Vertex Operator Algebras, Modular Tensor Categories and a Kazhdan-Lusztig Correspondence at a Non-negative Integral Level

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Abstract

There is a fascinating and, at present, poorly understood connection between affine vertex operator algebras and quantum groups. In this thesis, we start with vertex operator algebras and study categories of their modules. Using intertwining maps and intertwining operators, we work towards understanding the theory of canonically braided monoidal structures on such categories, as introduced by Huang, Lepowsky and Zhang. We discuss modular tensor categories and Huang's construction of modular tensor categories from vertex operator algebras, focusing on lattice vertex operator algebras as an example. After developing this background knowledge, we explicitly detail a Kazhdan-Lusztig correspondence. Our construction is for the case of the simple affine vertex operator algebra associated to $\mathfrak{sl}_2$ at level 1, and the Lusztig form quantum group associated to $\mathfrak{sl}_2$ at the primitive sixth root of unity.

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Contents

Chapter 1

Introduction

The Kazhdan-Lusztig correspondence bridges affine vertex operator algebras to quantum groups. Both of these algebraic structures form braided monoidal categories made from their representations. This "bridge" is formed as an equivalence between these braided monoidal categories.

Let us start on the side of vertex operator algebras. One can think of an algebra as a vector space V with a multiplication defined by a linear map from V to $\operatorname{End} V$. There are additional conditions to be satisfied depending on whether the algebra is associative, commutative, unital, a Lie algebra, etc. In a similar fashion, one can think of a *vertex algebra* as a vector space V with a linear map from V to $(\operatorname{End} V)[[z, z^{-1}]]$. Here, $(\operatorname{End} V)[[z, z^{-1}]]$ is the space of formal series $\sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, where each a_n is an endomorphism of V. So, roughly speaking, a vertex algebra keeps track of *many* multiplications by assigning, to each element, a series of endomorphisms. There are conditions a vertex algebra must satisfy, providing analogues for multiplication, associativity, commutativity and a unit.

From a conformal-field-theoretic perspective, a vertex algebra is simultaneously each of the following:

(i) a space V of states,

(ii) a collection of fields, where each field is a series $\sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ in $(\operatorname{End} V)[[z, z^{-1}]]$, corresponding to a state a in V,

(iii) a symmetry algebra generated by the coefficients of the fields.

In a conformal field theory, the vertex algebra must include conformal symmetry, giving rise to the notion of a *vertex operator algebra*. The traditional string-theoretic interpretation views the fields of a vertex operator algebra as being inserted at z, a complex coordinate that locally parametrises the two-dimensional world-sheet of a string embedded in space-time.

It is worth remarking that vertex algebras have a historically "mathematical" motivation as well. The monster group is the automorphism group of the *moonshine module*, a vertex algebra constructed in [FLM88]. This vertex algebra was used by Borcherds, in [Bor92], to prove the Moonshine Conjectures, linking the monster group to the j function of number theory. Even though this thesis will not focus on the physical motivation, we will also not be approaching vertex algebras from a "monstrous" perspective.

The conformal-field-theoretic perspective mentioned above is, however, not the full picture. For one, the vertex operator algebra only describes the *holomorphic* symmetry–but this is not a problem, since the *antiholomorphic* symmetry is typically a copy of the holomorphic part. Secondly, the full state space should be a *module* for the vertex operator algebra. This motivates a perspective for us to view the representation theory of a vertex operator algebra as equally important as the vertex operator algebra itself.

Similarly to the idea that a module for an algebra V is a vector space M and a linear map from V to End M, subject to some conditions, one can think of a module for a vertex algebra V as a vector space M and a linear map from V to (End $M)[[z, z^{-1}]]$, subject to some conditions. Hence, we can think of a vertex algebra module and its fields as keeping track of *many* actions on a space.

In conformal field theory, by use of fields, it was observed that two modules can "fuse" together, in a tensor-product-like process, to create a third module which can then be decomposed giving *fusion rules*. In [Ver88], Verlinde conjectured that the fusion rules from a *rational* conformal field theory can be diagonalised by a modular transformation matrix. Moore and Seiberg were able to demonstrate in [MS88], on a physical level of rigour, that the Verlinde conjecture holds for rational conformal field theories. Here, it was observed that these modules obey fusing and braiding relations similar to those of braided monoidal categories, previously introduced in [JS86]. In [MS89], Moore and Seiberg developed this theory, making the first steps towards a structure, known today as a *modular tensor category*. It was understood that there is enough data in a rational vertex operator algebra to canonically define a modular tensor category of its modules. Hence, one can think of a modular tensor category as a categorical structure that encodes certain data from a rational conformal field theory. The underlying structures of a modular tensor category include a *rigid* braided monoidal category, a *ribbon twist* and a compatible abelian structure.

There are other reasons for being interested in modular tensor categories—they serve as an input datum for topological quantum field theories and, related, they are used as tools to compute invariants for knots and 3-manifolds. We will not explore either of these perspectives here and, instead, we refer the reader to [Tur94].

Quantum groups form a family of algebraic structures, seemingly unrelated to vertex operator algebras, but they can also produce modular tensor categories. Their underlying structure is the Hopf algebra. In some sense, Hopf algebras are designed to produce rigid monoidal categories–each datum of a Hopf algebra directly defines a part of the rigid monoidal structure on its category of finite-dimensional modules. When equipped with a universal R-matrix and a ribbon element, Hopf algebras very naturally produce ribbon tensor categories, a key ingredient of modular tensor categories. Even without a universal

R-matrix or a ribbon element, ribbon tensor categories can still be canonically produced from certain Hopf algebras.

In [Dri90], Drinfeld produced rigid braided monoidal categories from quasi-triangular quasi-Hopf algebras using the Knizhnik-Zamolodchikov equations from [KZ84], which are a system of differential equations abstracted from the conformal-field-theoric Wess-Zumino-Witten models of [Wit84]. In [KL91]–[KL94b], Kazhdan and Lusztig were inspired by [MS88], [MS89] and [Dri90] to construct equivalences between certain rigid braided monoidal categories similar to Moore and Seiberg's, and rigid braided monoidal categories built from quantum groups.

Given any simply-laced simple Lie algebra $\mathfrak{g}$ and its affinisation $\widehat{\mathfrak{g}}$, they constructed equivalences between rigid braided monoidal categories of $\widehat{\mathfrak{g}}$-modules at certain levels k and a rigid braided monoidal categories of modules of a quantum group associated to $\mathfrak{g}$. This *Kazhdan-Lusztig correspondence*, as we call it today, was originally proven for only k satisfying $k+h^\vee \notin \mathbb{Q}_{\geq 0}$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. The original construction did not extend to the case where $k + h^\vee \in \mathbb{Q}_{\geq 0}$, which includes the motivating case from [MS88] and [MS89], where k is a non-negative integer. Soon after Kazhdan and Lusztig's work, Finkelberg constructed correspondences for nearly all the simple Lie algebras and non-negative integral levels, in [Fin96] and included rigidity in [Fin13]. Here, the category of quantum group modules was replaced with a certain "semisimple subquotient". An overview of the Kazhdan-Lusztig correspondence, and many current conjectures, can be found in [Hua], where Huang also states the Kazhdan-Lusztig correspondence from a vertex-operator-algebraic perspective, which we use here.

Finkelberg's construction uses Kazhdan and Lusztig's original construction, which is beyond the scope of this thesis. Furthermore, these works can be interpreted as constructions of the rigid braided monoidal category on categories of $\widehat{\mathfrak{g}}$-modules, instead of using a pre-existing canonical rigid braided monoidal structure from the vertex operator algebra data. This is understandable since Kazhdan and Lusztig's work came before Huang and Lepowsky finished developing their tensor theory of rational vertex operator algebras in [HL95a]–[HL95c], furthermore the vertex operator algebras corresponding to Kazhdan and Lusztig's work are not rational.

There is an open problem to construct a direct equivalence between these two independent constructions of rigid braided monoidal categories: one on the vertex operator algebra side and one on the quantum group side. This construction should hold for all simple Lie algebras and non-negative integral levels, *without* using the original construction by Kazhdan and Lusztig. We will attempt to construct such an equivalence for a single case.

And now we arrive at the main problem of this thesis:

> Construct a modular tensor category of modules of the vertex operator algebra $L_1(\mathfrak{sl}_2)$ and construct a modular tensor category of modules of some $\mathfrak{sl}_2$-quantum group, both in a canonical fashion. Then construct an equivalence of rigid braided monoidal categories between these two categories. Does this equivalence extend to modular structures?

But first, we must unpack this problem. As our starting point, we assume the reader has some experience with braided monoidal categories, Hopf algebras and quantum groups. Summaries of these topics can be found in Appendix B, Appendix C and Appendix D, respectively. In Chapter 2, we will discuss vertex operator algebras and their modules, with the free boson (Heisenberg) vertex operator algebra as a guiding example. A summary of formal algebra, which forms the vertex-operator-algebraic language, can be found in Appendix A. Vertex operator algebras will be our main source for constructing modular tensor categories.

Next, in Chapter 3, we discuss the $P(w)$*-tensor product* for vertex operator algebra modules. To motivate the definition, which uses intertwining maps and a universal property, we will use the tensor product of Lie algebras as an analogy. The $P(w)$-tensor product defines the *fusion product*, which is used as the tensor product bifunctor in the monoidal category constructed from certain vertex operator algebra modules.

We will then work towards the definition of a modular tensor category in Chapter 4, where several levels of structure will be presented. At each of these levels, we will explore what is meant by an equivalence. The notion of a modular equivalence will be needed when extending our Kazhdan-Lusztig correspondence to the level of modular categories. This will be a natural thing to check, since each of our categories will have a canonical modular tensor structure.

Unfortunately, at this point, we will not have seen any examples of modular tensor categories constructed from vertex operator algebras, let alone any examples of modular tensor categories. Thankfully, Chapter 5 will discuss how certain vertex operator algebras have categories of modules with a canonical modular tensor structure. Here, we will also explicitly compute the modular-categorical data given by lattice vertex operator algebras. This chapter will provide insight to readers who want an explicit computation as guidance for understanding the general proofs in [Hua08] and [HLZ14]–[HLZg]. One of these examples will go on to provide the modular tensor category of $L_1(\mathfrak{sl}_2)$-modules to be used in Chapter 6 for solving our main problem.

Despite the fact that our $\mathfrak{sl}_2$-quantum group comes canonically equipped with a ribbon tensor structure, discussed in Appendix D, we will see that additional machinery is still needed. Appendix E will present the machinery that *semisimplifies* a pivotal tensor category by recontextualising its objects, while still retaining a lot of its structure. This will be used in Chapter 6 to produce a modular tensor category from a specific $\mathfrak{sl}_2$-quantum group. We will then compare the monoidal structure of our modular tensor category of $L_1(\mathfrak{sl}_2)$-modules with our modular tensor category constructed from an $\mathfrak{sl}_2$-quantum group. Using the explicitness of our constructions, we will construct a monoidal equivalence between these two modular tensor categories. Our final chapter will serve as an example of a direct construction for a Kazhdan-Lusztig correspondence of non-negative integral level. We will show that this equivalence is braided, as expected, and is also a *modular equivalence*.

Chapter 2

Vertex operator algebras and their modules

There are various, non-trivially equivalent, definitions for vertex algebras. This thesis will use the definition from [LL04], which uses the *Jacobi identity*. This definition may not be the most field-theoretically motivated, but it transparently shows parallels between the vertex algebra, its modules and its fusion product. Subsequent chapters will focus on additional structures on categories of vertex algebra modules.

A comparison of equivalent definitions of vertex algebras can be found in Section 1 of [DK06]. A field-theoretic definition is used by [Kac98] and [FB04], where the Jacobi identity is replaced by *translation* and *locality* axioms. A motivation, starting from conformal field theory, can be found in [Sch08].

Vertex operator algebras are vertex algebras that contain a representation of the Virasoro algebra. They form part of the symmetry algebra of a conformal field theory. Not all definitions for vertex operator algebras and their modules are exactly equivalent, however, we will only use those used in [HLZ14] and [Hua08]. In subsequent chapters, this will allow us to produce modular tensor categories from their modules.

The theory of vertex algebras requires formal calculus. A summary of the required formal calculus can be found in Appendix A. In this chapter, we assume that all vector spaces and linear maps are complex.

2.1 Vertex algebras and vertex operator algebras

We will immediately present the definition of a vertex algebra.

DEFINITION 2.1. A *vertex algebra* $(V, Y, \mathbf{1})$ consists of the following data:

(i) a vector space V,

(ii) a linear map

$$Y(\cdot, z) : V \to (\mathrm{End}\, V)[[z, z^{-1}]], \quad v \mapsto Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, \tag{2.1}$$

called the *vertex operator map* or the *state field correspondence*, which can be equivalently expressed as a bilinear map $V \times V \to V[[x, x^{-1}]]$,

(iii) a distinguished vector $\mathbf{1}$ in V called the *vacuum*,

satisfying the following conditions, for all $u, v \in V$:

(i) (*truncation condition*)

$$u_n v = 0 \qquad \text{for all } n \text{ sufficiently large}, \tag{2.2}$$

or equivalently

$$Y(u, z)v \in V((z)), \tag{2.3}$$

(ii) (*Jacobi identity*)

$$\begin{aligned} &x^{-1}\delta\left(\frac{y-z}{x}\right) Y(u, y)Y(v, z) - x^{-1}\delta\left(\frac{z-y}{-x}\right) Y(v, z)Y(u, y) \\ &= z^{-1}\delta\left(\frac{y-x}{z}\right) Y(Y(u, x)v, z), \end{aligned} \tag{2.4}$$

(iii) (*vacuum property*)

$$Y(\mathbf{1}, z) = \mathrm{id}_V, \tag{2.5}$$

(iv) (*creation property*)

$$Y(v, z)\mathbf{1} \in V[[z]] \qquad \text{and} \qquad \lim_{z\to 0} Y(v, z)\mathbf{1} = v. \tag{2.6}$$

For a vector v in V, we call $Y(v, z)$ the *field* or *vertex operator* corresponding to v. The endomorphisms v_n are called the *modes* of v. For clarity, we may sometimes write the modes as v_n^V to indicate that they are endomorphisms of V.

REMARK 2.2. The truncation condition ensures the existence of normal ordered products of two fields with the same formal variable. Recall the definition of normal ordering from Appendix A and consider the normal ordered product ${}^\circ_\circ Y(u, y)Y(v, z) {}^\circ_\circ$. If we formally allow the two formal variables y and z to be equal, then the normal ordered product of fields $Y(u, z)$ and $Y(v, z)$, for any $u, v \in V$, is

$$\begin{aligned} {}^\circ_\circ Y(u, z)Y(v, z) {}^\circ_\circ &= \sum_{n\in\mathbb{Z}} \sum_{m<0} u_m v_n z^{-m-n-1} + \sum_{n\in\mathbb{Z}} \sum_{m\geq 0} v_n u_m z^{-m-n-1} \\ &= \sum_{k\in\mathbb{Z}} \left(\sum_{\substack{n\in\mathbb{Z}, m<0 \\ m+n=k}} u_m v_n + \sum_{\substack{n\in\mathbb{Z}, m\geq 0 \\ n+m=k}} v_n u_m \right) z^{-k-1}. \end{aligned} \tag{2.7}$$

Given any $w \in V$, by the truncation condition, there are integers M and N such that

$$u_m w = 0 \quad \text{for all } m > M \qquad \text{and} \qquad v_n w = 0 \quad \text{for all } n > N.$$

So, we have a finite sum

$$\sum_{\substack{n\in\mathbb{Z},m<0\\ m+n=k}} u_m v_n w + \sum_{\substack{n\in\mathbb{Z},m\geq 0\\ n+m=k}} v_n u_m w = \sum_{k<n\leq N} u_m v_n w + \sum_{0\leq m\leq M} v_{k-m} u_m w \in V. \tag{2.8}$$

Hence, ${}^{\circ}_{\circ} Y(u,z)Y(v,z) {}^{\circ}_{\circ} \in (\mathrm{End}\, V)[[z,z^{-1}]]$. Furthermore, for sufficiently large k, (2.8) is zero. So, we can inductively repeat the previous steps to obtain the normal ordered product of multiple fields, given by

$${}^{\circ}_{\circ} Y(v^1,z)\cdots Y(v^n,z) {}^{\circ}_{\circ} \in (\mathrm{End}\, V)[[z,z^{-1}]] \qquad \text{for } v^1,\ldots,v^n \in V. \tag{2.9}$$

The vertex operator map acts as a $\mathbb{Z}$-graded analogue to the multiplication in an algebra and we can "multiply" these fields with the normal ordered product or by indexing them with distinct formal variables. The Jacobi identity resembles the associativity condition for associative algebras and the Jacobi identity for Lie algebras, as seen in Examples 2.14 and 2.16 below. The former example will also show how the vacuum and creation property resemble the unit conditions for unital algebras. △

EXAMPLE 2.3. Let A be a commutative associative unital algebra. Define the function

$$Y(\cdot,z) : A \to (\mathrm{End}\, A)[[z,z^{-1}]], \qquad a \mapsto a\cdot = (a\cdot)z^0, \tag{2.10}$$

where $a\cdot$ denotes the linear endomorphism of A given by $b \mapsto a\cdot b$. Let $\mathbf{1}$ be the identity element in A. Then, $(A,Y,\mathbf{1})$ is a vertex algebra. The Jacobi identity is satisfied since

$$\begin{aligned} x^{-1}\delta\left(\frac{y-z}{x}\right) & Y(a,y)Y(b,z) - x^{-1}\delta\left(\frac{z-y}{-x}\right) Y(b,z)Y(a,y) \\ &= \left(x^{-1}\delta\left(\frac{y-z}{x}\right) - x^{-1}\delta\left(\frac{z-y}{-x}\right)\right)(a\cdot)(b\cdot) \\ &= z^{-1}\delta\left(\frac{y-x}{z}\right)(a\cdot)(b\cdot) = z^{-1}\delta\left(\frac{y-x}{z}\right)(a\cdot b)\cdot \\ &= z^{-1}\delta\left(\frac{y-x}{z}\right) Y(Y(a,x)b,z), \end{aligned}$$

where we have used the result from Example A.20 in Appendix A to go from the second to the third line.

The algebra A can be chosen to be finite dimensional. However, the examples of interest to this thesis are infinite dimensional and noncommutative. ♢

EXAMPLE 2.4. In Section 2.4, a vertex algebra called the Heisenberg vertex algebra will be built from the affine Lie algebra $\widehat{\mathfrak{gl}_1}$. This will be a guiding example that shows the key features of the types of vertex algebras that we consider. ♢

DEFINITION 2.5. Let $(V,Y_V,\mathbf{1}_V)$ and $(W,Y_W,\mathbf{1}_W)$ be vertex algebras. A *vertex algebra homomorphism* from $(V,Y_V,\mathbf{1}_V)$ to $(W,Y_W,\mathbf{1}_W)$ is a linear map $f : V \to W$ satisfying the following conditions:

(i) (*vertex operator map is preserved*)

$$f(u_n v) = f(u)_n f(v) \qquad \text{for all } u,v \in V \text{ and } n \in \mathbb{Z}, \tag{2.11}$$

or equivalently for the canonical extension $f : V[[z, z^{-1}]] \to W[[z, z^{-1}]]$

$$f(Y_V(u, z)v) = Y_W(f(u), z)f(v) \qquad \text{for all } u, v \in V, \tag{2.12}$$

(ii) (*vacuum is preserved*)

$$f(\mathbf{1}_V) = \mathbf{1}_W. \tag{2.13}$$

Our focus is on the notion of a *vertex operator algebra*, another algebraic structure built from a vertex algebra. Vertex operator algebras have a distinguished vector whose modes generate a representation of the Virasoro algebra.

DEFINITION 2.6. The *Virasoro algebra* is the complex Lie algebra $\mathcal{L}$ spanned by the basis $\{L_n : n \in \mathbb{Z}\} \cup \{\mathbf{c}\}$ with the Lie bracket relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\mathbf{c} \qquad \text{for all } m, n \in \mathbb{Z}, \tag{2.14}$$

$$[\mathbf{c}, L_n] = 0 \qquad \text{for all } n \in \mathbb{Z}. \tag{2.15}$$

The Virasoro algebra arises in two-dimensional conformal field theory as the symmetry algebra of holomorphic infinitesimal conformal transformations on the punctured complex plane. For this reason, related definitions are also known as *conformal vertex algebras* in [FB04] and [Kac98].

DEFINITION 2.7. A *vertex operator algebra* $(V, Y, \mathbf{1}, \omega)$ consists of the following data:

(i) a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}$, graded by *weights* $\mathrm{wt}\, v = n$, for all $v \in V_{(n)}$,

(ii) a vertex operator map $Y(\cdot, z) : V \to (\mathrm{End}\, V)[[z, z^{-1}]]$,

(iii) a vacuum vector $\mathbf{1}$ in V,

(iv) a distinguished vector ω in $V_{(2)}$, called the *conformal vector*, with modes given by

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1} =: \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}, \tag{2.16}$$

satisfying the following conditions:

(i) the triple $(V, Y, \mathbf{1})$ is a vertex algebra,

(ii) (*grading restrictions*)

$$\dim V_{(n)} < \infty \qquad \text{for all } n \in \mathbb{Z}, \tag{2.17}$$

$$V_{(n)} = 0 \qquad \text{for all } n \text{ sufficiently negative}, \tag{2.18}$$

(iii) ($L(0)$-*eigenspace decomposition by grading property*)

$$L(0)v = (\mathrm{wt}\, v)v = nv \qquad \text{for all } n \in \mathbb{Z} \text{ and } v \in V_{(n)}, \tag{2.19}$$

(iv) (*Virasoro algebra relations*)

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c_V \qquad \text{for all } m, n \in \mathbb{Z}, \tag{2.20}$$

for some complex number c_V called the *central charge* of V,

(v) ($L(-1)$-*derivative property*)

$$Y(L(-1)v, z) = \frac{d}{dz} Y(v, z) \qquad \text{for all } v \in V. \tag{2.21}$$

DEFINITION 2.8. A *vertex operator algebra homomorphism* from a vertex operator algebra $(V, Y_V, \mathbf{1}_V, \omega_V)$ to a vertex operator algebra $(W, Y_W, \mathbf{1}_W, \omega_W)$ is a vertex algebra homomorphism $f : V \to W$ such that the conformal vector is preserved. That is,

$$f(\omega_V) = \omega_W. \tag{2.22}$$

REMARK 2.9. A *conformal vertex algebra* or a *vertex operator algebra without grading restrictions* $(V, Y, \mathbf{1}, \omega)$ is the same as in the definition of a vertex operator algebra but without the grading restriction conditions (2.17) and (2.18). Other literature may use the term "vertex operator algebra" for similar definitions, for example in [FZ92], condition (2.18) is required but not (2.17). △

EXAMPLE 2.10. Recall Example 2.3, explaining how commutative associative unital algebras are vertex algebras. The algebra can be graded with every non-zero vector having zero weight. This gives a conformal vertex algebra with a conformal vector of zero. All of the Virasoro modes are zero, so the Virasoro relations are immediately satisfied with zero central charge. The $L(0)$-eigenspace property is satisfied by the trivial grading. All fields are constants with respect to z, hence satisfy the $L(-1)$-derivative property. If the algebra is finite dimensional, then it is also a vertex operator algebra. ◊

EXAMPLE 2.11. In Section 2.4, the Heisenberg vertex algebra will be given the structure of a vertex operator algebra. ◊

EXAMPLE 2.12. Vertex operator algebras can be constructed from positive definite even lattices. In Chapter 5, these will be used to explicitly compute examples of modular tensor categories. ◊

EXAMPLE 2.13. Vertex operator algebras can be constructed from affinisations of simple Lie algebras. These are the holomorphic symmetry algebras in Wess-Zumino-Witten models from conformal field theory, originating in [Wit84]. In Chapter 6, we will see a specific example for $\mathfrak{sl}(2)$ and show its relation to an $\mathfrak{sl}(2)$-quantum group through equivalences of (pre-)modular categories. ◊

2.2 Modules

We now have a theory for vertex (operator) algebras—an algebraic structure. A natural next step would be to study how vertex (operator) algebras can be represented as linear endomorphisms, that is, study their representation theory.

But first, we will draw some analogies between associative unital algebras, Lie algebras and vertex operator algebras, and their respective notions of modules. (For simplicity, we will continue to assume that all vector spaces and linear maps are over $\mathbb{C}$.)

Consider an associative unital algebra $(A, \cdot, 1)$. A representation of A is defined by transferring its structure into endomorphisms of a vector space. Given a vector space M, we use the canonical associative unital structure (composition as the multiplication and the identity map as the unit) on the vector space End M. A representation of A is an associative unital algebra homomorphism from $(A, \cdot, 1)$ to (End $M, \circ, \mathrm{id}_M)$. Representations are analogously defined for Lie algebras by endowing End M with the Lie bracket $[f, g] = f \circ g - g \circ f$, for $f, g \in$ End M.

If we want to find an analogy for vertex algebras, one could try to endow End M or (End $M)[[z, z^{-1}]]$ with a vertex algebra structure so that we can take vertex algebra morphisms from V to End M or (End $M)[[z, z^{-1}]]$. This works to some extent as shown in Chapter 5 of [LL04], but we will use a different, and more standard, approach.

As with associative unital algebras, we can instead redefine these notions via actions on a vector space, and then require that all conditions analogous to the defining conditions hold (at least the conditions that make sense). This produces the notion of an associative unital algebra *module*. Modules of Lie algebras can be defined in a similar way as well. This is the approach taken to define vertex algebra *modules* instead of representations.

As a guiding analogy, we first give an equivalent definition of an associative unital algebra.

DEFINITION 2.14. An *associative unital algebra* $(A, Y, 1)$ consists of the following data:

(i) a vector space A,

(ii) a linear map $Y : A \to$ End A (or equivalently a bilinear map $\cdot : A \times A \to A$),

(iii) a distinguished element 1 in A,

satisfying the following conditions:

(i) (*associativity*)

$$Y(a)Y(b) = Y(Y(a)b) \qquad \text{for all } a, b \in A \tag{2.23}$$

(or equivalently $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in A$),

(ii) (*identity*)

$$Y(1) = \mathrm{id}_A \quad \text{and} \quad Y(a)1 = a \qquad \text{for all } a \in A \tag{2.24}$$

(or equivalently $1 \cdot a = a = a \cdot 1$ for all $a, b, c \in A$).

DEFINITION 2.15. Let $(A, Y, 1)$ be an associative unital algebra. An A-*module* (M, Y_M) consists of the following data:

(i) a vector space M,

(ii) a linear map $Y_M : A \to$ End M (or equivalently a bilinear map $\bullet : A \times M \to M$),

satisfying the following conditions:

(i) (*associativity*)

$$Y_M(a)Y_M(b) = Y_M(Y(a)b), \qquad \text{for all } a, b \in A \tag{2.25}$$

(or equivalently $a \bullet (b \bullet m) = (a \cdot b) \bullet m$ for all $a, b \in A,\ m \in M$),

(ii) (*identity*)

$$Y_M(1) = \mathrm{id}_M, \tag{2.26}$$

(or equivalently $1 \bullet m = m$ for all $m \in M$).

To create Definition 2.15, the conditions for the associative unital algebra in Definition 2.14 have been copied, replacing Y with Y_M where it is appropriate. We note that the inner Y in the associativity condition is still the map for the associative unital algebra A because it would not make sense for this to be Y_M instead. Furthermore, we have dropped the second identity condition because there is no unit element in M. This definition agrees with the usual definition of an associative unital algebra module or representation.

We can give similarly styled definitions for Lie algebras and representations. At the same time we will illuminate the reason for the name of the Jacobi identity for vertex operator algebras.

DEFINITION 2.16. A *Lie algebra* $(\mathfrak{g}, Y)$ consists of the following data:

(i) a vector space $\mathfrak{g}$,

(ii) a linear map $Y : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$ (or equivalently a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$),

satisfying the following conditions:

(i) (*anticommutativity*)

$$Y(x)y = -Y(y)x \qquad \text{for all } x, y \in \mathfrak{g}, \tag{2.27}$$

(ii) (*Jacobi identity*)

$$Y(x)Y(y) - Y(y)Y(x) = Y(Y(x)y) \qquad \text{for all } x, y \in \mathfrak{g}. \tag{2.28}$$

DEFINITION 2.17. Let $(\mathfrak{g}, Y)$ be a Lie algebra. An $\mathfrak{g}$-module (M, Y_M) consists of the following data:

(i) a vector space M,

(ii) a linear map $Y_M : \mathfrak{g} \to \mathrm{End}\,M$ (or equivalently a bilinear map $\bullet : \mathfrak{g} \times M \to M$),

satisfying the following conditions:

(i) (*Jacobi identity*)

$$Y_M(x)Y_M(y) - Y_M(y)Y_M(x) = Y_M(Y(x)y) \qquad \text{for all } x, y \in \mathfrak{g}. \tag{2.29}$$

We have simply replaced the traditional Lie bracket $[\cdot,\cdot]$ with the adjoint representation $Y = \mathrm{ad}$. Similarly to the associative unital algebra, we have copied the conditions for the Lie algebra and replaced Y with Y_M where it is appropriate. We note that the inner Y in the Jacobi condition is still the map for $\mathfrak{g}$ because it would not make sense for this to be Y_M instead. Further, we have dropped the anticommutivity condition because $Y_M(x)$ does not necessarily act on $\mathfrak{g}$ nor, if we instead require $y \in M$, can we assign an action on $\mathfrak{g}$. This definition agrees with the usual definition of a Lie algebra module or representation.

