On the twisted sector of Wess-Zumino-Witten models

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Abstract

We introduce the required knowledge to understand the Wess-Zumino-Witten (WZW) models of two dimensional conformal field theory. We focus on the integrable representations of affine Lie algebras that appear in the spectrum of a WZW model at positive integer level. The twisted sector of the WZW model is constructed from outer automorphisms of the horizontal subalgebra, the resulting symmetry algebra is a twisted affine Lie algebra. We analyse the restrictions imposed by singular vectors on the allowed modules in the twisted sector. For the $\widehat{\mathfrak{sl}(3)}_k$ WZW model at positive integer levels, we show that the allowed modules are restricted to the integrable highest weight $\mathbf{A}_2^{(2)}$-modules. We extend these arguments to a general WZW model at positive integer level and propose that the allowed modules in the twisted sector are likewise restricted to the integrable highest weight modules of the corresponding twisted affine Lie algebra.

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Contents

Chapter 1

Introduction

The study of conformal invariance has been of significant interest in the development of mathematical physics in the past thirty years. A conformal map is a transformation that preserves angles. The properties of such maps have been studied for many years, with early contributions from Lie [1] and Boltzmann [2] dating back to the 1880's.

In 1984, a landmark paper by Belavin, Polyakov and Zamolodchikov (BPZ) [3] generated considerable interest in conformally invariant quantum field theories. This paper showed that conformal invariance in two dimensions gives rise to an infinite dimensional symmetry algebra, called the Virasoro algebra. They introduced the conformal structure of theories that arise from considering the Virasoro algebra as the symmetry algebra, these are called minimal models. In particular, they showed that all the fields in these models could be grouped into a finite set of conformal families, each corresponding to a highest weight representation of the Virasoro algebra.

Since then, conformal field theory in two dimensions has found many important physical applications. The properties of conformal invariance are central to statistical mechanics as they allow one to predict universal quantities in many systems at their critical point [4]. For example, this has led to results in condensed matter physics [5]. In addition, conformal field theories provide a fundamental building block to understand string theory [6], a candidate theory of quantum gravity.

The technique applied by BPZ led to the development of a much larger class of rational conformal field theories, see for example [7]. In each of these cases, the conformal field theory possesses an infinite dimensional symmetry algebra that in some way contains a copy of the Virasoro algebra. One such conformal field theory is the Wess-Zumino-Witten model, which we will study in detail in this thesis.

The Wess-Zumino-Witten (WZW) model is derived from the non-linear sigma model, which was exactly solved by Polyakov and Wiegmann in 1983 [8]. However, to fully realise the conformal symmetry of the model an extra term in the action was introduced by Witten in 1984 [9]. This term is called the Wess-Zumino term after the researchers who discovered it in 1971, in the context of Ward identities [10]. In [9], Witten derived the commutation relations for the WZW symmetry algebra to show that it is given by an affine Lie algebra. Furthermore, Witten was the first to present evidence that the WZW model demonstrated conformal invariance.

Knizhnik and Zamolodchikov are responsible for developing the conformal structure of the WZW model [11]. In particular, the conformal invariance of the model is exhibited through the existence of an energy-momentum tensor via the Sugawara construction. This construction originates in the 1960s from work done independently by Sugawara [12] and Sommerfield [13] on the theory of current algebras.

The work done by Knizhnik and Zamolodchikov was extended by Gepner and Witten [14], who determined the structure of the WZW primary fields. Gepner and Witten construct the spectrum of the WZW model through the analysis of affine singular vectors. These ideas were developed further by Felder, Gawedzki and Kupiainen [15]. At positive integer levels, the spectrum of a WZW model is given by the integrable highest weight representations of the corresponding affine Lie algebra.

In order to understand the WZW model, it is clearly important to understand affine Lie algebras and their representations. Affine Lie algebras were discovered independently by Kac [16] and Moody [17]. They first appeared in a physical context through the study of the dual quark model [18], a precursor to string theory. Since then there has been a multitude of research regarding affine Lie algebras, much of this is described in textbooks such as that by Kac [19], Moody and Pianzola [20] and Fuchs and Schweigert [22].

Kac was responsible for constructing the affine Lie algebras as centrally extended loop algebras of the finite dimensional simple Lie algebras. Similar to finite dimensional simple Lie algebras, they can also be constructed from a Cartan matrix via the Chevalley-Serre relations. Moody demonstrated this procedure under more generalised requirements on the Cartan matrix. The broad class of matrices that arise from a generalised Cartan matrix are referred to as the Kac-Moody Lie algebras.

The theory of finite dimensional simple Lie algebras underpins both these constructions. The concept of Lie groups and their corresponding algebras was introduced by Lie in 1874, for a comprehensive history on the development of Lie theory see [23]. A full classification of the finite dimensional complex simple Lie algebras is attributed to Killing [24] and Cartan [25]. A fundamental result of this construction is that all complex finite dimensional semisimple Lie algebras possess a non-degenerate invariant bilinear form, called the Killing form. A basis of particular importance to the finite dimensional simple Lie algebra is a Chevalley basis [26].

Chapter 2 of this thesis will summarise the important properties of the finite dimensional complex simple Lie algebras and construct the affine Lie algebras as centrally extended loop algebras. We introduce the twisted affine Lie algebras by studying outer automorphisms of the finite dimensional simple Lie algebras and present an explicit construction of the simplest twisted affine Lie algebra $\mathbf{A}_2^{(2)}$. An emphasis will be placed on results that are important for understanding subsequent sections of the thesis.

In Chapter 3 we will outline the key concepts involved in the representations of both the finite-dimensional simple Lie algebras and their affine extensions. A focus will be placed on understanding the integrable highest weight modules since these form a fundamental component of the WZW models.

Chapter 4 will introduce the basic constructions of conformal field theory, before illustrating these concepts through some of the most well-known examples: the free boson and the free fermion. This chapter will end with a detailed discussion of WZW models. We focus on analysing the spectrum through the restrictions imposed by the affine singular vectors as presented in [14] and [27].

Throughout the first three chapters, the concepts introduced will be illustrated with examples that are important for later sections of the thesis. A reader unfamiliar with the topics introduced should pay particular attention to the examples presented.

The concept of the twisted sector can be understood through the introduction of boundary conditions to a conformal field theory. This concept was introduced by Ramond [28] in the context of the free fermion model. The introduction of boundary conditions leads to a symmetry algebra whose modes are indexed differently to the original symmetry algebra, but satisfy the same (anti)commutation relations. We introduce the concept of a twisted sector in Chapter 4 and use a generalised commutation relation for the free fermion to determine normal ordering in the Ramond (twisted) sector.

The aim of this thesis is to understand the twisted sector of the WZW models, this will be the focus of Chapter 5. The twisted sector arises from automorphisms on the horizontal subalgebra of the corresponding untwisted affine Lie algebra. In particular, twisted affine symmetry algebras can be realised as non-trivial outer automorphisms of the WZW fields through the appropriate boundary conditions:

$$J(ze^{2\pi i}) = \sigma(J(z)). \tag{1.1}$$

The goal of this thesis was somewhat inspired by a paper by Nepomechie [29]. In this paper Nepomechie proposes a construction of the Virasoro algebra from the modes of a twisted affine Lie algebra, indicating the existence of a conformal field theory with a twisted affine symmetry algebra. Here, we interpret Nepomechie's results as a twisted sector of an existing WZW model.

In Chapter 5 we introduce the concept of a twisted sector of the WZW models and derive a generalised commutation relation to determine the normal ordered product of two fields in the twisted sector. This formula also provides an expansion for the Virasoro algebra (this method is different to that used by Nepomechie) in terms of the modes of the twisted affine Lie algebra and we use this to determine the conformal weight of a highest weight vector in the twisted sector.

We extend these ideas to determine the structure of the spectrum in the twisted sector of the WZW model. We first deal with the simplest case of a non-trivial outer automorphism, this arises from the Dynkin diagram automorphism of $\mathfrak{sl}(3)$. This outer automorphism gives rise to a twisted sector of the $\widehat{\mathfrak{sl}(3)}_k$ WZW model. The twisted sector has a symmetry algebra given by the twisted affine Lie algebra $\mathbf{A}_2^{(2)}$. Making use of the WZW affine singular vectors, we determine that the spectrum of the WZW in the twisted sector is restricted to the integrable highest weight representations of $\mathbf{A}_2^{(2)}$ at $k = 1$.

In the case of $\widehat{\mathfrak{sl}(3)}_k$, we show that this result generalises to any positive integer level k. Finally, under any non-trivial automorphism of a WZW model at positive integer level k, we propose that the spectrum is restricted to the integrable highest weight representations of the corresponding twisted affine Lie algebra. We outline an argument of this proposal, but have not completed all details of the proof.

The twisted sector of the WZW models has been determined by Li [30], who presents his results in the context of vertex operator algebras. Since the mathematical formalism of vertex operator algebras is beyond the scope of this masters project, we will not attempt to interpret Li's results here.

It should be noted that throughout this thesis an emphasis is placed on making the material presented accessible to a graduate level student. For this reason most ideas are introduced at a relatively fundamental level and the corresponding theory built up from there. This does not mean that the content will be entirely understandable to a graduate student as a stand alone document. It is highly recommended that a reader new to the concepts presented, work through the ideas in conjunction with a textbook. References to relevant texts are provided at the beginning of each chapter.

In the conclusion we will give an overview of the concepts introduced in this thesis. We will also summarise the results found for the twisted sector of the WZW models and indicate areas for potential future research related to this topic.

Chapter 2

Affine Lie algebras

This chapter will act as an introduction to the theory of affine Lie algebras. We will briefly summarise the key concepts of finite dimensional simple Lie algebras. For a thorough treatment of the theory of finite dimensional Lie algebras we refer the reader to [31].

We will then introduce the broader class of Kac-Moody Lie algebras and the special case of these Lie algebras which are known as the affine Lie algebras. We will then construct the untwisted affine Lie algebras via the loop construction on finite dimensional simple Lie algebras. The twisted affine Lie algebras are constructed in a similar way, but the loop construction is performed with respect to a gradation on a finite dimensional simple Lie algebra. In order to understand the relevant gradations we introduce the class of automorphisms known as the outer automorphisms. The section of most relevance to the material presented later in this thesis is the section on twisted affine Lie algebras and specifically the example of $\mathbf{A}_2^{(2)}$. For a detailed treatment of the construction of affine Lie algebras the reader should refer to [19, 21, 22].

2.1 Finite dimensional simple Lie algebras

Lie algebra definitions

An *algebra* is a vector space endowed with a binary product that is bilinear. A *Lie algebra* is an algebra $\mathfrak{g}$, over a field F, whose binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called the *Lie bracket* and satisfies the following criteria:

1. *Alternativity*:

$$[x, x] = 0 \quad \text{for all } x \in \mathfrak{g}. \tag{2.1}$$

2. *Jacobi Identity*:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in \mathfrak{g}. \tag{2.2}$$

Since $\mathbb{C}$ is algebraically closed, the study of Lie algebras over the field $\mathbb{C}$ is the easiest and most well known. Lie algebras over the field $\mathbb{C}$ are also physically important and play a large role in the study of quantum physics [32]. For the purpose of this thesis we will restrict our view to the case $F = \mathbb{C}$.

We define for any element $x \in \mathfrak{g}$ the *adjoint map of* x, $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ given by

$$y \mapsto \mathrm{ad}_x(y) := [x, y]. \tag{2.3}$$

If ad_x is diagonalisable on $\mathfrak{g}$, then x is called a *semisimple* element.

Suppose $\mathfrak{g}$ is a Lie algebra. A subspace $\mathfrak{h} \subset \mathfrak{g}$ which itself is a Lie algebra is called a Lie *subalgebra*. This is equivalent to $\mathfrak{g}$ being closed under the Lie bracket. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$. Or equivalently

$$[\mathbf{h}, \mathfrak{g}] \subseteq \mathfrak{h}, \tag{2.4}$$

where $[\mathfrak{h}, \mathfrak{g}] := \{[x, y] | x \in \mathfrak{h}, y \in \mathfrak{g}\}$.

A *proper* ideal of $\mathfrak{g}$ is an ideal that is neither $\{0\}$ or $\mathfrak{g}$ itself. A Lie algebra that contains no proper ideals is called *simple*. A Lie algebra that can be written as a direct sum of simple Lie algebras is called *semisimple*. We will now turn our attention to the details of finite dimensional simple Lie algebras.

Example: An important series of finite dimensional simple Lie algebras is given by

$$\mathfrak{sl}(n) = \{A \in M(n \times n, \mathbb{C}) | \mathrm{tr}(A) = 0\}, \qquad n \geq 2. \tag{2.5}$$

A basis for $\mathfrak{sl}(2)$, the space all traceless 2×2 complex matrices, is given by the elements

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2.6}$$

With respect to this basis the Lie brackets of $\mathfrak{sl}(2)$ are given by

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H. \tag{2.7}$$

It follows that the adjoint maps of the elements H, E and F with respect to the basis $\{H, E, F\}$ are represented by the matrices

$$\mathrm{ad}_H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad \mathrm{ad}_E = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{ad}_F = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}. \tag{2.8}$$

Cartan subalgebra and roots

Every finite dimensional simple Lie algebra $\mathfrak{g}$ has a *Cartan subalgebra* $\mathfrak{g}_0$, which is defined to be a maximal abelian subalgebra consisting entirely of semisimple elements. The Cartan subalgebras of $\mathfrak{g}$ are all related by automorphisms of $\mathfrak{g}$, which implies that the dimension of all Cartan subalgebras is the same. This common dimension is referred to as the *rank* of $\mathfrak{g}$:

$$r = \mathrm{rank}(\mathfrak{g}) := \dim(\mathfrak{g}_0). \tag{2.9}$$

It follows from the definition of a Cartan subalgebra that the maps $\{\mathrm{ad}_H | H \in \mathfrak{g}_0\}$ are simultaneously diagonalisable. Hence, there exists a basis for $\mathfrak{g}$ with respect to which the maps ad_H are all diagonalised. For any x in this basis we have

$$\mathrm{ad}_H(x) = [H, x] = \alpha_x(H)x \quad \forall H \in \mathfrak{g}_0, \tag{2.10}$$

where $\alpha_x \in \mathfrak{g}_0^*$ is called the *root* of $x \in \mathfrak{g}$. We define the root space of $\alpha \in \mathfrak{g}_0^*$ as

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} | [H, x] = \alpha(H)x \ \forall H \in \mathfrak{g}_0\}. \tag{2.11}$$

We refer to the set of all non-zero roots with non-zero root spaces as the set of *roots* or the *root system* of $\mathfrak{g}$, which we denote by Φ. Every root space of a finite dimensional simple Lie algebra is one-dimensional, so there exists an element $E^\alpha \in \mathfrak{g}$ that spans $\mathfrak{g}_\alpha$. Hence, there exists a basis for $\mathfrak{g}$ given by

$$\mathcal{B} = \{H^1, H^2, ..., H^r | r = \mathrm{rank}(\mathfrak{g})\} \cup \{E^\alpha | \alpha \in \Phi\}. \tag{2.12}$$

Any basis of this form is called a *Cartan-Weyl* basis of $\mathfrak{g}$. An important property of a Cartan-Weyl basis is that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \quad \forall \alpha, \beta \in \mathfrak{g}_0^*. \tag{2.13}$$

Example: From the brackets (2.7) of $\mathfrak{sl}(2)$, $\{H\}$ forms a basis for a Cartan subalgebra of $\mathfrak{sl}(2)$. Thus, $\mathrm{rank}(\mathfrak{sl}(2)) = 1$. The Lie brackets (2.7) imply that the roots of $\mathfrak{sl}(2)$ are $\Phi = \{\pm\alpha\}$, where $\alpha(H) = 2$. Furthermore,

$$\mathfrak{sl}(2)_\alpha = \mathrm{span}\{E\}, \qquad \mathfrak{sl}(2)_{-\alpha} = \mathrm{span}\{F\}. \tag{2.14}$$

Example: We can define a basis for $\mathfrak{sl}(3)$ by

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad E_{ij} \text{ for } i \neq j, \tag{2.15}$$

where E_{ij} is the matrix with a one in the i^{th} row and j^{th} column and all other entries are zero. Then $\{H_1, H_2\}$ forms a basis for the Cartan subalgebra of $\mathfrak{sl}(3)$, $\mathrm{rank}(\mathfrak{sl}(3)) = 2$ and

$$\mathrm{ad}_{H_1} = \mathrm{diag}(0, 0, 2, -2, 1, -1, -1, 1), \ \mathrm{ad}_{H_2} = \mathrm{diag}(0, 0, -1, 1, 1, -1, 2, -2), \tag{2.16}$$

with respect to the basis $\{H_1, H_2, E_{12}, E_{21}, E_{13}, E_{31}, E_{23}, E_{32}\}$.

Since $\text{rank}(\mathfrak{sl}(3)) = 2$, the root system of $\mathfrak{sl}(3)$ is two-dimensional. From the adjoint maps of the elements H_1 and H_2 the root system of $\mathfrak{sl}(3)$ is given by

$$\Phi = \{\pm\alpha, \pm\beta, \pm\theta | \alpha = (2, -1), \beta = (-1, 2), \theta = (1, 1)\}, \tag{2.17}$$

where the roots are given as vectors whose entries are determined by the values of the root on the basis elements $\{H_1, H_2\}$.

We can construct a basis for the root spaces by defining $E^\alpha = E_{12}$, $E^{-\alpha} = E_{21}$, $E^\beta = E_{23}$, $E^{-\beta} = E_{32}$, $E^\theta = E_{13}$, $E^{-\theta} = E_{31}$. From now on the elements of $\mathfrak{sl}(3)$ will be referred to by their roots rather than their matrix entries.

Killing form

We define the Killing form on a finite dimensional Lie algebra $\mathfrak{g}$ by

$$\kappa(x, y) := tr(\text{ad}_x \circ \text{ad}_y). \tag{2.18}$$

Up to a multiplicative constant the Killing form is the unique bilinear, invariant form ($\kappa([x, y], z) = \kappa(x, [y, z])$) on $\mathfrak{g}$. There is a standard convention for setting this multiplicative constant which we will determine later. The Killing form is a useful tool in proving many important properties of finite dimensional simple Lie algebras; in particular, the Killing form is non-degenerate if and only if $\mathfrak{g}$ is semisimple.

For simple $\mathfrak{g}$ the restriction of the Killing form to the Cartan subalgebra $\mathfrak{g}_0$ is non-degenerate. Hence, we can use the Killing form to define an isomorphism between the Cartan subalgebra and its dual space $\mathfrak{g}_0^*$. All finite dimensional simple Lie algebras satisfy $\text{span}(\Phi) = \mathfrak{g}_0^*$, so this isomorphism can be defined in terms of the roots of $\mathfrak{g}$. For each $\alpha \in \Phi$ we can define $H^\alpha \in \mathfrak{g}_0$ by

$$\alpha(h) = c_\alpha \kappa(H^\alpha, h), \tag{2.19}$$

where c_α is some normalisation constant. There is a standard convention for setting this normalisation constant that we will introduce later. We can use this isomorphism to define an inner product on $\mathfrak{g}_0^*$ by

$$(\alpha, \beta) := c_\alpha c_\beta \kappa(H^\alpha, H^\beta). \tag{2.20}$$

Example: The normalised Killing form on $\mathfrak{sl}(2)$ with respect to the basis $\{H, E, F\}$ is

$$\kappa = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.21}$$

For $\mathfrak{sl}(2)$ we have $2 = \alpha(H) = \kappa(H^\alpha, H)$ and $\kappa(H, H) = 2$ from which it follows that $\alpha \mapsto H^\alpha = H$. The inner product on the root system is determined by

$$(\alpha, \alpha) = \kappa(H, H) = 2. \tag{2.22}$$

Example: The restriction of the normalised Killing form on $\mathfrak{sl}(3)$ to the Cartan subalgebra is represented by the matrix

$$\kappa = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \tag{2.23}$$

with respect to the basis $\{H_1, H_2\}$. We take $\{\alpha, \beta\}$ to be a basis for the two dimensional root system of $\mathfrak{sl}(3)$. From (2.17) it follows that $\alpha \mapsto H^\alpha = H_1$ and $\beta \mapsto H^\beta = H_2$. The inner product on the root system is given by

$$(\alpha, \alpha) = 2, \qquad (\alpha, \beta) = -1, \qquad (\beta, \beta) = 2. \tag{2.24}$$

Note that $(\theta, \theta) = (\alpha + \beta, \alpha + \beta) = 2$.

Root space decomposition

Since $\mathfrak{g}$ is finite dimensional it has finitely many roots, so there exists a hyperplane $\mathcal{H}$ in $\mathfrak{g}_0^*$, such that $\mathcal{H}$ contains no roots. Thus, $\mathcal{H}$ separates $\mathfrak{g}_0^*$ into two sections which we will denote by $\mathcal{H}_\pm$. With respect to $\mathcal{H}$ we can define the *positive roots* of $\mathfrak{g}$ to be

$$\Phi_+ := \Phi \cap \mathcal{H}_+, \tag{2.25}$$

and similarly the *negative roots* of $\mathfrak{g}$ to be

$$\Phi_- := \Phi \cap \mathcal{H}_- = \Phi \backslash \Phi_+. \tag{2.26}$$

From (2.13) we can define the following two subalgebras of $\mathfrak{g}$:

$$\mathfrak{g}_+ := \mathrm{span}\{E^\alpha | \alpha \in \Phi_+\}, \qquad \mathfrak{g}_- := \mathrm{span}\{E^\alpha | \alpha \in \Phi_-\}. \tag{2.27}$$

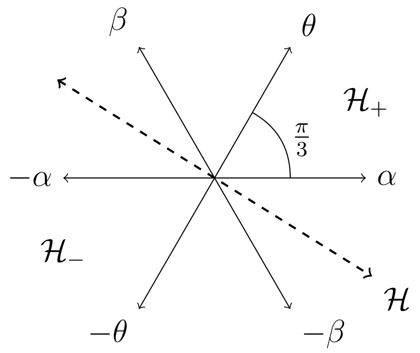
Since $\Phi = \Phi_+ \cup \Phi_-$, $\mathfrak{g}$ can now be expressed as the vector space direct sum

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+. \tag{2.28}$$

A decomposition of this form is called a *triangular* or *Gauss decomposition* of $\mathfrak{g}$. An important property of a triangular decomposition is that $[\mathfrak{g}_0, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm$.

Example: Since $\mathfrak{sl}(2)$ has a Cartan subalgebra of dimension one the only hyperplane is trivial. Choosing α to lie in the positive root space gives $\Phi_+ = \{\alpha\}$ and $\Phi_- = \{-\alpha\}$.

Example: The root system of $\mathfrak{sl}(3)$ is represented in the following diagram, with the canonical choice of hyperplane depicted.



It follows that the positive roots are given by $\Phi_+ = \{\alpha, \beta, \theta\}$ and the negative roots by $\Phi_- = \{-\alpha, -\beta, -\theta\}$, this motivates our labelling of the roots earlier (2.17). Note that the angle on the root space diagram is calculated using the inner product on the root space, similarly all other angles between adjacent roots are $\pi/6$.

We define a *simple root* of $\mathfrak{g}$ to be a positive root that cannot be expressed as a linear combination with positive coefficients of other positive roots. Denote by Φ_s the set of all simple roots. An important property of the root system is that all positive roots can be written as a positive integral linear combination of the simple roots. Hence, the set of all simple roots form a basis for $\mathfrak{g}_0^*$. It follows that there must be $r = \text{rank}(\mathfrak{g})$ simple roots which we denote by

$$\Phi_s = \{\alpha^{(i)} | i = 1, 2, ..., r\}. \tag{2.29}$$

It is a nice property of the simple roots that $\alpha^{(i)} - \alpha^{(j)} \notin \Phi$ for all $i \neq j$.

With respect to a root space decomposition we define the *highest root* as the root $\theta \in \Phi_+$ that satisfies

$$\theta + \alpha \notin \Phi \quad \forall \alpha \in \Phi_+. \tag{2.30}$$

All finite dimensional simple Lie algebras possess a unique highest root. For $\mathfrak{sl}(2)$ we have $\theta = \alpha$ and for $\mathfrak{sl}(3)$ we have labelled θ so that it is the highest root with respect to the positive roots defined above. The convention for normalising the Killing form is to set $(\theta, \theta) = 2$.

Coroots and the Cartan matrix

For each root $\alpha \in \Phi$ we define the *coroot* by

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)}. \tag{2.31}$$

The *Cartan matrix* A of $\mathfrak{g}$ is defined as the $r \times r$ matrix with entries given by

$$A^{ij} := \frac{2(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(j)}, \alpha^{(j)})} = (\alpha^{(i)}, \alpha^{(j)\vee}). \tag{2.32}$$

Example: $\mathfrak{sl}(2)$ has one positive root α, so the Cartan matrix is $A(\mathfrak{sl}(2)) = [2]$.

Example: $\mathfrak{sl}(3)$ has two simple roots $\alpha^{(1)} = \alpha$ and $\alpha^{(2)} = \beta$ and a third positive root $\theta = \alpha^{(1)} + \alpha^{(2)}$. The entries of the Cartan matrix are given by $A^{11} = A^{22} = 2$ and

$$A^{12} = \frac{2(\alpha,\beta)}{(\beta,\beta)} = -1, \qquad A^{21} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = -1, \tag{2.33}$$

so that the Cartan matrix is given by

$$A(\mathfrak{sl}(3)) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \tag{2.34}$$

Chevalley-Serre relations

A Chevalley basis of $\mathfrak{g}$ is a Cartan-Weyl basis that also satisfies the properties

$$[H^\alpha, E^{\pm\alpha}] = \pm 2E^{\pm\alpha}, \qquad [E^\alpha, E^{-\alpha}] = H^\alpha. \tag{2.35}$$

It has the nice property that H^α is identified with its coroot via

$$\alpha^\vee(h) = \kappa(H^\alpha, h). \tag{2.36}$$

Given a Chevalley basis it is useful to analyse the Lie bracket on the elements associated to the simple roots. To each simple root we associate the elements $\{E^i, E^{-i}, H^i\}$ such that $E^{\pm i} \in \mathfrak{g}_{\pm\alpha^{(i)}}$, $H^i \in \mathfrak{g}_0$ and together $\{E^i, E^{-i}, H^i\}$ satisfy the relations (2.7) of the standard $\mathfrak{sl}(2)$ basis. For all $i, j \in \{1, ..., r\}$ we have

$$\begin{aligned} [H^i, H^j] &= 0, \\ [H^i, E^j_\pm] &= \pm A^{ji} E^\alpha, \\ [E^i, E^j] &= \delta_{ij} H^i, \\ (\mathrm{ad}_{E^i_\pm})^{1-A^{ji}} E^j_\pm &= 0. \end{aligned} \tag{2.37}$$

These relations are known as the *Chevalley-Serre* relations and the set of elements $\{E^{\pm i}, H^i\}_{i=1}^r$ are known as the *Chevalley-Serre generators*, their importance is demonstrated in the following theorem.

Theorem 2.1.1 *A simple Lie algebra $\mathfrak{g}$ with Cartan matrix A of dimension $r \times r$ is isomorphic to the Lie algebra that is algebraically generated by the elements $\{E^{\pm i}, H^i\}_{i=1}^r$ subject to the Chevalley-Serre relations.*

This theorem implies that a simple Lie algebra $\mathfrak{g}$ is fully determined up to isomorphism by its Cartan matrix and all simple Lie algebras can be constructed from a Cartan matrix via the Chevalley-Serre relations. We will refer to this construction as the *Chevalley-Serre construction*.

Properties of a Chevalley basis

We now note some properties of a Chevalley basis that will be important in the construction of twisted affine Lie algebras. Suppose that the Killing form of a Lie algebra has been fixed and we wish to normalise a Cartan-Weyl basis to get a Chevalley basis. The relations (2.35) can be shown to satisfy

$$[E^\alpha, E^{-\alpha}] = c_\alpha \kappa(E^\alpha, E^{-\alpha}) H^\alpha = H^\alpha \qquad \Longrightarrow \qquad c_\alpha = \frac{1}{\kappa(E^\alpha, E^{-\alpha})}, \tag{2.38}$$

$$[H^\alpha, E^{\pm\alpha}] = \pm c_\alpha \kappa(H^\alpha, H^\alpha) E^{\pm\alpha} = \pm 2E^{\pm\alpha} \qquad \Longrightarrow \qquad c_\alpha = \frac{2}{\kappa(H^\alpha, H^\alpha)}. \tag{2.39}$$

Equation (2.39) fully specifies the normalisation of c_α in a Chevalley basis and (2.38) determines an appropriate normalisation of the elements $E^{\pm\alpha}$.

Classification of finite dimensional simple Lie algebras

To fully classify all finite dimensional simple Lie algebras up to isomorphism it remains to find all possible Cartan matrices up to permutations of the rows and columns. The following proposition is important in achieving this classification.

