An Introduction to String Theory

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Declaration

This thesis is an account of research undertaken between July 2012 and February 2013 at The Department of Physics, Faculty of Science, The Australian National University, Canberra, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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February, 2013

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Abstract

In this thesis, we will start with a discussion of the bosonic string theory. Although this theory is not realistic, many techniques which will be developed there are very useful for the superstirng theory as well. After defining the bosonic string action (Polyakov action), we will study symmetries for this action and then, conserved quantities for the theory. These will be used for quantizing the theory. Calculating its mass spectrum, we will see that the bosonic string theory has unphysical ghost states. However, these states can be removed at the cost of fixing the spacetime dimension at 26.

Next, we will see the bosonic string theory in a different point of view: conformal field theories.

Finally, we will discuss the RNS superstring theory along almost the same way as we do in the bosonic string theory and we will see that there are two types of superstring theories: the type IIA and IIB (of course, there are other types as well). We will end with a discussion of T-duality, which relates the type IIA theory to the type IIB theory and vice versa.

Contents

List of Figures

Chapter 1

Introduction

String theory arose at the end of sixties in an attempt to describe the theory of strong interactions (The Veneziano formula by Veneziano in 1969). Therewith, Nambu and Susskind realized that the fundamental dynamical object from which the Veneziano formula can be derived is a relativistic string, which is an extended one-dimensional line or loop.

However, further investigations revealed that it was difficult to treat string theory as the theory of strong interactions. The first difficulty is the existence of a critical dimension. The construction of the quantum mechanics of relativistic strings leads to mathematically consistent theory if and only if the spacetime dimension is 26, which is called the critical dimension. The second difficulty was the that string theory predicts the existence of a massless spin-2 particle which is absent in the hadronic world.

In 1974, a proposal to change the view on string theory was made (by Scherk and Schwarz). Their suggestion was to regard the massless spin-2 particle as the quantum of the gravitational interaction (the graviton). Thus, according to their point of view, string theory could give the unifying description of all the particles and matter forces including gravity.

Even if we accept that string theory can be defined in the unusual 26-dimensional spacetime, we have another problem. Such a string does not include fermionic degrees of freedom in the theory and it predicts the existence if a particle with negative mass squared, called tachyon, which is known as a source of instability and its existence implies that the theory is not well-defined. String theory had faced the problematic features of critical dimensions, absence of fermion and existence of tachyon.

However, by the discovery of supersymmetry, this situation of string theory greatly changed. Supersymmetry is a symmetry between bosons and fermions (by Wess and Zumino 1974). All universe is made of two fundamental types of particles: bosons and fermions. Bosons mediate interactions of the matter particles, which are fermions. Many physicists hope that supersymmetry could provide an underlying principle for unification of all interactions.

The incorporation of supersymmetry in string theory was achieved in 1971 by Ramond and shortly after, by Neveu and Schwarz. This was dawn of the RNS (Ramond-Neveu-Schwarz) superstring. This theory is consistent in 10 dimensions ,instead of 26 for bosonic strings. In 1977, Gliozzi, Scherk and Olive realized that further conditions (the GSO projection) should be imposed on the spectrum of the RNS string which leads to the removal of tachyon and, as a result, to the spacetime supersymmetry. It also turned out that the GSO projection can be imposed in two different ways, which produce two different types of superstrings, called the type IIA and type IIB. However, it was priven that these two superstring theory are equivalent to each other by T-duality transformation.

In this thesis, we will start with a discussion of the bosonic string theory. Although this theory is not realistic, many techniques which will be developed there are very useful for the superstirng theory as well. After defining the bosonic string action (Polyakov action), we will study symmetries for this action and then, conserved quantities for the theory. These will be used for quantizing the theory. Calculating its mass spectrum, we will see that the bosonic string theory has unphysical ghost states. However, these states can be removed at the cost of fixing the spacetime dimension at 26.

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Finally, we will discuss the RNS superstring theory in almost the same way as we do in the bosonic string theory and we will see that there are two types of superstring theories: the type IIA and IIB (of course, there are other types as well). We will end with a discussion of T-duality, which relates the type IIA theory to the type IIB theory and vice versa.

Chapter 2

Classical Bosonic String Theory

2.1 Classical Action for Relativistic Point Particles

In classical physics, the evolution of a theory is described by its field equations, or its equations of motion. When we have a point particle, the field equations for spacetime coordinates of the particle $X(t)$ results from extremizing the action, which is given by

$$S = \int dtL, \tag{2.1}$$

where $L = T - V = \frac{1}{2}m\dot{X}(t)^2 - V(X(t))$. By setting the variation of S with respect to $X(t)$ equal to zero, we get the equations of motion for $X(t)$.

Now, we apply this method to a relativistic point particle moving through a D-dimensional spacetime. The relativistic action is given, in the system of natural units ($c = \hbar = 1$), by the invariant length of its world-line,

$$S_0 = -m \int ds, \tag{2.2}$$

where m is the mass of the particle, which was included in order for S_0 to be dimensionless. In this equation, the line element ds is given by

$$ds^2 = -g_{\mu\nu}(X)dx^\mu dx^\nu, \tag{2.3}$$

where the metric $g_{\mu\nu}$, with $\mu, \nu = 0, 1, \dots, D-1$, describes the geometry of the background spacetime and is chosen to have Minkowski signature $(-, +, \dots, +)$. The minus sign in front of $g_{\mu\nu}$ has been introduced so that ds is real for a time-like trajectory.

If we parametrize the particle's path (the world-line of the particle) by a real parameter τ, then we can rewrite (2.3) as

$$-g_{\mu\nu}(X)dX^\mu dX^\nu = -g_{\mu\nu}(X)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}d\tau^2, \tag{2.4}$$

which gives the following expression for the action S_0

$$S_0 = -m \int d\tau \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu}, \tag{2.5}$$

where $\dot{X}^\mu \equiv \frac{dX^\mu(\tau)}{d\tau}$. An important property of the action is that it is independent of the choice of parametrization (Proposition 1.1.1).

Proposition 2.1.1. *The action (2.5) remains unchanged if the parameter τ is replaced by another one $\tau' = f(\tau)$.*

Proof. Under the reparametrization $\tau \to \tau' = f(\tau)$, we have

$$d\tau \to d\tau' = \frac{df}{d\tau} d\tau,$$

and therefore,

$$\frac{dX^\mu(\tau')}{d\tau} = \frac{dX^\mu(\tau')}{d\tau'}\frac{d\tau'}{d\tau} = \frac{dX^\mu(\tau')}{d\tau'}\frac{df}{d\tau}$$

With these relations, we can obtain the following result:

$$\begin{aligned} S_0' &\equiv -m \int d\tau' \sqrt{g_{\mu\nu}(X)\frac{dX^\mu(\tau')}{d\tau'}\frac{dX^\nu(\tau')}{d\tau'}} \\ &= -m \int \frac{\partial f}{\partial \tau} d\tau \sqrt{g_{\mu\nu}(X)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\left(\frac{\partial f}{\partial \tau}\right)^{-2}} \\ &= -m \int d\tau \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu} \\ &= S_0, \end{aligned}$$

which means that the proposition was proved. □

Due to the above proposition, one can choose a proper parametrization in order to simplify the action and, as a result, the equations of motion. Now, this parametrization freedom will be used to simplify the action (2.5).

The action S_0 has two disadvantages. The first one is that it contains a square root, which causes difficulty when quantizing. And the second is that this action cannot be used to describe a massless particle. These problems can be avoided by introducing an action which is equivalent to the previous one. The equivalent action is given by

$$\tilde{S}_0 = \frac{1}{2}\int d\tau \Big[e(\tau)^{-1}\dot{X}^2 - m^2 e(\tau)\Big], \tag{2.6}$$

where $\dot{X}^2 \equiv g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$ and $e(\tau)$ is an auxiliary field.

Now, to see that this action is equivalent to (2.5), consider the variation of $\tilde{S}_0$ with respect to the field $e(\tau)$.

$$\delta\tilde{S}_0 = -\frac{1}{2}\int d\tau \frac{\delta e}{e^2}(\dot{X}^2 + m^2 e^2).$$

By setting $\delta\tilde{S}_0 = 0$, the field equation for $e(\tau)$ can be obtained as

$$e^2 = -\frac{\dot{X}^2}{m^2} \;\Rightarrow\; e = \sqrt{-\frac{\dot{X}^2}{m^2}}. \tag{2.7}$$

Substituting this back into (2.6) gives

$$\begin{aligned} \tilde{S}_0 &= \frac{1}{2}\left[\left(-\frac{\dot{X}^2}{m^2}\right)^{-1/2}\dot{X}^2 - m^2\left(-\frac{\dot{X}^2}{m^2}\right)^{1/2}\right] \\ &= -m\int d\tau (-\dot{X}^2)^{1/2} \\ &= S_0, \end{aligned}$$

which means if the equation of motion holds, then $\tilde{S}_0$ is equivalent to S_0.

2.1.1 Reparametrization Invariance of $\tilde{S}_0$

$\tilde{S}_0$ should be invariant under a reparametrization (diffeomorphism) of τ. To see this, one needs to know how the fields $X^\mu(\tau)$ and $e(\tau)$ transform under an infinitesimal change of parametrization $\tau \to \tau' = \tau - \xi(\tau)$.

Since the fields $X^\mu(\tau)$ are scalar fields, they transform according to $X'^\mu(\tau') = X^\mu(\tau)$. Therefore,

$$\begin{aligned}\delta X^\mu &\equiv X'^\mu(\tau) - X^\mu(\tau) = X'^\mu(\tau' + \xi(\tau)) - X^\mu(\tau) \\ &= X^\mu(\tau) + \xi(\tau)\dot{X}^\mu - X^\mu(\tau) \\ &= \xi(\tau)\dot{X}^\mu. \end{aligned} \tag{2.8}$$

Also, under the reparametrization, the auxiliary field is required to transform as

$$e'(\tau')d\tau' = e(\tau)d\tau, \tag{2.9}$$

which ensures that $\tilde{S}_0$ is invariant as it will be seen later. Thus,

$$\begin{aligned} e'(\tau')d\tau' &= e'(\tau - \xi(\tau))(d\tau - \dot{\xi}d\tau) \\ &= \left(e'(\tau) - \xi\dot{e}(\tau) + \mathcal{O}(\xi^2)\right)(d\tau - \dot{\xi}d\tau) \\ &= e'(\tau)d\tau - \frac{d}{d\tau}(\xi e)d\tau, \end{aligned} \tag{2.10}$$

where in the last line, we have replaced $e'\dot{\xi}$ by $e\dot{\xi}$ since they are equal up to second order in ξ. Now, equating (2.9) with (2.10), we get

$$\frac{d}{d\tau}(\xi(\tau)e(\tau)) = e'(\tau) - e(\tau) \equiv \delta e(\tau). \tag{2.11}$$

With these results, we will show that the action $\tilde{S}_0$ is invariant under the reparametrization, for simplicity, in the case of a flat-spacetime metric $g_{\mu\nu}(X) = \eta_{\mu\nu}$. Firstly, the variation of $\tilde{S}_0$ under a change both in $X^\mu(\tau)$ and $e(\tau)$ is given by

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau\left[-\frac{\delta e}{e^2}\dot{X}^2 + \frac{2}{e}\dot{X}\delta\dot{X} - m^2\delta e\right].$$

Here $\delta\dot{X}^\mu$ is given by

$$\delta\dot{X}^\mu = \frac{d}{d\tau}\delta X^\mu = \dot{\xi}\dot{X}^\mu + \xi\ddot{X}^\mu.$$

Substituting this back into $\delta\tilde{S}_0$ together with (2.11), we obtain

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau\frac{d}{d\tau}\left[\frac{\xi}{e}\dot{X}^2 - m^2\xi e\right],$$

which is the integral of a total derivative. Therefore, it can be dropped, which means $\tilde{S}_0$ is invariant under reparametization. This invariance can be used to set the auxiliary field to unity,[1] thereby simplifying the action, and it leads to the mass-shell condition. In fact,

$$\left.\frac{\delta\tilde{S}_0}{\delta e}\right|_{e(\tau)=1} = -\frac{1}{2}(\dot{X}^2 + m^2), \tag{2.13}$$

[1] Let us find the transformation rule for $e(\tau)$, (2.9). Under reparametrizations $\tau \to \tau' = f(\tau)$, $d\tau$ and $\dot{X}^\mu$ transform as

$$d\tau \to d\tau' = d\tau\dot{f}, \quad \dot{X}^\mu \to \frac{dX^\mu}{d\tau'} = \frac{\dot{X}^\mu}{\dot{f}}.$$

which implies that $\dot{X}^2 + m^2 = 0$ and this is the mass-shell condition.[2]

2.2 Generalization to p-Branes

A p-brane is a p-dimensional object moving through a D ($D \geq p$) dimensional spacetime. For example, a 0-brane is a point particle, a 1-brane is a string, and a 2-brane is a membrane etc.

The notion of an action for a point particle (0-brane) can be generalized to an action for a p-brane.

The generalization of $S_0 = -m \int ds$ to a p-brane in a $D(\geq p)$ dimensional background spacetime is given by

$$S_p = -T_p \int d\mu_p, \tag{2.14}$$

where T_p is the p-brane tension, which has units of mass/vol and $d\mu_p$ is the $(p+1)$ dimensional volume element given by

$$d\mu_p = \sqrt{-\det\left(G_{\alpha\beta}(X)\right)}d^{p+1}\sigma, \tag{2.15}$$

where $G_{\alpha\beta}$ is the induced metric on the world-surface (i.e. the world-sheet for $p = 1$), which is given by

$$G_{\alpha\beta}(X) = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta} g_{\mu\nu}(X), \quad \alpha, \beta = 0, 1, \ldots p, \tag{2.16}$$

where $\sigma^0 = \tau$ and σ^i $(i = 1, \ldots p)$ are the p spacelike coordinates.

Also, the action (2.14) is invariant under a reparametrization of σ^α

Proposition 2.2.1. *The action of a p-brane* (2.14) *is invariant under reparamerization of the* $p+1$ *world-volume coordinates* $\sigma^\alpha \to \sigma^\alpha(\tilde{\sigma})$.

Proof. Under this change of variables, the induced metric in (2.14) transforms in the following way:

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta} g_{\mu\nu} = (f^{-1})_\alpha^{\ \gamma}\frac{\partial X^\mu}{\partial \tilde{\sigma}^\gamma}(f^{-1})_\beta^{\ \delta}\frac{\partial X^\nu}{\partial \tilde{\sigma}^\delta} g_{\mu\nu},$$

where

$$f^\alpha_{\ \beta}(\tilde{\sigma}) = \frac{\partial \sigma^\alpha}{\partial \tilde{\sigma}^\beta}.$$

Thus, the determinant appearing in the action, by using the Jacobian of the world-volume coordinate transformation $J = \det f^\alpha_\beta$, becomes

$$\det\left(g_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\right) = J^{-2}\det\left(g_{\mu\nu}\frac{\partial X^\mu}{\partial \tilde{\sigma}^\gamma}\frac{\partial X^\nu}{\partial \tilde{\sigma}^\delta}\right).$$

If e transforms as $e \to e'$, then the transformation of the action (2.6) is

$$\tilde{S}_0 \to \frac{1}{2}\int d\tau \left((e'\dot{f})^{-1}\dot{X}^2 - m^2 e'\dot{f}\right).$$

For this to be equal to (2.6), we need

$$e' = \frac{e}{\dot{f}} = \frac{d\tau}{d\tau'}e. \tag{2.12}$$

This is the transformation rule for e. Then, we want to find a transformation $e \to e'$ such as $e'(\tau') = 1$. From (2.12), if $e' = 1$, we have $\dot{f} = e$. So, we can choose f to be $f(\tau) = \int d\tau e(\tau)$. Thus, given an arbitrary $e(\tau)$, we can always appropriately choose f to go to a gauge where $e' = 1$.

[2] The canonical momentum conjugate to $X^\mu(\tau)$ is defined by

$$P^\mu(\tau) \equiv \frac{\partial L}{\partial \dot{X}^\mu}.$$

In the case at the moment, the canonical momentum is given by $P^\mu(\tau) = \dot{X}^\mu(\tau)$. By using this, we see that the vanishing of (2.13) is nothing more than the mass-shell condition of a particle with mass m, $P^\mu P_\mu + m^2 = 0$.

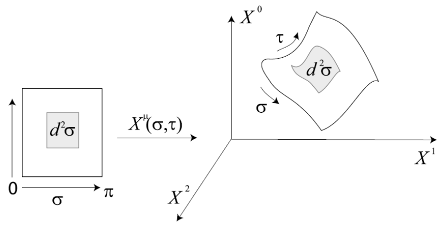

Figure 2.1: The functions $X^\mu(\tau,\sigma)$ describe the embedding of the string worldsheet in spacetime.

The measure of the integral transforms according to

$$d^{p+1}\sigma = J d^{p+1}\tilde{\sigma},$$

so that the action becomes

$$\tilde{S}_p = -T_p \int d^{p+1}\tilde{\sigma}\sqrt{-\det\left(g_{\mu\nu}\frac{\partial X^\mu}{\partial\tilde{\sigma}^\gamma}\frac{\partial X^\nu}{\partial\tilde{\sigma}^\delta}\right)},$$

which means that the action is invariant under reparametriztion of the world-volume coordinates. □

We will now specialize the p-brane action to the $p = 1$ case, i.e the string action.

2.2.1 The String Action

The string action is a 1-brane action, which describes a string in a D-dimensional spacetime. We will parametrize the world-sheet of the string, which is the two- dimensional extension of the world-line, by the two coordinates $\sigma^0 \equiv \tau$ and $\sigma^1 \equiv \sigma$, with τ being time-like and σ space-like. The embedding of the string into the D-dimensional background spacetime is given by the functions (or the fields) $X^\mu(\tau,\sigma)$, as drawn in Fig 2.1.

If the variable σ is periodic, then the embedding gives a closed string in the spacetime.

Now, we specialize to the case of Minkowski background spacetime. Then, (2.16) becomes

$$\begin{aligned} G_{00} &= \frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\tau}\eta_{\mu\nu} \equiv \dot{X}^2 \\ G_{11} &= \frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\sigma}\eta_{\mu\nu} \equiv X'^2 \\ G_{01} = G_{10} &= \frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\sigma}\eta_{\mu\nu} \equiv \dot{X}\cdot X' \end{aligned}$$

Thus, we obtain

$$G_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix}. \tag{2.17}$$

As a result, the string action, from (2.14), can be written as

$$S_{NG} = -T\int d\tau d\sigma\sqrt{(\dot{X}\cdot X')^2 - (\dot{X}^2)(X'^2)}, \tag{2.18}$$

which is known as the Nambu-Goto action. The integral in this action can be physically interpreted as the area of the world-sheet swept out by the string.

2.2.2 The String Sigma Model Action (Polyakov Action)

In order to remove the square root in S_{NG}, we can introduce an auxiliary field $h_{\alpha\beta}(\tau,\sigma)$, as we did before in the point particle case. This is the kind of metric that we use in two-dimensional general relativity, which is therefore intrinsic metric on the world-sheet. The resulting action is called the string sigma model action, and is given by

$$S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta} g_{\mu\nu}, \tag{2.19}$$

where $h \equiv \det(h_{\alpha\beta})$. This action is equivalent to the Nambu-Goto action at the classical level.

Proposition 2.2.2. *The string sigma model action S_σ is equivalent to the Nambu-Goto action S_{NG}.*

Proof. To begin with, recall that varying any action with respect to the metric yields the stress-energy tensor $T_{\alpha\beta}$:

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_\sigma}{\delta h^{\alpha\beta}}. \tag{2.20}$$

The equations of motion for $h^{\alpha\beta}$ can be obtained by setting the variation of the action S_σ with respect to $h^{\alpha\beta}$ equal to zero. That is,

$$\begin{aligned} \delta S_\sigma &\equiv \int d\tau d\sigma \frac{\delta S_\sigma}{\delta h^{\alpha\beta}} \delta h^{\alpha\beta} \\ &= -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\delta h^{\alpha\beta} T_{\alpha\beta} = 0, \end{aligned} \tag{2.21}$$

which holds when $T_{\alpha\beta} = 0$.

On the other hand, from the variation of S_σ in (2.19), we have by $h^{\alpha\beta}$ is

$$\begin{aligned} \delta S_\sigma &= -\frac{T}{2}\int d\tau d\sigma \left(\delta\sqrt{-h} h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X\right) \\ &= -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\delta h^{\alpha\beta}\left(-\frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X + \partial_\alpha X \cdot \partial_\beta X\right). \end{aligned}$$

In the second line, we have used the formula$\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}\delta h^{\alpha\beta}h_{\alpha\beta}$. Setting this equal to zero yields the equations of motion for $h^{\alpha\beta}$. Compared with (2.21), this gives

$$T_{\alpha\beta} \equiv -\frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X + \underbrace{\partial_\alpha X \cdot \partial_\beta X}_{G_{\alpha\beta}} = 0. \tag{2.22}$$

Thus, taking the determinant gives

$$\frac{1}{2}\sqrt{-h}h^{\gamma\delta}\partial_\gamma X \dot{\partial}_\delta X = \sqrt{-\det(G_{\alpha\beta})},$$

which means that S_σ is classically equivalent to S_{NG}. □

2.3 Symmetries and Field Equations

In this section, we will discuss the symmetries which the string sigma model action has. We will consider the background spacetime to be Minkowskian, that is, the action which we will discuss is

$$S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}. \tag{2.23}$$

2.3.1 Global Symmetries

A global transformation in some spacetime is a transformation whose parameter/parameters do not depend on where in the spacetime the transformation is performed. Invariance of a theory under global transformations gives conserved currents and charges via Noether's theorem.

By contrast, if a theory has local symmetries, we can these symmetries to cope with a redundancy which the theory has in its degrees of freedom. This is known as gauge fixing.

Poincaré Transformations

These global transformations are of the form:

$$\delta X^\mu(\tau,\sigma) = a^\mu{}_\nu X^\nu(\tau,\sigma) + b^\mu, \tag{2.24}$$
$$\delta h_{\alpha\beta}(\tau,\sigma) = 0, \tag{2.25}$$

where fields $X^\mu(\tau,\sigma)$ are defined on the world-sheet and $a^\mu{}_\nu$ (with $a_{\mu\nu} = -a_{\nu\mu}$) describes infinitesimal Lorentz transformations and b^μ spacetime translations. Here, S_σ in (2.23) is invariant under the Poincaré transformations.

Proposition 2.3.1. *S_σ in* (2.23) *is invariant under the Poincaré transformations in* (2.24) *and* (2.25).

Proof. Consider the following,

$$\delta S_\sigma = -T\int d\tau d\sigma \sqrt{-h}h^{\alpha\beta}\partial_\alpha(\delta X^\mu)\partial_\beta X^\mu \eta_{\mu\nu},$$

where we have used the fact that $h^{\alpha\beta}$ is invariant under the transformation (2.25). Substituting δX^μ from (2.24) gives

$$\begin{aligned}\delta S_\sigma &= -T\int d\tau d\sigma \sqrt{-h}h^{\alpha\beta}\partial_\alpha(a^\mu{}_\kappa X^\kappa + b^\mu)\partial_\beta X^\nu \eta_{\mu\nu}\\ &= -T\int d\tau d\sigma \sqrt{-h}\,\underbrace{a_{\nu\kappa}}_{anti}\,\underbrace{h^{\alpha\beta}\partial_\alpha X^\kappa \partial_\beta X^\nu}_{symmetric}\\ &= 0.\end{aligned}$$

In the last line, we have used the fact that the contraction of an antisymmetric tensor with a symmetric tensor equals zero. This result means that this action is invariant under the Poincaré transformations. □

2.3.2 Local Symmetries

Reparametrization Invariance (diffeomorphism)

This is a local symmetry for the world-sheet parametrized by two coordinates τ and σ. Under a coordinate transformation $\sigma^\alpha \to f^\alpha(\sigma) = \sigma'^\alpha$, which is a reparametrization of the world-sheet, the metric $h_{\alpha\beta}$ transforms as

$$h_{\alpha\beta}(\sigma) = \frac{\partial f^\gamma}{\partial\sigma^\alpha}\frac{\partial f^\delta}{\partial\sigma^\beta}h'_{\gamma\delta}(\sigma'). \tag{2.26}$$

Proposition 2.3.2. *S_σ in* (2.23) *is invariant under the reparametrization* (2.26).