With these two analogies in mind, we now present the definition for a vertex algebra module by modifying Definition 2.1.

DEFINITION 2.18. Let $(V, Y, \mathbf{1})$ be a vertex algebra. A *V-module* (M, Y_M) consists of the following data:

(i) a complex vector space M,

(ii) a linear map, also called the *vertex operator map*,

$$Y_M(\cdot, z) : V \to (\text{End } M)[[z, z^{-1}]], \quad v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, \tag{2.30}$$

which can be equivalently expressed as a bilinear map $V \times M \to M[[x, x^{-1}]]$,

satisfying the following conditions for all $u, v \in V$ and $m \in M$:

(i) (*truncation condition*)

$$u_n m = 0 \qquad \text{for all } n \text{ sufficiently large,} \tag{2.31}$$

or equivalently

$$Y_M(u, z)m \in M((z)), \tag{2.32}$$

(ii) (*Jacobi identity*)

$$\begin{aligned} &x^{-1}\delta\left(\frac{y-z}{x}\right) Y_M(u, y) Y_M(v, z) - x^{-1}\delta\left(\frac{z-y}{-x}\right) Y_M(v, z) Y_M(u, y) \\ &= z^{-1}\delta\left(\frac{y-x}{z}\right) Y_M(Y(u, x)v, z), \end{aligned} \tag{2.33}$$

(iii) (*vacuum property*)

$$Y_M(\mathbf{1}, z) = \text{id}_M . \tag{2.34}$$

For v in V, the endomorphisms v_n are still called the *modes* of v. For clarity, we may sometimes write the modes as v_n^M to indicate that they are endomorphisms of M.

REMARK 2.19. The truncation condition now has u_n acting on elements in M. The inner vertex operator map in the Jacobi identity remains the map on V. The creation property is dropped since there is no vacuum in M.

The vertex operator map of V allows elements in V to act on V in many ways, indexed by $\mathbb{Z}$. This can be equivalently thought of as assigning an (End V)-valued field (i.e. a series satisfying the truncation condition) that acts on V. The vertex operator map for the module assigns an (End M)-valued field to each element in V, hence acting on M in $\mathbb{Z}$-many ways. △

REMARK 2.20. The notion of a vertex algebra module is most commonly used in the literature, rather than some concept of a *representation* $\rho : V \to \text{End } M$. There is a definition of *vertex algebra representations* found in Section 5.3 of [LL04]. This definition of a representation of a vertex algebra V includes a vector space M similar to the definition for modules. However, it does not endow End M with the structure of a vertex algebra. Instead, a *weak* vertex algebra structure is given to $\mathcal{E}(M) := \text{End }(M, M((x)))$ and a representation is defined as a weak vertex algebra homomorphism from V to $\mathcal{E}(M)$. It is then shown that this notion of representation is equivalent to the notion of modules. △

A definition of vertex operator algebra modules can also be produced by writing down module analogues for each vertex operator algebra condition.

DEFINITION 2.21. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. A *V-module* (M, Y_M) consists of the following data:

(i) a $\mathbb{C}$-graded vector space $M = \bigoplus_{h \in \mathbb{C}} M_{(h)}$, graded by *(conformal) weights* $\operatorname{wt} v = h$, for all $v \in V_{(h)}$,

(ii) a vertex operator map $Y(\cdot, z) : V \to (\operatorname{End} M)[[z, z^{-1}]]$, with modes of the conformal vector defined by

$$Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} \omega_n z^{-n-1} =: \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \tag{2.35}$$

satisfying the following conditions:

(i) the pair (M, Y_M) is a vertex algebra module of the vertex algebra $(V, Y, \mathbf{1})$,

(ii) (*grading restrictions*)

$$\dim M_{(h)} < \infty \qquad \text{for all } h \in \mathbb{C}, \tag{2.36}$$

$$M_{(h)} = 0 \qquad \text{for all } h \text{ with real part sufficiently negative}, \tag{2.37}$$

(iii) (*$L(0)$-eigenspace decomposition by grading property*)

$$L(0)m = (\operatorname{wt} m)m = hm \qquad \text{for all } h \in \mathbb{C} \text{ and } m \in M_{(h)}. \tag{2.38}$$

REMARK 2.22. Recall that, in Definition 2.7, vertex operator algebras had integral grading but now their modules have complex grading. This is done to allow $L(0)$ to have complex eigenvalues, that is, to allow for complex weights. Other similar definitions require rational, integral or non-negative integer grading and then change the $L(0)$-eigenspace decomposition condition to be $L(0)m = (n + h)m$ for $n \in \mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$, and some constant $h \in \mathbb{C}$. $\triangle$

When copying the definition for the vertex operator algebra to the definition for the vertex operator algebra module, we see that the last two conditions have been removed. The Virasoro algebra relations and the $L(-1)$-derivative property in fact follow from Definition 2.21 (see Proposition 4.1.5 of [LL04]).

PROPOSITION 2.23. Let (M, Y_M) be a module for vertex operator algebra $(V, Y, \mathbf{1}, \omega)$. Then, the following conditions are satisfied:

(iv) (*Virasoro algebra relations*)

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n} c_V \qquad \text{for all } m, n \in \mathbb{Z}, \tag{2.39}$$

where c_V is the central charge of V,

(v) (*$L(-1)$-derivative property*)

$$Y_M(L(-1)v, z) = \frac{d}{dz} Y_M(v, z) \qquad \text{for all } v \in V. \tag{2.40}$$

We will not prove this result here. It follows from the fact that the commutator with $L(-1)$ acts as a derivative and that $L(-2)\mathbf{1} = \omega$.

REMARK 2.24. Consider a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$. Definition 2.21 for vertex operator algebra modules is more restrictive than Definition 2.18 for vertex algebra modules because we carry across the grading restrictions from the definition of a vertex operator algebra. So, the collection of $(V, Y, \mathbf{1}, \omega)$-modules is smaller than the collection of $(V, Y, \mathbf{1})$-modules; in the sense that there is an assignment including every $(V, Y, \mathbf{1}, \omega)$-module as a $(V, Y, \mathbf{1})$-module. (Note that the grading is unique by (2.38) so $(V, Y, \mathbf{1}, \omega)$-modules only have additional conditions and no additional data, hence the assignment is indeed injective.) This inclusion is proper since there can be a $(V, Y, \mathbf{1})$-module with infinite-dimensional $L(0)$-eigenspaces, whereas there can be no such $(V, Y, \mathbf{1}, \omega)$-module due to (2.36) and (2.38). This more restrictive notion of vertex operator algebra modules is used to form modular tensor categories, see Chapter 5. △

EXAMPLE 2.25. Given a commutative associative unital algebra A, the A-modules are also vertex algebra modules when A is viewed as a vertex algebra. If A is finite-dimensional, then the finite-dimensional A-modules are vertex operator algebra modules. ◊

EXAMPLE 2.26. Given any vertex algebra $(V, Y, \mathbf{1})$, we have the following examples of V-modules:

(a) The pair (V, Y), referred to as the *vacuum module* or *adjoint module* of V.

(b) The zero vector space together with the map $Y(v, z) = 0$, for all $v \in V$.

(c) Let (M, Y_M) be a V-module with linear subspace $N \subseteq M$. Assume that N satisfies the *submodule condition*:

$$v_i n \in N \qquad \text{for all } v \in V,\ n \in N,\ i \in \mathbb{Z}, \tag{2.41}$$

on modes, or equivalently for vertex operators

$$Y_M(v, z)n \in N[[z, z^{-1}]] \qquad \text{for all } v \in V,\ n \in N. \tag{2.42}$$

Then, (N, Y_M) is a V-module. In this case we say that N is a *submodule* of M. If the only submodules of M itself and the zero module, then we call M *irreducible*.

(d) Let (M, Y_M) be a V-module with submodule N. Define the vertex operator map $Y_{M/N}(\cdot, z) : V \to (\mathrm{End}\ M/N)[[z, z^{-1}]]$ by:

$$Y_{M/N}(v, z)[m] = [Y_M(v, z)m] \qquad \text{for all } v \in V,\ [m] \in M/N, \tag{2.43}$$

for vertex operators, or equivalently for modes

$$v_i^{M/N}[m] = [v_i^M m] \qquad \text{for all } v \in V,\ [m] \in M/N,\ i \in \mathbb{Z}. \tag{2.44}$$

Then, $(M/N, Y_{M/N})$ is a V-module called the *quotient module* of M by N.

(e) The previous examples also hold for vertex operator algebras if the subspaces are taken to have the same grading, that is, subspaces are replaced with graded subspaces. ◊

EXAMPLE 2.27. Let (M, Y_M) be a module for a vertex operator algebra V. Define the *restricted dual* M' of $M = \bigoplus_{h\in\mathbb{C}} M_{(h)}$ to be the $\mathbb{C}$-graded vector space

$$M' = \bigoplus_{h\in\mathbb{C}} M^*_{(h)}, \tag{2.45}$$

where $M^*_{(h)}$ denotes the vector space dual of the finite-dimensional vector space $M_{(h)}$. Define the linear map

$$Y_{M'}(\cdot, z) : V \to (\text{End } M')[[z, z^{-1}]], \qquad Y_{M'}(v, z) = \sum_{n\in\mathbb{Z}} v'_n z^{-n-1}, \tag{2.46}$$

determined by the condition

$$\langle Y_{M'}(v, z)m', m\rangle = \langle m', Y(e^{zL(1)}(-z^{-2})^{L(0)}v, z^{-1})m\rangle, \tag{2.47}$$

for all $m' \in M'$ and $m \in M$.

It is shown in Theorem 5.2.1 of [FHL93] that $(M', Y_{M'})$ is a V-module. This is actually shown for the case of $\mathbb{Q}$-graded modules, but as mentioned in Remark 2.33 of [HLZ14], the proof carries over to the $\mathbb{C}$-graded case. We call $(M', Y_{M'})$ the *contragredient module* of (M, Y_M). $\Diamond$

Explicit examples of vertex operator algebra modules will be given in Section 2.4, Chapter 5 and Chapter 6.

REMARK 2.28. Given a fixed vertex operator algebra V, the V-modules form a category $V-$Mod. As is expected for notions of representations of some algebraic object, this category will be abelian. The abelian structure is the first of many additional structures that will be endowed on $V-$Mod in subsequent chapters. To construct this category, we need a notion of V-module homomorphisms. $\triangle$

DEFINITION 2.29. Let V be a vertex (operator) algebra. Let (M, Y_M) and (N, Y_N) be V-modules. A *V-module homomorphism* from (M, Y_M) to (N, Y_N) is a linear map $f : M \to N$ satisfying the following condition:

(i) (*compatibility of the action of modes/vertex operators*)

$$f(v_i^M m) = v_i^N f(m) \qquad \text{for all } v \in V,\ m \in M,\ i \in \mathbb{Z}, \tag{2.48}$$

or equivalently for the canonical extension $f : M[[z, z^{-1}]] \to N[[z, z^{-1}]]$,

$$f(Y_M(v, z)m) = Y_N(v, z)f(m) \qquad \text{for all } v \in V,\ m \in M. \tag{2.49}$$

REMARK 2.30. Vertex operator algebra modules have, a priori, more data compared to vertex algebra modules, namely, the $\mathbb{C}$-grading. Hence, we should also impose that $f(M_{(h)}) \subseteq N_{(h)}$, for all $h \in \mathbb{C}$. However, this condition is satisfied by the compatibility of the action of modes and $L(0)$-eigenspace decomposition:

$$L(0)f(m) = f(L(0)m) = hf(m) \qquad \text{for all } h \in \mathbb{C} \text{ and } m \in M_{(h)}. \tag{2.50}$$ $\triangle$

REMARK 2.31. Vertex (operator) algebra module homomorphisms provide a good notion for the morphisms in the category $V-$Mod of V-modules. To see this, we observe that the identity map id_M is a V-module homomorphism for the V-module (M, Y_M). The composition of two V-module homomorphisms is a V-module homomorphism and the associativity and identity conditions come from the category of complex vector spaces. $\triangle$

REMARK 2.32. As expected, the category of V-modules has an abelian structure, which will be utilised in subsequent chapters. The hom-sets in $V-\mathsf{Mod}$ naturally have $\mathbb{C}$-linear structure inherited from $\mathbb{C}-\mathsf{Vect}$. Then, composition is $\mathbb{C}$-bilinear with respect to the structure, so $V-\mathsf{Mod}$ is a preadditive category.

We have a binary biproduct of (M, Y_M) and (N, Y_N) consisting of the vector space $M \oplus N$ and vertex operator map

$$Y_{M\oplus N}(v,z)(m,n) = (Y_M(v,z)m, Y_N(v,z)n) \quad \text{for all } (m,n) \in M \oplus N,\ v \in V, \tag{2.51}$$

or equivalently for modes

$$v_i^{M\oplus N}(m,n) = (v_i^M m, v_i^N n) \qquad \text{for all } (m,n) \in M \oplus N,\ i \in \mathbb{Z},\ v \in V. \tag{2.52}$$

The biproduct embedding and projection homomorphisms are the same as the ones for coproducts and products of vector spaces; these are indeed V-module homomorphisms. The zero module is an initial and terminal object, hence a zero object. Combined with preadditivity and the biproduct, $V-\mathsf{Mod}$ is an additive category. Given a V-module homomorphism $f : M \to N$, the kernel of f as a linear map is a submodule of M. This submodule, together with its inclusion into M, is a kernel of f in $V-\mathsf{Mod}$. The image of f is a submodule of N. The quotient of N by $\mathrm{im} f$, together with the canonical projection map, is a cokernel of f in $V-\mathsf{Mod}$. Since $V-\mathsf{Mod}$ is a concrete category, injective morphisms are mono and surjective morphisms are epi. Inherited from $\mathbb{C}-\mathsf{Vect}$, any monomorphism is a kernel of its cokernel and any epimorphism is a cokernel of its kernel. This gives $V-\mathsf{Mod}$ its abelian structure. $\triangle$

Finally, we can use the language of modules to succinctly define the following properties for vertex (operator) algebras.

DEFINITION/PROPOSITION 2.33. A vertex (operator) algebra V is *simple* if its vacuum module is irreducible. An *ideal* of V is a submodule I of its vacuum module. The vector space quotient V/I has the structure of a vertex (operator) algebra with its vertex operator map defined by

$$Y_{V/I}([v],z) = \sum_{n\in\mathbb{Z}} [v]_n z^{-n-1}, \quad \text{where } [v]_n[u] = [v_n u] \text{for } Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1},$$

for all $[u], [v] \in V/I$. Then, $(V/I, Y_{V/I})$ is equipped with the vacuum $[\mathbf{1}]$ and the conformal vector $[\omega]$. In fact, V/I has the same central charge as V and the canonical quotient map $V \to V/I$ is a vertex operator algebra homomorphism.

This proposition can be proven directly from the definition of a vertex (operator) algebra, but we will not show this here. The notion of simple vertex operator algebras will eventually be used in Chapter 6 to construct the *simple affine vertex operator algebras* from simple Lie algebras.

2.3 Constructing vertex algebras

The setting for vertex algebras and their modules has now been established. However, we are yet to introduce any non-trivial examples. To provide more richly structured examples, there is a theorem that constructs the state-field correspondence from a vector space and a chosen vacuum vector. But first, we need to give an equivalent definition of a vertex algebra.

PROPOSITION 2.34. In Definition 2.1, the Jacobi identity can be replaced with the following conditions:

(i) (*translation*)

$$[d, Y(v, z)] = \frac{d}{dz} Y(a, z) \qquad \text{for all } v \in V, \tag{2.53}$$

where d is the endomorphism $v \mapsto v_{-2}\mathbf{1}$ of V (and $d = L(-1)$ for vertex operator algebras),

(ii) (*locality*) for all $u, v \in V$, there exists $k \in \mathbb{Z}_{\geq 0}$ such that

$$(y - z)^k [Y(u, y), Y(v, z)] = 0. \tag{2.54}$$

REMARK 2.35. A proof for Proposition 2.34 can be found in the Section 1 of [DK06], where actually, the *Borcherds identity* is used instead of the Jacobi identity (but these can be seen to be equivalent after explicitly expanding the series and equating coefficients). Definition 2.1 is natural for motivating the notions of modules, intertwining operators and maps, and the fusion product. However, it is inconvenient in practice when trying to construct vertex algebras. Fortunately, the second formulation gives us the following theorem. △

THEOREM 2.36. *The Construction Theorem.* (Theorem 5.7.1 [LL04]) Let V be a vector space equipped with a distinguished vector $\mathbf{1}$ and d an endomorphism of V with $d\mathbf{1} = 0$. Let T be a subset of V equipped with a map

$$Y_0(\cdot, z) : T \to (\mathrm{End}\, V)[[z, z^{-1}]], \quad Y_0(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}. \tag{2.55}$$

We call $Y_0(a, z)$, for $a \in T$, the *generating fields* because we assume that the following holds:

(i) the vector space V is spanned by

$$\{a^{(1)}_{n_1} \cdots a^{(r)}_{n_r} \mathbf{1} : r \in \mathbb{Z}_{\geq 0},\ a^{(i)} \in T,\ n_i \in \mathbb{Z}_{>0}\}, \tag{2.56}$$

(ii) the truncation condition (2.2) and creation property (2.6) hold for all $a \in T$,

(iii) (*translation*)

$$[d, Y_0(a, z)] = \frac{d}{dz} Y_0(a, z) \qquad \text{for all } a \in T, \tag{2.57}$$

(iv) (*locality*) for all $a, b \in T$, there exists $k \in \mathbb{Z}_{\geq 0}$ such that

$$(y - z)^k [Y_0(a, y), Y_0(b, z)] = 0. \tag{2.58}$$

Then, there is a unique extension of Y_0 to a linear map

$$Y(\cdot, z) : V \mapsto (\mathrm{End}\, V)[[z, z^{-1}]] \tag{2.59}$$

$$Y(a^{(1)}_{n_1} \cdots a^{(r)}_{n_r}\mathbf{1}, z) = \mathop{\circ}\limits^{\circ} \frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1} a^{(1)}(z) \cdots \frac{1}{(n_\ell-1)!}\left(\frac{d}{dz}\right)^{n_\ell-1} a^{(r)}(z) \mathop{\circ}\limits^{\circ}$$

such that $(V, Y, \mathbf{1})$ is a vertex algebra.

We will see an application of the Construction Theorem in the next section.

REMARK 2.37. The translation and locality conditions are reflective of the equivalent formulation for vertex algebras, given by Proposition 2.34. Note that $d\mathbf{1} = 0$ needs to be satisfied since $\mathbf{1}_{-2}\mathbf{1} = 0$. Also note that the Construction Theorem can be used as a *reconstruction* theorem to show that (2.59) is the unique form for a vertex operator map given a vertex algebra. △

In Section 5.7 of [LL04], a general construction theorem for modules is given. It requires that the vertex algebra is constructed at the same time, with a construction theorem similar to Theorem 2.36. Generating fields for the module are required and conditions must be shown, one of them again being an analogue of locality. However, this is not the only means of constructing modules. Certain vertex (operator) algebras can be constructed using Lie algebra module induction, and in these cases, some of the vertex algebra modules can also be constructed by induction. See for example, [FZ92], where vertex operator algebras and their modules are constructed using affine Lie algebras and the Virasoro algebra. This relates an associative algebra to the vertex algebra, in this case it is the (completion of the) universal enveloping algebra of the Lie algebra.

2.4 An example: the Heisenberg vertex operator algebra

In the days of early string theory, attempts were made to model a free, massless, spinless bosonic string in flat space-time. The following vertex operator algebra arises as the holomorphic symmetry algebra for the free boson in each dimension.

Let $\mathfrak{h} = \mathbb{C}a \cong \mathfrak{gl}_1$ be an abelian Lie algebra. Let $\widehat{\mathfrak{h}}$ be the affinisation of $\mathfrak{h}$. That is, let $\widehat{\mathfrak{h}} = (\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}\mathbf{k}$ be the Lie algebra with central element $\mathbf{k}$ and bracket relations

$$[a_m, a_n] = m\delta_{m+n,0}\mathbf{k} \qquad \text{for all } m, n \in \mathbb{Z},$$

where we have denoted $a \otimes t^n$ by a_n. Consider the Lie subalgebra

$$\widehat{\mathfrak{h}}_{\geq 0} = (\mathfrak{h} \otimes \mathbb{C}[t]) \oplus \mathbb{C}\mathbf{k}, \tag{2.60}$$

of $\mathfrak{h}$ and consider the $\widehat{\mathfrak{h}}_{\geq 0}$-module $\mathbb{C}_0 = \mathbb{C}$ with the actions

$$\mathbf{k} \cdot 1 = 1 \qquad \text{and} \qquad a_n \cdot 1 = 0 \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{2.61}$$

Inducing by $\mathbb{C}_0$ gives the $\widehat{\mathfrak{h}}$-module

$$\mathsf{H} = \mathcal{U}(\widehat{\mathfrak{h}}) \otimes_{\mathcal{U}(\widehat{\mathfrak{h}}_{\geq 0})} \mathbb{C}_0. \tag{2.62}$$

To use Theorem 2.36, we define the distinguished vectors in H,

(2.63) $$\mathbf{1} = 1 \otimes 1 \qquad \text{and} \qquad a = a_{-1} \otimes 1 = a_{-1}\mathbf{1},$$

and define the single generating field

(2.64) $$Y_0(a, z) = a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

where the $a_n \in \widehat{\mathfrak{h}}$ are viewed as endomorphisms on H via their $\widehat{\mathfrak{h}}$-module actions. Finally, we define the following endomorphism on H:

(2.65) $$d = \sum_{n>0} a_{-n-1} a_n.$$

(i) We can write $\mathsf{H} = \operatorname{span}\{a_{-n_1} \cdots a_{-n_\ell}\mathbf{1} \mid \ell \in \mathbb{Z}_{\geq 0},\ n_1 \geq \cdots \geq n_\ell > 0\}$ as a span of Poincaré–Birkhoff–Witt-basis (PBW-basis) vectors. That is, we can use the Poincaré–Birkhoff–Witt theorem to say that $\mathsf{H} \cong \mathcal{U}(\mathfrak{h} \otimes \mathbb{C}[t^{-1}]t^{-1})$ is isomorphic to the symmetric algebra of $\mathfrak{h} \otimes \mathbb{C}[t^{-1}]t^{-1}$, as vector spaces.

(ii) Note that every basis vector $a_{-n_1} \cdots a_{-n_\ell}\mathbf{1}$ is annihilated by a_m, for all $m > n_1$. So, the truncation condition is satisfied. Also, (2.65) is indeed an endomorphism on H, since it becomes a finite sum after acting on any vector in H. Furthermore,

$$\lim_{z \to 0} Y_0(a, z)\mathbf{1} = \lim_{z \to 0} \sum_{n \in \mathbb{Z}} a_n \mathbf{1} z^{-n-1} = \lim_{z \to 0} \sum_{n \leq -1} a_n \mathbf{1} z^{-n-1} = a_{-1}\mathbf{1} = a.$$

Hence, the creation condition is satisfied by the generating field.

(iii) Let $v \in \mathsf{H}$. Then,

$$\begin{aligned}
[d, a_n]v &= \left[\sum_{m>0} a_{-m-1}a_m, a_n\right] v = \sum_{m>0} [a_{-m-1}a_m, a_n]v \\
&= \sum_{m>0} \left(a_{-m-1}[a_m, a_n] + [a_{-m-1}, a_n]a_m\right) v \\
&= \sum_{m>0} \left(m\delta_{m+n,0} a_{-m-1} + (-m-1)\delta_{-m-1+n,0} a_m\right) v = -n a_{n-1} v.
\end{aligned}$$

Hence, the translation condition holds:

$$\begin{aligned}
\left[d, \sum_{n \in \mathbb{Z}} a_n z^{-n-1}\right] v &= \sum_{n \in \mathbb{Z}} [d, a_n] z^{-n-1} v = \sum_{n \in \mathbb{Z}} -n a_{n-1} z^{-n-1} v \\
&= \sum_{n \in \mathbb{Z}} (-n-1) a_n z^{-n-2} v = \frac{d}{dz} Y_0(a, z) v.
\end{aligned}$$

(iv) Observe that, for all $v \in \mathsf{H}$,

$$\begin{aligned}
(y - z)\left[Y_0(a, y), Y_0(a, z)\right] v &= (y - z) \sum_{m,n \in \mathbb{Z}} [a_m, a_n] y^{-m-1} z^{-n-1} v \\
&= (y - z) \sum_{m \in \mathbb{Z}} m y^{-m-1} z^{m-1} v = z^{-1}\delta\left(\frac{z}{y}\right) v.
\end{aligned}$$

So,

$$(y-z)^2\,[Y_0(a,y),Y_0(a,z)] = (y-z)z^{-1}\delta\left(\frac{z}{y}\right) = (z-z)z^{-1}\delta\left(\frac{z}{y}\right) = 0,$$

hence the generating field is local with itself.

By The Construction Theorem 2.36, we have the following vertex algebra.

DEFINITION 2.38. The *free boson* or *(rank-1) Heisenberg vertex algebra* $(\mathsf{H}, Y, \mathbf{1})$ consists of:

(i) the vector space (and $\widehat{\mathfrak{h}}$-module) H

(ii) the distinguished vacuum vector $\mathbf{1} = 1 \otimes 1$,

(iii) the state field correspondence map

$$Y(\cdot, z) : \mathsf{H} \to (\mathrm{End}\,\mathsf{H})[[z, z^{-1}]] \tag{2.66}$$

$$Y(a_{-n_1}\cdots a_{-n_\ell}\mathbf{1}, z) = {\circ\atop\circ}\,\frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1} a(z)\cdots\frac{1}{(n_\ell-1)!}\left(\frac{d}{dz}\right)^{n_\ell-1} a(z)\,{\circ\atop\circ},$$

for all $\ell \in \mathbb{Z}_{\geq 0}$ and $n_1 \geq \cdots \geq n_\ell > 0$.

In fact, H can also be given the structure of a vertex operator algebra.

DEFINITION 2.39. The *(rank-1) Heisenberg vertex operator algebra* $(\mathsf{H}, Y, \mathbf{1}, \omega)$ consists of:

(i) the (rank-1) Heisenberg vertex algebra $(\mathsf{H}, Y, \mathbf{1})$,

(ii) the conformal vector

$$\omega = \frac{1}{2}a_{-1}^2\mathbf{1}. \tag{2.67}$$

For the moment, we will not prove that $\omega = \frac{1}{2}a_{-1}^2\mathbf{1}$ is a conformal vector, but it can be verified directly, similarly to what we will do below for its modules. The choice of conformal vector for $(\mathsf{H}, Y, \mathbf{1})$ is not unique, and different choices give different central charges in general. The central charge corresponding to (2.67) is $c = 1$.

From now on, we use H to refer to the Heisenberg vertex *operator* algebra $(\mathsf{H}, Y, \mathbf{1}, \omega)$. Similarly to the construction of H, induction of $\widehat{\mathfrak{h}}_{\geq 0}$-modules gives H-modules.

DEFINITION 2.40. For each $\lambda \in \mathbb{C}$, define (F^λ, Y_λ) consisting of:

(i) the vector space (and $\widehat{\mathfrak{h}}$-module)

$$F^\lambda = \mathcal{U}(\widehat{\mathfrak{h}}) \otimes_{\widehat{\mathfrak{h}}_{\geq 0}} \mathbb{C}_\lambda, \tag{2.68}$$

where $\mathbb{C}_\lambda = \mathbb{C}$ is the $\widehat{\mathfrak{h}}_{\geq 0}$ module with action $\mathbf{k}\cdot 1 = 1$, $a_n \cdot 1 = 0$, for $n \in \mathbb{Z}_{>0}$, and $a_0 \cdot 1 = \lambda$ (we denote by $v_\lambda = 1 \otimes 1$, the highest weight vector),

(ii) the map

$$Y_\lambda(\cdot, z) : \mathsf{H} \to (\mathrm{End}\,F^\lambda)[[z, z^{-1}]] \tag{2.69}$$

$$Y_\lambda(a_{-n_1}\cdots a_{-n_\ell}\mathbf{1},z) = \mathop{\circ}\limits^{\circ} \frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1} a(z)\cdots \frac{1}{(n_\ell-1)!}\left(\frac{d}{dz}\right)^{n_\ell-1} a(z) \mathop{\circ}\limits^{\circ}.$$

Note that we will always write vectors in F^λ with respect to the PBW-basis

$$\{a_{-n_1}\cdots a_{-n_k}v_\nu \mid n_1 \geq \cdots \geq n_k \geq 1,\ k \in \mathbb{Z}_{\geq 0}\}. \tag{2.70}$$

We can see that F^0 is the vacuum module of H. In fact, the family $\{(F^\lambda, Y_\lambda)\}_{\lambda\in\mathbb{C}}$ uniquely classifies all the irreducible H-modules, as vertex operator algebra modules (Definition 2.7), up to isomorphism. Note that this classification does not hold for the vertex algebra modules of H, viewed as a vertex algebra, since the definition for vertex operator algebra modules is much more restrictive.