Proposition 2.1.2 *A finite dimensional simple Lie algebra $\mathfrak{g}$ has a Cartan matrix A that satisfies*

1. $A^{ii} = 2$
2. $A^{ij} = 0 \iff A^{ji} = 0$
3. $A^{ij} \in \mathbb{Z}_{\leq 0}$ *for* $i \neq j$
4. $\det A > 0$
5. *A is indecomposable in the sense that it can't be written as a block diagonal matrix constructed from two or more matrices satisfying 1 - 4 by reordering of the rows and columns.*

The converse is also true in the sense that any matrix A of dimension $r \times r$ that satisfies 1 - 5 also generates a finite dimensional simple Lie algebra of rank r via the Chevalley-Serre construction.

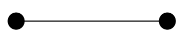
For a table of Cartan matrices of finite dimensional simple Lie algebras see [22, p117]. Furthermore, for any finite dimensional semisimple Lie algebra, the Cartan matrix decomposes as per criterion 5, and the diagonal blocks are given by the Cartan matrices of the simple ideals.

Another useful tool in the classification of finite dimensional simple Lie algebras is the Dynkin diagram. Suppose $\mathfrak{g}$ is a Lie algebra with Cartan matrix A. To $\mathfrak{g}$ we prescribe the *Dynkin diagram* as follows:

1. Draw $r = \text{rank}(\mathfrak{g})$ vertices and label them $1, 2..., r$.
2. Connect each pair of vertices (i, j) by $\max\{|A^{ij}|, |A^{ji}|\}$ edges.
3. If $|A^{ij}| > |A^{ji}|$, then affix an arrow to the edges between i and j pointing from i to j.

This is not the only convention for prescribing Dynkin diagrams, but we have chosen this convention since it is adopted in [22]. In this way each Cartan matrix has a unique Dynkin diagram and all finite dimensional simple Lie algebras can be completely determined from their Dynkin diagram. For a complete list of Dynkin diagrams for the finite dimensional simple Lie algebras see [22, p144].

Example: From the Cartan matrix of $\mathfrak{sl}(3)$ (2.34) we can construct the Dynkin diagram for $\mathfrak{sl}(3)$:



2.2 Kac-Moody Lie algebras

In the previous section we have observed that we can construct all finite dimensional simple Lie algebras from matrices that satisfy the requirements of (2.1.2). This can be seen through the Chevalley-Serre construction presented in Theorem 2.1.1.

In order to start developing a theory of infinite dimensional Lie algebras without losing all the nice properties of the finite dimensional Lie algebras we can relax the requirements on the Cartan matrix. A *generalised Cartan matrix* is a matrix A that satisfies all of the requirements in (2.1.2) except for $\det A > 0$. The class of Lie algebras generated by the generalised Cartan matrices is called the *Kac-Moody* Lie algebras.

Currently, the most important class of Kac-Moody algebras is obtained if instead of completely removing the requirement $\det A > 0$ we replace it with

$$\det A_{\{i\}} > 0 \quad \text{for all } i = 0, 1, ..., r. \tag{2.40}$$

The notation $A_{\{i\}}$ denotes the matrix obtained from A by removing the i^{th} row and column. The labelling convention for the rows and the columns has also been changed so that the index starts from zero instead of one. The reasons for this will become apparent soon. In this way A is an $(r + 1) \times (r + 1)$ matrix. The above requirement implies that the Cartan matrix has rank at least r. Matrices of this form are called *affine Cartan matrices* and their corresponding Lie algebras are called *affine Lie algebras*.

The classification of affine Lie algebras can be achieved by finding all affine Cartan matrices [22, p125]. In the next section we provide an alternative construction of the affine Lie algebras which will prove to be very useful in understanding the root system as well as understanding many important applications of these algebras.

2.3 Untwisted affine Lie algebras

The simplest class of affine Lie algebras to construct are the untwisted affine Lie algebras, these are constructed via loop algebras of the finite dimensional Lie algebras. This construction of the affine Lie algebras is called the *loop construction*. We will first outline the loop construction of the untwisted affine Lie algebras before determining important properties of the root system such as the simple roots and the inner product on the root system.

2.3.1 Loop construction

In this section the Lie algebra which is the starting point of the loop construction will be referred to as $\mathfrak{g}$. The Killing form on $\mathfrak{g}$ will be denoted by $\bar{\kappa}$. The affine Lie algebra that is the final goal of the loop construction will be denoted by $\hat{\mathfrak{g}}$.

Loop Algebras

The *loop algebra over* $\mathfrak{g}$ consists of the space of polynomial maps from the circle S^1 to $\mathfrak{g}$ and is denoted by $\mathfrak{g}_{\text{loop}}$. We can consider S^1 as the unit circle in the complex plane with coordinate $z = e^{2\pi i t}$. The loop algebra is given by

$$\mathfrak{g}_{\text{loop}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]. \tag{2.41}$$

If $\{T_n^a | a = 1, ..., d\}$ is a basis for $\mathfrak{g}$ then the loop algebra has a basis given by

$$\mathcal{B}_{\text{loop}} = \{T_n^a | a = 1, ..., d; n \in \mathbb{Z}\}, \qquad T_n^a := \bar{T}^a \otimes z^n. \tag{2.42}$$

This space is a Lie algebra when endowed with the Lie bracket inherited from $\mathfrak{g}$:

$$[T_n^a, T_m^b] = [\bar{T}^a \otimes z^n, \bar{T}^b \otimes z^m] := [\bar{T}^a, \bar{T}^b] \otimes z^{n+m} \tag{2.43}$$

The index n of T_n^a is additive under the Lie bracket, hence the Lie bracket provides a $\mathbb{Z}$-gradation of $\mathfrak{g}_{\text{loop}}$. Furthermore the subalgebra of $\mathfrak{g}_{\text{loop}}$ spanned by generators of the form T_0^a is called the *horizontal subalgebra* and is isomorphic to the original Lie algebra $\mathfrak{g}$.

Central Extension

The construction of a loop algebra is well defined for all Lie algebras $\mathfrak{g}$. However, the special case of taking $\mathfrak{g}$ to be a finite dimensional simple Lie algebra will lead us to a construction of the untwisted affine Lie algebras. First, we must also introduce another element called the central extension.

For a finite dimensional simple Lie algebra, $\mathfrak{g}$, its loop algebra described in section 2.3.1 has no central element. In order to construct the untwisted affine Lie algebras we must first introduce a central extension to $\mathfrak{g}_{\text{loop}}$. This element is denoted by K, since it is central it has Lie bracket given by

$$[K, K] = 0, \quad [K, T_n^a] = 0 \quad \forall T_n^a \in \mathfrak{g}_{\text{loop}}. \tag{2.44}$$

The brackets of generators of $\mathfrak{g}_{\text{loop}}$ are now altered to include a factor of K:

$$[T^a_n, T^b_m] = [\bar{T}^a, \bar{T}^b] \otimes z^{n+m} + f^{ab}{}_{mn} K, \tag{2.45}$$

where $f^{ab}{}_{mn}$ is a structure constant which is determined by the properties (2.1) and (2.2) of the Lie bracket. Up to a multiplicative constant the result is

$$f^{ab}{}_{mn} = m\delta_{m+n,0}\bar{\kappa}^{ab}, \tag{2.46}$$

where $\bar{\kappa}$ is the Killing form on the horizontal algebra $\mathfrak{g}$ [22].

The derivation

To complete the construction we must introduce one further generator called the *derivation*, denoted by D, that satisfies the following relations

$$[D, T^a_n] = nT^a_n, \qquad [D, K] = 0. \tag{2.47}$$

In this way the generator D measures the index n of the generators T^a_n. We call the negative of this index the *grade* of the element T^a_n. The role of D is similar to that of L_0 in the Virasoro algebra, see Section 4.1, and thus is often identified with L_0 in applications pertaining to conformal field theory. For this reason the operator D will not appear in Chapters 4 and 5.

The affine Lie algebra $\hat{\mathfrak{g}}$ has is given by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}K + \mathbb{C}D. \tag{2.48}$$

The Killing form

One of the important roles played by the derivation D is to make the Killing form κ on $\hat{\mathfrak{g}}$ non-degenerate. It can be shown [22] that the Killing form on $\hat{\mathfrak{g}}$ satisfies the relations

$$\begin{gathered} \kappa(T^a_n, T^b_m) = \delta_{n+m,0}\bar{\kappa}(T^a, T^b), \\ \kappa(D, K) = 1 = \kappa(K, D), \qquad \kappa(D, D) = 0 = \kappa(K, K), \end{gathered} \tag{2.49}$$

where $\bar{\kappa}$ is the Killing form on the finite dimensional simple Lie algebra $\mathfrak{g}$. Hence, the non-degeneracy of the Killing form $\bar{\kappa}$ on the finite dimensional simple Lie algebra $\mathfrak{g}$ implies that the Killing form on the affine Lie algebra $\hat{\mathfrak{g}}$ is non-degenerate. If the derivation D was not included the Killing form would be non-degenerate since $\kappa(K, X) = 0$ for all basis elements of $\hat{\mathfrak{g}}$ except $X = D$.

2.3.2 The root system

To construct a Cartan-Weyl basis for the affine Lie algebra $\hat{\mathfrak{g}}$ we begin by choosing a Cartan-Weyl basis (2.12) for $\mathfrak{g}$

$$\mathcal{B}(\mathfrak{g}) = \{H^i | i = 1, 2, ..., r = \text{rank}(\mathfrak{g})\} \cup \{E^\alpha | \alpha \in \Phi(\mathfrak{g})\}. \tag{2.50}$$

The corresponding Cartan-Weyl basis for the affine Lie algebra is given by

$$\mathcal{B}(\hat{\mathfrak{g}}) = \{H_n^i | i \in \{1, ..., r\}, n \in \mathbb{Z}\} \cup \{E_n^\alpha | \alpha \in \Phi(\mathfrak{g}), n \in \mathbb{Z}\} \cup \{K, D\}. \tag{2.51}$$

The canonical Cartan subalgebra is given by $\hat{\mathfrak{g}}_0 = \mathrm{span}\{H_0^i, K, D | i = 1, ..., r\}$. Clearly this set of elements is commuting and their adjoint actions are diagonal with respect to the Cartan-Weyl basis for $\hat{\mathfrak{g}}$. This set is maximal since all other generators have non-zero bracket with either D or at least one of the H_0^i.

The adjoint action of the Cartan subalgebra basis $\{H_0^i, K, D | i = 1, ..., r\}$ on the elements E_n^α and H_n^j of $\hat{\mathfrak{g}}$ is given by

$$[H_0^i, E_n^\alpha] = \alpha(H^i) E_n^\alpha, \quad [K, E_n^\alpha] = 0, \quad [D, E_n^\alpha] = n E_n^\alpha, \tag{2.52}$$

$$[H_0^i, H_n^j] = 0, \quad [K, H_n^j] = 0, \quad [D, H_n^j] = n H_n^j. \tag{2.53}$$

From (2.52) we represent the root of $E_n^\alpha \in \hat{\mathfrak{g}}$ as $(\alpha, 0, n)$. From (2.53) we represent the root of H_n^i as $(0, 0, n)$. It follows that the root system of $\hat{\mathfrak{g}}$ is given by

$$\Phi(\hat{\mathfrak{g}}) = \{(\alpha, 0, n) | \alpha \in \Phi(\mathfrak{g}), n \in \mathbb{Z}\} \cup \{(0, 0, n) | n \in \mathbb{Z} \backslash \{0\}\}. \tag{2.54}$$

Since $\mathfrak{g}$ is a finite dimensional simple Lie algebra we can separate the root space $\Phi(\mathfrak{g})$ into positive roots, $\Phi_+(\mathfrak{g})$, and negative roots, $\Phi_-(\mathfrak{g})$. We can then define the positive roots of $\hat{\mathfrak{g}}$ to be all those with positive index (i.e. $n > 0$) as well as the roots $(\alpha, 0, 0)$ with $\alpha \in \Phi_+(\mathfrak{g})$:

$$\Phi_+(\hat{\mathfrak{g}}) := \{(\alpha, 0, n) | \alpha \in \Phi(\mathfrak{g}) \text{ or } \alpha = 0, n > 0\} \cup \{(\alpha, 0, 0) | \alpha \in \Phi_+(\mathfrak{g})\} \tag{2.55}$$

Furthermore, if $\alpha^{(i)}, ..., \alpha^{(r)}$ are the simple roots of $\mathfrak{g}$ and θ is the highest root of $\mathfrak{g}$, then we can define the simple roots of $\hat{\mathfrak{g}}$ to be

$$\Phi_s(\hat{\mathfrak{g}}) := \{(-\theta, 0, 1)\} \cup \{(\alpha^{(i)}, 0, 0) | i = 1, ..., r\}. \tag{2.56}$$

The root $(-\theta, 0, 1)$ in the first set of this union is called the *affine* simple root of $\hat{\mathfrak{g}}$ and the roots in the second set are called the *non-affine* simple roots. We label the simple roots of $\hat{\mathfrak{g}}$ by $\alpha^{(0)} := (-\theta, 0, 1)$ and $\alpha^{(i)} := (\alpha^{(i)}, 0, 0)$ for $i = 1, ..., r$ (clearly this is an abuse of notation, but the context in which the root appears should make it clear whether the root belongs to the affine Lie algebra or the corresponding finite dimensional simple Lie algebra).

It is clear now that an untwisted affine Lie algebra that is constructed via the loop construction from a finite dimensional Lie algebra $\mathfrak{g}$ has $r + 1$ simple roots, where $r = \mathrm{rank}(\mathfrak{g})$. This explains why the simple roots are indexed from zero to r. Furthermore, if the first row and column are removed from the Cartan matrix of the affine Lie algebra, the remaining matrix is given by the Cartan matrix of $\mathfrak{g}$. In order to be able to compute the Cartan matrix of an untwisted affine Lie algebra we first need to introduce the inner product on the root system.

Inner product on the affine root space

Suppose an isomorphism between $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ as in (2.19) is fixed. For each $\alpha \in \Phi(\mathfrak{g})$, denote by H^α the element of the Cartan subalgebra such that $\alpha \mapsto H^\alpha$ under this isomorphism. From the Killing form (2.49) on $\hat{\mathfrak{g}}$, this isomorphism induces an isomorphism between the root space of $\hat{\mathfrak{g}}$ and the Cartan subalgebra $\hat{\mathfrak{g}}_0$ given by

$$(\alpha, k, n) \mapsto H_0^\alpha + kD + nK. \tag{2.57}$$

From this isomorphism it follows that the inner product on the affine root space is given by

$$\langle(\alpha, k, n), (\alpha', k', n')\rangle = (\alpha, \alpha')_{\mathfrak{g}} + k'n + n'k, \tag{2.58}$$

where $(\cdot, \cdot)_{\mathfrak{g}}$ is the inner product on the root space of $\mathfrak{g}$. Taking the inner product of a root $(\alpha, 0, n)$ with itself gives

$$\langle(\alpha, 0, n), (\alpha, 0, n)\rangle = (\alpha, \alpha). \tag{2.59}$$

Thus, all roots of the form $(0, 0, n)$ have zero norm, a root with this property is called *imaginary*. All other roots have non-zero norm since the inner product on $\mathfrak{g}$ is non-degenerate.

As in the case of the finite dimensional simple Lie algebras we can define the simple coroots of $\hat{\mathfrak{g}}$ by the formula (2.31) and the Cartan matrix by the formula (2.32), with the inner product being defined in (2.58).

If the inner product on $\mathfrak{g}$ is normalised as discussed in Section 2.1 then the coroots of $\hat{\mathfrak{g}}$ are given by

$$\alpha^{(0)^\vee} = (-\theta, 0, 1), \qquad \alpha^{(i)^\vee} = (\alpha^{(i)^\vee}, 0, 0). \tag{2.60}$$

From equation (2.36) the resulting Chevalley-Serre generators of $\hat{\mathfrak{g}}$ are

$$\begin{aligned} H^0 &= -H_0^\theta + K, & E^{\pm 0} &= E_{\pm 1}^{\mp\theta}, & \\ H^i &= H_0^i, & E^{\pm i} &= E_0^{\pm i}, & \forall i = 1, ..., r \end{aligned} \tag{2.61}$$

where $\{E^{\pm i}, H^i\}_{i=1}^r$ are the Chevalley-generators of $\mathfrak{g}$.

Coxeter labels

If A is the Cartan matrix of the affine Lie algebra then we define the Coxeter labels a_i and the dual Coxeter labels $a_i^\vee$ of an affine Lie algebra by requiring that

$$\sum_{j=0}^{r} a_j A^{ji} = 0 = \sum_{j=0}^{r} A^{ij} a_j^\vee, \tag{2.62}$$

with the added normalisation condition

$$\max\{a_1, ..., a_r\} = 1 = \max\{a_1^\vee, ..., a_r^\vee\}. \tag{2.63}$$

Such numbers exist and are unique since all affine Cartan matrices have eigenvalue zero with multiplicity one. The *Coxeter number* g and *dual Coxeter number* $g^\vee$ are defined as

$$g := \sum_{i=0}^{r} a_i, \qquad g^\vee := \sum_{i=0}^{r} a_i^\vee. \tag{2.64}$$

These numbers will be important in the representation theory of affine Lie algebras as well as in the WZW models.

We can also observe the need to add a central element in the loop construction by noting that all affine Lie algebras have a central element given by

$$K := \sum_{i=0}^{r} a_i^\vee H^i, \tag{2.65}$$

where H^i are the Chevalley-generators of the affine Lie algebra, they satisfy the relations given in (2.37).

Example: $\widehat{\mathfrak{sl}(2)}$

The simplest case of an untwisted affine Lie algebra is $\widehat{\mathfrak{sl}(2)}$. The Cartan-Weyl basis elements are given by

$$\{E_n, H_n, F_n | n \in \mathbb{Z}\} \cup \{K, D\}. \tag{2.66}$$

With respect to the basis elements the non-zero Lie brackets of $\widehat{\mathfrak{sl}(2)}$ are given by

$$\begin{aligned} [E_n, F_m] &= H_{n+m} + n\delta_{n+m,0}K, & [H_n, E_m] &= 2E_{n+m}, \\ [H_n, F_m] &= -2F_{n+m}, & [H_n, H_m] &= 2n\delta_{n+m,0}K, \end{aligned}$$

as well as those defined in (2.44) and (2.47) for the central element K and the derivation D respectively.

The canonical Cartan subalgebra has basis given by $\{H_0, K, D\}$. The roots of the elements E_n, F_n and H_n are represented by $(\alpha, 0, n)$, $(-\alpha, 0, n)$ and $(0, 0, n)$ where $\alpha(H) = 2$ is the simple root of $\mathfrak{sl}(2)$. It follows that the root system of $\widehat{\mathfrak{sl}(2)}$ is given by

$$\Phi = \{(\pm\alpha, 0, n) | n \in \mathbb{Z}\} \cup \{(0, 0, n) | 0 \neq n \in \mathbb{Z}\}. \tag{2.67}$$

With respect to the H_0 and D eigenvalue, the root system for grades between -3 and 3 is given in figure 2.1.

The hyperplane $\mathcal{H}$ in figure 2.1 is the canonical hyperplane used to separate the root system into positive and negative roots. Using this hyperplane the simple roots are

$$\Phi_s = \{(-\alpha, 0, 1), (\alpha, 0, 0)\}. \tag{2.68}$$

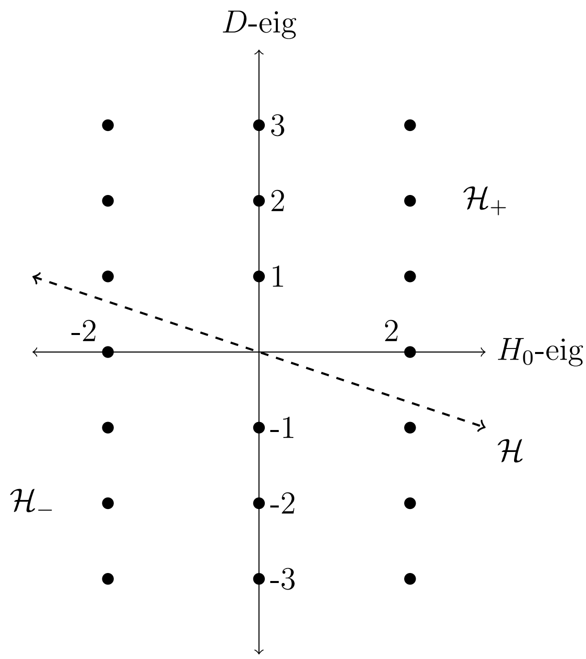


Figure 2.1: Root system for $\widehat{\mathfrak{sl}(2)}$ between grades -3 and 3.

With respect to the inner product defined in (2.58) the norm of both of these roots is $(\alpha, \alpha) = 2$. By definition (2.31) the simple coroots of $\widehat{\mathfrak{sl}(2)}$ are equal to the simple roots:

$$\alpha^{(0)} = \alpha^{(0)^\vee} = (-\alpha, 0, 1), \qquad \alpha^{(1)} = \alpha^{(1)^\vee} = (\alpha, 0, 0). \tag{2.69}$$

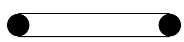
Taking the inner product of the two simple roots gives

$$(\alpha^{(0)}, \alpha^{(1)}) = \langle(-\alpha, 0, 1), (\alpha, 0, 1)\rangle = -(\alpha, \alpha) = -2. \tag{2.70}$$

Putting these results together, definition (2.32) implies that the Cartan matrix is

$$A(\widehat{\mathfrak{sl}(2)}) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}. \tag{2.71}$$

Equivalently, $\widehat{\mathfrak{sl}(2)}$ has the following Dynkin digram:



2.4 Twisted affine Lie algebras

The twisted affine Lie algebras are constructed in a similar way to the untwisted affine Lie algebras. However, the loop construction is now done with respect to a gradation of a finite dimensional Lie algebra.

Every automorphism on a finite dimensional simple Lie algebra gives rise to a gradation of $\mathfrak{g}$. The automorphisms that are important in the construction of the twisted affine Lie algebras are the outer automorphisms. Each outer automorphism is associated with a non-trivial automorphism of the Dynkin diagram of $\mathfrak{g}$. We first review a few results about automorphisms on finite dimensional simple Lie algebras, before dealing with the specific case of outer automorphisms.

2.4.1 Automorphisms of finite dimensional simple Lie algebras

An *automorphism* σ of a Lie algebra $\mathfrak{g}$ is a bijective linear map from $\mathfrak{g}$ to itself that obeys the Lie bracket

$$\sigma([x,y]) = [\sigma(x), \sigma(y)]. \tag{2.72}$$

The set of all automorphisms of $\mathfrak{g}$ form a group under composition which will be denoted by $\mathrm{Aut}(\mathfrak{g})$.

An automorphism σ is said to be of *finite order* if there exists $N \in \mathbb{N}$ such that σ^N is the identity map. In this case N is called the *order* of σ. If σ is of finite order N, then $\mathfrak{g}$ can be written as the vector space direct sum of eigenspaces of σ:

$$\mathfrak{g} = \bigoplus_{j=0}^{N-1} \mathfrak{g}_{[j]}, \qquad \mathfrak{g}_{[j]} = \{x \in \mathfrak{g} | \sigma(x) = e^{2\pi i j/N} x\}. \tag{2.73}$$

The condition (2.72) applied to the two eigenspaces in the decomposition (2.73) gives

$$[\mathfrak{g}_{[i]}, \mathfrak{g}_{[j]}] \subseteq \mathfrak{g}_{[i+j \mod N]}. \tag{2.74}$$

Any decomposition of the form (2.73) that satisfies (2.74) is called a $\mathbb{Z}_N$-*gradation* of $\mathfrak{g}$. Hence any automorphism of a Lie algebra of finite order N generates a $\mathbb{Z}_N$-gradation of the Lie algebra. These gradations have a few important properties [22].

Proposition 2.4.1 *Suppose σ is an automorphism of finite order N on a finite dimensional simple Lie algebra $\mathfrak{g}$. Then*

1. *Only the eigenspace $\mathfrak{g}_{[0])}$ with eigenvalue 1 is a subalgebra of $\mathfrak{g}$, this is often called the fixed point subalgebra of σ.*

2. *If κ is a Killing form on $\mathfrak{g}$ then κ is invariant under σ.*

3. *If $x_i \in \mathfrak{g}_{[i]}$ and $x_j \in \mathfrak{g}_{[j]}$ then the Killing form satisfies*

$$\kappa(x_i, x_j) = e^{2\pi i \frac{(i+j)}{N}} \kappa(x_i, x_j) \delta_{(i+j,0 \mod N)} \tag{2.75}$$

Eigenspaces of outer automorphisms

Let $\{\alpha^{(i)}|i = 1, ..., r = \text{rank}(\mathfrak{g})\}$ be a basis of simple roots for a finite dimensional simple Lie algebra $\mathfrak{g}$. Suppose ω is a non-trivial automorphism of the Dynkin diagram of $\mathfrak{g}$ and thus can be extended to a map on the simple roots:

$$\omega(\alpha_i) = \alpha^{\omega(i)}. \tag{2.76}$$

This Dynkin diagram automorphism ω generates an *outer automorphism* σ on $\mathfrak{g}$ by defining σ to act on the Chevalley-Serre generator, E^i, (2.1.1) for each simple root by

$$\sigma(E^i) = E^{\sigma(i)}. \tag{2.77}$$

The requirement (2.72) implies that

$$\sigma(E^{-i}) = E^{-\sigma(i)}, \qquad \sigma(H^i) = H^{\sigma(i)}. \tag{2.78}$$

The general form of the automorphism σ on an arbitrary root α is

$$\sigma(E^\alpha) = e_\alpha E^{\sigma(\alpha)}, \tag{2.79}$$

where $e_\alpha = \pm 1$ and $e_\alpha = e_{-\alpha}$. The exact value of e_α is determined by the Lie brackets of $\mathfrak{g}$. For example, suppose α^i and α^j are two simple roots which are exchanged under the automorphism ω, and are such that $\alpha = \alpha^i + \alpha^j$ is an element of the root system. Then $\omega(\alpha) = \alpha$ and we find that $e_\alpha = -1$ by the calculation

$$\sigma(E^\alpha) = \sigma([E^{\alpha^i}, E^{\alpha^j}]) = [E^{\alpha^j}, E^{\alpha^i}] = -E^\alpha. \tag{2.80}$$

The complete evaluation of all e_α for $\alpha \in \Phi$ is in general a difficult process, but can be done without too much difficulty for simple Lie algebras of small rank or by the use of computer programs. Having calculated these values we are able to construct the eigenspaces of σ.

From [22, p188], all outer automorphisms of the finite dimensional simple Lie algebras, except for D_4, are of order two. Thus, we will assume σ has order two, under which there are two eigenspaces $\mathfrak{g}_{[0]}$ and $\mathfrak{g}_{[1]}$ with eigenvalues 1 and -1 respectively. In this case we can construct bases for $\mathfrak{g}_{[0]}$ and $\mathfrak{g}_{[1]}$ as follows.

1. Beginning with the simple coroots we have $H^i \in \mathfrak{g}_{[0]}$ if $\sigma(i) = i$, otherwise $H^i + H^{\sigma(i)} \in \mathfrak{g}_{[0]}$ and $H^i - H^{\sigma(i)} \in \mathfrak{g}_{[0]}$

2. Extending to a Cartan Weyl basis of $\mathfrak{g}$ and making use of the constants e_α. The rest of $\mathfrak{g}$ decomposes into eigenspaces as follows. If $\sigma(\alpha) = \sigma(\alpha)$ then $E^\alpha \in \mathfrak{g}_{[0]}$ if $e_\alpha = 1$ and $E^\alpha \in \mathfrak{g}_{[1]}$ if $e_\alpha = -1$. Otherwise $E^\alpha + e_\alpha E^{\sigma(\alpha)} \in \mathfrak{g}_{[0]}$ and $E^\alpha - e_\alpha E^{\sigma(\alpha)} \in \mathfrak{g}_{[1]}$

The treatment of the order three automorphisms of the Dynkin diagram for D_4 need to be treated more carefully, but we will not consider this case.