Proof. We have the following relations under this reparametrization:

$$\frac{\partial}{\partial\sigma'^\alpha} = \frac{\partial\sigma^\rho}{\partial\sigma'^\alpha}\frac{\partial}{\partial\sigma^\rho} \quad \text{and} \quad X^\mu(\tau,\sigma) = X'^\mu(\tau',\sigma').$$

Thus, it follows that

$$h^{\alpha\beta}(\tau,\sigma)\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X_\mu}{\partial\sigma^\beta} = h'^{\rho\lambda}(\tau',\sigma')\frac{\partial X'^\mu}{\partial\sigma'^\rho}\frac{\partial X'_\mu}{\partial\sigma'^\lambda}.$$

The Jacobian for a change in the coordinates is given by

$$J = \det\left(\frac{\partial\sigma'^\alpha}{\partial\sigma^\beta}\right),$$

and the determinant of the metric $h'_{\alpha\beta}$ is, from (2.26),

$$\det(h^{\alpha\beta}) = J^2\det(h'_{\alpha\beta}).$$

Also, the integration measure transforms as

$$d^2\sigma' = Jd^2\sigma$$

Therefore, these cancel out with each other in the following:

$$\begin{aligned} d^2\sigma'\sqrt{-\det h'} &= Jd^2\sigma\sqrt{-J^{-2}\det h} \\ &= d^2\sigma\sqrt{-\det h} \end{aligned}$$

Putting all of these results together, we see that a reparametrization leaves S_σ invariant. □

Weyl Symmetry

Weyl transformations are those that change the scale of the metric $h_{\alpha\beta}$:

$$h_{\alpha\beta}(\tau,\sigma) \to h'_{\alpha\beta}(\tau,\sigma) = e^{2\phi(\sigma)}h_{\alpha\beta}(\tau,\sigma), \tag{2.27}$$

$$\delta X^\mu(\tau,\sigma) = 0. \tag{2.28}$$

Proposition 2.3.3. *S_σ in (2.23) is invariant under a Weyl transformation.*

Proof. First, we need to know how $\sqrt{-h}$ and $\sqrt{-h}h^{\alpha\beta}$ transform. The transformation of $\sqrt{-h}$ is given by

$$\begin{aligned} \sqrt{-h'} &\equiv \sqrt{-\det(h'_{\alpha\beta})} \\ &= e^{2\phi(\sigma)}\sqrt{-h}. \end{aligned}$$

Whereas, since $h^{\alpha\beta}h_{\beta\gamma} = \delta^\alpha_\gamma$, it follows from (2.27) that $h'^{\alpha\beta} = e^{-2\phi(\sigma)}h^{\alpha\beta}$. Thus, the transformation of $\sqrt{-h}h^{\alpha\beta}$ is given by

$$\sqrt{-h'}h'^{\alpha\beta} = e^{2\phi(\sigma)}\sqrt{-h}e^{-2\phi(\sigma)}h^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}.$$

Therefore, under a Weyl transformation, S_σ does not change. □

This invariance under a Weyl transformation leads to the fact that the stress-energy tensor is traceless, $h^{\alpha\beta}T_{\alpha\beta} = 0$. To begin, from (2.20), we obtain

$$\delta S_\sigma \equiv \int\frac{\delta S_\sigma}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta} = -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}\delta h^{\alpha\beta}T_{\alpha\beta}.$$

If we restrict to a Weyl transformation, the above variation of S_σ becomes[3]

$$\delta S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}(-2\phi)h^{\alpha\beta}T_{\alpha\beta},$$

which must be equal to zero since $\delta S_\sigma = 0$ under a Weyl transformation. Now, it follows from the arbitrariness of $\sqrt{-h}$ and ϕ that

$$h^{\alpha\beta}T_{\alpha\beta} = 0, \tag{2.29}$$

which means that for a Weyl invariant classical theory, the corresponding stress-energy tensor must be traceless.

From now on, it will be shown that we can fix a gauge so that the intrinsic metric $h_{\alpha\beta}$ becomes flat $\eta_{\alpha\beta}$ if our theory is invariant under diffeomorphism and Weyl transformations.

First, we note that the metric $h_{\alpha\beta}$ has only three independent components since it is symmetric by its definition:

$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} \quad \text{with} \quad h_{01} = h_{10}. \tag{2.30}$$

A diffeomorphism can be used to change the metric into a form that is proportional to the two-dimensional flat Minkowski metric $\eta_{\alpha\beta}$ as follows:[4]

$$h_{\alpha\beta} \to e^{2\phi(\sigma)}\eta_{\alpha\beta} = e^{2\phi(\sigma)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Next, we can use Weyl transformations to remove the factor $e^{2\phi(\sigma)}$:

$$h_{\alpha\beta} \to \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.31}$$

Here, we should note that since gauge transformations are local ones, this transformation from the metric $h_{\alpha\beta}$ to the flat metric $\eta_{\alpha\beta}$ is generally only possible locally. However, it is known that in the case that the Euler characteristic of the world-sheet is zero, we can extend a locally flat metric to a globally flat metric.

With this gauge fixed flat metric, the string sigma model action S_σ becomes

$$\begin{aligned} S_\sigma &= -\frac{T}{2}\int d\tau d\sigma \sqrt{-\eta}\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \\ &= \frac{T}{2}\int d\tau d\sigma \left(\dot{X}^2 - X'^2\right). \end{aligned} \tag{2.32}$$

2.3.3 Equation of Motion for S_σ and Boundary Conditions

Let us suppose that the world-sheet topology allows the locally flat metric to be extended globally. In this case, for a closed string, an infinite cylinder is chosen as the world-sheet and for an open string, an infinite strip. For convenience, we will choose the coordinate σ to have the range $0 \le \sigma \le \pi$. The equations of motion for the field $X^\mu(\tau,\sigma)$ are described by setting the variation of S_σ with respect to

[3] From (2.27), we have $h'^{\alpha\beta} = e^{-2\phi}h^{\alpha\beta} = (1 - 2\phi + \cdots)h^{\alpha\beta}$. Thus, we obtain $\delta h^{\alpha\beta} \equiv h'^{\alpha\beta} - h^{\alpha\beta} = -2\phi h^{abeta}$.

[4] A diffeomorphism allows us to change two of the independent components by using two coordinate transformations, to set $h_{10} = h_{01} = 0$ and $h_{00} = -h_{11}$.

X^μ equal to zero. This gives, from (2.32),

$$\begin{aligned}\delta S_\sigma &= \frac{T}{2}\int d\tau d\sigma\left(2\dot{X}\delta\dot{X} - 2X'\delta X'\right)\\ &= T\int d\tau d\sigma\Big[(-\partial_\tau^2+\partial_\sigma^2)X^\mu\Big]\delta X_\mu + T\int d\sigma \dot{X}^\mu\delta X_\mu\Big|_{\partial\tau}\\ &\quad - \Big[T\int d\tau X'^\mu\delta X_\mu\Big|_{\sigma=\pi} + T\int d\tau X'^\mu\delta X_\mu\Big|_{\sigma=0}\Big],\end{aligned}$$

where $\partial\tau$ represents the boundary in the τ-direction.[5] In the second line, we integrated both terms by parts. We will assume that fields vanish at $\tau \to \pm\infty$. As a result, the terms which are left become

$$\begin{aligned}\delta S_\sigma = T\int d\tau d\sigma\Big[(-\partial_\tau^2+\partial_\sigma^2)X^\mu\Big]\delta X_\mu\\ - T\int d\tau\Big[X'^\mu\delta X_\mu\Big|_{\sigma=\pi} + X'^\mu\delta X_\mu\Big|_{\sigma=0}\Big].\end{aligned} \tag{2.33}$$

In addition to the equations of motion, which come from the first term, there are the σ boundary terms (the second and third term), which must vanish.

Closed string

For closed strings, we take X^μ to be periodic:

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma). \tag{2.34}$$

By imposing this condition, the boundary terms appearing in the variation of S_σ vanish. As a result, we obtain the equations of motion for the closed string:

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)X^\mu = 0 \tag{2.35}$$

with the boundary conditions (2.34).

Open string with Neumann boundary conditions

In this case, the derivative of X^μ with respect to σ at the boundaries vanish (see Fig 2.2):

$$\partial_\sigma X^\mu(\tau,\sigma) = 0 \quad \text{at } \sigma = 0, \pi. \tag{2.36}$$

These conditions make the σ boundary terms vanish and the equations of motion become

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)X^\mu = 0 \tag{2.37}$$

with the boundary conditions (2.36). Note that the Neumann boundary conditions preserve Poincaré invariance because

$$\partial_\sigma X'^\mu\Big|_{\sigma=0,\pi} = \partial_\sigma\left(a^\mu{}_\nu X^\nu + b^\mu\right)\Big|_{\sigma=0,\pi} = 0.$$

Open string with Dirichlet boundary conditions

[5] In a strict sense, this statement is not correct because we are taking the cylinder and strip to be infinite. The exact meaning is infinitely distant area where fields vanish.

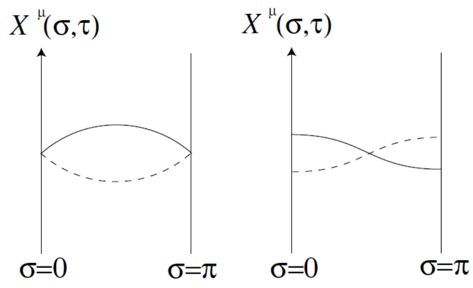

Figure 2.2: Neumann (right) and Dirichlet (left) boundary conditions

For the Dirichlet boundary conditions, we set the value of X^μ at the boundary to a constant:

$$X^\mu(\tau,\sigma)\Big|_{\sigma=0} \equiv X_0^\mu \quad \text{and} \quad X^\mu(\tau,\sigma)\Big|_{\sigma=\pi} \equiv X_\pi^\mu, \tag{2.38}$$

where X_0^μ and X_π^μ are constants (see Fig 2.2). Under these conditions, the σ boundary terms vanish. Thus, the equations of motion become

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0 \tag{2.39}$$

with the boundary conditions (2.38). Different from the Neumann case, Dirichlet boundary conditions do not preserve Poincaré invariance because

$$X'^\mu\Big|_{\sigma=0,\pi} = \left(a^\mu{}_\nu X^\nu + b^\mu\right)\Big|_{\sigma=0,\pi} \neq X_{0,\pi}^\mu.$$

This means that the ends of the string change under Poincaré transformations.

We have seen that the equations of motion are all the same under three different boundary conditions. Furthermore, as additional constraint, the equations of motion of the world-sheet metric $h_{\alpha\beta}$ must be imposed. These were given in (2.22):

$$0 = T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X.$$

In the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$, the above equations can be written for each components as follows:

$$0 = T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right), \tag{2.40}$$

$$0 = T_{01} = T_{10} = \dot{X} \cdot X'. \tag{2.41}$$

2.3.4 Solution to the equations of motion

We will solve the equations of motion by introducing light-cone coordinates for the world-sheet, which are defined as

$$\sigma^\pm = \tau \pm \sigma. \tag{2.42}$$

The derivatives, then, in terms of the light-cone coordinates become

$$\partial_+ \equiv \frac{\partial}{\partial\sigma^+} = \frac{\partial\tau}{\partial\sigma^+}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial\sigma^+}\frac{\partial}{\partial\sigma} = \frac{1}{2}\left(\partial_\tau + \partial_\sigma\right),$$

$$\partial_- \equiv \frac{\partial}{\partial\sigma^-} = \frac{\partial\tau}{\partial\sigma^-}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial\sigma^-}\frac{\partial}{\partial\sigma} = \frac{1}{2}\left(\partial_\tau - \partial_\sigma\right).$$

Since the metric transforms as

$$\eta'_{\alpha'\beta'} = \frac{\partial \sigma^\gamma}{\partial \sigma^{\alpha'}} \frac{\partial \sigma^\delta}{\partial \sigma^{\beta'}} \eta_{\gamma\delta},$$

we obtain the metric in terms of light-cone coordinates, which is given by [6]

$$\eta_{\alpha\beta} = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(in l-c coordinate).} \tag{2.43}$$

Then the inverse $\eta^{\alpha\beta}$ is calculated as

$$\eta^{\alpha\beta} = -2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(in l-c coordinate).} \tag{2.44}$$

With the relation $\partial_+ \partial_- = \frac{1}{4}\left(\partial_\tau^2 - \partial_\sigma^2\right)$, the equations of motion in terms of light-cone coordinates become

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0 \quad \Rightarrow \quad \partial_+ \partial_- X^\mu = 0 \ \ \text{(in l-c coordinate).} \tag{2.45}$$

Also, the equations of motion for the metric $h_{\alpha\beta}$ become

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0 \tag{2.46}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu = 0. \tag{2.47}$$

We have to solve these equations (2.45) $\sim$ (2.47). The general solution to the wave equation (2.45) is given by a linear combination of two arbitrary functions whose component depends on one of the light-cone coordinates:

$$\begin{aligned} X^\mu(\sigma^+, \sigma^-) &= X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+) \\ &= X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma), \end{aligned} \tag{2.48}$$

which is a sum of right-moving and left-moving waves, which are called right-movers and left-movers, respectively. These are expanded as follows:[7]

$$X_R^\mu(\tau - \sigma) = \frac{x^\mu}{2} + \frac{l_s^2 p^\mu}{2}(\tau - \sigma) + \frac{i l_s}{2} \sum_{k \neq 0} \frac{\alpha_k^\mu}{k} e^{-ik(\tau - \sigma)}, \tag{2.49}$$

$$X_L^\mu(\tau + \sigma) = \frac{x^\mu}{2} + \frac{l_s^2 \bar{p}^\mu}{2}(\tau + \sigma) + \frac{i l_s}{2} \sum_{k \neq 0} \frac{\bar{\alpha}_k^\mu}{k} e^{-ik(\tau + \sigma)}. \tag{2.50}$$

We have introduced some new terms here: x^μ and p^μ are constants and are the center of mass coordinate and the total momentum of the string, respectively. Also, l_s is the length of the string, which is related to the string tension T and the Regge slope parameter α' via

$$T = \frac{1}{2\pi\alpha'} \quad \text{and} \quad \frac{1}{2} l_s^2 = \alpha'.$$

[6] For example, the $(+,+)$ component is calculated as follows:

$$\eta_{++} = -\left(\frac{\partial \tau}{\partial \sigma^+}\right)^2 + \left(\frac{\partial \sigma}{\partial \sigma^+}\right)^2 = -\frac{1}{4} + \frac{1}{4} = 0$$

The rest of the components can be obtained in a similar fashion.

[7] We should expand them with integral symbol because k is an integer. But for future convenience, we expanded them with summation symbol.

The α_k^μ and $\bar{\alpha}_k^\mu$ are arbitrary modes. The function $X^\mu(\tau,\sigma)$ must be real, so that x^μ, p^μ and $\bar{p}^\mu$ are real and so that positive and negative modes are conjugated to one another,

$$\alpha_{-k}^\mu = (\alpha_k^\mu)^* \,,$$
$$\bar{\alpha}_{-k}^\mu = (\bar{\alpha}_k^\mu)^* \,.$$

Our next task is to apply the boundary conditions.

Closed strings

The closed string coordinates satisfy the periodicity boundary condition

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma), \tag{2.51}$$

which implies that the propagation of a closed string sweeps out a cylinder in spacetime. Imposing this condition on the general solution (2.49) and (2.50), we get

$$X_R^\mu(\tau-\sigma) = \frac{x^\mu}{2} + \frac{l_s^2 p^\mu}{2}(\tau-\sigma) + \frac{il_s}{2}\sum_{n\in\mathbb{Z},\neq 0}\frac{\alpha_n^\mu}{n}e^{-2in(\tau-\sigma)}, \tag{2.52}$$

$$X_L^\mu(\tau+\sigma) = \frac{x^\mu}{2} + \frac{l_s^2 \bar{p}^\mu}{2}(\tau+\sigma) + \frac{il_s}{2}\sum_{n\in\mathbb{Z},\neq 0}\frac{\bar{\alpha}_n^\mu}{n}e^{-2in(\tau+\sigma)}, \tag{2.53}$$

from which we get

$$\partial_- X_R^\mu = l_s \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-2in(\tau-\sigma)},$$
$$\partial_+ X_L^\mu = l_s \sum_{n\in\mathbb{Z}} \bar{\alpha}_n^\mu e^{-2in(\tau+\sigma)}.$$

We defined

$$\alpha_0^\mu \equiv \frac{l_s}{2}p^\mu \quad \text{and} \quad \bar{\alpha}_0^\mu \equiv \frac{l_s}{2}\bar{p}^\mu,$$

In addition, the periodicity condition imposes

$$p^\mu = \bar{p}^\mu.$$

Thus, the general solution for closed strings is

$$\begin{aligned} X^\mu(\tau,\sigma) &= X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma) \\ &= x^\mu + l_s^2 p^\mu \tau + \frac{il_s}{2}\sum_{n\neq 1}\frac{1}{n}\{\alpha_n^\mu e^{2in\sigma} + \bar{\alpha}_n^\mu e^{-2in\sigma}\}e^{-2in\tau}. \end{aligned} \tag{2.54}$$

In preparation for the quantization, we will calculate the Poisson brackets which the modes obey. With the definition of the canonical momentum $P^\mu(\tau,\sigma) \equiv \frac{\partial L}{\partial \dot{X}^\mu}$, we can get the mode expansion of the canonical momentum on the world-sheet:

$$\begin{aligned} P^\mu(\tau,\sigma) &= T\dot{X}^\mu = \frac{\dot{X}^\mu}{\pi l_s^2} \\ &= \frac{p^\mu}{\pi} + \frac{1}{\pi l_s}\sum_{n\neq 0}\left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \bar{\alpha}_n^\mu e^{-2in(\tau+\sigma)}\right) \end{aligned} \tag{2.55}$$

The fields X^μ and its canonical momentum P^μ satisfy the following Poisson brackets

$$\begin{aligned}\Big[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\Big]_{P.B.} &= \Big[P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\Big]_{P.B.} = 0,\\ \Big[P^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\Big]_{P.B.} &= \eta^{\mu\nu}\delta(\sigma-\sigma').\end{aligned}$$

Substituting the mode expansions for X^μ and P^μ to get the Poisson brackets in terms of $\alpha_n^\mu, \bar{\alpha}_n^\mu, x^\mu$ and p^μ, we obtain

$$\big[\alpha_m^\mu, \alpha_n^\nu\big]_{P.B.} = \big[\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\big]_{P.B.} = im\eta^{\mu\nu}\delta_{m+n,0}, \tag{2.56}$$

$$\big[\alpha_m^\mu, \bar{\alpha}_n^\nu\big]_{P.B.} = 0 \tag{2.57}$$

$$\big[p^\mu, x^\nu\big]_{P.B.} = \eta^{\mu\nu}. \tag{2.58}$$

Open Strings with Neumann boundary conditions

Imposing the Neumann boundary conditions (2.36) at the $\sigma = 0$ end on the equations (2.49) and (2.50) yields

$$X'^\mu\Big|_{\sigma=0} = \frac{l_s^2}{2}(p^\mu - \bar{p}^\mu) + \frac{l_s}{2}\sum_{k\neq 0} e^{-ik\tau}(\alpha_k^\mu - \bar{\alpha}_k^\mu) = 0,$$

from which we can see

$$p^\mu = \bar{p}^\mu \quad \text{and} \quad \alpha_k^\mu = \bar{\alpha}_k^\mu,$$

which means that the left- and right-movers get mixed by the boundary condition. The other boundary condition, which is imposed at the $\sigma = \pi$ end gives

$$X'^\mu\Big|_{\sigma=\pi} = il_s\sum_{k\neq 0}\alpha_k^\mu e^{-ik\tau}\sin(k\pi) = 0,$$

which implies that k must be an integer, denoting it by n. Thus, the general solution for an open string with Neumann boundary conditions is

$$(NN) \quad X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu\tau + il_s\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos(n\sigma), \tag{2.59}$$

from which we get

$$\begin{aligned}2\partial_\pm X^\mu &= l_s\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in(\tau\pm\sigma)}\\ \alpha_0^\mu &\equiv l_s p^\mu.\end{aligned}$$

The Poisson brackets for the modes are the same as that of closed strings.

Open Strings with Dirichlet conditions

The Dirichlet conditions (2.38) implies

$$\partial_\tau X^\mu(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0.$$

Substitution of the equations (2.49) and (2.50) into the boundary conditions yields

$$\bar{p}^\mu = -p^\mu, \quad \bar{\alpha}_k^\mu = -\alpha_k^\mu \quad \text{and} \quad k\in\mathbb{Z}.$$

Thus, denoting k by n and imposing (2.38), the general solution for open strings with Dirichlet conditions is

$$(DD) \quad X^\mu(\tau,\sigma) = x^\mu + \frac{\sigma}{\pi}(X^\mu_\pi - X^\mu_0) + l_s \sum_{n\neq 0} \frac{\alpha^\mu_n}{n} e^{-in\tau} \sin{(n\sigma)}, \tag{2.60}$$

from which we get

$$2\partial_\pm X^\mu = \pm l_s \sum_{n\in\mathbb{Z}} \alpha^\mu_n e^{-in(\tau\pm\sigma)},$$
$$\alpha^\mu_0 \equiv \frac{X^\mu_\pi - X^\mu_0}{\pi l_s}.$$

The Poisson brackets for the modes are the same as that of closed strings.

We can also impose mixed boundary conditions, i.e. different boundary conditions at the two ends of the open string. For Neumann boundary conditions at $\sigma = 0$ and Dirichlet boundary conditions at $\sigma = \pi$, the general solution reads

$$(ND) \quad X^\mu(\tau,\sigma) = x^\mu + il_s \sum_{r\in\mathbb{Z}+1/2} \frac{\alpha^\mu_r}{r} e^{-ir\tau} \cos{(\pi r\sigma)},$$

where x^μ is the position of the $\sigma = \pi$ end of the open string.

For completeness, we also give the last possible combination of boundary conditions.

$$(DN) \quad X^\mu(\tau,\sigma) = x^\mu + l_s \sum_{r\in\mathbb{Z}+1/2} \frac{\alpha^\mu_r}{r} e^{-ir\tau} \sin{(\pi r\sigma)},$$

where x^μ is the position of the $\sigma = 0$ end of the open string.

Chapter 3

Quantized Bosonic String Theory

According to Noether's theorem, associated with any global symmetry of a theory. there exists a conserved current $\mathcal{J}$ and a conserved charge Q:

$$\partial_\alpha \mathcal{J}^\alpha = 0,$$
$$\frac{dQ}{d\tau} = \frac{d}{d\tau}\left(\int d\sigma \mathcal{J}^0\right) = 0.$$

3.1 Conserved Quantities on the World-sheet

In the last chapter, we saw that our bosonic theory had Poincaré symmetry (translations and Lorentz transformations). we will discuss the conserved quantities associated with this symmetry.

Translations

Translations are given by $\delta X^\mu = b^\mu(\sigma^\alpha)$ and the variation of the action in a Minkowski spacetime (2.32) is given by

$$\delta S_\sigma = -T \int d\tau d\sigma \partial_\alpha \left(b^\mu(\sigma^\alpha)\right) \partial^\alpha X_\mu,$$

which implies that the current is given by

$$\mathcal{J}^{\alpha\mu} = -T\partial^\alpha X^\mu \tag{3.1}$$

In fact, this current is conserved only when the equations of motion hold:

$$\partial_\alpha \mathcal{J}^{\alpha\mu} = \partial^\alpha \mathcal{J}_\alpha{}^\mu = -T\partial^\alpha \partial_\alpha X^\mu = T\underbrace{(\partial_\tau^2 - \partial_\sigma^2)X^\mu}_{=0} = 0.$$

Then, the corresponding charge is given by

$$p^\mu \equiv \int d\sigma \mathcal{J}^{0\mu} = -\int_0^\pi d\sigma T\partial^0 X^\mu = \int_0^\pi d\sigma T\partial_0 X^\mu = \int_0^\pi d\sigma P^\mu,$$

where P^μ is the canonical momentum, conjugate to the field X^μ. This charge p^μ is called the total momentum and it is the same as the term appearing in the mode expansion (2.49) and (2.50).

Lorentz transformations

Lorentz transformations are given by

$$\delta X^\mu = a^\mu{}_k X^k,$$

Therefore,

$$\begin{aligned}\delta S_\sigma &= -T\int d\tau d\sigma \partial_\alpha (a^\mu{}_k X^k)\partial^\alpha X^\nu \eta_{\mu\nu}\\ &= -T\int d\tau d\sigma (\partial_\alpha a^\mu{}_k) X^k \partial^\alpha X^\nu \eta_{\mu\nu}\\ &= -T\int d\tau d\sigma (\partial_\alpha a_{\nu k}) X^k \partial^\alpha X^\nu,\end{aligned}$$

from which we can read off the current. However, considering the fact that $a_{\nu k}$ is antisymmetric, the current $\mathcal{J}_\alpha^{\mu\nu}$ becomes

$$\mathcal{J}_\alpha^{\mu\nu} = -\frac{T}{2}\left(X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu\right), \tag{3.2}$$

which is antisymmetric and conserved only when the equations of motion hold:

$$\partial^\alpha \mathcal{J}_\alpha^{\mu\nu} = -\frac{T}{2}(X^\mu \underbrace{\partial^\alpha\partial_\alpha X^\nu}_{=0} - X^\nu \underbrace{\partial^\alpha\partial_\alpha X^\mu}_{=0}) = 0.$$

3.2 Hamiltonian and Energy-Momentum Tensor

The time evolution of a system is generated by the Hamiltonian. Thus, the world-sheet time evolution is generated by the Hamiltonian defined by

$$H = \int_0^\pi d\sigma \left(\dot{X}_\mu P^\mu - \mathcal{L}\right). \tag{3.3}$$

In the case of the bosonic string theory, we have $P^\mu = T\dot{X}^\mu$ and $\mathcal{L} = \frac{T}{2}(\dot{X}^2 - X'^2)$. Substituting these into (3.3) gives

$$H = \frac{T}{2}\int_0^\pi d\sigma \left(\dot{X}^2 + X'^2\right). \tag{3.4}$$

Inserting the mode expansions (2.55) for closed strings, the closed string Hamiltonian is

$$H = \sum_{n=-\infty}^{\infty} (\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n) \quad \text{with} \quad \alpha_0^\mu = \bar{\alpha}_0^\mu = \frac{l_s p^\mu}{2}, \tag{3.5}$$

while for open strings we have

$$H = \frac{1}{2}\sum_{n=-\infty}^{\infty} (\alpha_{-n}\cdot\alpha_n) \quad \text{with} \quad \alpha_0^\mu = l_s p^\mu. \tag{3.6}$$

These results only hold in the classical theory. In the quantum theory, we will have order ambiguities when we promote the modes to operators.

Next, let us consider the mode expansions of the energy-momentum tensor, in terms of a closed string theory. The result for open strings follows from an analogous procedure to the closed string theory.