REMARK 2.41. We do not have room to prove that the F^λ, $\lambda \in \mathbb{C}$, are H-modules and furthermore exhaust the irreducible H-modules up to isomorphism. This can be found in Section 6.3 of [LL04], but our preferred method involves computing the *Zhu algebra* (see [Zhu96] for the definition) corresponding to H. Results from [FZ92] and [Zhu96] can be used create a one-to-one correspondence between the irreducible vertex *operator* algebra modules of H and the finite-dimensional irreducible modules of $\mathcal{U}(\mathfrak{gl}_1) \cong \mathbb{C}[a_0]$, the Zhu algebra of H. This correspondence is given by the induction (2.68), with inverse its restriction to its highest-weight space. $\triangle$

In order to develop intuition for vertex operator algebras, we will now demonstrate some key features of the Heisenberg vertex operator algebra and its modules. By use of the commutation relations of $\mathcal{U}(\widehat{\mathfrak{h}})$, the normal ordered product of the field $a(z)$ with itself gives the following normal ordering on modes:

$$\mathop{\circ}\limits^{\circ} a_m a_n \mathop{\circ}\limits^{\circ} = \begin{cases} a_m a_n & \text{if } m < 0, \\ a_n a_m & \text{if } m \geq 0. \end{cases} \tag{2.71}$$

It then follows that

$$\mathop{\circ}\limits^{\circ} a_m a_n \mathop{\circ}\limits^{\circ} = a_m a_n = a_n a_m, \qquad \text{when } m+n \neq 0, \tag{2.72}$$

$$\mathop{\circ}\limits^{\circ} a_{-n} a_n \mathop{\circ}\limits^{\circ} = \mathop{\circ}\limits^{\circ} a_n a_{-n} \mathop{\circ}\limits^{\circ} = a_{-n}a_n, \qquad \text{when } n > 0, \tag{2.73}$$

where we have use the commutation relations of $\widehat{\mathfrak{h}}$ for (2.72).

Let $\lambda \in \mathbb{C}$. We can find the Virasoro modes by computing

$$\begin{aligned} Y_\lambda(\omega,z) &= \frac{1}{2} \mathop{\circ}\limits^{\circ} \sum_{m\in\mathbb{Z}} a_m z^{-m-1} \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \mathop{\circ}\limits^{\circ} = \sum_{k\in\mathbb{Z}} \sum_{\substack{m,n\in\mathbb{Z}\\ m+n=k}} \frac{1}{2} \mathop{\circ}\limits^{\circ} a_m a_n \mathop{\circ}\limits^{\circ} z^{-k-2} \\ &= \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{1}{2} \mathop{\circ}\limits^{\circ} a_m a_{k-m} \mathop{\circ}\limits^{\circ} z^{-k-2} = \sum_{k\in\mathbb{Z}} L(k) z^{-k-2}. \end{aligned}$$

Specifically,

$$L(0) = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n \qquad \text{and} \qquad L(k) = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_m a_{k-m} \quad \text{for } k \neq 0. \tag{2.74}$$

Note that $L(-1) = d$, as required. Recall that the sums in (2.74) become finite when acting on any $v \in F^\lambda$ and, hence, belong to End F^λ. When performing calculations with the $L(n)$ as elements in the associative algebra End F^λ, we should always evaluate them acting on an arbitrary vector in F^λ. However, for brevity, we will often omit this and understand that there is an implicit test vector being acted on.

We can also see how $L(0)$ grades F^λ. Let $m \in \mathbb{Z}_{>0}$. Then,

$$\sum_{n>0} a_{-n}a_n a_{-m} = \sum_{\substack{n>0\\ n\neq m}} a_{-n}a_n a_{-m} + a_{-m}a_m a_{-m} = a_{-m}\sum_{n>0} a_{-n}a_n + m.$$

So, a basis vector $a_{-n_1}\cdots a_{-n_\ell}v_\lambda$ in F^λ has conformal weight given by

$$L(0)a_{-n_1}\cdots a_{-n_\ell}v_\lambda = \left(\frac{1}{2}\lambda^2 + \sum_{i=1}^{\ell} n_i\right) a_{-n_1}\cdots a_{-n_\ell}v_\lambda. \tag{2.75}$$

We can also see that the grading restrictions (2.36) and (2.38) are satisfied.

The explicit expressions (2.74) can be used to directly verify that the Virasoro modes satisfy the Virasoro relations. We will omit the intermediate steps, but this is outlined as follows. First, compute for $m, n \in \mathbb{Z}$,

$$[L(m), a_n] = -na_{m+n}, \quad \text{when } m \neq 0 \quad \text{and} \quad [L(0), a_n] = \begin{cases} 0 & \text{if } n = 0, \\ -na_n & \text{if } n < 0, \\ -na_n & \text{if } n > 0. \end{cases}$$

That is, $[L(m), a_n] = -na_{m+n}$, for all $m, n \in \mathbb{Z}$. The Virasoro relations

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n} \qquad \text{for all } m, n \in \mathbb{Z}, \tag{2.76}$$

can then be checked in two cases: $n \neq 0$, $m + n \neq 0$, and $m > 0$, $n = -m$. In the second case, using $\sum_{k=0}^{m-1}(m-k)k = \frac{m^3-m}{6}$ shows where the second term in (2.76) comes from.

Note that in the case for F^0, this shows that ω is indeed a conformal vector. Hence, the same calculation can be used to show that $(\mathsf{H}, Y, \mathbf{1}, \omega)$ is a vertex operator algebra with central charge $c_{\mathsf{H}} = 1$.

A physical interpretation for the free boson is that the a_0-eigenvalue λ is the momentum of the bosonic string. The $L(0)$-eigenvalue is its energy with the $\frac{1}{2}\lambda^2$ term corresponding to the kinetic energy and the $\sum_{i=1}^{\ell} n_i$ term corresponding to the vibrational energy.

Further computations using H will be shown in the next chapter in Section 3.3 when computing intertwining maps. The Heisenberg vertex operator algebra provides intuition for the lattice vertex operator algebras, which we will use in Chapter 5 to construct modular tensor categories.

Chapter 3

The fusion product

In a conformal field theory, the *fields* correspond to vectors in the modules of the theory's holomorphic symmetry algebra V, a vertex operator algebra. A *primary field* corresponds to a highest-weight vector in a V-module, and there is a notion of "fusing" two primary fields. The result can be decomposed into a sum of primary fields and their descendants (fields corresponding to vectors generated from the highest-weight vector). This decomposition can expressed as *fusion rules*:

$$\Phi_i \times \Phi_j = \sum_k \mathcal{N}_{ij}^k \Phi_k \qquad \text{for some } \mathcal{N}_{ij}^k \in \mathbb{Z}_{\geq 0}, \tag{3.1}$$

where i, j, k index the primary fields in the theory, and the *fusion coefficient* $\mathcal{N}_{ij}^k$ is the number of times Φ_k occurs in the *fusion product* of Φ_i and Φ_j.

The notion of fusion has motivated the *fusion product* in a category of V-modules:

$$M_i \boxtimes M_j \cong \bigoplus_k \mathcal{N}_{ij}^k M_k \qquad \text{for some } \mathcal{N}_{ij}^k \in \mathbb{Z}_{\geq 0}, \tag{3.2}$$

where M_i and M_j are V-modules and M_k are irreducible highest-weight V-modules.

Attempts to define a fusion product $\boxtimes$ include [Gab94], a construction using comultiplication and quotients. However, we will use a universal-property-based definition of the $P(w)$-tensor (fusion) product originating from [HL95a], [HL95b] and [HL94].

3.1 Motivation by Lie algebra modules

The idea of motivating vertex operator algebra modules via a Lie algebraic analogy is given in Section 3 of [HL94]. We will give a roundabout way of defining the tensor product for Lie algebra modules. This method motivates the definition of the $P(w)$-tensor product (and the fusion product) for vertex operator algebra modules. It also illustrates how a universal definition for a tensor product can produce a bifunctor.

Recall Definition 2.16 for a Lie algebra $(\mathfrak{g}, Y)$. The Jacobi identity was expressed as

$$Y(x)Y(y)z - Y(y)Y(x)z = Y(Y(x)y)z \qquad \text{for all } x, y, z \in \mathfrak{g}. \tag{3.3}$$

The definition for a $\mathfrak{g}$-module (M, Y_M) came from replacing $z \in \mathfrak{g}$ with $m \in M$ and Y with Y_M where appropriate. Consider the introduction of a second module (N, Y_N). If $y \in \mathfrak{g}$ were to be replaced with $n \in N$, then $Y(y)$ can be replaced with a module map from M to a possibly new module (L, Y_L). Or equivalently, $Y(y)z$ is replaced with $I(y \otimes z)$ where $I : N \otimes M \to L$ is a linear map. Finally, $Y(x)$ needs to be replaced with the appropriate Y_M, Y_N or Y_L, depending on the context. The Jacobi identity then becomes the condition

$$Y_L(x)I(n \otimes m) - I(n \otimes Y_M(x)m) = I(Y_N(x)n \otimes m), \tag{3.4}$$

for all $x \in \mathfrak{g},\ n \in N,\ m \in M$.

Note that the anticommutativity condition (2.27) does not meaningfully translate into a condition. We call a linear map $I : N \otimes M \to L$ satisfying (3.4) an *intertwining map* of type $\binom{L}{N\,M}$. We call the pair $((L, Y_L), I)$ a *product* of N and M. A *product homomorphism* from $((L_1, Y_1), I_1)$ to $((L_2, Y_2), I_2)$ is a $\mathfrak{g}$-module map $\eta : L_1 \to L_2$ such that $\eta \circ I_1 = I_2$. Define the *tensor product* of N and M to be a product $((L_0, Y_0), I_0)$ of N and M satisfying the following universal property.

> For all products $((L, Y_L), I)$ of N and M, there exists a unique product homomorphism from $((L_0, Y_0), I_0)$ to $((L, Y_L), I)$, that is, there exists a unique module homomorphism $\eta : L_0 \to L$ such that the following diagram commutes
>
> $$\begin{tikzcd} N \otimes M \arrow[r, "I_0"] \arrow[dr, "I"'] & L_0 \arrow[d, "\eta"] \\ & L \end{tikzcd} . \tag{3.5}$$

If such a universal tensor product exists, then it is unique up to a unique isomorphism of products. Suppose there are two tensor products $((L_0, Y_0), I_0)$ and $((L_0', Y_0'), I_0')$ of N and M. Then, the universal property gives two unique module homomorphisms $\eta : L_0 \to L_0'$ and $\varepsilon : L_0' \to L_0$ that are inverse to each other since their compositions are uniquely identities, as summarised in the following commutative diagrams.

$$\begin{tikzcd} & L_0 \arrow[d, "\eta"] \arrow[dd, bend left=60, "\mathrm{id}_{L_0}"] \\ N \otimes M \arrow[ur, "I_0"] \arrow[r, "I_0'"] \arrow[dr, "I_0"'] & L_0' \arrow[d, "\varepsilon"] \\ & L_0 \end{tikzcd} \qquad \begin{tikzcd} & L_0' \arrow[d, "\varepsilon"] \arrow[dd, bend left=60, "\mathrm{id}_{L_0'}"] \\ N \otimes M \arrow[ur, "I_0'"] \arrow[r, "I_0"] \arrow[dr, "I_0'"'] & L_0 \arrow[d, "\eta"] \\ & L_0' \end{tikzcd} \tag{3.6}$$

We know that we can endow the vector space $N \otimes M$ with a $\mathfrak{g}$-module structure given by

$$Y_{N \otimes M}(x)(n \otimes m) = Y_N(x)n \otimes m + n \otimes Y_M(x)m \quad \text{for all } x \in \mathfrak{g},\ n \in N,\ m \in M.$$

Then, the Jacobi identity (3.4), for an intertwining map $I : N \otimes M \to L$, becomes the condition for a $\mathfrak{g}$-module homomorphism. Hence, $L_0 = N \otimes M$ with $I_0 = \mathrm{id}_{N\otimes M}$ satisfies (3.4) and the product $(N \otimes M, \mathrm{id}_{N\otimes M})$ is a tensor product of N and M. Given an arbitrary product $((L, Y_L), I)$ of N and M, the unique module homomorphism is an intertwining map I, as seen by

$$\begin{tikzcd} N \otimes M \arrow[r, "I_0=\mathrm{id}_{N\otimes M}"] \arrow[dr, "I"'] & N \otimes M \arrow[d, "\eta=I"] \\ & L \end{tikzcd} . \tag{3.7}$$

It can be seen as only a “fortunate coincidence” that the tensor product of $\mathfrak{g}$-modules coincides with the tensor product of vector spaces.

This definition for the tensor product emulates the standard universal property definition for a tensor product $(X \otimes Y, \otimes)$ of $\mathbb{C}$-vector spaces X and Y. Recall that for each bilinear map $f : X \times Y$, there is a unique linear map $\widetilde{f} : X \otimes Y \to Z$ such that we have the commutative diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\otimes"] \arrow[dr, "f"'] & X \otimes Y \arrow[d, "\widetilde{f}"] \\ & Z \end{tikzcd} . \tag{3.8}$$

However, in the Lie algebraic case, the bilinear maps (or equivalently a linear map from the tensor product) also need to respect the $\mathfrak{g}$-module structure.

The universal property also motivates how to produce a bifunctor from the tensor product. We fix a choice of model $((N \boxtimes M, Y_{N\boxtimes M}), I_{N\boxtimes M})$ for each pair

$$((N, Y_N), (M, Y_M)) \in \mathrm{ob}(\mathfrak{g}\mathsf{-Mod} \times \mathfrak{g}\mathsf{-Mod}).$$

Let $(f, g) \in \hom_{\mathfrak{g}\mathsf{-Mod}\times\mathfrak{g}\mathsf{-Mod}}((N, M), (N', M'))$. Then, $I_{N'\boxtimes M'} \circ (f \otimes g)$ is an intertwining map of type $\binom{L}{N\ M}$ since for all $x \in \mathfrak{g}$, $n \in N$, $m \in M$, we have

$$\begin{aligned} Y_L(x)I_{N'\boxtimes M'}(f \otimes g)(n \otimes m) &= Y_L(x)I_{N'\boxtimes M'}(fn \otimes gm) \\ &= I_{N'\boxtimes M'}(Y_{N'}(x)fn \otimes gm + fn \otimes Y_{M'}(x)gm) \\ &= I_{N'\boxtimes M'}(fY_N(x)n \otimes gm + fn \otimes gY_M(x)m) \\ &= I_{N'\boxtimes M'}(f \otimes g)(Y_N(x)n \otimes m + n \otimes Y_M(x)m). \end{aligned}$$

So, by the universal property of $\mathfrak{g}$-module tensor products, there exists a unique $\mathfrak{g}$-module homomorphism $f \boxtimes g : N \boxtimes M \to N' \boxtimes M'$ such that the following diagram commutes.

$$\begin{tikzcd} N \otimes M \arrow[r, "I_{N\boxtimes M}"] \arrow[d, "f\otimes g"'] & N \boxtimes M \arrow[d, "f\boxtimes g"] \\ N' \otimes M' \arrow[r, "I_{N'\boxtimes M'}"] & N' \boxtimes M' \end{tikzcd} \tag{3.9}$$

We then define the assignment

$$
\begin{aligned}
-\boxtimes- : \mathfrak{g}\text{–}\mathsf{Mod} \times \mathfrak{g}\text{–}\mathsf{Mod} &\to \mathfrak{g}\text{–}\mathsf{Mod} \\
(N, M) \mapsto N \boxtimes M, \quad (f, g) &\mapsto f \boxtimes g.
\end{aligned} \tag{3.10}
$$

It can be shown that $-\boxtimes-$ is a bifunctor similarly as in the proof of Definition/Proposition 3.13. Essentially, $-\boxtimes-$ inherits its functoriality from $-\otimes-$ (which inherits its functoriality from the Cartesian product $-\times-$). If we choose the assignment $-\boxtimes-$ on objects to be that of the usual $\mathfrak{g}$-module tensor product, then $f \boxtimes g = f \otimes g$, as expected.

3.2 The $P(w)$-tensor product

(We will throughout refer to the eight part series 'Logarithmic Tensor Category Theory' [HLZ14; HLZa; HLZb; HLZc; HLZd; HLZe; HLZf; HLZg] as HLZ.)

We now have a method for defining the tensor product of Lie algebra modules using intertwining maps, products and a universal property, albeit in a roundabout way. HLZ uses this as motivation to define analogues of these notions for vertex operator algebra modules. In certain cases, the "tensor product" exists and the category of vertex operator algebra modules naturally has the structure of braided monoidal category.

The target of the intertwining maps will require the following definition.

DEFINITION 3.1. (Definition 2.18 of [HLZ14]) Let $W = \bigoplus_{h\in\mathbb{C}} W_{(h)}$ be a $\mathbb{C}$-graded vector space. The *formal completion* $\overline{W}$ of W with respect to the $\mathbb{C}$-grading is the vector space

$$
\overline{W} = \prod_{h\in\mathbb{C}} W_{(h)}. \tag{3.11}
$$

The projection of $\overline{W}$ onto $W_{(h)}$ will be denoted by π_h. Note that this is a purely formal algebraic notion of completion.

REMARK 3.2. HLZ constructs a theory for *Möbius vertex algebras* and conformal vertex algebras (recall Remark 2.9). The precise definitions can be found in Section 2 of [HLZ14]. In the general case, the algebras are strongly graded with respect to an abelian group. This strong grading breaks up the $\mathbb{C}$-grading into finite dimensional subspaces. The examples in this thesis will only use vertex operator algebras (i.e. conformal vertex algebras *with* grading restrictions). This is exactly the HLZ notion of a conformal vertex algebra strongly graded with respect to the trivial group. Furthermore, HLZ develops the tensor theory for *generalised modules*, where $L(0)$ grades by generalised eigenvalues. Any definitions or theorems taken from HLZ will be presented here in terms of vertex operator algebras and their ordinary modules only.[1] △

[1] Even though we will not be using HLZ to its full general and "logarithmic" potential, we still use HLZ because it is the most current, and moreover complete, exposition of the tensor theory at this present time.

DEFINITION 3.3. (Definition 4.2 of [HLZb]) Let w be a non-zero complex number. Let (M_i, Y_i), for $i = 1, 2, 3$, be V-modules for a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$. A $P(w)$-*intertwining map of type* $\binom{M_3}{M_1\, M_2}$ is a linear map $I : M_1 \otimes M_2 \to \overline{M_3}$ satisfying the following conditions:

(i) (*truncation condition*) for all $m_1 \in M_1,\ m_2 \in M_2,\ h \in \mathbb{C}$,

$$\pi_{h-n} I(m_1 \otimes m_2) = 0 \qquad \text{for } n \in \mathbb{Z} \text{ sufficiently large,} \tag{3.12}$$

(ii) (*Jacobi identity*) for all $v \in V,\ m_1 \in M_1,\ m_2 \in M_2$,

$$\begin{aligned} &x^{-1}\delta\left(\frac{y-w}{x}\right) Y_3(v, y) I(m_1 \otimes m_2) - x^{-1}\delta\left(\frac{w-y}{-x}\right) I(m_1 \otimes Y_2(v, y)m_2) \\ &= w^{-1}\delta\left(\frac{y-x}{w}\right) I(Y_1(v, x)m_1 \otimes m_2). \end{aligned} \tag{3.13}$$

Note that x and y are still formal variables. We will denote the vector space of $P(w)$-*intertwining maps of type* $\binom{M_3}{M_1\, M_2}$ by $I\binom{M_3}{M_1\, M_2}$.

REMARK 3.4. We can compare this to the case of Lie algebra modules for which intertwining maps are defined by adapting the conditions in the definition for modules. Note that obtaining (3.13) from the Jacobi identity (2.4) is the vertex operator algebra theoretic analogue of obtaining (3.4) from the Lie algebra Jacobi identity (3.3). The main difference is that that the intertwining maps for vertex operator algebra modules are defined in terms of a non-zero complex number w. One may expect that the Jacobi identity would have three formal variables, and this is the case for *intertwining operators*, given in Definition 5.1. As discussed in [HL94], the non-zero complex number w represents the third puncture in the Riemann sphere with punctures at 0, ∞ and w. We will not discuss Huang's geometric interpretation of vertex algebra theory in this thesis and, instead, we keep to the algebraic formulation only. $\triangle$

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra.

EXAMPLE 3.5. Let (M_1, Y_1) and (M, Y_M) be V-modules. Consider the zero module $(0, 0)$. Then the linear map

$$0 : 0 \otimes M_1 = M_1 \otimes 0 = 0 \to \overline{M} \tag{3.14}$$

is a $P(w)$-intertwining map of type $\binom{M}{0\, M_1}$ or $\binom{M}{M_1\, 0}$. Note that it is the only $P(w)$-intertwining map of this type since 0 is an initial object for vector spaces. $\diamondsuit$

EXAMPLE 3.6. Let (M_i, Y_i^M), (N_i, Y_i^N), for $i = 1, 2$, and (M, Y_M) be V-modules. Let $I : N_1 \otimes N_2 \to \overline{M}$ be a $P(w)$-intertwining map. Let $f : M_1 \to N_1$ and $g : M_2 \to N_2$ be V-module homomorphisms. Then, the linear map $I \circ (f \otimes g) : M_1 \otimes M_2 \to \overline{M}$ satisfies the following for all $m_1 \in M_1$, $m_2 \in M_2$:

(i) for all $h \in \mathbb{C}$ and sufficiently large $n \in \mathbb{Z}$,

$$\pi_{h-n} I(f \otimes g)(m_1 \otimes m_2) = \pi_{h-n} I(fm_1 \otimes gm_2) = 0,$$

(ii) for all $v \in V$,

$$\begin{aligned}
&x^{-1}\delta\left(\tfrac{y-w}{x}\right)Y_M(v,y)I(f\otimes g)(m_1\otimes m_2) - x^{-1}\delta\left(\tfrac{w-y}{-x}\right)I(f\otimes g)(m_1\otimes Y_2^M(v,y)m_2)\\
&\quad = x^{-1}\delta\left(\tfrac{y-w}{x}\right)Y_M(v,y)I(fm_1\otimes gm_2) - x^{-1}\delta\left(\tfrac{w-y}{-x}\right)I(fm_1\otimes Y_2^N(v,y)gm_2)\\
&\quad = w^{-1}\delta\left(\tfrac{y-x}{w}\right)I(Y_1^N(v,x)fm_1\otimes gm_2)\\
&\quad = w^{-1}\delta\left(\tfrac{y-x}{w}\right)I(fY_1^M(v,x)m_1\otimes gm_2)\\
&\quad = w^{-1}\delta\left(\tfrac{y-x}{w}\right)I(f\otimes g)(Y_1^M(v,x)m_1\otimes m_2).
\end{aligned}$$

In the second and fourth line, we use the property that V-module homomorphisms preserve vertex operators (2.49). The second and third line is the Jacobi identity for the intertwining map I.

So, $I\circ(f\otimes g): M_1\otimes M_2 \to \overline{M}$ is also a $P(w)$-intertwining map. $\Diamond$

The definition of the $P(w)$-tensor product will be in terms of a universal property. As such, it will be category dependent. Let $\mathscr{C}$ be a full subcategory of $V-$Mod.

DEFINITION 3.7. Let $(M_1, Y_1),\ (M_2, Y_2) \in$ ob$(\mathscr{C})$. A *$P(w)$-product* $((M_3, Y_3), I_3)$ *of M_1 and M_2 in $\mathscr{C}$* is a pair consisting of an object (M_3, Y_3) in $\mathscr{C}$ together with a $P(w)$-intertwining I_3 map of type $\binom{M_3}{M_1\ M_2}$.

DEFINITION 3.8. Let $((M_3, Y_3), I_3)$ and $((M_4, Y_4), I_4)$ be $P(w)$-products of M_1 and M_2 in $\mathscr{C}$. A *$P(w)$-product homomorphism* from $((M_3, Y_3), I_3)$ to $((M_4, Y_4), I_4)$ is a V-module homomorphism $\eta: M_3 \to M_4$ such that the extension $\overline{\eta}: \overline{M_3} \to \overline{M_4}$ satisfies the following commutative diagram.

$$\begin{tikzcd}
M_1\otimes M_2 \arrow[r, "I_3"] \arrow[dr, "I_4"'] & \overline{M_3} \arrow[d, "\overline{\eta}"] \\
 & \overline{M_4}
\end{tikzcd} \tag{3.15}$$

The $P(w)$-tensor product will be a $P(w)$-product that is universal in $\mathscr{C}$, put precisely in the following definition.

DEFINITION 3.9. Let $(M_1, Y_1),\ (M_2, Y_2) \in$ ob$(\mathscr{C})$. A *$P(w)$-tensor product of M_1 and M_2 in $\mathscr{C}$* is a $P(w)$-product $((M_0, Y_0), I_0)$ of M_1 and M_2 in $\mathscr{C}$ satisfying the following universal property.

For all $P(w)$-products $((M, Y_M), I)$ of M_1 and M_2 in $\mathscr{C}$, there exists a unique $P(w)$-product homomorphism η from $((M_0, Y_0), I_0)$ to $((M, Y_M), I)$, that is, there exists a unique V-module homomorphism $\eta: M_0 \to M_1$ such that the following diagram commutes.

$$\begin{tikzcd}
M_1\otimes M_2 \arrow[r, "I_0"] \arrow[dr, "I"'] & \overline{M_0} \arrow[d, "\overline{\eta}"] \\
 & \overline{M}
\end{tikzcd} \tag{3.16}$$

If such a model exists we denote it by $((M_1 \boxtimes_{P(w)} M_2, Y_{P(w)}), \boxtimes_{P(w)})$. Since this is unique up to a unique isomorphism, we will usually fix a construction and call it *the* $P(w)$-tensor product of M_1 and M_2 in $\mathscr{C}$. When we are dealing with $P(w)$-tensor products of multiple pairs of objects, we may use superscripts on the intertwining map $\boxtimes_{P(w)}$ so that the maps can be clearly distinguished.

EXAMPLE 3.10. Assume $\mathscr{C}$ contains $(0,0)$, a zero object in $V-$Mod. Let (M_1, Y_1) be in $\mathscr{C}$. Example 3.5 showed that $((0,0),0)$ was a $P(w)$-product of $(0,0)$ and (M_1, Y_1). Then for all $P(w)$-products $((M, Y_M), I)$ of $(0,0)$ and (M_1, Y_1) in $\mathscr{C}$ there is unique $P(w)$-product homomorphism $0 : 0 \to M$ such that the following diagram commutes.

$$\begin{tikzcd} M_1 \otimes M_2 \arrow[r, "0"] \arrow[dr, "I=0"'] & \overline{0} \arrow[d, "\overline{\eta}=0"] \\ & \overline{M} \end{tikzcd} \tag{3.17}$$

Hence $0 \boxtimes_{P(w)} M_1 = 0$ is the $P(w)$-tensor product. Similarly $M_1 \boxtimes_{P(w)} 0 = 0$ as well. $\Diamond$

EXAMPLE 3.11. Assume that the $P(w)$-tensor product exists for all pairs of objects in $\mathscr{C}$. Then the $P(w)$-tensor product distributes over direct sums up to isomorphism (recall Remark 2.32 about the abelian structure of $V-$Mod). This structure is inherited from the fact that vector space tensor products distribute over direct sums. This is summarised in the following commutative diagram:

$$\begin{tikzcd} (\bigoplus_i M_i) \otimes (\bigoplus_j N_j) \arrow[r, "f \cong"] \arrow[drr, "I"'] & \bigoplus_{i,j} M_i \otimes N_j \arrow[r, "\bigoplus_{i,j} \boxtimes^{i,j}_{P(w)}"] & \overline{\bigoplus_{i,j} M_i \boxtimes_{P(w)} N_j} \arrow[d, "\sum_{i,j} \overline{\eta_{i,j}}"] \\ & & \overline{M} \end{tikzcd}, \tag{3.18}$$

where the $\eta_{i,j}$ are defined by

$$\begin{tikzcd} M_i \otimes N_j \arrow[r, "\boxtimes^{i,j}_{P(w)}"] \arrow[d, hook] & \overline{M_i \boxtimes_{P(w)} N_j} \arrow[d, "\eta_{i,j}"] \\ \bigoplus_{i,j} M_i \otimes N_j \arrow[r, "I \circ f^{-1}"] & \overline{M} \end{tikzcd}.$$

$\Diamond$

3.2.1 The $P(w)$-tensor product as a bifunctor

In this section, we demonstrate how to produce a bifunctor out of $P(w)$-tensor products. Assume that the $P(w)$-tensor product exists for all pairs of objects in $\mathscr{C}$ and we have fixed our choice of $P(w)$-tensor product models. We use the universal property (3.16) to define the $P(w)$-tensor product on homomorphisms. This does not seem to be mentioned in the literature, probably because it not needed explicitly for demonstrating the existence of monoidal categories. However, we need it because we are interested in explicitly computing monoidal data in future examples.

