite-dimensional L_0^{sl}-eigenspaces. The irreducible characters in $\mathscr{KL}$, which are precisely those of the $\mathcal{L}_{i,0}$, have linearly independent meromorphic continuations. The proof is complete.

The induction of an $\mathsf{M}(p,u)\otimes A_1(u,v)$-module $\mathcal{M}$ to a $\mathsf{B}_{0|1}(p,v)$-module $\mathcal{M}\uparrow$, is defined by

$$\mathcal{M}\uparrow = \mathsf{B}_{0|1}(p,v)\times\mathcal{M} \quad \Rightarrow \quad \mathcal{M}\uparrow\downarrow \cong \bigoplus_{i=1}^{u-1}(\mathcal{V}_{1,i}\otimes\mathcal{L}_{i,0})\times\mathcal{M}, \tag{6.1.7}$$

where $\times$ denotes the fusion product of $\mathsf{M}(p,u)\otimes A_1(u,v)$-modules. In this section, we shall use induction to construct $\mathsf{B}_{0|1}(p,v)$-modules from $\mathsf{M}(p,u)\otimes A_1(u,v)$-modules and identify them as level-k $\widehat{\mathfrak{osp}}(1|2)$-modules. This is an instance of what we call 'inverting the coset'.

We start by recalling the branching rule (6.1.5), in which $\mathsf{B}_{0|1}(p,v)$ is decomposed into irreducible $\mathsf{M}(p,u)\otimes A_1(u,v)$-modules, and exploring the results of inducing its direct summands $\mathcal{V}_{1,i}\otimes\mathcal{L}_{i,0}$. If $i=1$, then it is straightforward to identify the result, as an $\mathsf{M}(p,u)\otimes A_1(u,v)$-module, using the fusion rules (3.2.14) and (3.3.41):

$$\begin{aligned}(\mathcal{V}_{1,1}\otimes\mathcal{L}_{1,0})\uparrow\downarrow &\cong \bigoplus_{i=1}^{u-1}(\mathcal{V}_{1,i}\otimes\mathcal{L}_{i,0})\times(\mathcal{V}_{1,1}\otimes\mathcal{L}_{1,0}) \cong \bigoplus_{i=1}^{u-1}(\mathcal{V}_{1,i}\times\mathcal{V}_{1,1})\otimes(\mathcal{L}_{i,0}\times\mathcal{L}_{1,0}) \\ &\cong \bigoplus_{i=1}^{u-1}\mathcal{V}_{1,i}\otimes\mathcal{L}_{i,0} \cong \mathsf{B}_{0|1}(p,v)\downarrow\end{aligned} \tag{6.1.8}$$

which suggests

$$(\mathcal{V}_{1,1}\otimes\mathcal{L}_{1,0})\uparrow = \mathsf{B}_{0|1}(p,v). \tag{6.1.9}$$

This follows immediately from the definition of induction because $\mathcal{V}_{1,1}\otimes\mathcal{L}_{1,0}$ is just the vacuum module of $\mathsf{M}^{\mathsf{N}=2}(p,u)\otimes A_1(u,v)$.

A consistent result is obtained if we induce the $\mathsf{M}(p,u)\otimes A_1(u,v)$-module with $i=u-1$,

$$(\mathcal{V}_{1,u-1}\otimes\mathcal{L}_{u-1,0})\uparrow\downarrow \cong \bigoplus_{i=1}^{u-1}(\mathcal{V}_{1,i}\otimes\mathcal{L}_{i,0})\times(\mathcal{V}_{1,u-1}\otimes\mathcal{L}_{u-1,0}) \tag{6.1.10a}$$

$$\cong \bigoplus_{i=1}^{u-1}(\mathcal{V}_{1,i}\times\mathcal{V}_{1,u-1})\otimes(\mathcal{L}_{i,0}\times\mathcal{L}_{u-1,0})$$

$$\cong \bigoplus_{i=1}^{u-1}\left(\bigoplus_{s''=1}^{u-1}\mathsf{N}_{i,u-1}^{[u]\,s''}\mathcal{V}_{1,s''}\right)\otimes\left(\bigoplus_{r''=1}^{u-1}\mathsf{N}_{i,u-1}^{[u]\,r''}\mathcal{L}_{r'',0}\right) \tag{6.1.10b}$$

$$\cong \bigoplus_{i=1}^{u-1}\mathcal{V}_{1,u-i}\otimes\mathcal{L}_{u-i,0} \cong \bigoplus_{i=1}^{u-1}\mathcal{V}_{1,i}\otimes\mathcal{L}_{i,0} \cong \mathsf{B}_{0|1}(p,v)\downarrow, \tag{6.1.10c}$$

where the Virasoro fusion coefficients in (6.1.10b) are computed using identity (3.2.16). However, (6.1.10) does not by itself allow us to conclude that we have the corresponding isomorphism of $\mathsf{B}_{0|1}(p,v)$-modules, since the parities of induced modules may not be the same. This issue with identifying inductions is less trivial for other modules. We shall therefore analyse the simple case of inducing from modules with $i = 1$ in detail, describing a methodology that generalises straightforwardly to all modules.

Before commencing this analysis, we note that the induction is quite different for all $i \neq 1, u-1$. For example, when $u > 3$, we have

$$\begin{aligned}&(\mathcal{V}_{1,2} \otimes \mathcal{L}_{2,0}) \uparrow\downarrow \\ &\cong \bigoplus_{i=1}^{u-1} \mathcal{V}_{1,i} \otimes \mathcal{L}_{i,0} \oplus \bigoplus_{i=2}^{u-2} (\mathcal{V}_{1,i} \otimes \mathcal{L}_{i,0} \oplus \mathcal{V}_{1,i-1} \otimes \mathcal{L}_{i+1,0} \oplus \mathcal{V}_{1,i+1} \otimes \mathcal{L}_{i-1,0}) \\ &\cong \mathsf{B}_{0|1}(p,v) \downarrow \oplus \mathcal{M} \downarrow,\end{aligned} \tag{6.1.11}$$

where $\mathcal{M}$ is some other, as yet uncharacterised, $\mathsf{B}_{0|1}(p,v)$-module. These results are consistent with Theorem A.1 in Appendix A, which applies when i is such that the $(\mathcal{V}_{1,i} \otimes \mathcal{L}_{i,0}) \times (\mathcal{V}_{1,j} \otimes \mathcal{L}_{j,0})$ are inequivalent and irreducible for all j. If this holds, then the result of inducing is an irreducible $\mathsf{B}_{0|1}(p,v)$-module (which is clearly not the case in (6.1.11)).

6.2 The Rational $\mathfrak{osp}(1|2)$ Minimal Models $\mathsf{B}_{0|1}(p,1)$

We start with the non-negative integer-level models $\mathsf{B}_{0|1}(p,1)$. Recall from Section 4.4.2 that the modules of such minimal models are irreducible highest-weight modules. In order to satisfy (4.4.9), the parameter p here is odd and greater than 1, so $u = \mathsf{k}+2 = \frac{p+1}{2} \geq 2$. In the tensored model $\mathsf{M}(p,u) \otimes A_1(u,v)$, the only irreducible modules available for induction are the $\mathcal{V}_{r,s} \otimes \mathcal{L}_{r',0}$, where $r = 1,\dots,p-1$ and $r', s = 1,\dots,u-1$. Inspecting the fusion rules involving these irreducibles and the $\mathcal{V}_{1,i} \otimes \mathcal{L}_{i,0}$, using (3.2.14) and (3.3.43a), it is easy to see that the result will be irreducible if $r', s \in \{1, u-1\}$.

Taking $r' = s = 1$, we detail the determination of the decomposition of the induced module $(\mathcal{V}_{r,1} \otimes \mathcal{L}_{1,0}) \uparrow$, which we shall denote by $\mathcal{A}_{r,0}$ for brevity:

$$\mathcal{A}_{r,0} \downarrow \cong \bigoplus_{i=1}^{u-1} (\mathcal{V}_{1,i} \otimes \mathcal{L}_{i,0}) \times (\mathcal{V}_{r,1} \otimes \mathcal{L}_{1,0}) \cong \bigoplus_{i=1}^{u-1} \mathcal{V}_{r,i} \otimes \mathcal{L}_{i,0}. \tag{6.2.1}$$

The summands on the right-hand side are clearly inequivalent (and irreducible), hence Theorem A.1 in Appendix A applies and we conclude that $\mathcal{A}_{r,0}$ is an irreducible $\mathsf{B}_{0|1}(p,1)$-module as claimed. However, taking $r' = u-1$ and $s = 1$, $r' = 1$ and $s = u-1$, or $r' = s = u-1$ gives inductions whose decompositions are identical to that in (6.2.1), though perhaps