Example: Outer automorphism on $\mathfrak{sl}(3)$

The simplest Dynkin diagram with a non-trivial automorphism is that of $\mathfrak{sl}(3)$. The non-trivial diagram automorphism ω corresponds to switching the two simple roots, α and β:

$$\omega(\alpha) = \beta, \qquad \omega(\beta) = \alpha. \tag{2.81}$$

This generates an automorphism σ of $\mathfrak{sl}(3)$ that has finite order 2 and satisfies

$$\sigma(E^{\pm\alpha}) = E^{\pm\beta}, \quad \sigma(E^{\pm\beta}) = E^{\pm\alpha}, \quad \sigma(H^{\alpha}) = H^{\beta}, \quad \sigma(H^{\beta}) = H^{\alpha}, \tag{2.82}$$

$$\sigma(E^{\pm\theta}) = \sigma(\pm[E^{\pm\alpha}, E^{\pm\beta}]) = \pm[E^{\pm\beta}, E^{\pm\alpha}] = -E^{\pm\theta}. \tag{2.83}$$

Bases for the eigenspaces of σ are given by

$$\begin{aligned} \mathfrak{sl}(3)_{[0]} &= \mathrm{span}\{H^{\alpha} + H^{\beta}, E^{\alpha} + E^{\beta}, E^{-\alpha} + E^{-\beta}\}, \\ \mathfrak{sl}(3)_{[1]} &= \mathrm{span}\{H^{\alpha} - H^{\beta}, E^{\alpha} - E^{\beta}, E^{-\alpha} - E^{-\beta}, E^{\theta}, E^{-\theta}\}. \end{aligned} \tag{2.84}$$

2.4.2 Loop construction with twisted boundary conditions

In constructing the untwisted affine Lie algebras we considered only polynomials from S^1 to $\mathfrak{g}$. These satisfy the periodic boundary condition

$$\mathcal{P}(e^{2\pi i}z) = \mathcal{P}(z). \tag{2.85}$$

To obtain the twisted affine Lie algebras we instead apply the twisted boundary condition

$$x \otimes \mathcal{P}(e^{2\pi i}z) = \sigma(x) \otimes \mathcal{P}(z) \tag{2.86}$$

where σ is an outer automorphism of $\mathfrak{g}$ with finite order N.

Using the properties of finite order automorphisms we know that σ creates a $\mathbb{Z}_N$-gradation of $\mathfrak{g}$ whose eigenstates satisfy

$$\sigma(x_j) = e^{2\pi i j/N} x_j. \tag{2.87}$$

In order for the product of a map $\mathcal{P}$ with x_j to satisfy the twisted boundary conditions we require

$$\mathcal{P}(e^{2\pi i}z) = e^{2\pi i j/N}\mathcal{P}(z). \tag{2.88}$$

All such maps can be written as a series of the form

$$\mathcal{P}(z) = \sum_{k\in\mathbb{Z}} a_i z^{k+\frac{j}{N}}, \tag{2.89}$$

with coefficients a_i. Suppose that we have a basis for $\mathfrak{g}$ that respects the eigenspace decomposition of σ (i.e. all elements of the basis belong to an eigenspace of σ):

$$\mathcal{B} = \{T^a_j | T^a_j \in \mathfrak{g}_{[j]}; a = 1, 2, ..., d = \dim(\mathfrak{g}), j = 1, ..., N\}. \tag{2.90}$$

From (2.89), we define the *twisted loop algebra* to be the space with basis

$$\mathcal{B}_{\text{loop}} = \{T^a_{k+\frac{j}{n}} | T^a_j \in \mathfrak{g}_{[j]}; a = 1, 2, ..., d; j = 1, ..., N\}, \tag{2.91}$$

$$T^a_{k+\frac{j}{n}} := T^a_j \otimes z^{k+\frac{j}{n}}. \tag{2.92}$$

Since the gradation of $\mathfrak{g}$ with respect to σ respects the Lie bracket (2.74), the twisted loop algebra of $\mathfrak{g}$ can be endowed with a Lie bracket via

$$[T^a_{n+\frac{j}{N}}, T^b_{m+\frac{j'}{N}}] = [T^a_j, T^b_{j'}] \otimes z^{n+m+\frac{j+j'}{N}}. \tag{2.93}$$

Note that this Lie bracket has the same form as the Lie bracket on the (untwisted) loop algebra over $\mathfrak{g}$ (2.43), except that the indices n of the elements T^a_n are no longer integer valued. Hence, we can express the Lie bracket on the twisted loop algebra as

$$[T^a_n, T^b_m] = [T^a, T^b]_{n+m}, \tag{2.94}$$

where $T^a \in \mathfrak{g}_{[i]}$, $T^b \in \mathfrak{g}_{[j]}$ implies that $n \in \mathbb{Z} + i/N$ and $m \in \mathbb{Z} + j/N$.

To complete the construction of the twisted affine Lie algebra we must again add a central extension, K, and a derivation, D, to the twisted loop algebra. This process is the same as for the untwisted case, see Section 2.3. As such, the non-zero Lie brackets on the basis elements of the twisted affine Lie algebra are

$$[T^a_n, T^b_m] = [T^a, T^b]_{n+m} + n\bar{\kappa}^{ab}\delta_{n+m,0}K, \tag{2.95}$$

$$[D, T^a_n] = nT^a_n, \tag{2.96}$$

where $\bar{\kappa}$ is a matrix representation of the Killing form on $\mathfrak{g}$ with respect to the basis (2.90). The resulting algebra is called the twisted affine Lie algebra of $\mathfrak{g}$ with respect to σ and is denoted by $\hat{\mathfrak{g}}_\sigma$.

Root system

As in the untwisted case we will call the subalgebra spanned by elements of the form T^a_0 the *horizontal subalgebra*. Note that instead of being isomorphic to $\mathfrak{g}$ the horizontal subalgebra is isomorphic to $\mathfrak{g}_{[0]}$, a subalgebra of $\mathfrak{g}$. From examining the table of outer automorphisms appearing in [22, p215], the horizontal subalgebra is always a semisimple Lie algebra.

Let $\mathfrak{h}_0$ denote a Cartan subalgebra for $\mathfrak{g}_{[0]}$ and let $\{H^1_0, H^2_0, ...H^r_0\}$ be a basis for $\mathfrak{h}_0$. Here $r = \text{rank}(\mathfrak{g}_{[0]})$. Then a basis for a Cartan subalgebra of $\hat{\mathfrak{g}}_\sigma$ is

$$\mathcal{B} = \{H^1_0, H^2_0, ...H^r_0, K, D\}. \tag{2.97}$$

To deal with the root system in generality requires the use of representation theory of finite dimensional semisimple Lie algebras, which appears in Section 3.4.

We will briefly discuss this structure, before carefully demonstrating the root system in the example of $\mathbf{A}_2^{(2)}$.

Since $\mathfrak{g}_{[0]}$ is a semisimple Lie algebra, we can find a Cartan-Weyl basis. If E^α is an element of this basis with root α, then the eigenvalues of E^α_n ($n \in \mathbb{Z}$) with respect to the Cartan-Weyl basis are

$$[H^i_0, E^\alpha_n] = \alpha(H^i)E^\alpha_n, \qquad [K, E^\alpha_n] = 0, \qquad [D, E^\alpha_n] = nE^\alpha_n. \tag{2.98}$$

We thus represent the root of E^α_n by $(\alpha, 0, n)$ where $\alpha \in \Phi(\mathfrak{g}_{[0]})$. Similarly the roots of the elements H^i_n, where $0 \neq n \in \mathbb{Z}$ are represented by $(0, 0, n)$.

For $j \neq 0$, Proposition 2.4.1 implies

$$[\mathfrak{g}_{[0]}, \mathfrak{g}_{[j]}] \subseteq \mathfrak{g}_{[j]}. \tag{2.99}$$

In this way $\mathfrak{g}_{[j]}$ forms a $\mathfrak{g}_{[0]}$-module under the adjoint action of $\mathfrak{g}_{[0]}$. We denote this module by $V_{(j)}$. Since $V_{(j)}$ is finite dimensional, it decomposes into weight spaces where the weights are linear functionals on the Cartan subalgebra of $\mathfrak{g}_{[0]}$: $\lambda \in {\mathfrak{h}_0}^*$. Thus, if $E^\lambda \in \mathfrak{g}_{(j)}$ with weight λ, then the eigenvalues of E^λ_n ($n \in \mathbb{Z} + j/N$) with respect to the Cartan subalgebra (2.97) are given by

$$[H^i_0, E^\lambda_n] = \lambda(H^i)E^\lambda_n, \qquad [K, E^\lambda_n] = 0, \qquad [D, E^\lambda_n] = nE^\lambda_n. \tag{2.100}$$

Hence, the root of E^λ_n is represented by $(\lambda, 0, n)$.

If we denote by $W_{(j)}$ the set of all non-zero weight spaces in the decomposition of $V_{(j)}$, then the entire root space of $\hat{\mathfrak{g}}_\sigma$ is

$$\begin{aligned}\Phi(\hat{\mathfrak{g}}_\sigma) = \{(\alpha, 0, n)|\alpha \in \Phi(\mathfrak{g}_{[0]}), n \in \mathbb{Z}\} \cup \{(0, 0, n)|0 \neq n \in \mathbb{Z}\} \\ \cup \{(\lambda, 0, n)|j = 1, .., N-1, \lambda \in W_{(j)}, n \in \mathbb{Z} + j/N\}.\end{aligned} \tag{2.101}$$

2.4.3 Example: $\mathbf{A}_2^{(2)}$

$\mathbf{A}_2^{(2)}$ is the simplest case of a twisted affine Lie algebra and is constructed from the outer automorphism on $\mathfrak{sl}(3)$ introduced in section (2.4.1). It is of order two and the eigenspace decomposition of $\mathfrak{sl}(3)$ with respect to this automorphism is given in (2.84). Since $\mathfrak{g}_{[0]}$ has dimension three it must be isomorphic to $\mathfrak{sl}(2)$ and we can normalise a basis of $\mathfrak{g}_{[0]}$ to give the same brackets as the standard $\mathfrak{sl}(2)$ basis (2.7):

$$\mathfrak{g}_{[0]} = \{\sqrt{2}(E^\alpha + E^\beta), 2(H^\alpha + H^\beta) =: H^1, \sqrt{2}(E^{-\alpha} + E^{-\beta})\}. \tag{2.102}$$

A basis for a Cartan subalgebra for $\mathfrak{g}_{[0]}$ is $\{H^1\}$. The adjoint action of H^1 on $\mathfrak{g}_{[1]}$ is given by

$$\begin{aligned}&[H^1, E^\theta] = 4E^\theta, \qquad [H^1, E^\alpha - E^\beta] = 2(E^\alpha - E^\beta), \quad [H^1, H^\alpha - H^\beta] = 0, \\ &[H^1, E^{-\theta}] = -4E^{-\theta}, \quad [H^1, E^{-\alpha} - E^{-\beta}] = -2(E^{-\alpha} - E^{-\beta}).\end{aligned} \tag{2.103}$$

It follows that $\mathfrak{g}_{[1]}$ is isomorphic to the irreducible highest weight $\mathfrak{sl}(2)$-module with highest weight 4, $\mathcal{L}_{(4)}$. The root system of $\mathbf{A}_2^{(2)}$ is

$$\begin{aligned}\Phi(\mathbf{A}_2^{(2)}) = \{(\alpha, 0, n)|\alpha \in \Phi(\mathfrak{sl}(2)), n \in \mathbb{Z}\} \cup \{(0, 0, n)|0 \neq n \in \mathbb{Z}\} \\ \cup \{(\lambda, 0, n)|\lambda(H^1) \in \{-4, -2, 0, 2, 4\}, n \in \mathbb{Z} + 1/2\}. \end{aligned} \tag{2.104}$$

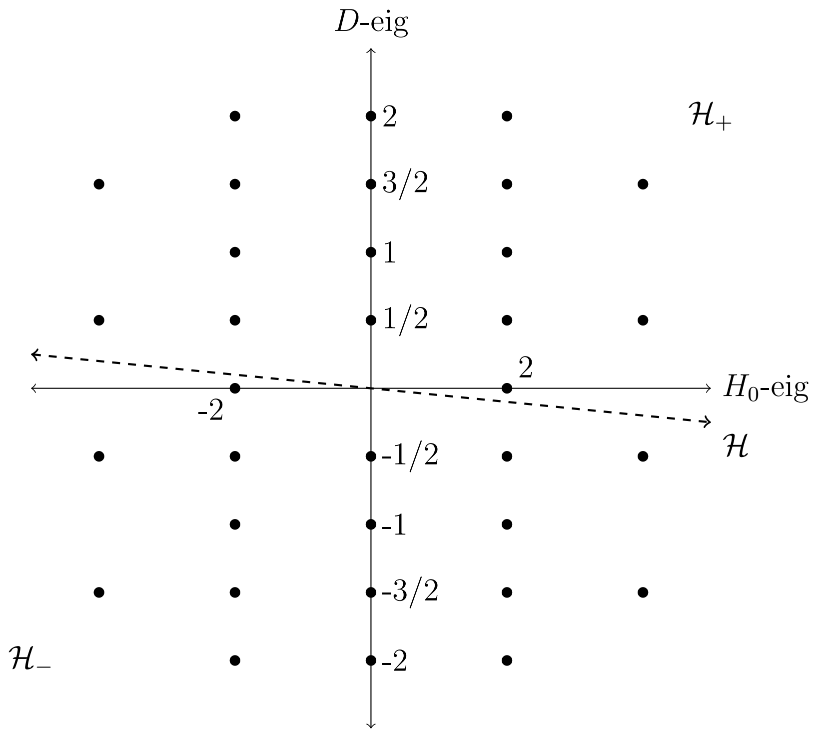


Figure 2.2: Root system for $\mathbf{A}_2^{(2)}$ for grades between -2 and 2.

The hyperplane $\mathcal{H}$ in Figure 2.2 is used to separate the negative and positive roots. From this it is clear that the simple roots are

$$\Phi_s = \{(-\Lambda, 0, 1/2), (\alpha, 0, 0)\}, \tag{2.105}$$

where $\Lambda(H^1) = 4$ is the highest weight of $\mathfrak{g}_{[1]}$ when considered as a $\mathfrak{g}_{[0]}$ module and $\alpha(H^1) = 2$ is the positive $\mathfrak{g}_{[0]}$ root. The simple root vectors are $E^{-\theta}_{\frac{1}{2}}$ and $\sqrt{2}(E_0^\alpha + E_0^\beta)$ with respect to Φ_s. The normalisations are chosen to give a Chevalley basis for $\mathbf{A}_2^{(2)}$.

We choose the normalisation of the inner product on $\mathbf{A}_2^{(2)}$ by satisfying the equations (2.38) and (2.39):

$$\langle(\lambda_1, k_1, n_1), (\lambda_2, k_2, n_2)\rangle = \frac{1}{4}\langle\lambda_1, \lambda_2\rangle_{\mathfrak{sl}(2)} + k_1 n_2 + k_2 n_1,$$

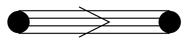
where $\langle \cdot, \cdot \rangle_{\mathfrak{sl}(2)}$ is the normalised inner product on $\mathfrak{sl}(2)$ (i.e. $\langle \alpha, \alpha \rangle_{\mathfrak{sl}(2)} = 2$). It follows that the coroots are

$$\alpha^{(0)^\vee} = \alpha^{(0)} = (-4, 0, 1/2), \qquad \alpha^{(1)^\vee} = 4\alpha^{(1)} = (8, 0, 0).$$

The Cartan matrix for $\mathbf{A}_2^{(2)}$ is then

$$\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}. \tag{2.106}$$

The corresponding Dynkin diagram is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Chapter 3

Representations of Lie algebras

This chapter outlines the properties of important types of representations in the theory of Lie algebras. Most relevant to this thesis are the integrable highest weight modules. For example, the integrable highest weight representations of untwisted affine Lie algebras are vital for understanding the untwisted spectrum of the Wess-Zumino-Witten conformal field theories [14]. In order to analyse the twisted sector of the Wess-Zumino-Witten models, which is the aim of Chapter 5, we must also understand the theory of integrable highest weight modules for twisted affine Lie algebras. The goal of this chapter is to present the theory of such representations. We explicitly construct the components required to understand the representations of $\mathbf{A}_2^{(2)}$ which will be used in Chapter 5.

W introduce key definitions in Section 3.3 that will be used throughout the chapter. In Section 3.2 we discuss in detail the representation theory of $\mathfrak{sl}(2)$, which is fundamental to understanding the representation theory of a much broader class of Lie algebras, namely the Kac-Moody Lie algebras which are constructed from a Cartan matrix via the Chevalley-Serre relations (2.37). The concept of integrable highest weight modules are introduced in Section 3.3.2 and then applied to representations of finite dimensional simple Lie algebras in Section 3.4. We then extend these ideas to the case of affine Lie algebras in Section 3.5.

In Section 3.6 we develop a more detailed analysis of the mathematical structure behind the integrable highest weight modules. We introduce important concepts such as Verma modules and singular vectors before using these to provide an explicit construction of the integrable highest weight modules. We will draw on some of these results in Chapters 4 and 5.

For a concise treatment of the concepts introduced in this chapter the reader is referred to Fuchs [22] (Chapters 13 and 14). A detailed treatment of the representation theory of semisimple Lie algebras can be found in either of [31, 33]. For detailed results on the representation theory of affine Lie algebras the reader should consult [19, 20].

3.1 Basic definitions

A *representation* of a Lie algebra on a vector space V is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. In other words, π is a linear map that satisfies

$$\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x), \tag{3.1}$$

for all $x, y \in \mathfrak{g}$. Every representation generates a $\mathfrak{g}$-module structure by defining the action of $x \in \mathfrak{g}$ on a vector $v \in V$ as

$$x \cdot v = \pi(x)v. \tag{3.2}$$

Thus, we can treat the representation theory of a Lie algebra $\mathfrak{g}$ as a theory of $\mathfrak{g}$-modules over vector spaces. This is the convention adopted in this thesis. Since we are assuming that $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$, the vector spaces that we are dealing with are complex vector spaces. The requirement (3.1) can be written as

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v). \tag{3.3}$$

From here on we will drop the parentheses and $\cdot$ in the above notation and it will be assumed that the “product” xyv is the action of y on the vector v followed by the action of x.

For clarity, we define the following algebraic concepts as usual. A *submodule* W of a $\mathfrak{g}$-module V is a subspace of V which is itself a $\mathfrak{g}$-module (under the same action). This is equivalent to the subspace W being preserved under the action of $\mathfrak{g}$, that is for all $w \in W$ and $x \in \mathfrak{g}$ we have $xw \in W$. A submodule W of V is *proper* if it is not zero or all of V. A $\mathfrak{g}$-module is *irreducible* if it contains no proper submodules (this is also sometimes called *simple*).

If W is a subset of a $\mathfrak{g}$-module, then we call the intersection of all submodules containing W the submodule *generated* by W. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ and V is a $\mathfrak{g}$-module, then V is endowed with a $\mathfrak{h}$-module structure under the Lie bracket inherited from $\mathfrak{g}$.

Suppose V and W are $\mathfrak{g}$-modules. A module *homomorphism* is a linear map $\psi : V \to W$ that commutes with the action of $\mathfrak{g}$:

$$\psi(xv) = x\psi(v) \qquad \text{for all } v \in V. \tag{3.4}$$

3.2 Representation theory of $\mathfrak{sl}(2)$

Finite dimensional representations of $\mathfrak{sl}(2)$ are a fundamental building block for understanding a much larger class of representations which includes all integrable highest weight modules of affine Lie algebras. Recall from Section 2.1, that $\mathfrak{sl}(2)_1$ has a standard basis given by $\{E, H, F\}$ that satisfies the relations (2.7. We will present the representation theory of $\mathfrak{sl}(2)$ here and then draw on these results throughout the thesis.

Let V be an $\mathfrak{sl}(2)$-module. To begin with we assume that there exists a vector v_λ with eigenvalue λ with respect to the action of H:

$$Hv_\lambda = \lambda v_\lambda. \tag{3.5}$$

We define the λ-eigenspace of H as

$$V_\lambda := \{v \in V | H(v) = \lambda v\}. \tag{3.6}$$

We now wish to see how E and F act on this eigenspace. Consider the action of H on the (possibly zero) vector Ev_λ,

$$HEv_\lambda = ([H,E] + EH)v_\lambda = 2Ev_\lambda + \lambda Ev_\lambda = (\lambda+2)Ev_\lambda. \tag{3.7}$$

Hence, if non-zero, Ev_λ is an eigenvector of H with eigenvalue $\lambda+2$. Similarly, one can show that Fv_λ is either zero or an eigenvector under the action of H with eigenvalue $\lambda - 2$. Since this is true for all eigenvalues λ, acting with E or F multiple times will continue to shift the eigenvalue by +2 or -2 respectively:

$$E^n V_\lambda \subseteq V_{\lambda+2n}, \qquad F^n V_\lambda \subseteq V_{\lambda-2n}. \tag{3.8}$$

Therefore, each non-zero eigenvector v_λ is part of a chain created by acting with E and F a possibly infinite number of times. This is summarised in the following diagram:

$$\begin{tikzcd}
\cdots & V_{\lambda-2} \arrow[l, bend right, "F"'] \arrow[r, bend right, "E"'] \arrow[loop above, "H"] & V_\lambda \arrow[l, bend right, "F"'] \arrow[r, bend right, "E"'] \arrow[loop above, "H"] & V_{\lambda+2} \arrow[l, bend right, "F"'] \arrow[r, bend right, "E"'] \arrow[loop above, "H"] & \cdots
\end{tikzcd}$$

Suppose now that V is irreducible, since each chain of the form above is invariant under the action of $\mathfrak{sl}(2)$ it follows that V must be given by such a chain. In other words, we can write V as the vector space direct sum

$$V = \bigoplus_{n \in \mathbb{Z}} V_{\lambda+2n}.$$

Furthermore, each eigenspace must be one-dimensional; otherwise, V would decompose into two chains, each invariant under the action of $\mathfrak{sl}(2)$ (refer to Theorem 3.4.3). We are particularly interested in the finite-dimensional modules. For the chain above to be finite there must exist an eigenvalue Λ and non-zero eigenvector v_Λ that is annihilated by the action of E:

$$Ev_\Lambda = 0.$$

It follows that V is spanned by the elements $\{F^n v_\Lambda | n \in \mathbb{Z}_{\geq 0}\}$. Thus, for the module to be finite we require $F^N v_\Lambda = 0$ for some minimal $N \in \mathbb{N}$. If this is the case then we obviously have $EF^N v_\Lambda = 0$. However it can be shown inductively that

$$EF^n v_\Lambda = n(\Lambda - (n-1))F^{n-1} v_\Lambda. \tag{3.9}$$

Since $N > 0$ and $F^{N-1} v_\Lambda \neq 0$ by the minimality of N, the only way we can have $EF^N v_\Lambda = 0$ is if $N = \Lambda + 1$. Since $N \in \mathbb{N}$ this implies that $\Lambda \in \mathbb{Z}_{\geq 0}$. In this case V has a basis given by $\{F^n v_\Lambda | 0 \leq n \leq \Lambda\}$. Hence, V has dimension $\Lambda + 1$ and decomposes into one-dimensional H eigenspaces with eigenvalues $\Lambda, \Lambda - 2, \cdots, -\Lambda + 2$. This finite dimensional irreducible $\mathfrak{sl}(2)$-module is denoted by $\mathcal{L}_\Lambda$.

Of course, we began with the assumption that there exists a vector v_λ with H eigenvalue λ. Since we are considering complex modules, if V is finite dimensional there exists at least one eigenvector for every linear operator on V, for example the action of H. This assumption is thus justified for finite dimensional modules. Thus, we can conclude that the finite dimensional irreducible $\mathfrak{sl}(2)$-modules are given by $\mathcal{L}_\Lambda$ for $\Lambda \in \mathbb{Z}_{\geq 0}$.

3.3 Fundamental Concepts

This section will introduce the fundamental concepts of the representation theory of Lie algebras such as weights and highest weight modules. We will introduce the important concept of integrability and determine the implications of integrability on highest weight representations.

The concepts introduced here apply to a large class of Lie algebras, in particular to all Kac-Moody Lie algebras. In this section $\mathfrak{g}$ will denote a Kac-Moody Lie algebra. Recall from Section 2.2 that a Kac-Moody Lie algebra has a set of simple roots and can be constructed from set of generators via the Chevalley-Serre construction presented in Theorem 2.1.1. It follows that all Kac-Moody Lie algebras possess a triangular decomposition (2.28).

3.3.1 Weights

Let V be a $\mathfrak{g}$-module. For $\lambda \in \mathfrak{g}_0^*$, a linear functional on the Cartan subalgebra, we define the eigenspace with respect to λ as

$$V_\lambda := \{v \in V |\ Hv = \lambda(H)v\ \forall H \in \mathfrak{g}_0\}. \tag{3.10}$$

If $V_\lambda \neq 0$ we call λ a *weight* of the module V and V_λ the *weight space* of λ. The set of all weights is referred to as the *weight system* or *weight space* of V and is a subset of $\mathfrak{g}_0^*$. Recall from (2.20) that $\mathfrak{g}_0^*$ is endowed with an inner product (2.20), which we will denote by $\langle \cdot, \cdot \rangle$ to avoid confusion.

We have already dealt with a specific type of weight in detail, namely the roots of $\mathfrak{g}$. These are the non-zero weights of $\mathfrak{g}$ when considered as a $\mathfrak{g}$-module under the adjoint action (2.3) of $\mathfrak{g}$. The representation corresponding to this module is called the *adjoint representation* of $\mathfrak{g}$.

In order to understand the action of $\mathfrak{g}$ on the weight spaces we consider the action of a root vector E^α on a vector $v_\lambda \in V_\lambda$,

$$HE^\alpha v_\lambda = ([H, E^\alpha] + E^\alpha H^i)v_\lambda = (\lambda(H) + \alpha(H))E^\alpha v_\lambda = (\lambda + \alpha)(H)E^\alpha v_\lambda.$$

Recalling that $\alpha \in \mathfrak{g}_0^*$ (2.10), the action of E^α increments the weight by α. Hence for all $\alpha \in \Phi$ and $\lambda \in \mathfrak{g}_0^*$ we have

$$E^\alpha V_\lambda \subseteq V_{\lambda+\alpha}. \tag{3.11}$$

A class of $\mathfrak{g}$-modules of particular importance are those that can be written as a decomposition of their weight spaces. That is, V can be written as the vector space direct sum

$$V = \bigoplus_\lambda V_\lambda. \tag{3.12}$$

In this case we say V is the direct sum of its weight spaces or simply V is a *weight module*. This class incorporates a large number of physically and mathematically important modules. These include all finite dimensional modules of finite dimensional semisimple Lie algebras [33], the adjoint module of a Kac-Moody Lie algebra and all highest weight modules of Kac-Moody Lie algebras [21] which we will define later in this section.

Dynkin labels

Suppose that $\Phi_s = \{\alpha^{(1)}, ..., \alpha^{(r)}\}$ is a set of simple roots for $\mathfrak{g}$. For every weight λ of V we define

$$\lambda_i := \langle \lambda, \alpha^{(i)^\vee} \rangle. \tag{3.13}$$

The λ_i are called the *Dynkin labels* of the weight λ. The Chevalley-Serre generator (2.37), H^i, satisfies

$$\lambda_i = \lambda(H^i). \tag{3.14}$$

Hence, a weight λ is fully determined by it's Dynkin labels and is often written as vector of eigenvalues $\lambda = (\lambda_1, ..., \lambda_r)$ with respect to the Cartan subalgebra basis given by the Chevalley-Serre generators $\{H^i | i = 1, ..., r\}$.

To analyse representations of Lie algebras we construct a useful basis for the weight space. Let $\{\Lambda_{(1)}, ..., \Lambda_{(r)}\}$ be the dual basis of the coroots $\{\alpha^{(1)^\vee}, ..., \alpha^{(r)^\vee}\}$, that is

$$\Lambda_{(i)}(\alpha^{(j)^\vee}) := \delta_{ij}. \tag{3.15}$$

While $\Lambda_{(i)}$ are technically elements of the dual of $\mathfrak{g}_0^*$, we can identify them as elements of $\mathfrak{g}_0^*$ via

$$\langle \Lambda_{(i)}, \alpha^{(j)^\vee} \rangle := \Lambda_{(i)}(\alpha^{(j)^\vee}) = \delta_{ij}. \tag{3.16}$$

From here on we will recognise $\Lambda_{(i)}$ as elements of $\mathfrak{g}_0^*$ called the *fundamental weights*. In this way all weights can be written as a linear combination of the fundamental weights,

$$\lambda = \sum_{i=1}^{r} \lambda_i \Lambda_{(i)}, \tag{3.17}$$

where the coefficients are given by the Dynkin labels of the weight λ. The *Dynkin* basis of $\mathfrak{g}_0^*$ is given by

$$\mathcal{B}^\star = \{\Lambda_{(i)} | i = 1, ..., r\}, \tag{3.18}$$

is referred to as the *Dynkin basis*. Note that the Dynkin labels of the simple roots are given by the rows of the Cartan matrix. Since these entries are integral, it follows that all roots can be written as an integral linear combination of the fundamental weights.

Highest weight modules

The study of highest weight modules is very important to the representation theory of Lie algebras. They also possess significant physical importance since the highest weight can be thought of as a lowest energy state in many applications such as conformal field theory.