DEFINITION 3.12. Let $M_1,\ M_2,\ N_1$ and N_2 be objects in $\mathscr{C}$ with $P(w)$-tensor products $(M_1 \boxtimes_{P(w)} M_2, \boxtimes_{P(w)}^M)$ and $(N_1 \boxtimes_{P(w)} N_2, \boxtimes_{P(w)}^N)$. Let $f_1 : M_1 \to N_1$ and $f_2 : M_2 \to N_2$ be morphisms in $\mathscr{C}$. Define $f_1 \boxtimes_{P(w)} f_2 : M_1 \boxtimes_{P(w)} M_2 \to N_1 \boxtimes_{P(w)} N_2$ as the unique morphism in $\mathscr{C}$ such that the following commutes:

$$\begin{tikzcd} M_1 \otimes M_2 \arrow[r, "\boxtimes_{P(w)}^M"] \arrow[d, "f_1 \otimes f_2"'] & \overline{M_1 \boxtimes_{P(w)} M_2} \arrow[d, "\overline{f_1 \boxtimes_{P(w)} f_2}"] \\ N_1 \otimes N_2 \arrow[r, "\boxtimes_{P(w)}^N"] & \overline{N_1 \boxtimes_{P(w)} N_2} \end{tikzcd} . \tag{3.19}$$

Note that, $\boxtimes_{P(w)}^N \circ (f_1 \otimes f_2)$ is an intertwining map by Example 3.6, so $f_1 \boxtimes_{P(w)} f_2$ exists by the universal property of the $P(w)$-tensor product.

DEFINITION/PROPOSITION 3.13. Assume that, for all pairs of objects in $\mathscr{C}$, their $P(w)$-tensor product exists and we have fixed our choice of $P(w)$-tensor products. Then, the *$P(w)$-tensor product bifunctor* is the functor

$$\begin{gathered} - \boxtimes_{P(w)} - : \mathscr{C} \times \mathscr{C} \to \mathscr{C} \\ (M_1, M_2) \mapsto M_1 \boxtimes_{P(w)} M_2 \qquad (f_1, f_2) \mapsto f_1 \boxtimes_{P(w)} f_2. \end{gathered} \tag{3.20}$$

Proof. We show that $- \boxtimes_{P(w)} -$ is indeed a functor. Assume $M_1, M_2 \in \mathrm{ob}(\mathscr{C})$. Then, the following commutes.

$$\begin{tikzcd} M_1 \otimes M_2 \arrow[r, "\boxtimes_{P(w)}^M"] \arrow[d, "\mathrm{id}_{M_1} \otimes \mathrm{id}_{M_2}"'] & \overline{M_1 \boxtimes_{P(w)} M_2} \arrow[d, "\overline{\mathrm{id}_{M_1 \boxtimes_{P(w)}^M M_2}}"] \\ M_1 \otimes M_2 \arrow[r, "\boxtimes_{P(w)}^M"] & \overline{M_1 \boxtimes_{P(w)} M_2} \end{tikzcd} \tag{3.21}$$

So, $\mathrm{id}_{M_1} \boxtimes_{P(w)} \mathrm{id}_{M_2} = \mathrm{id}_{M_1 \boxtimes_{P(w)} M_2}$.

Assume $f_i : L_i \to M_i, g_i : M_i \to N_i \in \hom(\mathscr{C})$, for $i = 1, 2$. Then, the following commutes.

$$\begin{tikzcd} L_1 \otimes L_2 \arrow[r, "\boxtimes_{P(w)}^L"] \arrow[d, "f_1 \otimes f_2"'] \arrow[dd, bend right=60, "(g_1 \circ f_1) \otimes (g_2 \circ f_2)"'] & \overline{L_1 \boxtimes_{P(w)} L_2} \arrow[d, "\overline{f_1 \boxtimes_{P(w)} f_2}"] \arrow[dd, bend left=60, "\overline{(g_1 \boxtimes_{P(w)} g_2) \circ (f_1 \boxtimes_{P(w)} f_2)}"] \\ M_1 \otimes M_2 \arrow[r, "\boxtimes_{P(w)}^M"] \arrow[d, "g_1 \otimes g_2"'] & \overline{M_1 \boxtimes_{P(w)} M_2} \arrow[d, "\overline{g_1 \boxtimes_{P(w)} g_2}"] \\ N_1 \otimes N_2 \arrow[r, "\boxtimes_{P(w)}^N"] & \overline{N_1 \boxtimes_{P(w)} N_2} \end{tikzcd} \tag{3.22}$$

So, $(g_1 \circ f_1) \boxtimes_{P(w)} (g_2 \circ f_2) = (g_1 \boxtimes_{P(w)} g_2) \circ (f_1 \boxtimes_{P(w)} f_2)$. $\square$

EXAMPLE 3.14. Inherited from the $\mathbb{C}$-vector space tensor product, the $P(w)$-tensor product is bilinear on morphisms. Let $f, g : M_1 \to N_1$ and $h : M_2 \to N_2$ be morphisms in $\mathscr{C}$ and let $\alpha, \beta \in \mathbb{C}$. Then, the following diagram commutes:

$$\begin{tikzcd}[ampersand replacement=\&] M_1 \otimes M_2 \arrow[r, "\boxtimes^M_{P(w)}"] \arrow[d, "(\alpha f+\beta g)\otimes h=\alpha(f\otimes h)+\beta(g\otimes h)"'] \& \overline{M_1 \boxtimes_{P(w)} M_2} \arrow[d, "\overline{\alpha(f\boxtimes h)+\beta(g\boxtimes h)}"] \\ N_1 \otimes N_2 \arrow[r, "\boxtimes^N_{P(w)}"] \& \overline{N_1 \boxtimes_{P(w)} N_2} \end{tikzcd} . \tag{3.23}$$

So, $(\alpha f + \beta g) \boxtimes_{P(w)} h = \alpha(f \boxtimes_{P(w)} h) + \beta(g \boxtimes_{P(w)} h)$. Linearity in the second argument also follows from the bilinearity of $\otimes$. Important examples include

$$\alpha\,\mathrm{id}_{M_1} \boxtimes_{P(w)} \beta\,\mathrm{id}_{M_2} = \alpha\beta\,\mathrm{id}_{M_1 \boxtimes_{P(w)} M_2}, \quad 0 \boxtimes_{P(w)} f = 0 \quad \text{and} \quad f \boxtimes_{P(w)} 0 = 0.$$

Since $\mathscr{C}$ has $\mathbb{C}$-enriched hom-spaces, by Schur's lemma, we can combine the bilinearity with Example 3.11 to know the $P(w)$-tensor product on all morphisms of a semisimple category. $\Diamond$

3.3 An example: the Heisenberg vertex operator algebra

We discuss the $P(w)$-products in the category $\mathsf{H}-\mathsf{Mod}_{\mathsf{SS}}$, the semisimple category of finite direct sums of Heisenberg vertex operator algebra modules. (We do this so as to not worry about any modules that may not by completely reducible.) On a "physics level of rigour", the $P(w)$-intertwining maps between the irreducible free boson modules are readily found (see Section 6.3.2 of [DMS97]) as

$$I(v_\lambda \otimes v_\mu) = \Phi_\lambda(w) v_\mu = w^{\lambda\mu} \exp\left(\lambda \sum_{n=1}^{\infty} \frac{a_{-n}}{n} w^n\right) v_{\lambda+\mu}, \tag{3.24}$$

for any $\lambda, \mu \in \mathbb{C}$. That is, I is the only $P(w)$-intertwining map of type $\binom{F^{\lambda+\mu}}{F^\lambda \;\; F^\mu}$, up to scalar multiple. Any $P(w)$-intertwining map of type $\binom{F^{\nu}}{F^\lambda \;\; F^\mu}$, for $\nu \neq \lambda + \mu$, is shown to be zero. With all $P(w)$-intertwining maps known, we can find that $F^\lambda \boxtimes_{P(w)} F^\mu = F^{\lambda+\mu}$ is a suitable model for the $P(w)$-tensor product.

The proof that we give for the following proposition demonstrates an explicit computation using the Jacobi identity. It sets upper bounds for the dimensions of the spaces of $P(w)$-intertwining maps.

PROPOSITION 3.15. For $\lambda, \mu, \nu \in \mathbb{C}$, the following holds:

(i) If $I \in I\binom{F^{\nu}}{F^\lambda \;\; F^\mu}$, then I is uniquely determined by $I(v_\lambda \otimes v_\mu)$.

(ii) If $I \in I\binom{F^{\nu}}{F^\lambda \;\; F^\mu}$, then $I(v_\lambda \otimes v_\mu) = C \exp\left[\lambda \sum_{n>0} \frac{a_{-n}}{n} w^n\right] v_\nu$ for a constant $C \in \mathbb{C}$.

(iii) If $\lambda + \mu \neq \nu$, then $I\binom{F^{\nu}}{F^\lambda \;\; F^\mu} = 0$.

Proof. Let $\lambda, \mu, \nu \in \mathbb{C}$.

(i) Let $I \in I\binom{F^\nu}{F^\lambda\ F^\mu}$, which contains at least 0. Then, I satisfies the Jacobi identity (3.13) for all $v \in \mathsf{H}$. Choosing $v = a = a_{-1}\mathbf{1}$ and explicitly writing out the delta functions in three variables (from Definition A.18) gives

$$\begin{aligned} x^{-1}\sum_{n\in\mathbb{Z}}\sum_{k\geq 0}\binom{n}{k}y^{n-k}(-w)^k x^{-n}\sum_{i\in\mathbb{Z}}a_i y^{-i-1}I(m_{(1)}\otimes m_{(2)}) \\ - x^{-1}\sum_{n\in\mathbb{Z}}\sum_{k\geq 0}\binom{n}{k}w^{n-k}(-y)^k(-x)^{-n}\sum_{i\in\mathbb{Z}}I(m_{(1)}\otimes a_i m_{(2)})y^{-i-1} \\ = w^{-1}\sum_{n\in\mathbb{Z}}\sum_{k\geq 0}\binom{n}{k}y^{n-k}(-x)^k w^{-n}\sum_{i\in\mathbb{Z}}I(a_i m_{(1)}\otimes m_{(2)})x^{-i-1}. \end{aligned}$$

Then, after some simplification, we have

$$\begin{aligned} \sum_{n\in\mathbb{Z}}\sum_{k\geq 0}\sum_{i\in\mathbb{Z}}(-1)^k w^k\binom{n}{k}a_i I(m_{(1)}\otimes m_{(2)})x^{-n-1}y^{n-k-i-1} \\ - \sum_{n\in\mathbb{Z}}\sum_{k\geq 0}\sum_{i\in\mathbb{Z}}(-1)^{k-n}w^{n-k}\binom{n}{k}I(m_{(1)}\otimes a_i m_{(2)})x^{-n-1}y^{k-i-1} \\ = \sum_{n\in\mathbb{Z}}\sum_{k\geq 0}\sum_{i\in\mathbb{Z}}(-1)^k w^{-n-1}\binom{n}{k}I(a_i m_{(1)}\otimes m_{(2)})x^{k-i-1}y^{n-k}. \end{aligned} \tag{3.25}$$

Let $j \in \mathbb{Z}_{>0}$. In (3.25), we substitute $m_{(1)} = v_\lambda$ and equate the coefficients of $x^{-1}y^{j-1}$ to obtain

$$a_{-j}I(v_\lambda\otimes m_{(2)}) - I(v_\lambda\otimes a_{-j}m_{(2)}) = \lambda w^{-j}I(v_\lambda\otimes m_{(2)}). \tag{3.26}$$

We can start with $m_{(2)} = v_\mu$ and inductively obtain $I(v_\lambda \otimes a_{-j_1}\cdots a_{-j_\ell}v_\mu)$ in terms of $I(v_\lambda\otimes v_\mu)$, for all $a_{-j_1}\cdots a_{-j_\ell}v_\mu$ in our PBW-basis for F^μ. Also, in (3.25), we can leave $m_{(1)}$ and $m_{(2)}$ as arbitrary, and equate the coefficients of $x^{j-1}y^{-1}$ to obtain

$$\begin{aligned} \sum_{k\geq 0}(-1)^k w^k\binom{-j}{k}a_{-j-k}I(m_{(1)}\otimes m_{(2)}) \\ - \sum_{k\geq 0}(-1)^{k+j}w^{-j-k}\binom{-j}{k}I(m_{(1)}\otimes a_k m_{(2)}) = I(a_{-j}m_{(1)}\otimes m_{(2)}). \end{aligned} \tag{3.27}$$

We can start with $m_{(1)} = v_\lambda$ and inductively obtain $I(a_{-j_1}\cdots a_{-j_\ell}v_\mu\lambda\otimes m_{(2)})$ in terms of $I(v_\lambda\otimes -)$, for all $a_{-j_1}\cdots a_{-j_\ell}v_\lambda$ in the PBW-basis (2.70) for F^λ. Since $I(v_\lambda\otimes -)$ is completely determined by $I(v_\lambda\otimes v_\mu)$, then so is $I(-\otimes -)$.

(ii) Let $j\in\mathbb{Z}_{>0}$. In (3.25), substitute $m_{(1)} = v_\lambda$ and $m_{(2)} = v_\mu$, and equate the coefficients of $x^{-1}y^{-j-1}$ to obtain

$$a_j I(v_\lambda\otimes v_\mu) = \lambda w^j I(v_\lambda\otimes v_\mu). \tag{3.28}$$

We will write $I(v_\lambda\otimes v_\mu)$ in terms of the PBW-basis (2.70), giving

$$I(v_\lambda\otimes v_\mu) = (C + C(1)a_{-1} + C(1,1)a_{-1}a_{-1} + C(2)a_{-2} + \cdots)v_\nu. \tag{3.29}$$

That is, the coefficient of $a_{-n_1}\cdots a_{-n_k}v_\nu$ is $C(n_1,\ldots,n_k)$. For computational convenience, we define

$$D(k_\ell,\ldots,k_1) := C(n_1,\ldots,n_k), \qquad \text{where } a_{-n_1}\cdots a_{-n_k} = a_{-\ell}^{k_\ell}\cdots a_{-1}^{k_1}, \tag{3.30}$$

for some $\ell \geq 1$, $k_1,\ldots,k_\ell \geq 0$, with ℓ chosen minimally so that $k_\ell \neq 0$. Since $[a_n, a_{-n}^k] = kna_{-n}^{k-1}$, we have

$$a_n^k a_{-n}^k v_\nu = a_n^{k-1}(a_{-n}^k a_n + kna_{-n}^{k-1})v_\nu = k!n^k v_\nu. \tag{3.31}$$

Hence, for $\ell \geq 1$, for $k_1,\ldots,k_\ell \geq 0$, we have

$$a_\ell^{k_\ell}\cdots a_1^{k_1}a_{-\ell}^{k_\ell}\cdots a_{-1}^{k_1}v_\nu = k_\ell!\ell^{k_\ell}\cdots k_1!1^{k_1}v_\nu, \tag{3.32}$$

Then,

$$\begin{aligned}(w^\ell\lambda)^{k_\ell}\cdots(w^1\lambda)^{k_1}I[v_\lambda\otimes v_\mu] &= a_\ell^{k_\ell}\cdots a_1^{k_1}I[v_\lambda\otimes v_\mu]\\ &= 0+\cdots+0+k_\ell!\ell^{k_\ell}\cdots k_1!1^{k_1}D(k_\ell,\ldots,k_1)v_\nu+\cdots.\end{aligned}$$

So, the coefficient of $a_{-\ell}^{k_\ell}\cdots a_{-1}^{k_1}v_\nu$ can be related to C by

$$w^{\ell k_\ell+\cdots+1k_1}\lambda^{k_\ell+\cdots+k_1}C = k_\ell!\cdots k_1!\ell^{k_\ell}\cdots 1^{k_1}D(k_\ell,\ldots,k_1)\,, \tag{3.33}$$

or equivalently

$$w^{n_1+\cdots+n_k}\lambda^k C = \frac{k!}{N(n_1,\ldots,n_k)}n_1\cdots n_k C(n_1,\ldots,n_k)\,, \tag{3.34}$$

where $N(n_1,\ldots,n_k) = \frac{k!}{k_1!\cdots k_\ell!}$ is the number of unique ways to order the numbers $(n_1,\ldots,n_k) \in \mathbb{Z}_{>0}^k$. Hence,

$$C(n_1,\ldots,n_k) = N(n_1,\ldots,n_k)\frac{\lambda^k}{k!}\frac{w^{n_1+\cdots+n_k}}{n_1\cdots n_k}C\,. \tag{3.35}$$

Thus, we arrive at the unique expression

(3.36)

$$I(v_\lambda\otimes v_\mu) = C\sum_{k=0}^\infty\frac{\lambda^k}{k!}\sum_{n_1,\ldots,n_k=1}^\infty\frac{w^{n_1+\cdots+n_k}}{n_1\cdots n_k}a_{-n_1}\cdots a_{-n_k}v_\nu = C\exp\left[\lambda\sum_{n=1}^\infty\frac{a_{-n}}{n}w^n\right]v_\nu.$$

(iii) Assume $\lambda+\mu\neq\nu$. In equation (3.25), we equate the coefficients of $x^{-1}y^{-1}$ to obtain

$$\begin{aligned}&\nu I(m_{(1)}\otimes m_{(2)}) - \mu I(m_{(1)}\otimes m_{(2)}) = a_0 I(m_{(1)}\otimes m_{(2)}) - I(m_{(1)}\otimes a_0 m_{(2)})\\ &= \sum_{k\geq 0}\left((-1)^k w^k\binom{0}{k}a_{-k}I(m_{(1)}\otimes m_{(2)}) - (-1)^k w^{-k}\binom{0}{k}I(m_{(1)}\otimes a_k m_{(2)})\right)\\ &= \sum_{k\geq 0}(-1)^k w^{-k}\binom{k-1}{k}I(a_k m_{(1)}\otimes m_{(2)}) = I(a_0 m_{(1)}\otimes m_{(2)}) = \lambda I(m_{(1)}\otimes m_{(2)}).\end{aligned}$$

So, we have the relation

$$(\nu-\mu-\lambda)I(m_{(1)}\otimes m_{(2)}) = 0. \tag{3.37}$$

Since $\nu-\mu-\lambda\neq 0$, we have $I = 0$. □

The previous proof highlights that the Jacobi identity is, after simplifying and equating coefficients, an infinite number of equations. It is restrictive enough to force a unique form for an intertwining map of type $\binom{F^\nu}{F^\lambda\ F^\mu}$, up to scalar multiples, that is completely determined by its image on the highest weight vectors of F^μ and F^ν. Moreover, it shows that this map is zero when $\lambda + \mu \neq \nu$. However, the Jacobi identity is so strong that it becomes impractical to verify that $I : F^\lambda \otimes F^\mu \to \overline{F^\nu}$, defined by (3.26), (3.27) and (3.36), is actually an intertwining operator when $\lambda + \mu = \nu$. It is worth remarking here that the truncation condition (3.12) is satisfied by I since F^ν has zero weight spaces $F^\nu_{(h)}$ when $\mathrm{Re}(h) < \mathrm{Re}(\frac{\nu^2}{2})$. Hence, we are left to verify that I satsifies the Jacobi identity. We will not prove this—instead we refer the reader to the proof in Sections 3 and 4 of [TZ12].

PROPOSITION 3.16. The space $I\binom{F^\nu}{F^\lambda\ F^\mu}$ is one dimensional when $\nu = \lambda + \mu$, with basis vector $I : v_\lambda \otimes v_\mu \mapsto \exp\left[\lambda \sum_{n=1}^{\infty} \frac{a_{-n}}{n} w^n\right] v_{\lambda+\mu}$, and zero dimensional when $\nu \neq \lambda + \mu$.

REMARK 3.17. Trying to verify the Jacobi identity *directly* was not fruitful for us. It was too computationally difficult without a different approach as in [TZ12]. We also tried two other methods for computing the $P(w)$-intertwining maps for the Heisenberg vertex operator algebra.

The Zhu algebra $A(V)$ is an associative algebra derived from a vertex operator algebra V, introduced in [Zhu90]. We used the definitions and results in [FZ92] and [Zhu96] to compute the Zhu algebra for the Heisenberg vertex operator algebra. It was not too difficult to compute that $A(\mathsf{H}) \cong \mathcal{U}(\mathfrak{gl}_1)$, as algebras over $\mathbb{C}$. However, the next step of this method is to compute the dimensions of the space of intertwining maps by computing an isomorphic vector space using left-, right- and bi-modules of the $A(\mathsf{H})$. In our attempts, we were only able to find an upper bound for dimensions of the spaces, but not the dimension itself. Hence, we arrived at the same result as (ii) and (iii) of Proposition 3.15. For more on the functoriality of the Zhu algebra construction, see [FB04]. Equivalences between certain categories of V-modules and $A(V)$-modules are also discussed. We do not know when or if this equivalence can be extended to a monoidal equivalence.

The double-dual construction, as used in HLZ, was not any simpler than trying to verify the Jacobi identity directly. When attempting to use the construction explicitly for the free boson, there were too many equations to have to verify directly. This highlights the difficulty of *computing* $P(w)$-intertwining maps and tensor products. Unlike the Lie algebra case, there is no explicit general construction, nor guaranteed existence, of the tensor product of two modules. $\triangle$

3.4 Constructing the monoidal data

We have now seen that the $P(w)$-tensor product can produce the bifunctor $- \boxtimes_{P(w)} -$ on a suitable category of vertex operator algebra modules. Compare this to the case for the tensor product of $\mathfrak{g}$-modules, for a Lie algebra $\mathfrak{g}$. Here, the category $\mathfrak{g}-\mathrm{Mod}$ can be given the structure of a symmetric braided monoidal category (recall Example B.23). Keeping to the "Lie algebra – vertex operator algebra" analogy, it is then natural to ask if $(\mathscr{C}, \boxtimes_{P(w)})$ has a canonical (possibly non-symmetric) braided monoidal category structure.[2]

HLZ gives a natural construction of braided monoidal structure for certain categories of modules of vertex operator algebras. In fact, the theory applies to the more general setting of generalised modules and Möbius vertex algebras. Since our examples are for vertex operator algebras, we will present the relevant theorems specialised to the vertex operator algebra case with modules. We will briefly discuss the procedure in HLZ in what follows.

In [HLZc], models for the $P(w)$-tensor products are constructed with duals. Given two V-modules M_1 and M_2, consider $(M_1 \otimes M_2)^*$, the algebraic dual of their vector space tensor product. A subspace $M_1 \boxbackslash_{P(w)} M_2$ of $(M_1 \otimes M_2)^*$ is found by imposing certain *compatibility conditions* that ensure the subspace has the structure of a V-module. The restricted dual is then taken to give the object model $M_1 \boxtimes_{P(w)} M_2 = (M_1 \boxbackslash_{P(w)} M_2)'$. The universal $P(w)$-intertwining map is defined using duals and $(M_1 \boxtimes_{P(w)} M_2, \boxtimes_{P(w)})$ is shown to satisfy the universal property for the $P(w)$-tensor product in $V\!-\!\mathsf{Mod}$. One consequence of this construction is that the full subcategory $\mathscr{C}$ of V-modules must be assumed to be closed under restricted duals. The double-dual construction utilises formal completions since the graded formal completion is the dual of the restricted dual. It is explained in [HLZb] that $P(w)$-intertwining maps correspond to intertwining operators, a formal analogue of $P(w)$-intertwining maps, which are independent of w. As a result, the $P(w)$-tensor product is independent, up to isomorphism, of the choice of $w \in \mathbb{C}^\times$. Hence, for concreteness, the following definition is made.

DEFINITION 3.18. The *fusion product* $\boxtimes = \boxtimes_{P(1)}$ is chosen to be a $P(w)$-tensor product bifunctor at $w = 1$.

Other attempts to define the fusion product include Gaberdiel's construction in [Gab94]. This model attempts to define the fusion product as a quotient of $M_1 \otimes M_2$ by certain relations. The double-dual approach formalises this idea since the dual notion of a quotient object is a subobject. An explanation of the connection between these two methods, as well as an overview of the fusion product, can be found in [KR19].

In [HLZe], the category $\mathscr{C}$ of V-modules is assumed to satisfy the following:

(A1) The category $\mathscr{C}$ is a full subcategory of $V\!-\!\mathsf{Mod}$.

(A2) Every module in $\mathscr{C}$ only has real weights.

(A3) The category $\mathscr{C}$ is closed under images, closed under restricted duals, closed under finite direct sums and under the $P(w)$-tensor product for some $w \in \mathbb{C}^\times$.

In [HLZg], the category $\mathscr{C}$ is assumed to further satisfy the following:

(A4) The adjoint module of V is an object of $\mathscr{C}$.

[2]It is also natural to ask this question when motivating vertex operator algebras from a conformal-field-theoretic perspective. The possibility of braided monoidal structure was first remarked in [MS88]

(A5) Every object of $\mathscr{C}$ is a direct sum of irreducible objects of $\mathscr{C}$ and there are only finitely many irreducible C_1-cofinite objects of $\mathscr{C}$, up to isomorphism.

DEFINITION 3.19. (Definition 11.5 [HLZf]) Let V be a vertex operator algebra and let W be a V-module. Let

$$V_+ = \bigoplus_{n=1}^{\infty} V_{(n)}, \tag{3.38}$$

$$C_1(W) = \text{span}\{u_{-1}w : u \in V_+,\ w \in W\}. \tag{3.39}$$

If W satisfies the C_1-*cofiniteness condition*

$$\dim\left(W/C_1(W)\right) < \infty, \tag{3.40}$$

then we say that W is C_1-*cofinite*.

REMARK 3.20. Condition (A5) is actually stronger than the condition used in HLZ. Originally, this condition, Assumption 12.12 of [HLZg], is an analytic property used for defining the associators. We, however, do not have the space nor need to go into these details. △

As suggested by the use of a non-zero complex number w in the $P(w)$-tensor product, there is a departure from the formal algebraic world and an entrance into an analytic one. In [HLZa], the notion of an *intertwining operator* is defined (see Definition 5.1). These operators capture the monodromy of paths in $\mathbb{C}^\times$. The associators, unitors and braiding are constructed using analytic notions. The $P(w)$-tensor product (for all $w \in \mathbb{C}^\times$) is used to compute the associator, even though the final fusion product has $w = 1$.

THEOREM 3.21. (Theorem 12.15 of [HLZg]) Let V be a vertex operator algebra. Any category $\mathscr{C}$ of V-modules satisfying assumptions **(A1)** – **(A5)** is canonically a braided monoidal category.

The canonical construction for the associator, unitors and braiding will be discussed in Section 5.2. But for now, we give an example.

EXAMPLE 3.22. Recall, for the Heisenberg vertex operator algebra, the $P(w)$-intertwining maps of type $\binom{F^\nu}{F^\lambda\ F^\mu}$, for $\lambda, \mu, \nu \in \mathbb{C}$, are uniquely determined by

$$v_\lambda \otimes v_\mu \mapsto \delta_{\lambda+\mu=\nu} \exp\left(\lambda \sum_{n=1}^{\infty} \frac{a_{-n}}{n} w^n\right) v_\nu, \tag{3.41}$$

up to scalar multiples. It follows that

$$\left(F^\lambda \boxtimes_{P(w)} F^\mu = F^{\lambda+\mu},\ \boxtimes_{P(w)} : v_\lambda \boxtimes_{P(w)} v_\mu = \exp\left(\lambda \sum_{n=1}^{\infty} \frac{a_{-n}}{n} w^n\right) v_{\lambda+\mu}\right) \tag{3.42}$$

are $P(w)$-tensor products in the category of H-modules.

Define $\mathsf{H}{-}\mathsf{Mod}'$ to be the full category of H-modules $\{F^\lambda : \lambda \in \mathbb{C}\}$, the zero module and the finite direct sums

$$\bigoplus_{\lambda \in L} F^\lambda, \quad \text{with } L \text{ a finite multiset of } \mathbb{C}. \tag{3.43}$$

Note that we chose these objects so $\mathsf{H}{-}\mathsf{Mod}'$ is skeletal and closed under the fusion product, hence we can easily understand the explicit braided monoidal data.

We have used the HLZ method to produce the following braided monoidal structure for $\mathsf{H}{-}\mathsf{Mod}'$ (defined on the simple objects and then extended semisimply):

(i) the tensor product is the fusion product $-\boxtimes- : F^\lambda \boxtimes F^\mu = F^{\lambda+\mu}$,

(ii) the vacuum module F^0 is the unit object,

(iii) the associator, left unitor and right unitor are respectively

$$\mathcal{A}_{F^\lambda, F^\mu, F^\nu} = \mathrm{id}_{F^{\lambda+\mu+\nu}} : F^\lambda \boxtimes (F^\mu \boxtimes F^\nu) \to (F^\lambda \boxtimes F^\mu) \boxtimes F^\nu, \tag{3.44}$$

$$l_{F^\lambda} = \mathrm{id}_{F^\lambda} : F^0 \boxtimes F^\lambda \to F^\lambda \quad \text{and} \quad r_{F^\lambda} = \mathrm{id}_{F^\lambda} : F^\lambda \boxtimes F^0 \to F^\lambda, \tag{3.45}$$

(iv) the braiding is

$$\mathcal{R}_{F^\lambda, F^\mu} = e^{i\pi\lambda\mu}\,\mathrm{id}_{F^{\lambda+\mu}} : F^\lambda \boxtimes F^\mu \to F^\mu \boxtimes F^\lambda. \tag{3.46}$$

Observe that this braided monoidal structure is strict, but not symmetric. In Chapter 5, we will see non-strict monoidal structures arising from lattice vertex operator algebras. ◊

In Chapter 5, we will exploit the HLZ construction to explicitly compute braided monoidal data for the case of lattice vertex operator algebras. We omit our computation for the braided monoidal data in Example 3.22, however, it can be computed using similar methods to those we will detail in Chapter 5.