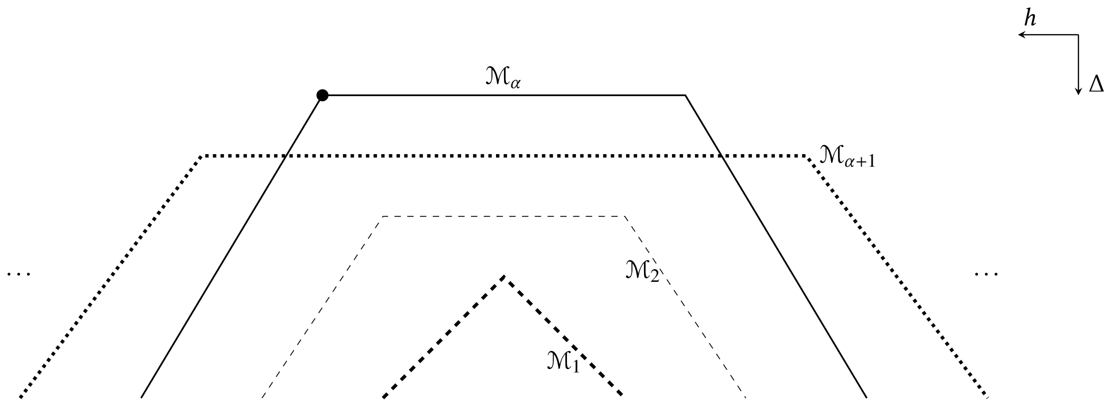

Figure 6.1: Ramond coset module $\mathcal{A}_{r,0}$ constructed from $\mathsf{M}(p,v)$ and $A_1(u,v)$ modules according to (6.2.1). $\mathcal{M}_i$ represents $\mathcal{V}_{r,i} \otimes \mathcal{L}_{i,0}$. The highest-weight state of $\mathcal{M}_\alpha$ (labelled by the solid dot), which has the minimum conformal dimension among all $\mathcal{M}_i$ is identified as the highest-weight state of $\mathcal{A}_{r,0}$, whose h_0-eigenvalue and conformal dimension are denoted by $\lambda^{\text{osp}}_{r,0}$ and $\Delta^{\text{osp}}_{r,0}$, respectively. In the Neveu-Schwarz sector, there are two modules $\mathcal{M}_{(r+s+1)/2}$ and $\mathcal{M}_{(r+s-1)/2}$ both with a minimum conformal dimension.

with r replaced by $p-r$. For example, writing $\widetilde{\mathcal{A}}_{r,0}$ for the irreducible $\mathsf{B}_{0|1}(p,1)$-module $(\mathcal{V}_{r,1} \otimes \mathcal{L}_{u-1,0}) \uparrow$, we have

$$\widetilde{\mathcal{A}}_{p-r,0} \downarrow \cong \bigoplus_{i=1}^{u-1} \mathcal{V}_{p-r,i} \otimes \mathcal{L}_{u-i,0} = \bigoplus_{i=1}^{u-1} \mathcal{V}_{p-r,u-i} \otimes \mathcal{L}_{i,0} = \bigoplus_{i=1}^{u-1} \mathcal{V}_{r,i} \otimes \mathcal{L}_{i,0}. \tag{6.2.2}$$

As before, however, this need not imply that $\mathcal{A}_{r,0}$ and $\widetilde{\mathcal{A}}_{p-r,0}$ are isomorphic as $\mathsf{B}_{0|1}(p,1)$-modules.

To answer this question of possible isomorphisms, and to identify the induced modules $\mathcal{A}_{r,0}$ as $\widehat{\mathfrak{osp}}(1|2)$-modules, we again exploit the method of extremal states, which in this case, relies on explicitly identifying the ground states of the (irreducible) induced module. This has the added advantage of allowing us to compare with the list of irreducible $\widehat{\mathfrak{osp}}(1|2)$-modules given in Section 4.4.2 and thereby identify the induced module completely.

Considering the branching rule (6.2.1), the coset module must be a highest-weight module because the modules in its decomposition are. The ground states of $\mathcal{A}_{r,0}$ are therefore the ground states of the summand $\mathcal{V}_{r,i} \otimes \mathcal{L}_{i,0}$ with the lowest conformal dimension. The construction of $\mathcal{A}_{r,0}$ from the sum of the tensored modules of $\mathsf{M}(p,v)$ and $A_1(u,v)$ is depicted in Figure 6.1. By (3.1.19) and (3.3.18), the conformal dimension of the ground states of the i-th summand on the right-hand side of (6.2.1) is

$$\Delta^{\text{Vir}}_{r,i} + \Delta^{\text{sl}}_{i,0} = \frac{1}{2}i^2 - \frac{r}{2}i + \frac{(r^2-1)u}{4p}. \tag{6.2.3}$$

The global minimum therefore occurs when $i = \frac{1}{2}r$, if r is even, and when $i = \frac{1}{2}(r \pm 1)$, if r

is odd. This minimal conformal dimension may now be written in the form

$$\Delta^{\text{osp}}_{r,0} = \frac{r^2-1}{8p} - \frac{1+(-1)^r}{16}. \tag{6.2.4}$$

Moreover, there is a highest-weight vector among the ground states of minimal conformal dimension whose h_0-charge is

$$\lambda^{\text{osp}}_{r,0} = \frac{r-1}{2} - \frac{1+(-1)^r}{4}. \tag{6.2.5}$$

The module $\mathcal{A}_{r,0}$ is therefore an irreducible highest-weight $\widehat{\mathfrak{osp}}(1|2)$-module of h_0-charge $\lambda^{\text{osp}}_{r,0}$. To determine its sector, note that the conformal dimensions of the ground states of the i-th and j-th summands in (6.2.1) differ by $\frac{1}{2}(i-j)(i+j-r)$. If r is odd, then this difference is always an integer, and $\mathcal{A}_{r,0}$ belongs to the Neveu-Schwarz sector. Likewise, $\mathcal{A}_{r,0}$ belongs to the Ramond sector when r is even. The sector information is also reflected in (6.2.4) and (6.2.5), the conformal dimension and the charge of a module are respectively shifted by $\frac{1}{8}$ and $\frac{1}{2}$ in the Ramond sector due to the Ramond vacuum.

It only remains to determine the parity of the highest-weight vector of $\mathcal{A}_{r,0}$. To do so, note that the h_0-charges of the summands $\mathcal{V}_{r,i} \otimes \mathcal{L}_{i,0}$ in (6.2.1) are equal to $i-1$ (mod 2). Since $\mathcal{V}_{r,i}$ has zero eigenvalue with respect to h_0, this charge comes from the $\mathcal{L}_{i,0}$ component, which was denoted by $\lambda_{i,0}$ in (3.3.18). Since we are inducing from $\mathcal{V}_{r,1} \otimes \mathcal{L}_{1,0}$, we shall define the states in this module to be bosonic. It follows that $\mathcal{V}_{r,i} \otimes \mathcal{L}_{i,0}$ is bosonic for i odd and fermionic for i even. In the Neveu-Schwarz sector where r is odd, the highest-weight vector corresponds to $i = \frac{r+1}{2}$, hence it is bosonic if $r = 1 \pmod 4$ and fermionic if $r = 3 \pmod 4$. For a Ramond module where r is even, we similarly conclude that we have a bosonic highest-weight vector if $r = 2 \pmod 4$ and a fermionic one if $r = 0 \pmod 4$. Comparing with the list of irreducible $\widehat{\mathfrak{osp}}(1|2)$-modules given in Section 4.4.2, this then completes the identification of the $\mathcal{A}_{r,0}$.

$r \pmod 4$	1	2	3	4
$\mathcal{A}_{r,0}$	${}^{\text{NS}}\mathcal{A}_{\lambda^{\text{osp}}_{r,0}}$	${}^{\text{R}}\mathcal{A}_{\lambda^{\text{osp}}_{r,0}}$	$\Pi^{\text{NS}}\mathcal{A}_{\lambda^{\text{osp}}_{r,0}}$	$\Pi^{\text{R}}\mathcal{A}_{\lambda^{\text{osp}}_{r,0}}$

We recall that Π denotes parity reversal, meaning that the module has had its bosonic and fermionic subspaces swapped. Also recall that the $\mathcal{A}$-modules given in the list of Section 4.4.2 are all defined as bosonic. Note that, according to this identification, $(\mathcal{V}_{1,1} \otimes \mathcal{L}_{1,0}) \uparrow = \mathcal{A}_{1,0} \cong {}^{\text{NS}}\mathcal{A}_0$ is indeed the bosonic vacuum module of $\mathsf{B}_{0|1}(p,1)$, as expected.

If we repeat this analysis with the $\widetilde{\mathcal{A}}_{r,0}$, which were induced from $\mathcal{V}_{r,1} \otimes \mathcal{L}_{u-1,0}$ in (6.2.2), we do not obtain any new $\mathsf{B}_{0|1}(p,1)$-modules except perhaps for parity reversals. Indeed, the identification is as follows.

$r \pmod 4$	1	2	3	4
$\widetilde{\mathcal{A}}_{r,0}$	${}^{\text{R}}\mathcal{A}_{\lambda^{\text{osp}}_{p-r,0}}$	${}^{\text{NS}}\mathcal{A}_{\lambda^{\text{osp}}_{p-r,0}}$	$\Pi^{\text{R}}\mathcal{A}_{\lambda^{\text{osp}}_{p-r,0}}$	$\Pi^{\text{NS}}\mathcal{A}_{\lambda^{\text{osp}}_{p-r,0}}$

In particular, $\mathcal{A}_{r,0}$ is isomorphic to $\widetilde{\mathcal{A}}_{p-r-1,0}$ if $p = 0 \pmod 4$, to $\Pi\widetilde{\mathcal{A}}_{p-r,0}$, if $p = 1 \pmod 4$, to $\Pi\widetilde{\mathcal{A}}_{p-r-1,0}$, if $p = 2 \pmod 4$, and to $\widetilde{\mathcal{A}}_{p-r,0}$, if $p = 3 \pmod 4$. We remark that the fact that no new modules are encountered (except parity reversals) was guaranteed because the spectral flow automorphisms of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{osp}}(1|2)$ are consistent with the coset construction. Thus,

$$\widetilde{\mathcal{A}}_{r,0} = (\mathcal{V}_{r,1} \otimes \mathcal{L}_{u-1,0}) \uparrow \cong \left(\mathcal{V}_{r,1} \otimes \sigma_{\mathrm{sl}}\left(\mathcal{L}_{1,0}\right)\right) \uparrow \cong \sigma_{\mathrm{osp}}((\mathcal{V}_{r,1} \otimes \mathcal{L}_{1,0}) \uparrow) = \sigma_{\mathrm{osp}}(\mathcal{A}_{r,0}), \quad (6.2.6)$$

a relation that is easy to verify directly. We conclude that inducing the $\mathcal{V}_{r,1} \otimes \mathcal{L}_{1,0}$ and applying parity reversal will give all the irreducibles that can be obtained by inducing an arbitrary $\mathsf{M}(p,u) \otimes A_1(u,v)$-module and parity-reversing.