So then, substituting the mode expansions for X_R^μ and X_L^μ of a closed string into the components of the stress-energy tensor given by (2.46) and (2.47) gives

$$\begin{aligned} T_{--} &= l_s^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n e^{-2im(\tau-\sigma)} \\ &\equiv 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau-\sigma)}, \end{aligned} \tag{3.7}$$

where we have defined L_m as

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n. \tag{3.8}$$

On the other hand,

$$\begin{aligned} T_{++} &= l_s^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \bar{\alpha}_{m-n} \cdot \bar{\alpha}_n e^{-2im(\tau+\sigma)} \\ &\equiv 2l_s^2 \sum_{m=-\infty}^{\infty} \bar{L}_m e^{-2im(\tau+\sigma)}, \end{aligned} \tag{3.9}$$

where $\bar{L}_m$ has been defined as

$$\bar{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \bar{\alpha}_{m-n} \cdot \bar{\alpha}_n. \tag{3.10}$$

These Fourier coefficients L_m and $\bar{L}_m$ are called the Virasoro generators. Comparing the Hamiltonian with L_m and $\bar{L}_m$, we note that for a closed string

$$H = 2\left(L_0 + \bar{L}_0\right) = \sum_{n=-\infty}^{\infty} \left(\alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n\right), \tag{3.11}$$

while for an open string

$$H = L_0 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n. \tag{3.12}$$

Next, we will consider a mass formula with these mode expansions for the Hamiltonian and the energy-momentum tensor.

3.3 Classical Mass Formula for a Bosonic String

Classically, all the components of the stress-energy tensor vanish. This implies that classically, all the Fourier modes also vanish:

$$L_n = 0 \quad \text{and} \quad \bar{L}_n = 0 \quad \text{for} \quad \forall n \in \mathbb{Z}.$$

Also, recall that the mass-shell condition is

$$M^2 = -p_\mu p^\mu,$$

where p^μ is the total momentum of the string, which is given by

$$p^\mu = \int_0^\pi d\sigma P^\mu = T \int_0^\pi d\sigma \dot{X}^\mu = \begin{cases} \dfrac{2\alpha_0^\mu}{l_s} & \text{for a closed string} \\ \dfrac{\alpha_0^\mu}{l_s} & \text{for an open string} \end{cases} \tag{3.13}$$

Thus,

$$p^\mu p_\mu = \begin{cases} \frac{2\alpha_0\cdot\alpha_0}{\alpha'} & \text{for a closed string} \\ \frac{\alpha_0\cdot\alpha_0}{\alpha'} & \text{for an open string,} \end{cases}$$

where $\alpha' = l_s^2/2$. Then, for an open string, the vanishing of L_0 becomes

$$\begin{aligned} 0 = L_0 &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot\alpha_n = \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0\cdot\alpha_0 \\ &= \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \alpha' p^\mu p_\mu = \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n - \alpha' M^2. \end{aligned}$$

Hence, for an open string, we obtain the following mass formula:

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n. \tag{3.14}$$

In a similar way, for a closed string, we can get

$$M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}\left(\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n\right). \tag{3.15}$$

These mass-shell conditions are only valid classically. In the quantized theory, they will get quantum corrections.

3.4 Canonical Quantization

First, we will quantize the bosonic string theory using canonical quantization.

In the canonical quantization procedure, we quantize the theory by promoting Poisson brackets to commutators

$$[\,\cdot\,,\cdot\,]_{P.B.} \mapsto i[\,\cdot\,,\cdot\,] \tag{3.16}$$

and the field X^μ to an operator. This results in promoting the modes α_n^μ, the constant x^μ and their barred versions and hence $L_n,\ \bar{L}_n$ to operators.

In particular, the results for (2.56) $\sim$ (2.58), via the quantization procedure (3.16), become

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu] &= [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \\ [\alpha_m^\mu, \bar{\alpha}_n^\nu] &= 0, \\ [p^\mu, x^\nu] &= -i\eta^{\mu\nu}. \end{aligned} \tag{3.17}$$

If we define new operators as

$$a_m^\mu \equiv \frac{1}{\sqrt{m}}\alpha_m^\mu \quad \text{and} \quad a_m^{\mu\dagger} \equiv \frac{1}{\sqrt{m}}\alpha_{-m}^{\mu\dagger} \quad \text{for } m>0,$$

then the commutation relations are rewritten as

$$[a_m^\mu, a_n^{\nu\dagger}] = [\bar{a}_m^\mu, \bar{a}_n^{\nu\dagger}] = \eta^{\mu\nu}\delta_{m,n} \quad \text{for } m,n>0.$$

This looks like the same algebraic structure as the algebra obeyed by the creation and annihilation operators of harmonic oscillator except for the $\mu=\nu=0$ case. In the case of $\mu=\nu=0$, we get

$$[a_m^0, a_n^0] = \eta^{00}\delta_{m,n} = -\delta_{m.n}. \tag{3.18}$$

We will see later that this minus sign in the right hand side leads to negative norm states (or ghost states), which is physically unacceptable.

The ground state, which is denoted by $|0\rangle$, is defined as the state which is annihilated by the lowering operators a_m^μ:

$$a_m^\mu|0\rangle = 0 \quad \text{for } m > 0. \tag{3.19}$$

Other physical states $|\phi\rangle$ can be constructed by acting on the ground states with the raising operators $a_m^{\mu\dagger}$:

$$|\phi\rangle = a_{m_1}^{\mu_1\dagger} a_{m_2}^{\mu_2\dagger} \cdots a_{m_n}^{\mu_n\dagger}|0; k^\mu\rangle, \tag{3.20}$$

which are eigenstates of the momentum operator p^μ,[1]

$$p^\mu|\phi\rangle = k^\mu|\phi\rangle. \tag{3.21}$$

To see the existence of negative norm states, let us consider, for example, the state $|\psi\rangle = a_m^{0\dagger}|0; k^\mu\rangle$ for $m > 0$. Then the norm of this state becomes

$$\begin{aligned} \langle\psi|\psi\rangle &= \langle 0; k^\mu|a_m^0 a_m^{0\dagger}|0; k^\mu\rangle \\ &= \langle 0; k^\mu|[a_m^0, a_m^{0\dagger}]|0; k^\mu\rangle = -\langle 0; k^\mu|0; k^\mu\rangle. \end{aligned}$$

If we define $\langle 0; k^\mu|0; k^\mu\rangle$ to be positive, then we will have negative norm states in the theory. These negative norm states cause a problem because they are unphysical. Fortunately, we can remove these states. However, this removal will put a constraint on the number of dimensions of the background spacetime. This will be shown later.

3.5 Virasoro Algebra

In the quantum theory, the modes α_m^μ become operators, which implies that the generators L_m will also become operators. These operators are defined to be normal ordered, that is, (for an open string)

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}\cdot\alpha_n:, \tag{3.22}$$

where the symbol : denotes the normal ordering operator. According to (3.17), only when $m = 0$, normal ordering ambiguity arises and L_0 is given by

$$\begin{aligned} L_0 &= \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{-n}\cdot\alpha_n: \\ &= \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n=-\infty}^{-1} :\alpha_{-n}\cdot\alpha_n: + \frac{1}{2}\sum_{n=1}^{\infty} :\alpha_{-n}\cdot\alpha_n: \\ &= \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n=-\infty}^{-1} \alpha_n\cdot\alpha_{-n} + \frac{1}{2}\sum_{n=-\infty}^{-1} \alpha_{-n}\cdot\alpha_n \\ &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n. \end{aligned} \tag{3.23}$$

Actually, this is the only Virasoro operator for which normal ordering matters. We have to choose an ordering but we do not know which one is a correct ordering. They all differ from the normal ordered

[1] This result implies that the state $|0; k^\mu\rangle$ is a ground state with center of mass momentum k^μ.

one by a constant, because $[\alpha_n^\mu, \alpha_{-n}^\mu]$ is a constant. Thus, the L_0 defined in (**??**) should be $L_0 - a$ (a is a constant).

Also, these Virasoro operators satisfy the relation[2]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{3.24}$$

where $c = D$ is the spacetime dimension (called the central charge). The second term in (3.24) is called a central extension.

3.6 Physical States

Here we will discuss the physical states in terms of the Virasoro operators. As was mentioned above, when we quantized the theory, we should add an constant to L_0. Therefore, for an open string, the vanishing of the L_0 constraint transforms into

$$(L_0 - a)|\phi\rangle = 0, \tag{3.25}$$

where a is a unknown constant and $|\phi\rangle$ is any physical on-shell state in the theory.

For a closed string, we instead have

$$(L_0 - a)|\phi\rangle = (\bar{L}_0 - a)|\phi\rangle = 0. \tag{3.26}$$

This normal ordering ambiguity also contributes to the mass formula. For an open string, the mass formula becomes

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - a \equiv N - a, \tag{3.27}$$

where

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n \tag{3.28}$$

is called the number operator.[3]

For a closed string, we have the mass formula

$$\frac{1}{4}\alpha' M^2 = \underbrace{\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n}_{\equiv N} - a = \underbrace{\sum_{n=1}^{\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n}_{\equiv \bar{N}} - a \tag{3.29}$$

Also, from (3.26), we can get $(L_0 - \bar{L}_0)|\phi\rangle = 0$, which implies

$$N = \bar{N}. \tag{3.30}$$

This is known as the level matching condition and it is the constraint that relates the left moving and the right moving modes.

[2] This relation can be obtained directly by calculating $[L_m, L_n]$ from $[a_m^\mu, a_n^\nu]$ but it is quite messy. We want to derive it in the next chapter by using the OPE of the energy-momentum tensor.

[3] By using this number operator, we can see the mass spectrum.

$N = 0 \quad \alpha' M^2 = -a$ (ground state)

$N = 1 \quad \alpha' M^2 = -a + 1$ (first excited state)

$N = 2 \quad \alpha' M^2 = -a + 2$ (second excited state)

3.6.1 Virasoro Generators and Physical States

In the rest of this chapter, we will not discuss closed strings. The results for closed strings can be obtained from the results for open strings since a closed string state is a tensor product of a left- and right-moving state. In the quantum theory, we cannot demand that the operator L_m annihilates all the physical states, for all $m \neq 0$, since this is incompatible with the Virasoro algebra.

Instead, we will only impose on physical states

$$L_{m>0}|\phi\rangle = 0 = \langle\phi|L^\dagger_{m>0}. \tag{3.31}$$

3.7 Removing Ghost States

Now that we have the physical open string states $|\phi\rangle$ defined as

$$|\phi\rangle = a^{\mu_1\dagger}_{m_1} a^{\mu_2\dagger}_{m_2} \cdots a^{\mu_n\dagger}_{m_n} |0; k^\mu\rangle,$$

and obeying the following two constraints

$$(L_0 - a)|\phi\rangle = 0$$
$$L_{m>0}|\phi\rangle = 0.$$

We also saw that there were some states whose norm was negative. However, we can remove these unphysical negative norm states by constraining the constant a and the central charge of the Virasoro algebra (the number of dimensions of the spacetime).

In order to construct a theory which does not include negative norm states, we will search for zero-norm states which satisfy the physical state conditions. Thus, we need to introduce new states which are called spurious states.

3.7.1 Spurious States

A state $|\psi\rangle$ is called spurious if it satisfies the mass-shell condition

$$(L_0 - a)|\psi\rangle = 0 \tag{3.32}$$

and is orthogonal to all physical states

$$\langle\phi|\psi\rangle = 0 \quad \forall \text{physical states } |\phi\rangle, \tag{3.33}$$

which means the set of all spurious states is an orthogonal subspace to the space of all physical states.

In general, a spurious state can be written as

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle, \tag{3.34}$$

where $|\chi_n\rangle$ satisfies

$$(L_0 - a + n)|\chi_n\rangle = 0. \tag{3.35}$$

This follows from the definition of a spurious state.[4] However, any spurious state can be simplified to

$$|\psi\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle. \tag{3.36}$$

[4]The reason is as follws: From (3.32), we have

$$L_0|\psi\rangle - a|\psi\rangle = 0 \Rightarrow \sum_{n=1}^{\infty} \left([L_0, L_{-n}] + L_{-n}L_0\right)|\chi_n\rangle - a|\psi\rangle = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(nL_{-n} + L_{-n}L_0 - aL_{-n}\right)|\chi_n\rangle = 0 \Rightarrow \sum_{n=1}^{\infty} L_{-n}\left(L_0 - a + n\right)|\chi_n\rangle = 0,$$

As an example, let us consider the state $|\psi\rangle = L_{-3}|\chi_3\rangle$. We will prove that the state $L_{-3}|\chi_3\rangle$ can be written as the form of (3.36). From (3.24), we have $L_{-3} = [L_{-1}, L_{-2}]$. Therefore,

$$\begin{aligned} L_{-3}|\chi_3\rangle &= [L_{-1}, L_{-2}]|\chi_3\rangle \\ &= L_{-1}L_{-2}|\chi_3\rangle - L_{-2}L_{-1}|\chi_3\rangle, \end{aligned}$$

which is of the form $L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle$ if we regard $L_{-2}|\chi_3\rangle$ and $-L_{-1}|\chi_3\rangle$ as $|\chi_1\rangle$ and $|\chi_2\rangle$, respectively. Now then, we need to check that

$$(L_0 - a + 1)L_{-2}|\chi_3\rangle = 0 \quad \text{and} \quad (L_0 - a + 2)L_{-1}|\chi_3\rangle = 0.$$

Here, note that L_{-n} raises the eigenvalue of the operator L_0 by the amount n.[5] Now, from (3.35), $|\chi_3\rangle$ satisfies $(L_0 - a + 3)|\chi_3\rangle = 0$, which implies $L_0|\chi_3\rangle = (a-3)|\chi_3\rangle$. Thus,$L_0L_{-2}|\chi_3\rangle = (a-3+2)L_{-2}|\chi_3\rangle = (a-1)L_{-2}|\chi_3\rangle$, which implies $(L_0 - a + 1)L_{-2}|\chi_3\rangle = 0$. This equation is just what we wanted to check. Similarly, we can obtain $L_0L_{-1}|\chi_3\rangle = (a-3+1)L_{-1}|\chi_3\rangle$, which implies $(L_0 - a + 2)L_{-1}|\chi_3\rangle = 0$. We have shown that the state $L_{-3}|\chi_3\rangle$ can be written as the linear combination (3.36). As for other general states, we can prove them in a similar manner.

The spurious states defined above are orthogonal to any physical state $|\phi\rangle$ since

$$\begin{aligned} \langle\phi|\psi\rangle &= \sum_{n=1}^{\infty} \langle\phi|L_{-n}|\chi_n\rangle \\ &= \sum_{n=1}^{\infty} \langle\chi_n|L_n|\phi\rangle^* \\ &= 0, \end{aligned}$$

where the second line follows from $L_{-n}^\dagger = L_n$ and the third line from $L_{n>0}|\phi\rangle = 0$.

If a state $|\psi\rangle$ is spurious and physical, then it is orthogonal to all physical states including itself

$$\langle\psi|\psi\rangle = \sum_{n=1}^{\infty} \langle\chi_n|L_n|\psi\rangle = 0.$$

As a result, such a state has zero-norm.

Thus, we have constructed physical states whose norm is zero. We will remove the negative norm states by using these zero-norm states.

3.7.2 Removing the negative norm states (determination of a and c)

In order to find the suitable a value, we will start with the following physical spurious state

$$|\psi\rangle = L_{-1}|\chi_1\rangle,$$

where $|\chi_1\rangle$ satisfies $(L_0 - a + 1)|\chi_1\rangle = 0$ and $L_{m>0}|\chi_1\rangle = 0$, where the last equation comes because we have assumed $|\psi\rangle$ to be physical. Since we are assuming that $|\psi\rangle$ is physical, it must satisfy the condition

which holds for all states $|\chi_n\rangle$. thus we can get (3.35).

[5]Let $|k\rangle$ be a state such that $L_0|k\rangle = k|k\rangle$. Then,

$$\begin{aligned} L_0L_{-n}|k\rangle &= ([L_0, L_{-n}] + L_{-n}L_0)\,|k\rangle \\ &= (n+k)L_{-n}|k\rangle. \end{aligned}$$

Therefore, $L_{-n}|k\rangle$ is an eigenstate of L_0 with eigenvalue $n + k$.

$(L_0 - a)|\psi\rangle = 0$ along with the condition $L_{m>0}|\psi\rangle = 0$. The second condition implies that $L_1|\psi\rangle = 0$, from which we have $0 = L_1 L_{-1}|\chi_1\rangle = ([L_1, L_{-1}] + L_{-1}L_1)|\chi_1\rangle = [L_1, L_{-1}]|\chi_1\rangle = 2L_0|\chi_1\rangle = 2(a-1)|\chi_1\rangle$. Thus, we can determine the value of a as 1.

Next, in order to determine the value of c, we need to construct the following physical spurious state

$$|\psi\rangle = (L_{-2} + \gamma L_{-1}L_{-1})|\chi_2\rangle, \tag{3.37}$$

where γ is a constant, which will have to be fixed to ensure that $|\psi\rangle$ has a zero-norm (i.e. physical) and $|\chi_2\rangle$ obeys the relations: $(L_0 - a + 2)|\chi_2\rangle = (L_0 + 1)|\chi_2\rangle = 0$ and $L_{m>0}|\chi_2\rangle = 0$. Now, since we are assuming that $|\psi\rangle$ is physical, we have $L_1|\psi\rangle = 0$, which implies that $0 = L_1(L_{-2} + \gamma L_{-1}L_{-1})|\chi_2\rangle = ([L_1, L_{-2} + \gamma L_{-1}L_{-1}])|\chi_2\rangle = \{(3 - 2\gamma)L_{-1} + 4\gamma L_0 L_{-1}\}|\chi_2\rangle$. The second term vanishes[6] Thus, $L_1|\psi\rangle = (3 - 2\gamma)L_{-1}|\chi_2\rangle = 0$. The equality holds only for $\gamma = 3/2$. As a result, the equation (3.37) becomes

$$|\psi\rangle = (L_{-2} + \frac{3}{2}L_{-1}L_{-1})|\chi_2\rangle.$$

So then, let us consider the constraint $L_2|\psi\rangle = 0$, which implies that $0 = L_2(L_{-2} + 3/2L_{-1}L_{-1})|\chi_2\rangle = [L_2, L_{-2} + 3/2L_{-1}L_{-1}]|\chi_2\rangle = (13L_0 + c/2)|\chi_2\rangle = (-13 + c/2)|\chi_2\rangle$. Thus, if we assume that $|\psi\rangle$ is physical and spurious, then we must have $c = 26$.

Overall, if we would like to remove the negative norm states, then we have to, at least, fix the value of a, γ and c at 1, 3/2 and 26, respectively. Since the central charge c is equal to the dimension of the background spacetime, our theory is only physically acceptable for the case that it lives in a space of 26 dimensions. The $a = 1$, $c = 26$ bosonic string theory is called critical, where the critical dimension is 26. (Although there can exist bosonic string theories with non-negative norm physical states for $a \leq 1$ and $c \leq 25$, which are called non-critical.)

3.8 Light-Cone Gauge Quantization

In the canonical quantization, we kept the bosonic theory manifestly Lorentz invariant but it predicted the existence of negative norm states. By contrast, light-cone quantization does not predict negative norm states but it is no longer manifestly Lorentz invariant. When we impose the Lorentz invariance of the theory, we will see the theory has $a = 1$ and $c = D = 26$ again.

To proceed, let us define light-cone coordinates for spacetime as

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right). \tag{3.38}$$

Then, the D spacetime coordinates X^μ consist of the null coordinates $X^\pm$ and the $D-2$ transverse coordinates X^i:

$$\{X^-, X^+, X^i\}_{i=1}^{D-2}.$$

In these coordinates, the inner product of two vectors v and w is given by

$$v \cdot w = -v^+ w^- - v^- w^+ + \sum_{i=1}^{D-2} v^i w^i$$

and indices are raised and lowered by the following rules

$$v^- = -v_+, \quad v^+ = -v_- \quad \text{and} \quad v^i = v_i,$$

[6] $L_0 L_{-1}|\chi_2\rangle = ([L_0, L_{-1}] + L_{-1}L_0)|\chi_2\rangle = L_{-1}(1 + L_0)|\chi_2\rangle = 0$.

which implies that the theory is no longer manifestly Lorentz invariant in this coordinates since the two coordinates $X^{\pm}$ are treated differently from the others.

In the light-cone gauge, we choose

$$X^{+}(\tau,\sigma) = x^{+} + l_s^2 p^{+}\tau. \tag{3.39}$$

In terms of the modes, this gauge corresponds to setting $\alpha_n^{+} = \bar{\alpha}_n^{+} = 0$ for $\forall n \neq 0$ (see (2.54) and (2.59)). The light-cone gauge eliminates the oscillator modes of X^{+}.

Next, we will consider what happens to the oscillator modes of X^{-} in the light-cone gauge. Recall the Virasoro constraints (**??**), these are equivalent to

$$(\dot{X} \pm X')^2 = 0. \tag{3.40}$$

In terms of the light-cone coordinates, (3.40) becomes

$$\dot{X}^{-} \pm X^{-\prime} = \frac{1}{2l_s^2 p^{+}}(\dot{X}^i \pm X^{i\prime})^2. \tag{3.41}$$

These equations are used to determine X^{-} and for an open string with Neumann boundary conditions, we have

$$X^{-}(\tau,\sigma) = x^{-} + l_s^2 p^{-}\tau + \sum_{n\neq 0}\frac{1}{n}\alpha_n^{-} e^{-in\tau}\cos(n\sigma).$$

Substituting this into (3.41) gives

$$\alpha_n^{-} = \frac{1}{p^{+}l_s}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i\alpha_m^i: - a\delta_{n,0}\right). \tag{3.42}$$

In the light-cone gauge, only the zero modes for X^{-} survive as independent degrees of freedom as was the case for X^{+}. Thus, we can express the bosonic string theory in terms of transverse oscillators only.

3.8.1 Mass-Shell Condition (Open Bosonic String)

In the light-cone coordinates, the mass-shell condition is given by

$$M^2 = -p^{\mu}p_{\mu} = 2p^{+}p_{-} - \sum_{i=1}^{D-2} p^i p_i. \tag{3.43}$$

From (3.42), we have for $n = 0$

$$p^{-}l_s \equiv \alpha_0^{-} = \frac{1}{p^{+}l_s}\left(\frac{1}{2}(\alpha_0^i)^2 + \underbrace{\sum_{i=1}^{D-2}\sum_{m>0} :\alpha_{-m}^i\alpha_m^i:}_{\equiv N} - a\right), \tag{3.44}$$

which can be rewritten as

$$2p^{+}p^{-} - p^i p_i = \frac{2}{l_s^2}(N-a).$$

Thus, together with (3.43), we can get $M^2 = 2/l_s^2(N-a)$, that is,

$$M^2 = \frac{2}{l_s^2}\sum_{i=1}^{D-2}\sum_{n>0} :\alpha_{-n}^i\alpha_n^i: - a. \tag{3.45}$$

3.8.2 Mass Spectrum (Open Bosonic String)

In the light-cone gauge, all the excitations are generated by transverse oscillators α_n^i. In the canonical quantization, we had to include all the oscillators, which cause the problem of negative norm states. However, the commutator of the transverse oscillators does not have the negative value, which comes from $\eta^{00} = -1$. Thus, we do not have negative norm states in the light-cone gauge.

The first excited state is given by $\alpha_{-1}^i|0;k^\mu\rangle$, which belongs to a $(D-2)$-component vector representation of the $SO(D-2)$ in the transverse space.

In general, it is known that Lorentz invariance implies that physical states form a representation of $SO(D-1)$ for massive states and $SO(D-2)$ for massless states. Hence, since $\alpha_{-1}^i|0;k^\mu\rangle$ belongs to a representation of $SO(D-2)$, it must correspond to a massless state if the theory is Lorentz invariant.

By using this fact, the value of a in (3.45) can be determined. Acting on the first excited state with the mass operator,

$$M^2\left(\alpha_{-1}^i|0;k^\mu\rangle\right) = \frac{2}{l_s^2}(N-a)\left(\alpha_{-1}^i|0;k^\mu\rangle\right) = \frac{2}{l_s^2}(1-a)\left(\alpha_{-1}^i|0;k^\mu\rangle\right).$$

Thus, in order for the first excited state to have an eigenvalue of zero for the mass operator, and not to contradict with Lorentz invariance, the condition $a = 1$ must be imposed.

After fixing the a value, the next task is to determine the spacetime dimension D, which is conducted by calculating the normal ordering constant a directly. The normal ordering constant a occurs when we normal order the expression

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty}\alpha_{-m}^i\alpha_m^i.$$

Normal ordering this expression yields

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty}\alpha_{-m}^i\alpha_m^i = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty}:\alpha_{-m}^i\alpha_m^i: + \frac{D-2}{2}\sum_{m=1}^{\infty}m \tag{3.46}$$

since $[\alpha_m^i, \alpha_{-m}^j] = m\delta_{ij}$. The second sum of the right hand side is divergent and needs to be regularized by using ζ-function regularization.

Firstly, consider the sum $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, which is defined for any complex number s. For $Re(s) > 1$, this sum converges to the Riemann zeta function $\zeta(s)$. This zeta function has the following value for $s = -1$: $\zeta(-1) = -1/12$. Thus, the second sum can be written as $-(D-2)/24$. Using the earlier result that a should be equal to one, we get

$$\frac{D-2}{24} = 1,$$

from which we can conclude $D = 24$.

3.8.3 Analysis of the Spectrum

Open String

For convenience, the mass operator is written below.

$$\alpha' M^2 = N - 1, \quad \text{where } \alpha' \equiv \frac{l_s^2}{2}.$$

At the first few mass levels, the physical states of the open string are as follows.

- For $N = 0$

  There is a tachyon, which has an imaginary mass and whose mass is given by $\alpha' M^2 = -1$.

- For $N = 1$

  There is a vector boson $\alpha^i_{-1}|0; k^\mu\rangle$ which is massless because of the Lorentz invariance. This state gives a vector representation of $SO(26 - 2)$.

- For $N = 2$

  We have the first states with positive mass. The states are

  $$\alpha^i_{-2}|0; k^\mu\rangle \quad \text{and} \quad \alpha^i_{-1}\alpha^j_{-1}|0; k^\mu\rangle \quad \text{with } \alpha' M^2 = 1.$$

  These are 24 and $24 * 25/2$ states, respectively. Thus, the total number of states is 324, which is the dimensionality of the symmetric traceless second-rank tensor representation of $SO(25)$. So, in this sense, the spectrum consists of a single massive spin-two state at the $N = 2$ mass level.