REMARK 3.23. Note that in Example 3.22, the category $\mathsf{H}{-}\mathsf{Mod}'$ has modules with non-real weights and infinitely many irreducible objects. That is, not all assumptions **(A1)** – **(A5)** are satisfied. Nonetheless, the Heisenberg vertex operator algebra was able to produce a braided monoidal category using the same methods as HLZ. This would suggest that the results of HLZ can be generalised. △

HLZ gives a proof of existence of braided monoidal structure for categories satisfying assumptions **(A1)** – **(A5)**. However, it does not prove the existence of structures such as rigidity and modularity (which will be defined in the next chapter). Rigidity and modularity are guaranteed when stronger assumptions are made, as in [Hua08], which will be discussed in Chapter 5. In the next chapter, we will discuss such additional structures on (braided) monoidal categories.

Chapter 4

Modular tensor categories

In [MS88], rational conformal field theories were noticed to produce structures, similar to those of braided monoidal categories, with polynomial equations that imply the Verlinde formula. These structures came to be known as *modular tensor categories*. In [Hua08], it was proven that a certain class of vertex operator algebras have categories of modules with a natural modular structure.

This chapter aims to give a concise presentation of the definitions needed to build modular tensor categories, following [EGNO16]. Along the way, we will provide small examples using Hopf algebras (see Appendix C for the necessary definitions). In Section 4.5, we will define homomorphisms for modular tensor categories.

Some of the following proofs are readily found using string diagrams, for example [Kas95]. This graphical calculus is a powerful tool for computations and also illuminates the "algebra" of (modular) tensor categories. However, we choose to present proofs using commutative diagrams or equations so as to not suppress associators and unitors as well as to clarify when naturality, functoriality and other conditions are being used. It should be noted that we originally computed these diagrams by first using graphical techniques, then converted to commutative diagrams and finally added back in the associators and unitors. But unfortunately, due to a lack of space, we will not include any string diagrams.

4.1 Rigidity

Finite dimensional vector spaces have duals with evaluation and coevalution morphisms. Left and right duals bring this notion to a general monoidal categorical setting.

DEFINITION 4.1. Let $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category and let X be an object in $\mathscr{C}$. A *left dual* of X is the triple $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ consisting of the following data:

(i) an object X^* in $\mathscr{C}$,

(ii) a morphism $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$ in $\mathscr{C}$, called the *evaluation*,

(iii) a morphism $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$, called the *coevaluation*,

satisfying the following conditions:

(i) the following composes to id_X

$$X \xrightarrow{\lambda_X^{-1}} \mathbb{1} \otimes X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha^{-1}_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes \mathbb{1} \xrightarrow{\rho_X} X, \tag{4.1}$$

(ii) the following composes to id_{X^*}

$$X^* \xrightarrow{\rho_{X^*}^{-1}} X^* \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} \mathbb{1} \otimes X^* \xrightarrow{\lambda_{X^*}} X^*. \tag{4.2}$$

We will define right duals below in Definition 4.8. But first, we will give some examples of left duals.

EXAMPLE 4.2. In any monoidal category, the left dual of the unit object $\mathbb{1}$ can be chosen to be $\mathbb{1}$ with evaluation $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ and coevaluation $\lambda_{\mathbb{1}}^{-1} = \rho_{\mathbb{1}}^{-1}$. The left dual conditions are satisfied by coherence. $\Diamond$

EXAMPLE 4.3. Let $(H, \nabla, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra over a field $\Bbbk$. Denote by $H-\mathsf{Mod}$ the category of modules of (H, ∇, η) as an associative unital algebra, together with H-module homomorphisms. We can endow $H-\mathsf{Mod}$ with a monoidal structure as follows:

(i) given objects M and N in $H-\mathsf{Mod}$, the tensor product $-\otimes-$ assigns the H-module with underlying vector space $M \otimes_{\Bbbk} N$ and H-action defined by the $(H \otimes H)$-module action through the coproduct, that is,

$$h \cdot (m \otimes_{\Bbbk} n) = \Delta(h)(m \otimes_{\Bbbk} n) = \sum_{(h)} (h' \cdot m) \otimes_{\Bbbk} (h'' \cdot n) \qquad h \in H, m \in M, n \in N. \tag{4.3}$$

On morphisms f and g in $H-\mathsf{Mod}$, the tensor product is $f \otimes g = f \otimes_{\Bbbk} g$.

(ii) the unit $\mathbb{1}$ is $\Bbbk$ equipped with the *trivial* module structure through the counit:

$$h \cdot 1 = \varepsilon(h)1 \qquad \text{for all } h \in H, \text{where } 1 \in \Bbbk. \tag{4.4}$$

(iii) the associator and unitors are the canonical isomorphisms adopted from $\Bbbk-\mathsf{Vect}$:

$$\begin{aligned} \alpha_{L,M,N} : l \otimes_{\Bbbk} (m \otimes_{\Bbbk} n) \mapsto (l \otimes_{\Bbbk} m) \otimes_{\Bbbk} n \qquad & l \in L,\ m \in M,\ n \in N, \\ \lambda_M : 1 \otimes_{\Bbbk} m \mapsto m \quad \text{and} \quad \rho_M : m \otimes_{\Bbbk} 1 \mapsto m \qquad & m \in M, \end{aligned} \tag{4.5}$$

for all H-modules L, M, N.

It follows that $(H-\mathsf{Mod}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category.

The forgetful functor $F : H-\mathsf{Mod} \to \Bbbk-\mathsf{Vect}$, equipped with $J : F(-) \otimes_{\Bbbk} F(-) \to F(- \otimes -)$ and $\varphi : \Bbbk \to F(\Bbbk)$ as identities, is a monoidal functor because we equipped $H-\mathsf{Mod}$ with the same monoidal structure as $\Bbbk-\mathsf{Vect}$.

So far, we have only utilised the coproduct for constructing the tensor product and the counit for constructing the unit object, so the previous construction for the monoidal structure on $H-\mathsf{Mod}$ holds for bialgebras $(H, \nabla, \eta, \Delta, \varepsilon)$ as well.

To utilise the antipode, we restrict to the full subcategory $H-\mathsf{Mod}_{\mathsf{fd}}$ of finite dimensional H-modules. Observe that $(H-\mathsf{Mod}_{\mathsf{fd}}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal subcategory. For each object M in $H-\mathsf{Mod}_{\mathsf{fd}}$ we give the vector space dual $M^* = \hom_{\Bbbk-\mathsf{Vect}}(M, \Bbbk)$ the structure of an H-module by defining

$$(h \cdot \phi)(m) = \phi(S(h) \cdot m) \qquad \text{for all } h \in H,\ \phi \in M^*,\ m \in M. \tag{4.6}$$

Define the evaluation map

$$\mathrm{ev}_M : M^* \otimes M \to \Bbbk, \quad \phi \otimes_{\Bbbk} m \mapsto \phi(m) \qquad \text{for all } \phi \in M^*,\ m \in M, \tag{4.7}$$

to be the evaluation of m by ϕ on the left. Fix a basis $\{v_i\}_{i=1}^{\dim M}$ of M with dual basis $\{v^i\}_{i=1}^{\dim M}$ of M^*. Define the coevaluation map

$$\mathrm{coev}_M : \Bbbk \to M \otimes M^*, \quad 1 \mapsto \sum_{i=1}^{\dim M} v_i \otimes_{\Bbbk} v^i. \tag{4.8}$$

This definition is independent of the choice of basis. The maps ev_M and coev_M are H-module homomorphisms and satisfy the left dual conditions (4.1) and (4.2). It follows that $(M^*, \mathrm{ev}_M, \mathrm{coev}_M)$ is a left dual of M in $H-\mathsf{Mod}_{\mathsf{fd}}$. $\Diamond$

Even though left duals are not defined via a universal property, we still have the following fact.

PROPOSITION 4.4. Left duals are unique up to a unique isomorphism.

Proof. Let $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ and $(\overline{X^*}, \overline{\mathrm{ev}_X}, \overline{\mathrm{coev}_X})$ be left duals of X. Define the morphisms

$$\begin{gathered}\phi : X^* \xrightarrow{\rho_{X^*}^{-1}} X^*\mathbb{1} \xrightarrow{\mathrm{id}_{X^*} \otimes \overline{\mathrm{coev}_X}} X^*(X\overline{X^*}) \xrightarrow{\alpha_{X^*,X,\overline{X^*}}} (X^*X)\overline{X^*} \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{\overline{X^*}}} \mathbb{1}\overline{X^*} \xrightarrow{\lambda_{\overline{X^*}}} \overline{X^*}, \\ \psi : \overline{X^*} \xrightarrow{\rho_{\overline{X^*}}^{-1}} \overline{X^*}\mathbb{1} \xrightarrow{\mathrm{id}_{\overline{X^*}} \otimes \mathrm{coev}_X} \overline{X^*}(XX^*) \xrightarrow{\alpha_{\overline{X^*},X,X^*}} (\overline{X^*}X)X^* \xrightarrow{\overline{\mathrm{ev}_X} \otimes \mathrm{id}_{X^*}} \mathbb{1}X^* \xrightarrow{\lambda_{X^*}} X^*.\end{gathered} \tag{4.9}$$

(Here and below, we shall often omit $\otimes$ symbols, to be deduced from context, for brevity.) Then, ϕ and ψ are inverse to each other. These isomorphisms are unique in the sense that they satisfy

$$\mathrm{ev}_X = \overline{\mathrm{ev}_X} \circ (\phi \otimes \mathrm{id}_X) \qquad \text{and} \qquad \mathrm{coev}_X = (\mathrm{id}_X \otimes \psi) \circ \overline{\mathrm{coev}_X}. \tag{4.10}$$

We will omit the commutative diagrams needed for this proof because they are very large and do not demonstrate techniques different than the other proofs shown. $\square$

DEFINITION 4.5. Let X and Y be objects in $\mathscr{C}$ with left duals. Let $f : X \to Y$ be a morphism in $\mathscr{C}$. The *left dual* of f is the morphism $f^* : Y^* \to X^*$ defined by the composition

$$Y^* \xrightarrow{\rho^{-1}_{Y^*}} Y^*\mathbb{1} \xrightarrow{\mathrm{id}_{Y^*}\otimes \mathrm{coev}_X} Y^*(XX^*) \xrightarrow{\alpha_{Y^*,X,X^*}} (Y^*X)X^* \xrightarrow{(\mathrm{id}_{Y^*}\otimes f)\otimes \mathrm{id}_{X^*}} (Y^*Y)X^* \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}} \mathbb{1}X^* \xrightarrow{\lambda_{X^*}} X^*. \tag{4.11}$$

REMARK 4.6. This notation and terminology is appropriate because the properties of the vector space dual, such as $(g \circ f)^* = f^* \circ g^*$ and $(\mathrm{id}_X)^* = \mathrm{id}_{X^*}$, generalise nicely. Hence, if a $\mathscr{C}$ is a monoidal category such that every object has a left dual, then a choice of left duals gives a contravariant functor $(-)^* : \mathscr{C} \to \mathscr{C}$. △

PROPOSITION 4.7. Let (F, J, φ) be a monoidal functor from a monoidal category $\mathscr{C}$ to another monoidal category $\mathscr{D}$. Let X be an object in $\mathscr{C}$ with a left dual object X^*. Then, $F(X^*)$ is a left dual object of $F(X)$.

Proof. Let $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$ be a left dual of X. Define

$$\mathrm{ev}_{F(X)} = \varphi^{-1} \circ F(\mathrm{ev}_X) \circ J_{X^*,X} \quad \text{and} \quad \mathrm{coev}_{F(X)} = J^{-1}_{X,X^*} \circ F(\mathrm{coev}_X) \circ \varphi.$$

Then, the following diagram commutes.

(4.12)

$$\begin{tikzcd}
FX \arrow[r, "F\lambda^{-1}"] \arrow[d, "\lambda^{-1}"'] & F(\mathbb{1}X) \arrow[r, "F(\mathrm{coev}_X\otimes \mathrm{id})"] & F((XX^*)X) \arrow[r, "F\alpha^{-1}"] \arrow[d, "J^{-1}"] & F(X(X^*X)) \arrow[r, "F(\mathrm{id}\otimes \mathrm{ev}_X)"] & F(X\mathbb{1}) \arrow[r, "F\rho"] \arrow[d, "J^{-1}"] & FX \\
\mathbb{1}FX \arrow[r, "\varphi\otimes \mathrm{id}"] \arrow[drr, bend right, "\mathrm{coev}_{FX}\otimes \mathrm{id}"'] & F\mathbb{1}FX \arrow[u, "J"] \arrow[r, "F\mathrm{coev}_X\otimes \mathrm{id}"] & F(XX^*)FX \arrow[d, "J^{-1}\otimes \mathrm{id}"] & FXF(X^*X) \arrow[u, "J"] \arrow[r, "\mathrm{id}\otimes F\mathrm{ev}_X"] & FXF\mathbb{1} \arrow[r, "\mathrm{id}\otimes\varphi^{-1}"] & FX\mathbb{1} \arrow[u, "\rho"'] \\
 & & (FXFX^*)FX \arrow[r, "\alpha^{-1}"'] & FX(FX^*FX) \arrow[u, "\mathrm{id}\otimes J"] \arrow[urr, bend right, "\mathrm{id}\otimes \mathrm{ev}_{FX}"'] & &
\end{tikzcd}$$

Subscripts can be deduced from context. Here we have used the compatibility of associators and unitors, as well as the naturality of J. The top composes to $\mathrm{id}_{F(X)}$. Similarly,

$$\lambda_{F(X^*)} \circ (\mathrm{ev}_{F(X)} \otimes \mathrm{id}_{F(X)}) \circ \alpha_{F(X^*),F(X),F(X^*)} \circ (\mathrm{id}_{F(X^*)} \otimes \mathrm{coev}_{F(x)}) \circ \rho^{-1}_{F(X^*)} = \mathrm{id}_{F(X^*)}\,.$$

So, $(F(X^*), \mathrm{ev}_{F(X)}, \mathrm{coev}_{F(X)})$ is a left dual of $F(X)$ in $\mathscr{D}$. □

We can also define right duals to satisfy the condition that an object is a right dual of its left dual, and vice versa.

DEFINITION 4.8. Let $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. Let X be an object in $\mathscr{C}$. A *right dual* of X is the triple $({}^*X, \mathrm{ev}'_X, \mathrm{coev}'_X)$ consisting of the following data:

(i) an object ${}^*X$ in $\mathscr{C}$,

(ii) a morphism $\mathrm{ev}'_X : X \otimes {}^*X \to \mathbb{1}$ in $\mathscr{C}$, called the *(right dual) evaluation*,

(iii) a morphism $\mathrm{coev}'_X : \mathbb{1} \to {}^*X \otimes X$, called the *(right dual) coevaluation*,

satisfying the following conditions:

(i) the following composes to id_X

$$X \xrightarrow{\rho^{-1}_X} X \otimes \mathbb{1} \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}'_X} X \otimes ({}^*X \otimes X) \xrightarrow{\alpha_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{\mathrm{ev}'_X \otimes \mathrm{id}_X} \mathbb{1} \otimes X \xrightarrow{\lambda_X} X, \tag{4.13}$$

(ii) the following composes to id_{X^*}

(4.14)
$$^*X \xrightarrow{\lambda^{-1}_{^*X}} \mathbb{1} \otimes {}^*X \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{id}_{^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{\alpha^{-1}_{^*X, X, ^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\mathrm{id}_{^*X} \otimes \mathrm{ev}'_X} {}^*X \otimes \mathbb{1} \xrightarrow{\rho_{^*X}} {}^*X.$$

DEFINITION 4.9. Let X and Y be objects in $\mathscr{C}$ with right duals. Let $f : X \to Y$ be a morphism in $\mathscr{C}$. The *right dual* of f is the morphism ${}^*f : {}^*Y \to {}^*X$ defined by the composition

(4.15)
$${}^*Y \xrightarrow{\lambda^{-1}_{^*Y}} \mathbb{1}{}^*Y \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{id}_{^*Y}} ({}^*XX){}^*Y \xrightarrow{\alpha^{-1}_{^*X, X, ^*Y}} {}^*X(X{}^*Y) \xrightarrow{\mathrm{id}_{^*X} \otimes (f \otimes \mathrm{id}_{^*Y})} {}^*X(Y{}^*Y) \xrightarrow{\mathrm{id}_{^*X} \otimes \mathrm{ev}'_X} {}^*X\mathbb{1} \xrightarrow{\rho_{^*X}} {}^*X.$$

All statements for left duals have analogues for right duals.

EXAMPLE 4.10. We consider the setting of Example 4.3, but further assume that the Hopf algebra H has an invertible antipode. For each object M in $H-\mathsf{Mod}_{\mathsf{fd}}$, we give the vector space dual ${}^*M = \hom_{\mathbb{k}-\mathsf{Vect}}(M, \mathbb{k})$ the structure of an H-module by

$$(h \cdot \phi)(m) := \phi(S^{-1}(h) \cdot m) \qquad \text{for all } h \in H,\ \phi \in {}^*M,\ m \in M. \tag{4.16}$$

This is a right dual with evaluation and coevaluation

$$\mathrm{ev}'_M : m \otimes_{\mathbb{k}} \phi \mapsto \phi(m) \quad \text{and} \quad \mathrm{coev}'_M : 1 \mapsto \sum_i v^i \otimes_{\mathbb{k}} v_i. \tag{4.17}$$

The inverse of S is needed in (4.16) so that the evaluation and coevaluation are H-module homomorphisms. For example, for $h \in H$, $m \in M$, $\phi \in {}^*M$,

$$\begin{aligned}
\mathrm{ev}'_M\big(\Delta(h)(m \otimes_{\mathbb{k}} \phi)\big) &= \mathrm{ev}'_M\Bigg(\sum_{(h)} h' \cdot m \otimes_{\mathbb{k}} h'' \cdot \phi\Bigg) = \sum_{(h)} (h'' \cdot \phi)(h' \cdot m) \\
&= \sum_{(h)} \big((S(h')h'') \cdot \phi\big)(m) = \varepsilon(h)\phi(m).
\end{aligned}$$
◊

DEFINITION 4.11. An object in a monoidal category is called *rigid* if it has both left and right duals. If every object in monoidal category is rigid, we say that the monoidal category is *rigid*.

EXAMPLE 4.12. If a Hopf algebra H has an invertible antipode, then $H-\mathsf{Mod}$ is a rigid monoidal category. ◊

REMARK 4.13. Left (and right) duality is not a categorification of invertible elements in a monoid since evaluation and coevaluation are not necessarily isomorphisms. They can, however, be used to define a categorified notion of inverses that we will not need here. We will use the notion of left or right duality as a generalisation of the dual of finite-dimensional vector spaces. As such, we should also have a generalised notion of trace and dimension. △

DEFINITION 4.14. Let $\mathscr{C}$ be a rigid monoidal category. Let X be an object in $\mathscr{C}$ and f a morphism in $\hom_{\mathscr{C}}(X, X^{**})$. The *left categorical trace*, or *left quantum trace*, of f is an endomorphism of the unit object defined by the composition

$$\mathrm{Tr}^L(f) : \mathbb{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{f \otimes \mathrm{id}_{X^*}} X^{**} \otimes X^* \xrightarrow{\mathrm{ev}_{X^*}} \mathbb{1}. \tag{4.18}$$

We can similarly define the *right categorical trace*.

REMARK 4.15. Consider the case when $\mathrm{End}_{\mathscr{C}}(\mathbb{1}) \cong_{\Bbbk-\mathsf{Vect}} \Bbbk$, for example, when $\mathscr{C}$ is $\Bbbk$-linear abelian with $\mathbb{1}$ a simple object and $\Bbbk$ is algebraically closed. We can identify the traces $\mathrm{Tr}^L(f)$, for $f : X \to X^{**}$, with elements in $\Bbbk$. Canonically, this is done by identifying $\mathrm{id}_{\mathbb{1}}$ with $1 \in \Bbbk$. Hence, this notion of the left categorical trace generalises the notion of the vector space trace for an *endomorphism* of a finite dimensional vector space. To further refine this analogy, we would need to associate each endomorphism, of each object X, to a morphism from X to X^{**}. △

DEFINITION 4.16. Let $\mathscr{C}$ be a rigid monoidal category. Assume we have a prescription for assigning a specific left dual object X^* to each object X in $\mathscr{C}$. A *pivotal structure* on $\mathscr{C}$ is a natural isomorphism $a : \mathrm{id}_{\mathscr{C}} \Rightarrow (-)^{**}$ satisfying

$$a_{X \otimes Y} = a_X \otimes a_Y \qquad \text{for all } X, Y \in \mathrm{ob}(\mathscr{C}). \tag{4.19}$$

A similar definition can be made by replacing rigidity with *left rigidity*, the property of having all left duals. However, we will just assume that $\mathscr{C}$ is rigid.

DEFINITION 4.17. A rigid monoidal category $\mathscr{C}$ is called *pivotal* if it is equipped with a pivotal structure a. We call $(\mathscr{C}, a)$ a *pivotal category*.

DEFINITION 4.18. Let $(\mathscr{C}, a)$ be a pivotal category. Let X be an object in $\mathscr{C}$. The *left categorical dimension*, or *left quantum dimension*, of X with respect to a is defined as

$$\dim_a^L(X) = \mathrm{Tr}^L(a_X). \tag{4.20}$$

DEFINITION 4.19. Let X be an object in a pivotal category $(\mathscr{C}, a)$. The *left categorical trace* or *left quantum trace* of an endomorphism f of X is defined as

$$\mathrm{Tr}_a^L(f) : \mathbb{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{f \otimes \mathrm{id}_{X_*}} X \otimes X^* \xrightarrow{a_X \otimes \mathrm{id}_{X_*}} X^{**} \otimes X^* \xrightarrow{\mathrm{ev}_{X^*}} \mathbb{1}. \tag{4.21}$$

That is, $\mathrm{Tr}_a^L(f) = \mathrm{Tr}^L(a_X \circ f)$ and $\dim_a^L(X) = \mathrm{Tr}_a^L(\mathrm{id}_X)$.

EXAMPLE 4.20. Consider the category $\Bbbk-\mathsf{Vect}_{\mathsf{fd}}$ of finite dimensional $\Bbbk$-vector spaces. The left dual of $X \in \mathrm{ob}(\Bbbk-\mathsf{Vect}_{\mathsf{fd}})$ can be chosen to be the dual vector space X^* together with the evaluation and coevaluation morphisms defined in (4.7) and (4.8). (The right duals can be defined similarly.) Define the following pivotal structure for $\Bbbk-\mathsf{Vect}$:

$$a_X^{\Bbbk-\mathsf{Vect}} : X \to X^{**}, \quad x \mapsto a_X^{\Bbbk-\mathsf{Vect}}(x) : \varphi \mapsto \varphi(x), \ \varphi \in X^*. \tag{4.22}$$

This is the canonical identification of a finite dimensional vector space with its double dual.

Given an endomorphism $f : X \to X$ in $\Bbbk-\mathsf{Vect}_{\mathsf{fd}}$, the left categorical trace of f is

$$1 \mapsto \sum_{i=1}^{\dim X} v_i \otimes_{\Bbbk} v^i \mapsto \sum_{i=1}^{\dim X} f v_i \otimes_{\Bbbk} v^i \mapsto \sum_{i=1}^{\dim X} a_X^{\Bbbk-\mathsf{Vect}}(f v_i) \otimes_{\Bbbk} v^i \mapsto \sum_{i=1}^{\dim X} v^i(f v_i),$$

where $\{v_i\}_{i=1}^{\dim X}$ is a basis of X with dual basis $\{v^i\}_{i=1}^{\dim X}$. After identification with $\Bbbk$, this agrees with the usual notion of the trace of f. Moreover, the left categorical dimension can be seen to agree with the dimension of X. $\Diamond$

EXAMPLE 4.21. Consider a pivotal category where the component of the pivotal structure at the unit can be chosen to be the identity, such as in Example 4.20. Recall that in Example 4.2, it was shown that a left dual of the unit is the unit with unitors as evaluation and coevaluation morphisms. Hence, the left categorical dimension of the unit is the identity. Moreover, if $\mathrm{End}(\mathbb{1}) \cong_{\Bbbk-\mathsf{Vect}} \Bbbk$, then we identify the dimension with 1. $\Diamond$

PROPOSITION 4.22. Let $f : X \to Y$ and $g : Y \to X$ be morphisms in pivotal category $(\mathscr{C}, a)$. Then,

$$\mathrm{Tr}_a^L(g \circ f) = \mathrm{Tr}_a^L(f \circ g), \tag{4.23}$$

which generalises a property of traces for linear maps.

To prove this, we will need a preparatory lemma.

LEMMA 4.23. Let $f : X \to Y$ be a morphism in a rigid monoidal category $\mathscr{C}$. Then, the following diagrams commute.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\mathrm{coev}_X"] \arrow[d, "\mathrm{coev}_Y"'] & X \otimes X^* \arrow[d, "f \otimes \mathrm{id}_{X^*}"] \\
Y \otimes Y^* \arrow[r, "\mathrm{id}_Y \otimes f^*"'] & Y \otimes X^*
\end{tikzcd}
\qquad
\begin{tikzcd}
Y^* \otimes X \arrow[r, "\mathrm{id}_{Y^*} \otimes f"] \arrow[d, "f^* \otimes \mathrm{id}_X"'] & Y^* \otimes Y \arrow[d, "\mathrm{ev}_Y"] \\
X^* \otimes X \arrow[r, "\mathrm{ev}_X"'] & \mathbb{1}
\end{tikzcd} \tag{4.24}$$

Proof. The commutativity of the coevaluation diagram is a consequence of the following commutative diagrams.

$$\begin{tikzcd}
XX^* \arrow[rr, "f\otimes\mathrm{id}"] & & YX^* \arrow[r, "\lambda^{-1}\otimes\mathrm{id}"] \arrow[dr, "\lambda^{-1}"'] & (\mathbb{1}Y)X^* \arrow[r, "(\mathrm{coev}_Y\otimes\mathrm{id})\otimes\mathrm{id}"] \arrow[d, "\alpha^{-1}"] & ((YY^*)Y)X^* \arrow[dd, "\alpha^{-1}"] \\
 & & \mathbb{1}(XX^*) \arrow[r, "\mathrm{id}\otimes(f\otimes\mathrm{id})"'] & \mathbb{1}(YX^*) \arrow[dr, "\mathrm{coev}_Y\otimes(\mathrm{id}\otimes\mathrm{id})"] & \\
\mathbb{1} \arrow[uu, "\mathrm{coev}_X"] \arrow[r, "\lambda^{-1}=\rho^{-1}"] \arrow[dd, "\mathrm{coev}_Y"'] & \mathbb{1}\mathbb{1} \arrow[ur, "\mathrm{id}\otimes\mathrm{coev}_X"] \arrow[dr, "\mathrm{coev}_Y\otimes\mathrm{id}"'] & & & (YY^*)(YX^*) \arrow[dd, "\alpha^{-1}"] \\
 & & (YY^*)\mathbb{1} \arrow[r, "(\mathrm{id}\otimes\mathrm{id})\otimes\mathrm{coev}_X"] \arrow[d, "\alpha^{-1}"] & (YY^*)(XX^*) \arrow[ur, "(\mathrm{id}\otimes\mathrm{id})\otimes(f\otimes\mathrm{id})"] & \\
YY^* \arrow[urr, "\rho^{-1}"] \arrow[rr, "\mathrm{id}\otimes\rho^{-1}"'] & & Y(Y^*\mathbb{1}) \arrow[r, "\mathrm{id}\otimes(\mathrm{id}\otimes\mathrm{coev}_X)"'] & Y(Y^*(XX^*)) \arrow[r, "\mathrm{id}\otimes(\mathrm{id}(f\otimes\mathrm{id}))"'] & Y(Y^*(YX^*))
\end{tikzcd} \tag{4.25}$$

$$
\begin{tikzcd}
((YY^*)Y)X^* \arrow[r, "\alpha^{-1}\otimes\mathrm{id}"] \arrow[d, "\alpha^{-1}"'] & (Y(Y^*Y))X^* \arrow[r, "(\mathrm{id}\otimes\mathrm{ev}_Y)\otimes\mathrm{id}"] \arrow[dd, "\alpha^{-1}"] & (Y\mathbb{1})X^* \arrow[dd, "\alpha^{-1}"] \arrow[dr, "\rho\otimes\mathrm{id}"] & \\
(YY^*)(YX^*) \arrow[d, "\alpha^{-1}"'] & & & YX^* \\
Y(Y^*(YX^*)) \arrow[r, "\mathrm{id}\otimes\alpha"'] & Y((Y^*Y)X^*) \arrow[r, "\mathrm{id}\otimes(\mathrm{ev}_Y\otimes\mathrm{id})"'] & Y(\mathbb{1}X^*) \arrow[ur, "\mathrm{id}\otimes\rho"'] &
\end{tikzcd}
\tag{4.26}
$$

Here we have used the naturality of the associator and unitors, and the coherence and bifunctoriality of the tensor product. The right side of (4.25) attaches to the left side of (4.26). The top composes to $(f\otimes \mathrm{id}_{X^*})\circ \mathrm{coev}_X$ and the bottom composes to $(\mathrm{id}_Y \otimes f^*)\circ \mathrm{coev}_Y$. Similar techniques can be used for the evaluations. $\square$

Proof of Proposition 4.22. We use the functoriality of the tensor product, the naturality of the pivot and Lemma 4.23:

$$
\begin{aligned}
\mathrm{Tr}^L_a(g\circ f) &= \mathrm{ev}_{X^*}\circ (a_X\otimes \mathrm{id}_{X^*})\circ (g\otimes \mathrm{id}_{X^*})\circ (f\otimes \mathrm{id}_{X^*})\circ \mathrm{coev}_X\\
&= \mathrm{ev}_{X^*}\circ (a_X\otimes \mathrm{id}_{X^*})\circ (g\otimes \mathrm{id}_{X^*})\circ (\mathrm{id}_Y\otimes f^*)\circ \mathrm{coev}_Y\\
&= \mathrm{ev}_{X^*}\circ (\mathrm{id}_{X^{**}}\otimes f^*)\circ (a_X\otimes \mathrm{id}_{Y^*})\circ (g\otimes \mathrm{id}_{Y^*})\circ \mathrm{coev}_Y\\
&= \mathrm{ev}_{Y^*}\circ (f^{**}\otimes \mathrm{id}_{Y^*})\circ (a_X\otimes \mathrm{id}_{Y^*})\circ (g\otimes \mathrm{id}_{Y^*})\circ \mathrm{coev}_Y\\
&= \mathrm{ev}_{Y^*}\circ (a_Y\otimes \mathrm{id}_{Y^*})\circ (f\otimes \mathrm{id}_{Y^*})\circ (g\otimes \mathrm{id}_{Y^*})\circ \mathrm{coev}_Y\\
&= \mathrm{Tr}^L_a(f\circ g).
\end{aligned}
$$
$\square$

EXAMPLE 4.24. The following definition can be found in [BBG] for finite dimensional Hopf algebras, though it holds for infinite dimensional Hopf algebras with invertible antipodes as well.