The characters of the $\mathcal{A}_{r,0}$ are now obtained by taking characters of modules on both sides of the branching rule (6.2.1). This gives

$$\mathrm{Ch}[\mathcal{A}_{r,0}](\mathsf{z};\mathsf{q}) = \mathrm{Tr}_{\mathcal{A}_{r,0}} \mathsf{z}^{h_0} \mathsf{q}^{L_0^{\mathrm{osp}} - \mathsf{c}^{\mathrm{osp}}/24} = \sum_{i=1}^{u-1} \chi_{r,i}^{(p,u)}(\mathsf{q})\,\mathrm{Ch}[\mathcal{L}_{i,0}](\mathsf{z};\mathsf{q}). \quad (6.2.7)$$

One can expand this using the explicit forms (3.2.12) and (3.3.20) for the irreducible $\mathsf{M}(p,u)$- and $A_1(u,v)$-characters. Since $\widehat{\mathfrak{osp}}(1|2)$ is a superalgebra, it is appropriate to consider its supercharacters as well. As the highest-weight vector of the module $\mathcal{V}_{r,1} \otimes \mathcal{L}_{1,0}$ is bosonic, its h_0-charge differs from those of the fermionic states by an odd integer. The supercharacter of $\mathcal{A}_{r,0}$ is therefore simply given by

$$\mathrm{Sch}[\mathcal{A}_{r,0}](\mathsf{z};\mathsf{q}) = \mathrm{Tr}_{\mathcal{A}_{r,0}}(-1)^F \mathsf{z}^{h_0} \mathsf{q}^{L_0^{\mathrm{osp}} - \mathsf{c}^{\mathrm{osp}}/24} = \sum_{i=1}^{u-1} (-1)^{i-1} \chi_{r,i}^{(p,u)}(\mathsf{q})\,\mathrm{Ch}[\mathcal{L}_{i,0}](\mathsf{z};\mathsf{q}). \quad (6.2.8)$$

where F acts as 0 on a bosonic state and as multiplication by 1 on a fermionic one.

6.3 The Logarithmic $\mathfrak{osp}(1|2)$ Minimal Models $\mathsf{B}_{0|1}(u,v)$ with $v \neq 1$

Following a similar method as in the $v = 1$ case, we construct irreducible $\mathsf{B}_{0|1}(p,v)$-modules from those of $\mathsf{M}(p,u)$ and $A_1(u,v)$ through induction. These modules are then identified as $\widehat{\mathfrak{osp}}(1|2)$-modules using the list presented in Section 4.4.2. This identification uses h_0-charges and conformal dimensions and is therefore straightforward for all cases except that of the Neveu-Schwarz relaxed highest-weight modules ${}^{\mathrm{NS}}\mathcal{C}_{\lambda,\Sigma}$ for which the super-Casimir eigenvalue Σ on bosonic eigenstates is only determined by the conformal dimension up to a sign, see (4.4.13).

To fix this sign, we must realise Σ in terms of $\mathsf{M}(p,u)$ and $A_1(u,v)$ data. Recall that the super-Casimir ς, defined in (4.4.3), of $\mathfrak{osp}(1|2)$ (embedded in $\widehat{\mathfrak{osp}}(1|2)$ as the horizontal subalgebra) commutes with e_0, h_0 and f_0, but anticommutes with x_0 and y_0. We therefore introduce the field

$$\varsigma(z) = {:}xy{:}(z) - {:}yx{:}(z), \tag{6.3.1}$$

noting that its zero mode ς_0 acts on Neveu-Schwarz ground states as multiplication by $\pm\Sigma - \frac{1}{2}$, where the sign is positive for bosonic ground states and negative for fermionic ones. To realise $\varsigma(z)$ in terms of $\mathsf{M}(p,u)$ and $A_1(u,v)$ fields, recall the Sugawara construction for $\mathfrak{osp}(1|2)$ (4.4.7), which in terms of states is given by

$$L^{\text{osp}}_{-2}|0\rangle = \frac{1}{2(\mathsf{k}+3/2)}\left(\frac{1}{2}h_{-1}^2 + e_{-1}f_{-1} + f_{-1}e_{-1} - \frac{1}{2}x_{-1}y_{-1} + \frac{1}{2}y_{-1}x_{-1}\right)|0\rangle, \tag{6.3.2}$$

which leads to

$$\begin{aligned}(x_{-1}y_{-1} - y_{-1}x_{-1})|0\rangle &= 2\left(\frac{1}{2}h_{-1}^2 + e_{-1}f_{-1} + f_{-1}e_{-1}\right)|0\rangle - 4(\mathsf{k}+\frac{3}{2})L^{\text{osp}}_{-2}|0\rangle \\ &= 4(\mathsf{k}+2)L^{\text{sl}}_{-2}|0\rangle - 4(\mathsf{k}+\frac{3}{2})\left(L^{\text{sl}}_{-2} + L^{\text{Vir}}_{-2}\right) \\ &= 2L^{\text{sl}}_{-2}|0\rangle - 2(2\mathsf{k}+3)L^{\text{Vir}}_{-2}|0\rangle.\end{aligned} \tag{6.3.3}$$

In terms of fields, (6.3.3) takes the form

$$\varsigma(z) = 2T^{\text{sl}}(z) - \frac{2p}{v}T^{\text{Vir}}(z), \tag{6.3.4}$$

under the embedding (6.1.3). It follows that Σ may be computed in terms of the action of the zero modes of $T^{\text{sl}}(z)$ and $T^{\text{Vir}}(z)$ on a bosonic Neveu-Schwarz ground state w:

$$\Sigma w = \left(2L_0^{\text{sl}} - \frac{2p}{w}L_0^{\text{Vir}} + \frac{1}{2}\right)w. \tag{6.3.5}$$

Having dealt with this minor subtlety, we can now follow the same procedure as in the $v = 1$ case and construct irreducible $\mathsf{B}_{0|1}(p,v)$-modules by inducing certain modules of $\mathsf{M}(p,u) \otimes A_1(u,v)$. We shall adopt the following convention in defining our $\mathsf{B}_{0|1}(p,v)$-modules:

$$\begin{aligned}\mathcal{A}_{r,0} &= (\mathcal{V}_{r,1} \otimes \mathcal{L}_{1,0})\uparrow, & \mathcal{B}^{\pm}_{r,s} &= (\mathcal{V}_{r,1} \otimes \mathcal{D}^{\pm}_{1,s})\uparrow, \\ \mathcal{C}_{\lambda;(r,s)} &= (\mathcal{V}_{r,1} \otimes \mathcal{E}_{\lambda;(1,s)})\uparrow, & \mathcal{C}^{\pm}_{r,s} &= (\mathcal{V}_{r,1} \otimes \mathcal{E}^{\pm}_{1,s})\uparrow.\end{aligned} \tag{6.3.6}$$

Here, $r = 1,\dots,p-1$ and $s = 1,\dots,v-1$, while $\lambda \in \mathbb{C}$ satisfies $\lambda \neq \lambda^{\text{sl}}_{1,s}, \lambda^{\text{sl}}_{u-1,v-s} \pmod 2$. The corresponding branching rules are computed as in (6.2.1). Taking $\mathcal{B}^{\pm}_{r,s}$ as an example,

$$\begin{aligned}\mathcal{B}^{\pm}_{r,s}\downarrow = (\mathcal{V}_{r,1} \otimes \mathcal{D}^{\pm}_{1,s})\uparrow\downarrow &= \left(\mathcal{V}_{r,1} \otimes \mathcal{D}^{+}_{1,s}\right) \times \left(\bigoplus_{i=1}^{u-1} \mathcal{V}_{1,i} \otimes \mathcal{L}_{i,0}\right) \\ &= \bigoplus_{i=1}^{u-1} (\mathcal{V}_{r,1} \times \mathcal{V}_{1,i}) \otimes \left(\mathcal{D}^{+}_{1,s} \times \mathcal{L}_{i,0}\right) = \bigoplus_{i=1}^{u-1} \mathcal{V}_{r,i} \otimes \mathcal{D}^{+}_{i,s}\end{aligned} \tag{6.3.7}$$

The rest of the branching rules are calculated in a similar fashion to be

$$\begin{aligned} \mathcal{A}_{r,0}\downarrow &\cong \bigoplus_{i=1}^{u-1} \mathcal{V}_{r,i}\otimes\mathcal{L}_{i,0}, & \mathcal{B}^{\pm}_{r,s}\downarrow &\cong \bigoplus_{i=1}^{u-1} \mathcal{V}_{r,i}\otimes\mathcal{D}^{\pm}_{i,s}, \\ \mathcal{C}_{\lambda;(r,s)}\downarrow &\cong \bigoplus_{i=1}^{u-1} \mathcal{V}_{r,i}\otimes\mathcal{E}_{\lambda+i-1;(i,s)}, & \mathcal{C}^{\pm}_{r,s}\downarrow &\cong \bigoplus_{i=1}^{u-1} \mathcal{V}_{r,i}\otimes\mathcal{E}^{\pm}_{i,s}. \end{aligned} \tag{6.3.8}$$

It is now easy to check that the Theorem A.1 in Appendix A applies to the $\mathcal{A}_{r,0}$, $\mathcal{B}^{\pm}_{r,s}$ and $\mathcal{C}_{\lambda;(r,s)}$, hence that these are irreducible $\mathsf{B}_{0|1}(p,v)$-modules.