A non-zero vector $v_\Lambda \in V_\Lambda$ is called a *highest weight vector* and $\Lambda \in \mathfrak{g}_0^*$ a *highest weight* if v_Λ is annihilated by the action of all positive roots:

$$E^\alpha v_\Lambda = 0 \qquad \forall \alpha \in \Phi_+. \tag{3.19}$$

Given a highest weight vector we would like to consider the $\mathfrak{g}$-module that it generates. Suppose V is a $\mathfrak{g}$-module with highest weight vector v_Λ, if $v \in V$ can be written as

$$v = x_1 x_2 \cdots x_n v_\Lambda, \tag{3.20}$$

for some $n \in \mathbb{Z}_{\geq 0}$ and $x_i \in \mathfrak{g}$ then we say v is a *descendent* of the highest weight vector v_Λ. The $\mathfrak{g}$-submodule of V generated by v_Λ is the span of all descendants of v_Λ and is denoted by $\langle v_\Lambda \rangle$. If there exists a highest weight vector $v_\Lambda \in V$ such that $V = \langle v_\Lambda \rangle$ then we say that V is a *highest weight module* with highest weight Λ. If a highest weight module is also irreducible we denote it by $\mathcal{L}_\Lambda$.

3.3.2 Integrable highest weight modules

For physical reasons related to bounded energy states we are interested in highest weight representations. There is also a more subtle property of a representation called integrability that is of significant physical importance. Since we will not deal explicitly with the nature of this physical importance it suffices to say that the property of integrability enables the local behaviour described by a Lie algebra representation to be in some way 'integrated' to a representation of the global Lie group from which the Lie algebra is derived [22].

Integrable modules

For our purposes we define an *integrable module* to be any $\mathfrak{g}$-module V that is the direct sum of its weight spaces and satisfies the following requirement: for every $v \in V$ and every simple root $\alpha^{(i)}$ there exists positive integers p and q such that $(E^i)^p v = 0$ and $(E^{-i})^q v = 0$.

From (2.37), for each simple root the Chevalley-Serre generators $\{E^{-i}, H^i, E^i\}$ satisfy

$$[E^i, E^{-i}] = H^i, \qquad [H^i, E^{\pm i}] = \pm 2E^{\pm i}. \tag{3.21}$$

Hence $\{E^{-i}, H^i, E^i\}$ span a subalgebra isomorphic to $\mathfrak{sl}(2)$.

Suppose that $v_\lambda \in V_\lambda$ has weight λ. Then for some $p, q \in \mathbb{N}$ we have

$$H^i(v_\lambda) = \lambda_i v_\lambda, \qquad (E^i)^p v_\lambda = 0, \qquad (E^{-i})^q v_\lambda = 0. \tag{3.22}$$

This implies that the module generated by the action of $\{E^{-i}, H^i, E^i\}$ on v_λ is finite dimensional and therefore must be isomorphic to an irreducible finite dimensional $\mathfrak{sl}(2)$-module. Note that $\lambda_i \in \mathbb{Z}$ for all λ_i.

We can summarise this result by saying that V decomposes into finite dimensional irreducible $\mathfrak{sl}(2)$-modules under the action of the elements $\{E^{-i}, H^i, E^i\}$ for all $i = 1, ..., r$.

Implications of integrability on a highest weight

Suppose now that $v_\Lambda \in V$ is a highest weight vector with highest weight Λ. Then for all $i \in \{0, .., r\}$ there exists a minimal positive integer N_i such that

$$E^i v_\Lambda = 0, \qquad H^i v_\Lambda = \Lambda_i v_\Lambda, \qquad (E^{-i})^{N_i} v_\Lambda = 0. \tag{3.23}$$

Since $\{E^i, H^i, E^{-i}\}$ satisfy the $\mathfrak{sl}(2)$ relations (3.21), using our knowledge of the representation theory of $\mathfrak{sl}(2)$ (see (3.9) in Section 3.2)) we have

$$E^i(E^{-i})^n v_\Lambda = n(\Lambda_i - (n-1))(E^{-i})^{n-1} v_\Lambda \qquad \forall n. \tag{3.24}$$

So if $(E^{-i})^{N_i} v_\Lambda = 0$ we require

$$N_i(\Lambda_i - (N_i - 1))(E^{-i})^{N_i - 1} v_\Lambda = 0. \tag{3.25}$$

Since N_i is minimal, $(E^{-i})^{N_i-1} v_\Lambda \neq 0$ which implies $\Lambda_i = N_i - 1$. As $N_i \in \mathbb{N}$ this implies that $\Lambda_i \in \mathbb{Z}_{\geq 0}$ for all i. We summarise this result with the following theorem.

Theorem 3.3.1 *Let $\mathfrak{g}$ be a Kac-Moody Lie algebra with simple roots $\alpha^{(1)}, .., \alpha^{(r)}$. Let V be a $\mathfrak{g}$-module with a unique highest weight vector v_Λ with Dynkin label Λ_i with respect to the simple root $\alpha^{(i)}$. Then V is integrable if and only if $\Lambda_i \in \mathbb{Z}_{\geq 0}$ and*

$$(E^{-i})^{\Lambda_i+1} v_\Lambda = 0, \tag{3.26}$$

for all $i \in \{1, ..., r\}$. Moreover, in this case V is irreducible, so that $V = \mathcal{L}_\Lambda$.

We have only shown the forwards implication of this theorem, the converse is more difficult to prove, but has been done, for example, in [19].

3.4 Finite dimensional simple Lie algebras

We now turn to the analysis of finite dimensional modules of finite dimensional simple Lie algebras. These modules all decompose into their weight spaces [33]. Furthermore, since V is finite and the action of $E^{\pm i}$ shifts the weight spaces by $\pm\alpha^{(i)}$ it follows that V must also be integrable.

We will restrict ourselves initially to the case where V is irreducible. Since there are finitely many weight spaces there must exist a highest weight vector v_Λ with highest weight Λ. Since V is irreducible, $V = \langle v_\Lambda \rangle$. Furthermore, it is a well known consequence of irreducibility that the dimension of V_Λ must be one [33].

We will now introduce a method to construct the weight system of the irreducible highest weight representations of the finite dimensional simple Lie algebras. This is most easily seen through examples, but first we will outline the general approach.

Since V is integrable every vector $v_\lambda \in V_\lambda$ belongs to an $\mathfrak{sl}(2)$ module with respect to the action of each set of Chevalley-Serre generators, $\{E^i, H^i, E^{-i}\}$. Hence, beginning with the highest weight Λ we construct a sequence of weight spaces corresponding to the finite dimensional $\mathfrak{sl}(2)$-module generated by the action of the generators $\{E^i, H^i, E^{-i}\}$ on the highest weight vector. Specifically, for each i these weights are given by

$$\{\Lambda - n\alpha_i | n \in \{0, 1, ..., \Lambda_i\}\}, \tag{3.27}$$

since for integrability (and irreducibility) we require

$$(E^{-i})^{\Lambda_i+1} v_\Lambda = 0. \tag{3.28}$$

Each new weight space then has a new non-zero vector which belongs to a finite dimensional $\mathfrak{sl}(2)$-module with respect to each simple root. In this way we construct the weight spaces of V iteratively by constructing the weight spaces of the $\mathfrak{sl}(2)$-modules of each new weight with respect to all the simple roots. This procedure continues until it terminates when all the new weight have negative Dynkin labels. This process terminates in the case of the finite dimensional simple Lie algebras, but not in the case of the affine Lie algebras [21].

Example: Consider $\mathfrak{sl}(3)$ whose Cartan matrix is given by

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \tag{3.29}$$

There are two simple roots whose Dynkin labels are described by the rows of the Cartan matrix, $\alpha^{(1)} = 2\Lambda_{(1)} - \Lambda_{(2)}$ and $\alpha^{(2)} = -\Lambda_{(1)} + 2\Lambda_{(2)}$. Figure 3.1 demonstrates the weight space for two irreducible highest weight $\mathfrak{sl}(3)$-modules, $\mathcal{L}_{(0,1)}$ and $\mathcal{L}_{(1,1)}$. Notice that the weight space for $\mathcal{L}_{(1,1)}$ has the same structure as the root space of $\mathfrak{sl}(3)$ (except $(0,0)$ is not included as a root) shown in Section 2.1. Hence, the representation corresponding to $\mathcal{L}_{(1,1)}$ is isomorphic to the adjoint representation.

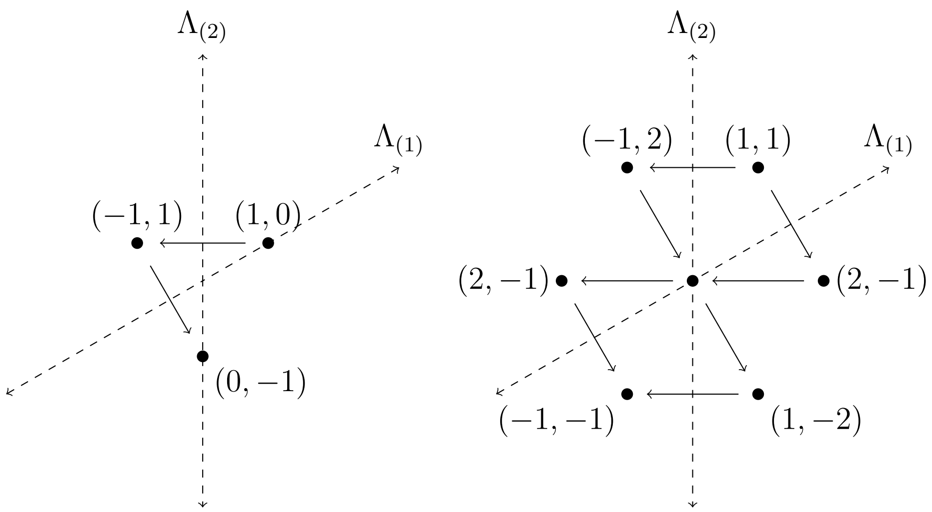

Figure 3.1: Weight space diagrams for the irreducible highest weight $\mathfrak{sl}(3)$-modules with highest weight $\Lambda = (1,0)$ (left diagram) and $\Lambda = (1,1)$ (right diagram).The left pointing arrows indicate the action of E^{-1} (the negative root vector of the first simple root $\alpha^{(1)}$) while the other arrows indicate the action of E^{-2} (the negative root vector of the second simple root $\alpha^{(2)}$).

It should be noted that while this method is useful in constructing the weight spaces of integrable highest weight modules it does not determine the dimension of V_λ for each weight λ. For example, in the $\mathcal{L}_{(1,1)}$ $\mathfrak{sl}(3)$-module we know from the adjoint module that the weight space for $\lambda = (0,0)$ has dimension two, while all other weight spaces have dimension one. To determine the dimension of each weight space requires the introduction of more complex representation theoretic concepts such as the character of a representation. The resulting formula is called Freudenthal's multiplicity formula, but we will not introduce it here. For more information the reader can consult any textbook on the representation theory of Lie algebras including [34].

Having demonstrated how to construct the finite dimensional $\mathfrak{g}$-modules we finish the section with a few important theorems about the properties of finite dimensional representations. Proofs of these theorems are omitted, but can be found in most textbooks on the representation theory of Lie algebra such as [31, 33]. Firstly we summarise our results in the following theorem.

Theorem 3.4.1 *If V is a finite dimensional irreducible $\mathfrak{g}$-module, then $V \cong \mathcal{L}_\Lambda$ where the Dynkin labels of Λ are non-negative integers.*

It is not immediately obvious that the converse of this theorem is true. This requires that applying the process described above will always terminate (and that the multiplicities are finite) and thus yield a finite dimensional module.

Theorem 3.4.2 *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with simple roots $\alpha^{(1)}, \cdots, \alpha^{(r)}$. Let Λ be a weight with Dynkin labels that satisfy $\Lambda_i \in \mathbb{Z}$ for all i. Then $\mathcal{L}_\Lambda$ is finite dimensional.*

With a bit more work the results of Theorem 3.4.1 can be generalised to any finite dimensional $\mathfrak{g}$-module.

Theorem 3.4.3 (Weyl's Theorem) *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra $\mathfrak{g}$. Let V be a finite dimensional $\mathfrak{g}$-module. Then there exists weights $\Lambda^1, \cdots, \Lambda^n$ such that*

$$V = \bigoplus_{i=1}^{n} \mathcal{L}_{\Lambda^i}. \tag{3.30}$$

Furthermore, the Λ^i are unique up to permutation.

3.5 Affine Lie algebras

We now generalise our results for finite dimensional simple Lie algebras to affine Lie algebras. Since all affine Lie algebras can be constructed from Cartan matrices via the Chevalley-Serre relations, the concepts introduced in Section 3.3 apply to affine Lie algebras. The process that we introduced in Section 3.4 can also be applied to construct the weight spaces for integrable highest weight representations of affine Lie algebras.

Hence, we focus now on the subtleties of a highest weight representations of affine Lie algebras that make them from the integrable highest weight representation of finite dimensional Lie algebras. The section finishes with a detailed discussion of integrable highest weight $\mathbf{A}_2^{(2)}$-modules, these modules will be fundamental to the analysis presented in Chapter 5.

3.5.1 The role of the derivation

From the loop construction of an affine Lie algebra, $\hat{\mathfrak{g}}$, in Section 2.3 we know that each affine Lie algebra has simple roots $\alpha^{(0)}, \alpha^{(1)}, \cdots \alpha^{(r)}$ and that the Cartan subalgebra is spanned by the elements $\{H_0^1, H_0^2, \cdots, H_0^r, K, D\}$. Hence, we can see that the Cartan subalgebra has dimension one greater than the number of simple roots and so there will be difficulty equating the fundamental weights with a basis for the dual of the Cartan subalgebra. This difficulty and also the subtlety of the affine weight space can be explained by the derivation, D.

Consider a highest weight represented by $(\bar{\Lambda}, k, n_0) \in \hat{\mathfrak{g}}_0^*$, where $\bar{\Lambda}$ is a weight of the horizontal subalgebra whose Cartan subalgebra is spanned by the elements $\{H_0^1, H_0^2, \cdots, H_0^r\}$. Assuming $k \neq 0$, we may set $D \to D - (n_0/k)K$ which gives

$$Dv_\Lambda \to (D - (n_0/k)K)v_\Lambda = (n_0 - n_0)v_\Lambda = 0. \tag{3.31}$$

Since K is central, this map doesn't change the commutation relations of $\hat{\mathfrak{g}}$, so we are free to set $n_0 = 0$. Thus, a highest weight is completely determined by the eigenvalues of the elements H_0^i and an eigenvalue k of the element K. Since K is central, it commutes with all other elements and so the K eigenvalue is the same for all elements in an irreducible module. We call this value the *level* of the highest weight module. Furthermore, the module is graded with respect to the action D. Consider an element $T_{-n}^a v_\Lambda$, then

$$DT_{-n}^a v_\Lambda = ([D, T_{-n}^a] + T_{-n}^a D)v_\Lambda = -nT_{-n}^a v_\Lambda. \tag{3.32}$$

Hence, the action of an element T_{-n}^a decreases the D eigenvalue by n. In particular, the action of the negative simple untwisted affine root vector, E_{-1}^θ, decreases the D eigenvalue by 1. We refer to the negative of the D eigenvalue as the *grade* (or *energy* when D is identified with the Virasoro operator L_0 in the context of conformal field theory) of a particular vector in a highest weight module.

In this way the derivation endows a highest weight module of an untwisted affine Lie algebra with a $\mathbb{Z}_{\geq 0}$-gradation. The rest of the structure is fairly similar to that of the finite dimensional Lie algebras. Suppose we have a weight (λ, k, n) then the action of an element E_m^α on this weight space satisfies

$$E_m^\alpha V_{(\lambda,k,n)} \subseteq V_{(\lambda+\alpha,k,n+m)}. \tag{3.33}$$

By analysing the action of the simple root on each weight space we can decompose an integrable module into finite dimensional $\mathfrak{sl}(2)$-modules with respect to the simple roots. However, this process will no longer terminate (provided the highest weight is non-zero) as it did in the case of finite-dimensional simple Lie algebras. This can be seen through the action of the negative imaginary roots, these root vectors will act infinitely many times without annihilating a vector if $k \neq 0$. This also implies that there is no upper-bound on the grade of the eigenvectors appearing in an integrable highest weight module. This makes it difficult to draw a diagram for the weight space, so we usually restrict the weight space diagram to the first few grades.

3.5.2 Untwisted affine Lie algebras

We will illustrate an integrable highest weight module for an untwisted affine Lie algebra through the example of $\widehat{\mathfrak{sl}(2)}$. First we determine the fundamental weights in more detail, this will provide a useful formula for the level of a highest weight module.

Suppose V is a highest weight $\hat{\mathfrak{g}}$-module with highest weight $\Lambda = (\bar{\Lambda}, k, 0)$. The fundamental weights (3.16) of the untwisted affine Lie algebra are determined by solving

$$\langle \Lambda_{(0)}, (-\theta, 0, 1) \rangle = 1, \qquad \langle \Lambda_{(i)}, (\alpha^{(i)\vee}, 0, 0) \rangle = 1, \tag{3.34}$$

$$\langle \Lambda_{(0)}, (\alpha^{(i)\vee}, 0, 0) \rangle = 0, \qquad \langle \Lambda_{(i)}, (-\theta, 0, 1) \rangle = 0, \tag{3.35}$$

for all $i \in \{1, ..., r\}$. In terms of the fundamental weights, $\bar{\Lambda}_{(i)}$, of the horizontal subalgebra, $\mathfrak{g}$, we find that the fundamental weights of $\hat{\mathfrak{g}}$ are

$$\Lambda_{(0)} = (0, 1, 0), \qquad \Lambda_{(i)} = (\bar{\Lambda}_{(i)}, a_i^\vee, 0), \tag{3.36}$$

where $a_i^\vee$ are the dual Coxeter labels of $\hat{\mathfrak{g}}$ as defined in (2.62) .

Since we have enforced the grade of the highest weight to be zero, it follows that all highest weights can be written in the form

$$\Lambda = \sum_{i=0}^{r} \Lambda_i \Lambda_{(i)} = (\bar{\Lambda}, \sum_{i=0}^{r} a_i^\vee \Lambda_i, 0), \tag{3.37}$$

where $\bar{\Lambda}$ is the highest weight of the horizontal algebra with Dynkin labels given by $\Lambda_1, ..., \Lambda_r$. This gives a formula for the level k of a highest weight representation:

$$k = \sum_{i=0}^{r} a_i^\vee \Lambda_i. \tag{3.38}$$

The requirement that the module be integrable enforces $\Lambda_i \in \mathbb{Z}_{\geq 0}$ for all $i \in \{0, 1, ..., r\}$. Furthermore, the dual Coxeter labels of an affine Lie algebra are always positive integers. Therefore, the level of an integrable highest weight $\hat{\mathfrak{g}}$-module must be a non-negative integer.

For the module to be integrable it must again satisfy (3.26). To illustrate the above ideas we provide an example of an integrable highest weight $\widehat{\mathfrak{sl}(2)}$-module.

Example: $\widehat{\mathfrak{sl}(2)}$ has Cartan matrix given by

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}. \tag{3.39}$$

All of the dual Coxeter labels are one so we have

$$\Lambda_{(0)} = (0, 1, 0), \qquad \Lambda_{(1)} = (1, 1, 0), \qquad k = \Lambda_1 + \Lambda_2. \tag{3.40}$$

Expressing the simple roots in terms of the fundamental weights gives

$$\alpha^{(0)} = 2\Lambda_{(0)} - 2\Lambda_{(1)}, \qquad \alpha^{(1)} = -2\Lambda_{(0)} + 2\Lambda_{(1)}. \tag{3.41}$$

Since there are only two fundamental weights, whenever we write a fundamental weight with respect to its Dynkin labels it will only have two entries. The weights, up to grade three, of the integrable highest weight module with Dynkin labels of the highest weight given by $(0, 1)$ is shown in Figure 3.2.

$$
\begin{tikzcd}[column sep=small]
 & & & & \text{grade} \\
 & (2,-1) \arrow[dr] & (0,1) \arrow[l] & & 0 \\
 & (2,-1) \arrow[dr] & (0,1) \arrow[l] \arrow[dr] & & 1 \\
(-4,-3) \arrow[dr] & (-2,-1) \arrow[l] \arrow[dr] & (0,1) \arrow[l] \arrow[dr] & (-2,3) \arrow[l] & 2 \\
(-4,-3) & (-2,-1) \arrow[l] & (0,1) \arrow[l] & (-2,3) \arrow[l] & 3
\end{tikzcd}
$$

Figure 3.2: Integrable highest weight $\widehat{\mathfrak{sl}(2)}$-module for highest weight $\Lambda = (0, 1)$. The weight space is shown for grades from zero to three. The left arrows indicate the action of the non-affine negative simple root, while the downwards right pointing arrows indicate the action of the affine negative root (note that this action increases the grade by one). All weights are written with respect to their Dynkin labels.

3.5.3 Twisted affine Lie algebras

From Section 2.4.2 we know that a twisted affine Lie algebra is constructed from a finite dimensional Lie algebra $\mathfrak{g}$ and an outer automorphism σ of $\mathfrak{g}$. Recall the definition of the eigenspaces $\mathfrak{g}_{[j]}$ for an automorphism from Section 2.4.1. N denotes the order of the automorphism σ.

The theory of integrable highest weight modules of twisted affine Lie algebras is almost identical to that of the untwisted case. The derivation, D, provides a gradation of the module. The fundamental weights are defined as a dual basis to the coroots and the requirement of integrability enforces the Dynkin labels of the highest weight to take non-negative integral values. The main difference is that the affine simple root is now $(-\Lambda_1, 0, 1/N)$, where Λ_1 is the highest weight of $\mathfrak{g}_{[1]}$ when considered as a $\mathfrak{g}_{[0]}$-module under the adjoint action.

Hence, the action of the negative affine root vector $E^{\Lambda_1}_{-1/N}$ increases the level by $1/N$, so that the gradation is now over non-negative integer multiples of $1/N$. The formula for the fundamental weights will no longer be the same as in section 3.5.2, we demonstrate this through a specific example rather than deriving a general formula. This example constructs the integral highest weight representations for $\mathbf{A}_2^{(2)}$ and is fundamental to the results that are presented in Chapter 4.

3.5.4 Integrable highest weight $\mathbf{A}_2^{(2)}$-modules

Consider the twisted affine Lie algebra $\mathbf{A}_2^{(2)}$ which was in Section 2.4.3. We found the Cartan matrix to be given by

$$A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}. \tag{3.42}$$

k	Integrable highest weight $\mathbf{A}_2^{(2)}$-modules
0	(0,0)
1	(0,1)
2	(0,2),(1,0)
3	(0,3),(1,1)
4	(0,4),(1,2),(2,0)
5	(0,5),(1,3),(2,1)

Figure 3.3: The highest weights in terms of their Dynkin labels for the integrable highest weight $\mathbf{A}_2^{(2)}$-modules at integer levels $k = 0, 1, ..., 5$.

$$\begin{tikzcd}
 & & & & & & \text{grade} \\
 & & (1,-1) \arrow[drr] & (0,1) \arrow[l] & & & 0 \\
 & (2,-3) \arrow[drr] & (1,-1) \arrow[l] \arrow[drr] & (0,1) \arrow[l] & (-1,3) \arrow[l] & & \frac{1}{2} \\
 & (2,-3) \arrow[drr] & (1,-1) \arrow[l] \arrow[drr] & (0,1) \arrow[l] \arrow[drr] & (-1,3) \arrow[l] & & 1 \\
(3,-5) & (2,-3) \arrow[l] & (1,-1) \arrow[l] & (0,1) \arrow[l] & (-1,3) \arrow[l] & (-2,5) \arrow[l] & \frac{3}{2}
\end{tikzcd}$$

Figure 3.4: Weight space for the integrable highest weight $\mathbf{A}_2^{(2)}$-module for $\Lambda = (0, 1)$ at grades from zero to 3/2. The left arrow indicates the action of the non-affine negative simple root vector, while the downward right arrow indicates the action of the negative affine simple root vector (note that it increases the grade by 1/2).

The simple coroots were given by $\alpha^{(0)} = (-4, 0, 1/2)$ and $\alpha^{(1)} = (2, 0, 0)$. Recall also the formula for the inner product on the root space of $\mathbf{A}_2^{(2)}$ given by (2.106). We solve the system of equations given in (3.16) to find the fundamental weights

$$\Lambda_{(0)} = (0, 2, 0), \qquad \Lambda_{(1)} = (1, 1, 0). \tag{3.43}$$

A formula for the level k in terms of the Dynkin labels (Λ_0, Λ_1) of the highest weight is then given by

$$k = 2\Lambda_0 + \Lambda_1. \tag{3.44}$$

Figure 3.3 lists the highest weights of the integrable highest weight modules for integer levels $k = 0, 1, ..., 5$. In Chapter 4 we are most interested in the integrable highest weight modules at level $k = 1$, there is only one such integrable module and we provide the weight space diagram in Figure 3.4 for grades $0, \frac{1}{2}, 1, \frac{3}{2}$.

3.6 A general construction

We now present a more rigorous mathematical framework to understand the concepts introduced in this chapter so far. To begin we define the universal enveloping algebra associated to a Lie algebra, this is useful tool in the analysis of modules of Lie algebras. We then introduce the concept of Verma modules and singular vectors. These ideas are used to summarise a specific construction of the integrable highest weight modules for Kac-Moody algebras.

3.6.1 Universal enveloping algebras

A module of a Lie algebra $\mathfrak{g}$ generated by a vector v consists of all descendants (3.20) of v, that is all elements of the form

$$x_1 x_2 ... x_n v \qquad n \in \mathbb{Z}_{\geq 0} \quad x_i \in \mathfrak{g} \quad \forall i. \tag{3.45}$$

We would those like to construct a useful way of thinking about elements of the form $x_1 x_2 ... x_n$. To do this we construct an associative algebra called the universal enveloping algebra of $\mathfrak{g}$.

The n^{th} tensor power of a Lie algebra $\mathfrak{g}$ is denoted by $\mathfrak{g}^n := \mathfrak{g}^{\otimes n}$. It's elements are of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ where $x_i \in \mathfrak{g}$. At the moment $\mathfrak{g}^{\otimes n}$ is nothing more than a vector space. We define the *tensor algebra* of $\mathfrak{g}$ as the direct product of all n^{th} tensor powers of $\mathfrak{g}$,

$$\mathcal{T}(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} \mathfrak{g}^n, \tag{3.46}$$

where $\mathfrak{g}^0 = \mathbb{C}$ since for our purposes $\mathfrak{g}$ is a complex vector space. The tensor product endows the space $\mathcal{T}(\mathfrak{g})$ with a natural multiplication that satisfies $\mathfrak{g}^n \otimes \mathfrak{g}^m \subseteq \mathfrak{g}^{n+m}$. In this way $\mathcal{T}(\mathfrak{g})$ is an associative algebra graded by $\mathbb{Z}_{\geq 0}$.

Being an associative algebra we can endow $\mathcal{T}(\mathfrak{g})$ with a Lie bracket, namely that defined by $[x^n, y^m] = x^n \otimes y^m - y^m \otimes x^n$, where $x^n \in \mathfrak{g}^n$ and $y^m \in \mathfrak{g}^m$. To ensure relation (3.3), we need to identify the element $[x, y] \in \mathfrak{g}^1$ with the element $x \otimes y - y \otimes x \in \mathfrak{g}^2$. Hence, we define the subset $I = \{x \otimes y - y \otimes x - [x, y] | x, y \in \mathfrak{g}\}$ of $\mathcal{T}(\mathfrak{g})$. Since all elements of this set should be set to zero, we would like to take the quotient of $\mathcal{T}(\mathfrak{g})$ with I, but I is not yet an ideal. Hence, we define $\mathcal{I}$ as the smallest ideal of $\mathcal{T}(\mathfrak{g})$ containing I, this is

$$\mathcal{I} := \mathcal{T}(\mathfrak{g}) \otimes I \otimes \mathcal{T}(\mathfrak{g}). \tag{3.47}$$

The *universal enveloping algebra* is the associative algebra given by the quotient of $\mathcal{T}(\mathfrak{g})$ with $\mathcal{I}$,

$$\mathcal{U}(\mathfrak{g}) := \mathcal{T}(\mathfrak{g}) / \mathcal{I}. \tag{3.48}$$

To observe the implications of this quotient space we consider an element of $\mathcal{U}(\mathfrak{g})$ given by $x_1 \otimes x_2 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n$ and we now wish to permute the elements x_i and x_{i+1}. Making use of the relation in I we get

$$\begin{aligned} &x_1 \otimes x_2 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n \\ &\qquad = x_1 \otimes x_2 \otimes \cdots \otimes ([x_i, x_{i+1}] + x_{i+1} \otimes x_i) \otimes \cdots \otimes x_n \end{aligned} \tag{3.49}$$

From here on we will drop the $\otimes$ from our expressions and simply write the tensor product of n elements as $x_1 x_2 \cdots x_n$.