DEFINITION 4.25. Let $(H,\nabla,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra over a field $\Bbbk$. A *pivot* of H is an element g in H that satisfies the following conditions:

(i) the element g is *group like*, that is $\Delta(g) = g\otimes g$,

(ii) $S^2(x) = gxg^{-1}$ for all $x\in H$.

If g is a pivot, we call $(H,\nabla,\eta,\Delta,\varepsilon,S,g)$ a *pivotal Hopf algebra*.

Given a pivot g of H, the rigid monoidal category $H\mathrm{-Mod}_{\mathsf{fd}}$ is endowed with the pivotal structure a^g defined by

$$
a^g_X : X\to X^{**},\qquad x\mapsto g\cdot a^{\Bbbk-\mathsf{Vect}}_X(x),
\tag{4.27}
$$

where $a^{\Bbbk-\mathsf{Vect}}$ is the pivotal structure for $\Bbbk\mathrm{-Vect}$ defined in (4.20). The assignment a^g is indeed a natural isomorphism and pivotal.

As an explicit example, consider the Hopf algebra $\mathbb{C}[\mathbb{Z}_2]$, where $\mathbb{C}[\mathbb{Z}_2]$ is the group algebra of $\mathbb{Z}_2 = \{e,s\}$ over $\mathbb{C}$. Denote the simple modules by

$$
\begin{aligned}
\mathbb{1} &= \mathbb{C}, & s\cdot 1 &= 1 & &\text{(trivial representation)},\\
M &= \mathbb{C}v_M, & s\cdot v_M &= -v_M & &\text{(alternating/non-trivial representation)}.
\end{aligned}
$$

Then, $\mathbb{1}^* \cong \mathbb{1}$ and $M^* \cong M$. We can choose pivots $g = e$ or $g = s$. The quantum dimensions are

$$(4.28) \qquad \dim_{a^g}(\mathbb{1}) = \mathrm{id}_{\mathbb{1}}, \text{ for } g = e, s \quad \text{and} \quad \dim_{a^g}(M) = \begin{cases} \mathrm{id}_{\mathbb{1}} & \text{if } g = e, \\ -\mathrm{id}_{\mathbb{1}} & \text{if } g = s. \end{cases}$$

For example, we can compute these dimensions as

$$(4.29) \qquad \mathrm{Tr}^L(a^s_M) : 1 \mapsto v_M \otimes v_M^* \mapsto (s \cdot a_M^{\Bbbk-\mathsf{Vect}}(v_M)) \otimes v_M^* \mapsto v_M^*(s \cdot v_M) = -1.$$

Note that the choice of pivot was not unique and that the quantum dimension depends on the choice of pivotal structure. $\Diamond$

DEFINITION 4.26. A pivotal structure a on a rigid monoidal category $\mathscr{C}$ is *spherical* if $\dim_a^L(X) = \dim_a^L(X^*)$ for all objects X in $\mathscr{C}$, or equivalently, if $\mathrm{Tr}_a^L(f) = \mathrm{Tr}_a^R(f)$ for all endomorphisms in $\mathscr{C}$. A rigid monoidal category $\mathscr{C}$ is called *spherical* if it is equipped with a spherical structure a.

EXAMPLE 4.27. Let $(H, \nabla, \eta, \Delta, \epsilon, S, g)$ be a pivotal Hopf algebra satisfying $S(g) = g$, like in Example 4.24. Let X an finite-dimensional H-module, with basis $\{v_i\}_{i=1}^{\dim X}$, dual basis $\{v^i\}_{i=1}^{\dim X}$ and double dual basis identified with the basis. Then,

$$\dim_{a^g}(X) = \sum_{i=1}^{\dim X} v^i(gv_i) = \sum_{i=1}^{\dim X} v^i(S(g)v_i) = \sum_{i=1}^{\dim X} v_i(gv^i) = \dim_{a^g}(X^*).$$

So, $(H{-}\mathsf{Mod}_{\mathsf{fd}}, a^g)$ is spherical. $\Diamond$

In a spherical category, where left and right traces coincide, we shall write $\dim_a = \dim_a^L$ and $\mathrm{Tr}_a = \mathrm{Tr}_a^L$. In Section 4.4, we will see spherical structures arising in *ribbon fusion categories*.

4.2 Ribbon categories

Ribbon categories combine duals with braiding by "twisting" morphisms. They can be used to produce pivotal and spherical structures. The ribbon structure contributes as an underlying structure of modular tensor categories.

DEFINITION 4.28. A *twist* (or a *balancing transformation*) on a braided rigid monoidal category $\mathscr{C}$ is an natural isomorphism $\theta : \mathrm{id}_{\mathscr{C}} \Rightarrow \mathrm{id}_{\mathscr{C}}$ such that

$$(4.30) \qquad \theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y} \qquad \text{for all } X, Y \in \mathrm{ob}(\mathscr{C}).$$

A twist is called a *ribbon structure* if

$$(4.31) \qquad (\theta_X)^* = \theta_{X^*} \qquad \text{for all } X \in \mathrm{ob}(\mathscr{C}).$$

A *ribbon category* is a braided rigid monoidal category equipped with a ribbon structure.

EXAMPLE 4.29. Recall the definition of quasi-triangular Hopf algebras from Appendix C. A quasi-triangular Hopf algebra $(H, \nabla, \eta, \Delta, \varepsilon, S, R)$ admits a canonical braiding on the monoidal category $H-\mathsf{Mod}_{\mathsf{fd}}$, given by

$$c_{X,Y}(x\otimes y)=\tau_{X,Y}(R\cdot(x\otimes y)),\qquad \text{where } \tau_{X,Y}(x\otimes y)=y\otimes x, \tag{4.32}$$

for all $x\in X,\ y\in Y$ and $X,Y\in \mathrm{ob}(H-\mathsf{Mod}_{\mathsf{fd}})$.

Continuing Example 4.24, we endow the Hopf algebra $\mathbb{C}[\mathbb{Z}_2]$ with the universal R-matrix

$$R=\frac{1}{2}(e\otimes e+e\otimes s+s\otimes e-s\otimes s). \tag{4.33}$$

One can check that (4.33) is a universal R-matrix by explicit computation. The braiding on the simple objects of $\mathbb{C}[\mathbb{Z}_2]-\mathsf{Mod}_{\mathsf{fd}}$, up to isomorphism, is given by

$$\begin{aligned} c_{\mathbb{1}\otimes\mathbb{1}}&:1\otimes 1\mapsto 1\otimes 1, & c_{\mathbb{1}\otimes M}&:1\otimes v_M\mapsto v_M\otimes 1,\\ c_{M\otimes M}&:v_M\otimes v_M\mapsto -v_M\otimes v_M, & c_{M\otimes\mathbb{1}}&:v_M\otimes 1\mapsto 1\otimes v_M. \end{aligned} \tag{4.34}$$

Notice that the braiding c is symmetric. So, $\theta=\mathrm{id}:\mathrm{id}_{\mathscr{C}}\Rightarrow\mathrm{id}_{\mathscr{C}}$ is a twist for $\mathbb{C}[\mathbb{Z}_2]-\mathsf{Mod}_{\mathsf{fd}}$. Since the dual of the identity is the identity of the dual, this twist is also a ribbon structure. A different ribbon structure can be given by defining θ' on the isomorphism classes of simple objects as follows:

$$\theta'_{\mathbb{1}}=\mathrm{id}_{\mathbb{1}}\qquad\text{and}\qquad \theta'_M=-\,\mathrm{id}_M\,. \tag{4.35}$$

This is a twist since

$$\mathbb{1}\otimes\mathbb{1}\cong\mathbb{1},\quad \mathbb{1}\otimes M\cong M,\quad M\otimes\mathbb{1}\cong M,\quad M\otimes M\cong\mathbb{1}, \tag{4.36}$$

and a ribbon structure since $\mathbb{1}^*\cong\mathbb{1}$ and $M^*\cong M$. This in fact extends to all objects in $\mathbb{C}[\mathbb{Z}_2]-\mathsf{Mod}_{\mathsf{fd}}$ since it is a semisimple abelian category. Recalling Definition C.11 for ribbon Hopf algebras, θ and θ' can be equivalently described as the ribbon structures given by the ribbon elements $\nu=e$ and $\nu=s$, respectively. $\Diamond$

4.3 Tensor categories

The rigid monoidal categories that are realised in terms of representations of certain algebraic objects should naturally have some abelian structure. Furthermore, this abelian structure should be compatible with the monoidal structure, that is, the tensor product should distribute over direct sums, up to isomorphism. Tensor categories generalise this notion. Firstly, we specify what we mean by the following definitions about $\Bbbk$-linear abelian categories.

Definition 4.30. Let $\Bbbk$ be a field. Let $\mathscr{C}$ be a $\Bbbk$-linear abelian category. Then, $\mathscr{C}$ is *locally finite* (or *artinian*) if:

(i) for any two objects X, Y in $\mathscr{C}$, the $\Bbbk$-vector space $\hom_{\mathscr{C}}(X,Y)$ is finite dimensional,

(ii) every object in $\mathscr{C}$ has finite Jordan-Hölder length.

Furthermore, $\mathscr{C}$ is *finite* if:

(i) $\mathscr{C}$ is locally finite,

(ii) $\mathscr{C}$ has finitely many isomorphism classes of simple objects.

Abelian categories can be interpreted as a categorified version of abelian groups, with the biproduct being the analogue of addition.[1] Monoidal categories are a categorified version of monoids, with the tensor product being the analogue of the product. Tensor categories can be roughly thought of as a kind of categorified unital ring.

In what follows, let $\Bbbk$ be an algebraically closed field.

DEFINITION 4.31. A *tensor category* over $\Bbbk$ is a $\Bbbk$-linear abelian rigid monoidal category $\mathscr{C}$ satisfying the following conditions:

(i) $\mathscr{C}$ is locally finite,

(ii) the tensor bifunctor $\otimes$ is bilinear on morphisms,

(iii) $\mathrm{End}_{\mathscr{C}}(\mathbb{1}) \cong_{\Bbbk-\mathsf{Vect}} \Bbbk$.

Importantly, the tensor product distributes over biproducts (in this case direct sums) as given by the bilinearity of the tensor product. The local finiteness condition is an analogue of taking finite sums. The tensor category is a model of how a 'very nice' representation theory should behave. When the unit object is simple, it has a one dimensional space of endomorphisms. This follows from Schur's lemma (for general abelian categories) when $\Bbbk$ is algebraically closed.

PROPOSITION 4.32. Let $\mathscr{C}$ be a tensor category with unit $\mathbb{1}$ and zero object 0. Then,

(i) $X \otimes (Y \oplus Z) \cong (X \otimes Y) \oplus (X \otimes Z)$ and $(X \oplus Y) \otimes Z \cong (X \otimes Z) \oplus (Y \otimes Z)$, for all objects X, Y, Z in $\mathscr{C}$,

(ii) $X \otimes 0 \cong 0 \cong 0 \otimes X$, for all objects X in $\mathscr{C}$,

(iii) $\mathbb{1}$ is a simple object.

Proof. Recall that the biproduct $X_1 \oplus X_2$ comes equipped with projection morphisms $\pi_{X_i} : X_1 \oplus X_2 \to X_i$ and embedding morphisms $\iota_{X_i} : X_i \to X_1 \oplus X_2$, satisfying

$$\pi_{X_i} \circ \iota_{X_j} = \begin{cases} \mathrm{id}_{X_i} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \qquad \text{and} \qquad \iota_{X_1} \circ \pi_{X_1} + \iota_{X_2} \circ \pi_{X_2} = \mathrm{id}_{X_1 \oplus X_2}\,.$$

Let X, Y and Z be objects in $\mathscr{C}$. Consider the following morphisms

$$\begin{tikzcd}
 & X \otimes Y \arrow[dl, shift left, "\mathrm{id}_X \otimes \iota_Y"] \arrow[dr, shift left, "\iota_{X\otimes Y}"] & \\
X \otimes (Y \oplus Z) \arrow[ur, shift left, "\mathrm{id}_X \otimes \pi_Y"] \arrow[dr, shift left, "\mathrm{id}_X \otimes \pi_Z"] & & (X \otimes Y) \oplus (X \otimes Z) \arrow[ul, shift left, "\pi_{X\otimes Y}"] \arrow[dl, shift left, "\pi_{X\otimes Z}"] \\
 & X \otimes Z \arrow[ul, shift left, "\mathrm{id}_X \otimes \iota_Z"] \arrow[ur, shift left, "\iota_{X\otimes Z}"] &
\end{tikzcd}$$

[1] Given a non-zero object X, there is no "additive inverse" object $-X$ such that $X \oplus (-X) \cong 0$. So, more accurately, abelian categories are categorified abelian monoids and the Grothendieck group is the abelian group.

Then, using bilinearity of the composition and tensor bifunctor, we have

$$\big(\iota_{X\otimes Y} \circ (\mathrm{id}_X \otimes \pi_Y) + \iota_{X\otimes Z} \circ (\mathrm{id}_X \otimes \pi_Z)\big) \circ \big((\mathrm{id}_X \otimes \iota_Y) \circ \pi_{X\otimes Y} + (\mathrm{id}_X \otimes \iota_Z) \circ \pi_{X\otimes Z}\big)$$
$$= \iota_{X\otimes Y} \circ \pi_{X\otimes Y} + \iota_{X\otimes Z} \circ \pi_{X\otimes Z} = \mathrm{id}_{(X\otimes Y)\oplus(X\otimes Z)},$$

$$\big((\mathrm{id}_X \otimes \iota_Y) \circ \pi_{X\otimes Y} + (\mathrm{id}_X \otimes \iota_Z) \circ \pi_{X\otimes Z}\big) \circ \big(\iota_{X\otimes Y} \circ (\mathrm{id}_X \otimes \pi_Y) + \iota_{X\otimes Z} \circ (\mathrm{id}_X \otimes \pi_Z)\big)$$
$$= \mathrm{id}_X \otimes (\iota_Y \circ \pi_Y + \iota_Z \circ \pi_Z) = \mathrm{id}_{X\otimes(Y\oplus Z)}\,.$$

So, $X \otimes (Y \oplus Z) \cong (X \otimes Y) \oplus (X \otimes Z)$.

The bilinearity of the tensor bifunctor also gives

$$\mathrm{id}_{X\otimes 0} = \mathrm{id}_X \otimes \mathrm{id}_0 = \mathrm{id}_X \otimes 0 = 0(\mathrm{id}_X \otimes 0) = 0.$$

So, for any object Y, and morphisms $f : X \otimes 0 \to Y$ and $g : Y \to X \otimes 0$, we have

$$f = f \circ (\mathrm{id}_{X\otimes 0}) = 0(f \circ (\mathrm{id}_{X\otimes 0})) = 0 \quad \text{and} \quad g = (\mathrm{id}_{X\otimes 0}) \circ g = 0((\mathrm{id}_{X\otimes 0}) \circ g) = 0.$$

That is, f and g are unique. So, $X \otimes 0$ is a zero object.

A proof that $\mathbb{1}$ is simple is given in Theorem 4.3.8 of [EGNO16] for *ring categories*, a generalisation of tensor categories. In summary, a consequence of the rigidity is that, for each $X \in \mathscr{C}$, $X^* \otimes -$ is left adjoint to $X \otimes -$ and $- \otimes X^*$ is right adjoint to $- \otimes X$. This makes $- \otimes -$ biexact and the dual functor $(-)^*$ exact. It follows that, after some work, any subobject of $\mathbb{1}$ is isomorphic to either 0 or $\mathbb{1}$. □

REMARK 4.33. In Proposition 4.32, the statements (i) and (ii) are, respectively, category-theoretic analogues for the ring properties of distributivity of multiplication over addition and the result that multiplication with zero is zero. △

EXAMPLE 4.34. Recall from Example 4.10 that if a Hopf algebra H has an invertible antipode, then the monoidal category of its finite dimensional modules is rigid. The tensor product is compatible with the abelian structure and it is locally finite because it consists of only finite dimensional modules. The unit is the base field with a one-dimensional endomorphism space. So, $H-\mathsf{Mod}_{\mathsf{fd}}$ is a tensor category. ◊

4.4 Pre-modular and modular tensor categories

There is an even 'nicer' model for how representations should behave.

DEFINITION 4.35. A *fusion category* is a semisimple tensor category with finitely many isomorphism classes of simple objects.

Now assume that $\Bbbk$ is algebraically closed and has characteristic zero. In all vertex operator algebra applications, we have $\Bbbk = \mathbb{C}$.

DEFINITION 4.36. A *pre-modular category* is a ribbon fusion category.

DEFINITION/PROPOSITION 4.37. A ribbon tensor category $\mathscr{C}$ with twist θ can be equipped with the *canonical pivotal structure*

$$a_X = u_X \circ \theta_X : X \to X^{**}, \qquad X \in \mathrm{ob}(\mathscr{C}), \tag{4.37}$$

where the *Drinfeld morphism* u_X is defined to be the composition

(4.38)

$$X \xrightarrow{\rho_X^{-1}} X\mathbb{1} \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}_{X^*}} X(X^*X^{**}) \xrightarrow{\alpha_{X,X^*,X^{**}}} (XX^*)X^{**} \xrightarrow{c_{X,X^*} \otimes \mathrm{id}_{X^{**}}} (X^*X)X^{**} \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^{**}}} \mathbb{1}X^{**} \xrightarrow{\lambda_{X^{**}}} X^{**}.$$

If $\mathscr{C}$ is moreover a fusion category, then a is the *canonical spherical structure*.

The proof can be found in Propositions 8.9.3, 8.10.6 and 8.10.12 of [EGNO16].

REMARK 4.38. In this sense, ribbon fusion categories are regarded as having a spherical structure. In fact, pre-modular categories can be equivalently defined as braided fusion categories equipped with spherical structure. In this case, the canonical ribbon strucure is defined as $\theta = u^{-1} \circ a$ by rearranging (4.37) (see Proposition 8.10.12 of [EGNO16]). $\triangle$

DEFINITION 4.39. Let $\mathscr{C}$ be a pre-modular category with spherical structure a. The S*-matrix* of $\mathscr{C}$ is defined by

$$S := (s_{XY})_{X,Y \in \mathcal{O}(\mathscr{C})}, \quad \text{where} \quad s_{XY} = \mathrm{Tr}_a(c_{Y,X} \circ c_{X,Y}), \tag{4.39}$$

and $\mathcal{O}(\mathscr{C})$ denotes the set of isomorphism classes of simple objects of $\mathscr{C}$.

We identify $\mathrm{End}_{\mathscr{C}}(\mathbb{1})$ with the base field $\mathbb{k}$, via $\mathrm{id}_{\mathbb{1}} \mapsto 1$, and then regard S as a $|\mathcal{O}(\mathscr{C})| \times |\mathcal{O}(\mathscr{C})|$-matrix over $\mathbb{k}$.

DEFINITION 4.40. A *modular tensor category*, or *modular category*, is a pre-modular category that has a non-degenerate S-matrix.

EXAMPLE 4.41. We continue Examples 4.24, 4.34 and 4.29. The abelian category $\mathbb{C}[\mathbb{Z}_2]-\mathsf{Mod}_{\mathsf{fd}}$ has two simple objects, up to isomorphism. It is semisimple since $\mathbb{Z}_2$ is a finite group and the representations are over $\mathbb{C}$.

The Hopf algebra $\mathbb{C}[\mathbb{Z}_2]$ was endowed with pivots $e, s \in \mathbb{Z}_2$ and a universal R-matrix (4.33). We then equipped $\mathbb{C}[\mathbb{Z}_2]-\mathsf{Mod}_{\mathsf{fd}}$ with two ribbon structures $\theta = \mathrm{id}$ and θ' defined in (4.35). So, $\mathbb{C}[\mathbb{Z}_2]-\mathsf{Mod}_{\mathsf{fd}}$, together with θ or θ', is a ribbon fusion category. That is, a pre-modular category.

Evaluating equation (4.38), we have

$$u_{\mathbb{1}} : 1 \mapsto 1^{**} \qquad \text{and} \qquad u_M : v_M \mapsto -v_M^{**}. \tag{4.40}$$

So the canonical spherical structures θ and θ', from Example 4.29, correspond to a^s and a^e, respectively.

Since the braiding is symmetric, the elements of the S-matrix are given by $s_{X,Y} = \mathrm{Tr}_{a^g}(\mathrm{id}_{X\otimes Y}) = \dim_{a^g}(X \otimes Y)$. That is,

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{when } g = e \quad \text{and} \quad S = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{when } g = s. \tag{4.41}$$

Here we have taken the order of the simple objects to be $(\mathbb{1}, M)$. We have identified $\mathrm{End}(\mathbb{1})$ with $\mathbb{C}$ by $\mathrm{id}_{\mathbb{1}} \mapsto 1$.

Neither of these S matrices are invertible. So, these examples are pre-modular, but not modular. We will see examples of modular categories generated by vertex operator algebras in the next chapter. $\Diamond$

4.5 Homomorphisms

In Chapter 6, our main goal is to produce an *equivalence of modular tensor categories*. For the various kinds of categories we defined in the previous sections, we will need to define a notion of homomorphism in order to obtain a notion of equivalence. As in Appendix B, we will define an equivalence (of a certain type of structured category) to be a homomorphism with an underlying functor that is an equivalence of categories. We should expect that the quasi-inverse functor is also a homomorphism.

We were unable to find published notions for all of the following homomorphisms, so the names of the following definitions may not be standard. But nonetheless, they are straight forward to define as monoidal functors that preserve all necessary structure.

Firstly, rigidity is purely an existence condition on monoidal categories. That is, there is no extra data needed. Furthermore, by Proposition 4.7, monoidal functors preserve the left (and right) dual structures. Hence, the correct notion of a *homomorphism of rigid monoidal categories* is just a monoidal functor between rigid monoidal categories.

PROPOSITION 4.42. Let $\mathscr{C}$ be a rigid monoidal category and let $\mathscr{D}$ be a monoidal category. If (F, J, φ) is a monoidal equivalence from $\mathscr{C}$ to $\mathscr{D}$, then $\mathscr{D}$ is rigid.

Before we can prove Proposition 4.42, we require the following lemma.

LEMMA 4.43. Let X and Y be objects in a monoidal category $\mathscr{C}$. Assume X has a left dual $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$. Assume $f : X \to Y$ is an isomorphism. Then, X^* is a left dual object of Y. Similarly for right duals.

Proof. Define

$$\mathrm{ev}_Y = \mathrm{ev}_X \circ (\mathrm{id}_{X^*} \otimes f) \quad \text{and} \quad \mathrm{coev}_Y = (f^{-1} \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X.$$

Then, the following diagrams commute.

$$\begin{tikzcd}
X \arrow[r, "\lambda^{-1}"] & \mathbb{1}X \arrow[r, "\mathrm{coev}_X \otimes \mathrm{id}"] & (XX^*)X \arrow[r, "\alpha^{-1}"] \arrow[d, "(f^{-1}\otimes \mathrm{id})\otimes f^{-1}"] & X(X^*X) \arrow[r, "\mathrm{id}\otimes \mathrm{ev}_X"] & X\mathbb{1} \arrow[r, "\rho"] \arrow[d, "f^{-1}\otimes \mathrm{id}"] & X \arrow[d, "f^{-1}"] \\
Y \arrow[u, "f"] \arrow[r, "\lambda^{-1}"'] & \mathbb{1}Y \arrow[u, "\mathrm{id}\otimes f"'] \arrow[r, "\mathrm{coev}_Y \otimes \mathrm{id}"'] & (YX^*)Y \arrow[r, "\alpha^{-1}"'] & Y(X^*Y) \arrow[u, "f\otimes(\mathrm{id}\otimes f)"'] \arrow[r, "\mathrm{id}\otimes \mathrm{ev}_Y"'] & Y\mathbb{1} \arrow[r, "\rho"'] & Y
\end{tikzcd} \tag{4.42}$$

$$
\begin{tikzcd}
X^* \arrow[r, "\rho^{-1}"] & X^*\mathbb{1} \arrow[r, "\mathrm{id}\otimes\mathrm{coev}_X"] \arrow[dr, "\mathrm{id}\otimes\mathrm{coev}_Y"'] & X^*(XX^*) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}\otimes(f^{-1}\otimes\mathrm{id})"] & (X^*X)X^* \arrow[r, "\mathrm{ev}_X\otimes\mathrm{id}"] & \mathbb{1}X^* \arrow[r, "\lambda"] & X^* \\
& & X^*(YX^*) \arrow[r, "\alpha"] & (X^*Y)X^* \arrow[u, "(\mathrm{id}\otimes f)\otimes\mathrm{id}"'] \arrow[ur, "\mathrm{ev}_Y\otimes\mathrm{id}"'] & &
\end{tikzcd}
\tag{4.43}
$$

In each diagram, the top composes to the identity. So, $(X^*, \mathrm{ev}_Y, \mathrm{coev}_Y)$ is a left dual of Y. □

Proof of Proposition 4.42 . Let $Y \in \mathrm{ob}(\mathscr{D})$. Since F is an equivalence, F is essentially surjective. Hence, there is $X \in \mathrm{ob}(\mathscr{C})$ such that $F(X) \cong Y$. Since $\mathscr{C}$ is rigid, it follows that X is rigid, hence $F(X)$ has left and right duals. Then, by Lemma 4.43, Y has left and right duals. So, every object in $\mathscr{D}$ is rigid and hence $\mathscr{D}$ is rigid. □

REMARK 4.44. Since the quasi-inverse of a monoidal equivalence is a monoidal equivalence (see Proposition B.16), it also follows that the rigidity of the target category of a monoidal equivalence imposes rigidity of the source category. That is to say, rigidity is an equivalence invariant in MonCat. △

Now we give a definition for functors that preserve ribbon structures. A *homomorphism of braided rigid monoidal categories* is a braided monoidal functor. The twists should satisfy 'the image of the twist is the twist of the image'. No additional data is needed.

DEFINITION 4.45. Let $\mathscr{C}$ and $\mathscr{D}$ be ribbon categories with ribbon structures $\theta^{\mathscr{C}}$ and $\theta^{\mathscr{D}}$, respectively. A *ribbon functor* (F, J, φ) from $\mathscr{C}$ to $\mathscr{D}$ is a braided monoidal functor that satisfies the condition:

(i) (*compatibility of twists*)

$$
F(\theta^{\mathscr{C}}_X) = \theta^{\mathscr{D}}_{F(X)} \qquad \text{for all } X \in \mathrm{ob}(\mathscr{C}). \tag{4.44}
$$

REMARK 4.46. The compatibility of twists is not satisfied by every braided monoidal functor. Consider the two distinct twists, θ and θ', given in Example 4.29 for the same braided monoidal category. The identity monoidal functor is a braided monoidal functor from $(\mathbb{C}[\mathbb{Z}_2]-\mathsf{Mod}, \theta)$ to $(\mathbb{C}[\mathbb{Z}_2]-\mathsf{Mod}, \theta')$, but it does not satisfy (4.44). That is, we have a braided functor that is not a ribbon functor. So, we must always verify the compatibility of twists condition for ribbon functors. △

REMARK 4.47. The ribbon structure condition should be preserved by ribbon functors. Indeed,

$$
\theta^{\mathscr{D}}_{(FX)^*} = \theta^{\mathscr{D}}_{F(X^*)} = F(\theta^{\mathscr{C}}_{X^*}) = F((\theta^{\mathscr{C}}_X)^*) = (F\theta^{\mathscr{C}}_X)^* = (\theta^{\mathscr{D}}_{(FX)})^* \qquad \text{for all } X \in \mathrm{ob}(\mathscr{C}).
$$

Here we have chosen $F(X^*)$ to be the left dual object of FX and used $F(f^*) = F(f)^*$, which can be proven with the following diagrams, which commute by the compatibility conditions and the naturality of J.