As before, the states in the $\mathsf{M}(p,u)\otimes A_1(u,v)$-module being induced are bosonic in the resulting $\mathsf{B}_{0|1}(u,v)$-module, hence the states of the summands of (6.3.8) with i odd (even) are bosonic (fermionic). The ground states of each summand has conformal dimension

$$\Delta^{\text{Vir}}_{r,i} + \Delta^{\text{sl}}_{i,s} = \frac{(ur-pi)^2-(u-p)^2}{4pu} + \frac{(vi-us)^2-v^2}{4uv}, \tag{6.3.9}$$

where we take $s=0$ for the summand in $\mathcal{A}_{r,0}\downarrow$. In each branching rule, we determine the indices i for which the conformal dimension of the ground states of the $\mathsf{M}(p,u)\otimes A_1(u,v)$-module is minimised. In the Neveu-Schwarz sector, where $r+s\in 2\mathbb{Z}+1$, the global minimum occurs for $i=\frac{r+s\pm1}{2}$, while in the Ramond sector, where $r+s\in 2\mathbb{Z}$, the minimum is at $i=\frac{r+s}{2}$. (We take $s=0$ for the $\mathcal{A}_{r,0}$.) The conformal dimensions of the ground states of the induced modules (6.3.6) are thereby found to be given by

$$\Delta^{\text{osp}}_{r,s} = \frac{(vr-ps)^2-v^2}{8pv} - \frac{1+(-1)^{r+s}}{16}. \tag{6.3.10}$$

This clearly reduces to (6.2.4) when $v=1$ (forcing $s=0$).

The $\mathcal{A}_{r,0}$ and $\mathcal{B}^{+}_{r,s}$ are highest-weight $\mathsf{B}_{0|1}(p,v)$-modules and the h_0-charges of their highest-weight vectors are easily seen to be

$$\lambda^{\text{osp}}_{r,s} = \frac{1}{2}\Big(r-1-\frac{p}{v}s\Big) - \frac{1+(-1)^{r+s}}{4}. \tag{6.3.11}$$

This likewise reduces to (6.2.5) when $v=1$ and $s=0$. The $\mathcal{B}^{-}_{r,s}$ are clearly the conjugates of the $\mathcal{B}^{+}_{r,s}$, so it remains to identify the $\mathcal{C}_{\lambda;(r,s)}$ and the $\mathcal{C}^{\pm}_{r,s}$. In the Neveu-Schwarz sector, we use (6.3.5) to show that the super-Casimir eigenvalue on the bosonic ground states is

$$\Sigma_{r,s} = \frac{1}{2}(-1)^{(r+s-1)/2}\Big(r-\frac{p}{v}s\Big), \tag{6.3.12}$$

which is easily checked to be consistent with (4.4.13) and (6.3.10). In the Ramond sector, (4.4.13) and (6.3.10) lead to the eigenvalue of the $\widehat{\mathfrak{sl}}_2$ Casimir on the ground states being

$$q_{r,s} = \frac{1}{8}\left(r - \frac{p}{v}s\right)^2 - \frac{1}{2}. \tag{6.3.13}$$

We now summarise the properties of the induced coset modules (6.3.6) in the following list, thereby identifying them as $\mathsf{B}_{0|1}(p,v)$-modules. Modules with $r+s$ odd (even), where s is understood to be 0 for the $\mathcal{A}_{r,0}$, belong to the Neveu-Schwarz (Ramond) sector. The global parities of these induced modules are determined as in Section 6.2.

- The $\mathcal{A}_{r,0}$, with $1 \le r \le p-1$, are irreducible highest-weight modules whose ground state spaces are finite-dimensional. The highest-weight vector of each module has h_0-charge $\lambda^{\text{osp}}_{r,0}$ and conformal dimension $\Delta^{\text{osp}}_{r,0}$. The sectors and global parities are found to follow the same pattern as for the case where $v = 1$.

$r \pmod 4$	1	2	3	4
$\mathcal{A}_{r,0}$	${}^{\text{NS}}\mathcal{A}_{\lambda^{\text{osp}}_{r,0}}$	${}^{\text{R}}\mathcal{A}_{\lambda^{\text{osp}}_{r,0}}$	$\Pi\,{}^{\text{NS}}\mathcal{A}_{\lambda^{\text{osp}}_{r,0}}$	$\Pi\,{}^{\text{R}}\mathcal{A}_{\lambda^{\text{osp}}_{r,0}}$

- The $\mathcal{B}^+_{r,s}$, with $1 \le r \le p-1$ and $1 \le s \le v-1$, are irreducible highest-weight modules whose ground state spaces are infinite-dimensional. The highest-weight vector has charge $\lambda^{\text{osp}}_{r,s}$ and conformal dimension $\Delta^{\text{osp}}_{r,s}$. The $\mathcal{B}^-_{r,s}$ are the conjugates of the $\mathcal{B}^+_{r,s}$.

$r+s \pmod 4$	1	2	3	4
$\mathcal{B}^\pm_{r,s}$	${}^{\text{NS}}\mathcal{B}^\pm_{\lambda_{r,s}}$	${}^{\text{R}}\mathcal{B}^\pm_{\lambda_{r,s}}$	$\Pi\,{}^{\text{NS}}\mathcal{B}^\pm_{\lambda_{r,s}}$	$\Pi\,{}^{\text{R}}\mathcal{B}^\pm_{\lambda_{r,s}}$

- The $\mathcal{C}_{\lambda;(r,s)}$, with $1 \le r \le p-1$, $1 \le s \le v-1$ and $\lambda \ne \lambda^{\text{sl}}_{1,s}, \lambda^{\text{sl}}_{u-1,v-s} \pmod 2$ are irreducible relaxed highest-weight modules whose ground state spaces are infinite-dimensional. There is a bosonic ground state of charge λ that is characterised by its super-Casimir eigenvalue $\Sigma_{r,s}$ (if $r+s$ is odd) or its $\mathfrak{sl}_2$ Casimir eigenvalue $q_{r,s}$ (if $r+s$ is even). In either case, the conformal dimension of the ground states is $\Delta^{\text{osp}}_{r,s}$.

$r+s \pmod 4$	1	2	3	4
$\mathcal{C}_{\lambda;(r,s)}$	${}^{\text{NS}}\mathcal{C}_{\lambda,\Sigma_{r,s}}$	${}^{\text{R}}\mathcal{C}_{\lambda,q_{r,s}}$	${}^{\text{NS}}\mathcal{C}_{\lambda,\Sigma_{r,s}}$	$\Pi\,{}^{\text{R}}\mathcal{C}_{\lambda+1,q_{r,s}}$

It is easy to check that the restriction $\lambda \ne \lambda^{\text{sl}}_{1,s}, \lambda^{\text{sl}}_{u-1,v-s} \pmod 2$ translates into $\lambda \ne \xi^\pm_{r,s} \pmod 2$, where

$$\xi^\pm_{r,s} = \begin{cases} \pm(\Sigma_{r,s} - \frac{1}{2}), & \text{if } r+s \text{ is odd,} \\ -1 \pm \sqrt{1+2q_{r,s}}, & \text{if } r+s \text{ is even.} \end{cases} \tag{6.3.14}$$

For example, $r+s = 1 \pmod 4$ implies that

$$\lambda^{\text{sl}}_{1,s} = -\frac{u}{v}s = \frac{1}{2}(-s - \frac{p}{v}s) = \frac{1}{2}(r - 1 - \frac{p}{v}s) = \Sigma_{r,s} - \frac{1}{2} \pmod 2 \tag{6.3.15}$$

and, similarly, $\lambda^{\text{sl}}_{u-1,v-s} = -(\Sigma_{r,s} - \frac{1}{2}) \pmod 2$.

- The $\mathcal{C}^{\pm}_{r,s}$, with $1 \le r \le p-1$ and $1 \le s \le v-1$, are reducible relaxed highest-weight modules with a bosonic ground state of charge $\lambda^{\text{osp}}_{r,s}$ and conformal dimension $\Delta^{\text{osp}}_{r,s}$. They are characterised by the following short exact sequences:

$$0 \longrightarrow \mathcal{B}^{\pm}_{r,s} \longrightarrow \mathcal{C}^{\pm}_{r,s} \longrightarrow \Pi^{u}\mathcal{B}^{\mp}_{p-r,v-s} \longrightarrow 0, \tag{6.3.16}$$

  recalling $u = (p+v)/2$. Unpacking this, we find that the submodule $\mathcal{S}$ and quotient $\mathcal{Q}$ of $\mathcal{C}^{\pm}_{r,s}$ are identified as follows,

$r+s \pmod 4$	1	2	3	4
$\mathcal{S}$	${}^{\text{NS}}\mathcal{B}^{\pm}_{\lambda_{r,s}}$	${}^{\text{R}}\mathcal{B}^{\pm}_{\lambda_{r,s}}$	$\Pi^{\text{NS}}\mathcal{B}^{\pm}_{\lambda_{r,s}}$	$\Pi^{\text{R}}\mathcal{B}^{\pm}_{\lambda_{r,s}}$
$\mathcal{Q}$	$\Pi^{\text{R}}\mathcal{B}^{\mp}_{\lambda_{p-r,v-s}}$	${}^{\text{R}}\mathcal{B}^{\mp}_{\lambda_{p-r,v-s}}$	${}^{\text{NS}}\mathcal{B}^{\mp}_{\lambda_{p-r,v-s}}$	$\Pi^{\text{R}}\mathcal{B}^{\mp}_{\lambda_{p-r,v-s}}$

where we simplify the notation by writing $\lambda^{\text{sl}}_{r,s}$ as $\lambda_{r,s}$. We emphasise that the parity reversals of the $\mathcal{A}_{r,0}$, $\mathcal{B}^{\pm}_{r,s}$, $\mathcal{C}_{\lambda;(r,s)}$ and $\mathcal{C}^{\pm}_{r,s}$ are also $\mathsf{B}_{0|1}(p,v)$-modules, as are their images under spectral flow.