Example: Consider the basis for $\mathfrak{sl}(2)$ given by $\{E, H, F\}$ as introduced in (3.21). By commuting the order of the $\mathfrak{sl}(2)$ elements we find that

$$\begin{aligned} EF &= [E, F] + FE = H + FE, \\ E^2 F &= EH + EFE = -2E + 2HE + FE^2. \end{aligned} \tag{3.50}$$

Notice that by permuting elements and applying the Lie brackets we are able to express these elements as linear combinations of elements of the form $E^i H^j F^k$, $(i, j, k \in \mathbb{Z}_{\geq 0})$. This is in fact a very important result that holds for all Lie algebras with a countable basis. It is known as the Poincaré-Birkhoff-Witt theorem, we state it now and outline its importance through the following corollaries and examples.

Theorem 3.6.1 (Poincaré-Birkhoff-Witt Theorem) *Let $\mathfrak{g}$ be a Lie algebra with a basis $\{T^i | i \in I\}$ where I is a totally ordered set. Then a basis for $\mathcal{U}(\mathfrak{g})$ is given by*

$$\mathcal{B} = \{T^{a_1} T^{a_2} \cdots T^{a_k} | a_i \in I \text{ such that } a_1 \leq a_2 \leq \cdots \leq a_k, k \in \mathbb{Z}_{\geq 0}\} \tag{3.51}$$

Corollary 3.6.2 *The canonical map of $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$ is an inclusion.*

In other words, $\mathfrak{g}$ can be considered as a subspace of $\mathcal{U}(\mathfrak{g})$. Furthermore, the Lie bracket defined on $\mathcal{U}(\mathfrak{g})$ restricted to $\mathfrak{g}$ is equivalent to the Lie bracket on $\mathfrak{g}$: $[x, y]_{\mathcal{U}(\mathfrak{g})} = x \otimes y - y \otimes x = [x, y]_{\mathfrak{g}} \in \mathfrak{g}$. This means that the universal enveloping algebra is an extension of $\mathfrak{g}$ to an associative algebra. It follows that every $\mathcal{U}(\mathfrak{g})$-module V, determines a unique $\mathfrak{g}$-module structure on V. The universal enveloping algebra has the additional property that every $\mathfrak{g}$-module extends to a unique module of the universal enveloping algebra. This result indicates the importance of the universal enveloping algebra to the representation theory of a Lie algebra, it is summarised in Theorem 3.6.4.

Example: Consider the basis $\{F, H, E\}$ for $\mathfrak{sl}(2)$. Applying the Poincaré-Birkhoff-Witt Theorem, a basis for $\mathcal{U}(\mathfrak{sl}(2))$ is given by

$$\{F^i H^j E^k | i, j, k \in \mathbb{Z}_{\geq 0}\}. \tag{3.52}$$

In particular this means that $\mathcal{U}(\mathfrak{sl}(2)) = \mathcal{U}(\mathfrak{sl}(2)_{-\alpha}) \otimes \mathcal{U}(\mathfrak{sl}(2)_0) \otimes \mathcal{U}(\mathfrak{sl}(2)_\alpha)$ for the triangular decomposition of $\mathfrak{sl}(2)$ presented in (2.14). This result can be generalised to any Lie algebra that admits a triangular decomposition.

Corollary 3.6.3 *Suppose that $\mathfrak{g}$ is a Lie algebra that admits a triangular decomposition, $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$. Then the universal enveloping algebra decomposes as*

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_-) \otimes \mathcal{U}(\mathfrak{g}_0) \otimes \mathcal{U}(\mathfrak{g}_+). \tag{3.53}$$

Theorem 3.6.4 *Let $\mathfrak{g}$ be a Lie algebra as in Theorem 3.6.1. There is a bijective correspondence between $\mathcal{U}(\mathfrak{g})$-modules and $\mathfrak{g}$-modules.*

3.6.2 Verma modules

Having introduced the concept of highest weight modules in Section 3.3, we now use the universal enveloping algebra to study the properties of highest weight modules.

Let V be a $\mathfrak{g}$-module of a Lie algebra $\mathfrak{g}$ with triangular decomposition (2.28). Suppose there exists a highest weight vector $v_\Lambda \in V$. By definition we have that $\mathfrak{g}_+(v_\Lambda) = 0$ and that each element $H \in \mathfrak{g}_0$ acts on v_Λ as scalar multiplication by some eigenvalue $\Lambda(H)$. If we consider V as a $\mathcal{U}(\mathfrak{g})$-module then Corollary 3.6.3 implies that

$$\mathcal{U}(\mathfrak{g})v_\Lambda = \mathcal{U}(\mathfrak{g}_-) \otimes \mathcal{U}(\mathfrak{g}_0) \otimes \mathcal{U}(\mathfrak{g}_+)v_\Lambda = \mathcal{U}(\mathfrak{g}_-)v_\Lambda. \tag{3.54}$$

Note that by definition (3.20), $\langle v_\Lambda \rangle = \mathcal{U}(\mathfrak{g})v_\Lambda$. In the case that V is a highest weight module generated by v_Λ, we have $V = \mathcal{U}(\mathfrak{g})v_\Lambda = \mathcal{U}(\mathfrak{g}_-)v_\Lambda$. It follows that V is generated by the action of the elements of $\mathcal{U}(\mathfrak{g}_-)$ on v_Λ.

The *Verma module* V_Λ is the highest weight module generated by v_Λ such that none of the non-zero elements of $\mathcal{U}(\mathfrak{g}_-)$ annihilate v_Λ. This module can be thought of as the largest possible highest weight module generated by v_Λ, in the sense that each element in a basis for $\mathcal{U}(\mathfrak{g}_-)$ generates a linearly independent vector in V_Λ.

Example: Consider a highest weight $\mathfrak{sl}(2)$-module with highest weight vector v_Λ. By definition $Hv_\Lambda = \Lambda(H)v_\Lambda$ where $\Lambda(H) \in \mathbb{C}$ and $Ev_\Lambda = 0$. The Verma module V_Λ then has a basis given by

$$\{F^n v_\Lambda | n \in \mathbb{Z}_{\geq 0}\}. \tag{3.55}$$

3.6.3 Singular vectors

Let Λ be a highest weight of a Lie algebra with triangular decomposition and V_Λ be the corresponding Verma module. A *singular vector* is a non-zero element, $s \in V_\Lambda$, such that s has weight $\lambda \neq \Lambda$ and is annihilated by all positive root vectors of $\mathfrak{g}$:

$$\mathfrak{g}_+ s = 0. \tag{3.56}$$

Notice that this is similar to the definition of a highest weight vector, except, instead of generating the Verma module, s is a *descendent* of the highest weight vector. Not all Verma modules contain singular vectors, but we will demonstrate the existence of singular vectors in certain Verma modules of Kac-Moody algebras. The existence of these singular vectors is a crucial tool that we will use to analyse the allowed modules of the WZW models in Chapters 4 and 5.

Example: Let $\Lambda = n$ be the highest weight of an $\mathfrak{sl}(2)$-module as introduced in Section 3.2. To find a singular vector we consider the action of E on these basis elements (3.55). Recalling equation (3.9), for $EF^m v_\Lambda$ to be zero we require $m = 0$ or $\Lambda - (m-1) = 0$. Setting $m = 0$ returns the requirement $Ev_\Lambda = 0$, however enforcing the second requirement gives $m = \Lambda + 1$. Thus, V_Λ has a singular vector given by $F^{\Lambda+1}v_\Lambda$.

We now construct a general example of a singular vector which will prove very powerful in constructing the integrable representations of a Kac-Moody Lie algebra. Suppose we have a Lie algebra $\mathfrak{g}$ with triangular decomposition and simple roots $\alpha^{(1)}, \cdots, \alpha^{(r)}$ where $r = \text{rank}(\mathfrak{g})$. Let $\Lambda = (\Lambda_1, \cdots, \Lambda_r)$ be a highest weight and v_Λ be a highest weight vector of weight Λ. For each simple root, $\alpha^{(i)}$, we have the Chevalley-Serre generators $\{E^i, H^i, E^{-i}\} \subset \mathfrak{g}$ that satisfy (3.21). For $1 \leq i \leq r$, we have shown in section 3.3.2 that

$$E^i(E^{-i})^n v_\Lambda = n(\Lambda_i - (n-1))(E^{-i})^{n-1}v_\Lambda \qquad \forall n \in \mathbb{Z}_{\geq 0}, \tag{3.57}$$

so if $\Lambda_i \in \mathbb{Z}_{\geq 0}$ then for $n = \Lambda_i + 1$ we have

$$E^i(E^{-i})^{\Lambda_i+1}v_\Lambda = 0. \tag{3.58}$$

Furthermore, from the Chevalley-Serrre relations (2.37) we have $[E^j, E^{-i}] = 0$ for all $j \neq i$. This gives

$$E^j(E^{-i})^n v_\Lambda = (E^{-i})^n E^j v_\Lambda = 0 \qquad \forall n \in \mathbb{Z}_{\geq 0}. \tag{3.59}$$

Hence $(E^{-i})^{\Lambda_i+1}v_\Lambda$ is annihilated by all the simple root vectors. Since these generate all of the positive root vectors, it follows that $(E^{-i})^n v_\Lambda$ is a singular vector in the Verma module V_Λ, provided that $\Lambda_i \in \mathbb{Z}_{\geq 0}$.

If $\Lambda_i \in \mathbb{Z}_{\geq 0}$ for all i, then the Verma module contains r singular vectors given by

$$s_i = (E^{-i})^{\Lambda_i+1}v_\Lambda, \quad \forall i \in \{1, 2, ..., r\}. \tag{3.60}$$

These need not be all the singular vectors contained in a Verma module with non-negative integral Dynkin labels, but they are the most important ones, in the sense that every other singular vector is a descendent of at least one of the singular vectors in (3.60) [19]. We use this information to construct the integrable highest weight modules.

3.6.4 Irreducible integrable highest weight modules

To construct the irreducible representations we consider submodules generated by singular vectors. By the definition a singular vector, the action of $\mathcal{U}(\mathfrak{g})$ on s is given by

$$\mathcal{U}(\mathfrak{g})s = \mathcal{U}(\mathfrak{g}_-) \otimes \mathcal{U}(\mathfrak{g}_0) \otimes \mathcal{U}(\mathfrak{g}_+)s = \mathcal{U}(\mathfrak{g}_-)s.$$

Hence, the submodule generated by s is given by $\langle s\rangle = \mathcal{U}(\mathfrak{g}_-)s$. Since $s \notin V_\Lambda$, $\langle s\rangle$ is a proper submodule of V_Λ.

Given a submodule of V_Λ, we may define a new module by taking the quotient of V_Λ by that submodule. Furthermore, the quotient of a module by a maximal submodule yields an irreducible module. The construction of the integrable highest weight modules is based on the following non-trivial arguments. If the highest weight has non-negative integral Dynkin labels then all singular vectors are contained in a module of the form $\langle s_i\rangle$. The module formed by taking the sum of all such modules is a maximal submodule of V_Λ. We summarise these results on the construction of the irreducible integrable highest weight modules through the following theorem. It is attributed to Kac [16] and can be found in [19].

Theorem 3.6.5 *Let $\mathfrak{g}$ be a Kac-Moody Lie algebra with r simple roots $\alpha^{(1)}, \ldots, \alpha^{(r)}$. Suppose V_Λ is a Verma module such that the Dynkin labels of Λ satisfy $\Lambda_i \in \mathbb{Z}_{\geq 0}$. Then $S_\Lambda := \sum_{i=1}^{r}\langle s_i\rangle$ is a maximal submodule of V_Λ. Moreover, the irreducible integrable highest weight module with highest weight Λ is given by*

$$\mathcal{L}_\Lambda = \frac{V_\Lambda}{S_\Lambda}. \tag{3.61}$$

Chapter 4

Conformal field theory

This chapter will begin by introducing the basic concepts of two dimensional conformal field theory. For a detailed overview of conformal field theory the reader is directed to the comprehensive text [7]. These concepts will then be illustrated through the well-known examples of the free boson and the free fermion conformal field theories. We do not introduce the concept of primary fields or vertex operator algebras, but rather treat the spectrum of a conformal field theory from the perspective of representations of the symmetry algebra (see Section 4.2.7).

We will also introduce the concept of the twisted sector in the context of normal ordering (Section 4.2.3) and illustrate how to analyse the properties of a twisted sector through the use of a generalised commutation relation in the free fermion theory (Section 4.4).

Finally, we will introduce the Wess-Zumino-Witten (WZW) models as conformal field theories with symmetry algebras given by untwisted affine Lie algebras. We will not present their derivation from a physical action as introduced by Witten [9], for this the reader should see Chapter 15 of [7]. We will present the concept of singular vectors in the WZW models and outline how they restrict the modules that appear in the spectrum of a Wess-Zumino-Witten model at integer level k.

4.1 Conformal invariance in two dimensions

In two-dimensions the property of conformal invariance gives rise to an infinite dimensional symmetry algebra. This is the Virasoro algebra. We first introduce the Virasoro algebra and then motivate the reasons behind its appearance in conformal field theory.

The Virasoro algebra

The Virasoro algebra is a complex Lie algebra defined by the basis elements $\{L_n | n \in \mathbb{Z}\} \cup \{C\}$ and the relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}n(n^2-1)\delta_{n+m,0}C, \quad [L_m, C] = 0. \tag{4.1}$$

This algebra plays a pivotal role in the analysis of conformal mappings in two dimensions and consequently conformal field theory. We will show how a copy of this algebra arises from considering infinitesimal conformal transformations on the plane.

Generators of infinitesimal conformal transformations

It can be shown [7] that an infinitesimal transformation on the plane is determined by a holomorphic and an antiholomorphic function under an appropriate transformation of variables. Any holomorphic infinitesimal transformation can be written in terms of an infinitesimal Taylor expansion

$$z' = z + \epsilon(z), \qquad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1}, \tag{4.2}$$

for some coefficients $c_n \in \mathbb{C}$. Similarly with an antiholomorphic infinitesimal transformation we have (for some $\bar{c}_n \in \mathbb{C}$)

$$\bar{z}' = \bar{z} + \epsilon(\bar{z}), \qquad \epsilon(\bar{z}) = \sum_{-\infty}^{\infty} \bar{c}_n \bar{z}^{n+1}. \tag{4.3}$$

Applying these transformations to a function $\phi(z, \bar{z})$ on the plane gives

$$\phi(z,\bar{z}) = \phi(z',\bar{z}') - \epsilon(z')\partial_{z'}\phi(z',\bar{z}') - \bar{\epsilon}(\bar{z}')\partial_{\bar{z}'}\phi(z',\bar{z}'), \tag{4.4}$$

$$\implies \qquad \delta\phi = -\epsilon(z)\partial_z\phi - \bar{\epsilon}(\bar{z})\partial_{\bar{z}}\phi = -\sum_{-\infty}^{\infty}[c_n z^{n+1}\partial_z\phi + \bar{c}_n\bar{z}^{n+1}\partial_{\bar{z}}\phi]. \tag{4.5}$$

Hence the generators of the infinitesimal conformal transformation on ϕ are

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \tag{4.6}$$

where $n \in \mathbb{Z}$. These operators satisfy the commutation relations

$$[l_n, l_m] = (n-m)l_{n+m}, \qquad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}, \qquad [l_n, \bar{l}_m] = 0. \tag{4.7}$$

Hence, these generators form two commuting copies of the Virasoro algebra with $C = 0$. The values of C is called is called the *central charge* of the conformal field theory. Here $C = 0$ because we have considered a function in its classical sense. However, it is a hallmark of quantum systems that the central charge is non-zero.

In a general two dimensional conformal field theory the conformal generators form two commuting copies of the Virasoro algebra for some value $C \in \mathbb{C}$.

Since the two copies of the Virasoro algebra commute the field theory decomposes into a holomorphic and antiholomorphic sector. In general the properties of these two sectors are related and so we will restrict our view to the holomorphic sector, and express our fields only in terms of the holomorphic component: $\phi(z)$.

4.2 Formalism of conformal field theory

The formalities of conformal field theory are complex and detailed. Here we build up the fundamental concepts of a conformal field theory from the perspective of operations on a space of fields. We emphasise connections between operator products of fields and the properties of the symmetry algebra. The concept of an untwisted and twisted sector is introduced in the context of normal ordering. We also introduce important concepts such as singular vectors and outline their restriction on the spectrum of a conformal field theory.

4.2.1 Fields

A *generating field* is an expression of the form

$$A(z) = \sum_n a_n z^{-n-h_A}, \tag{4.8}$$

where the coefficients a_n are elements of a graded algebra over $\mathbb{C}$ called the *symmetry algebra* and h_A is a real number called the *conformal dimension* of $A(z)$. The labels n of the elements a_n will be referred to as the *modes* of $A(z)$ and usually range over integral or half integral values. The above definition is also equivalent to the set of relations given by

$$a_n = \oint_0 A(z) z^{n+h_A-1} \frac{dz}{2\pi i}. \tag{4.9}$$

For the purposes of this thesis we will primarily consider the case where the symmetry algebra is a infinite dimensional Lie algebra. However, in many cases, the symmetry algebra may have a more complex structure. The only case we will consider outside of the infinite dimensional Lie algebras is that of the free fermion. For this reason we will assume from here on that the symmetry algebra is an infinite dimensional Lie algebra. We will provide the required extensions of the theory when we deal with the free fermion in Section 4.4.

Under the operations of addition and scalar multiplication these fields generate a vector space. We can also define the derivative on any of these fields in the usual way

$$\frac{d}{dz}(A(z)) = \sum_n (-n - h_A) a_n z^{-n-h_A-1}. \tag{4.10}$$

It follows that taking the derivative increases the conformal dimension by one. The space of fields is also equipped with an identity field $\Omega = 1$.

There another operation on the space of fields, called the normal ordered product of two fields (see Section 4.2.3). This describes how to order the modes of two fields $A(z)$ and $B(z)$ when taking their product. Under these operations we call all the fields generated by the generating fields the *space of fields*. Firstly, we need to introduce some rules which govern the behaviour of these fields.

4.2.2 Operator product expansion

One of the motivating reasons for studying a field theory is to understand how fields interact. The way this information is encoded in a conformal field theory is through an operator product expansion. For a conformal field theory to behave nicely (we will not define what this means precisely since it is beyond the scope of this thesis) the operator product expansion must satisfy certain properties.

Before we define these properties we must introduce the concept of radial ordering which is analogous to the property of time ordering developed in the analysis of quantum field theory. In the case that the symmetry algebra is a Lie algebra the *radially-ordered* product of two fields is defined as

$$\mathcal{R}\{A(z)B(w)\} = \begin{cases} A(z)B(w) & |z| > |w|, \\ B(w)A(z) & |w| < |z|. \end{cases} \tag{4.11}$$

In the context of this thesis, for a conformal field theory to be well defined, all fields $A(z)$ and $B(w)$ must have an *operator product expansion* (OPE) of the form

$$\mathcal{R}\{A(z)B(w)\} = \sum_{j \leq N} c^j(w)(z-w)^{-j}, \tag{4.12}$$

where N is an integer and j sums over integers strictly less than N. The elements $c^j(w)$ must also belong to the space of fields.

Through the definition of radial ordering it can be seen that the singular part of

(4.12) is equivalent to the commutation relations of the symmetry algebra:

$$\begin{aligned}
[a_n, b_m] = &\oint_0 \oint_0 A(z)B(w) z^{n+h_A-1} w^{m+h_B-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&\qquad - \oint_0 \oint_0 B(w)A(z) z^{n+h_A-1} w^{m+h_B-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
= &\oint_0 \oint_{|z|>|w|} A(z)B(w) z^{n+h_A-1} w^{m+h_B-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&\qquad - \oint_0 \oint_{|z|<|w|} B(w)A(z) z^{n+h_A-1} w^{m+h_B-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
= &[\oint_0 \oint_{|z|>|w|} - \oint_0 \oint_{|z|<|w|}] \mathcal{R}\{A(z)B(w)\} z^{n+h_A-1} w^{m+h_B-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
= &\oint_0 \oint_w \mathcal{R}\{A(z)B(w)\} z^{n+h_A-1} w^{m+h_B-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i}.
\end{aligned} \tag{4.13}$$

The equivalence between commutation relations and operator product expansions is a fundamental property of conformal field theory. For this reason we outline the steps taken. The first step of the derivation is an application of the definition (4.9) and the commutator. Line two involves a simple contour manipulation and three is given by the definition of radial ordering. Step four follows from the contour manipulation outlined in Figure 4.1.

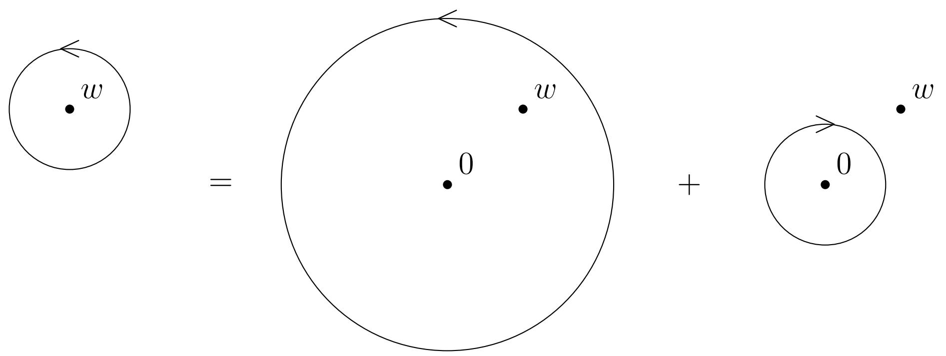

Figure 4.1: Contour manipulation used in (4.13)

Since we always assume that the product of two fields is radially-ordered, we will often write $\mathcal{R}\{A(z)B(w)\}$ as $A(z)b(w)$, in which case radial-ordering is implicitly assumed. Because the singular terms of the product (4.12) determine the commutation relations of the symmetry algebra we will often summarise the OPE of two fields as

$$A(z)B(w) \sim \sum_{j=1}^{N} \frac{c^j(w)}{(z-w)^j}. \tag{4.14}$$

Since this expansion fully specifies the commutation relations between the modes of the symmetry algebra, we may also define a field theory to be generated by certain fields that satisfy OPEs of the form (4.14). In this case we will refer to these fields as the *defining fields* of the field theory.

4.2.3 Normal ordering

When we introduced the operator product expansion in (4.12) we required that all the $c^j(w)$ terms belonged to the space of fields. We have seen that the commutation relations of the symmetry algebra determine the singular part of the operator product expansion, but this still leaves the $c^j(w)$ terms for $j \leq 0$, that form the regular part of (4.12). To understand these terms we introduce the notion of normal ordering. We define the *normally-ordered* product of two fields $A(z)$ and $B(w)$ as the analytic part of the operator expansion

$$: A(z)B(w) : \; := \sum_{j \leq 0} c^j(w)(z-w)^{-j}. \tag{4.15}$$

Taking an expansion of the field $: A(z)B(w) :$ about the point $z = w$ gives

$$: A(z)B(w) : \; =: A(w)B(w) : + : \partial_w A(w)B(w) : (z-w) + \cdots . \tag{4.16}$$

We thus define the *normally-ordered* product of two fields, $A(w)$ and $B(w)$, with the same argument, as the constant term in the operator product expansion of the two fields

$$: A(w)B(w) : \; := c^0(w). \tag{4.17}$$

From the OPE (4.12) it is clear that an explicit formula for the normally ordered product is given by

$$: A(w)B(w) : \; = \oint_w \frac{A(z)B(w)}{z-w} \frac{dz}{2\pi i}. \tag{4.18}$$

This formula proves very useful if the modes of the fields satisfy certain properties. We are thus lead to define two different sectors of a conformal field theory.

The *untwisted sector* of a conformal field theory satisfies the property that all generating fields are of the form

$$A(z) = \sum_{n \in \mathbb{Z} - h_A} a_n z^{-n-h_A}. \tag{4.19}$$

The fields in this sector may all be written as power series in z and z^{-1}. The *twisted sector* of a conformal field theory occurs when any of the generating fields do not satisfy (4.19). Such a sector may not exist and there could also be multiple different twisted sectors corresponding to different mode values. It is important to note that despite taking different mode values the commutation relations in the untwisted and twisted sector have the same form. The terms untwisted and twisted sector are often replaced with Neveu-Schwarz [35] and Ramond [28] sector when the fields are fermionic.

Since a conformal field theory is described by the relationships between the fields, it is possible for a conformal field theory to have both an untwisted sector and a twisted sector. In general the untwisted sector is the easy sector to work with since nice formulas hold in this sector, see Appendix A. The primary aim of this thesis is to understand the twisted sector of the Wess-Zumino-Witten models. For this reason we will focus on the properties and formulation of the twisted sector which will be introduced in Sections 4.3 and 4.4. Firstly, we outline the formulas that determine normal ordering in the untwisted sector.

Normal ordering in the untwisted sector

In Appendix A we derive the following important formula for the normal ordering of two fields in the untwisted sector.

$$:A(z)B(z): \ = \sum_{m\in\mathbb{Z}-h_B-h_A} \Big[\sum_{n\leq -h_A} a_n b_{m-n} + \sum_{n>-h_A} b_{m-n}a_n\Big] z^{-m-h_A-h_B}. \tag{4.20}$$

It is natural to define the normal-ordering of modes by

$$:a_n b_m: \ = \begin{cases} a_n b_m & \text{if } n \leq -h_A, \\ b_m a_n & \text{if } n > -h_A. \end{cases} \tag{4.21}$$

In this way we have

$$:A(z)B(z): \ = \sum_{m\in\mathbb{Z}-h_B} \sum_{n\in\mathbb{Z}-h_A} :a_n b_m: z^{-n-h_A} z^{-m-h_B}. \tag{4.22}$$

Some nice consequences of these expressions for normal ordering are outlined in Appendix A, in particular, equation (A.17) determines an easy method to calculate operator product expansions using the commutation relations of the symmetry algebra.

Normal ordering in the twisted sector

In the twisted sector the derivation of the normal ordering formula (4.20) no longer applies, see Appendix A. In order to determine the normally-ordered product of two fields a more general approach is required. This method will be introduced in Section 4.4 in the context of the Free Fermion. We shall apply this is in Chapter 5 for the WZW model.

4.2.4 The energy-momentum tensor

So far we have introduced the idea of a space of fields arising from some generating fields with coefficients taken from a symmetry algebra. However as motivated in

Section 4.1, conformal invariance of a two dimensional field theory is equivalent to the existence of a copy of the Virasoro algebra. Specifically, this corresponds to the existence of a field called the *energy-momentum tensor* denoted by $T(z)$

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{4.23}$$

whose coefficients are the operators of the Virasoro algebra described in Section 4.1:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{4.24}$$

where $c \in \mathbb{C}$ is called the *central charge* of the conformal field theory. Note that the conformal dimension of the energy-momentum tensor is necessarily 2. It can be derived from (4.13) that the operator product expansion of the energy-momentum tensor with itself is given by

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{4.25}$$

The existence of an energy-momentum tensor is the requirement of a conformal field theory that is responsible for conformal invariance. Since an OPE is equivalent to the commutation relations (4.13) of its modes the existence of an energy-momentum tensor is equivalent to the existence of a field that satisfies (4.25).

4.2.5 The state-field correspondence

So far we have considered the operations of fields upon themselves and other fields. Now we would also like to analyse the action of these fields on vector spaces and thus the possible modules related to a conformal field theory. To do this we introduce a particularly important module of the symmetry algebra (the vacuum module) and a map from the space of fields to this module (the state-field correspondence).

The vacuum module belongs to the untwisted sector of the conformal field theory, hence the generating fields are of the form (4.19). We define a vector $|0\rangle$ called the *vacuum* with the following properties

$$a_n|0\rangle = 0 \qquad \forall \quad n > -h_A, \tag{4.26}$$

where a_n are the coefficients of the generating fields. The *universal vacuum module*, denoted by $\mathcal{V}_{|0\rangle}$, is the largest module generated by $|0\rangle$ under the action of the symmetry algebra. It consists of all elements of the form

$$\{x_1 x_2 \cdots x_n|0\rangle | x_i \in \text{symmetry algebra}\}. \tag{4.27}$$

In particular, if the symmetry algebra is a Lie algebra $\mathfrak{g}$, then the universal vacuum module consists of all descendants of the vacuum and is given by $\mathcal{U}(\mathfrak{g})|0\rangle$ subject to the relations (4.26). $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, see Section 3.6.1.

We define a map called the *state-field correspondence* between the space of fields and elements of the universal vacuum module

$$A(z) \mapsto \lim_{z\to 0} A(z)|0\rangle. \tag{4.28}$$

On the right-hand side of this map, $A(z)$ is taken in the untwisted sector since the vacuum module is in the untwisted sector. Fields with conformal dimension h_A then map as follows

$$A(z) \mapsto a_{-h_A}|0\rangle. \tag{4.29}$$

Of importance to this thesis is the mapping of a normal ordered product:

$$:A(z)B(z): \mapsto \Big(\sum_{n\leq -h_a} a_n b_{h_A+h_B-n} + \sum_{n>-h_a} b_{h_A+h_B-n}a_n\Big)|0\rangle = a_{-h_A}b_{-h_B}|0\rangle. \tag{4.30}$$

It is also important to note that this map is linear and so the zero field is mapped to the zero vector. The state-field correspondence also has an inverse map from the vacuum module to the space of fields which is injective [36]. These facts will prove to be very important in the discussion of singular vectors.