Closed String

The spectrum of the closed string can be obtained from the spectrum of the open string since a closed string state is a tensor product of a left-moving state and a right-moving state. In addition, we must consider the level matching condition (3.30) for left-moving and right-moving modes.

The mass operator of the closed string is given by

$$\alpha' M^2 = 4(N - 1) = 4(\bar{N} - 1)$$

The physical state of the closed string at the first two mass levels are as follows.

- For $N = 0 = \bar{N}$

  We have $\alpha' M^2 = -4$. Thus, the ground state $|0; k^\mu\rangle$ is again a tachyon.

- For $N = 1 = \bar{N}$

  We have the massless states

  $$\alpha^i_{-1}\bar{\alpha}^j_{-1}|0; k^\mu\rangle,$$

  which corresponds to the tensor product of two massless vectors (one left-mover and one right-mover). The number of the states is $24^2 = 576$ states.

Chapter 4

Conformal Field Theory

4.1 Role of CFT in String Theory

We will discuss why conformal field theories are important in string theory. It will turn out that two-dimensional conformal field theories are very important to describe the world-sheet dynamics.

A string has internal degrees of freedom described by its vibrational modes. The different vibrational modes of the string are interpreted as particles.

The vibrational modes of the string can be studied by investigating the world-sheet, which is a two-dimensional surface. It will turn out that when studying the world-sheet, the vibrational modes of the string are described by a two-dimensional conformal field theory.

4.2 Conformal Group in d dimensions

The conformal group is defined as follows. If one has a metric $g_{\alpha\beta}(x)$ in d dimensional spacetime, then under a coordinate change $x \to x'$ (such that $x^\mu = f^\mu(x'^\nu)$ we have, since the metric is a second-rank tensor,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}\left(f(x')\right). \tag{4.1}$$

The conformal group is defined to be the subgroup of coordinate transformations that leaves the metric unchanged, up to a scale factor $\Omega(x)$, (i.e. preserves angles)

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{4.2}$$

The transformation $x \to x'$ is called a conformal transformation and, roughly speaking, a conformal field theory is a field theory which respects these transformations.

The transformation of (4.2) has a different interpretation depending on whether we are considering a fixed background metric or a dynamical background metric. When the background is dynamical, the transformation is a diffeomorphism, which is a gauge symmetry. When the background is fixed, the transformation is a global symmetry, with a corresponding current. We will see later that the corresponding charges for this current are the Virasoro generators.

We will start with a flat background metric[1]. We can find the infinitesimal generators of the conformal group. For an infinitesimal conformal transformation $x^\mu \to f^\mu(x'^\nu) = x^\mu + \epsilon^\mu$, we have, from (4.1),

$$g'_{\mu\nu}(x^\alpha + \epsilon^\alpha) = g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu,$$

[1] In two dimensions, this is not a restriction on the theory since we have seen that if the theory has a Weyl symmetry and reparametrization invariance, then we can make the metric flat with this symmetry.

which must be equal to (4.2). Thus, we get

$$(\Omega(x) - 1)\, g_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \tag{4.3}$$

Multiplying $g^{\mu\nu}$ on the both sides and taking the trace gives

$$\Omega(x) - 1 = \frac{2}{d}(\partial \cdot \epsilon).$$

Substituting this back into (4.3) yields

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}. \tag{4.4}$$

The solutions to the above equation correspond to infinitesimal conformal transformations.

For $d > 2$, we obtain the following solutions.

1. $\epsilon^\mu = a^\mu$ (a^μ is a constant.)

   These correspond to translations.

2. $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$ ($\omega^\mu{}_\nu$ is an antisymmetric tensor.)

   These correspond to Lorentz transformations.

3. $\epsilon^\mu = \lambda x^\mu$ (λ is a number.)

   These correspond to scale transformations.

4. $\epsilon^\mu = b^\mu x^2 - 2x^\mu b \cdot c$

   These are known as the special conformal transformations.

Then, the finite transformations for each of them are given as

1. $x'^\mu = x^\mu + a^\mu$.
2. $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ ($\Lambda \in SO(1, d)$).
3. $x'^\mu = \lambda x^\mu$.
4. $x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}$.

These transformations form the conformal group in d dimensions. The generators for them are

1. $P^\mu = \partial_\mu$.
2. $M_{\mu\nu} = \frac{1}{2}(x_\mu \partial_\nu - x_\nu \partial_\mu)$.
3. $D = x^\mu \partial_\mu$.
4. $k_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu$.

4.3 Two dimensional Conformal Algebra

We will take $d = 2$ and $g_{\mu\nu} = \delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is the two dimensional Euclidean metric.[2] Under this situation, the equation (4.4) becomes

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (\partial \cdot \epsilon)\, \delta_{\mu\nu}. \tag{4.5}$$

- For $\mu = \nu = 1$ and $\mu = \nu = 2$,
  we have
  $$\partial_1 \epsilon_1 = \partial_2 \epsilon_2. \tag{4.6}$$
- For $\mu = 1$ and $\nu = 2$,
  we have
  $$\partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \tag{4.7}$$

Thus, we see that the equation (4.4), in the case of two dimensions, are the Cauchy-Riemann equations. That is, in the two dimensional case, infinitesimal conformal transformations are functions which obey the C-R equations.

In terms of the two dimensional complex coordinates

$$z, \bar{z} = x^1 \pm i x^2,$$

defining $\epsilon = \epsilon^1 + i\epsilon^2$ and $\bar{\epsilon} = \epsilon^1 - \epsilon^2$, the equations (4.6) and (**??**) can be rewritten as

$$\partial_{\bar{z}} \epsilon = 0 \quad \text{and} \quad \partial_z \bar{\epsilon} = 0.$$

Thus, in two dimensions, conformal transformations coincide with the holomorphic and the antiholomorphic coordinate transformations given by

$$\begin{aligned} z &\to f(z) \\ \bar{z} &\to \bar{f}(\bar{z}), \end{aligned}$$

where $\partial_{\bar{z}} f(z) = \partial_z \bar{f}(\bar{z}) = 0$.

In order to obtain the generators, let us consider the infinitesimal coordinate transformations

$$z \to z' = z + \epsilon(z) \quad \text{and} \quad \bar{z} \to \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}),$$

where $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ can be expanded as

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1} \tag{4.8}$$

$$\bar{\epsilon}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n \bar{z}^{n+1}. \tag{4.9}$$

Then, the corresponding infinitesimal generators can be obtained as

$$l_n = -z^{n+1} \partial_z \tag{4.10}$$

$$\bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}. \tag{4.11}$$

[2] Until now it has been assumed that the string world-sheet has a Lorentzian signature metric. However, it is convenient to make a Wick rotation $\tau \to -i\tau$, so as to obtain a world-sheet with Euclidean signature, and thereby make the world-sheet metric $h_{\alpha\beta}$ positive definite.

The set $\{l_n, \bar{l}_n\}_{n\in\mathbb{Z}}$ forms an algebra and its commutators are given by

$$[l_m, l_n] = (m-n)l_{m+n} \tag{4.12}$$
$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \tag{4.13}$$
$$[l_m, \bar{l}_n] = 0. \tag{4.14}$$

These commutation relations are modified a little after the quantization.

The generators $l_{0,\pm1}$ and $\bar{l}_{0,\pm1}$ are a special case because these form a subalgebra and generate finite conformal transformations. Specifically,

1. Translation

$$l_{-1} = -\partial_z, \quad \bar{l}_{-1} = -\partial_{\bar{z}}$$

2. Scaling + Rotation

$$l_0 = -z\partial_z, \quad \bar{l}_0 = -\bar{z}\partial_{\bar{z}}$$

To be exact, $l_0 + \bar{l}_0$ is the generator of scaling and $i(l_0 - \bar{l}_0)$ is the generator of rotation.

3. SCF

$$l_1 = -z^2\partial_z, \quad \bar{l}_1 = -\bar{z}^2\partial_{\bar{z}}$$

4.4 Conformal Field Theories in Two Dimensions

A conformal field theory is a field theory that respects conformal transformations (4.2). Here, we will adopt a Euclidean metric:

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2. \tag{4.15}$$

In terms of the complex coordinates $z = x^1 + ix^2$, the above equation can be rewritten as

$$ds^2 = dzd\bar{z}. \tag{4.16}$$

Under a coordinate transformation $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$, ds^2 transforms as

$$ds^2 \to \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} ds^2. \tag{4.17}$$

A field $\Phi(z,\bar{z})$ is called a primary field with a conformal weight $(h, \bar{h})$ if, under a conformal transformation $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$, it transforms as

$$\Phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})). \tag{4.18}$$

Especially, under an infinitesimal transformation $f(z) = z + \epsilon(z)$ and $\bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z})$, we can get

$$\left(\frac{\partial f}{\partial z}\right)^h = (1 + \partial_z \epsilon(z))^h = 1 + h\partial_z\epsilon(z) + \mathcal{O}(\epsilon^2)$$
$$\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} = (1 + \partial_{\bar{z}} \bar{\epsilon}(\bar{z}))^{\bar{h}} = 1 + \bar{h}\partial_{\bar{z}}\bar{\epsilon}(\bar{z}) + \mathcal{O}(\bar{\epsilon}^2)$$

Then, the infinitesimal transformation of the field $\Phi(z,\bar{z})$ is given by

$$\begin{aligned}\Phi(z,\bar{z}) &\to (1+h\partial_z\epsilon(z)+\mathcal{O}(\epsilon^2))(1+\bar{h}\partial_{\bar{z}}\bar{\epsilon}(\bar{z})+\mathcal{O}(\bar{\epsilon}^2))\Phi(z+\epsilon(z),\bar{z}+\bar{\epsilon}(\bar{z}))\\ &= \Phi(z,\bar{z})+\big[(h\partial_z\epsilon(z)+\epsilon(z)\partial_z)+(\bar{h}\partial_{\bar{z}}\bar{\epsilon}(\bar{z})+\bar{\epsilon}(\bar{z})\partial_{\bar{z}})\big]\Phi(z,\bar{z})+\mathcal{O}(\epsilon^2,\bar{\epsilon}^2,\epsilon\bar{\epsilon}).\end{aligned}$$

Thus, the variation of the field $\Phi(z,\bar{z})$ becomes, up to the first order in ϵ and $\bar{\epsilon}$,

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(z,\bar{z}) = \big[(h\partial_z\epsilon(z)+\epsilon(z)\partial_z)+(\bar{h}\partial_{\bar{z}}\bar{\epsilon}(\bar{z})+\bar{\epsilon}(\bar{z})\partial_{\bar{z}})\big]\Phi(z,\bar{z}). \tag{4.19}$$

4.4.1 Correlation Functions

The transformation rule of two primary fields in (4.19) gives constraints on the correlation functions of the primary fields.

Since a two-point correlation function of primary fields, $G^{(2)}(z_i,\bar{z}_i) \equiv \langle\Phi_1(z_1,\bar{z}_1\Phi_1(z_2,\bar{z}_2\rangle$, is invariant under an infinitesimal transformation $\delta_{\epsilon,\bar{\epsilon}}$, now if we assume that the transformation $\delta_{\epsilon,\bar{\epsilon}}$ is a derivation, then we have

$$\begin{aligned}0 &= \delta_{\epsilon,\bar{\epsilon}}G^{(2)}(z_i,\bar{z}_i)\\ &= \langle(\delta_{\epsilon,\bar{\epsilon}}\Phi_1)\Phi_2\rangle+\langle\Phi_1(\delta_{\epsilon,\bar{\epsilon}}\Phi_2)\rangle\\ &= \langle(\epsilon(z_1)\partial_{z_1}\Phi_1+h_1\partial_{z_1}\epsilon(z_1)\Phi_1)\Phi_2\rangle+\langle(\bar{\epsilon}(\bar{z}_1)\partial_{\bar{z}_1}\Phi_1+\bar{h}_1\partial_{\bar{z}_1}\bar{\epsilon}(\bar{z}_1)\Phi_1)\Phi_2\rangle\\ &\qquad+\langle\Phi_1(\epsilon(z_2)\partial_{z_2}\Phi_2+h_2\partial_{z_2}\epsilon(z_2)\Phi_2)\rangle+\langle\Phi_1(\bar{\epsilon}(\bar{z}_2)\partial_{\bar{z}_2}\Phi_2+\bar{h}_2\partial_{\bar{z}_2}\bar{\epsilon}(\bar{z}_2)\Phi_2)\rangle\\ &= \big[(\epsilon(z_1)\partial_{z_1}+h_1\partial_{z_1}\epsilon(z_1))+(\epsilon(z_2)\partial_{z_2}+h_2\partial_{z_2}\epsilon(z_2))\\ &\qquad+(\bar{\epsilon}(\bar{z}_1)\partial_{\bar{z}_1}+\bar{h}_1\partial_{\bar{z}_1}\bar{\epsilon}(z_1))+(\bar{\epsilon}(\bar{z}_2)\partial_{\bar{z}_2}+\bar{h}_2\partial_{\bar{z}_2}\bar{\epsilon}(z_2))\big]G^{(2)}(z_i,\bar{z}_i).\end{aligned}$$

Thus, we get

$$\begin{aligned}0 = \big[&(\epsilon(z_1)\partial_{z_1}+h_1\partial_{z_1}\epsilon(z_1))+(\epsilon(z_2)\partial_{z_2}+h_2\partial_{z_2}\epsilon(z_2))\\ &+\text{anti part}\qquad\big]G^{(2)}(z_i,\bar{z}_i).\end{aligned}$$

For an infinitesimal translation $\epsilon(z)=\epsilon_{-1},\ \bar{\epsilon}(\bar{z})=\bar{\epsilon}_{-1}$, the two-point function $G^{(2)}$ satisfies

$$(\partial_{z_1}+\partial_{z_2})G^{(2)} = (\partial_{\bar{z}_1}+\partial_{\bar{z}_2})G^{(2)} = 0, \tag{4.20}$$

from which we see that $G^{(2)}$ is the function of $z_{12}\equiv z_1-z_2$ and $\bar{z}_{12}\equiv\bar{z}_1-\bar{z}_2$. Then, for an infinitesimal dilatation $\epsilon(z)=\epsilon_0 z,\ \bar{\epsilon}(\bar{z})=\bar{\epsilon}_0\bar{z}$, $G^{(2)}$ satisfies

$$(h_1+z_1\partial_{z_1}+h_2+z_2\partial_{z_2})G^{(2)} = (\bar{h}_1+\bar{z}_1\partial_{\bar{z}_1}+\bar{h}_2+\bar{z}_2\partial_{\bar{z}_2})G^{(2)} = 0. \tag{4.21}$$

By using (4.20), (4.21) becomes

$$(h_1+h_2+z_{12}\partial_{z_1}G^{(2)} = (\bar{h}_1+\bar{h}_2+\bar{z}_{12}\partial_{\bar{z}_1}G^{(2)} = 0.$$

Solving this equation, we can get

$$G^{(2)}(z_i,\bar{z}_i) = \frac{C}{z_{12}^{h_1+h_2}\bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}. \tag{4.22}$$

Finally, the differential equations for an infinitesimal SCT $\epsilon(z)=\epsilon_1 z^2,\ \bar{\epsilon}(\bar{z})=\bar{\epsilon}_1\bar{z}^2$, are

$$(2h_1z_1+z_1^2\partial_{z_1}+2h_2z_2+z_2^2\partial_{z_2})G^{(2)} = (2\bar{h}_1\bar{z}_1+\bar{z}_1^2\partial_{\bar{z}_1}+2\bar{h}_2\bar{z}_2+\bar{z}_2^2\partial_{\bar{z}_2})G^{(2)} = 0. \tag{4.23}$$

Consider the holomorphic part. By using (4.20) and substituting (4.22) into (4.23), we can see that $G^{(2)}$ satisfies

$$(2h_1z_1+2h_2z_2-(h_1+h_2)(z_1+z_2))G^{(2)} = 0.$$

For this equation to hold for any z_1 and z_2, the following must be satisfied.

$$\begin{cases} 2h_1 = h_1 + h_2 \\ 2h_2 = h_1 + h_2 \end{cases} \implies h_1 = h_2.$$

In a similar way, we can get $\bar{h}_1 = \bar{h}_2$. Thus, we see that the two-point function is constrained to take the form

$$G^{(2)}(z_i, \bar{z}_i) = \begin{cases} \frac{C}{z_{12}^{2h} \bar{z}_{12}^{2h}} & \text{if } h_1 = h_2 \text{ and } \bar{h}_1 = \bar{h}_2, \\ 0 & \text{if } h_1 \neq h_2 \text{ or } \bar{h}_1 \neq \bar{h}_2. \end{cases} \tag{4.24}$$

4.5 Radial Quantization

We will begin with a flat two-dimensional Euclidean surface with the coordinates labeled by σ^0 for a time-like coordinate and σ^1 for a space-like coordinate. Then, the metric on the surface is given by

$$ds^2 = (d\sigma^0)^2 + (d\sigma^1)^2. \tag{4.25}$$

Let us consider the worldsheet of a free closed string (cylinder) parametrized by $\sigma^1 \in [0, 2\pi]$ and $\sigma^0 \in (-\infty, \infty)$. Then, we can regard this Euclidean surface as the product space $\mathbb{R} \times S^1$, where S^1 denotes a circle. Now, we define light-cone coordinates for this Euclidean surface by

$$\zeta, \bar{\zeta} = \sigma^0 \pm i\sigma^1,$$

which are Wick rotation of the light-cone coordinates used for Minkowski string worldsheet in the previous chapter. In terms of these coordinates, the metric becomes

$$ds^2 = d\zeta d\bar{\zeta}. \tag{4.26}$$

We can introduce the complex plane with coordinates z using ζ and $\bar{\zeta}$ via the map

$$\begin{aligned} \zeta &\to z = e^{2\zeta} = e^{2(\sigma^0 + i\sigma^1)} \\ \bar{\zeta} &\to \bar{z} = e^{2\bar{\zeta}} = e^{2(\sigma^0 - i\sigma^1)}, \end{aligned} \tag{4.27}$$

where the factor of 2 in the exponents reflects the earlier convention of choosing the periodicity of the closed-string parametrization to be $\sigma \to \sigma + \pi$. We can see, from the mapping, that the infinite past ($\sigma^0 = -\infty$) and future ($\sigma^0 = \infty$) of the cylinder are mapped to the point $z = 0$ and the point $z = \infty$, respectively. Also, equal time slices of the cylinder, i.e. the surface defined by $\sigma^0 = const$ and $\sigma^1 \in [0, \pi)$, become circles of constant radius $\exp 2\sigma^0$ in the complex plane (see Fig 4.1). Also, time translations, $\sigma^0 + a$ (a is a constant), are the dilatations in the complex plane. Recall that the Hamiltonian generates time translations. Thus, we can see that the dilatation generator on the complex plane corresponds to the Hamiltonian on the cylinder. Therefore, the Hilbert space defined on the cylinder is built up of constant time slices while the Hilbert space defined on the complex plane is built up of circles of constant radius. In the end,this procedure of quantizing a theory on a manifold where geometry is given by the complex plane is known as radial quantization.

4.6 Conserved Charges and Symmetry Generators

Before we begin to study the conserved charges on the complex z plane, we need to know the components of the metric and the stress energy tensor in the complex coordinates z and $\bar{z}$. The metric on the Euclidean surface (cylinder) was given by, in the light-cone coordinates, (4.26). According to (4.27), we see that

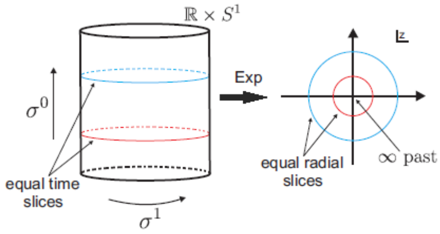

Figure 4.1: Mapping from the cylinder to the complex plane.

$\zeta = 1/2 \ln z$ and $\bar{\zeta} = 1/2 \ln \bar{z}$, which give us $ds^2 = \frac{1}{4}\frac{1}{|z|^2} dz d\bar{z}$. The scaling factor of $1/(4|z|^2)$ can be removed via a conformal transformation and since the theory which we are considering is a CFT, it should be invariant under this transformation. Therefore, we can take the metric on the complex plane to be

$$ds^2 = dz d\bar{z}, \tag{4.28}$$

from which we can find that the components of the metric in the complex coordinates z is given by

$$g_{zz} = g_{\bar{z}\bar{z}} = 0$$
$$g_{z\bar{z}} = g_{\bar{z}z} = 1/2.$$

Next, we would like to find the components of the stress energy tensor in terms of the complex coordinates z and $\bar{z}$. The components are given by

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \tag{4.29}$$

$$T_{\bar{z}\bar{z}z} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \tag{4.30}$$

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = \frac{1}{4}T^{\mu}{}_{\mu}. \tag{4.31}$$

Now, by the translational invariance, we have $\partial^{\nu} T_{\mu\nu} = 0$,[3] which implies that

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0 \tag{4.32}$$
$$\partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0. \tag{4.33}$$

Also, imposing dilatational invariance gives us that the stress energy tensor is traceless, $T^{\mu}_{\ \mu} = 0$.[4] This implies, from (4.31), that $T_{\bar{z}z} = T_{z\bar{z}} = 0$. Combining this result with (4.32) and (4.33) gives us

$$\partial_z T_{\bar{z}\bar{z}} = 0 \tag{4.34}$$
$$\partial_{\bar{z}} T_{zz} = 0, \tag{4.35}$$

[3] If the theory is invariant under an infinitesimal coordinate transformation, $x^{\mu} \to x^{\mu} + \epsilon^{\mu}$, then the corresponding conserved current is given by $j^{\mu} = T_{\mu\nu}\epsilon^{\nu}$, where $T_{\mu\nu}$ is the stress energy tensor. In particular, for translations along x^{α} by a, we have $\epsilon^{\mu}_{\alpha} = a\delta^{\mu}_{\alpha}$. Thus, the current is given by $j^{\mu\alpha} = aT_{\mu\alpha}$. If the theory is translationally invariant, this current gets conserved, $\partial^{\alpha} T_{\mu\alpha} = 0$.

[4] For dilatations (scaling), we have $\epsilon^{\mu} = bx^{\mu}$, where b is the constant of proportionality. Thus, the current corresponding to this transformation is $bT_{\mu\nu}x^{\nu}$. If the theory is invariant under dilatation, we have $\partial^{\mu} j_{\mu} = 0$, which implies $0 = \partial^{\mu}(bT_{\mu\nu}x^{\nu}) = bT_{\mu}{}^{\mu}$. The second equality, in fact, is not correct but it becomes correct in the sense that a total derivative vanishes in the action of the theory. Thus, in a conformally invariant theory, the stress energy tensor is traceless.

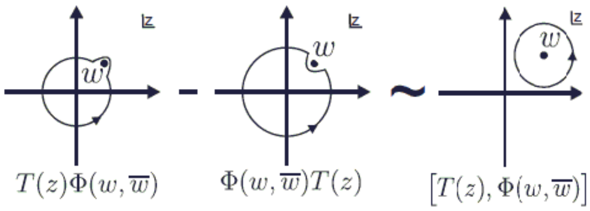

Figure 4.2: The illustration of the new contour $\mathcal{C}'$.

which tell us that $T_{\bar{z}\bar{z}}$ is a holomorphic function of $\bar{z}$ only and T_{zz} is an anti-holomorphic function of z,

$$T_{zz} \equiv T(z) \quad \text{and} \quad T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}),$$

which are the only non-vanishing components of the stress energy tensor for two-dimensional CFT.

Now, we are ready to study symmetries and their corresponding conserved charges for two-dimensional CFT. For an infinitesimal conformal transformation $\delta z = \epsilon(z)$ and $\delta \bar{z} = \bar{\epsilon}(\bar{z})$, the associated conserved charge is given by

$$Q = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(dz T(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \right), \tag{4.36}$$

where the contour $\mathcal{C}$ is over a circle in the complex plane. The variation of a field $\Phi(w, \bar{w})$ with respect to the above transformation is then given by the equal-radius commutator

$$\begin{aligned} \delta_{\epsilon\bar{\epsilon}}\Phi(w,\bar{w}) &\equiv [Q, \Phi(w,\bar{w})] \\ &= \frac{1}{2\pi i}\oint_{\mathcal{C}} dz\epsilon(z)\Big[T(z),\Phi(w,\bar{w})\Big] + (\text{the anti-holomorphic part}). \end{aligned} \tag{4.37}$$

Products of fields are only defined if we put them in radial order. Radial order is defined in analogy with time order in QFT as follows:

$$\mathcal{R}[A(z)B(w)] = \begin{cases} A(z)B(w) & \text{for } |w| < |z| \\ B(w)A(z) & \text{for } |z| < |w|, \end{cases} \tag{4.38}$$

where $\mathcal{R}$ is the radial ordering operator. Then, we can rewrite $\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w})$ in (4.37) as

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) &= \frac{1}{2\pi i}\left(\oint_{|w|<|z|} - \oint_{|z|<|w|}\right)\big(dz\epsilon(z)\mathcal{R}\big[T(z)\Phi(w,\bar{w})\big] + (\text{anti-holomorphic part})\big) \\ &= \frac{1}{2\pi i}\oint_{\mathcal{C}'}\Big(dz\epsilon(z)\mathcal{R}\big[T(z)\Phi(w,\bar{w})\big] + (\text{anti-holomorphic part})\Big), \end{aligned} \tag{4.39}$$

where the new contour $\mathcal{C}'$ is the contour enclosing the point w and is shown in Fig 4.2.