The characters and supercharacters of the induced $\mathsf{B}_{0|1}(p,v)$-modules follow from (6.3.8) as in the $v = 1$ case. The characters are given by

$$\text{Ch}[\mathcal{A}_{r,0}](\mathsf{z};\mathsf{q}) = \sum_{i=1}^{u-1} \chi^{(p,u)}_{r,i}(\mathsf{q})\,\text{Ch}[\mathcal{L}_{i,0}](\mathsf{z};\mathsf{q}), \tag{6.3.17a}$$

$$\text{Ch}[\mathcal{B}^{\pm}_{r,s}](\mathsf{z};\mathsf{q}) = \sum_{i=1}^{u-1} \chi^{(p,u)}_{r,i}(\mathsf{q})\,\text{Ch}[\mathcal{D}^{\pm}_{i,s}](\mathsf{z};\mathsf{q}), \tag{6.3.17b}$$

$$\text{Ch}[\mathcal{C}_{\lambda;(r,s)}](\mathsf{z};\mathsf{q}) = \sum_{i=1}^{u-1} \chi^{(p,u)}_{r,i}(\mathsf{q})\,\text{Ch}[\mathcal{E}_{\lambda+i-1,\Delta_{i,s}}](\mathsf{z};\mathsf{q}), \tag{6.3.17c}$$

$$\text{Ch}[\mathcal{C}^{\pm}_{r,s}](\mathsf{z};\mathsf{q}) = \sum_{i=1}^{u-1} \chi^{(p,u)}_{r,i}(\mathsf{q})\,\text{Ch}[\mathcal{E}^{\pm}_{i,s}](\mathsf{z};\mathsf{q}). \tag{6.3.17d}$$

And again, since we define states in summand of (6.3.8) with odd (even) i as bosonic (fermionic), the supercharacters are given by the same formulae as above, but with $(-1)^{i-1}$ inserted into each sum:

$$\text{Sch}[\mathcal{A}_{r,0}](\mathsf{z};\mathsf{q}) = \sum_{i=1}^{u-1} (-1)^{i-1}\chi^{(p,u)}_{r,i}(\mathsf{q})\,\text{Ch}[\mathcal{L}_{i,0}](\mathsf{z};\mathsf{q}), \tag{6.3.18a}$$

$$\text{Sch}[\mathcal{B}^{\pm}_{r,s}](\mathsf{z};\mathsf{q}) = \sum_{i=1}^{u-1} (-1)^{i-1}\chi^{(p,u)}_{r,i}(\mathsf{q})\,\text{Ch}[\mathcal{D}^{\pm}_{i,s}](\mathsf{z};\mathsf{q}), \tag{6.3.18b}$$

$$\text{Sch}[\mathcal{C}_{\lambda;(r,s)}](\mathsf{z};\mathsf{q}) = \sum_{i=1}^{u-1} (-1)^{i-1}\chi^{(p,u)}_{r,i}(\mathsf{q})\,\text{Ch}[\mathcal{E}_{\lambda+i-1,\Delta_{i,s}}](\mathsf{z};\mathsf{q}), \tag{6.3.18c}$$

$$\mathrm{Sch}[\mathcal{C}^{\pm}_{r,s}](\mathsf{z};\mathsf{q}) = \sum_{i=1}^{u-1} (-1)^{i-1} \chi^{(p,u)}_{r,i}(\mathsf{q})\, \mathrm{Ch}[\mathcal{E}^{\pm}_{i,s}](\mathsf{z};\mathsf{q}) . \tag{6.3.18d}$$

More explicit formulae may now be obtained by substituting (3.2.12), (3.3.38) and (3.3.41). As usual, the characters and supercharacters of parity reversals are obtained from

$$\mathrm{Ch}[\Pi\mathcal{M}] = \mathrm{Ch}[\mathcal{M}], \qquad \mathrm{Sch}[\Pi\mathcal{M}] = -\mathrm{Sch}[\mathcal{M}]. \tag{6.3.19}$$

We remark that substituting the formula (3.3.39) for the irreducible relaxed $A_1(u,v)$-characters gives the following form for the irreducible relaxed $\mathsf{B}_{0|1}(p,v)$-characters:

$$\mathrm{Ch}[\mathcal{C}_{\lambda;(r,s)}](\mathsf{z};\mathsf{q}) = \frac{1}{\eta(\mathsf{q})^2} \sum_{i=1}^{u-1} \mathsf{z}^{\lambda+i-1} \chi^{(p,u)}_{r,i}(\mathsf{q})\, \chi^{(u,v)}_{i,s}(\mathsf{q}) \sum_{j\in\mathbb{Z}} \mathsf{z}^{2j}. \tag{6.3.20}$$

Comparing with the character formulae recently proved in [55], we deduce the following remarkable identities:

$$\sum_{i=1}^{u-1} \chi^{(p,u)}_{r,i}(\mathsf{q})\, \chi^{(i,s)}_{u,v}(\mathsf{q}) = \begin{cases} \mathrm{Ch}\left[\mathcal{W}^{(P,v)}_{r,s}\right](\mathsf{q}) \sqrt{\dfrac{\vartheta_3(1;\mathsf{q})}{\eta(\mathsf{q})}}, & \text{if } r+s \in 2\mathbb{Z}, \\ 2\mathrm{Ch}\left[\mathcal{W}^{(p,v)}_{r,s}\right](\mathsf{q}) \sqrt{\dfrac{\vartheta_2(1;\mathsf{q})}{2\eta(\mathsf{q})}}, & \text{if } r+s \in 2\mathbb{Z}+1, \end{cases} \tag{6.3.21a}$$

$$\sum_{i=1}^{u-1} (-1)^{i-1} \chi^{(p,u)}_{r,i}(\mathsf{q})\, \chi^{(u,v)}_{i,s}(\mathsf{q}) = \begin{cases} \mathrm{Sch}\left[\mathcal{W}^{(p,v)}_{r,s}\right](\mathsf{q}) \sqrt{\dfrac{\vartheta_4(1;\mathsf{q})}{\eta(\mathsf{q})}}, & \text{if } r+s \in 2\mathbb{Z}, \\ 0, & \text{if } r+s \in 2\mathbb{Z}+1. \end{cases} \tag{6.3.21b}$$

Here, $\mathrm{Ch}\left[\mathcal{W}^{(p,v)}_{r,s}\right]$ and $\mathrm{Sch}\left[\mathcal{W}^{(p,v)}_{r,s}\right]$ denote the characters and supercharacters of the $N=1$ superconformal minimal model $\mathsf{M}^{\mathsf{N=1}}(p,v)$ of central charge $\frac{3}{2} - \frac{3(v-p)^2}{pv}$, which we have computed in (4.2.15).

The identities (6.3.21) may be understood as resulting from the branching rules associated with the coset

$$\mathsf{M}(p,u) \cong \mathrm{Com}(\mathsf{M}(u,v), \mathsf{M}^{\mathsf{N=1}}(p,v)\otimes\mathsf{F}) \equiv \frac{\mathsf{M}^{\mathsf{N=1}}(p,v)\otimes\mathsf{F}}{\mathsf{M}(u,v)} \qquad (p+v=2u), \tag{6.3.22}$$

where F denotes the free fermion vertex operator superalgebra. Indeed, the embedding

$$\mathsf{M}(p,u)\otimes\mathsf{M}(u,v) \hookrightarrow \mathsf{M}^{\mathsf{N=1}}(p,v)\otimes\mathsf{F} \tag{6.3.23}$$

is strongly suggested by the character decomposition (6.3.21a) with $r=s=1$ and is easily confirmed by explicitly constructing the two commuting Virasoro subalgebras. The identification of the generating fields of the subalgebra is found to be

$$\begin{aligned} T^{(p,u)}(z) &= \frac{p}{2u} T^{\mathrm{N}=1}(z) - \sqrt{\frac{p(2u-p)}{4u^2}} :\psi G:(z) + \frac{4u-3p}{4u} :\partial\psi\psi:(z) \\ T^{(u,v)}(z) &= \frac{v}{2u} T^{\mathrm{N}=1}(z) - \sqrt{\frac{v(v-2u)}{4u^2}} :\psi G:(z) + \frac{4u-3v}{4u} :\partial\psi\psi:(z), \end{aligned} \tag{6.3.24}$$

where $T^{(n,m)}(z)$ is the energy-momentum tensor of the Virasoro minimal model $\mathsf{M}(n,m)$, $G(z)$ is the fermionic generating field of conformal dimension $\frac{3}{2}$ of $\mathsf{M}^{\mathrm{N}=1}(p,v)$ and $\psi(z)$ generates the free fermion F.

A version of this coset was previously considered, but deduced heuristically, in [125, 126] — however, there F was incorrectly replaced by its bosonic orbifold $\mathsf{M}(3,4)$. From our perspective, it is natural to regard this beautiful coset as the quantum hamiltonian reduction of the coset (6.1.2) (this is explained in [127, Thm. 2.10] and [128]).

6.4 Remarks on Completeness

So far, we have constructed several families of irreducible $\mathsf{B}_{0|1}(p,v)$-modules using $\mathsf{M}(p,u)$- and $A_1(u,v)$-modules as building blocks. A natural question to ask is whether this procedure has in fact constructed *all* the irreducible $\mathsf{B}_{0|1}(p,v)$-modules, up to isomorphism. The answer to this is surely no, because one expects to be able to similarly construct irreducible Whittaker modules for $\mathsf{B}_{0|1}(p,v)$ from those known for $A_1(u,v)$ when $v > 1$ [54]. However, we can refine our question to instead ask whether we have constructed all the irreducible $\mathsf{B}_{0|1}(p,v)$-modules in some physically relevant, and hopefully consistent, class (category) of $\widehat{\mathfrak{osp}}(1|2)$-modules.

When $v = 1$, this question was asked and answered in [122] using the notion of Perron-Frobenius dimensions, which relied crucially on there being only finitely many irreducible highest-weight $\mathsf{B}_{0|1}(p,v)$-modules, up to isomorphism. As such, this dimension argument should also succeed when $v > 1$ as long as we only want to know if we have constructed all the irreducible highest-weight $\mathsf{B}_{0|1}(p,v)$-modules with finite-dimensional L_0^{osp}-eigenspaces. It will not obviously help with the completeness question for more general classes of modules.