4.2.6 Singular vectors

Elements of the universal vacuum module that are particularly important are the singular vectors. A vector $|s\rangle$ in the vacuum module is called a *singular vector* if it is not a multiple of the vacuum and it satisfies

$$a_n|s\rangle = 0 \qquad \forall n > 0. \tag{4.31}$$

The module consisting of this vector and all of its descendants under the action of the symmetry algebra is a submodule of the universal vacuum module. We will call this module the *singular module* generated by $|s\rangle$ and it will be denoted by $\langle s\rangle$. In order to obtain a physically consistent conformal field theory we shall set all the singular vectors to zero, if we set a singular vector to zero it follows that we must set the entire module $\langle s\rangle$ to zero.

The importance of this identification can be seen through the state-field correspondence. Suppose $|v\rangle \in \langle s\rangle$ is a descendent of a singular vector. Since the state field correspondence is bijective there exists a field $A(z)$ that maps to $|v\rangle$. Since the state-field correspondence is linear, by identifying $|v\rangle$ with zero we must also identify $A(z)$ with zero. In this case we call $|v\rangle$ a null vector and $A(z)$ a *null field*.

In Section 3.6.4 we discussed the procedure of setting singular vectors to zero in terms of quotient modules in the context of irreducible integrable highest weight modules. We will use these results in Section 4.5.5 when we discuss the general structure of the universal vacuum module of a Wess-Zumino-Witten model. For now it is important to note that once all singular vectors have been identified with zero the remaining vectors in the vacuum module form an irreducible module which is called the *irreducible vacuum module*, denoted by $\mathcal{L}_{|0\rangle}$.

In order to understand the importance of singular vectors we must first introduce another component of a conformal field theory, namely the spectrum.

4.2.7 The spectrum

For a conformal field theory to provide useful information (e.g. measurements of observables) we would like to know how the fields act on vector spaces. Suppose we have a vector space V, then for an element $|v\rangle \in V$ a generating field $A(z)$ will act upon $|v\rangle$ as

$$A(z)|v\rangle = \sum_n a_n|v\rangle z^{-n-h_A}.$$

Hence, we can consider the action of $A(z)$ on V by considering V as a module of the symmetry algebra.

Now we can realise the importance of null fields. If $A(z)$ is a null field, then it must act on every vector in V as the zero field. In other words $A(z)|v\rangle = 0$ for all $|v\rangle \in V$. We summarise this result in the following proposition.

Proposition 4.2.1 *Let $|s\rangle$ be a singular vector and $|a\rangle \in \langle s\rangle$. Let $A(z) = \sum_n a_n z^{-n-h_A}$ be the field that maps to $|a\rangle$ under the state-field correspondence. If V is a module of fields then*

$$a_n|v\rangle = 0 \qquad \forall n. \tag{4.32}$$

We call any module of the symmetry algebra that satisfies all requirements of the form (4.32) for all null fields an *allowed module*. For the purposes of this thesis we define the *spectrum* of a conformal field theory to be the set of all allowed irreducible highest weight modules.

The spectrum of a conformal field theory is closely related to the space of states. The space of states is a Hilbert space constructed from all the allowed modules of a conformal field theory and describes all the possible states of the conformal field theory. Its construction is non-trivial and usually involves tensoring holomorphic and antiholomorphic copies of allowed modules subject to certain constraints. This is beyond the scope of this thesis and so we will restrict our view to the allowed modules.

Since the symmetry algebra takes different mode values in the untwisted and twisted sector, the allowed modules will be different in each of these sectors. We have already been introduced to one allowed module in the untwisted sector, namely the irreducible vacuum module. Hence, the spectrum of every conformal field theory contains the irreducible vacuum module as an allowed module in the untwisted sector. The aim of this thesis is to determine the allowed modules in the twisted sector of the Wess-Zumino-Witten conformal field theories, this will be the focus of Chapter 4. Before we are able to do this we will highlight the concepts introduced in Section 4.2 through the example of the free boson and the free fermion conformal field theory. We will describe how to determine normal ordering of fields in the twisted sector of the free fermion conformal field theory via a generalised commutation relation.

4.3 The free boson

The current algebra for the free boson conformal field theory is given by the Heisenberg algebra whose central element is the scalar one. The Lie brackets are

$$[a_n, a_m] = n\delta_{n+m,0}. \tag{4.33}$$

In the untwisted sector the generating field is given by

$$\partial\psi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \tag{4.34}$$

where the partial derivative is motivated by physical connections to an action [7]. By (4.13), it follows that the operator product expansion for this field theory is

$$\partial\psi(z)\partial\psi(w) = \frac{1}{(z-w)^2} + :\partial\psi(z)\partial\psi(w): \;\sim \frac{1}{(z-w)^2}. \tag{4.35}$$

To show that this field theory is conformal we must find a field that satisfies the requirements of the energy-momentum tensor (4.25). For this we consider the field

$$T(z) = \frac{1}{2} :\partial\psi(z)\partial\psi(z): \;= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} :a_n a_{m-n}: z^{-m-2}, \tag{4.36}$$

where the second equality holds only in the untwisted sector. Using Wick's Theorem [7] the operator product expansion of $T(z)$ with itself gives

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{4.37}$$

It follows that a copy of the Virasoro algebra with central charge $c = 1$ is given by

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :a_n a_{m-n}: . \tag{4.38}$$

By the Poincaré-Birkhoff-Witt Theorem (Theorem 3.6.1) the universal vacuum module is spanned by elements of the form

$$a_{-n_1}^{k_1} a_{-n_2}^{k_2} \cdots a_{-n_l}^{k_l}|0\rangle, \tag{4.39}$$

where $n_1 > n_2 > \cdots > n_l > 0$ and $k_1, k_2, ..., k_l, l \in \mathbb{N}$. It is a well-known result that a Verma module of a Heisenberg algebra is irreducible and hence the universal vacuum module contains no singular vectors. Thus there are no null fields (excluding the zero field) in the free boson conformal field theory.

The spectrum of this conformal field theory contains all highest weight modules of the Heisenberg algebra. Let $|p\rangle$ be a highest weight vector of weight p, that is

$$a_0|p\rangle = p|p\rangle, \qquad a_n|p\rangle = 0 \qquad \forall n > 0. \tag{4.40}$$

In order to calculate the conformal dimension of the highest weight vector we use the normal ordering formula (4.20) for L_0

$$L_0 = \frac{1}{2}\sum_n : a_n a_{-n} : \; = \frac{1}{2}a_0 a_0 + \sum_{n>0} a_{-n}a_n. \tag{4.41}$$

Apply L_0 to a highest weight vector $|p\rangle$ gives

$$L_0|p\rangle = \frac{1}{2}a_0 a_0|p\rangle = \frac{1}{2}p^2|p\rangle. \tag{4.42}$$

Hence, the conformal dimension of the highest weight vector $|p\rangle$ is $\frac{1}{2}p^2$. The irreducible module generated by the highest weight $|p\rangle$ consists of all descendent vectors, this is the Verma module for the Heisenberg algebra with highest weight p. We denote this module by V_p. Since the field theory should provide physically meaningful results we restrict p to the real numbers. The untwisted spectrum of the free boson conformal field theory is given by $\{V_p | p \in \mathbb{R}\}$.

4.4 The free fermion

The free fermion is an example of a conformal field theory whose symmetry algebra is endowed with an anticommutation relation. In this case the fields in the conformal field theory can be fermionic or bosonic. The formalism we have introduced so far applies only to bosonic fields. We will introduce the formalism for fermionic fields as required. In order to relate anticommutation relations with operator product expansions we need to define the radial ordering of two fermionic fields by

$$\mathcal{R}\{A(z)B(w)\} = \begin{cases} A(z)B(w) & \text{if } |z| > |w|, \\ -B(w)A(z) & \text{if } |z| < |w|. \end{cases} \tag{4.43}$$

Now the formula (4.13) can be replaced by

$$\{a_n, b_m\} = \oint_0 \oint_w \mathcal{R}\{A(z)B(w)\} z^{n+h_A-1} w^{m+h_B-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i}. \tag{4.44}$$

The free fermion has defining field $\psi(z)$, with conformal dimension $\frac{1}{2}$, which satisfies the operator product expansion

$$\psi(z)\psi(w) \sim \frac{1}{z-w}. \tag{4.45}$$

The field $T(z) = \frac{1}{2} : \partial\psi(z)\psi(z) :$ then satisfies the operator product expansion

$$T(z)T(w) \sim \frac{\frac{1}{4}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{4.46}$$

Hence, the central charge of the free fermion theory is $c = \frac{1}{2}$. Equation (4.44) implies that the modes of the symmetry algebra satisfy

$$\{b_n, b_m\} = \delta_{n+m,0}. \tag{4.47}$$

Since the fields have conformal dimension $\frac{1}{2}$ in the Neveu-Schwarz (untwisted) sector, the modes of the fields must take half integral values

$$\psi(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n z^{-n-\frac{1}{2}}. \tag{4.48}$$

Defining the vacuum vector as usual, the state-field correspondence implies that

$$\psi(z) \mapsto b_{-\frac{1}{2}}|0\rangle. \tag{4.49}$$

A basis for the universal vacuum module consists of elements of the form

$$b_{-n_1}b_{-n_2}\cdots b_{-n_k}|0\rangle, \tag{4.50}$$

where $n_i \in \mathbb{Z}+\frac{1}{2}$, $n_1 > n_2 > \cdots n_k > 0$ and $k \in \mathbb{Z}_{\geq 0}$. Each term may appear only once due to the anticommutation relation $\{b_{-n}, b_{-n}\} = 0$. This phenomenon is referred to as the Pauli Exclusion Principle [32]. There no singular vectors in the universal vacuum module. As there are no zero modes in the Neveu-Schwarz sector, the irreducible vacuum module, $\mathcal{L}_{|0\rangle}$, is the only allowed highest weight module.

In the Ramond sector the modes of the fields take integer values

$$\psi(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-\frac{1}{2}}. \tag{4.51}$$

A vector $|p\rangle_T$, which we assume to have no parity, is a highest weight vector in the Ramond sector if it satisfies

$$b_0|p\rangle_T = p|p\rangle_T, \qquad b_n|p\rangle_T = 0 \quad \forall n > 0. \tag{4.52}$$

Since $\{b_0, b_0\} = 2b_0b_0 = 1$, it follows that $p = \pm\frac{1}{\sqrt{2}}$ and so the only allowed highest weight module in the twisted sector is that generated by $|\pm\frac{1}{\sqrt{2}}\rangle_T$. We would like to be able to compute the energy of this highest weight vector and its descendants. To do this we need to be able to find an expansion for L_0. In order to describe normal ordering in the Ramond sector, we will derive a new formula called a generalised commutation relation.

4.4.1 Generalised commutation relation

A generalised commutation relation is a formula derived from a double contour integral involving the operator product expansion of two fields. To illustrate a useful generalised commutation relation for the Free Fermion we begin with the expression

$$\oint_0 \oint_w \mathcal{R}\{\psi(z)\psi(w)\} z^{n+\frac{3}{2}} w^{m-\frac{1}{2}} (z-w)^{-2} \frac{dz}{2\pi i}\frac{dw}{2\pi i}. \tag{4.53}$$

By inserting the OPE of the two fields we may compute this integral directly:

$$
\begin{aligned}
&\oint_0\oint_w [\frac{1}{z-w} + :\psi(w)\psi(w): + :\partial\psi(w)\psi(w): + \cdots] z^{n+\frac{3}{2}} w^{m-\frac{1}{2}} (z-w)^{-2} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \oint_0 [\frac{1}{2}(n+\frac{3}{2})(n+\frac{1}{2}) w^{n+m-1} + :\partial\psi(w)\psi(w): w^{n+m+1}] \frac{dw}{2\pi i} \\
&= \frac{1}{2}(n+\frac{3}{2})(n+\frac{1}{2})\delta_{n+m,0} + :\partial\psi\psi:_{n+m} \\
&= \ :\partial\psi\psi:_{n+m} + \frac{(2n+1)(2n+3)}{8}\delta_{n+m,0}.
\end{aligned} \tag{4.54}
$$

Here we have made use of the fact that $:\psi(w)\psi(w): \ = 0$ and the expansion

$$
:\partial\psi(w)\psi(w): \ = \sum_n :\partial\psi\psi:_n w^{-n-2}. \tag{4.55}
$$

In a similar fashion to the derivation of the commutation relation in (4.13) we can apply the definition of radial ordering to compute a relation in terms of the modes from (4.53). We will make use of the sums:

$$
\begin{aligned}
\text{if } |z| > |w| : \qquad & (z-w)^{-2} = \sum_{r=1}^{\infty} r z^{-r-1} w^{r-1}, \\
\text{if } |z| < |w| : \qquad & (z-w)^{-2} = -\sum_{r=1}^{\infty} r z^{r-1} w^{-r-1}.
\end{aligned}
$$

The integral (4.53) then becomes

$$
\begin{aligned}
&\oint_0\oint_{|z|>|w|} \psi(z)\psi(w) z^{n+\frac{3}{2}} w^{m-\frac{1}{2}} \sum_{r=1}^{\infty} r z^{-r-1} w^{r-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&\qquad + \oint_0\oint_{|z|<|w|} \psi(w)\psi(z) z^{n+\frac{3}{2}} w^{m-\frac{1}{2}} \sum_{r=1}^{\infty} r z^{r-1} w^{-r-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \sum_{r=1}^{\infty} r \oint_0 \psi(z) z^{n-r+\frac{1}{2}} \frac{dz}{2\pi i} \oint_0 \psi(w) w^{m+r-\frac{3}{2}} \frac{dw}{2\pi i} \\
&\qquad + \sum_{r=1}^{\infty} r \oint_0 \psi(w) w^{m-r-\frac{3}{2}} \frac{dw}{2\pi i} \oint_0 \psi(z) z^{n+r+\frac{1}{2}} \frac{dz}{2\pi i} \\
&= \sum_{r=1}^{\infty} r [b_{n-r+1} b_{m+r-1} + b_{m-r-1} b_{n+r+1}] \\
&= \sum_{r=0}^{\infty} (r+1) [b_{n-r} b_{m+r} + b_{m-r-2} b_{n+r+2}].
\end{aligned} \tag{4.56}
$$

Combining equations (4.54) and (4.56) we arrive at a generalised commutation relation,

$$\sum_{r=0}^{\infty}(r+1)[b_{n-r}b_{m+r}+b_{m-r-2}b_{n+r+2}] = \,:\partial\psi\psi:_{n+m} + \frac{(2n+1)(2n+3)}{8}\delta_{n+m,0}. \tag{4.57}$$

Since $T(z)=\frac{1}{2}:\partial\psi(w)\psi(w):$, the above formula provides an expansion for the Virasoro operators,

$$L_{n+m} = \frac{1}{2}\sum_{r=0}^{\infty}(r+1)[b_{n-r}b_{m+r}+b_{m-r-2}b_{n+r+2}] - \frac{(2n+1)(2n+3)}{16}\delta_{n+m,0}. \tag{4.58}$$

Note that these formulas hold in both the Ramond and the Neveu-Schwarz sector. In the Ramond sector, by setting $m=1$ and $n=-1$, we can write L_0 as

$$L_0 = \frac{1}{2}\sum_{r=0}^{\infty}(r+1)[b_{-r-1}b_{r+1}+b_{-r-1}b_{r+1}] + \frac{1}{16}. \tag{4.59}$$

If we act with L_0 on the highest weight vector $|p\rangle_T$, all the positive modes act on the right annihilating $|p\rangle_T$, leaving only the constant term:

$$L_0|p\rangle_T = \frac{1}{16}|p\rangle_T. \tag{4.60}$$

Thus, the conformal dimension of the twisted highest weight state is 1/16.

4.5 Wess-Zumino-Witten models

We first introduce the operator product expansions and energy-momentum tensor of the Wess-Zumino-Witten (WZW) model. These derivations are not produced here since they are quite lengthy, more detailed explanations of these derivations can be found in [37] or [7]. This thesis focuses much more heavily on the representation theory aspects of the WZW model. In Section 4.5.2 we determine the singular vectors that appear in the universal vacuum module. Sections 4.5.3 and 4.5.4 apply the restrictions imposed by the singular vector to the specific WZW models $\widehat{\mathfrak{sl}(2)}$ and $\widehat{\mathfrak{sl}(3)}$ at level $k=1$. Finally, we present a theory for the spectrum of a general WZW model at non-negative integer level k. The final result is that the allowed modules in the spectrum are the integrable highest weight modules of the corresponding affine Lie algebra.

4.5.1 Operator product expansions

A WZW model is a conformal field theory whose symmetry algebra is an untwisted affine Lie algebra. Recall from the construction of the untwisted affine Lie algebras in Section 2.3 that an untwisted affine Lie algebra $\hat{\mathfrak{g}}$ is constructed from a finite dimensional Lie algebra $\mathfrak{g}$. The elements of $\hat{\mathfrak{g}}$ have Lie bracket given by

$$[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + n\delta_{n+m,0}\bar{\kappa}^{ab}K, \tag{4.61}$$

where K is a central element of $\hat{\mathfrak{g}}$, $\bar{\kappa}$ is the Killing form on $\mathfrak{g}$ and $[J^a, J^b]$ is the Lie bracket of $\mathfrak{g}$ on the elements $J^a, J^b \in \mathfrak{g}$. The *level* of a WZW model is the value k that satisfies $K|0\rangle = k|0\rangle$ and we denote by $\hat{\mathfrak{g}}_k$ the WZW model associated to the affine Lie algebra $\hat{\mathfrak{g}}$ at level k.

The generating fields are constructed as series expansions of the elements of $\hat{\mathbf{g}}$,

$$J^a(z) = \sum_n J^a_n z^{-n-1}. \tag{4.62}$$

The operator product expansion of these fields is given by

$$J^a(z)J^b(w) \sim \frac{[J^a, J^b](w)}{z-w} + \frac{k\bar{\kappa}^{ab}}{(z-w)^2}. \tag{4.63}$$

If we take the basis $\mathcal{B} = \{J^a | a = 1, ..., \dim(\mathbf{g})\}$ of $\mathbf{g}$ such that the Killing form is orthogonal with respect ot this basis, $\bar{\kappa}^{ab} = \delta^{ab}$, then the energy-momentum tensor is given by the Sugawara construction in the form [7]:

$$T(z) = \gamma \sum_{J^a \in \mathcal{B}} : J^a(z)J^a : (z), \qquad \gamma = \frac{1}{2(k+g)}, \tag{4.64}$$

where g is the dual Coxeter number of $\mathbf{g}$ (2.64). Evaluating the operator product expansion of $T(z)$ with itself gives

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{z-w}, \qquad c = \frac{k\dim(\mathbf{g})}{k+g}. \tag{4.65}$$

It follows that in the untwisted sector the elements of the Virasoro algebra can be expressed as

$$L_m = \gamma \sum_{J^a \in \mathcal{B}} \sum_n : J^a_n J^a_{m-n} : . \tag{4.66}$$

4.5.2 Singular vectors

We will consider the universal vacuum module for a general WZW model $\hat{\mathfrak{g}}_k$ at integer level k. By definition, the vacuum vector satisfies

$$J^a_n|0\rangle = 0 \qquad \forall n \geq 0, \quad \forall J^a \in \mathfrak{g}. \tag{4.67}$$

The requirement (4.67) implies that $|0\rangle$ is a highest weight vector in an a $\hat{\mathfrak{g}}$-module. Let $\hat{\mathfrak{g}}$ have simple roots $\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(r)}$. Suppose $|0\rangle$ has weight Λ, then writing Λ in terms of Dynkin labels for $\hat{\mathfrak{g}}$ gives

$$\Lambda = (\Lambda_0, \Lambda_1, \cdots \Lambda_r), \qquad \Lambda_i = \langle {\alpha^{(i)}}^\vee, \Lambda\rangle. \tag{4.68}$$

Equation (4.67) implies that the $E_0^{-\alpha^{(i)}}|0\rangle = 0$ for all non-affine simple roots (i.e. $i = 1, 2, \ldots, r$). Hence, the Dynkin label $\Lambda_i = 0$ for $i \in \{1, \cdots, r\}$. By equation (3.38) this implies that $k = \Lambda_0$. In summary, $|0\rangle$ has weight given by the Dynkin labels $(k, 0, \ldots, 0)$.

From Section 3.6.3 we know that the element $(E_{-1}^{\theta})^{k+1}|0\rangle$ is annihilated by all the positive simple roots of $\hat{\mathfrak{g}}$. By definition (4.31), $(E_{-1}^{\theta})^{k+1}|0\rangle$ is therefore a singular vector of the universal vacuum module.

A particularly important descendent of this singular vector is the null vector

$$(E_{-1}^{-\theta})^{k+1}|0\rangle. \tag{4.69}$$

This vector can be seen to be a descendent of $(E_{-1}^{\theta})^{k+1}|0\rangle$ by acting with $(E_0^{-\theta})^{2(k+1)}$, or by analysing the structure of the $\mathfrak{g}$-module generated by the action of the horizontal subalgebra on $(E_{-1}^{\theta})^{k+1}|0\rangle$.

By equation (4.30) the state-field correspondence maps

$$:(E^{-\theta})^{k+1}(z): \;\mapsto (E_{-1}^{-\theta})^{k+1}|0\rangle, \tag{4.70}$$

where $:(E^{-\theta})^{k+1}(z):$ is the normal ordered product of $k+1$ copies of $E^{-\theta}(z)$. Hence, $:(E^{-\theta})^{k+1}(z):$ is a null field. If V is an allowed module, by Proposition 4.2.1 we require

$$:(E^{-\theta})^{k+1}:_n |v\rangle = 0 \qquad \forall n, \quad \forall |v\rangle \in V. \tag{4.71}$$

In order to determine the allowed highest weight modules in the untwisted sector we must analyse the implications of (4.71). We will do so for specific examples in the next two sections before developing the theory for the generalised case in Section 4.5.5. These examples will form the basis for our approach to understanding the allowed modules in the twisted sector in Chapter 5.

4.5.3 Allowed highest weight modules: $\widehat{\mathfrak{sl}(2)}_1$

Recall the construction of $\widehat{\mathfrak{sl}(2)}$ in Section 2.3. The simple roots are given by $\alpha^{(0)} = (-\alpha, 0, 1)$ and $\alpha^{(1)} = (\alpha, 0, 0)$ where α is the positive root of the horizontal subalgebra which is isomorphic to $\mathfrak{sl}(2)$. Since α is also the highest root of the horizontal subalgebra, we have $\theta = \alpha$. The null vector from (4.69) is given by

$$(E_{-1}^{-\alpha})^2|0\rangle. \tag{4.72}$$

Under the state-field correspondence $:E^{-\alpha}(z)E^{-\alpha}(z): \;\mapsto (E_{-1}^{-\alpha})^2|0\rangle$ so that the corresponding null field is $:E^{-\alpha}(z)E^{-\alpha}(z):$.

Let us consider an $\widehat{\mathfrak{sl}(2)}$-module V. Applying the formula for normal ordering in the untwisted sector (4.20), the requirement (4.71) for V to be an allowed module becomes

$$\Big(\sum_{n\le -1} E_n^{-\alpha}E_{m-n}^{-\alpha} + \sum_{n\ge 0} E_{m-n}^{-\alpha}E_n^{-\alpha}\Big)|v\rangle = 0 \qquad \forall m \in \mathbb{Z}, \quad \forall |v\rangle \in V. \tag{4.73}$$

This provides infinitely many requirements on every element of the module V. Fortunately, we only need to consider this requirement for certain values of n on specific vectors of V to fully determine the allowed modules. We will show through this example which requirements are important and in Section 4.5.5 provide a discussion about why these conditions suffice.

A highest weight vector $|\lambda\rangle \in V$ with weight λ satisfies the requirements

$$E_n^{\alpha}|\lambda\rangle = E_n^{-\alpha}|\lambda\rangle = H_n^{\alpha}|\lambda\rangle = 0 \quad \forall n > 0, \qquad E_0^{\alpha}|\lambda\rangle = 0,$$
$$H_0^{\alpha}|\lambda\rangle = \lambda_1|\lambda\rangle, \qquad K|\lambda\rangle = k|\lambda\rangle,$$

where λ has Dynkin labels (λ_0, λ_1). By equation (3.38) it follows that $\lambda_0 = k - \lambda_1$. Hence, for k fixed there is only one degree of freedom in the possible Dynkin labels of $|\lambda\rangle$.

Setting $m = 0$ in the requirement (4.73) with $|v\rangle = |\lambda\rangle$ we find

$$E_0^{-\alpha}E_0^{-\alpha}|\lambda\rangle = 0. \tag{4.74}$$

By the analysis of $\mathfrak{sl}(2)$ representations in Section 3.5.2 this requirement implies that $\lambda_1 = 0$ or $\lambda_1 = 1$. So there are two possible allowed highest weights

$$|\lambda\rangle = (1, 0) \quad \text{and} \quad |\lambda\rangle = (0, 1)$$

The first weight is that of the vacuum vector for $k = 1$ and the second is the highest weight of the other integrable highest weight $\widehat{\mathfrak{sl}(2)}$ module at $k = 1$.

We have shown that $E_0^{-\alpha}$ acts a finite number of times on the highest weight vector $|\lambda\rangle$. We now consider whether or not E_{-1}^{α} must act a finite number of times. For $\lambda = (1, 0)$, from Section 3.3.2 we know that the singular vector $(E_{-1}^{\alpha})^2|\lambda\rangle$ satisfies

$$J_n^a(E_{-1}^{\alpha})^2|\lambda\rangle = 0 \quad \forall n > 0.$$

Hence, requirement (4.73) gives

$$E_0^{-\alpha}E_0^{-\alpha}(E_{-1}^{\alpha})^2|\lambda\rangle = 0. \tag{4.75}$$

If $|\lambda\rangle$ has Dynkin labels $(1, 0)$, then $(E_{-1}^{\alpha})^2|\lambda\rangle$ must have Dynkin labels $(-3, 4)$ since $\alpha^{(0)} = 2\Lambda_{(0)} - 2\Lambda_{(1)}$. If $(E_{-1}^{\alpha})^2|\lambda\rangle$ is non-zero, $E_0^{-\alpha}$ will be able to act on $(E_{-1}^{\alpha})^2|\lambda\rangle$ at least four times without annihilating. This contradicts (4.75) and so we require $(E_{-1}^{\alpha})^2|\lambda\rangle = 0$. From our study of integrable highest weight representations in Section 3.3.2, this implies that the only allowed highest weight module with highest weight Dynkin labels $(1, 0)$ is the integrable highest weight module. This is the irreducible vacuum module $\mathcal{L}_{|0\rangle}$.

We now repeat this process for the highest weight vector, $|\lambda\rangle$, with Dynkin labels $(1, 0)$. Applying (4.73) to the singular vector $E_{-1}^{\alpha}|\lambda\rangle$ gives

$$E_0^{-\alpha}E_0^{-\alpha}(E_{-1}^{\alpha})|\lambda\rangle = 0. \tag{4.76}$$

The Dynkin labels of $(E^{\alpha}_{-1})|\lambda\rangle$ are $(-1,2)$. We require $(E^{\alpha}_{-1})|\lambda\rangle = 0$, otherwise $E^{-\alpha}_0$ will be able to act two times upon $(E^{\alpha}_{-1})|\lambda\rangle$ without giving zero. Therefore, the only allowed highest weight module with highest weight Dynkin labels $(0,1)$ is the integrable highest weight module.

To summarise, we have shown for $\widehat{\mathfrak{sl}(2)}_1$ that the allowed highest weight modules are restricted to the integrable highest weight modules at level $k=1$. This is true for all WZW models at any integer level k, these results will be presented in Section 4.5.5. First, we will repeat this process to determine the allowed highest weight modules in the $\widehat{\mathfrak{sl}(3)}$ WZW model at level one. This example will be important when we determine the allowed modules in the twisted sector of $\widehat{\mathfrak{sl}(3)}$ in Chapter 5.

4.5.4 Allowed highest weight modules: $\widehat{\mathfrak{sl}(3)}_1$

From Section 2.3 we know that $\widehat{\mathfrak{sl}(3)}$ has simple roots given by $\alpha^{(0)} = (-\theta, 0, 1)$, $\alpha^{(1)} = (\alpha, 0, 0)$ and $\alpha^{(2)} = (\beta, 0, 0)$. Here α and β are the simple roots of the horizontal subalgebra which is isomorphic to $\mathfrak{sl}(3)$ and $\theta = \alpha + \beta$ is the highest root. The coroots of $\widehat{\mathfrak{sl}(3)}$ are all one so that (3.38) implies that a highest weight module with highest weight Dynkin labels $(\lambda_0, \lambda_1, \lambda_2)$ has level $k = \lambda_0 + \lambda_1 + \lambda_2$.