We know the infinitesimal transformation of the primary field under conformal transformations was given by (4.19). Then, by equating (4.19) with (4.39) and using the Cauchy-Riemann formula, we can infer any primary field must have the following radial-ordered operator product with $T(z)$.

$$\mathcal{R}\big[T(z)\Phi(w,\bar{w})\big] = \frac{h\Phi(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\Phi(w,\bar{w})}{z-w} + \text{regular terms} \tag{4.40}$$

$$\mathcal{R}\big[\bar{T}(\bar{z})\Phi(w,\bar{w})\big] = \frac{\bar{h}\Phi(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}}\Phi(w,\bar{w})}{\bar{z}-\bar{w}} + \text{regular terms}, \tag{4.41}$$

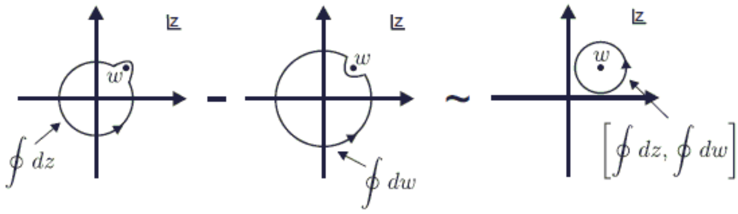

Figure 4.3: Subtraction of contours.

where $(h, \bar{h})$ are called the conformal weights of the primary field $\Phi(w, \bar{w})$. We have seen that the transformation property of primary fields leads to a short distance operator product expansion (OPE) for the holomorphic and anto-holomorphic stress energy tensor with the field Φ. From now on, we will drop the $\mathcal{R}$ symbol as a shorthand notation.

As an aside, we would like to relate OPEs to commutators here. Let $a(z)$ and $b(z)$ be two holomorphic fields and consider the integral $\oint_w dza(z)b(w)$, where the integration contour circles counterclockwise around w. This expression has an operator meaning within correlation functions as long as it is radially ordered. Accordingly, we split the contour into two fixed-time circles (see Fig 4.3.) going in opposite directions. Thus, the above integral is seen to be a commutator,

$$\oint_w dza(z)b(w) = \oint_{C_1} dza(z)b(w) - \oint_{C_2} dzb(w)a(z)$$
$$\equiv [A, b(w)], \tag{4.42}$$

where the operator A is the integral over space of fixed-time of the field $a(z)$, $A = \oint a(z)dz$ and C_1 and C_2 are fixed-time contours of radii respectively equal to $|w+\epsilon|$ and $|w-\epsilon|$, with ϵ being infinitesimal. The commutator obtained is then, in a sense, an equal time commutator.

The commutator $[A, B]$ of two operators, each of which is the integral of a holomorphic field, is obtained by integrating (4.42) over w,

$$[A, B] = \oint_o dw \oint_w dza(z)b(w), \tag{4.43}$$

where the integral over z is taken around w and the integral over w around the origin and $A = \oint a(z)dz$, $B = \oint b(z)dz$.

4.7 The Free Massless Bosonic Field

In this section, we will calculate the OPE's for some specific quantities in the free bosonic field theory. Then, we will show that the corresponding charges for the conserved current arising from global conformal transformations are the Virasoro generators L_m.

The action for the bosonic theory is given by

$$S = \frac{1}{2\pi} \int dzd\bar{z}\partial_z X(z, \bar{z})\partial_{\bar{z}} X(z, \bar{z}). \tag{4.44}$$

The equation of motion is given by

$$\partial_z \partial_{\bar{z}} X(z, \bar{z}) = 0, \tag{4.45}$$

from which, we can see that the field $X(z,\bar{z})$ decomposes into a holomorphic and anti-holomorphic part,

$$X(z,\bar{z}) = X(z) + \bar{X}(\bar{z}). \tag{4.46}$$

To find the propagator $\overbracket{X(z,\bar{z})X}(w,\bar{w})$ for this theory, we calculate the following equation.

$$\begin{aligned} 0 &= \frac{1}{Z}\int \mathcal{D}X \frac{\delta}{\delta X(z,\bar{z})}\left(e^{-S}X(w,\bar{w})\right) \\ &= \frac{1}{Z}\int \mathcal{D}X e^{-S}\left(\delta^{(2)}(z-w,\bar{z}-\bar{w}) + \frac{1}{2\pi}\partial_z\partial_{\bar{z}}X(z,\bar{z})X(w,\bar{w})\right). \end{aligned}$$

Thus, we have

$$0 = \left\langle \delta^{(2)}(z-w,\bar{z}-\bar{w})\right\rangle + \frac{1}{2\pi}\partial_z\partial_{\bar{z}}\,\overbracket{X(z,\bar{z})X}(w,\bar{w}).$$

By using $\partial_z(1/\bar{z}) = 2\pi\delta^{(2)}(z,\bar{z}) = \partial_{\bar{z}}(1/z)$, we can integrate the above equation to become

$$\begin{aligned} \overbracket{X(z,\bar{z})X}(w,\bar{w}) &= -\log|z-w|^2 \\ &= -\log(z-w) - \log(\bar{z}-\bar{w}). \end{aligned} \tag{4.47}$$

On the other hand,

$$\overbracket{X(z,\bar{z})X}(w,\bar{w}) = \overbracket{X(z)X}(w) + \overbracket{\bar{X}(\bar{z})X}(w) + \overbracket{X(z)\bar{X}}(\bar{w}) + \overbracket{\bar{X}(\bar{z})\bar{X}}(\bar{w}). \tag{4.48}$$

Since the correlation function of two dimensional CFTs has translation invariance, the contraction $\overbracket{X(z)\bar{X}}(\bar{w})$ is equal to some function of $z-\bar{w}$. Thus, we can infer that, compared with (4.47) and (4.48),

$$\overbracket{X(z)\bar{X}}(\bar{w}) = 0, \qquad \overbracket{\bar{X}(\bar{z})X}(w) = 0, \tag{4.49}$$

$$\overbracket{X(z)X}(w) = -\log(z-w), \qquad \overbracket{\bar{X}(\bar{z})\bar{X}}(\bar{w}) = -\log(\bar{z}-\bar{w}). \tag{4.50}$$

Next, we will define the stress energy tensor. From the action in (4.44), the holomorphic[5] part of it should be given by

$$\begin{aligned} T(z) &= -\frac{1}{2} :\partial_z X(z)\partial_z X(z): \\ &\equiv -\frac{1}{2}\lim_{z\to w}\left(\partial_z X(z)\partial_w X(w) - \text{singularity}\right), \end{aligned}$$

where the singularity is, from Wick's theorem, given by

$$\begin{aligned} \text{singularity} &= \partial_z\,\overbracket{X(z)\partial_w X}(w) = \partial_z\partial_w\left(\overbracket{X(z)X}(w)\right) \\ &= -\frac{1}{(z-w)^2}. \quad (\because (4.50)) \end{aligned} \tag{4.51}$$

[5]The anti-holomorphic part can inferred from the holomorphic part easily.

Thus, the holomorphic part of the stress energy tensor is defined to be

$$T(z) = -\frac{1}{2} : \partial_z X(z)\partial_z X(z) :\equiv -\frac{1}{2}\lim_{z\to w}\left(\partial_z X(z)\partial_w X(w) + \frac{1}{(z-w)^2}\right). \tag{4.52}$$

Finally, we can calculate the OPEs of some fields with the stress energy tensor to see whether or not these fields are primary in the free bosonic field theory.

- The OPE of $T(z)X(w)$

$$\begin{aligned} T(z)X(w) &= \frac{1}{2} : \partial_z X(z)\partial_z X(z) : X(w) \\ &= -\partial_z X(z)\partial_z \left(\langle X(z)X(w)\rangle\right) + \text{regular terms} \\ &= -\partial_z X(z)\partial_z\left(-\log(z-w)\right) + \text{regular terms} \\ &\sim \frac{\partial_z X(z)}{z-w}, \end{aligned}$$

where $\sim$ means that equivalence holds up to regular terms. By expanding $\partial_z X(z)$ around the point w, we get

$$\begin{aligned} T(z)X(w) &= \left(\partial_w X(w) + \partial_w^2 X(w)(z-w) + \cdots\right)\left(\frac{1}{z-w}\right) \\ &\sim \frac{\partial_w X(w)}{z-w}, \end{aligned}$$

which means that the field $X(w)$ is not a primary field[6] Therefore, $X(w,\bar{w})$ is also not a primary field.

- The OPE of $T(z)\partial_w X(w)$

$$\begin{aligned} T(z)\partial_w X(w) &= -\frac{1}{2} : \partial_z X(z)\partial_Z X(z) : \partial_w X(w) \\ &\sim -\partial_z X(z)\partial_z\partial_w\left(\langle X(z)X(w)\rangle\right) \\ &\sim \frac{\partial_z X z}{(z-w)^2} \\ &\sim \left(\partial_w X(w) + (z-w)\partial_w^2 X w + \cdots\right)\frac{1}{(z-w)^2} \\ &\sim \frac{\partial_w X(w)}{(z-w)^2} + \frac{\partial_w^2 X(w)}{z-w}, \end{aligned}$$

which means that $\partial_w X(w)$ is a primary field with the conformal weight $h = 1$. Thus, $T(z)\partial_w X(w,\bar{w})$ is also a primary field.

Here, as an aside, we would like to calculate the commutation relation of bosonic modes (3.17) in terms of OPE calculation. Generally, a primary field $\phi(z,\bar{z})$ of conformal weights $(h,\bar{h})$ may be

[6] Recall that the OPE of a primary field with the conformal weight h with the stress energy tensor was given by (4.40).

mode expanded as follows:

$$\phi(z,\bar{z}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n}, \tag{4.53}$$

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz z^{m+h-1}\frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+\bar{h}-1}, \tag{4.54}$$

where the second equation can be obtained by Cauchy's theorem. Thus, from the above result, we can see that $\partial_z X(z)$ is mode expanded as [7]

$$\partial_z X^\mu(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}\alpha_n^\mu, \tag{4.56}$$

$$\alpha_n^\mu = \frac{1}{2\pi i}\oint dz z^n \partial_x X^\mu(z). \tag{4.57}$$

By using the formula (4.43), we can finally calculate the commutator as follows.

$$\begin{aligned}
[\alpha_m^\mu, \alpha_n^\nu] &= \oint_0 \frac{dw}{2\pi i}\oint_w \frac{dz}{2\pi i}\partial_z X^\mu(z)\partial_w X^\nu(w) z^m z^n \\
&= \oint_0 \frac{dw}{2\pi i}\oint_w \frac{dz}{2\pi i}\frac{z^m w^n}{(z-w)^2}\eta^{\mu\nu} \qquad (\because (4.55)) \\
&= m\delta_{m+n,0}\eta^{\mu\nu},
\end{aligned}$$

which is the same as (3.17).

- The OPE of $T(z)T(w)$

$$\begin{aligned}
T(z)T(w) &= \frac{1}{4} :\partial_z X(z)\partial_z X(z) :: \partial_w X(w)\partial_w X(w): \\
&\sim \frac{1}{(z-w)^2} :\partial_z X(z)\partial_w X(w): + \frac{1/2}{(z-w)^4} \\
&\sim \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \frac{1/2}{(z-w)^4},
\end{aligned}$$

from which we see that the stress energy tensor is not strictly a primary field because of the anomalous term $(1/2)/(z-w)^4$. Suppose we have d scalar fields X^μ with $\mu = 0, \ldots, d-1$. Then, we get the following expression for $T(z)$,

$$T(z) = -\frac{1}{2} :\partial_z X^\mu(z)\partial_z X_\mu(z): . \tag{4.58}$$

The OPE of $T(z)T(w)$ can be obtained in a similar fashion,

$$T(z)T(w) = \frac{d/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}, \tag{4.59}$$

in which d is called the central charge, usually labeled by c. The central charge is the dimensionality of the spacetime.

[7] We will write this mode expansion in terms of d scalar fields in the Minkowski flat worldsheet. As a result, the OPE of the equation (4.51) is rewritten as

$$\partial_z X^\mu(z)\partial_w X^\nu(w) \sim \frac{\eta^{\mu\nu}}{(z-w)^2}, \tag{4.55}$$

which will be used for calculating th commutation relation.

4.8 Charges of the Conformal Symmetry Currents

The classical current corresponding to conformal symmetry was given by

$$j(z) = T(z)\epsilon(z), \tag{4.60}$$

where $\epsilon(z)$ is a holomorphic function. Similarly, the anti-holomorphic part of the current is $\bar{j}(\bar{z}) = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$. Thus, substituting (4.8) and (4.9) into the above $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$, the charges L_n and $\bar{L}_n$ corresponding to $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ are given by

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \quad \text{and} \quad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} \bar{T}(\bar{z}), \tag{4.61}$$

where the contour is taken to enclose the origin $z, \bar{z} = 0$. Then, by using the Cauchy's theorem, the mode expansions for $T(z)$ and $\bar{T}(\bar{z})$ are given by

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad \text{and} \quad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n. \tag{4.62}$$

Promoting these to operators, we will find the algebra of commutators satisfied by the operator L_n by using the formula (4.43).

$$\begin{aligned}
[L_m, L_n] &= \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} T(z)T(w) z^{m+1} w^{n+1} \\
&= \oint_o \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{m+1} w^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg} \right) \ \text{(use (4.59) with } d \to c) \\
&= \frac{1}{2\pi i} \oint_o dw w^{n+1} \left(\frac{1}{12} c(m+1)m(m-1) w^{m-2} + 2(m+1) w^m T(w) + w^{m+1} \partial_w T(w) \right) \\
&= \frac{1}{12} cm(m^2-1)\delta_{m+n,0} + 2(m+1) L_{n+m} - \frac{1}{2\pi i} \oint_o dw (m+n+2) w^{m+n+1} T(w) \\
&= \frac{1}{12} cm(m^2-1)\delta_{m+n,0} + (m-n) L_{m+n},
\end{aligned}$$

where, in the fourth line, the last term has been integrated by parts. In a similar way, we can get

$$\begin{aligned}
[L_m, \bar{L}_n] &= 0, \\
[\bar{L}_m, \bar{L}_n] &= \frac{1}{12} cm(m^2-1)\delta_{m+n,0} + (m-n)\bar{L}_{m+n}.
\end{aligned}$$

These commutators have the same algebraic structure obeyed by the Virasoro algebra.

4.9 Free Massless Fermion Field

The action for this theory is given by, in two component spinor $\Psi = (\psi, \bar{\psi})$,

$$S_F = \frac{1}{2\pi} \int dz d\bar{z} (\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi}). \tag{4.63}$$

The equations of motion, resulting from the above action, are given by

$$\partial_z \bar{\psi} = 0 \quad \text{and} \quad \partial_{\bar{z}} \psi = 0, \tag{4.64}$$

which tell us that the field ψ is holomorphic, while the field $\bar{\psi}$ is antiholomprphic. Therefore, we can only focus on one part, either holomorphic or antiholomorphic, since the other can be obtained by removing bars. Then, we will focus on the holomorphic part.

In order to derive the OPE of the fermionic field with itself, we begin with the partition function and the assumption that the path integral of a total functional derivative vanishes, just as we do for ordinary integrals. Thus, we can write

$$\begin{aligned}0 &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\frac{\delta}{\delta\psi(z)}(e^{-S_F}\psi(w)) \\ &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\Big[\left(\frac{\delta}{\delta\psi(z)}e^{-S_F}\right)\psi(w) + e^{-S_F}\left(\frac{\delta}{\delta\psi(z)}\psi(w)\right)\Big] \\ &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S_F}\left(-\frac{1}{2\pi}\partial_{\bar{z}}\psi(z)\psi(w) + \delta(z-w)\right),\end{aligned}$$

which implies that

$$\partial_{\bar{z}}\psi(z)\psi(w) = 2\pi\delta(z-w) = \partial_{\bar{z}}\left(\frac{1}{z-w}\right).$$

Integrating the both sides of the above equation, we obtain the OPE of the fermion with itself,

$$\psi(z)\psi(w) \sim \frac{1}{z-w}. \tag{4.65}$$

In a similar way we also get

$$\bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}. \tag{4.66}$$

Now that we have obtained the OPE of the fermionic field with itself, we want to calculate the OPE of the energy momentum tensor with the fermionic field and itself. The energy momentum tensor can be defined as

$$T_F(z) = -\frac{1}{2} :\psi(z)\partial_z\psi(z):, \tag{4.67}$$

where we have used the normal-ordered product,

$$:\psi(z)\partial_z\psi(w) \equiv \lim_{w\to z}\left(\psi(z)\partial_w\psi(w) - \overbrace{\psi(z)\partial_w\psi(w)}\right).$$

Then, the OPE between T_F and ψ can be calculated by using Wick's theorem:

$$\begin{aligned}T_F(z)\psi(w) &= -\frac{1}{2} :\psi(z)\partial_z\psi(z):\psi(w) \\ &= -(-1)\frac{1}{2} :\overbrace{\psi(z)\partial_z\psi(z):\psi(w)} - \frac{1}{2} :\psi(z)\partial_z \overbrace{\psi(z):\psi(w)} \\ &\sim \frac{\partial_z\psi(z)}{2(z-w)} + \frac{\psi(z)}{2(z-w)^2} \\ &\sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial_w\psi(w)}{z-w}.\end{aligned}$$

We see, from this result, that the fermionic field ψ is a primary field with a conformal weight $h = 1/2$. Thus, from (4.53) and (4.54), we can mode expand this ψ as

$$\psi^\mu(z) = \sum_{k\in\mathbb{Z}+a} z^{-k-1/2} b^\mu_k, \tag{4.68}$$

$$b^\mu_k = \frac{1}{2\pi i}\oint dz z^{k-1/2}\psi^\mu(z), \tag{4.69}$$

where we have introduced the parameter a to distinguish NS and R sectors. Integer modes ($a = 0$) corresponds to the R sectors and half-integer modes ($a = 1/2$) to the NS sectors, which will be discussed later in the next chapter. Also, note that, as we did before, we mode expanded ψ in terms of d fermions in the Minkowski flat worldsheet. As a result, the OPE of (4.65) can be rewritten as[8]

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}.$$

Then, by using the formula (4.43), we can derive the commutator which the modes b^μ_k obey.

$$\begin{aligned}
\{b^\mu_k, b^\nu_q\} &= \oint_0 \frac{dw}{2\pi i}\oint_w \frac{dz}{2\pi i}\psi^\mu(z)\psi^\nu(w) z^{k-1/2} w^{q-1/2} \\
&= \oint_0 \frac{dw}{2\pi i}\oint_w \frac{dz}{2\pi i}\frac{z^{k-1/2}w^{q-1/2}}{z-w}\eta^{\mu\nu} \\
&= \eta^{\mu\nu}\delta_{m+n,0},
\end{aligned}$$

which will be the same as the fermionic commutator we will see later although b will be replaced by d when k and q take integer values.

Next, the OPE of T_F with itself is calculated in the same way, and the result is given by

$$\begin{aligned}
T_F(z)T_F(w) &= \frac{1}{4} :\psi(z)\partial_z\psi(z)::\psi(w)\partial_w\psi(w): \\
&\sim \frac{\frac{1}{4}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}.
\end{aligned}$$

4.10 The Ghost System

The action for this theory is given by

$$S_{gh} = \frac{1}{2\pi}\int dz d\bar{z}(b\partial_{\bar{z}}c + \bar{b}\partial_z\bar{c}), \tag{4.70}$$

where b and c fields are called ghosts and are fermions. These fields arise from a change of variables in some functional integrals in the Faddeev-Popov analysis of the path integral.

This ghost action gives the following equations of motion,

$$\partial_z\bar{b} = \partial_{\bar{z}}b = \partial_z\bar{c} = \partial_{\bar{z}}c = 0, \tag{4.71}$$

which tells us that the fields b and c are holomorphic and the fields $\bar{b}$ and $\bar{c}$ are antiholomorphic. Thus, as usual, we will focus on the holomorphic part.

[8]Note that we, in addition to d fermions extension rescaled ψ to $i\psi$.

In order to calculate the OPE of b and c fields, we start with, as we did in the last section, the ghost partition function and assume that the path integral of a total functional derivative vanishes. Namely,

$$\begin{aligned} 0 &= \int \mathcal{D}b\mathcal{D}c \frac{\delta}{\delta b(z)} \left(e^{-S_{gh}} b(w) \right) \\ &= \int \mathcal{D}b\mathcal{D}c \left(-\frac{1}{2\pi} \partial_{\bar{z}} c(z) b(w) + \delta(z-w) \right), \end{aligned}$$

which implies that

$$\partial_{\bar{z}} c(z) b(w) = 2\pi\delta(z-w) = \partial_{\bar{z}} \left(\frac{1}{z-w} \right).$$

Integrating the both sides gives the OPE for $c(z)b(w)$,

$$c(z)b(w) \sim \frac{1}{z-w}. \tag{4.72}$$

In the above calculation, using $\delta/\delta c(z)$ instead of $\delta/\delta b(z)$, gives the OPE for $b(z)c(w)$,

$$b(z)c(w) \sim \frac{1}{z-w}, \tag{4.73}$$

from which we can immediately derive the following.

$$b(z)\partial_w c(w) \sim \frac{1}{(z-w)^2}, \tag{4.74}$$

$$\partial_z b(z) c(w) \sim -\frac{1}{(z-w)^2}. \tag{4.75}$$

The energy momentum tensor for this theory is defined as

$$T_{gh} = 2 : \partial_z c(z) b(z) : + : c(z) \partial_z b(z) : . \tag{4.76}$$

Thus, we can calculate the OPE of T_{gh} with the field $c(w)$ as follows.

$$\begin{aligned} T_{gh}(z)c(w) &= \left(2 : \partial_z c(z) b(z) : + : c(z) \partial_z b(z)\right) c(w) \\ &\sim \frac{2\partial_z c(z)}{z-w} - \frac{c(z)}{(z-w)^2} \\ &\sim \frac{-c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{z-w}. \end{aligned}$$

On the other hand, for the b field, we get

$$\begin{aligned} T_{gh}(z)b(w) &= \left(2 : \partial_z c(z) b(z) : + : c(z) \partial_z b(z)\right) b(w) \\ &\sim \frac{2b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{z-w}. \end{aligned}$$

From these results, we can see that $c(z)$ is a primary field of conformal weight $h = -1$, while $b(z)$ is a primary field of conformal weight $h = 2$.

Also, the expression for the OPE of the ghost energy momentum tensor with itself can be calculated in the exactly same way.

$$T_{gh}(z)T_{gh}(w) \sim \frac{-13}{(z-w)^4} + \frac{2T_{gh}(w)}{(z-w)^2} + \frac{\partial_w T_{gh}(w)}{z-w},$$

which implies that the central charge is $c_{gh} = -26$, which precisely cancels the conformal anomaly that arises from the matter energy momentum tensor T_X (, which is the bosonic energy momentum tensor here).

Chapter 5

Superstring Theories

The bosonic string theory which we discussed so far is unsatisfactory in two aspects. First, the string spectrum contains a tachyon. Tachyons are unphysical. The second is that the spectrum does not contain fermions, which include quarks and leptons in the standard model. Thus, if we want to describe nature by the string theory, we have to incorporate fermions into it.

The inclusion of fermions into the string theory turns out to be require supersymmetry, which is a symmetry that relates bosons $X^\mu(\tau,\sigma)$ to fermions $\Psi^\mu(\tau,\sigma)$. The latter are two-component spinors given by

$$\Psi^\mu(\tau,\sigma) = \begin{pmatrix} \psi^\mu_-(\tau,\sigma) \\ \psi^\mu_+(\tau,\sigma) \end{pmatrix}, \tag{5.1}$$

where we call $\psi^\mu_A(\tau,\sigma)$ $(A = \pm)$ the chiral components of the spinor. This resulting supersymmetric string theory is called superstring theory.

In this thesis, we will discuss superstring theory with RNS formalism.

- Ramond-Neveu-Schwarz (RNS) Formalism: It uses two-dimensional worldsheet supersymmetry and it requires the Gliozzi-Scherck-Olive (GSO) projection to remove unphysical states and make the theory supersymmetric..

5.1 RNS strings

In the RNS formalism, we add D free Majorana fermion fields[1] $\Psi^\mu(\tau,\sigma)$ to our D-dimensional bosonic string theory. The fields $\Psi^\mu(\tau,\sigma)$ are two-component spinors which describe fermions living on the worldsheet and they transform as vectors under a Lorentz transformation of the D-dimensional background spacetime. We incorporate these fermions by modifying the bosonic action.

The new action is now given by adding to the bosonic action S_B (see (2.32)) the Dirac action for D free massless fermions S_F, for the string tension $T = 1/\pi$,

$$\begin{aligned} S &= S_B + S_F \\ &= -\frac{1}{2\pi}\int d\tau d\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu - \frac{1}{2\pi}\int d\tau d\sigma \bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu, \end{aligned} \tag{5.2}$$

[1] A necessary condition for supersymmetric theory is that the number of bosonic degrees of freedom is equal to the number of fermionic degrees of freedom. Thus, we add D fermionic fields to pair up with the D bosonic fields.

where ρ^α with $\alpha = 0,\ 1$ is a two-dimensional representation of the Dirac algebra[2] and $\bar{\Psi}^\mu$ is the Dirac conjugate to Ψ^μ, defined by

$$\bar{\Psi}^\mu \equiv (\Psi^\mu)^\dagger\, i\rho^0. \tag{5.3}$$

We will choose a basis in which the matrices ρ^α take the form

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The above form of the Dirac matrices is called a Majorana representation. Also, in the Majorana representation, reality conditions on the spinors are imposed,

$$(\Psi^\mu)^T C = (\Psi^\mu)^\dagger\, i\rho^0, \tag{5.4}$$

where C is the charge conjugate matrix defined by $C \equiv i\rho^0$ in two dimensions, from which we can easily see that the reality condition (5.4) gives us the fact that the spinors have real components:

$$\left(\psi^\mu_\pm\right)^* = \psi^\mu_\pm, \tag{5.5}$$

which are called Majorana spinors.

Classically, the Majorana spinors are Grassman numbers, which imply that they obey the anti-commutation relations

$$\{\psi^\mu_A, \psi^\nu_B\} = 0. \tag{5.6}$$

This, of course, changes after quantization.