(4.45)
$$\begin{tikzcd}
FY^* \arrow[r, "F\rho^{-1}_{Y^*}"] \arrow[d, "\rho^{-1}"'] & F(Y^*\mathbb{1}) \arrow[r, "F(\mathrm{id}_{Y^*}\otimes \mathrm{coev}_X)"] & F(Y^*(XX^*)) \arrow[r, "F\alpha_{Y^*,X,X^*}"] \arrow[d, "J^{-1}"] & F((Y^*X)X^*) \\
FY^*\mathbb{1} \arrow[r, "\mathrm{id}\otimes\varphi"] \arrow[drr, bend right, "\mathrm{id}\otimes \mathrm{coev}_{FX}"'] & FY^*F\mathbb{1} \arrow[u, "J"] \arrow[r, "\mathrm{id}\otimes F\mathrm{coev}_X"] & FY^*F(XX^*) \arrow[d, "\mathrm{id}\otimes J^{-1}"] & F(Y^*\otimes X)FX^* \arrow[u, "J"'] \\
 & & FY^*(FXFX^*) \arrow[r, "\alpha^{-1}"'] & (FY^*FX)FX^* \arrow[u, "J\otimes \mathrm{id}"']
\end{tikzcd}$$

(4.46)
$$\begin{tikzcd}
F((Y^*X)X^*) \arrow[r, "F((\mathrm{id}_{Y^*}\otimes f)\otimes \mathrm{id}_{X^*})"] & F((Y^*Y)X^*) \arrow[r, "F(\mathrm{ev}_Y\otimes \mathrm{id}_{X^*})"] & F(\mathbb{1}X^*) \arrow[r, "F(\lambda_{X^*})"] \arrow[d, "J^{-1}"] & FX^* \\
F(Y^*X)FX \arrow[u, "J"] \arrow[r, "F(\mathrm{id}\otimes f)\otimes \mathrm{id}"] & F(Y^*Y)FX^* \arrow[u, "J"] \arrow[r, "F\mathrm{ev}_Y\otimes \mathrm{id}"] & F\mathbb{1}FX \arrow[r, "\varphi^{-1}\otimes \mathrm{id}"] & \mathbb{1}FX^* \arrow[u, "\lambda"'] \\
(FY^*FX)FX^* \arrow[u, "J\otimes \mathrm{id}"] \arrow[r, "(\mathrm{id}\otimes Ff)\otimes \mathrm{id}"'] & (FY^*FY)FX^* \arrow[u, "J\otimes \mathrm{id}"] \arrow[urr, bend right, "\mathrm{ev}_{FY}\otimes \mathrm{id}"'] & &
\end{tikzcd}$$

Note that the right side of (4.45) connects to the left side of (4.46). The top composes to $F(f^*)$ and the bottom outer perimeter composes to $F(f)^*$. $\triangle$

REMARK 4.48. Consider two ribbon functors

$$(F, J, \varphi) : \mathscr{C} \to \mathscr{D} \qquad \text{and} \qquad (G, K, \psi) : \mathscr{D} \to \mathscr{E}.$$

Their monoidal composition $(G, K, \psi) \bullet (F, J, \varphi)$ is a braided monoidal functor, as shown in Proposition B.27, and the ribbon structure is still preserved since

$$(G \circ F)(\theta^{\mathscr{C}}_X) = G(\theta^{\mathscr{D}}_{F(X)}) = \theta^{\mathscr{E}}_{(G\circ F)(X)} \qquad \text{for all } X \in \mathrm{ob}(\mathscr{C}).$$ $\triangle$

PROPOSITION 4.49. If a ribbon functor (F, J, φ) is an equivalence of categories, then its quasi-inverse has the canonical structure of a ribbon functor.

Proof. Let

(4.47) $$F : \mathscr{C} \to \mathscr{D}, \quad G : \mathscr{D} \to \mathscr{C}, \quad \varepsilon : FG \Rightarrow \mathrm{id}_{\mathscr{D}}, \quad \eta : \mathrm{id}_{\mathscr{C}} \Rightarrow GF,$$

be an adjoint equivalence. From Proposition B.29, we have that G has the canonical structure of a braided monoidal functor. We are left to show that G preserves the ribbon structure.

Let Y be an object in $\mathscr{D}$ and consider the following diagram.

(4.48)
$$\begin{tikzcd}
GY \arrow[r, "\eta_{GY}"] \arrow[d, "\theta^{\mathscr{C}}_{GY}"'] & GFGY \arrow[r, "G\varepsilon_Y"] \arrow[d, shift right, "GF\theta^{\mathscr{C}}_{GY}"'] \arrow[d, shift left, "G\theta^{\mathscr{D}}_{FGY}"] & GY \arrow[d, "G\theta^{\mathscr{D}}_Y"] \\
GY \arrow[r, "\eta_{GY}"'] & GFGY \arrow[r, "G\varepsilon_Y"'] & GY
\end{tikzcd}$$

The left square commutes by the naturality of η. The two middle arrows are equal since F is a ribbon functor. The right square commutes by the naturality of $\theta^{\mathscr{D}}$. The top and bottom arrows compose to the identity by the unit-counit zigzag identities. $\square$

Braided monoidal functors are a good notion of a morphism, as seen in Appendix B. So, the collection of ribbon categories, together with all ribbon functors between them, form a category RibCat. A ribbon functor (F, J, φ) where F is an equivalence of categories, has a quasi-inverse with ribbon functor structure. This motivates the following definition.

DEFINITION 4.50. A *ribbon equivalence* is a ribbon functor that is also an equivalence of categories.

The structure we need to preserve in a homomorphism of tensor categories is the $\Bbbk$-linear abelian monoidal structure. The locally finite condition, rigidity, tensor bilinearity, and the one-dimensional endomorphism space of the unit are all properties of the categories.

DEFINITION 4.51. A *homomorphism of tensor categories* is a $\Bbbk$-linear monoidal functor between tensor categories. We will not require left or right exactness.

Note that the functor is additive since it is $\Bbbk$-linear. If a homomorphism of tensor categories (F, J, φ) has F an equivalence of categories, then the quasi-inverse of F is also $\Bbbk$-linear and can be endowed with monoidal structure. So, we have the following definition.

DEFINITION 4.52. A *tensor equivalence* is a homomorphism of tensor categories that is also an equivalence of categories.

Of course, a *homomorphism of fusion categories* is a homomorphism of tensor categories between fusion categories. Since a pre-modular category is a ribbon fusion category, we have the following definition.

DEFINITION 4.53. A *pre-modular functor* is a $\Bbbk$-linear ribbon functor between pre-modular categories. A *pre-modular equivalence* is pre-modular functor (F, J, φ) with F an equivalence of categories.

REMARK 4.54. This notion of a homomorphism of pre-modular categories is enough to preserve (i.e. describe) all the structure of a pre-modular category. The collection of pre-modular categories and pre-modular functors form a category with the property that every pre-modular equivalence has a quasi-inverse pre-modular functor. The natural question is ‘do pre-modular equivalences preserve modularity?’. $\triangle$

PROPOSITION 4.55. Let $\mathscr{C}$ and $\mathscr{D}$ be pre-modular categories. Let (F, J, φ) be a pre-modular equivalence from $\mathscr{C}$ to $\mathscr{D}$. Then, $\mathscr{C}$ is modular if and only if $\mathscr{D}$ is modular.

We will need two lemmas to prove Proposition 4.55.

LEMMA 4.56. Let $\mathscr{C}$ and $\mathscr{D}$ be pre-modular categories with pivotal structures $a^{\mathscr{C}}$ and $a^{\mathscr{D}}$, respectively. Let (F, J, φ) be a pre-modular functor from $\mathscr{C}$ to $\mathscr{D}$. Then, F is *pivotal* in the sense that

$$F(a^{\mathscr{C}}_X) = a^{\mathscr{D}}_{F(X)} \qquad \text{for all } X \in \mathrm{ob}(\mathscr{C}). \tag{4.49}$$

Proof. Let X be an object in $\mathscr{C}$. Recall the definition of u_X from Remark 4.37 and consider the following commutative diagrams.

$$\begin{tikzcd}
FX \arrow[r, "F\rho^{-1}"] \arrow[d, "\rho^{-1}"'] & F(X\mathbb{1}) \arrow[r, "F(\mathrm{id}\otimes \mathrm{coev}_{X^*})"] & F(X(X^*X^{**})) \arrow[r, "F\alpha"] \arrow[d, "J^{-1}"] & F((XX^*)X^{**}) \\
FX\mathbb{1} \arrow[r, "\mathrm{id}\otimes\varphi"] \arrow[rr, bend right, "\mathrm{id}\otimes \mathrm{coev}_{FX^*}"'] & FXF\mathbb{1} \arrow[u, "J"'] \arrow[r, "\mathrm{id}\otimes F\mathrm{coev}_{X^*}"] & FXF(X^*X^{**}) \arrow[d, "\mathrm{id}\otimes J^{-1}"] & F(XX^*)FX^{**} \arrow[u, "J"'] \\
 & & FX(FX^*FX^{**}) \arrow[r, "\alpha"'] & (FXFX^*)FX^{**} \arrow[u, "J\otimes\mathrm{id}"']
\end{tikzcd} \tag{4.50}$$

$$\begin{tikzcd}
F((XX^*)X^{**}) \arrow[r, "F(c_{X,X^*}\otimes\mathrm{id})"] & F((X^*X)X^{**}) \arrow[r, "F(\mathrm{ev}_X\otimes\mathrm{id})"] & F(\mathbb{1}X^{**}) \arrow[r, "F\lambda"] & F(X^{**}) \\
F(XX^*)FX^{**} \arrow[u, "J"] \arrow[r, "Fc_{X,X^*}\otimes\mathrm{id}"] & F(X^*X)FX^{**} \arrow[u, "J"] \arrow[r, "F\mathrm{ev}_X\otimes\mathrm{id}"] & F\mathbb{1}FX^{**} \arrow[u, "J"] \arrow[r, "\varphi^{-1}\otimes\mathrm{id}"] & \mathbb{1}FX^{**} \arrow[u, "\lambda"'] \\
(FXFX^*)FX^{**} \arrow[u, "J\otimes\mathrm{id}"] \arrow[r, "c_{FX,FX^*}\otimes\mathrm{id}"'] & (FX^*FX)FX^{**} \arrow[u, "J\otimes\mathrm{id}"] \arrow[urr, bend right, "\mathrm{ev}_{FX}\otimes\mathrm{id}"'] & &
\end{tikzcd} \tag{4.51}$$

Commutativity comes from compatibility of associators and unitors, braiding and the naturality of J. The right side of (4.50) connects to the left side of (4.51). The top composes to $F(u^{\mathscr{C}}_X)$ and the bottom outer perimeter composes to $u^{\mathscr{D}}_{F(X)}$. So, we have $F(u^{\mathscr{C}}_X) = u^{\mathscr{D}}_{F(X)}$. Thus,

$$F(a^{\mathscr{C}}_X) = F(u^{\mathscr{C}}_X \circ \theta^{\mathscr{C}}_X) = F(u^{\mathscr{C}}_X) \circ F(\theta^{\mathscr{C}}_X) = u^{\mathscr{D}}_{F(X)} \circ \theta^{\mathscr{D}}_{F(X)} = a^{\mathscr{D}}_{F(X)}. \tag{4.52}$$

So, pre-modular functors respect the canonical pivotal (i.e. spherical) structure. $\square$

LEMMA 4.57. If (F, J, φ) is a pivotal functor, then

$$F(\mathrm{Tr}_{a^{\mathscr{C}}}(f)) = \varphi \circ \mathrm{Tr}_{a^{\mathscr{D}}}(F(f)) \circ \varphi^{-1} \qquad \text{for all } f \in \mathrm{End}_{\mathscr{C}}(X),\ X \in \mathrm{ob}(\mathscr{C}). \tag{4.53}$$

Proof. Let X be an object in $\mathscr{C}$ and f and endomorphism of X. Consider the following diagram.

$$\begin{tikzcd}
F\mathbb{1} \arrow[r, "F\mathrm{coev}_X"] & F(X\otimes X^*) \arrow[r, "F((a^{\mathscr{C}}_X\circ f)\otimes\mathrm{id})"] \arrow[d, "J^{-1}"] & F(X^{**}\otimes X^*) \arrow[r, "F\mathrm{ev}_{X^*}"] & F\mathbb{1} \arrow[d, "\varphi^{-1}"] \\
\mathbb{1} \arrow[u, "\varphi"] \arrow[r, "\mathrm{coev}_{FX}"'] & FX\otimes FX^* \arrow[r, "(Fa^{\mathscr{C}}_X\circ Ff)\otimes\mathrm{id} \,=\, (a^{\mathscr{D}}_{FX}\circ Ff)\otimes\mathrm{id}"'] & FX^{**}\otimes FX* \arrow[u, "J"'] \arrow[r, "\mathrm{ev}_{FX^*}"'] & \mathbb{1}
\end{tikzcd} \tag{4.54}$$

The middle square commutes by the naturality of J. The left and right squares are definitions. The top composes to the image of the trace of f. The bottom composes to the trace of the image of f. Reversing the unit isomorphisms obtains

$$F(\mathrm{Tr}_{a^{\mathscr{C}}}(f)) = \varphi \circ \mathrm{Tr}_{a^{\mathscr{D}}}(F(f)) \circ \varphi^{-1}. \qquad \square$$

Proof of Proposition 4.55. Since F is an equivalence between abelian categories, it provides a one-to-one correspondence between the sets of isomorphism classes of simple objects

$$\mathcal{O}(\mathscr{C}) \longleftrightarrow \mathcal{O}(\mathscr{D}), \qquad X \mapsto FX. \tag{4.55}$$

Let X and Y be simple objects in $\mathscr{C}$. Then,

$$\begin{aligned} F(s^{\mathscr{C}}_{X,Y}) &= F(\mathrm{Tr}_{a^{\mathscr{C}}}(c_{Y,X} \circ c_{X,Y})) \\ &= \varphi \circ \mathrm{Tr}_{a^{\mathscr{D}}}(F(c_{Y,X} \circ c_{X,Y})) \circ \varphi^{-1} \\ &= \varphi \circ \mathrm{Tr}_{a^{\mathscr{D}}}(J_{X,Y} \circ c_{FY,FX} \circ J^{-1}_{Y,X} \circ J_{Y,X} \circ c_{FX,FY} \circ J^{-1}_{X,Y}) \circ \varphi^{-1} \\ &= \varphi \circ \mathrm{Tr}_{a^{\mathscr{D}}}(c_{FY,FX} \circ c_{FX,FY}) \circ \varphi^{-1} \\ &= \varphi \circ s^{\mathscr{D}}_{FX,FY} \circ \varphi^{-1}. \end{aligned}$$

In the third line, we have used Proposition 4.22 to swap the order of the morphisms in the trace. So, we have the map

$$\varphi^{-1} \circ F(-) \circ \varphi : \mathrm{End}_{\mathscr{C}}(\mathbb{1}) \to \mathrm{End}_{\mathscr{D}}(\mathbb{1}). \tag{4.56}$$

Recall that we identify $s^{\mathscr{C}}_{X,Y}$ and $s^{\mathscr{D}}_{FX,FY}$ with elements in $\Bbbk$ via $\mathrm{id}_{\mathbb{1}} \mapsto 1$. Furthermore, $\varphi^{-1} \circ F(-) \circ \varphi$ is a $\Bbbk$-linear isomorphism with $\varphi^{-1} \circ F(\mathrm{id}_{\mathbb{1}}) \circ \varphi = \mathrm{id}_{\mathbb{1}}$. So, after identifying with the base field $\Bbbk$ we have

$$S^{\mathscr{C}} = \left(s^{\mathscr{C}}_{X,Y}\right)_{X,Y \in \mathcal{O}(\mathscr{C})} = \left(s^{\mathscr{D}}_{FX,FY}\right)_{X,Y \in \mathcal{O}(\mathscr{C})} = S^{\mathscr{D}}. \tag{4.57}$$

Thus, the S-matrix for $\mathscr{C}$ is invertible if and only if the S-matrix for $\mathscr{D}$ is invertible. $\square$

REMARK 4.58. Proposition 4.55 says that modularity is an equivalence invariant in the category of pre-modular categories. If two pre-modular categories are pre-modular equivalent and one of them is modular then they are *equivalent as modular categories* or *modular equivalent*. We will use this fact in Chapter 6 to produce a modular equivalence between pre-modular categories, one constructed from a vertex operator algebra, and the other constructed from a quantum group. $\triangle$

Throughout this chapter, we have seen examples of rigidity, pivotal and spherical structures, ribbon structures, tensor and fusion structures, and pre-modular structures. However, we have not seen any examples of modular tensor categories. We will dedicate the next chapter to providing some examples and, in accordance with the historical motivation of modular tensor categories, we will construct them from vertex operator algebras.

Chapter 5

Modular tensor categories from lattice vertex operator algebras

In this chapter we will briefly present a family of vertex operator algebras that are constructed from positive definite even lattices. We will also discuss their modules and intertwining maps, but we will not compute them here. Our aim is to demonstrate that the constructive proof of HLZ can be repurposed to explicitly compute the braided monoidal data for a category of lattice vertex operator algebra modules. We will then use the constructive proof of [Hua08] to compute the pre-modular data and verify that the S-matrix is invertible. The main result of this chapter is the explicit presentation of the data for the family of modular tensor categories constructed from positive definite even lattices.

5.1 Intertwining operators

The notion of an *intertwining operator* is used in HLZ to construct the associator, unitors and braiding. In fact, this notion is closely related to that of the intertwining maps of Definition 3.3 that were used in Section 3.2 to define the $P(w)$-tensor product.

The full theory of *logarithmic* intertwining operators and intertwining maps for generalised modules of Möbius algebras can be found in [HLZa] and [HLZb]. We will, however, only present the definitions and results needed for the computations in Section 5.5 below. That is, we will present these definitions and results for vertex operator algebras and their modules only.

DEFINITION 5.1. Let (M_i, Y_i), for $i = 1, 2, 3$, be modules for a vertex operator algebra V. An *intertwining operator* of type $\binom{M_3}{M_1\, M_2}$ is a linear map

$$\mathcal{Y}(\cdot, z)\cdot : M_1 \otimes M_2 \to M_3\{z\} \tag{5.1}$$
$$m_{(1)} \otimes m_{(2)} \mapsto \mathcal{Y}(m_{(1)}, z)m_{(2)} = \sum_{h\in\mathbb{C}} m_{(1)}{}^{\mathcal{Y}}_{h} m_{(2)} z^{-h-1}$$

satisfying the following conditions:

(i) (*truncation condition*) for all $m_{(1)} \in M_1$ and $m_{(2)} \in M_2$,

$$m_{(1)}{}^{\mathcal{Y}}_{h} m_{(2)} = 0 \qquad \text{for all } h \text{ whose real part is sufficiently large,} \tag{5.2}$$

(ii) (*Jacobi identity*) for all $v \in V$, $m_{(1)} \in M_1$ and $m_{(2)} \in M_2$,

$$\begin{aligned} &x^{-1}\delta\left(\frac{y-z}{x}\right) Y_3(u, y)\mathcal{Y}(m_{(1)}, z)m_{(2)} - x^{-1}\delta\left(\frac{z-y}{-x}\right)\mathcal{Y}(m_{(1)}, z)Y_2(v, y)m_{(1)} \\ &\quad = z^{-1}\delta\left(\frac{y-x}{z}\right)\mathcal{Y}(Y_1(v, x)m_{(1)}, z)m_{(2)}, \end{aligned} \tag{5.3}$$

(iii) ($L(-1)$*-derivative property*) for all $m_{(1)} \in M_1$ and $m_{(2)} \in M_2$,

$$\mathcal{Y}(L(-1)m_{(1)}, z)m_{(2)} = \frac{d}{dz}\mathcal{Y}(m_{(1)}, z)m_{(2)}. \tag{5.4}$$

We can see that the notion of an intertwining *operator* is similar to the notion of an intertwining *map* in Definition 3.3. The reason for using intertwining operators, as opposed to intertwining maps, is that the former can record monodromy in the punctured complex plane $\mathbb{C}^\times$. There is a correspondence (in fact, many correspondences) between intertwining operators and intertwining maps of the same type.

Recall our logarithm and substitution conventions outlined in Section A.3.

PROPOSITION 5.2. Fix an integer p. Let (M_i, Y_i), for $i = 1, 2, 3$, be modules of a vertex operator algebra V. Then, there is a linear isomorphism between the space of intertwining operators of type $\binom{M_3}{M_1\, M_2}$ and the space of intertwining maps of type $\binom{M_3}{M_1\, M_2}$:

$$\mathcal{Y} \mapsto I_{\mathcal{Y},p} : M_1 \otimes M_2 \to \overline{M_3} \quad \text{defined by} \tag{5.5}$$
$$I_{\mathcal{Y},p}(m_{(1)} \otimes m_{(2)}) = \mathcal{Y}(m_{(1)}, e^{l_p(w)})m_{(2)} \qquad \text{for all } m_{(1)} \in M_1,\ m_{(2)} \in M_2.$$

The inverse map is

$$I \mapsto \mathcal{Y}_{I,p} : M_1 \otimes M_2 \to M_3\{z\} \quad \text{defined by} \tag{5.6}$$
$$\mathcal{Y}_{I,p}(m_{(1)}, z)m_{(2)} = y^{L(0)}z^{L(0)}I(y^{-L(0)}z^{-L(0)}m_{(1)} \otimes y^{-L(0)}z^{-L(0)}m_{(2)})|_{y=e^{-l_p(w)}},$$

for all $m_{(1)} \in M_1,\ m_{(2)} \in M_2$.

The proof of Proposition 5.2, and an explanation for the existence of (5.5) and (5.6), can be found in Section 4.1 of [HLZb].

5.2 Modular tensor categories from vertex operator algebras

We will present the main results from HLZ and [Hua08], outlining how to construct pre-modular categories from certain types of vertex operator algebras. No proofs will be given here, instead we summarise what is needed for our computations in Section 5.5 and Section 5.6 below.

Recall the assumptions **(A1)** - **(A5)** in Section 3.4 for the construction of a braided monoidal category from a category of vertex operator algebra modules. What follows is a set of stronger assumptions that guarantee a category with a modular tensor structure.

In [Hua08], the vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ is assumed to satisfy the following conditions:

(**B1**) The vertex operator algebra V is simple.

(**B2**) The weight spaces satisfy $V_{(n)} = 0$, for all $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and the contragredient module V' is isomorphic to V in $V-\mathsf{Mod}$.

(**B3**) Every weak V-module is completely reducible.

As stated in [Hua08], condition **(B3)** is equivalent to satisfying both of the following:

(**B3a**) Every $\mathbb{Z}_{\geq 0}$-gradable weak V-module is completely reducible.

(**B3b**) The vertex operator algebra V is C_2-cofinite.

For completeness, we define these notions below, following Section 2 of [Li99].

DEFINITION 5.3. A *weak V-module* (M, Y_M) is a vertex algebra module for the vertex algebra $(V, Y, \mathbf{1})$. That is, a weak $(V, Y, \mathbf{1}, \omega)$-module is a $(V, Y, \mathbf{1})$-module.

As in the case of (vertex operator algebra) V-modules, the Virasoro relations and $L(-1)$-derivative property follow from the definition of a weak V-module.

REMARK 5.4. Recall that a vertex operator algebra module, as in Definition 2.21, is a vertex algebra module with additional conditions. Huang remarks that $\mathbb{Z}_{\geq 0}$-gradable weak V-modules naturally arise in the proofs and hence weak modules must be considered. Assumption **(B3)** is needed to then obtain V-modules, which arise in the Verlinde conjecture (i.e. the construction of modularity). △

DEFINITION 5.5. A $\mathbb{Z}_{\geq 0}$-gradable weak V-module is a weak V-module (M, Y_M) satisfying the following condition:

(i) there exists a $\mathbb{Z}_{\geq 0}$-grading $M = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} M_{(k)}$ such that

$$v_n M_{(k)} \subseteq M_{(\mathrm{wt}\ v+n-k+1)} \qquad \text{for all homogeneous } v \in V,\ n \in \mathbb{Z} \text{ and } k \in \mathbb{Z}_{\geq 0}. \tag{5.7}$$

DEFINITION 5.6. Define the subspace

$$C_2(V) = \mathrm{span}\{u_{-2}v \mid u, v \in V\} \tag{5.8}$$

of V. Then, V is *C_2-cofinite* if $C_2(V)$ has finite codimension, i.e. $\dim \big(V/C_2(V)\big) < \infty$.

We now state the main result of [Hua08].

THEOREM 5.7. (Theorem 4.6 of [Hua08]) If the assumptions **(B1)** - **(B3)** are satisfied by a vertex operator algebra V, then $V-\mathsf{Mod}$ has a natural modular tensor category structure.

REMARK 5.8. Note that $V-\mathsf{Mod}$ in Theorem 5.7 is the category of vertex operator algebra modules of $(V, Y, \mathbf{1}, \omega)$ and their homomorphisms. This is a subcategory of $(V, Y, \mathbf{1})-\mathsf{Mod}$, the category of vertex algebra modules for $(V, Y, \mathbf{1})$ and their homomorphisms. The main difference here is the finite-dimensional constraint on the weight spaces of $(V, Y, \mathbf{1}, \omega)$-modules. For one, this property is useful when obtaining dual objects using contragredient modules. (As an analogy, recall that rigidity is a property generalised from *finite-dimensional* vector spaces.) Furthermore, the requirement for finite-dimensional weight spaces can decrease the number of isomorphism classes of simple objects. If this is reduced to finitely many isomorphism classes of simple objects, then a category with objects of finite Jordan-Hölder length is also ensured to exist. Hence, one can see that the category of vertex operator modules is well-suited for pre-modular and modular tensor structures. $\triangle$

We now summarise how the modular tensor categorical data is constructed in [Hua08]. Let V be a vertex operator algebra satisfying **(B1)** - **(B3)**. Recall that a pre-modular category is a ribbon fusion category. So, the data we need to attach to $V-\mathsf{Mod}$ is:

(i) a tensor product $\boxtimes$,

(ii) a unit object $\mathbb{1}$,

(iii) an associator $\mathcal{A}$, left unitor l and right unitor r,

(iv) a braiding $\mathcal{R}$,

(v) a twist θ.

HLZ gives the construction for $(\boxtimes, \mathbb{1}, \mathcal{A}, l, r, \mathcal{R})$, while [Hua08] gives rigidity, θ and modularity. The *fusion product* $\boxtimes$ on $V-\mathsf{Mod}$ is constructed from a family of $P(w)$-tensor products $(M_1 \boxtimes_{P(w)} M_2, \boxtimes_{P(w)})$, for $w \in \mathbb{C}^\times$ and pairs (M_1, M_2) of V-modules. The fusion product is defined to be the $P(w)$-tensor product bifunctor at $w = 1$. The unit object is $\mathbb{1} = V$, as a V-module. In order to define $\mathcal{A}$ and $\mathcal{R}$, the isomorphism (5.9) below is used to compare the $P(w)$-tensor products at different points, whilst retaining the analytic properties of the punctured plane.

REMARK 5.9. The "double dual" construction in [HLZc] guarantees the existence of the $P(w)$-tensor products but since the $P(w)$-tensor product is defined by a universal property, we can choose any model that best suits our computation. One can verify that the construction of the associator and unitors is independent of the choice of model, up to monoidal equivalence. We will not show this here because it would take too long to present, but one can use the universal property to obtain the morphisms needed to construct the compatibility of associator and unitor commutative diagrams. It follows that the ribbon tensor structure is also model independent. $\triangle$

DEFINITION 5.10. Let M_1 and M_2 be V-modules and let $w_1, w_2 \in \mathbb{C}^\times$. Let γ be a path in $\mathbb{C}^\times$ from w_1 to w_2. Let $\mathcal{Y}$ be the intertwining operator associated to $M_1 \boxtimes_{P(w_2)} M_2$ by the isomorphism (5.6) with $p = 0$, that is, $\mathcal{Y} = \mathcal{Y}_{\boxtimes_{P(w_2)},0}$. Let $l(w_1)$ be the logarithm of w_1 uniquely determined by γ and $\log w_2$. The *parallel transport isomorphism associated to* γ is the V-module isomorphism

$$\mathcal{T}_\gamma : M_1 \boxtimes_{P(w_1)} M_2 \to M_1 \boxtimes_{P(w_2)} M_2, \tag{5.9}$$

characterised by the condition

$$\overline{\mathcal{T}_\gamma}(m_{(1)} \boxtimes_{P(w_1)} m_{(2)}) = \mathcal{Y}(m_{(1)}, z)m_{(2)}|_{z=e^{l(w_1)}}, \tag{5.10}$$

for all $m_{(1)} \in M_1,\ m_{(2)} \in M_2$, where $\overline{\mathcal{T}_\gamma} : \overline{M_1 \boxtimes_{P(w_1)} M_2} \to \overline{M_1 \boxtimes_{P(w_2)} M_2}$ is the extension of $\mathcal{T}_\gamma$.

DEFINITION 5.11. Let M_i, $i = 1, 2, 3$ be V-modules. Let $r_1 > r_2 > r_1 - r_2 > 0$ be real numbers and let

(i) γ_1 be a path in $(0, \infty)$ from 1 to r_1,

(ii) γ_2 be a path in $(0, \infty)$ from 1 to r_2,

(iii) γ_3 be a path in $(0, \infty)$ from r_2 to 1, and

(iv) γ_4 be a path in $(0, \infty)$ from $r_1 - r_2$ to 1.