From (4.70) the $\widehat{\mathfrak{sl}(3)}_1$ WZW model has a null field given by

$$:E^{-\theta}(z)E^{-\theta}(z): \tag{4.77}$$

Applying the normal ordering formula (4.20) in the untwisted sector, the requirement that $:E^{-\theta}(z)E^{-\theta}(z):$ is a null field can be written as

$$\Big(\sum_{n\leq -1} E^{-\theta}_n E^{-\theta}_{m-n} + \sum_{n\geq 0} E^{-\theta}_{m-n} E^{-\theta}_n\Big)|v\rangle = 0 \qquad \forall m \in \mathbb{Z}, \quad \forall |v\rangle \in V, \tag{4.78}$$

where V is an allowed module. If we let $|v\rangle$ satisfy the requirement

$$J^a_n|v\rangle = 0 \qquad \forall n > 0, \tag{4.79}$$

then equation (4.78) for $m=0$ gives

$$E^{-\theta}_0 E^{-\theta}_0 |v\rangle = 0. \tag{4.80}$$

Requirement (4.79) includes all elements in V that are generated by the action of the horizontal subalgebra of $\widehat{\mathfrak{sl}(3)}$ on a highest weight vector $|\lambda\rangle$. Since the horizontal subalgebra is isomorphic to $\mathfrak{sl}(3)$, it follows that the module generated by the action of the horizontal subalgebra on $|\lambda\rangle$ is an $\mathfrak{sl}(3)$-module. Furthermore, it must satisfy the requirement (4.80). The only highest weight $\mathfrak{sl}(3)$-modules that satisfy these requirements are the integrable highest weight modules with highest weights $\Lambda = (0,0)$, $\Lambda = (1,0)$ and $\Lambda = (0,1)$. Extending to an $\widehat{\mathfrak{sl}(3)}$ highest weight module at $k = 1$ the highest weight vector has three possible weights with Dynkin labels

$$\lambda = (1,0,0), \qquad \lambda = (0,1,0) \quad \text{or} \quad \lambda = (0,0,1). \tag{4.81}$$

To show that only the integrable highest weight modules are allowed, for each highest weight it remains to show that the singular vector $(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle$ is set to zero:

$$(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle = 0. \tag{4.82}$$

From Section 3.6.3 we know that $J^a_n(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle = 0$ for all $n > 0$. Thus $(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle$ must also satisfy (4.80),

$$(E^{-\theta}_0)^2(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle = 0. \tag{4.83}$$

The weight of $(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle$ is given by $\lambda - (\lambda_0+1)\alpha^{(0)}$. Since $\alpha^{(0)} = (2,-1,-1)$ with respect to the fundamental weights, it follows that the $H^{\theta}_0 = H^{\alpha}_0 + H^{\beta}_0$ eigenvalue of $(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle$ is $\lambda_1 + \lambda_2 + 2(\lambda_0 + 1)$. This is clearly greater than one for all of the above highest weights and so in order for (4.83) to hold we require (4.82). Thus, the only allowed modules are the integrable highest weight $\widehat{\mathfrak{sl}(3)}$-modules at level $k = 1$.

4.5.5 Allowed highest weight modules: general case

We now present the general case for $\widehat{\mathfrak{sl}(3)}_k$ Wess-Zumino-Witten model at positive integer level k. The universal vacuum module contains the null vector

$$(E^{\theta}_{-1})^{k+1}|0\rangle. \tag{4.84}$$

Under the state-field correspondence this vector corresponds to the $(k+1)$-th normally ordered product of $E^{-\theta}(z)$

$$:(E^{-\theta})^{k+1}(z):. \tag{4.85}$$

By iteration of the normal ordering formula this gives the following requirement on a highest weight vector:

$$(E^{-\theta}_0)^{k+1}|\lambda\rangle = 0. \tag{4.86}$$

It follows that the module generated by the action of the horizontal subalgebra on the highest weight vector $|\lambda\rangle$ is isomorphic to an $\mathfrak{sl}(3)$-module with highest weight $\Lambda = (\Lambda_1, \Lambda_2)$, where $\Lambda_1, \Lambda_2 \in \mathbb{Z}_{\geq 0}$ and $\Lambda_1 + \Lambda_2 \leq k$. The second requirement comes from the fact that the $H^{\theta}_0 = H^{\alpha}_0 + H^{\beta}_0$ eigenvalue on λ must be less than or equal to k. The possible weights of the highest weight vector $|\lambda\rangle$ in a highest weight $\widehat{\mathfrak{sl}(3)}$-module at level k are therefore

$$\{(k - \lambda_1 - \lambda_2, \lambda_1, \lambda_2) | \lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 0}, \Lambda_1 + \Lambda_2 \leq k\}. \tag{4.87}$$

To show that the only allowed modules are the integrable ones the singular vector, $(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle$, associated to each weight λ must satisfy

$$(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle = 0. \tag{4.88}$$

Since it is a singular vector, (4.71) implies that

$$(E^{-\theta}_{0})^{k+1}(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle = 0. \tag{4.89}$$

The Dynkin labels of the weight of the vector $(E^{\theta}_{-1})^{\lambda_0+1}|\lambda\rangle$ are $(-\lambda_0 - 2, \lambda_1 + \lambda_0 + 1, \lambda_2 + \lambda_0 + 1)$. The H^{θ}_0 eigenvalue is the sum of the second two Dynkin labels,

$$\lambda_1 + \lambda_2 + 2(\lambda_0 + 1). \tag{4.90}$$

Since this is clearly greater than k for all weights in (4.87), the requirement (4.89) implies that (4.88) must hold. Therefore, the only allowed modules at level k are the integrable highest weight modules.

We now present some theorems that are important in determining that we have considered all requirements imposed on the allowed module by the null fields of the $\widehat{\mathfrak{sl}(3)}_k$ model. We have considered only the requirement from (4.71) corresponding to $n = 0$. The following theorem which is a result of [38] implies that this is the only requirement we need to consider.

Proposition 4.5.1 *If V is a highest weight module of $\widehat{\mathfrak{sl}(3)}_1$ with highest weight vector $|\lambda\rangle$ that satisfies the requirement $:(E^{-\theta})^{k+1}:_0 |\lambda\rangle = 0$, then the requirement $:(E^{-\theta})^{k+1}:_n |v\rangle = 0$ is satisfied for all $|v\rangle \in V$ and $n \in \mathbb{Z}$.*

Furthermore, we are only considering one null field (4.70) in our analysis of the allowed modules of the spectrum. However, if this null field behaves as zero on an allowed module, all other null fields behave as zero [14].

Proposition 4.5.2 *Suppose $|v\rangle \in \langle (E^{-\theta}_{-1})^{k+1}|0\rangle\rangle$ and let $A(z)$ be the field associated with $|v\rangle$ under the state field correspondence. Then $A(z)$ is a null field, and for any vector $|w\rangle$ in a highest weight $\widehat{\mathfrak{sl}(3)}_k$ module we have*

$$A(z)_n|w\rangle = 0 \quad \forall n \in \mathbb{N} \qquad \text{if} \qquad :(E^{-\theta})^{k+1}(z):_n |w\rangle = 0 \quad \forall n \in \mathbb{N} \tag{4.91}$$

The properties of representations of affine Lie algebras as presented in [19] ensure that $\langle (E^{-\theta}_{-1})^{k+1}|0\rangle\rangle = \langle (E^{\theta}_{-1})^{k+1}|0\rangle\rangle$ in the universal vacuum module. Hence, by considering only the requirements arising from the null field $:(E^{-\theta})^{k+1}(z):$, the requirements imposed by all other null fields are satisfied. We have arrived at the conclusion that the allowed modules of the $\widehat{\mathfrak{sl}(3)}_k$ model for positive integer values of k are the integrable highest weight $\widehat{\mathfrak{sl}(3)}$-modules at level k.

In fact, a similar argument holds for all WZW models at positive integer levels [9, 27]. We conclude this section with the following theorem.

Theorem 4.5.3 *Let $k \in \mathbb{N}$ and $\hat{\mathfrak{g}}$ be an untwisted affine Lie algebra. Then the allowed highest weight modules in the untwisted sector of the WZW model given by $\hat{\mathfrak{g}}_k$ are exactly the integrable highest weight modules of $\hat{\mathfrak{g}}$ at level k.*

Chapter 5

Twisted sector of Wess-Zumino-Witten models

This chapter will introduce the twisted sector of the WZW models and analyse the allowed modules in the twisted sector at positive integer levels. We will introduce this process for the $\widehat{\mathfrak{sl}(3)}_k$ model and derive the result for level $k = 1$ before generalising to all positive integer levels.

Central to these calculations is the generalised commutation relation which describes the normal ordering in the twisted sector. We will highlight the consequences of this formula at level $k = 1$ to determine that the only allowed highest weight module in the twisted sector is the integrable highest weight $\mathbf{A}_2^{(2)}$-module. We use the generalised Wick theorem to extend the generalised commutation relation to arbitrary numbers of normally ordered fields under certain conditions. This will be sufficient to restrict the allowed highest weight modules at each level to the integrable highest weight $\mathbf{A}_2^{(2)}$-modules.

This method can then be extended to any $\hat{\mathfrak{g}}_k$ WZW model, where the finite dimensional simple Lie algebra $\mathfrak{g}$ has a non-trivial automorphism of its Dynkin diagram. We will outline this approach and determine the requirements needed to conclude that the only allowed modules in the twisted sector of a WZW model, for a given positive integer k, are the integrable highest weight modules of the corresponding twisted affine Lie algebra.

5.1 Introduction

Consider the WZW model, $\hat{\mathfrak{g}}_k$ of the untwisted affine Lie algebra $\hat{\mathfrak{g}}$ at level k. We have seen in the untwisted sector of the WZW model that the generating fields take integral mode values:

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}.$$

We would now like to consider whether a twisted sector of the WZW model exists. For this to be the case we require some of our fields to have non-integral indexed modes, but the Lie brackets must have the same form as the original affine Lie algebra $\hat{\mathfrak{g}}$.

We may construct the twisted sector via an automorphism of the finite dimensional Lie algebra $\mathfrak{g}$ from which the affine Lie algebra $\hat{\mathfrak{g}}$ was constructed (see Section 2.3). Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra automorphism of finite order N. As shown in Section 2.4.1, this automorphism creates a decomposition of $\mathfrak{g}$ into the vector space direct sum of σ eigenspaces:

$$\mathfrak{g} = \bigoplus_{j=0}^{N-1} \mathfrak{g}_{[j]}, \qquad \mathfrak{g}_{[j]} = \{x \in \mathfrak{g} | \sigma(x) = e^{2\pi i j/N} x\}, \qquad [\mathfrak{g}_j, \mathfrak{g}_{j'}] \subseteq \mathfrak{g}_{(j+j' \mod N)}. \tag{5.1}$$

By extending to a loop algebra and adding a central extension, we construct an affine Lie algebra, which we will denote by $\hat{\mathfrak{g}}_\sigma$, that is spanned by the elements

$$\{J^a_n | J^a \in \mathfrak{g}_{[j]}, n \in \mathbb{Z} + \frac{j}{N}\} \cup \{K\}, \tag{5.2}$$

where $J^a_n := J^a \otimes z^n$. The non-trivial Lie bracket on $\hat{\mathfrak{g}}_\sigma$ is given by

$$[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + n\delta_{n+m,0}\bar{\kappa}^{ab} K, \tag{5.3}$$

where $\bar{\kappa}^{ab} = \bar{\kappa}(J^a, J^b)$ is the Killing form on $\mathfrak{g}$. Hence, the Lie bracket on $\hat{\mathfrak{g}}_\sigma$ is the same as the Lie bracket for the untwisted affine Lie algebra $\hat{\mathfrak{g}}$ (4.61). Thus, we are able to define a twisted sector of the WZW model $\hat{\mathfrak{g}}_k$ where the generating fields are given by

$$J^a(z) = \sum_{n \in \mathbb{Z} + \frac{j}{N}} J^a_n z^{-n-1}, \qquad J^a \in \mathfrak{g}_{[j]}.$$

Since this construction holds for every automorphism on $\mathfrak{g}$, it is a reasonable question to ask: "Which automorphisms will yield a reasonable twisted sector?". The answer to this question is not entirely obvious. However, if we recall from Section 2.4.1 that the outer automorphisms (i.e. the automorphisms generated by Dynkin diagram automorphisms) generate the twisted affine Lie algebras, it is a natural question to ask what the spectrum in the sector defined by these automorphisms looks like. Since no other automorphisms generate new affine Lie algebras, it should be the case that the entire twisted sector can be understood from analysing the twisted sector corresponding to the outer automorphisms.

Before we continue along these lines it is worth noting that any twisted sector can be related to an automorphism σ on the Lie algebra $\mathfrak{g}$ via the identification

$$J^a(ze^{2\pi i}) = \sum_n \sigma(J^a)_n z^{-n-1}. \tag{5.4}$$

In this way, we highlight the correspondence between boundary conditions on the twisted sector and automorphisms on the Lie algebra $\mathfrak{g}$.

5.2 Generalised commutation relation

In order to analyse the properties of the twisted sector, we first need a relationship that describes how modes are expanded in a normal ordered product of fields. As shown in Section 4.4.1, one way of doing this is to determine a generalised commutation relation. We now determine a generalised commutation relation for the WZW models using the same method as for the free fermion in Section 4.4.1. We can write the operator product expansion of two fields $J^a(z)$ with $J^b(w)$ as follows

$$J^a(z)J^b(w) = \frac{\bar{\kappa}^{ab}k}{(z-w)^2} + \frac{J^{[a,b]}(w)}{z-w} + :J^a(w)J^b(w): + O((z-w)).$$

The normal ordered product $:J^a(w)J^b(w):$ is given by the constant term in the OPE above. To derive an equation involving this term we evaluate the contour integral

$$\oint_0 \oint_w R\{J^a(z)J^a(w)\}(z-w)^{-1}z^{n+1}w^m \frac{dz}{2\pi i}\frac{dw}{2\pi i}. \tag{5.5}$$

Following the same process as for the free fermion, we arrive at the relation

$$\begin{aligned}\sum_{r=0}^{\infty}[J^a_{n-r}J^b_{m+r} + J^b_{m-r-1}J^a_{n+r+1}] = \ :J^aJ^b:_{n+m} + \frac{\bar{\kappa}^{ab}kn(n+1)}{2}\delta_{m+n,0} \\ + (n+1)\sum_{c=1}^{d} f^{ab}{}_c J^c_{n+m}.\end{aligned} \tag{5.6}$$

This is a generalised commutation relation for the WZW models, it holds in both the twisted and untwisted sectors. Notice that in the case of $n=-1$ we retrieve the usual normal ordering formula for the untwisted sector:

$$\begin{aligned} :J^aJ^b:_{m-1} &= \sum_{r=0}^{\infty}[J^a_{-r-1}J^b_{m+r} + J^b_{m-r-1}J^a_r], \\ \Longrightarrow \qquad :J^aJ^b:_m &= \sum_{r\le -1} J^a_r J^b_{m-r} + \sum_{r>-1} J^b_{m-r}J^a_r. \end{aligned} \tag{5.7}$$

Application: Virasoro modes

For this section we will assume that the automorphism σ is of order 2. This is a reasonable assumption since only one Lie algebra, namely D_4, has an outer automorphism that is not of order 2 (see Table X on pg. 188 of [22]). We now outline some important calculations that can be performed with the generalised commutation relation. Recall that the energy-momentum tensor (4.64) is given by

$$T(z) = \gamma\sum_{J^a} :J^a(z)J^a(z):, \qquad \gamma = \frac{1}{2(k+g)}, \tag{5.8}$$

where the summation of J^a is over a basis of $\mathfrak{g}$ which is orthonormal under the Killing form on $\mathfrak{g}$. By Proposition 2.4.1 the Killing form satisfies $\bar{\kappa}(\mathfrak{g}_{[0]}, \mathfrak{g}_{[1]}) = 0$. Hence, the summation in (5.8) can be separated into a summation of elements from $\mathfrak{g}_{[0]}$ and another summation of elements from $\mathfrak{g}_{[1]}$:

$$T(z) = \gamma \sum_{J^a \in \mathfrak{g}_{[0]}} : J^a(z) J^a(z) : + \gamma \sum_{J^a \in \mathfrak{g}_{[1]}} : J^a(z) J^a(z) : . \tag{5.9}$$

Note that this is an abuse of notation since we are only summing over the basis elements from each of the eigenspaces $\mathfrak{g}_{[0]}$ and $\mathfrak{g}_{[1]}$, not the entire space. For simplicity we will tolerate this abuse of notation. We can find an expansion for the Virasoro modes by expanding each of the normal ordered products using (5.6).

We already have an expansion for the fields from the $\mathfrak{g}_{[0]}$ sum given in (5.7). So it remains to find an expansion for the normal ordered product of $: J^a J^a(z) :$, where $J^a \in \mathfrak{g}_{[1]}$. Since J^a is taken from an orthonormal basis with respect to the Killing form, we have $\bar{\kappa}^{aa} = 1$. The generalised commutation relation (5.6) gives

$$: J^a J^a :_{n+m} = \sum_{r=0}^{\infty} [J^a_{n-r} J^a_{m+r} + J^a_{m-r-1} J^a_{n+r+1}] - \frac{kn(n+1)}{2} \delta_{m+n,0}. \tag{5.10}$$

We can now construct the Virasoro algebra in terms of modes of the affine Lie algebra $\mathbf{A}_2^{(2)}$. Setting $n = -\frac{1}{2}$ and $m = \frac{1}{2}$ we find

$$: J^a J^a :_0 = 2 \sum_{r \in \mathbb{N} - \frac{1}{2}} J^a_{-r} J^a_r + \frac{k}{8}. \tag{5.11}$$

This gives an expansion for the Virasoro operator L_0

$$L_0 = \gamma (\sum_{J^a \in \mathfrak{g}_{[0]}} [\sum_{r \leq -1} J^a_r J^b_{-r} + \sum_{r > -1} J^b_{-r} J^a_r] + \sum_{J^a \in \mathfrak{g}_{[1]}} [2 \sum_{r \geq \frac{1}{2}} J^a_{-r} J^a_r + \frac{k}{8}]). \tag{5.12}$$

We can now compute the conformal weight of a given state. Suppose $|\lambda\rangle$ is a highest weight vector in a highest weight module of the twisted affine Lie algebra. Then $H^a_0 |\lambda\rangle = \lambda^a |\lambda\rangle$ for all Cartan subalgebra elements H^a that are included in the summation over the basis for $\mathfrak{g}_{[0]}$ and $J^a_n |\lambda\rangle = 0$ for all $n > 0$. The conformal weight of $|\lambda\rangle$ is given by

$$L_0 |v\rangle = \gamma (\sum_{H^a \in \mathfrak{g}_{[0]}} \lambda^a \lambda^a + \sum_{J^a \in \mathfrak{g}_{[1]}} \frac{k}{8}) |v\rangle \tag{5.13}$$

$$= \frac{\sum_{H^a \in \mathfrak{g}_{[0]}} (\lambda^a)^2}{2(k+g)} + \frac{\dim(\mathfrak{g}_{[1]}) k}{16(k+g)}. \tag{5.14}$$

Here, g is the dual Coxeter number of $\mathfrak{g}$. The reader interested in these results should compare this equation with equation (15.87) in [7]. We now move on to analyse the structure of the spectrum in the twisted sector.

5.3 Twisted sector for $\widehat{\mathfrak{sl}(3)}_k$

The simplest Lie algebra with an outer automorphism is $\mathfrak{sl}(3)$. This outer automorphism acts by switching the two simple roots, the twisted affine Lie algebra that it generates is $\mathbf{A}_2^{(2)}$. For a review of the structure of this Lie algebra see Section 2.4.3 and for a construction of its integrable highest weight modules see Section 3.5.4 .

Correspondingly, the simplest WZW model with a twisted sector has symmetry algebra $\widehat{\mathfrak{sl}(3)}$ and the twisted sector will have symmetry algebra $\mathbf{A}_2^{(2)}$. To begin with we will analyse the allowed $\mathbf{A}_2^{(2)}$-modules in the twisted sector at $k = 1$. Once we have done this we generalise the process to any positive integer level k by using induction.

5.3.1 Level $k = 1$

Recall from Section 4.5.2, that the universal vacuum module for $\widehat{\mathfrak{sl}(3)}_1$ contains a singular vector given by

$$E^\theta_{-1}E^\theta_{-1}|0\rangle. \tag{5.15}$$

By the state-field correspondence

$$:E^\theta(z)E^\theta(z): \;\mapsto E^\theta_{-1}E^\theta_{-1}|0\rangle, \tag{5.16}$$

so that $:E^\theta(z)E^\theta(z):$ is a null field. Thus for any vector $|v\rangle$ in an allowed module (in the twisted or untwisted sector) of $\widehat{\mathfrak{sl}(3)}_1$ we have

$$:E^\theta E^\theta:_n |v\rangle = 0 \qquad \forall n. \tag{5.17}$$

We have seen that in the untwisted sector this relation implies that the allowed highest weight modules are the integrable ones.

Let us now analyse the restriction (5.17) on the allowed modules in the twisted sector, to do this we will use the generalised commutation relation (5.6) to expand the normal ordered product $:E^\theta E^\theta:_n$. Note that OPE of $E^\theta(z)$ with itself is non-singular, that is

$$E^\theta(z)E^\theta(w) \sim 0, \tag{5.18}$$

so that generalised commutation relation (5.6) for $E^\theta(z)$ with itself is given by

$$\sum_{r=0}^{\infty}[E^\theta_{n-r}E^\theta_{m+r} + E^\theta_{m-r-1}E^\theta_{n+r+1}] = \;:E^\theta E^\theta:_{n+m}\,. \tag{5.19}$$

Applying this to (5.17) yields

$$\sum_{r=0}^{\infty}[E^\theta_{n-r}E^\theta_{m+r} + E^\theta_{m-r-1}E^\theta_{n+r+1}]|v\rangle = 0. \tag{5.20}$$

Suppose $|v\rangle$ is an element of a highest weight module in the twisted sector such that $|v\rangle$ is annihilated by all positive modes

$$J^a_n|v\rangle = 0 \qquad \forall n > 0. \tag{5.21}$$

Setting $n = m = -\frac{1}{2}$ the requirement (5.19) on $|v\rangle$ reduces to

$$E^\theta_{-\frac{1}{2}} E^\theta_{-\frac{1}{2}} |v\rangle = 0 \tag{5.22}$$

If we consider the submodule generated by the action of $E^\theta_{-\frac{1}{2}}$ upon $|v\rangle$, the requirement (5.21) and (5.22) imply that this module is isomorphic to an $\mathfrak{sl}(2)$-module with dimension one or two. Since $E^\theta_{-\frac{1}{2}}$ is the negative root vector of the affine root $\alpha^{(0)}$ of $\mathbf{A}_2^{(2)}$ it follows that the affine Dynkin label of $|v\rangle$ is either zero or one. So if the vector $|v\rangle$ has $\mathbf{A}_2^{(2)}$ Dynkin labels, (v_0, v_1), then requirement (5.22) implies that $v_0 = 0$ or $v_0 = 1$.

Suppose that $|\lambda\rangle$ is a highest weight vector in an $\mathbf{A}_2^{(2)}$-module:

$$J^a_n|\lambda\rangle = 0 \quad \forall n > 0, \qquad E^1_0|\lambda\rangle = 0, \tag{5.23}$$

where E^1_0 is the non-affine simple root vector of $\mathbf{A}_2^{(2)}$ in a Chevalley basis (2.35). Consider the module generated by the action of the horizontal subalgebra of $\mathbf{A}_2^{(2)}$ on $|\lambda\rangle$. We call this module the *zero grade module* and all vectors in this module *zero grade vectors* (note that all zero grade vectors satisfy (5.21)). Since the horizontal subalgebra is isomorphic to $\mathfrak{sl}(2)$, this module is a highest weight $\mathfrak{sl}(2)$-module generated by the action of E^{-1}_0 on $|\lambda\rangle$. If $|\lambda\rangle$ has highest weight λ then the weight space of this module is given by the following diagram.

$$\cdots \xleftarrow{E_0^{-1}} \lambda - 2\alpha^{(1)} \xleftarrow{E_0^{-1}} \lambda - \alpha^{(1)} \xleftarrow{E_0^{-1}} \lambda$$

If λ has Dynkin labels (λ_0, λ_1) then requirement (5.22) implies that $\lambda_0 = 0$ or $\lambda_0 = 1$. Since the formula for the level (3.44) is given by $k = 2\lambda_0 + \lambda_1$. This means that the possible highest weights are $(0, 1)$ or $(1, -1)$.

In the case of $(1, -1)$, the non-affine Dynkin label is -1 and E^{-1}_0 will not annihilate $|\lambda\rangle$. Since $\alpha^{(1)} = -\Lambda_{(0)} + 2\Lambda_{(1)}$, the affine Dynkin label of the weight $\lambda - \alpha^{(1)}$ is 2. However, $E^{-1}_0|\lambda\rangle$ satisfies (5.21) indicating that it should have affine Dynkin label 0 or 1. Thus, the highest weight $(1, -1)$ is not allowed.

This leaves only $(0, 1)$, in this case the integrable $\mathfrak{sl}(2)$-module generated by the action of E^{-1}_0 on $|\lambda\rangle$ has dimension two. The weight of the vector $E^{-1}_0|\lambda\rangle$ is given by $(1, -1)$ which is allowed since its affine Dynkin label is one. We must set $(E^{-1}_0)^2|\lambda\rangle = 0$, since the affine Dynkin label of the weight $\lambda - 2\alpha^{(1)}$ is two. Furthermore, (5.22) also implies $(E^\theta_{-1/2})^2|\lambda\rangle = 0$, the only way this can occur with

$\lambda_0 = 0$ is if $E^\theta_{-1/2}|\lambda\rangle = 0$. Hence, the only allowed module is the integrable $\mathbf{A}_2^{(2)}$-module with highest weight $\lambda = (0, 1)$.

We remark now that the condition of integrability arose from requirement (5.22). This enforced that the affine Dynkin label of all weights in the zero grade module must be non-negative integers less than $k + 1$. Since the action of the non-affine negative simple root vector increases the affine Dynkin label, it may only act a finite number of times (in fact it may act at most k times) without annihilating the highest weight vector. Requirement (5.22) also implies that we can only act with the negative affine root vector a finite number of times (again at most k times) before annihilating the highest weight vector. Thus, the module most be integrable. We will use this idea to extend our analysis to higher integer levels.

5.3.2 Positive integer levels

At level $k = 1$ we had a singular vector in our universal vacuum module given by $E^\theta_{-1}E^\theta_{-1}|0\rangle$,which corresponded to a null field $: E^\theta(z)E^\theta(z) :$. For a general positive integer level k the $\widehat{\mathfrak{sl}(3)}_k$ universal vacuum module has a singular vector $(E^\theta_{-1})^{k+1}|0\rangle$. The corresponding the null field is $: (E^\theta)^{k+1}(z) :$, the normal ordered product of $k+1$ copies of $E^\theta(z)$. Thus, in order to extend our analysis of the twisted sector to all positive integer levels we require a formula for the normal ordered product of an arbitrary number of copies of $E^\theta(z)$.