Now, substituting (5.1) into the action (5.2), we get, in world-sheet light-cone coordinates,

$$\begin{aligned} S = \frac{1}{\pi}\int d\sigma^+ d\sigma^- \partial_+ X^\mu(\sigma^-,\sigma^+)\partial_- X_\mu(\sigma^-,\sigma^+) \\ + \frac{1}{2\pi}\int d\sigma^+ d\sigma^- \Big(\psi^\mu_-(\sigma^-,\sigma^+)\partial_+\psi_{-\mu}(\sigma^-,\sigma^+) \\ + \psi^\mu_+(\sigma^-,\sigma^+)\partial_-\psi_{+\mu}(\sigma^-,\sigma^+)\Big). \end{aligned} \tag{5.7}$$

To see this expression is correct, we will show it only for the fermionic part. Since $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$, we have

$$\rho^\alpha \partial_\alpha = 2\begin{pmatrix} 0 & -\partial_- \\ \partial_+ & 0 \end{pmatrix}.$$

Next,

$$\Psi^\dagger i\rho^0 = i(\psi_+, -\psi_-),$$

where we used the fact that $\psi^*_A = \psi_A$ since Ψ is a Majorana spinor. Now, we need to calculate the Jacobian for the change of coordinates $(\tau,\sigma) \to (\sigma^-,\sigma^+)$. The Jacobian is given by $J = 1/2$. Thus, we have $d\tau d\sigma = J d\sigma^+ d\sigma^- = \frac{1}{2} d\sigma^+ \sigma^-$. Finally, substituting all of them into the fermionic action gives (5.7). By varying S_F, we see that the equations of motion for the two spinor components are given by the Dirac equation, which in the world-sheet light-cone coordinates, is given by

$$\partial_+\psi^\mu_- = 0 \quad \text{and} \quad \partial_-\psi^\mu_+ = 0, \tag{5.8}$$

which imply that the first equation describes a left-moving wave while the second one a right-moving wave.

[2] The Dirac matrices span a representation of a Clifford algebra, $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$, where $\{\ \cdot\ ,\ \cdot\ \}$ is the anti-commutator and $\eta^{\alpha\beta}$ is the Minkowskian flat metric

5.2 Global World-sheet Supersymmetry

The total action (5.7) has a global symmetry under the infinitesimal transformations

$$\delta X^\mu = \epsilon^\dagger i\rho^0\Psi^\mu \equiv \bar{\epsilon}\Psi^\mu, \tag{5.9}$$

$$\delta\Psi^\mu = \rho^\alpha\partial_\alpha X^\mu\epsilon, \tag{5.10}$$

where ϵ is a infinitesimal constant Majorana spinor, given by

$$\epsilon = \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix},$$

with the components ϵ_- and ϵ_+ being real Grassmann numbers. Since ϵ is not dependent on σ and τ, the transformations (5.9) and (5.10) are global symmetries and they are, in fact, supersymmetries since they mix the bosonic fields and fermionic fields. Thus, we can see that this RNS superstring theory is a theory which has supersymmetries on the world-sheet.

In terms of the spinor components, the transformations (5.9) and (5.10) are

$$\delta X^\mu = i(\epsilon_+\psi_-^\mu - \epsilon_-\psi_+^\mu), \tag{5.11}$$

$$\delta\psi_-^\mu = -2\partial_- X^\mu\epsilon_+, \tag{5.12}$$

$$\delta\psi_+^\mu = 2\partial_+ X^\mu\epsilon_-. \tag{5.13}$$

To see that the action (5.7) is invariant under the susy[3] transformations (5.9) and (5.10), let us see how the action varies under the transformations. The variation of the action is given by

$$\begin{aligned}\delta S = \frac{1}{\pi}\int d\sigma^+ d\sigma^- \Big(&2\partial_+(\delta X)\partial_- X + 2\partial_+ X\partial_-(\delta X) \\ &+ i(\delta\psi_-)\partial_+\psi_- + i\psi_-\partial_+(\delta\psi_-) \\ &+ i(\delta\psi_+)\partial_-\psi_+ + i\psi_+\partial_-(\delta\psi_+)\Big).\end{aligned} \tag{5.14}$$

Substituting (5.11) $\sim$ (5.13) into the above gives

$$\delta S = \frac{2i}{\pi}\int d\sigma^+ d\sigma^- \big\{\epsilon_+\left(\partial_+\partial_-(\psi_- X)\right) - \epsilon_-\left(\partial_+\partial_-(\psi_+ X)\right)\big\}.$$

Thus, if we assume that the boundary terms vanish, then we can finally see that under the susy transformations the RNS action (5.7) is invariant, which implies that there exists a supersymmetry in our theory.

5.3 Supercurrents and the Super-Virasoro Constraints

We will canonically quantize the RNS superstring and see that ghost states appear in the RNS theory. However, we will be able to eliminate them by using the super-Virasoro constraints which follow from the superconformal symmetry of the RNS theory, when the critical dimension is $D = 10$.

In order to derive the constraint equations, we begin with the two conserved currents associated to the two global symmetries of the RNS action (5.2). These two currents are the supercurrent, which comes form the supersymmetry of the action, and the stress-energy tensor, which comes from the translation symmetry of the action on the world-sheet. To begin, we will start with the supercurrent.

[3]Susy is a shorthand notation for supersymmetry or supersymmetric.

Since the supersymmetry is a global world-sheet symmetry, we get, via Noether's theorem, an associated conserved current, called the worldsheet supercurrent. The explicit form of the supercurrent is constructed as follows. By taking the supersymmetry spinor parameter ϵ to be dependent on world-sheet coordinates, we find that the total action (5.2) varies under this now local symmetry, as

$$\begin{aligned}\delta S &= \frac{1}{2\pi}\int d\tau d\sigma\Big[2\partial_\alpha(\delta X^\mu)\partial^\alpha X^\mu - (\delta\bar{\Psi}^\mu)\rho^\alpha\partial_\alpha\Psi_\mu + \bar{\Psi}^\mu\rho^\alpha\partial_\alpha(\delta\Psi_\mu)\Big]\\ &\sim \int d\tau d\sigma\left(\partial_\alpha\left(\frac{1}{2}\bar{\epsilon}\Psi^\mu\partial^\alpha X_\mu\right) - \partial_\alpha\bar{\epsilon}\left(\frac{1}{2}\rho^\beta\rho^\alpha\Psi_\mu\partial_\beta X^\mu\right)\right)\\ &= \int d\tau d\sigma\partial_\alpha\bar{\epsilon}\left(-\frac{1}{2}\rho^\beta\rho^\alpha\Psi_\mu\partial_\beta X^\mu\right),\end{aligned}$$

where we used the fact that a total derivative does not contribute to the variation of the action. Thus, we can see that the supercurrent is given by

$$J^\alpha_A = -\frac{1}{2}\left(\rho^\beta\rho^\alpha\Psi_\mu\right)_A\partial_\beta X^\mu, \tag{5.15}$$

where A indicates spinor components: $A \in \{+,-\}$. It can be shown that the supercurrent satisfies

$$(\rho_\alpha)_{AB}J^\alpha_B = 0, \tag{5.16}$$

where A and B indicate spinor components. This implies that the supercurrent has two independent components, labeled by j_- and j_+.

We, actually, want to have an expression for the supercurrent in terms of the worldsheet light-cone coordinates. Substituting the ϵ_- susy transformation[4] into (5.14), we get for the integrand

$$\begin{aligned}&2\partial_+(-i\epsilon_-\psi_+)\partial_-X + 2\partial_+X\partial_-(-i\epsilon_-\psi_+)\\ &\qquad + i(0)\partial_+\psi_- + i\psi_-\partial_+(0)\\ &\qquad\qquad + i(2\partial_+X\epsilon_-) + i\psi_+\partial_-(2\partial_+X\epsilon_-),\end{aligned}$$

which is equal to, up to a total derivative,

$$4i\epsilon_-\partial_-(\psi_+\partial_+X).$$

Substituting this back into (5.14) gives

$$\delta S = \frac{4i}{\pi}\int d\sigma^+d\sigma^-\epsilon_-\partial_-(\psi_+\partial_+X),$$

which becomes after integrating by parts,

$$\delta S = -\frac{4i}{\pi}\int d\sigma^+d\sigma^-(\partial_-\epsilon_-)(\psi_+\partial_+X).$$

Thus, choosing an appropriate normalization, the supercurrent associated with the ϵ_- transformation is given by

$$j_+ \equiv \psi^\mu_+\partial_+X_\mu. \tag{5.17}$$

[4] From (5.11) $\sim$ (5.13), we can easily read off the ϵ_- susy transformation as

$$\delta_-X^\mu = -i\epsilon_-\psi^\mu_+, \quad \delta_-\psi^\mu_- = 0, \quad \delta_-\psi^{|mu}_+ = 2\partial_+X^\mu\epsilon_-.$$

Similarly, the ϵ_+ transformation gives

$$j_- \equiv \psi_-^\mu \partial_- X_\mu. \tag{5.18}$$

It can be shown that the supercurrents are conserved as follows:

$$\begin{aligned}\partial_+ j_- &= \partial_+(\psi_-^\mu \partial_- X_\mu) \\ &= \partial_+ \psi_-^\mu \partial_- X_\mu + \psi_-^\mu \partial_+ \partial_- X_\mu \\ &= 0,\end{aligned}$$

where the last line follows by the field equations for ψ_- and X. Similarly, we can show that

$$\partial_- j_+ = 0.$$

Combining these results, we get

$$\partial_\alpha J_A^\alpha = 0, \tag{5.19}$$

which implies that the supercurrent is conserved.

The next current is the stress energy tensor and it is given by

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{1}{4}\bar{\Psi}^\mu \rho_\alpha \partial_\beta \Psi_\mu + \frac{1}{4}\bar{\Psi}^\mu \rho_\beta \partial_\alpha \Psi_\mu - (\text{trace}). \tag{5.20}$$

To derive this, let us consider an infinitesimal translation ϵ^α which is used to vary the world-sheet coordinates $\sigma^\alpha \to \sigma^\alpha + \epsilon^\alpha$. Then, we can write the change of the bosonic and fermionic fields by

$$X^\mu \to X^\mu + \epsilon^\alpha \partial_\alpha X^\mu, \tag{5.21}$$

$$\Psi^\mu \to \Psi^\mu + \epsilon^\alpha \partial_\alpha \Psi^\mu. \tag{5.22}$$

Following the Noether's method, we vary the action as if ϵ^α depended on the world-sheet coordinates. Since we have already considered the bosonic part, we will look at only the fermionic part here. Using (5.22), we vary S_F in (5.2) as follows:

$$\begin{aligned}\delta S_F &= -\frac{1}{2\pi}\int d\tau d\sigma \Big((\delta\bar{\Psi}^\mu)\rho^\alpha \partial_\alpha \Psi_\mu + \bar{\Psi}^\mu \rho^\alpha \partial_\alpha (\delta \Psi_\mu) \Big) \\ &= -\frac{1}{2\pi}\int d\tau d\sigma \Big(\epsilon^\beta \partial_\beta \bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi^\mu + \bar{\Psi}^\mu \rho^\alpha \partial_\alpha (\epsilon^\beta \partial_\beta \Psi^\mu) \Big) \\ &= -\frac{1}{2\pi}\int d\tau d\sigma \partial_\alpha \epsilon^\beta \left(\frac{1}{2}\bar{\Psi}^\mu \rho^\alpha \partial_\beta \Psi_\mu \right).\end{aligned}$$

The term in the parentheses should be the stress energy tensor. However, this is not correct because it should be symmetric. Thus, we take

$$\delta S_F = -\frac{1}{\pi}\int d\tau d\sigma \partial_\alpha \epsilon^\beta \left(\frac{1}{4}\bar{\Psi}^\mu \rho^\alpha \partial_\beta \Psi_\mu + \frac{1}{4}\bar{\Psi}^\mu \rho^\beta \partial_\alpha \Psi_\mu \right).$$

Together with the bosonic part, the stress energy tensor is given by (5.20).

We can rewrite this in terms of the world-sheet light-cone coordinates as

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\mu}, \tag{5.23}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{-\mu}, \tag{5.24}$$

$$T_{-+} = T_{+-} = 0. \tag{5.25}$$

Note that we can obtain conservation laws for the energy-momentum tensor: $\partial_- T_{++} = \partial_+ T_{--} = 0$. Equations (5.23) $\sim$ (5.25) can be obtained as follows. We consider the infinitesimal translation (5.21) and (5.22), focusing here on $\delta_+ X = \epsilon^+ \partial_+ X$ and $\delta_+ \Psi_A = \epsilon^+ \partial_+ \Psi_A$ since the ϵ^- transformation can be done in exactly the same way. From (5.14), the variation of the action under this transformation is given by

$$\begin{aligned}
\delta S &= \frac{1}{\pi} \int d\sigma^+ d\sigma^- \Big(2\partial_+(\delta_+ X)\partial_- X + 2\partial_+ X \partial_-(\delta_+ X) + i(\delta_+ \psi_-)\partial_+ \psi_- \\
&\qquad + i\psi_- \partial_+(\delta_+ \psi_-) + i(\delta_+ \psi_+)\partial_- \psi_+ + i\psi_+ \partial_-(\delta_+ \psi_+) \Big) \\
&= \frac{1}{\pi} \int d\sigma^+ d\sigma^- \epsilon^+ \Big(2\partial_+ \partial_+ X \partial_- X + 2\partial_+ X \partial_+ \partial_- X + i\partial_+ \psi_- \partial_+ \psi_- \\
&\qquad + i\partial_- \psi_+ \partial_+ \psi_- + i\partial_+ \psi_+ \partial_- \psi_+ + i\psi_+ \partial_+ \partial_- \psi_+ \Big) \\
&= \frac{1}{\pi} \int d\sigma^+ d\sigma^- \Big(-2\partial_-(\partial_+ X \partial_+ X) + i\partial_+(\psi_- \partial_+ \psi_-) - i\partial_-(\psi_+ \partial_+ \psi_+) \Big),
\end{aligned}$$

where the last line holds up to a total derivative. Identifying this with

$$-2\epsilon^+(\partial_- T_{++} + \partial_+ T_{-+})$$

gives us $T_{++} = \partial_+ X \partial_+ X + (i/2)\psi_+ \partial_+ \psi_+$ and $T_{-+} = -(i/2)\psi_- \partial_+ \partial_-$. However, T_{-+} vanishes by the equation of motion (5.8). Similarly, the ϵ^- variation gives (5.24).

Now, in the RNS theory, we set super-Virasoro constraints, which are given by

$$T_{++} = T_{--} = j_+ = j_- = 0. \tag{5.26}$$

In the next section, we will find the mode expansions of our fields.

5.4 Boundary Conditions and Mode Expansions

From now on, we will write our integration measure as $d^2\sigma = d\tau d\sigma$ instead of $d\sigma^+ d\sigma^-$ although we still write the integrand as functions of the world-sheet light-cone coordinates. Also, we will ignore the constant appearing due to the Jacobian of the coordinate transformation.

Since we have already studied the bosonic part, we will consider only the fermionic part here. Varying the fermionic action in (5.2) with respect to $\psi_\pm$ gives, if the equations of motion (5.8) are satisfied,

$$\delta S_F \sim \int d\tau (\psi_- \delta\psi_- - \psi_+ \delta\psi_+)\Big|_{\sigma=\pi} - \int d\tau (\psi_- \delta\psi_- - \psi_+ \delta\psi_+)\Big|_{\sigma=0}. \tag{5.27}$$

5.4.1 Open RNS Strings

In the case of open strings, the two boundary terms in (5.27) must vanish separately. We can achieve this by setting

$$\psi_+^\mu = \pm \psi_-^\mu \tag{5.28}$$

at each of the string's endpoints. The relative sign between ψ_+^μ and ψ_-^μ is a matter of convention. Thus, without loss of generality, we can choose to set

$$\psi_+^\mu \big|_{\sigma=0} = \psi_-^\mu \big|_{\sigma=0} = 0. \tag{5.29}$$

Then, the relative sign at the other end has meaning and there are two possible cases, which are called sectors.

Ramond Sector (R sector):

In this case, we choose, at the other end of the string,

$$\psi_+^\mu\big|_{\sigma=\pi} = \psi_-^\mu\big|_{\sigma=\pi}. \tag{5.30}$$

We will see later that the Ramond boundary condition (5.30) leads to fermions on the background spacetime. Then, because the field equations for the fermionic fields (5.8) implies that $\psi_- = \psi_-(\sigma^-)$ and $\psi_+ = \psi_+(\sigma^+)$. imposing the Ramond boundary condition gives us, for the mode expansion of the fermionic fields,

$$\psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)}, \tag{5.31}$$

$$\psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)}, \tag{5.32}$$

where the factor $1/\sqrt{2}$ has been chosen for future convenience. In addition, the Majorana condition of $\psi_\pm$ (5.5) gives us

$$d_{-n}^\mu = (d_n^\mu)^\dagger. \tag{5.33}$$

Neveu-Schwarz Sector (NS sector):

In this case, we choose, at the other end of the string,

$$\psi_+^\mu\big|_{\sigma=\pi} = -\psi_-^\mu\big|_{\sigma=\pi}. \tag{5.34}$$

The NS boundary condition will give rise to bosons on the background spacetime. In this sector, the mode expansion satisfying (5.34) is given by

$$\psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau-\sigma)}, \tag{5.35}$$

$$\psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau+\sigma)}, \tag{5.36}$$

In the following, the letters m and n are used for integers while r and s will denote half-integers.

5.4.2 Closed RNS Strings

Closed string boundary conditions give two sets of fermionic modes, corresponding to the left- and right-moving sectors. There are two possible periodicity conditions which make the boundary terms vanish:

$$\psi_\pm^\mu(\tau,\sigma) = \pm\psi_\pm^\mu(\tau,\sigma+\pi), \tag{5.37}$$

where the positive sign describes periodic boundary conditions (called R boundary conditions) while the negative sign describes anti-periodic boundary conditions (called NS boundary conditions). It is possible to impose either the R or the NS boundary conditions on the left- and the right-movers separately. This means that, for the right-movers, we can choose

$$\psi_-^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \quad \text{or} \quad \psi_-^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2in(\tau-\sigma)}, \tag{5.38}$$

while, for the left-movers, we can choose

$$\psi_+^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \bar{d}_n^\mu e^{-2in(\tau+\sigma)} \quad \text{or} \quad \psi_+^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{b}_r^\mu e^{-2in(\tau+\sigma)}, \tag{5.39}$$

Since real states are tensor products of a left- and right-mover, there are four different closed string sectors. We will see later that states in the NS-NS and R-R sectors are spacetime bosons, while states in the NS-R and R-NS sectors are spacetime fermions.

5.5 Canonical Quantization of the RNS superstring theory

By using the mode expansions, we will quantize the RNS theory.

We begin by promoting the modes α and $\bar{\alpha}$, which come from the bosonic fields and the modes b, $\bar{b}$, d and $\bar{d}$, which come from the fermionic fields to operators. The bosonic fields obey the same commutators as those we discussed already, which were given by (3.17). The fermionic fields obey the free Dirac equation on the worldsheet. As a result, the canonical anti-commutation relations are given by

$$\{\psi_A^\mu(\tau,\sigma), \psi_B^\nu(\tau,\sigma')\} = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma-\sigma'),$$

which implies that the modes satisfy

$$\{b_r^\mu, b_s^\nu\} = \{\bar{b}_r^\mu, \bar{b}_s^\nu\} = \delta_{r+s}\eta^{\mu\nu}, \tag{5.40}$$

$$\{d_m^\mu, d_n^\nu\} = \{\bar{d}_m^\mu, \bar{d}_n^\nu\} = \delta_{m+n}\eta^{\mu\nu}, \tag{5.41}$$

with the other anti-commutation relations vanishing. Since the spacetime metric $\eta^{\mu\nu}$ appears on the right hand side of the anti-commutators above, the time components of the fermionic oscillators give rise to ghost states, as well. However, we can remove these ghost states by using the super-Virasoro constraints (5.26). From here on, we will consider only the open string.

We will define the ground states of the RNS theory. Since we have two sectors, we are supposed to have two ground states, one for the R sector, labeled by $|0\rangle_R$, and one for the NS sector, labeled by $|0\rangle_{NS}$. They are defined by

$$\alpha_m^\mu|0\rangle_R = d_m^\mu|0\rangle_R = 0, \quad \text{for } m>0 \tag{5.42}$$

for the R sector and

$$\alpha_m^\mu|0\rangle_{NS} = b_r^\mu|0\rangle_{NS} = 0, \quad \text{for } m,r>0 \tag{5.43}$$

for the NS sector. The excited states are constructed by acting on the ground states with the negative mode oscillators. There are some important differences between the R ground state and the NS ground state.

5.5.1 R ground state and NS ground state

In the NS sector, the ground state is unique and it corresponds to a state of spin zero, i.e. a boson. Since all the oscillators α_n^μ and b_r^μ transform as spacetime vectors under a Lorentz transformation, the excited states in the NS sector will be spacetime bosons. Also, acting on a state with negative modes increases the mass of the state.

In the R sector, the ground state is degenerate, which can be understood as follows. The operators d_0^μ can act without effecting the mass of the state since they commute with the number operator N, which is defined by

$$N = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n + \sum_{n=1}^{\infty} n d_{-n}\cdot d_n, \tag{5.44}$$

whose eigenvalue determines the mass squared. Thus, $|0\rangle_R$ and $d_0^\mu|0\rangle_R$ are degenerate in mass. Now, from the oscillator algebra (5.41), we see that the d_0^μ obey the same algebraic relations as the Clifford algebra,up to a factor of 2:

$$\{d_0^\mu, d_0^\nu\} = \delta_{0,0}\eta^{\mu\nu} = \eta^{\mu\nu}.$$

Thus, since the above algebra is identical to the Dirac algebra, it implies that the set of degenerate ground states in the R sector must furnish a representation of the Dirac algebra. This implies that there is a set of degenerate ground states, which can be written in the form $|a\rangle$ with a being a spinor index, such that

$$d_0^\mu|a\rangle = \frac{1}{\sqrt{2}}\Gamma^\mu_{ba}|b\rangle, \tag{5.45}$$

where Γ^μ is an a dimensional matrix representation of d_0^μ, i.e. a Dirac matrix. Hence, the R sector ground state is a spacetime fermion. Now, since all of the oscillators α_n^μ and d_n^μ transform as spacetime vectors and since every state in the R sector can be obtained by acting with the negative modes on the ground state $|0\rangle_R$, we see that all the states in the R sector are spacetime fermions.

5.5.2 Super-Virasoro Constraints for Open Strings

The super-Virasoro generators are the modes of the stress energy tensor $T_{\alpha\beta}$ and the supercurrent J_A^α. Thus, for an open string, the super-Virasoro generators are given by

$$L_m = \frac{1}{\pi}\int_\pi^\pi d\sigma e^{im\sigma}T_{++} = L_m^{(b)} + L_m^{(f)}, \tag{5.46}$$

where the contribution from the bosonic modes was given by (**??**). Here, since we have two sectors for the fermionic modes of the RNS theory, we will get a different contribution to the super-Virasoro generators in each sector.

NS sector:

The contribution from the fermionic modes is given by

$$L_m^{(f)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(r + \frac{m}{2}\right) : b_{-r}\cdot b_{m+r} :, \quad m \in \mathbb{Z}, \tag{5.47}$$

and the modes of the supercurrent are

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^\pi e^{ir\sigma}j_+ = \sum_{n\in\mathbb{Z}}\alpha_{-n}\cdot b_{r+n}. \tag{5.48}$$

We can write the operator L_0 in the form

$$L_0 = \frac{1}{2}\alpha_0^2 + N, \tag{5.49}$$

where N is the number operator, defined by

$$N \equiv \sum_{n=1}^\infty \alpha_{-n}\cdot\alpha_n + \sum_{r=1/2}^\infty rb_{-r}\cdot b_r, \tag{5.50}$$

whose eigenvalues determine the mass squared of an excited state as in the bosonic theory.

R sector:

The contribution from the fermionic modes is given by

$$L_m^{(f)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(n+\frac{m}{2}\right) : d_{-n}\cdot d_{m+n} :, \quad m\in\mathbb{Z}, \tag{5.51}$$

and the modes of the supercurrent are given by

$$F_m = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} e^{im\sigma} j_+ = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot d_{m+n}. \tag{5.52}$$

Note that there is no normal-ordering ambiguity in the definition of F_0.

We will get a super-Virasoro algebra whose elements consist of the super-Virasoro generators and the modes of the supercurrent as we did in the bosonic theory. Since we have two different expressions for the super-Virasoro generator and the supercurrent, corresponding to the two sectors, we will get two sets of super-Virasoro algebras.

NS Sector:

The super-Virasoro algebra is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \tag{5.53}$$

$$[L_m, G_r] = \left(\frac{m}{2}-r\right)G_{m+r}, \tag{5.54}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}\left(4r^2-1\right)\delta_{r,-s}, \tag{5.55}$$

where the central charge is related to the spacetime dimension by $c = D + D/2$.

R Sector:

The super-Virasoro algebra is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m,-n}, \tag{5.56}$$

$$[L_m, F_n] = \left(\frac{m}{2}-n\right)F_{m+n}, \tag{5.57}$$

$$\{F_m, F_n\} = 2L_{m+n} + \frac{D}{2}m^2\delta_{m,-n}. \tag{5.58}$$

5.5.3 Physical State Condition

In the bosonic string theory, the physical states were characterized as such states $|\phi\rangle$ that

$$L_m^{(b)}|\phi\rangle = 0, \tag{5.59}$$

$$(L_0^{(b)} - a)|\phi\rangle = 0. \tag{5.60}$$

In the RNS superstring theory, we have analogous conditions. We will again get two different physical state conditions corresponding to two different sectors, i.e. quantum versions of $T(z) = 0$.