A different tool, Zhu's algebras, is used instead in [45] to prove that the lists of irreducible relaxed highest-weight $\mathsf{B}_{0|1}(p,v)$-modules constructed in this section are complete. We strongly believe that there is a physically consistent category for these vertex operator superalgebra in which the simple objects are precisely the spectral flows of the irreducible relaxed highest-weight modules. It therefore suffices to complete the classification of irreducible relaxed highest-weight $\mathsf{B}_{0|1}(p,v)$-modules. In [45, Sec. 4], an easy argument, independent of our constructions, is first presented for the $v = 1$ case. This trivially recovers the classification result of [122]. It is then followed by a slightly more involved argument for $v > 1$ that relies on our constructions and provides a quick proof of the general classification. This classification was originally proved in [111, Thm. 3.7] using free field realisations of $\widehat{\mathfrak{osp}}(1|2)$ and symmetric functions.

6.5 $\mathsf{B}_{0|1}(p,v)$ fusion rules

6.5.1 Grothendieck fusion rules

Recently, a fermionic version of the standard Verlinde formula was tested successfully in [41] for the $\mathfrak{osp}(1|2)$ minimal model $\mathsf{B}_{0|1}(2,4)$. We are thus confident that their result may be generalised straightforwardly to $\mathsf{B}_{0|1}(p,v)$ using the (super)character formulae derived here and the known S-matrices of the Virasoro and $\widehat{\mathfrak{sl}}_2$ minimal models [36].

We shall, again, adopt an alternative approach to compute the (Grothendieck) fusion rules using the coset (6.1.2) and the known (Grothendieck) fusion rules of the Virasoro and $\widehat{\mathfrak{sl}}_2$ minimal models $\mathsf{M}(p,u)$ and $A_1(u,v)$. We shall illustrate the idea by computing the fusion of $\mathcal{A}_{r,0}$ and $\mathcal{B}^+_{r',s'}$. Both of these modules are defined, see (6.3.8), as inductions of $\mathsf{M}(p,u)\otimes A_1(u,v)$-modules. Thus,

$$\mathcal{A}_{r,0}\times\mathcal{B}^+_{r',s'} = (\mathcal{V}_{r,1}\otimes\mathcal{L}_{1,0})\!\uparrow\times(\mathcal{V}_{r',1}\otimes\mathcal{D}^+_{1,s'})\!\uparrow\cong\Big((\mathcal{V}_{r,1}\otimes\mathcal{L}_{1,0})\times(\mathcal{V}_{r',1}\otimes\mathcal{D}^+_{1,s'})\Big)\!\uparrow, \quad (6.5.1)$$

as induction is preserved by fusion as discussed in Theorem A.2. Using the Virasoro fusion rules (3.2.14) and the $A_1(u,v)$ fusion rules (3.3.41), this becomes

$$\begin{aligned}\mathcal{A}_{r,0}\times\mathcal{B}^+_{r',s'} &\cong \Big((\mathcal{V}_{r,1}\times\mathcal{V}_{r',1})\otimes(\mathcal{L}_{1,0}\times\mathcal{D}^+_{1,s'})\Big)\!\uparrow\cong\bigoplus_{r''=1}^{p-1}\mathsf{N}^{[p]\,r''}_{r,r'}(\mathcal{V}_{r'',1}\otimes\mathcal{D}^+_{1,s'})\!\uparrow\\ &=\bigoplus_{r''=1}^{p-1}\mathsf{N}^{[p]\,r''}_{r,r'}\mathcal{B}^+_{r'',s'},\end{aligned} \quad (6.5.2)$$

where we have identified the final induced module using (6.3.6).

In an identical fashion, induction gives the following $\mathsf{B}_{0|1}(p,v)$ fusion rules:

$$\mathcal{A}_{r,0}\times\mathcal{A}_{r',0}=\bigoplus_{r''=1}^{p-1}\mathsf{N}^{[p]\,r''}_{r,r'}\mathcal{A}_{r'',0}, \quad (6.5.3a)$$

$$\mathcal{A}_{r,0}\times\mathcal{B}^\pm_{r',s'}=\bigoplus_{r''=1}^{p-1}\mathsf{N}^{[p]\,r''}_{r,r'}\mathcal{B}^\pm_{r'',s'}, \quad (6.5.3b)$$

$$\mathcal{A}_{r,0}\times\mathcal{C}_{\lambda';(r',s')}=\bigoplus_{r''=1}^{p-1}\mathsf{N}^{[p]\,r''}_{r,r'}\mathcal{C}_{\lambda';(r'',s')}. \quad (6.5.3c)$$

These fusion rules respect parity reversal and spectral flow in an analogous manner to the $N=2$ fusion rules (5.3.6):

$$\begin{aligned} \mathcal{M}\times\Pi\mathcal{N} &\cong \Pi(\mathcal{M}\times\mathcal{N}) \cong \Pi\mathcal{M}\times\mathcal{N}, \\ \mathcal{M}\times\sigma_{\text{osp}}(\mathcal{N}) &\cong \sigma_{\text{osp}}(\mathcal{M}\times\mathcal{N}) \cong \sigma_{\text{osp}}(\mathcal{M})\times\mathcal{N}. \end{aligned} \tag{6.5.4}$$

We remark that the fusion rules of the rational $\mathfrak{osp}(1|2)$ minimal models $\mathsf{B}_{0|1}(p,1)$ are given by (6.5.3a) alone.

The Grothendieck version of the rest of the fusion rules computed from induction are given by

$$\left[\mathcal{B}^+_{r,s}\right]\boxtimes\left[\mathcal{B}^+_{r',s'}\right] = \begin{cases} \displaystyle\sum_{r'',s''} \mathsf{N}^{[p,v]\,(r'',s'')}_{(r,s),(r',s')}\Big(\left[\sigma_{\text{osp}}(\mathcal{C}_{\lambda_{1,s+s'+1};(r'',s'')})\right]+\left[\mathcal{B}^+_{r'',s+s'}\right]\Big), & \text{if } s+s'<v, \\ \displaystyle\sum_{r'',s''} \mathsf{N}^{[p,v]\,(r'',s'')}_{(r,s+1),(r',s'+1)}\Big(\left[\sigma_{\text{osp}}(\mathcal{C}_{\lambda_{1,s+s'+1};(r'',s'')})\right] & \\ \qquad +\left[\sigma^2_{\text{osp}}(\mathcal{B}^+_{r'',2v-2-s-s'})\right]\Big), & \text{if } s+s'\geq v, \end{cases}$$

$$\begin{aligned} \left[\mathcal{B}^+_{r,s}\right]\boxtimes\left[\mathcal{C}_{\lambda';(r',s')}\right] &= \sum_{r'',s''} \mathsf{N}^{[p,v]\,(r'',s'')}_{(r,s+1),(r',s')}\left[\mathcal{C}_{\lambda'+\lambda_{1,s};(r'',s'')}\right] \\ &\qquad + \sum_{r'',s''} \mathsf{N}^{[p,v]\,(r'',s'')}_{(r,s),(r',s')}\left[\sigma_{\text{osp}}(\mathcal{C}_{\lambda'+\lambda_{1,s+1};(r'',s'')})\right], \end{aligned} \tag{6.5.5a}$$

$$\begin{aligned} \left[\mathcal{C}_{\lambda;(r,s)}\right]\boxtimes\left[\mathcal{C}_{\lambda';(r',s')}\right] &= \sum_{r'',s''} \mathsf{N}^{[p,v]\,(r'',s'')}_{(r,s),(r',s')}\Big(\left[\sigma_{\text{osp}}(\mathcal{C}_{\lambda+\lambda'-\mathsf{k};(r'',s'')})\right]+\left[\mathcal{C}_{\lambda+\lambda'+\mathsf{k};(r'',s'')}\right]\Big) \\ &\qquad + \sum_{r'',s''}\Big(\mathsf{N}^{[p,v]\,(r'',s'')}_{(r,s),(r',s'-1)}+\mathsf{N}^{[p,v]\,(r'',s'')}_{(r,s),(r',s'+1)}\Big)\left[\mathcal{C}_{\lambda+\lambda';(r'',s'')}\right]. \end{aligned} \tag{6.5.5b}$$

Here, the sums over r'' always run from 1 to $p-1$ while the sums over s'' run from 1 to $v-1$. These fusion rules can be extended to include parity reversals and spectral flows using the Grothendieck versions of (6.5.4).

6.5.2 Projective modules and their fusion rules

Following analogously from Section 5.3.2, we lift the projective modules $\mathcal{S}_{r,s}$ of $A_1(u,v)$ to conjectures for $\mathsf{B}_{0|1}(p,v)$. The lifts of these proposed projective modules will be denoted by $\mathcal{P}^\pm_{r,s}$ and are defined by

$$\mathcal{P}^\pm_{r,s} = (\mathcal{V}_{r,1}\otimes\mathcal{S}^\pm_{1,s})\uparrow, \qquad 1\leq r\leq p-1 \text{ and } 0\leq s\leq v-1. \tag{6.5.6}$$

Their restrictions are then

$$\mathcal{P}^\pm_{r,s}\downarrow \cong \bigoplus_{i=1}^{u-1}\mathcal{V}_{r,i}\otimes\mathcal{S}^\pm_{i,s} \tag{6.5.7}$$

and the corresponding Loewy diagrams take the form

$$\begin{array}{ccc} & \mathcal{B}^{\pm}_{r,s} & \\ \swarrow & & \searrow \\ \sigma_{\text{osp}}^{-1}(\mathcal{B}^{\pm}_{r,s-1}) & \mathcal{P}^{\pm}_{r,s} & \sigma_{\text{osp}}(\mathcal{B}^{\pm}_{r,s+1}) \\ \searrow & & \swarrow \\ & \mathcal{B}^{\pm}_{r,s} & \end{array} \qquad (s = 0, 1, \dots, v-1). \tag{6.5.8}$$

where we have introduced the following convenient notation:

$$\mathcal{B}^{\pm}_{r,-1} = \mathcal{B}^{\pm}_{r,1}, \qquad \mathcal{B}^{+}_{r,0} \equiv \mathcal{A}_{r,0} \equiv \mathcal{B}^{-}_{r,0} \qquad \text{and} \qquad \mathcal{B}^{\pm}_{r,v} = \sigma_{\text{osp}}^{\pm 1}(\mathcal{B}^{\pm}_{u-r,1}). \tag{6.5.9}$$

Of course, the $\mathcal{P}^{\pm}_{r,s}$ are staggered and are expected to be projective. There are also analogous statements obtained by applying parity reversal and spectral flow.