Extending the generalised commutation relation

In order to determine a formula for $: E^{k+1}(z) :$ we start by noting that $E^\theta(z)E^\theta(z) \sim 0$. By the application of the generalised Wick Theorem in Appendix B, this implies that

$$E^\theta(z) : (E^\theta)^n(w) : \sim 0, \tag{5.24}$$

for all $n \in \mathbb{N}$. This is very useful since we can apply this to evaluate the contour integral

$$\begin{aligned}
&\oint_0 \oint_w \frac{E^\theta(z) : (E^\theta)^k(w) :}{z-w} z^{n+1} w^{m+k-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \oint_0 \oint_w \frac{: E^\theta(w)(E^\theta)^k(w) :}{z-w} z^{n+1} w^{m+k-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \oint_0 : (E^\theta)^{k+1}(z) : w^{n+m+k} \frac{dw}{2\pi i} \\
&= : (E^\theta)^{k+1} :_{n+m} .
\end{aligned}$$

The last line follows since the conformal weight of $:(E^\theta)^{k+1}(z):$ is $k+1$. If we instead calculate the contour integral by applying radial ordering we get

$$\begin{aligned}
&\oint_0 \oint_w \frac{E^\theta(z) : (E^\theta)^k(w) :}{z-w} z^{n+1} w^{m+k-1} \frac{dz}{2\pi i}\frac{dw}{2\pi i} \\
&= \sum_{r=0}^{\infty} \oint_0 E^\theta(z) z^{n-r} \frac{dz}{2\pi i} \oint_0 : (E^\theta)^k(w) : w^{m+k+r-1} \frac{dw}{2\pi i} \\
&\qquad + \sum_{r=0}^{\infty} \oint_0 : (E^\theta)^k(w) : w^{m+k-r-2} \frac{dw}{2\pi i} \oint_0 E^\theta(z) z^{n+1+r} \frac{dz}{2\pi i} \\
&= \sum_{r=0}^{\infty} [E^\theta_{n-r} : (E^\theta)^k :_{m+r} + : (E^\theta)^k(w) :_{m-r-1} E^\theta_{n+r+1}].
\end{aligned}$$

Thus we have an iterated version of the generalised commutation relation:

$$: (E^\theta)^{k+1} :_{n+m} = \sum_{r=0}^{\infty} [E^\theta_{n-r} : (E^\theta)^k :_{m+r} + : (E^\theta)^k(w) :_{m-r-1} E^\theta_{n+r+1}]. \tag{5.25}$$

Allowed modules at positive integer levels

Following the steps taken at level $k=1$ we would like to analyse the action of $:(E^\theta)^{k+1}(z):$ on a zero grade vector $|v\rangle$. We make the hypothesis that

$$: (E^\theta)^{k+1} :_{\frac{-(k+1)}{2}} |v\rangle = (E^\theta_{-\frac{1}{2}})^{k+1} |v\rangle \tag{5.26}$$

This is obvious in the case $k=0$ and has already been proved for $k=1$ (5.22) . In order to proceed inductively we also make the assumption

$$: (E^\theta)^{k+1} :_n |v\rangle = 0, \tag{5.27}$$

for all $n > \frac{-(k+1)}{2}$. This is also readily verified for $k=1$ from the generalised commutation relation (5.19). We make the inductive hypothesis that for any given k, (5.26) and (5.27) hold for $k-1$. We may now apply the iterative generalised commutation relation (5.25) with $n = -\frac{1}{2}$ and $m = -\frac{k}{2}$

$$: (E^\theta)^{k+1} :_{\frac{-(k+1)}{2}} |v\rangle = E^\theta_{-\frac{1}{2}} : (E^\theta)^k :_{-\frac{k}{2}} |v\rangle = (E^\theta_{-\frac{1}{2}})^{k+1} |v\rangle.$$

Similarly for $n = -\frac{1}{2}$ and $m > -\frac{k}{2}$ we find

$$: (E^\theta)^{k+1} :_{m'} |v\rangle = 0,$$

for all $m' > -\frac{(k+1)}{2}$. Thus by induction (5.26) and (5.27) hold for all $k \in \mathbb{N}$.

We may now use formula (5.26) to analyse the allowed action of $E^\theta_{-\frac{1}{2}}$ on a zero grade vector. For $k \in \mathbb{N}$ the null field $:(E^\theta)^{k+1}(z):$ implies

$$(E^\theta_{-\frac{1}{2}})^{k+1} |v\rangle = 0. \tag{5.28}$$

Suppose the zero grade vector $|v\rangle$ has weight given by the Dynkin labels (v_0, v_1) in a highest weight $\mathbf{A}_2^{(2)}$-module. Then (5.28) means that the Dynkin label $v_0 \in \{0, 1, 2, ..., k\}$. Hence, all the weights in the zero grade submodule must have affine Dynkin label from $\{0, 1, .., k\}$. In particular if $|\lambda\rangle$ is a highest weight vector whose weight λ has Dynkin labels (λ_0, λ_1), we must have $\lambda_0 \in \{0, 1, 2, ..., k\}$. Furthermore, the zero grade module is an $\mathfrak{sl}(2)$-module with highest weight λ_1. Since the simple root $\alpha^1 = -\Lambda_{(0)} + 2\Lambda_{(1)}$ it follows that the Dynkin labels of the $\mathbf{A}_2^{(2)}$ weights of the zero grade module are given in the diagram:

$$\cdots \xleftarrow{E_0^{-1}} \underset{(\lambda_0+2, \lambda_1-4)}{\bullet} \xleftarrow{E_0^{-1}} \underset{(\lambda_0+1, \lambda_1-2)}{\bullet} \xleftarrow{E_0^{-1}} \underset{(\lambda_0, \lambda_1)}{\bullet}$$

The fact that the affine Dynkin labels of all the weights in the zero grade module must take values from $\{0, 1, 2, ..., k\}$, enforces that

$$(E_0^{-1})^{k-\lambda_0+1}|\lambda\rangle = 0, \implies (E_0^{-1})^{\lambda_1+1}|\lambda\rangle = 0 \ \text{ and } \lambda_1 \in \{0, 1, ..., k-\lambda_0\}. \tag{5.29}$$

We also have the requirement

$$(E^{\theta}_{-\frac{1}{2}})^{k+1}|\lambda\rangle = 0, \implies (E^{\theta}_{-\frac{1}{2}})^{\lambda_0+1}|\lambda\rangle = 0 \ \text{ and } \lambda_0 \in \{0, 1, ..., k\}. \tag{5.30}$$

By Theorem 3.3.1 these two requirements imply that the highest weight $\mathbf{A}_2^{(2)}$-module generated by $|\lambda\rangle$ must be integrable.

Of course, we must remember that not all highest weights that satisfy these requirements are allowed at level k. The level of the highest weight module is given by $k = 2\lambda_0 + \lambda_1$. However, it is clear from this formula that all integrable highest weight $\mathbf{A}_2^{(2)}$-modules at level k satisfy the requirements (5.29) and (5.30). Thus, for integer level k we have restricted the allowed modules in the twisted sector of the $\widehat{\mathfrak{sl}(3)}_k$ model to the integrable highest weight $\mathbf{A}_2^{(2)}$-modules at level k.

5.4 A general twisted sector for WZW models

Having completed an analysis of the twisted sector of the $\widehat{\mathfrak{sl}(3)}_k$ WZW model at positive integer levels we would now like to construct a general theory for the twisted sector of WZW models at positive integer levels. We begin by making the following proposition.

Proposition 5.4.1 *Suppose that $\mathfrak{g}$ is a Lie algebra with an outer automorphism σ. Let $\hat{\mathfrak{g}}_\sigma$ be the twisted affine Lie algebra corresponding to σ. Then the $\hat{\mathfrak{g}}_k$ WZW model for integer level k has a twisted sector whose allowed modules are the integrable highest weight $\hat{\mathfrak{g}}_\sigma$-modules at level k.*

In order to prove such a proposition we will follow the steps taken for $\widehat{\mathfrak{sl}(3)}_k$. The key to determining the allowed modules in the twisted sector was analysing the action of the negative affine root vector $E^\theta_{-\frac{1}{2}}$ on the zero grade elements of the highest weight $\mathbf{A}_2^{(2)}$-modules.

Firstly note that the generalised commutation relation (5.6) is derived in generality and so holds for all WZW models. So does the iterated version of this formula for $E^\theta(z)$ in equation (5.25). Thus, if we can prove that $E^\theta(z) \in \mathfrak{g}_{[1]}$ with respect to σ, then we can apply this formula to determine the action of the negative affine root vector on the zero grade elements of the $\hat{\mathfrak{g}}_\sigma$ highest weight modules.

Conjecture 5.4.2 *Let $\mathfrak{g}$ be a Lie algebra with highest root θ. Let σ be an outer automorphism on $\mathfrak{g}$. Then $E^\theta \in \mathfrak{g}_{[1]}$ with respect to the gradation of $\mathfrak{g}$ generated by σ.*

The proof of this is not yet complete. We know that as an automorphism of the Dynkin diagram, σ preserves the comarks of the simple roots so that $\sigma(\theta) = \theta$. This proves that E^θ is an eigenvector of σ, we are yet to prove that it must belong to the eigenspace $\mathfrak{g}_{[1]}$. Since all outer automorphisms on finite dimensional Lie algebras are known, it should just be a matter of computation to check that this property holds. So far we have checked the simplest cases of A_2 ,A_3, A_4 and A_5.

Given that $E^\theta \in \mathfrak{g}_{[1]}$ we may repeat the analysis of Section 5.3.2. Letting N be the order of σ, for a zero grade element $|v\rangle$ this yields

$$: (E^\theta)^{k+1} :_{-\frac{k+1}{N}} |v\rangle = (E^\theta_{-\frac{1}{N}})^{k+1}|v\rangle. \tag{5.31}$$

At integer level k the vacuum module of $\hat{\mathfrak{g}}_k$ has a singular vector given by $(E^\theta_{-1})^{k+1}|0\rangle$, which maps under the state field correspondence to the null field $: (E^\theta)^{k+1}(z) :$. Applying (5.31) then gives

$$(E^\theta_{-\frac{1}{N}})^{k+1}|v\rangle = 0. \tag{5.32}$$

This implies that $|v\rangle$ has affine Dynkin label v_0, where v_0 is a non-negative integer less than k. In particular a highest weight vector $|\lambda\rangle$ must satisfy

$$\lambda_0 \in \{0, 1, ..., k\} \tag{5.33}$$

The action of the non-affine negative simple root vectors upon $|\lambda\rangle$ generate the zero grade module. Since the untwisted affine algebra is simple, or equivalently the Dynkin diagram is connected, it follows that if any negative simple root can act infinitely many times the Dynkin labels of all simple roots will be unbounded. This can be seen by applying induction on the simple roots connected to a simple root whose negative root vector can act infinitely many times. However, as shown above the Dynkin label of the affine root is bounded on the zero grade module, hence the non-affine negative simple root vectors may only act a finite number of times upon the highest weight. Since the affine root may also only act a finite number of times on the highest weight by (5.32) this is enough to ensure that the module must be integrable.

So, short of proving the conjecture, we have proved that all the allowed modules must be integrable. However, Proposition 5.4.1 also asserts that all integrable modules at integer level k are allowed. The proof of this is a bit technical and has not been verified for all twisted affine Lie algebra, but we present an outline of it here for completeness. Combining some formulas from [22], we propose a formula for the level of a twisted affine Lie algebra as a sum of the highest weight Dynkin labels,

$$k = \frac{N}{a_0^\vee} \sum_{i=0}^{r} a_i^\vee \lambda_i, \tag{5.34}$$

where $a_i^\vee$ are the dual Coxeter labels for $\hat{\mathfrak{g}}_\sigma$ and N is the order of σ. Furthermore, the dual Coxeter labels, $a_i^\vee$, of the affine Lie algebras are all positive integers. We can also write the highest root θ of $\mathfrak{g}$ in terms of the dual Coxeter labels

$$\theta = \sum_{i=1}^{r} a_i^\vee \alpha^{(i)\vee} \tag{5.35}$$

Hence the H_0^θ eigenvalue of $|\lambda\rangle$ is given by

$$(\theta, \lambda) = \sum_{i=1}^{r} a_i^\vee \lambda_i. \tag{5.36}$$

Now, if Conjecture 5.4.2 is true, then the action of $E_0^{-\theta}$ on $|\lambda\rangle$ generates an $\mathfrak{sl}(2)$-module with highest weight (θ, λ). The vector in this module that is annihilated by $E_0^{-\theta}$ is also annihilated by all other non-affine negative roots and thus is a lowest weight vector of the zero grade module. It will therefore have the highest affine Dynkin label since the action of any negative simple root vector increases or doesn't change the affine Dynkin label. So to show that a given integrable module is allowed, it suffices to show that this vector has affine Dynkin label less than or equal to k. By (5.35) the affine Dynkin label of θ is given by

$$\theta = \sum_{i=1}^{r} a_i^\vee |A^{i0}|, \tag{5.37}$$

where A is the Cartan matrix of $\mathfrak{g}$. It follows that the affine Dynkin label of the lowest weight vector is

$$\lambda_0 + (\theta, \lambda) \sum_{i=1}^{r} a_i^\vee |A^{i0}|, \tag{5.38}$$

but from (5.34) we have

$$k = N\lambda_0 + \frac{N}{a_0^\vee} \sum_{i=1}^{r} a_i^\vee \lambda_i = N\lambda_0 + \frac{N}{a_0^\vee}(\theta, \lambda). \tag{5.39}$$

To show (5.38) is less than or equal to k it suffices to show

$$\frac{N}{a_0^\vee} \geq \sum_{i=1}^{r} a_i^\vee |A^{i0}|.$$

But this is easily verified to be true by checking the Dynkin diagrams of all the twisted affine Lie algebras, see [22].

This completes our discussion of the spectrum in the twisted sector of the WZW model. In the conclusion we will summarise the results found and indicate areas for future exploration.

Chapter 6

Conclusion

We began in Chapter 1 with an overview of the fundamental concepts of finite dimensional simple Lie algebras and an introduction to Kac-Moody Lie algebras. Importantly every finite dimensional simple Lie algebra can be determined from a Cartan matrix via the Chevalley-Serre construction (Theorem 2.1.1). The Kac-Moody Lie algebras are a more general class of Lie algebras generated via a Chevalley-Serre construction on a generalised Cartan matrix. The untwisted affine Lie algebras are a special example of a Kac-Moody Lie algebra and can be realised via a loop construction on a finite dimensional simple Lie algebra. The twisted affine Lie algebras are also constructed via a loop construction, but this time with respect to the gradation of a finite dimensional simple Lie algebra imposed by an outer automorphism.

In Chapter 2 we introduced the fundamental concepts of representations of the class of Kac-Moody Lie algebras. In particular we introduced the concept of integrable highest weight modules. We showed that if Λ is the highest weight of an integrable highest weight module then each Dynkin label, Λ_i, of Λ must be a non-negative integer and furthermore that

$$(E^{-i})^{\Lambda_i+1}v_\Lambda = 0. \tag{6.1}$$

In the case of finite dimensional simple Lie algebras all finite dimensional modules can be decomposed as a direct sum of integrable highest weight modules. We also determined the structure of an integrable highest weight module in the case of affine Lie algebras and gave examples for the untwisted affine Lie algebra $\widehat{\mathfrak{sl}(2)}$ as well as the twisted affine Lie algebra $\mathbf{A}_2^{(2)}$.

In Section 3.6 we introduced the concepts of the universal enveloping algebra, Verma modules and singular vectors. We showed that for a highest weight module with non-negative integral Dynkin labels the vector

$$(E^{-i})^{\Lambda_i+1}v_\Lambda, \tag{6.2}$$

is a singular vector of the Verma module V_Λ. We completed the section by introducing a theorem by Kac [19] that states that the quotient of the Verma module by the module generated by all singular vectors of the form (6.2) is the irreducible integrable highest weight module with highest weight Λ.

In Chapter 3 we began by motivating the existence of an infinite dimensional symmetry algebra called the Virasoro algebra in two dimensional conformal field theory. We then introduced the fundamental concepts of two dimensional conformal field theory such as the operator product expansion and normal ordering. In particular, we introduced the concept of an untwisted sector where the fields are Laurent polynomials in z and z^{-1} and a twisted sector where some of the fields have expansions in non-integral powers of z. We illustrated these concepts through the examples of the free boson and free fermion. Of importance was the use of a generalised commutation relation in the free fermion theory to describe normal ordering in the twisted sector.

In Section 4.5 we introduced the Wess-Zumino-Witten models which are a class of conformal field theories with symmetry algebras given by the untwisted affine Lie algebras. We used the results of Chapter 2 to show that a WZW model at level k has a singular vector in the universal vacuum module given by

$$(E_{-1}^{-\theta})^{k+1}|0\rangle \tag{6.3}$$

In the case of $\widehat{\mathfrak{sl}(2)}$ and $\widehat{\mathfrak{sl}(3)}$ at level $k = 1$ we demonstrated that this singular vector restricts the allowed modules in the spectrum to the integrable highest weight modules. We completed Chapter 3 by summarising a general approach to determine that the allowed modules of the WZW models at positive integer levels are given by the integrable highest weight modules of the symmetry algebra.

In Chapter 4 we introduced the twisted sector of the Wess-Zumino-Witten model $\hat{\mathfrak{g}}_k$. We showed that the twisted sector corresponds to automorphisms on the Lie algebra $\mathfrak{g}$ and that the twisted symmetry algebra is the affine Lie algebra constructed from the gradation imposed by these automorphisms. We then focused on the outer automorphisms which generate the twisted affine Lie algebras. We derived a generalised commutation relation for the WZW models and used this to construct a copy of the Virasoro algebra in terms of the modes of the twisted affine Lie algebra.

Using the singular vector from (6.3) and the generalised commutation relation we determined that the allowed modules in the twisted sector for $\widehat{\mathfrak{sl}(3)}_1$ are restricted to the integrable highest weight module of the twisted affine Lie algebra $\mathbf{A}_2^{(2)}$. We used the generalised Wick Theorem (Appendix B) to show that this result holds for all positive integer levels.

We finished by arguing that this result holds for all WZW models at positive integer levels. In particular, if we can show that $E^\theta \in \mathfrak{g}_{[1]}$ for all outer automorphisms on a finite dimensional simple Lie algebra then our argument is complete. This result may be able to be proved directly or could be determined by explicitly computing the eigenvalue of E^θ for all outer automorphisms since there are a finitely many series of outer automorphisms, see pg. 188 of [22] for a list. The result of these computations will either consolidate the arguments made in this thesis or provide examples where we may expect to see some more complex behaviour. This would be a good starting point for further research in this area.

Future work

For further investigation into the twisted sector of WZW models I suggest starting by considering the case of an automorphism other than the outer automorphisms. From the construction in Section 5.1 there is no reason that a general automorphism should not give rise to a twisted sector. It would be interesting to determine whether the twisted sector of a general automorphism exists, and if it does: "What does its spectrum look like?". The spectrum of a general automorphism should be able to be explained in terms of the allowed modules of the untwisted and twisted affine Lie algebras. How this occurs is not particularly obvious and is a question that needs to be answered in order to fully determine the twisted sectors of the WZW model.

As mentioned in Chapter 3 there are standard representation notions that have not been introduced here, such as characters of modules, which act as generating functions for dimensions of weight spaces. Understanding and computing these functions for the integrable highest weight modules of twisted affine Lie algebras will give a more complete understanding of the twisted sector.For example, in the untwisted sector the WZW characters span a unitary representation of $SL(2, \mathbb{Z})$. It is unclear whether this is still true if the twisted characters are included.

It is important to mention that we have restricted ourselves to considering the allowed modules enforced by a the existence of a singular vector in the universal vacuum module. To construct a conformal field theory there are other concepts that must also be determined. For example, to construct a space of states we must work out a fusion product that allows us to tensor copies of allowed modules together. This product is know for the untwisted case, but if this product can be carried over to the twisted case would need further investigation.

Appendix A

Normal ordering in the untwisted sector

We assume here that the symmetry algebra is a Lie algebra and so the definition of radial ordering is given in (4.11). From (4.18) the definition for normal ordering is given by

$$:A(w)B(w): \; = \oint_w \frac{\mathcal{R}\{A(z)B(w)\}}{z-w}\frac{dz}{2\pi i}. \tag{A.1}$$

In the untwisted sectors the fields $A(z)$ and $B(w)$ have the expansion

$$A(z) = \sum_{n\in\mathbb{Z}-h_A} a_n z^{-n-h_A}, \qquad B(z) = \sum_{n\in\mathbb{Z}-h_B} b_n z^{-n-h_B}. \tag{A.2}$$

These expansions are made with respect to the point $z = 0$ and so the coefficients a_n and b_n are really functions of z evaluated at $z = 0$. However, this point is not special and we can instead expand our fields as series about an arbitrary point $z = x$. Expanding both these fields about this arbitrary point gives

$$A(z) = \sum_{n\in\mathbb{Z}-h_A} a_n(x)(z-x)^{-n-h_A}, \qquad B(z) = \sum_{n\in\mathbb{Z}-h_B} b_n(x)(z-x)^{-n-h_B}. \tag{A.3}$$

Using the definition of radial ordering the integral in (A.1) becomes

$$\oint_w \frac{\mathcal{R}\{A(z)B(w)\}}{z-w}\frac{dz}{2\pi i} = \oint_{|z|>|w|} \frac{A(z)B(w)}{z-w}\frac{dz}{2\pi i} - \oint_{|z|<|w|} \frac{B(w)A(z)}{z-w}\frac{dz}{2\pi i}. \tag{A.4}$$

Dealing with the first term we expand the fields $A(z)$ and $B(w)$ about an arbitrary point x satisfying $|w| < |x| < |z|$. In this annulus we have the following sum

$$\frac{1}{z-w} = \sum_{l\geq 0} \frac{(w-x)^l}{(z-x)^{l+1}}. \tag{A.5}$$

The first term on the right hand side of (A.4) can then be evaluated as

$$\begin{aligned}&\sum_{m\in\mathbb{Z}-h_B}\sum_{n\in\mathbb{Z}-h_A}\sum_{l\geq 0}\oint_{|z|>|w|}(w-x)^{-l-m-h_B}(z-x)^{-n-1-h_A-l}a_n(x)b_m(x)\frac{dz}{2\pi i}\\ =&\sum_{m\in\mathbb{Z}-h_B}\sum_{n\in\mathbb{Z}-h_A}\sum_{l\geq 0}(w-x)^{-l-m-h_B}a_n(x)b_n(x)\delta_{l-n-h_A,0}\\ =&\sum_{m\in\mathbb{Z}-h_B}\sum_{n\leq -h_A}(w-x)^{-n-m-h_B-h_B}a_n(x)b_n(x).\end{aligned}\tag{A.6}$$

The contour integral has a pole at $w=x$ provided that $n+l+h_A=0$, this occurs when $n+h_A=-l\in\mathbb{Z}_{\leq 0}$. This implies that the modes of $A(z)$ satisfy $n+h_A\in\mathbb{Z}$ which is a requirement of the untwisted sector. Hence this analysis will only work for arbitrary fields $A(z)$ and $B(w)$ if we are in the untwisted sector. In the twisted sector, any field $A(z)$ that doesn't satisfy these requirements will produce branch cuts in the complex plane and so calculating the normal ordered product is much more difficult using this method.

Dealing with the region $|z|<|w|$ in a similar manner gives

$$\oint_{|z|<|w|}\frac{B(w)A(z)}{z-w}\frac{dz}{2\pi i}=-\sum_{m\in\mathbb{Z}-h_B}\sum_{n>-h_A}(w-x)^{-n-m-h_A-h_B}b_n(x)a_n(x).\tag{A.7}$$

Putting the two halves of equation (A.4) together we find

$$\begin{aligned}:A(w)B(w): &=\sum_{m\in\mathbb{Z}-h_B-h_A}[\sum_{n\leq -h_A}a_n(x)b_n(x)-\sum_{n>-h_A}b_n(x)a_n(x)](w-x)^{-m-h_A-h_B}\\ &:=\sum_{m\in\mathbb{Z}-h_A-h_B}:AB:_m(w-x)^{-m-h_A-h_B}.\end{aligned}\tag{A.8}$$

By equating terms we have

$$:AB:_m=\sum_{n\leq -h_A}a_nb_n+\sum_{n>-h_A}b_na_n.\tag{A.9}$$

Consequences of normal ordering in the untwisted sector

It is natural to define the normal-ordering of modes by

$$:a_nb_m:\ =\begin{cases}a_nb_m & \text{if } n\leq -h_A,\\ b_ma_n & \text{if } n>-h_A.\end{cases}\tag{A.10}$$

Summarising these definitions we have

$$\begin{aligned}:A(z)B(z): &=\sum_{m\in\mathbb{Z}-h_B-h_A}:AB:_m z^{-m-h_A-h_B}\\ &=\sum_{m\in\mathbb{Z}-h_B-h_A}\sum_{n\in\mathbb{Z}-h_A}:a_nb_{m-n}:z^{-m-h_A-h_B}\\ &=\sum_{m\in\mathbb{Z}-h_B}\sum_{n\in\mathbb{Z}-h_A}:a_nb_m:z^{-n-h_A}z^{-m-h_B}.\end{aligned}$$

Furthermore, if we have a field $A(z)$ as given in (4.8), then we define the following notions

$$A(z)_- = \sum_{n\leq h_A} a_n z^{-n-h_A}, \qquad A(z)_+ = \sum_{n>h_A} a_n z^{-n-h_A}. \tag{A.11}$$

It is clear that $A(z) = A(z)_- + A(z)_+$. Applying normal ordering gives

$$\begin{aligned} :A(z)B(w): &= \sum_{n,m} :a_n b_m: z^{-n-h_A} w^{-m-h_B} && \text{(A.12)} \\ &= \sum_{n\leq -h_A} \sum_m a_n b_m z^{-n-h_A} w^{-m-h_B} + \sum_{n>-h_A} \sum_m b_m a_m z^{-n-h_A} w^{-m-h_B} && \text{(A.13)} \\ &= A(z)_- B(w) + B(w) A(z)_+. && \text{(A.14)} \end{aligned}$$

Suppose $|z| > |w|$ then we have

$$A(z)B(w) = \ :A(z)B(w): \ + [A(z)_+, B(w)]. \tag{A.15}$$

Similarly with $|z| < |w|$ we have

$$A(z)B(w) = \ :A(z)B(w): \ + [B(w), A(z)_-]. \tag{A.16}$$

It follows that the singular part of the OPE is given by

$$\begin{aligned} A(z)B(w) &\sim [A(z)_+, B(w)] \qquad \text{where } |z| > |w|, \\ &= [B(w), A(z)_-] \qquad \text{where } |z| < |w|. \end{aligned} \tag{A.17}$$

These equations provide a useful tool in calculating the operator product expansions of specific conformal field theories.

Appendix B

The generalised Wick theorem

Here we develop a method for calculating the operator product expansion of multiple fields. This technique is called the generalised Wick theorem and will be the topic of this appendix. Of most importance to the thesis is the application of the generalised Wick theorem that is discussed at the end of this appendix. During this appendix we will use

$$\underbrace{A(z)B(w)}, \tag{B.1}$$

to denote the singular terms in the OPE of $A(z)$ with $B(w)$ (4.14).

For many calculations we would like an expression for $A(z) : B(w)C(w) :$, the operator product expansion of a field with a normally ordered product of two fields. Applying the definition for $: B(w)C(w) :$ we find

$$A(z) : B(w)C(w) : \; = A(z) \oint_w \frac{B(x)C(w)}{x-w} \frac{dx}{2\pi i} = \oint_w \frac{A(z)B(x)C(w)}{x-w} \frac{dx}{2\pi i}, \tag{B.2}$$

so it is important to understand the expansion of three fields. We may assume without loss of generality that $|z| > |x| > |w|$ so that the fields are radially ordered. We will restrict ourselves to bosonic fields, in which case we can permute the order of the fields using the commutation relations of the symmetry Lie algebra. Using the definitions introduced in Appendix (A) we find

$$A(z)B(x)C(w) = \; : A(z)B(x)C(w) : + \underbrace{A(z)B(x)}\, C(w) + B(x)\, \underbrace{A(z)C(w)} \tag{B.3}$$

$$+ A(z)_- \underbrace{B(x)C(w)} + \underbrace{B(x)C(w)}\, A(z)_+. \tag{B.4}$$

If the fields are *free* then the coefficients of their singular OPE terms are constants and (B.4) reduces to

$$A(z)B(x)C(w) = \; : A(z)B(x)C(w) : + \underbrace{A(z)B(x)}\, C(w) + \underbrace{A(z)C(w)}\, B(x)$$
$$+ \underbrace{B(x)C(w)}\, A(z).$$

This is a version of generalised Wick theorem that can be applied to free field expansions such as in the Free Boson of Free Fermion conformal field theories. Since the fields in conformal field theories of interest, such as the WZW models, are not free we seek a generalised generalised Wick theorem. Substituting (B.4) back into (B.2) we can evaluate the contour integral on each term. Since the first term in (B.4) has no poles the contour integral just replaces x with w. The terms containing $\underbrace{B(x)C(w)}$ only depend on x though poles at $x = w$, hence by multiplying by $(x-w)^{-1}$ and taking the contour integral these terms vanish. We thus arrive at the generalised Wick theorem

$$\underbrace{A(z) : B(w)C(w) :} = \oint_w \frac{1}{x-w}[\underbrace{A(z)B(x)}C(w) + B(x)\underbrace{A(z)C(w)}]\frac{dx}{2\pi i}. \tag{B.5}$$

Having found a formula for the operator product expansion of $A(z) : B(w)C(w) :$ it is natural to extend this to operator products of a larger number of fields. It is clear from the presentation of generalised Wick theorem that this can be done inductively. Of particular importance to the calculations in this thesis is the following application.

Application: Suppose $A(z)$ is a field that has a vanishing singular OPE with itself

$$A(z)A(z) \sim 0.$$

Computing the operator product expansion of $A(z)$ with $: A(w)A(w) :$ yields

$$A(z) : A(w)A(w) : \sim \oint_w \frac{1}{x-w}[\underbrace{A(z)A(x)}A(w) + A(x)\underbrace{A(z)A(w)}]\frac{dx}{2\pi i} = 0.$$

Furthermore, by the inductive hypothesis that

$$A(z) : A^n(w) : \sim 0,$$

we find that

$$A(z) : A^{n+1}(w) : \sim \oint_w \frac{1}{x-w}[\underbrace{A(z)A(x)} : A^n(w) : + A(x)\underbrace{A(z) : A^n(w) :}]\frac{dx}{2\pi i} = 0.$$

Hence, the singular part of the OPE of $A(z)$ with the normally ordered product of any number of copies of $A(w)$ is always zero.

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