NS Sector:

The physical state condition is characterized as follows. If $|\phi\rangle$ is a physical state, then it must satisfy

$$L_m|\phi\rangle = 0 \quad m > 0, \tag{5.61}$$
$$G_r|\phi\rangle = 0 \quad r > 0, \tag{5.62}$$
$$(L_0 - a_{NS})|\phi\rangle = 0, \tag{5.63}$$

where a_{NS} is a constant which arises due to the normal-ordering ambiguity of L_0. The last of these conditions implies that $\alpha' M^2 = N - a_{NS}$, where M is the mass of a state $|\phi\rangle$ and N is the eigenvalue of the number operator on the state $|\phi\rangle$.

R Sector:

The physical state condition is as follows.

$$L_m|\phi\rangle = 0 \quad m > 0, \tag{5.64}$$
$$F_n|\phi\rangle = 0 \quad n \geq 0, \tag{5.65}$$
$$(L_0 - a_R)|\phi\rangle = 0, \tag{5.66}$$

where a_R is a constant which arises due to the normal-ordering ambiguity of L_0.

5.5.4 Removing the Ghost States

Let us consider a few examples of zero-norm spurious states to calculate the values of a_{NS}, a_R and D. Recall that these are states that are orthogonal to physical states and decouple from the theory even though they satisfy physical conditions.

NS Sector:

To begin, we consider a state of the form

$$|\phi\rangle = G_{-1/2}|\chi\rangle,$$

where $|\chi\rangle$ satisfies the conditions

$$L_{m>0}|\chi\rangle = 0, \tag{5.67}$$
$$G_{1/2}|\chi\rangle = G_{3/2}|\chi\rangle = (L_0 - a_{NS} + \frac{1}{2})|\chi\rangle = 0, \tag{5.68}$$

where the last equality follows from (5.63).[5] It is, therefore, sufficient to show that $G_{1/2}|\phi\rangle = G_{3/2}|\phi\rangle = 0$ in order for $|\phi\rangle$ to be physical.[6] Since the $G_{3/2}$ condition obviously holds by the corresponding condition

[5] This is because of the fact that $G_{-1/2}$ raises the eigenvalue of L_0 by 1/2.

[6] This is because all G_r for $r > 3/2$ can be written in terms of the generators $L_{m>0}$, $G_{1/2}$ and $G_{3/2}$. For example, we can get, from (5.54),

$$G_{5/2} = G_{1+3/2} = \frac{1}{1/2 - 3/2}[L_1, G_{3/2}] = -[L_1, G_{3/2}].$$

for $|\chi\rangle$, we only have to check the $G_{1/2}$ condition. This is given by $G_{1/2}|\phi\rangle = G_{1/2}G_{-1/2}|\chi\rangle$. Now, since $G_{1/2}G_{-1/2} = \{G_{-1/2}, G_{1/2}\} - G_{-1/2}G_{1/2}$ and $\{G_{-1/2}, G_{1/2}\} = 2L_0$, we can see that $G_{1/2}|\phi\rangle$ becomes

$$\begin{aligned} G_{1/2}|\phi\rangle &= \left(2L_0 - G_{-1/2}G_{1/2}\right)|\chi\rangle = 2L_0|\chi\rangle \\ &= 2(a_{NS} - 1/2)|\chi\rangle. \end{aligned}$$

For this to vanish, we need $a_{NS} = 1/2$. This choice gives a family of zero-norm spurious state $|\phi\rangle$. Such a state satisfies the conditions for physical state conditions with $a_{NS} = 1/2$. Moreover, $|\phi\rangle$ is orthogonal to all physical states $|\alpha\rangle$, since $\langle\alpha|\phi\rangle = \langle\alpha|G_{-1/2}|\chi\rangle = \langle\chi|G_{1/2}|\alpha\rangle^* = 0$. Therefore, for $a_{NS} = 1/2$, these are zero-norm spurious states.

In order to calculate the critical dimension, let us consider a second class of zero-norm spurious state

$$|\phi\rangle = (G_{-3/2} + \lambda G_{-1/2}L_{-1})|\chi\rangle. \tag{5.69}$$

Also, suppose that

$$G_{1/2}|\chi\rangle = G_{3/2}|\chi\rangle = (L_0 + 1)|\chi\rangle = 0. \tag{5.70}$$

Next, we need to show, as before, $G_{1/2}|\phi\rangle = 0$ and $G_{3/2}|\phi\rangle = 0$. The $G_{1/2}$ condition becomes, by using (5.53) $\sim$ (5.55), $G_{1/2}|\phi\rangle = (2 - \lambda)L_{-1}|\chi\rangle$. For this to vanish, we need $\lambda = 2$. Then, the $G_{3/2}$ condition becomes, in a similar way, $G_{3/2}|\phi\rangle = (D - 2 - 4\lambda) = (D - 10)|\chi\rangle$. Thus, for $G_{3/2}$ to annihilate $|\phi\rangle$, we need to have the critical dimension $D = 10$.

R Sector:

We do not need to use spurious states to calculate the a_R value. we do this as follows. From (5.58), if $m = n = 0$, we obtain $L_0 = F_0^2$. Then, from (5.65) we obtain $F_0|\phi\rangle = 0$. Acting on this equation with F_0 gives $F_0(F_0|\phi\rangle) = F_0^2|\phi\rangle = L_0|\phi\rangle = 0$. However, we have (5.66), which leads to $a_R = 0$.

In order to calculate the critical dimension, consider the state

$$|\phi\rangle = F_0F_{-1}|\chi\rangle, \tag{5.71}$$

where $|\chi\rangle$ satisfies

$$F_1|\chi\rangle = (L_0 + 1)|\chi\rangle = 0. \tag{5.72}$$

Also, $F_0|\phi\rangle = 0$ holds. If the state $|\phi\rangle$ is also annihilated by L_1, then it is a zero-norm physical state. Thus, we have $L_1|\phi\rangle = (1/2F_1 + F_0F_1)F_{-1}|\chi\rangle = 1/4(D - 10)|\chi\rangle$. For this to vanish, we need $D = 10$.

5.6 Light-Cone Quantization

We imposed the light-cone gauge condition of (3.39) since we had a residual symmetry after fixing the gauge symmetry. Now, this is also true for the RNS theory. We have a residual fermionic symmetry, which will allow us to impose more conditions on our RNS theory. Namely, we will set, in addition to (3.39),

$$\Psi^+(\tau, \sigma) = 0. \tag{5.73}$$

Note that, in the R sector, we should keep the zero mode. In the light-cone gauge, the coordinates X^- and Ψ^-, due to the super-Virasoro constraints, are not independent degrees of freedom. This implies that all the physical states are given by acting with the transverse raising modes of the bosonic and fermionic fields.

5.6.1 Open RNS String Mass Spectrum

Let us analyze some open RNS superstring states in the light-cone gauge. As usual, the mass spectrum depends on the two different sectors.

- **NS Sector:**

The mass formula is given by

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i - \frac{1}{2}, \tag{5.74}$$

where we substituted $a_{NS} = 1/2$.

Ground State:

The NS ground state is annihilated by all positive modes,

$$\alpha_n^i |0; k^\mu\rangle_{NS} = b_r^i |0; k^\mu\rangle_{NS} = 0, \tag{5.75}$$

and

$$\alpha_0^\mu |0; k^\mu\rangle_{NS} = \sqrt{2\alpha'} k^\mu |0; k^\mu\rangle_{NS}, \tag{5.76}$$

where $\sqrt{2\alpha'}$ is from normalization. Calculating the mass of the NS ground state, we get, by using (5.75),

$$\alpha' M^2 |0; k^\mu\rangle_{NS} = -\frac{1}{2} |0; k^\mu\rangle_{NS}.$$

Thus, the mass is given by $\alpha' M^2 = -1/2$, which implies that the ground state is a tachyon. We will see later that there is a way to consistently truncate this state from the spectrum.

First Excited State:

The first excited state is given by

$$b_{-1/2}^i |0; k^\mu\rangle_{NS}. \tag{5.77}$$

Since this operator is a vector in spacetime and it is acting on a bosonic ground state, which is a spacetime scalar, the resulting state is a spacetime vector. Also, now we are working in the light-cone gauge, so the label i runs from 1 to $D - 2 = 8$, corresponding to the transverse directions. Thus, the first excited state has a total of 8 polarizations, which is required for a massless vector in ten dimensions. To see that this state is indeed massless, note that the mass of the state is given by $\alpha' M^2 = 1/2 - \alpha_{NS}$. Since $a_{NS} = 1/2$, the state is massless.

- **R Sector:**

The mass formula is given by

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{n=1}^{\infty} n d_{-n}^i d_n^i. \tag{5.78}$$

Ground State:

The R ground state satisfies

$$\alpha_n^i|0;k^\mu\rangle_R = d_n^i|0;k^\mu\rangle_R = 0 \quad n>0, \tag{5.79}$$

and

$$F_0|0;k^\mu\rangle_R = 0. \tag{5.80}$$

Calculating the mass squared of the R ground state gives $\alpha' M^2|0;k^\mu\rangle_R = 0$, which implies that the R ground state is massless. Then, the equation (5.80) implies, from (5.52), that

$$\begin{aligned} 0 &= \left(\alpha_0^i d_0^i + \sum_{n=1}^{\infty}(\alpha_{-n}^i d_n^i + d_{-n}^i \alpha_n^i)\right)|0;k^\mu\rangle_R \\ &= d_0^i\sqrt{2\alpha'}k^i)|0;k^\mu\rangle_R \\ &\propto d_0^i k^i|0;k^\mu\rangle_R, \end{aligned}$$

which is the Dirac equation in the momentum representation. As was mentioned earlier, the R ground state is not unique due to the fact that the zero modes satisfy a $D=10$ dimensional Dirac algebra. This implies that the ground state is a $spin(9,1)$ spinor. The operation of multiplying with the operator d_0^μ is the nothing more that multiplying with a 10 dimensional Dirac matrix. This, in turn, implies that the R ground state is a spinor with 32 components.[7] However, in 10 dimensions, it is known that we can impose the Majorana condition and the Weyl condition on spinors, which reduce the number of independent components by 1/2. As a result, our ground state is described by a spinor with 16 independent components. But, note that this spinor satisfies the Dirac equation, so that the number of independent components reduces to 8. Thus, the R ground state has 8 degrees of freedom corresponding to an irreducible spinor of $spin(8)$.

As was mentioned before, the states in the R sector correspond to fermions in the background spacetime. Also, since the first excited state of the NS sector is a bosonic state with 8 degrees of freedom and the R ground state is a fermionic state with 8 degrees of freedom, if we could shift the NS first excited state back to its ground state, then we could think of the total open RNS string theory as a hopefully supersymmetric theory which have 8 massless bosons and 8 massless fermions in the background spacetime. However, we still would have a problem even if we could do this. The problem is that the NS ground state is a tachyon and there is nothing corresponding to this in the R sector.

However, we will see later that we can impose a further condition on our states, called the GSO projection, which removes the NS tachyon state and shift the NS first excited state to the NS ground state.

First Excited State:

The excited states are obtained by acting on the R ground state with α_{-n}^i or d_{-n}^i. Since these operators are spacetime vectors, the resulting states are spacetime spinors. The possibilities for excited states are further restricted by the GSO projection, which will be discussed now.

[7]It is known that a spinor in a $2k$-dimensional space has 2^k components.

5.6.2 GSO projection

In the last section, we have seen that the spectrum of RNS string states has several problems. The NS ground state is a tachyon and also the spectrum is not spacetime supersymmetric.

We will explain here how to make the RNS string theory a consistent theory, by truncating the spectrum in a very specific way that eliminates the tachyon and gives supersymmetic theory in ten dimensional spacetime. This is called the GSO projection.

In order to describe the truncation of the spectrum, first let us define operators which count the number of b-oscillators in the NS sector, and d-oscillators in the R sector, which are given by, respectively,

$$F_{NS} = \sum_{r=1/2}^{\infty} b^i_{-r} b^i_r, \tag{5.81}$$

$$F_R = \sum_{n=1}^{\infty} d^i_{-n} d^i_n. \tag{5.82}$$

Then, by using these operators, we can construct new operators, called the G-parity operators, which is given by, in the NS sector,

$$G = (-1)^{F_{NS}+1} = (-1)^{\sum_{r=1/2}^{\infty} b^i_{-r} b^i_r + 1}. \tag{5.83}$$

In the R sector, we have

$$G = \Gamma_{11}(-1)^{F_R} = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d^i_{-n} d^i_n}, \tag{5.84}$$

where Γ_{11} is defined by

$$\Gamma_{11} \equiv \Gamma_0 \Gamma_1 \cdots \Gamma_9, \tag{5.85}$$

which can be thought of as the ten dimensional analog of the γ_5 Dirac matrix in four dimensions. Thus, just as the γ_5 matrix, the Γ_{11} satisfies the following relations.

$$(\Gamma_{11})^2 = 1 \quad \text{and} \quad \{\Gamma_{11}, \Gamma^\mu\} = 0. \tag{5.86}$$

Spinors Ψ^μ which satisfy

$$\Gamma_{11}\Psi^\mu = \Psi^\mu, \tag{5.87}$$

are said to have positive chirality, while spinors which satisfy

$$\Gamma_{11}\Psi^\mu = -\Psi^\mu, \tag{5.88}$$

are said to have negative chirality. A spinor with a definite chirality is called a Weyl spinor.

Now, in the NS sector, we will impose the GSO projection which consists of keeping only the states with a positive G-parity, i.e. we keep only the states $|\Omega\rangle$ such that

$$G|\Omega\rangle = (-1)^{F_{NS}+1}|\Omega\rangle = |\Omega\rangle,$$

which implies that we have

$$1 = (-1)^{F_{NS}+1} = (-1)^{F_{NS}}(-1).$$

For this to hold, F_{NS} has to take an odd number. Thus, in the NS sector, we keep only the states with an odd number of b oscillator excitations. In the R sector, we can project on states with positive or negative G-parity depending on the chirality of the spinor ground state. The choice is purely a matter of convention.

The GSO projection eliminates the open string tachyon from the spectrum since it has negative G-parity, $G|0\rangle_{NS} = -|0\rangle_{NS}$. The first excited state $b^i_{-1/2}|0\rangle_{NS}$ has positive G-parity and survives the projection. After the GSO projection, this massless vector boson becomes the ground state of the NS sector. This matches nicely with the fact that the ground state in the fermionic sector (the R sector) is a massless spinor. Also, the ground state of the NS sector after the GS projection is $b^i_{-1/2}|0\rangle_{NS}$, which has only eight degrees of freedom. This matches the number of degrees of freedom for the fermionic ground state. This is a first indication that the spectrum could be spacetime supersymmetric after the GSO projection.

We will now show that there are the same number of physical degrees of freedom in the NS sector and the R sector at the first massive level after performing the GSO projection. To begin, note that at this level, we have $N = 3/2$ for the NS sector states and $N = 1$ for the R sector states. Also, the G-parity constraint in the NS sector requires the states to have an odd number of b oscillator excitations, while in the R sector the constraint correlates the number of d oscillator excitations with the chirality of the spinor. Now, in the NS sector, the states which survive the GSO projection, at this level, are given by

$$\alpha^i_{-1}b^j_{-1/2}|0;k^\mu\rangle_{NS}, \quad b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}|0;k^\mu\rangle_{NS}, \quad b^i_{-3/2}|0;k^\mu\rangle_{NS}.$$

Counting the number of these states gives us a total of $64 + 56 + 8 = 128$. For the R sector, we have the allowed states,

$$\alpha^i_{-1}|\psi_0;k^\mu\rangle, \quad d^i_{-1}|\psi'_0;k^\mu\rangle,$$

where $|\psi_0;k^\mu\rangle$ and $|\psi'_0;k^\mu\rangle$ denote a pair of Majorana-Weyl spinors of the R ground state of opposite chirality and each has 16 real components. However, note that they have only 8 degrees of physical freedom since they must satisfy the Dirac equation. Thus, counting the number of independent states in the R sector gives $64 + 64 = 128$ As a result, for the first massive excited state we see that after the GSO projection, both the NS and R sector have the same degrees of freedom.

5.6.3 Closed RNS String Spectrum

The closed string has left- and right-mover. Also, there is the possibility that each mover has either NS or R boundary conditions, which implies that we must consider the four possible sectors: R-R, R-NS, NS-R and NS-NS. As before, by projecting onto states with a positive G-parity in the NS sector, we can remove the tachyon state. For the R sector, we can project onto states with positive or negative G-parity depending on the chirality of the ground state. Thus, two different theories, the type IIA and the type I IB superstring theory, can be obtained depending on whether the G-parity of the left- and right-moving R sector is the same or opposite.

In the type IIB theory, the left- and right-moving R ground states have the same chirality, chosen to be positive for the definiteness. Therefore, the two R sectors have the same G-parity. Let us denote each of them by $|+\rangle_R$. With these considerations, the massless states in the type IIB closed string spectrum are

$$\begin{gathered} |+\rangle_R \otimes |+\rangle_R, \\ \bar{b}^i_{-1/2}|0\rangle_{NS} \otimes b^i_{-1/2}|0\rangle_{NS}, \\ \bar{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R, \\ |+\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS}. \end{gathered}$$

Since the state $|+\rangle_R$ is a spinor with eight components, we see that each of the four sectors contains $8 \times 8 = 64$ states.

In the type IIA theory, the left- and right-moving R ground states have opposite chirality, which we labeled as $|+\rangle_R$ and $|-\rangle_R$. The massless states in the type IIA closed string spectrum are given by

$$
\begin{gathered}
|-\rangle_R \otimes |+\rangle_R, \\
\bar{b}^i_{-1/2}|0\rangle_{NS} \otimes b^i_{-1/2}|0\rangle_{NS}, \\
\bar{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R, \\
|-\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS}.
\end{gathered}
$$

We can see that each of the four sectors have 64 states as in the type IIB theory.

The different types of states in the massless sectors of the two theories are summarized as follows.

- R-R Sector: These states are bosons obtained by performing the tensor product of a pair of Majorana-Weyl spinors. In the type IIA, the two Majorana-Weyl spinors have opposite chirality, and we obtain a one-form (vector) gauge field (8 states) and a three-form gauge field (56 states). In the type IIB, the two Majorana-Weyl spinors have the same chirality, and we obtain a zero-form (scalar) gauge field (1 state), a two-form gauge field (28 states) and a four-form gauge field with a self-dual field strength (35 states).

- NS-NS Sector: This sector is the same for the type IIA and type IIB. We obtain a scalar called dilaton (1 state), an antisymmetric two-form gauge field (28 states) called the Kalb-Ramond firld and a symmetric traceless rank 2 tensor called the graviton (35 states).

- NS-R and R-NS Sector: Each spectrum of these sectors contains a spin 3/2 gravitino (56 states) and a spin 1/2 fermion called the dilatino (8 states). In the type IIB, the two gravitino have the same chirality, while in the type IIA, opposite chirality.

Chapter 6

T-duality and Dp-brane

6.1 T-duality and Closed Bosonic Strings

We will consider the bosonic string theory with one of its spatial dimensions compactified. Namely, we assume that this spatial dimension has a periodic boundary condition. This implies that our background spacetime is topologically equivalent to the space given by Cartesian product of 25-dimensional Minkowski spacetime and a circle of radius R, i.e. $\mathbb{R}^{24,1} \times S^1_R$. We describe this procedure as compactification on a circle of radius R. We will choose to compactify the $X^{25}(\tau, \sigma)$ coordinate. We can now think of our background spacetime as that given in Fig 6.1, We will study the changes in the bosonic string theory by introducing this compactification.

In the non-compact theory, which we have discussed so far, closed strings satisfied the periodic boundary condition (2.34), which was stated with the implicit assumption where the string was moving in a spacetime with non-compact dimensions, but now in our modified situation, we let the 25th dimension be a circle of radius R. This changes the boundary condition (2.34) as follows, but only for X^{25}:

$$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R W, \tag{6.1}$$

$$X^i(\tau, \sigma + \pi) = X^i(\tau, \sigma) \quad i = 0, \ldots, 24 \tag{6.2}$$

where W is called the winding number.[1] The equation (6.1) implies that now the string has winding states. Simply put, the string can wind around the compactified dimension any number of times.

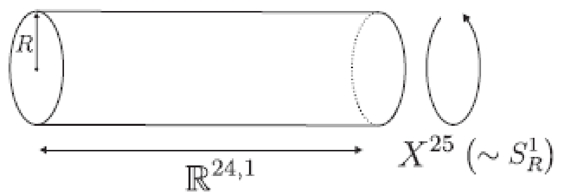

Figure 6.1: A compactified background spacetime.

[1]The winding number indicates how many times the string winds around the circle and its sign encodes the direction. For example, if we take each counterclockwise winding to be $+1$, then each clockwise winding is -1.

6.1.1 Mode Expansion for the Compactified Dimensions

The modified boundary condition (6.1) gives a modified mode expansion on X^{25}, while the mode expansions for X^i remain unchanged. The general solution for new boundary conditions for X^{25} is given by adding a term linear in σ to (2.54) in order to incorporate the boundary condition (6.1),

$$X^{25}(\tau,\sigma) = x^{25} + 2\alpha' p^{25}\tau + 2RW\sigma + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}(\alpha_n^{25}e^{2in\sigma} + \bar{\alpha}_n^{25}e^{-2in\sigma})e^{-2in\tau}. \tag{6.3}$$

Also, since one dimension is compact, the momentum eigenvalue along that direction, p^{25}, has to be quantized because, from quantum mechanics, we know that the wave function contains the factor $e^{ip^{25}x^{25}}$. As a result, if we increase x^{25} by the amount $2\pi R$, which corresponds to a winding number $W = 1$, the wave function should be mapped back to the initial value, i.e. the wave function should be single-valued on the circle. This implies that the momentum in the 25th direction has to be of the form

$$p^{25} = \frac{K}{R}, \quad K \in \mathbb{Z}, \tag{6.4}$$

where K is called the Kaluza-Klein excitation number. Thus, without the compactified dimensions, the center of mass momentum of the string was arbitrary, while compactifying one of the dimensions quantizes the center of mass momentum along that direction.

Now, splitting the expansion (6.3) into left- and right-movers gives

$$X^{25}(\tau,\sigma) = X_L^{25}(\tau+\sigma) + X_R^{25}(\tau-\sigma) \tag{6.5}$$

with

$$X_L^{25}(\tau+\sigma) = \frac{1}{2}(x^{25}+\bar{x}^{25}) + \left(\alpha'\frac{K}{R} + WR\right)(\tau+\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^{25}e^{-2in(\tau+\sigma)},$$
$$X_R^{25}(\tau-\sigma) = \frac{1}{2}(x^{25}-\bar{x}^{25}) + \left(\alpha'\frac{K}{R} - WR\right)(\tau-\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-2in(\tau-\sigma)},$$

where $\bar{x}^{25}$ is some constant which cancels in the sum to form $X^{25}(\tau,\sigma)$. Furthermore, defining the zero modes,

$$\sqrt{2\alpha'}\bar{\alpha}_0^{25} \equiv \left(\alpha'\frac{K}{R} + WR\right), \tag{6.6}$$
$$\sqrt{2\alpha'}\alpha_0^{25} \equiv \left(\alpha'\frac{K}{R} - WR\right), \tag{6.7}$$

we can rewrite the expression for the left- and right-movers as

$$X_L^{25}(\tau+\sigma) = \frac{1}{2}(x^{25}+\bar{x}^{25}) + \sqrt{2\alpha'}\bar{\alpha}_0^{25}(\tau+\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^{25}e^{-2in(\tau+\sigma)}, \tag{6.8}$$
$$X_R^{25}(\tau-\sigma) = \frac{1}{2}(x^{25}-\bar{x}^{25}) + \sqrt{2\alpha'}\alpha_0^{25}(\tau-\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-2in(\tau-\sigma)}. \tag{6.9}$$

We note that now we have generally $\alpha_0^{25} \neq \bar{\alpha}_n^{25}$.

6.1.2 Modified Mass Formula

The mass formula for the string with one dimension compactified on a circle can be interpreted from a 25-dimensional viewpoint in which we regard each of the Kaluza-Klein excitations on X^{25}, which are given by K, as distinct particles. In general, the mass formula is given by

$$M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu, \tag{6.10}$$

where note that we are only performing the sum over the non-compact dimensions.

On the other hand, we still have the requirement that all on-shell physical states are annihilated by the operators $L_0 - 1$ and $\bar{L}_0 - 1$, which imply that $L_0 = 1$ and $\bar{L}_0 = 1$. These equations become

$$\begin{aligned}\frac{1}{2}\alpha' M^2 &= (\bar{\alpha}_0^{25})^2 + 2N_L - 2 \\ &= (\alpha_0^{25})^2 + 2N_R - 2,\end{aligned}$$

where N_L and N_R are the number operators for the left- and right-movers, respectively. The above equations can be confirmed as follows. From (**??**), we have

$$\begin{aligned}L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n \\ &= \frac{1}{2}(\alpha_0^{25})^2 - \frac{\alpha'}{4}M^2 + N_R, \qquad (\because (3.3) \text{ and } (6.10))\end{aligned}$$

which is equal to 1. This is what we want to confirm.

Taking the difference of the above two expression for $(1/2)\alpha' M^2$ and using (6.6) and (6.7) gives

$$N_R - N_L = WK, \tag{6.11}$$

which is the modified level matching condition. In a similar way, taking the sum gives

$$\alpha' M^2 = \alpha' \left[\left(\frac{K}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2\right] + 2N_L + 2N_R - 4, \tag{6.12}$$

which is the modified mass formula for a string with one spatial dimension compactified.

6.1.3 T-duality of the Bosonic String

The level matching condition (6.11) and the mass formula (6.12) are invariant under interchange of W and K, provided that we simultaneously send R to $\tilde{R} = \alpha'/R$. This symmetry is called T-duality. It tells us that a theory compactified on a circle of radius R has the same mass spectrum as a theory which is compactified on a circle of radius $\tilde{R}$, and vice versa. (See Fig 6.2.) Also, note that the interchange of W and K implies that the momentum excitations K in one description corresponds to winding-mode excitations in the dual description, and vice versa.