For completeness, we also lift the conjectured $A_1(u,v)$ fusion rules (3.3.48) to $\mathsf{B}_{0|1}(p,v)$ fusion rules in order to show how the $\mathcal{P}^{\pm}_{r,s}$ arise. Let $\lambda \neq \xi^{\pm}_{1,1} \pmod 2$ and $\mu \neq \xi^{\pm}_{r,s} \pmod 2$, where we recall the definition in (6.3.14). Then, for all $1 \le r \le p-1$ and $2 \le s \le v-2$ (which requires that $v \ge 4$), we have the fusion rules

$$\mathcal{C}_{\lambda;(1,1)} \times \mathcal{C}_{\mu;(r,s)} \begin{cases} \mathcal{P}^{+}_{r,s-1} \oplus \sigma_{\text{osp}}^{-1}(\mathcal{C}_{\lambda+\mu+\mathsf{k};(r,s)}) \oplus \mathcal{C}_{\lambda+\mu;(r,s+1)}, & \text{if } \lambda+\mu = -\frac{p+v}{2v}(s-1), \\ \mathcal{P}^{+}_{u-r,v-s-1} \oplus \sigma_{\text{osp}}^{-1}(\mathcal{C}_{\lambda+\mu+\mathsf{k};(r,s)}) \oplus \mathcal{C}_{\lambda+\mu;(r,s-1)}, & \text{if } \lambda+\mu = \frac{p+v}{2v}(s+1), \\ \mathcal{P}^{-}_{u-r,v-s-1} \oplus \sigma_{\text{osp}}(\mathcal{C}_{\lambda+\mu-\mathsf{k};(r,s)}) \oplus \mathcal{C}_{\lambda+\mu;(r,s-1)}, & \text{if } \lambda+\mu = -\frac{p+v}{2v}(s+1), \\ \mathcal{P}^{-}_{r,s-1} \oplus \sigma_{\text{osp}}(\mathcal{C}_{\lambda+\mu-\mathsf{k};(r,s)}) \oplus \mathcal{C}_{\lambda+\mu;(r,s+1)}, & \text{if } \lambda+\mu = \frac{p+v}{2v}(s-1), \\ \sigma_{\text{osp}}(\mathcal{C}_{\lambda+\mu-\mathsf{k};(r,s)}) \oplus \sigma_{\text{osp}}^{-1}(\mathcal{C}_{\lambda+\mu+\mathsf{k};(r,s)}) & \\ \qquad \oplus \mathcal{C}_{\lambda+\mu;(r,s-1)} \oplus \mathcal{C}_{\lambda+\mu;(1,s+1)}, & \text{otherwise}, \end{cases} \tag{6.5.10}$$

where $\lambda+\mu$ is always understood mod 2. As in the $A_1(u,v)$ case from which the fusion rules are induced, when s takes the special value of 1 or $v-1$, we remove any modules $\mathcal{C}_{\lambda;(r,s)}$ with $s=0$ or v. We also remove any direct summands that do not appear in all expressions corresponding to the same value of $\lambda+\mu \pmod 2$.

In a similar fashion, one can induce (3.3.50) to $\mathsf{B}_{0|1}(p,v)$ fusion rules involving staggered modules as

$$\mathcal{C}_{\lambda;(1,1)} \times \mathcal{B}^{+}_{1,s} = \begin{cases} \mathcal{P}^{+}_{u-1,v-s-1}, & \text{if } \lambda = \frac{p+v}{2v}(2s+1), \\ \mathcal{C}_{\lambda+\lambda_{1,s};(1,s+1)} \oplus \sigma_{\text{sl}}\left(\mathcal{C}_{\lambda+\lambda_{1,s+1};(1,s)}\right), & \text{otherwise}. \end{cases} \tag{6.5.11a}$$

$$\mathcal{C}_{\lambda;(1,s)} \times \mathcal{B}^+_{1,1} = \begin{cases} \mathcal{P}^+_{1,s-1} \oplus \mathcal{C}_{\lambda+\lambda_{1,1};(1,s+1)}, & \text{if } \lambda = -\frac{p+v}{2v}(s-2), \\ \mathcal{P}^+_{u-1,v-s-1} \oplus \mathcal{C}_{\lambda+\lambda_{1,1};(1,s-1)}, & \text{if } \lambda = \frac{p+v}{2v}(s+2), \\ \mathcal{C}_{\lambda+\lambda_{1,1};(1,s-1)} \oplus \mathcal{C}_{\lambda+\lambda_{1,1};(1,s+1)} \oplus \sigma_{\text{osp}}(\mathcal{C}_{\lambda+\lambda_{1,2};(1,s)}), & \text{otherwise}, \end{cases} \tag{6.5.11b}$$

where λ is again understood mod 2.

Fusion rules (6.5.10) and (6.5.11) provide the basic cases from which the corresponding general fusions can be derived. The fusion between two $\mathcal{B}$-type modules, on the other hand, does not give rise to any staggered modules. This is because staggered modules are not involved in the fusion between two $\mathcal{D}$-type modules in $A_1(u,v)$, from which the $\mathcal{B}$-type modules are induced, The proper fusion rule therefore takes the same form as the Grothendieck fusion, with its sum replaced by direct sum:

$$\mathcal{B}^+_{r,s} \times \mathcal{B}^+_{r',s'} = \begin{cases} \displaystyle\bigoplus_{r'',s''} \mathsf{N}^{[p,v]\,(r'',s'')}_{(r,s),(r',s')} \Big(\sigma_{\text{osp}}(\mathcal{C}_{\lambda_{1,s+s'+1};(r'',s'')}) \oplus \mathcal{B}^+_{r'',s+s'}\Big), & \text{if } s+s' < v, \\ \displaystyle\bigoplus_{r'',s''} \mathsf{N}^{[p,v]\,(r'',s'')}_{(r,s+1),(r',s'+1)} \Big(\sigma_{\text{osp}}(\mathcal{C}_{\lambda_{1,s+s'+1};(r'',s'')}) & \\ \qquad \oplus \sigma^2_{\text{osp}}(\mathcal{B}^+_{r'',2v-2-s-s'})\Big), & \text{if } s+s' \geq v. \end{cases} \tag{6.5.12}$$

Chapter 7

Conclusion

In this thesis, we have developed some basic strategies for constructing the representation theories of the $\mathfrak{osp}(1|2)$ and $N = 2$ minimal models at admissible levels. For the $N = 2$ minimal models, our method relied on extracting the representation theory of its coset algebras. Whereas for the $\mathfrak{osp}(1|2)$ minimal models, we reconstructed its representation theory in terms of its subalgebras, and we call this method the *inverse coset construction*. We subsequently developed classifications of the irreducible modules of the minimal models, formulations of characters of the modules and fusion rules.

The thesis began with a brief introduction to conformal field theory, in which we discussed the richness of conformal symmetry in two dimensions. We described the basic setup and formalism which the thesis is based on. A general account of key concepts such as operator product expansions, fusion, characters and automorphisms were given.

We then illustrated the previous general discussion with examples of conformal field theories, in particular the theories from which we shall build both the $N = 2$ and the $\mathfrak{osp}(1|2)$ minimal models. The bosonic conformal field theories in this list include the free boson, the Virasoro algebra and $\widehat{\mathfrak{sl}}_2$, among which the latter plays a pivotal role in the construction of both the $\widehat{\mathfrak{osp}}(1|2)$ and $N = 2$ superalgebras. One of the interesting features of the $\mathfrak{sl}_2$ minimal models is that it has a continuous spectrum of modules, known as the relaxed highest-weight modules. We have seen how these modules can act as building blocks for the construction of other irreducible modules using resolutions.

Conformal field theories can be extended into more sophisticated theories by introducing supersymmetries. Other than the two fermionic conformal field theories we constructed, the thesis also described the $\mathfrak{bc}$-ghost and the $N = 1$ superconformal algebras. Different from bosonic theories, a fermionic theory is complicated by the concepts of sectors and parities. The conventional method of computing fusion rules, the Verlinde formula, turned out to be demanding and tedious for the fermionic theories we want to study. An alternative approach was therefore adopted. An overview of induction was given in the context of vertex algebras in Appendix A. Following from this, we proved branching formulae and module classifications. The fusion rules of the subalgebras, including the $\mathfrak{bc}$-ghosts, $\widehat{\mathfrak{sl}}_2$, the Heisenberg and Virasoro algebras, were then lifted to the fusions of $\mathfrak{osp}(1|2)$ and $N = 2$ superalgebras through the method of coset.