The T-duality transformation

$$\begin{aligned}T &: \mathbb{R}^{24,1} \times S^1_R \longleftrightarrow \mathbb{R}^{24,1} \times S^1_{\tilde{R}}, \\ T &: W \longleftrightarrow K,\end{aligned}$$

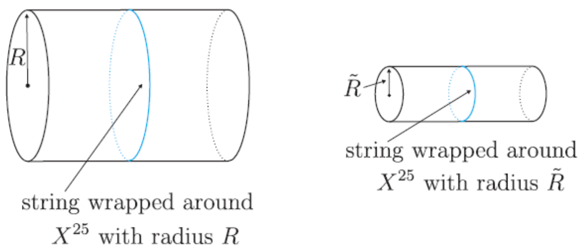

Figure 6.2: The T-duality of closed bosonic string theories.

is equivalent to the following transformation in terms of the modes in the expansion of the compactified dimension,

$$\alpha_0^{25} \longrightarrow -\alpha_0^{25}, \tag{6.13}$$

$$\bar{\alpha}_0^{25} \longrightarrow \bar{\alpha}_0^{25}, \tag{6.14}$$

which can be seen easily from (6.6) and (6.7). In fact, it is not just the zero mode, but the entire right-moving part of the compact coordinate that flips sign under the T-duality transformation,

$$T : X_R^{25} \longrightarrow -X_R^{25} \quad \text{and} \quad T : X_L^{25} \longrightarrow X_L^{25}. \tag{6.15}$$

Thus, we see that $X^{25}(\tau, \sigma)$ is mapped to, under the T-duality transformation,

$$T : X^{25}(\tau, \sigma) \to \tilde{X}^{25}(\tau, \sigma) = X_L(\tau + \sigma) - X_R(\tau - \sigma), \tag{6.16}$$

which has the expression,

$$\tilde{X}^{25}(\tau, \sigma) = \tilde{x}^{25} + 2\alpha' \frac{K}{R}\sigma + 2RW\tau + \frac{i}{2}\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} (\bar{\alpha}_{-n}^{25} e^{-2in\sigma} - \alpha_n^{25} e^{2in\sigma}) e^{-2in\tau}. \tag{6.17}$$

We should note that the coordinate x^{25}, which parametrizes the original circle with periodicity $2\pi R$, has been replaced by a coordinate $\tilde{x}^{25}$. It is clear that this parametrizes the dual circle with periodicity $2\pi\tilde{R}$, because the conjugate momentum is $\tilde{p}^{25} = RW/\alpha' = W/\tilde{R}$.

We will now see that T-duality interchanges X^{25} and $\tilde{X}^{25}$ from the viewpoint of the world-sheet. To begin, consider the following world-sheet action

$$S = \int d\tau d\sigma \left(\frac{1}{2} V^\alpha V_\alpha - \epsilon^{\alpha\beta} X^{25}(\tau, \sigma) \partial_\beta V_\alpha \right). \tag{6.18}$$

Varying this action with respect to X^{25} gives us the equations of motion

$$\epsilon^{\alpha\beta} \partial_\beta V_\alpha = 0, \Rightarrow \partial_1 V_2 - \partial_2 V_1 = 0,$$

whose solution is given by $V_\alpha = \partial_\alpha \tilde{X}^{25}$, where $\tilde{X}^{25}$ is an arbitrary function.

On the other hand, varying the action with respect to V_α gives the equation of motion

$$V_\alpha = -\epsilon_\alpha{}^\beta \partial_\beta X^{25}(\tau, \sigma). \tag{6.19}$$

Now, comparing the two expressions for V_α gives $\partial_\alpha \tilde{X}^{25} = -\epsilon_\alpha{}^\beta \partial_\beta X^{25}$, which implies

$$\partial_+ \tilde{X}^{25} = \partial_+ X^{25}, \quad \partial_- \tilde{X}^{25} = -\partial_- X^{25}.$$

The first and second equations imply $\tilde{X}_L^{25} = X_L^{25}$ and $\tilde{X}_R^{25} = -X_R^{25}$, respectively. That is, a T-duality transformation can be expressed in the compact dimension as $X_R^{25} \to -X_R^{25}$ and $X_L^{25} \to X_L^{25}$, as was stated previously. In addition, substituting (6.19) back into(6.18) gives

$$\frac{1}{2}\int d\tau d\sigma \partial^\alpha X^{25} \partial_\alpha X^{25},$$

which is the bosonic action (the Polyakov action) for the X^{25} component, which makes this discussion reasonable.

We will repeat the previous arguments for the case of open strings.

6.2 T-duality and Open Strings

Previously, we saw that when we varied the Polyakov action in conformal gauge, we got a bulk term whose vanishing gave the equation of motion and the boundary term

$$\delta S = \Big[-\frac{1}{2\pi\alpha'}\int d\tau d\sigma \partial_\sigma X_\mu \delta X^\mu\Big]_{\sigma=0}^{\sigma=\pi}.$$

Now, as before, we need to make this boundary term vanish. This is achieved by forcing the ends of the open string to obey either Neumann boundary conditions (2.36) or Dirichlet boundary conditions (2.38). However, in order to satisfy Poincaré invariance for all 26 dimensions, we choose Neumann boundary conditions for the open string.

Now, we will assume that we compactify the X^{25} coordinate. Then, we want to perform a T-duality transformation on the open string theory in the X^{25} direction.

But before we study the quantitative properties which arise from the transformation, we will see what kind of qualitative changes we will get. The first thing to notice is when we apply the T-duality transformation on the bosonic string which is compactified on a circle, we see that the winding number is meaningless. This is due to the fact that an open string cannot wind around the compact dimension. Hence, open strings do not have winding modes and so $W = 0$. Since the winding modes were critical to relate the closed string spectra of two bosonic theories by using T-duality, we then expect not to have open strings transform in this way.

In order to see how the open string transform under T-duality transformation, we first start with the mode expansion for the coordinate X^μ, which are given by setting $l_s = 1$ or $\alpha' = 1/2$ in (2.59), with the Neumann boundary conditions.

$$X^\mu(\tau,\sigma) = x^\mu + p^\mu\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma). \tag{6.20}$$

We can further split the mode expansions into two parts, left- and right-movers, just as was done for closed strings. They are given by

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}(x^\mu - \bar{x}^\mu) + \frac{1}{2}p^\mu(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}, \tag{6.21}$$

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}(x^\mu + \bar{x}^\mu) + \frac{1}{2}p^\mu(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau+\sigma)}. \tag{6.22}$$

$$\tag{6.23}$$

When we compactify the X^{25} coordinate and apply the T-duality transformation to this direction, we get, as was shown already,

$$T: X_L^{25} \to X_L^{25} \quad \text{and} \quad X_R^{25} \to -X_R^{25}.$$

(That is why we have written the open string modes in terms of left- and right-movers.) Thus, under the T-duality transformation, for the X^{25} coordinate we obtain

$$\begin{aligned} T: X^{25} \to \tilde{X}^{25} &\equiv X_L^{25} - X_R^{25} \\ &= \bar{x}^{25} + p^{25}\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\tau}\left(e^{-in\sigma} - e^{in\sigma}\right) \\ &= \bar{x}^{25} + p^{25}\sigma + \sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\tau}\sin(n\sigma), \end{aligned} \tag{6.24}$$

from which we can read off the properties of T-duality. First, note that since there is no linear term in τ, we see that the T-dual open string has no momentum in the X^{25} direction. Also, we see that the T-dual open string has fixed endpoints at $\sigma = 0, \pi$ since the oscillator term vanishes at these points. Note that this is equivalent to the Dirichlet boundary conditions for an open string. Thus, we see that the T-duality transformation maps the X^{25} coordinate with Neumann boundary conditions to that with Dirichlet boundary conditions (and vice versa). Explicitly, the boundary conditions of the X^{25} coordinate of the T-dual string, which are Dirichlet conditions, are given by

$$\tilde{X}^{25}(\tau, 0) = \bar{x}^{25}, \tag{6.25}$$

$$\tilde{X}^{25}(\tau, \pi) = \bar{x}^{25} + \frac{\pi K}{R} = \bar{x}^{25} + 2\pi K\tilde{R}, \tag{6.26}$$

where we have used $p^{25} = K/R$ and $\tilde{R} = \alpha'/R = 1/(2R)$. Hence, we get

$$\tilde{X}^{25}(\tau, \pi) - \tilde{X}^{25}(\tau, 0) = 2\pi K\tilde{R},$$

which tells us that the dual string winds around the dual dimension of radius $\tilde{R}$ with winding number K. This winding mode is topologically stable since the endpoints of the string are fixed by the Dirichlet boundary conditions. Therefore, this string cannot unwind without breaking.

Summarizing:

- T-duality transforms a bosonic open string with Neumann boundary conditions on a circle of radius R into a bosonic open string with Dirichlet boundary conditions on a circle of radius $\tilde{R}$.
- T-duality transforms a string that has momentum and no winding in the circular direction into a string that has winding but no momentum in the dual circular direction.
- The ends of the dual open string are attached to the 25-dimensional hyperplane $\tilde{X}^{25} = \bar{x}^{25}$ in spacetime.
- The ends of the dual open string can wrap around the circle an integer number of times.

This hyperplane is an example of Dp-brane, where D is for Dirichlet while p stands for the number of spatial dimensions of the hyperplane. In general, a Dp-brane is defined as a hyper-surface on which an open string can end, as illustrated in Fig 6.3, We should note that Dp-branes are also physical objects, i.e. they have their own dynamics. In the above example, the hyper-surface $\tilde{X}^{25} = \tilde{x}^{25}$ is a $D24$-brane.

By applying a T-duality transformation to open bosonic strings with Neumann boundary conditions in all directions, we learned that in the dual theory, the corresponding open strings have Dirichlet boundary conditions along the dual circle and therefore end on a $D24$-brane. This reasoning can be iterated by taking other directions to be circular and performing T-duality transformations on those directions. Starting with n such circles (or an n-torus), we end up with a T-dual description in which the open string has Dirichlet boundary conditions in the n directions. This implies that the string ends on a

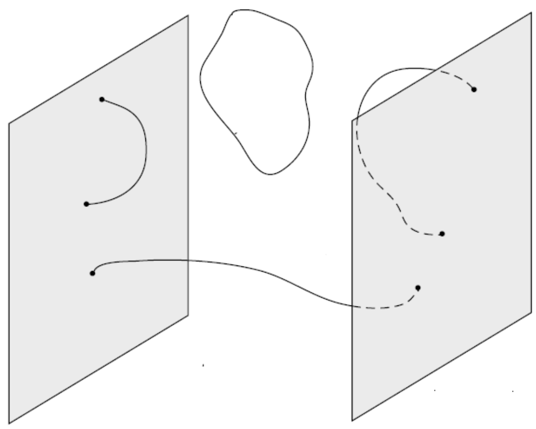

Figure 6.3: Dp-branes and open strings ending on them.

$D(25-n)$-brane. What this means for the open string in the original description, which had Neumann boundary conditions for all directions, is that this is clearly the $n=0$ case. So, the open string should be regarded as ending on a spacetime-filling $D25$-brane. In general, we can consider a set-up in which there are a number of D-branes of various dimensions. They are replaced by D-branes of other dimensions in T-dual formulations.

6.2.1 Mass spectrum of Open Strings on Dp-branes

Let us consider a configuration for our bosonic string theory in which the coordinates $X^0, X^1, \ldots, X^p \equiv X^\mu$ have Neumann boundary conditions and the coordinates $X^{p+1}, X^{p+2}, \ldots, X^{25} \equiv X^I$ have Dirichlet boundary conditions. Then, the mode expansion for X^μ is the same as (2.59) and is given by

$$X^\mu = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma), \tag{6.27}$$

where we note that μ runs from 0 to p. The mode expansion for X^I is the same as (2.60) and is given by

$$X^I = x^I + \frac{\sigma}{\pi}(x_j^I - x_i^I) + l_s \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \sin(n\sigma), \tag{6.28}$$

where i and j label the different Dp-branes. (See Fig 6.4.) Now, from these mode expansions, we can derive the mass-shell condition. Namely, we have, from (3.12),

$$\begin{aligned} L_0 = H &= \frac{T}{2}\int_0^\pi d\sigma \left(\dot{X}^2 + X'^2\right) \\ &= \frac{T}{2}\int_0^\pi d\sigma \big((\dot{X}^\mu)^2 + (X'^\mu)^2 + (\dot{X}^I)^2 + (X'^I)^2\big) \\ &= \sum_{n=1}^\infty \alpha_{-n}^\mu \alpha_{n\mu} + \alpha' p^\mu p_\mu + \sum_{n=1}^\infty \alpha_{-n}^I \alpha_{nI} + \frac{1}{4\alpha'}\left(\frac{x_i^I - x_j^I}{\pi}\right)^2. \end{aligned}$$

Thus, we can get

$$L_0 = N - \alpha' M + \frac{1}{4\alpha'}\left(\frac{x_i^I - x_j^I}{\pi}\right)^2, \tag{6.29}$$

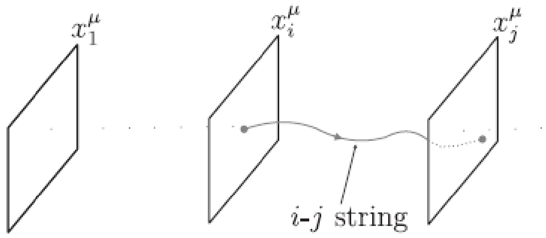

Figure 6.4: A compactified background spacetime.

where we have defined N as

$$N \equiv \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{n\mu} + \sum_{n=1}^{\infty} \alpha_{-n}^{I}\alpha_{nI}$$
$$= \sum_{n=1}^{\infty} \alpha_{-n}^{\nu}\alpha_{n\nu}, \tag{6.30}$$

with $\nu = 0, 1, \ldots, 25$. Furthermore, imposing the physical state condition, $L_0 - 1 = 0$, leads to

$$M^2 = \frac{N-1}{\alpha'} + T^2(x_i^I - x_j^I)^2, \tag{6.31}$$

where T is the tension, defined by $T = 1/(2\pi\alpha')$. Note that we can think of $(x_i^I - x_j^I)^2$ as the energy stored in the tension of a string stretched between a Dp-brane at x_i and a Dp-brane at x_j.

Let us now investigate the spectrum for our theory.

Massless State

- One Dp-brane:

For the case of one Dp-brane, the term $T^2(x_i^I - x_j^I)^2$ vanishes. This fact gives us the following expression for the mass

$$M^2 = \frac{N-1}{\alpha'},$$

which implies that there is still a tachyon for $N = 0$. Thus, we seen that for the massless states of the bosonic string theory with one Dp-brane, we must have $N = 1$. That is, the only massless states with one Dp-brane are level 1 states. These states are given by

$$\alpha_{-1}^{\mu}|0; k^{\nu}\rangle, \tag{6.32}$$

which corresponds to a $p+1$-dimensional vector, which we denote by A^{μ}, and also

$$\alpha_{-1}^{I}|0; k^{\nu}\rangle, \tag{6.33}$$

which corresponds to $25 - p$ scalars, which we denote by X^I. We can interpret A_μ as a gauge field and X^I as the position of the Dp-brane. This implies that a Dp-brane has a $U(1)$ gauge field on its world-volume and has a massless scalar for each direction normal to the Dp-brane.

- Two Dp-branes:

For this case, the mass formula is given by

$$M^2 = \frac{N-1}{\alpha'} + T^2(x_i^I - x_j^I)^2,$$

with $i, j = 1, 2$. Now, if the two Dp-branes are at the same position (i.e. $x_i = x_j$), then we get extra massless modes which arise from $1-2$ and $2-1$ strings, compared to $1-1$ and $2-2$ strings for one brane.[2] The states are given by

$$(A^\mu)_{\alpha\beta} \quad \text{and} \quad X^I_{\alpha\beta}, \tag{6.34}$$

where $\alpha, \beta = 1, 2$. The four component entity denoted by $(A^\mu)_{\alpha\beta}$ describes a $U(2)$ gauge theory. Thus, when the two branes are coincident, we get a $U(2)$ gauge theory living on them. In general, it is known that if we have N coincident Dp-branes, then we get a $U(N)$ gauge theory living on them and the fields are given by

$$(A^\mu)_{\alpha\beta} \quad \text{and} \quad X^I_{\alpha\beta}, \tag{6.35}$$

where $\alpha, \beta = 1, 2, \ldots, N$. Also, note that when one of the Dp-branes moves apart from the others, $U(N)$ divides into $U(N-1)\times U(1)$. Thus, for example, in the case of two Dp-branes, if they are not coincident, then the gauge symmetry is $U(1) \times U(1)$.

Dp-branes are not a story restricted to the bosonic theory. They can also exist in superstring theories.

6.3 Type II Superstrings and T-duality

We have seen that in the closed bosonic string theory, T-duality maps the theory which is compactified on a circle of radius R into an identical theory on a dual circle of radius $\tilde{R} = \alpha'/R$. Let us examine the same T-duality transformations for type II superstring theories. It will turn out that the type IIA theory is mapped to the type IIB theory and vice versa.

Consider the case of a single circle, i.e. consider that the X^9 coordinate of a type II is compactified on a circle of radius R and that a T-duality transformation is performed for this coordinate. The transformation for the bosonic coordinate is the same as for the bosonic string, namely,

$$X^9_L \to X^9_L \quad \text{and} \quad X^9_R \to -X^9_R, \tag{6.36}$$

which interchange momentum and winding numbers. In the RNS formalism, world-sheet supersymmetry requires the world-sheet fermion Ψ^9 to transform in the same way as its bosonic partner X^9, namely,

$$\Psi^9_L \to \Psi^9_L \quad \text{and} \quad \Psi^9_R \to -\Psi^9_R, \tag{6.37}$$

which imply that in particular, the zero mode of Ψ^9_R in the Ramond sector transforms

$$d^9_0 \to -d^9_0.$$

Now, we have the relation between Ramond sector zero modes and ten-dimensional Dirac matrices, $\Gamma^\mu = \sqrt{2} d^\mu_0$. Thus, a T-duality transformation yields, in terms of Γ^μ,

$$\Gamma^\mu \to \Gamma^\mu \text{ for } \mu \neq 9 \quad \text{and} \quad \Gamma^9 \to -\Gamma^9,$$

from which we can find that the chirality operator Γ^{11} behaves as

$$\Gamma_{11} \equiv \Gamma_0 \Gamma_1 \cdots \Gamma^9 \to -\Gamma_{11},$$

[2] We call a string which goes from a brane at x_i to a brane at x_j a $i-j$ string.

which means that the chirality of the right-moving Ramond ground state is reversed.

The relative chirality of the left-moving and right-moving ground states is what distinguishes the type IIA and type IIB theories. Then, we conclude that T-duality flips the chirality for right-moving spinors and thus transforms the various superstring sectors as

$$(R^+, R^\pm) \longrightarrow (R^+, R^\mp)$$
$$(NS^+, R^\pm) \longrightarrow (NS^+, R^\mp)$$

This exchanges type IIB theory and type IIA theory. The precise statement of this T-duality between both theories is

Type IIB on S^1 with radius $R \leftrightarrow$ Type IIA on S^1 with radius $\tilde{R}$.

6.3.1 Branes in Type II Superstring Theory

First, note that a point particle, i.e. $D0$-brane couples naturally to a gauge field A_μ as follows:

$$S_A = \int_{\mathcal{C}} A_\mu \dot{X}^\mu d\tau = \int_{\mathcal{C}} A_\mu \frac{dX^\mu}{d\tau} d\tau \equiv \int_{\mathcal{C}} A_{(1)}, \tag{6.38}$$

where $\mathcal{C}$ is the world-line of the particle. In this way, we can construct a one-form,[3] which is denoted by $A_{(1)}$, from a gauge field. For example, we know that a charges point particle couples to an electric or magnetic field, described by a gauge field A_μ, as it moves through spacetime. Also, given a one-form $A_{(1)}$, which is defined by a gauge field A_μ, then we can construct an object called the field strength $F_{(2)}$, which is associated to the one-form via the following definition:

$$F_{(2)} \equiv dA_{(1)}, \tag{6.41}$$

where d is the exterior derivative. Note that since $F_{(2)}$ is defined as the exterior derivative of a one-form, $F_{(2)}$ is a two-form.

The above discussion can be generalized to 1-brane. Namely, a 1-brane couples naturally to a two-form gauge potential $B_{(2)} = B_{\mu\nu} dX^\mu \wedge dX^\nu$ via the action

$$S_B = \int_{\mathcal{M}} d\tau d\sigma B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta} \equiv \int_{\mathcal{M}} B_{(2)}, \tag{6.42}$$

where $\mathcal{M}$ is the world-sheet mapped out by the 1-brane (i.e. the string). Indeed, the spectrum of closed superstrings contains an antisymmetric two-form $B_{\mu\nu}$, called the Kalb-Ramond field. This, in turn, can be used to define its associated field strength $H_{(3)} = dB_{(2)}$, which is a 3-form. Furthermore it is known that this can be generalized to p-branes as follows. A p-brane couples naturally to a $p+1$-form gauge potential $C_{(p+1)}$ via the action

$$S_C = \int_{\mathcal{M}} d\sigma^0 d\sigma^1 \cdots d\sigma^p C_{\mu_1\mu_2\cdots\mu_{p+1}} \partial_{\alpha_1} X^{\mu_1} \cdots \partial_{\alpha_{p+1}} X^{\mu_{p+1}} \epsilon^{\alpha_1\cdots\alpha_{p+1}} \equiv \int_{\mathcal{M}} C_{(p+1)}, \tag{6.43}$$

[3]Generally, given a gauge field A_μ, we can construct a one-form $A_{(1)}$ by defining it as

$$A_{(1)} = A_\mu dx^\mu. \tag{6.39}$$

Similarly, given a gauge field with n components $A_{\mu_1\mu_2\cdots\mu_n}$, we can construct a n-form $A_{(n)}$ via

$$A_{(n)} = \frac{1}{n!} A_{\mu_1\mu_2\cdots\mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}. \tag{6.40}$$

where $\mathcal{M}$ is the world-volume mapped out by the p-brane. Also, its associated $p+2$-form field strength $F_{(p+2)}$ is defined as $F_{(p+2)} \equiv dC_{(p+1)}$.

We can use the above information about the coupling of branes to gauge potentials in order to see what kind of stable branes we can expect in a string theory. Since we know what kind of gauge fields are present in the type IIA and IIB theories, we can see what kind of branes these two theories have by using this knowledge. To begin, let us start with the type IIA theory.

- Type IIA:

  In the type IIA theory, there exist 1-form and 3-form gauge potentials. Since $D0$-branes couple to 1-form gauge potentials and $D2$-branes couple to 3-form gauge potentials, we see that there exist $D0$-brane and $D2$-brane in the type IIA theory. However, this is not all because, given a field strength, which is associated to a form potential, we can construct another field strength by taking the Hodge dual. Foe example, if we have a gauge potential $C_{(1)}$, there is a field strength $F_{(2)} = dC_{(1)}$ and by taking the Hodge dual of $F_{(2)}$, we get another field strength $*F_{(2)}$. Thus, back to our case, we have 1- and 3-form potentials which defines 2- and 4-form field strengths and, since $D = 10$ for type IIA theories, there are a dual 8- and 6-form field strengths, corresponding to the 2- and 4-field strengths, respectively. In addition, associated to the dual 8- and 6-form field strengths, there are a 7- and 5-form gauge potential, respectively. They couple to $D6$- and $D4$-branes, respectively. As a result, the type IIA theory has $D0$-, $D2$-, $D4$ and $D6$-branes.

- Type IIB:

  The type IIB theory has 0-form (scalar), 2-form and 4-form gauge potentials, which are coupled to $D(-1)$-, $D1$- and $D3$-branes, respectively. The $D(-1)$-brane is an object which is localized in time as well as in space, i.e. it is interpreted as a D-instanton, In addition, by taking the Hodge dual as in the type IIA theory,we have dual $D5$- and $D7$-branes. Note that in the type IIA case, we had two types of branes ($D0$ and $D2$) and they gave rise to two types of dual branes ($D4$ and $D6$). But, in the type IIB case, we have three types of branes ($D(-1)$, $D1$ and $D3$), which give rise to only two types of dual branes ($D5$ and $D7$). The reason why this happen is as follows: the $D3$-brane couples to a 4-form gauge potential, which defines a 5-form field strength. But the Hodge dual of this field strength is a dual 5-form field strength, which gives a 4-form gauge potential. Then, it couples to a $D3$-brane. Thus, the Hodge dualizing of the $D3$-brane does not give any new branes. In that sense, $D3$-branes are called self-dual with respect to the Hodge star operator.

Summarizing: the type IIA theory has $D0$-, $D2$-, $D4$- and $D6$-branes, while the type IIB theory has $D(-1)$-, $D1$-, $D3$-, $D5$- and $D7$-branes. Lastly, T-duality maps a type IIA theory compactified on a circle of radius R to a type IIB theory compactified on a circle of radius $\tilde{R}$, sends the Neumann boundary conditions to Dirichlet boundary conditions and maps various Dp-branes (p even) into other Dq-branes (q odd), and vice versa.

Bibliography

[1] “String Theory and M-Theory A Modern Introduction”, by Kartin Becker, Melanie Becker, and John Schwarz, Cambridge University Press, 2007.

[2] “A First Course in String Theory”, by Barton Zwiebach, Cambridge University Press, 2009.

[3] “Conformal Field Theory”, by Philippe Di Francesco, Pierre Mathieu, and David Sénéchal, Springer, 1997.

[4] “String Theory Demystified”, by David McMahon, McGraw-Hill, 2009.

[5] “An Introduction to String Theory and D-Brane Dynamics”, by Richard J Szabo, Imperial College Press, 2011.

[6] “String Theory in a Nutshell”, by Elian Kiritsis, Princeton University Press, 2007.

[7] “String Theory”, by Joseph Polchinski, Cambridge University Press, 2001, 2005.

[8] “Introduction to String Theory”, by Timo Weigand, A lecture Note in University Heidelberg, 2011.

[9] “String Theory”, by David Tong, A lecture Note in University of Cambridge, 2009.