The study the $N = 2$ minimal models was based on the coset (5.1.3). The big superalgebra $A_1(u,v) \otimes \mathfrak{bc}$ is given as an extension of two subalgebras, $N = 2$ and the Heisenberg algebra $\mathcal{H}$, in such a way that the subalgebras, $N = 2$ and $\mathcal{H}$, form a commuting pair in $A_1(u,v) \otimes \mathfrak{bc}$. Since we have very good pictures of $A_1(u,v)$ and $\mathfrak{bc}$, we managed to extract the representation theory of the coset algebra from them with the help of an induction functor, which worked particularly well because one of the subalgebras is nothing but a Heisenberg algebra. In this case, there is a Schur-Weyl type duality between the coset modules and the $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$-modules. For every indecomposable $N = 2$-module, there exists a Fock space $\mathcal{F}_p$ such that its tensor product with the $N = 2$-module lifts to a $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$-module. Since the induction functor is monodial, the Grothendieck fusion rules of $N = 2$-modules were immediately given in terms of those of the $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{bc}$-modules together with the fusion of Fock spaces.

The minimal models of $\widehat{\mathfrak{osp}}(1|2)$ were investigated with a similar approach. The coset we employed was given in (6.1.2). The algebra of interest was extended from two commuting subalgebras, one being $\widehat{\mathfrak{sl}}_2$ again and the other being Virasoro algebra $\mathfrak{Vir}$ (or a Virasoro minimal model). This means that the modules of $\widehat{\mathfrak{osp}}(1|2)$ can simply be obtained by inducing certain modules of $\widehat{\mathfrak{sl}}_2 \otimes \mathfrak{Vir}$ (even this is quite non-trivial). With this setup, we provided a classification of the modules of the $\mathfrak{osp}(1|2)$ minimal models, which was shown to be complete in [45]. The Grothendieck fusion rules for $\widehat{\mathfrak{osp}}(1|2)$ follow from those of $\widehat{\mathfrak{sl}}_2$ and $\mathfrak{Vir}$, again, by induction. We have also conjectured the projective covers and their structures for the $\mathfrak{sl}_2$ minimal models. We further explained that they are consistent with the general expectations for fusion rules in tensor categories. This allowed us to construct conjectured projectives for the $\mathfrak{osp}(1|2)$ minimal models and state their (real) fusion rules.

It is, in general, a difficult problem to check if the conjectured projective modules in this thesis are indeed correct and exhaustive. Given the complexity of the algebras and the modules structures, it is not practical to carry out the Nahm-Gaberdiel-Kausch (NGK) fusion algorithm to check if the fusion algebra closes on the proposed models. One potential approach to the $\mathfrak{osp}(1|2)$ minimal models is motivated by one of its alternative constructions through $\mathfrak{sl}_2$ and $\beta\gamma$-ghosts [129]. In a study of $\mathfrak{sl}_2$ minimal models by Wakimoto free field realisations [35], projective modules are conjectured. One of our collaborators, Wood, is working on constructing projective $\beta\gamma$-modules by showing that certain category of modules is rigid. Once successful, one may be able to extend the result to the $\mathfrak{sl}_2$ minimal models, and show that their conjectured modules are indeed projective. It is then possible to propagate the results from $\mathfrak{sl}_2$ and $\beta\gamma$-ghosts to $\mathfrak{osp}(1|2)$ minimal models, for example, using the coset method we discussed in this thesis.

I would like to conclude this thesis with a motivation of project or an outlook of the whole picture. The thesis is part of a programme to understand the admissible-level Wess-Zumino-Witten models for a Lie algebra or superalgebra, and use them to study

$$
\begin{array}{ccccc}
\widehat{\mathfrak{sl}}_2 & \widehat{\mathfrak{osp}}(1|2) & \widehat{\mathfrak{sl}}(2|1) & \widehat{\mathfrak{osp}}(3|2) & \widehat{\mathfrak{psl}}(2|2) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathfrak{Vir} & N=1 & N=2 & N=3 & N=4
\end{array}
$$

Figure 7.1: (Super)conformal field theories (second row) and their corresponding affine Lie (super)algebras (first row), which are related by the method of quantum Hamiltonian reduction, as indicated by the arrows in the diagram.

the structures of algebras with supersymmetries, such as the $N = 1, 2, 3, 4$ superconformal algebras. Superconformal field theories have been shown to be related to many research areas including string theories, critical phenomena in two dimensions, integrable systems and the quantum Hall effect. Due to their very rich mathematical structures, the complete classification problems of some of these algebras are still open. Superconformal algebras are examples of so-called $\mathcal{W}$-*algebras*, which can be constructed from a vertex superalgebra via quantum Hamiltonian reduction. This relation is depicted in Figure 7.1 for the above mentioned superconformal field theories.

Among the algebras in the second row of Figure 7.1, the $\mathfrak{Vir}$ and the $N = 1$ theory have been well understood in the literature. Their constructions from the $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{osp}}(1|2)$ algebras were studied in [130] and [131]. The representation theories of the rest of the superalgebras, especially the non-unitary cases of the $N = 3$ and $N = 4$ theories, still remain unrevealed. Therefore, it would be exciting to investigate the theories via the Hamiltonian reduction of their corresponding affine Lie algebras, which are, unfortunately, also not yet well understood.

By solving the $N = 2$ and the $\widehat{\mathfrak{osp}}(1|2)$ superalgebras using coset constructions, this thesis has provided an approach which can be similarly applied to the study of $\widehat{\mathfrak{sl}}(2|1)$, and perhaps $\widehat{\mathfrak{osp}}(3|2)$ and $\widehat{\mathfrak{psl}}(2|2)$. Hopefully, these will lead to a possible resolution of, or at least shine some light on, the investigations of the corresponding superconformal field theories.

Appendix A

The method of induction and restriction

There exists an induction functor which extends a subalgebra to the corresponding algebra which preserves the fusion. In this appendix, we summarise this technique which is frequently used in the the study of branching rules and fusions throughout the thesis.

The setup is as follows. Let V be a simple vertex operator algebra with integer conformal weights and let W be a simple vertex operator superalgebra. Assume that we have a parity-preserving embedding $\mathsf{V} \hookrightarrow \mathsf{W}$, meaning that the image is contained in the bosonic subalgebra of W. This means that W is an extension of V and so it decomposes into V-modules as

$$\mathsf{W}\downarrow \cong \bigoplus_i \mathcal{W}_i. \tag{A.0.1}$$

Here and below, we assume that each of the $\mathcal{W}_i$ consists of either bosonic or fermionic states. An especially nice situation is when the $\mathcal{W}_i$ appearing in this decomposition are irreducible and inequivalent. The notion $\mathsf{W}\downarrow$ of the *restriction* of W to a module of the smaller vertex operator superalgebra V generalises to arbitrary W-modules $\mathcal{N}$ as we may also restrict them to V-modules:

$$\mathcal{N}\downarrow \cong \bigoplus_j \mathcal{N}_j. \tag{A.0.2}$$

The identification of a restricted W-module, as a V-module, is called a *branching rule*.

In this setup, there is a very closely related operation on modules called *induction*. For this, let $\mathcal{M}$ be a V-module and consider its fusion product with the V-module $\mathsf{W}\downarrow$. In many cases, the result has a natural structure as a W-module and this W-module is called the induction of $\mathcal{M}$, denoted by $\mathcal{M}\uparrow$. The restriction of an induced module decomposes as

$$\mathcal{M}\uparrow\downarrow \cong \bigoplus_i \mathcal{W}_i \times \mathcal{M}. \tag{A.0.3}$$

Not every module induces to a Neveu-Schwarz or Ramond module of W. Fortunately, there is a nice criterion to study the result of inducing, assuming that the conformal dimensions of the states of W are integers (which is the case we are interested in here). This criterion is different for the two cosets we study in this thesis: for the $\widehat{\mathfrak{osp}}(1|2)$ ($N = 2$) coset, an irreducible induced $\widehat{\mathfrak{osp}}(1|2)$- ($A_1(u,v) \otimes \widehat{\mathfrak{bc}}$-)module is Neveu-Schwarz (Ramond) if and only if the conformal dimensions of the bosonic and the fermionic states of the module all differ by integers. Moreover, an irreducible induced module is Ramond (Neveu-Schwarz) if and only if the conformal dimensions differ by $\frac{1}{2}$ modulo $\mathbb{Z}$. We warn the reader that this setup does not distinguish modules from their parity reversals.

We now come to the two most important statements of [65]; we formulate them as theorems. The first one gives a criterion that guarantees that induced modules are irreducible. We shall apply it frequently in what follows.

Theorem A.1 ([65, Prop. 4.4]) *Let* $\mathsf{V} \hookrightarrow \mathsf{W}$ *be an embedding of a simple vertex operator algebra* V *into a simple vertex operator superalgebra* W *under which* $\mathsf{W}\downarrow$ *decomposes into a direct sum of irreducible* V*-modules* $\mathcal{W}_i$ *as in* (A.0.1). *Suppose that* $\mathcal{M}$ *is an irreducible* V*-module for which the fusion products* $\mathcal{W}_i \times \mathcal{M}$ *are irreducible and inequivalent:* $\mathcal{W}_i \times \mathcal{M} \ncong \mathcal{W}_j \times \mathcal{M}$ *if* $i \neq j$. *Then, the induced* W*-module* $\mathcal{M}\uparrow = \mathsf{W} \times \mathcal{M}$ *is irreducible.*

Obviously, a necessary condition for the inequivalence of the $\mathcal{W}_i \times \mathcal{M}$ is that the $\mathcal{W}_i$ are all inequivalent.

The second theorem gives a way to easily determine the fusion rules of induced modules. The version below suffices for the application to the thesis.

Theorem A.2 ([65, Thm. 3.68]) *Let* $\mathsf{V} \hookrightarrow \mathsf{W}$ *be an embedding of a vertex operator algebra* V *into an vertex operator superalgebra* W *and let* $\mathcal{M}$ *and* $\mathcal{N}$ *be* V*-modules. Then, the fusion rules of the induced* W*-modules satisfy*

$$\mathcal{M}\uparrow \times \mathcal{N}\uparrow \cong (\mathcal{M} \times \mathcal{N})\uparrow . \tag{A.0.4}$$

This method for computing fusion rules from (A.0.4) has also been proposed in the physics literature, for example in [50, Eq. (3.3)].

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