The *associativity isomorphism associated to* r_1 *and* r_2 is the V-module isomorphism

$$\mathcal{A}^{P(r_1-r_2),P(r_2)}_{P(r_1),P(r_2)} : M_1 \boxtimes_{P(r_1)} (M_2 \boxtimes_{P(r_2)} M_3) \to (M_1 \boxtimes_{P(r_1-r_2)} M_2) \boxtimes_{P(r_2)} M_3, \tag{5.11}$$

characterised by the condition

$$\overline{\mathcal{A}^{P(r_1-r_2),P(r_2)}_{P(r_1),P(r_2)}} : m_{(1)} \boxtimes_{P(r_1)} (m_{(2)} \boxtimes_{P(r_2)} m_{(3)}) \mapsto (m_{(1)} \boxtimes_{P(r_1-r_2)} m_{(2)}) \boxtimes_{P(r_2)} m_{(3)}, \tag{5.12}$$

for all $m_{(i)} \in M_i$, $i = 1, 2, 3$. The *associator of* M_1, M_2 *and* M_3 is the V-module isomorphism $\mathcal{A}_{M_1,M_2,M_3}$ defined by the composition

$$\begin{tikzcd}
M_1 \boxtimes (M_2 \boxtimes M_3) \arrow[r, "\mathcal{A}_{M_1,M_2,M_3}"] \arrow[d, "\mathcal{T}_{\gamma_1}"'] & (M_1 \boxtimes M_2) \boxtimes M_3 \\
M_1 \boxtimes_{P(r_1)} (M_2 \boxtimes M_3) \arrow[d, "\mathrm{id}_{M_1} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}"'] & (M_1 \boxtimes M_2) \boxtimes_{P(r_2)} M_3 \arrow[u, "\mathcal{T}_{\gamma_3}"'] \\
M_1 \boxtimes_{P(r_1)} (M_2 \boxtimes_{P(r_2)} M_3) \arrow[r, "\mathcal{A}^{P(r_1-r_2),P(r_2)}_{P(r_1),P(r_2)}"'] & (M_1 \boxtimes_{P(r_1-r_2)} M_2) \boxtimes_{P(r_2)} M_3 \arrow[u, "\mathcal{T}_{\gamma_4} \boxtimes_{P(r_2)} \mathrm{id}_{M_3}"']
\end{tikzcd} . \tag{5.13}$$

DEFINITION 5.12. Let M be a V-module. The *left unitor of* M is the V-module isomorphism

$$l_M : V \boxtimes M \to M, \tag{5.14}$$

characterised by the condition

$$l_M(\mathbf{1} \boxtimes m) = m \qquad \text{for all } m \in M. \tag{5.15}$$

Note that an extension is not needed here since $\mathbf{1} \boxtimes m \in V \boxtimes M$. The *right unitor of* M is the V-module isomorphism

$$r_M : M \boxtimes V \to M, \tag{5.16}$$

characterised by the condition

$$\overline{r_M}(m \boxtimes \mathbf{1}) = e^{L(-1)}m \qquad \text{for all } m \in M. \tag{5.17}$$

DEFINITION 5.13. Let γ_1^- be the path from -1 to 1 along the unit circle in the upper half plane. Let M_1 and M_2 be V-modules. The *braiding of* M_1 *and* M_2 is the V-module isomorphism

$$\mathcal{R}_{M_1,M_2} : M_1 \boxtimes M_2 \to M_2 \boxtimes M_1, \tag{5.18}$$

characterised by the condition

$$\overline{\mathcal{R}_{M_1,M_2}}(m_{(1)} \boxtimes_{P(1)} m_{(2)}) = e^{L(-1)}\overline{\mathcal{T}_{\gamma_1^-}}(m_{(2)} \boxtimes_{P(-1)} m_{(1)}), \tag{5.19}$$

for all $m_{(1)} \in M_1$ and $m_{(2)} \in M_2$.

The associator, unitors and braiding become the natural isomorphisms with components as defined in Definitions 5.11, 5.12 and 5.13, respectively. The reasons for why these V-module homomorphisms exist can be found in [HLZe], [HLZf] and [HLZg].

The proof of the rigidity of the monoidal category $(V{-}\mathsf{Mod}, \boxtimes, V, \mathcal{A}, l, r, \mathcal{R})$, with an explicit construction of left and right duals and evaluation and coevaluation morphisms, can be found in Section 3 of [Hua08]. For a V-module M, the left and right dual object of M can be taken to be the contragredient module M'.

DEFINITION 5.14. Let M be a V-module. The *twist of* M is defined as the V-module isomorphism

$$\theta_M = e^{2\pi i L(0)}. \tag{5.20}$$

The twist is the natural isomorphism with components as defined in Definition 5.14. In Section 4 of [Hua08], it is shown that $(V{-}\mathsf{Mod}, \boxtimes, V, \mathcal{A}, l, r, \mathcal{R}, \theta)$ is a ribbon category. The $\mathbb{C}$-linear abelian structure is the usual structure in $V{-}\mathsf{Mod}$, finiteness is shown and $(V{-}\mathsf{Mod}, \boxtimes, V, \mathcal{A}, l, r, \mathcal{R}, \theta)$ is shown to be a modular tensor category.

EXAMPLE 5.15. Consider the braided monoidal category $\mathsf{H}-\mathsf{Mod}'$ from Example 3.22, constructed from the Heisenberg vertex operator algebra. Recall that assumptions **(A2)** and **(A5)** were violated, but we were still able to produce a braided monoidal category using HLZ. It is then natural to ask whether $\mathsf{H}-\mathsf{Mod}'$, following [Hua08], can be given a pre-modular structure.

We have computed the following additional structure on $(\mathsf{H}-\mathsf{Mod}', \boxtimes, \mathsf{H}, \mathcal{A}, l, r, \mathcal{R})$ (from Example 3.22):

(i) left and right duals $F^{-\lambda}$ of F^{λ} with evaluation and coevaluations identities, for $\lambda \in \mathbb{C}$,

(ii) the twist defined by $\theta_{F^\lambda} = e^{i\pi\lambda^2}\,\mathrm{id}_{F^\lambda}$, for $\lambda \in \mathbb{C}$,

(iii) the spherical structure $a = \mathrm{id}$ (canonically from θ as in Definition/Proposition 4.37).

(This computation is omitted but it is similar to the computation for the lattice vertex operator algebras below). Hence, by using the abelian structure adopted from the category of H-modules, $\mathsf{H}-\mathsf{Mod}'$ has the structure of a ribbon tensor category. However, $\mathsf{H}-\mathsf{Mod}'$ is not pre-modular since there are infinitely many isomorphism classes of simple objects.

There are no non-trivial finite subgroups of $(\mathbb{C}, +)$, the group that characterises the simple objects and the tensor product of $\mathsf{H}-\mathsf{Mod}'$, so $\mathsf{H}-\mathsf{Mod}'$ has no pre-modular subcategory other than the full subcategory of direct sums of the vacuum module. In this case, the S-matrix is $S = (1)$, hence the pre-modular category is also modular, but this trivial example of a pre-modular category is not of interest to us.

Semisimplification (see Appendix E) is no help either. The dimension of each simple object is $\dim_a(F^\lambda) = 1$, so the semisimplification of $\mathsf{H}-\mathsf{Mod}'$, nor any non-trivial tensor subcategory, still has infinitely many simple objects. $\Diamond$

We are yet to see any examples of modular tensor categories with more than one simple object. Fortunately, the lattice vertex operators will produce a family of non-trivial modular tensor categories, which we will explicitly compute below.

5.3 Lattice vertex operator algebras

In this section, we will describe the lattice vertex operator algebras, their modules and their intertwining operators. We will use the root lattices of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ as guiding examples. Only the former example will be used to in Chapter 6 to construct an explicit Kazhdan-Lusztig correspondence, however, other simply laced root lattices would be useful examples if we were to further explore generalisations of our results from Chapter 6.

Lattice vertex operator algebras are constructed from the data provided by a positive definite even lattice of finite rank. Roughly speaking, one can think of a lattice vertex operator algebra of rank d as d copies of the free boson vertex operator algebra and countably infinitely many of their modules. The free bosons act on each other, via their vertex operators, with an additional phase in order to satisfy the vertex operator algebra axioms. In what follows, we will use a central extension as the datum needed to describe this phase.

5.3.1 Setting

We first lay the setting for the notation and conventions using [DL93], Section 6.4 and Section 6.5 of [LL04], and [Don95].

DEFINITION 5.16. An *even lattice* $(L_0, \langle\cdot,\cdot\rangle)$ consists of the following data:

(i) a finitely generated free abelian group L_0,

(ii) a $\mathbb{Q}$-valued $\mathbb{Z}$-bilinear form $\langle\cdot,\cdot\rangle$ on L_0,

satisfying the following conditions:

(i) $\langle\cdot,\cdot\rangle$ is symmetric, non-degenerate,

(ii) $\langle\alpha,\alpha\rangle \in 2\mathbb{Z}$ for all $\alpha \in L_0$.

We call an even lattice *positive definite* if its bilinear form is positive definite.

REMARK 5.17. We are only interested in even lattices that are positive definite, as these produce vertex operator algebras, but we will still state this every time for clarity. $\triangle$

Let $(L_0, \langle\cdot,\cdot\rangle)$ be a positive definite even lattice. Define the vector space $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L_0$. Extend $\langle\cdot,\cdot\rangle$ to $\mathfrak{h}$ by $\mathbb{C}$-linearity. Viewing $\mathfrak{h}$ as an abelian Lie algebra of dimension $\operatorname{rank}(L_0)$, consider the affinisation

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}, \tag{5.21}$$

where we denote $a \otimes t^m$ by $a(m)$, for all $a \in \mathfrak{h}$ and $m \in \mathbb{Z}$. The Lie bracket is

$$[a(m), b(n)] = \langle a, b\rangle m\delta_{m+n,0}\mathbf{c} \qquad \text{and} \qquad [\mathbf{c}, \widehat{\mathfrak{h}}] = 0, \tag{5.22}$$

for $a, b \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. We decompose $\widehat{\mathfrak{h}}$ into subalgebras

$$\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}^+ \oplus \widehat{\mathfrak{h}}^- \oplus \mathfrak{h} \oplus \mathbb{C}\mathbf{c} = \widehat{\mathfrak{h}}_* \oplus \mathfrak{h}, \tag{5.23}$$

where

$$\widehat{\mathfrak{h}}^+ = \mathfrak{h} \otimes t\mathbb{C}[t], \quad \widehat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}], \quad \mathfrak{h} = \mathfrak{h} \otimes \mathbb{C}t^0 \quad \text{and} \quad \widehat{\mathfrak{h}}_* = \widehat{\mathfrak{h}}^+ \oplus \widehat{\mathfrak{h}}^- \oplus \mathbb{C}\mathbf{c}. \tag{5.24}$$

Define the $\widehat{\mathfrak{h}}$-module

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{c})} \mathbb{C}_0 \cong_{\mathbb{C}-\mathsf{Vec}} U(\widehat{\mathfrak{h}}^-), \tag{5.25}$$

where $\mathbb{C}_0 = \mathbb{C}$ is the $\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{c}$-module with trivial action of $\mathfrak{h} \otimes \mathbb{C}[t]$ and $\mathbf{c}$ acting as the identity.

Define the *dual lattice* of L_0 to be the lattice

$$(L_0)^\circ = \{\beta \in \mathfrak{h} : \langle \beta, L_0\rangle \subseteq \mathbb{Z}\}. \tag{5.26}$$

Let $\widehat{(L_0)^\circ}$ be a central extension of groups

$$1 \to \langle\omega_q\rangle \to \widehat{(L_0)^\circ} \xrightarrow{\bar{\cdot}} (L_0)^\circ \to 1, \tag{5.27}$$

for some $q \in 2\mathbb{Z}_{>0} \cup \{1\}$, where $\langle\omega_q\rangle$ is the group generated by the primitive q^{th}-root of unity $\omega_q \in \mathbb{C}^\times$. Assume that the *commutator map*

$$\mathsf{c} : (L_0)^\circ \times (L_0)^\circ \to \langle\omega_q\rangle \hookrightarrow \widehat{(L_0)^\circ}, \ \mathsf{c}(\overline{a}, \overline{b}) = aba^{-1}b^{-1} \qquad \text{for } a, b \in \widehat{(L_0)^\circ}, \tag{5.28}$$

satisfies

$$\mathsf{c}(\alpha, \beta) = (-1)^{\langle\alpha,\beta\rangle} \qquad \text{for all } \alpha, \beta \in L_0 \subseteq (L_0)^\circ. \tag{5.29}$$

REMARK 5.18. Note that the group operation in $(L_0)^\circ$ is written additively, even though we denote the trivial group by 1 in (5.27); we do this because $\widehat{(L_0)^\circ}$ may be non-abelian in general. In (5.29), in makes sense to raise -1 to the power of $\langle\alpha,\beta\rangle$ since $\langle\alpha,\beta\rangle = \frac{1}{2}(\langle\alpha+\beta,\alpha+\beta\rangle - \langle\alpha,\alpha\rangle - \langle\beta,\beta\rangle) \in \mathbb{Z}$ when α and β are in the even lattice L_0. This is why $q \in 2\mathbb{Z}_{>0}$ so $-1 \in \langle\omega_q\rangle$, or $q = 1$ when $\langle L_0, L_0\rangle \in 2\mathbb{Z}$ (as in the case of a one-dimensional lattice). $\triangle$

Define the $\mathbb{C}[\widehat{(L_0)^\circ}]$-module induced from the $\mathbb{C}[\langle\omega_q\rangle]$-module $\mathbb{C}$ on which ω_q acts by multiplication:

$$\mathbb{C}\{\widehat{(L_0)^\circ}\} = \mathbb{C}[\widehat{(L_0)^\circ}] \otimes_{\mathbb{C}[\langle\omega_q\rangle]} \mathbb{C} \cong_{\mathbb{C}-\mathsf{Vec}} \mathbb{C}[(L_0)^\circ]. \tag{5.30}$$

We write $\iota(a) = a \otimes_{\mathbb{C}[\langle\omega_q\rangle]} 1$, for each $a \in \widehat{(L_0)^\circ}$. Then, the $\widehat{(L_0)^\circ}$ action is given by

$$a \cdot \iota(b) = \iota(ab) \quad \text{and} \quad \omega_q \cdot \iota(b) = \omega_q \iota(b) \qquad \text{for all } a, b \in \widehat{(L_0)^\circ}. \tag{5.31}$$

For each $h \in \mathfrak{h}$ and a formal variable z, define the actions

$$h \cdot \iota(a) = \langle h, \overline{a}\rangle \iota(a) \quad \text{and} \quad z^h \cdot \iota(a) = z^{\langle h,\overline{a}\rangle}\iota(a) \qquad \text{for all } a \in \widehat{(L_0)^\circ}. \tag{5.32}$$

Define the vector space

$$V_{(L_0)^\circ} = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{\widehat{(L_0)^\circ}\} \cong_{\mathbb{C}-\mathsf{Vec}} U(\widehat{\mathfrak{h}}^-) \otimes \mathbb{C}[(L_0)^\circ]. \tag{5.33}$$

Let $\widehat{\mathfrak{h}}_*$ act on $V_{(L_0)^\circ}$ by acting on $M(1)$. Further let $\widehat{(L_0)^\circ}$, $\mathfrak{h}$ and $z^{\mathfrak{h}}$ act on $V_{(L_0)^\circ}$ by acting on $\mathbb{C}\{\widehat{(L_0)^\circ}\}$. Hence, the operators $\widehat{(L_0)^\circ}$, $\mathfrak{h}$ and $z^{\mathfrak{h}}$ commute with $\widehat{\mathfrak{h}}_*$.

REMARK 5.19. Looking at (5.33), it may appear that we have removed the dependence on the central extension (5.27) since $V_{(L_0)^\circ} \cong_{\mathbb{C}-\mathsf{Vec}} U(\widehat{\mathfrak{h}}^-) \otimes \mathbb{C}[(L_0)^\circ]$. But, this datum is still encoded in $V_{(L_0)^\circ}$ as a $\widehat{(L_0)^\circ}$-module. Consider $\widehat{(L_0)^\circ} :=_{\mathsf{Sets}} \langle\omega_q\rangle \times (L_0)^\circ$, a central extension of $(L_0)^\circ$ by $\langle\omega_q\rangle$ with the associated 2-cocycle $\epsilon : (L_0)^\circ \times (L_0)^\circ \to \langle\omega_q\rangle$. Then,

$$(1,\alpha)\iota(1,\beta) = \iota((1,\alpha)(1,\beta)) = \iota(\epsilon(\alpha,\beta), \alpha+\beta) = \epsilon(\alpha,\beta)\iota(1,\alpha+\beta),$$

for all $\alpha, \beta \in (L_0)^\circ$. So, even though $V_{(L_0)^\circ}$ has a basis of highest weight vectors corresponding to $(L_0)^\circ$, the central extension $\widehat{(L_0)^\circ}$ is still being used to track phases. $\triangle$

For any subset M of $(L_0)^\circ$, we denote the $\overline{\cdot}$-preimage of M by

$$\widehat{M} = \{b \in \widehat{(L_0)^\circ} : \overline{b} \in M\}, \tag{5.34}$$

and define the subspaces

$$\mathbb{C}\{M\} = \mathrm{span}_{\mathbb{C}}\{\iota(b) : b \in \widehat{M}\} \subseteq \mathbb{C}\{L\}, \tag{5.35}$$

$$V_M = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{\widehat{M}\} \subseteq V_{(L_0)^\circ}. \tag{5.36}$$

Choose representatives $\lambda_i \in (L_0)^\circ$, for each $i \in (L_0)^\circ/L_0$, such that $i = L_0 + \lambda_i$ and $\lambda_{L_0} = 0$. (We will need to fix these representatives as the braided monoidal data depends on their choice, however, we will show in Proposition 5.42 below that this dependence is only up to braided monoidal equivalence.) Denote, for each $i \in (L_0)^\circ/L_0$, the space

$$V(i) = V_{L_0+\lambda_i}. \tag{5.37}$$

We have the decomposition

$$V_{(L_0)^\circ} = \bigoplus_{i\in(L_0)^\circ/L_0} V(i). \tag{5.38}$$

We now define the *untwisted vertex operators* $Y(v,z)$ for $v \in V_{(L_0)^\circ}$. Define the series

$$\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha(n) z^{-n-1} \qquad \text{for } \alpha \in \mathfrak{h}.$$

For each $a \in \widehat{(L_0)}^\circ$, define

$$Y(\iota(a), z) = e^{\sum_{n>0} \bar{a}(-n)\frac{z^n}{n}} e^{\sum_{n>0} \bar{a}(n)\frac{z^{-n}}{-n}} a z^{\bar{a}}. \tag{5.39}$$

For $a \in \widehat{(L_0)}^\circ$, $\alpha_1, \dots, \alpha_\ell \in \mathfrak{h}$, $n_1, \dots, n_\ell \in \mathbb{Z}_{>0}$ and $\ell \in \mathbb{Z}_{\geq 0}$, denote by v the vector $\alpha_1(-n_1)\cdots\alpha_\ell(-n_\ell)\iota(a)$ and define

(5.40)
$$Y(v,z) = {}^\circ_\circ \left(\frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1}\alpha_1(z)\right)\cdots\left(\frac{1}{(n_\ell-1)!}\left(\frac{d}{dz}\right)^{n_\ell-1}\alpha_\ell(z)\right) Y(\iota(a),z) {}^\circ_\circ.$$

REMARK 5.20. The normal ordering in (5.40) is an extension of the normal ordering defined in Section A.2 to non-integral powers of z. The non-integral powers arise because $z^{\bar{a}}\iota(b) = z^{\langle \bar{a}, \bar{b}\rangle}\iota(b)$ with $\langle \bar{a}, \bar{b}\rangle$ not necessarily an integer. This normal ordering is defined in [LL04] by introducing *weak vertex operators*, but we will follow [DL93] by defining the normal ordering on "modes" (note that the operators in $z^{\mathfrak{h}}$ evaluate to a power of z, so are not actually endomorphisms of $V_{(L_0)^\circ}$, but rather serve to shift the indexing of the other modes). The *normal ordering* ${}^\circ_\circ \cdot {}^\circ_\circ$ is defined to be the reordering of a product of operators so that all operators in $\mathfrak{h}t^{-1}\mathbb{C}[t^{-1}]$ and $\widehat{(L_0)}^\circ$ are placed to the left of any operators in $\mathfrak{h}\mathbb{C}[t] \oplus \mathbb{C}\mathbf{c}$ and $z^{\mathfrak{h}}$ before the expression is evaluated. △

The series (5.39) gives the well-defined linear map

$$Y(\cdot, z) : V_{(L_0)^\circ} \to (\operatorname{End} V_{(L_0)^\circ})\{z\}, \qquad v \mapsto Y(v,z) = \sum_{n\in\mathbb{C}} v_n z^{-n-1}. \tag{5.41}$$

We call $Y(v,z)$ the *untwisted vertex operator associated with* v. These untwisted vertex operators will be used to define the intertwining operators in Subsection 5.3.3 below. The "vertex operator map" $Y(\cdot, z)$ will not be the vertex operator map of a vertex algebra since $V_{(L_0)^\circ}$ will not given be a vertex algebra structure,[1] but instead V_{L_0} will.

EXAMPLE 5.21. Consider the case when $L_0 = \mathbb{Z}\alpha$ is a rank-one positive definite even lattice with the $\mathbb{Z}$-bilinear form defined by $\langle \alpha, \alpha\rangle = 2$. Note that this is the root lattice of $\mathfrak{sl}_2$ where the Killing form has been normalised so that the (longest) root has a norm-squared of 2. Then, $\mathfrak{h} = \mathbb{C}\alpha$ and the dual lattice is $(L_0)^\circ = \frac{1}{2}\mathbb{Z}\alpha$. Note that the dual lattice

[1]The series $Y(v,z)$, for $v \notin V_{L_0}$, are vertex operators in the *generalised vertex algebra* $V_{(L_0)^\circ}$ (see [DL93]). We will not discuss generalised vertex algebras in any detail here, but $V_{(L_0)^\circ}$ is essentially a space containing a copy of each irreducible V_{L_0}-module, up to isomorphism, in which one can compute their intertwining operators.

is the weight lattice of $\mathfrak{sl}_2$. Choose $L = (L_0)^\circ$ and $\omega_q = 1$ so that the central extension

$$1 \to 1 \to \widehat{(L_0)^\circ} \xrightarrow{\bar{\cdot}} (L_0)^\circ \to 1 \tag{5.42}$$

is satisfied by $\widehat{(L_0)^\circ} = (L_0)^\circ$ and $\bar{\cdot} = \mathrm{id}_{(L_0)^\circ}$. In this case, we simply write $\widehat{(L_0)^\circ}$ additively, instead of multiplicatively. We will give the cosets in the quotient group

$$(L_0)^\circ/L_0 = (\tfrac{1}{2}\mathbb{Z}\alpha)/(\mathbb{Z}\alpha) = \{L_0, L_0 + \tfrac{1}{2}\alpha\} \cong_{\mathsf{Group}} \mathbb{Z}_2, \tag{5.43}$$

the representatives $\lambda_0 = 0$ and $\lambda_1 = \frac{1}{2}\alpha$. Here we have identified $L_0 = 0$ and $L_0 + \frac{1}{2}\alpha = 1$, and the addition of $i, j \in (L_0)^\circ/L_0 = \{0, 1\}$ will be modulo 2. We can write $V(0)$ and $V(1)$ as spans of the following bases, respectively,

$$\begin{aligned} &\{\alpha(-n_1)\cdots\alpha(-n_\ell)\iota(m\alpha) \mid n_1 \geq \cdots \geq n_\ell \geq 1,\ \ell \in \mathbb{Z}_{\geq 0},\ m \in \mathbb{Z}\}, \\ &\{\alpha(-n_1)\cdots\alpha(-n_\ell)\iota(m\alpha) \mid n_1 \geq \cdots \geq n_\ell \geq 1,\ \ell \in \mathbb{Z}_{\geq 0},\ m \in \mathbb{Z} + \tfrac{1}{2}\}. \quad \Diamond \end{aligned} \tag{5.44}$$

EXAMPLE 5.22. Consider the root lattice of $\mathfrak{sl}_3$, that is, $L_0 = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ with the symmetric $\mathbb{Z}$-bilinear form given by the A_2 Cartan matrix:

$$\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 2 \quad \text{and} \quad \langle \alpha_1, \alpha_2 \rangle = -1. \tag{5.45}$$

This bilinear form is non-degenerate, positive definite and even.

The dual lattice is

$$(L_0)^\circ = \mathrm{span}_{\mathbb{Z}}\{\alpha_1^*, \alpha_2^*\}, \quad \text{where } \alpha_1^* = \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2,\ \alpha_2^* = \tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2. \tag{5.46}$$

Since $\langle \alpha_1, \alpha_2 \rangle = -1$, we need a non-trivial and non-abelian central extension

$$1 \to \langle \omega_q \rangle \to \widehat{(L_0)^\circ} \xrightarrow{\bar{\cdot}} (L_0)^\circ \to 1 \tag{5.47}$$

in order to have a commutator satisfying $\mathsf{c}(\alpha_1, \alpha_2) = (-1)^{\langle \alpha_1, \alpha_2 \rangle} = -1$.

One such central extension is given as follows. We will choose q to be the smallest positive integer such that $q\langle \alpha, \beta \rangle \in 2\mathbb{Z}$, for all $\alpha, \beta \in (L_0)^\circ$. Use the $(L_0)^\circ$ and L_0 bases, respectively:

$$\{\beta_1, \beta_2\} \quad \text{and} \quad \{\beta_1, 3\beta_2\}, \quad \text{where } \beta_1 := \alpha_1,\ \beta_2 := -\frac{1}{3}\alpha_1 + \frac{1}{3}\alpha_2. \tag{5.48}$$

Since $\langle \alpha_1, \alpha_2 \rangle = -1$, we have $\langle L_0, L_0 \rangle = \mathbb{Z}$, hence $\langle (L_0)^\circ, (L_0)^\circ \rangle = \frac{1}{3}\mathbb{Z}$ and $q = 6$.

For all elements $\alpha = m_1\beta_1 + m_2\beta_2$ and $\beta = n_1\beta_1 + n_2\beta_2$ in $(L_0)^\circ$, define $\epsilon(\alpha, \beta) = \omega_6^{3n_2m_1} \in \{\pm 1\}$. Then, $\epsilon : (L_0)^\circ \times (L_0)^\circ \to \langle \omega_6 \rangle$ satisfies:

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma) \quad \text{and} \quad \epsilon(\alpha, 0) = \epsilon(0, \alpha) = 0, \tag{5.49}$$

for all $\alpha, \beta, \gamma \in (L_0)^\circ$. That is, ϵ is a normalised 2-cocycle for the trivial group action of $(L_0)^\circ$ on $\langle \omega_6 \rangle$. Hence, ϵ corresponds to a central extension of $(L_0)^\circ$ by $\langle \omega_6 \rangle$ defined as $\widehat{(L_0)^\circ} =_{\mathsf{Sets}} \langle \omega_6 \rangle \times (L_0)^\circ$ with group multiplication given by

$$(\omega_6^n, \alpha) = (\omega_6^n, 0)(1, \alpha) \quad \text{and} \quad (1, \alpha)(1, \beta) = (\epsilon(\alpha, \beta), \alpha + \beta),$$

for all $\alpha, \beta \in (L_0)^\circ$ and $n \in \{0, \ldots, 5\}$, with $(\langle \omega_6 \rangle, 0)$ central. The commutator is $\mathsf{c}(\alpha, \beta) = \epsilon(\alpha, \beta)/\epsilon(\beta, \alpha) = \omega_6^{3n_2m_1 + 3m_2n_1}$, which indeed satisfies $\mathsf{c}(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$, for all $\alpha, \beta \in L_0$.

Lastly, the quotient lattice is

$$(L_0)^\circ/L_0 = \{L_0, L_0 + \beta_1, L_0 + 2\beta_2\} = \{L_0, L_0 + \alpha_1^*, L_0 + \alpha_2^*\}. \qquad \Diamond$$

REMARK 5.23. To find the central extension in Example 5.22 above, we used the commutator construction from Remark 6.4.12 of [LL04], together with the 2-cocycle construction from Proposition 5.2.3 of [FLM88]. Both of these references give general constructions for arbitrary positive-definite even lattices. That is, every positive definitive even lattice has a dual lattice with a central extension (5.27) satisfying (5.29). $\triangle$

5.3.2 The lattice vertex operator algebra and its modules

The "vertex operator map" $Y(\cdot, z)$, as defined in (5.40), restricts to the subspace $V(0) = V_{L_0} \subseteq V_{(L_0)^\circ}$ as

$$Y(\cdot, z) : V_{L_0} \to (\text{End } V_{L_0})[[z, z^{-1}]], \quad v \mapsto Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}. \tag{5.50}$$

REMARK 5.24. Note that (5.50) is restricted from (End V_{L_0})$\{z\}$ to (End V_{L_0})$[[z, z^{-1}]]$. We can do this since $z^{\overline{a}}\iota(b) = z^{\langle \overline{a}, \overline{b}\rangle}\iota(b)$ only has integral powers of z because $\langle \overline{a}, \overline{b}\rangle \in \mathbb{Z}$, for all $a, b \in \widehat{L_0}$. $\triangle$

We define the vacuum

$$\mathbf{1} = 1 \otimes \iota(1) \in V_{L_0}, \tag{5.51}$$

and the conformal vector

$$\omega = \frac{1}{2}\sum_{k=1}^{d} u^k(-1)u^k(-1) \otimes \iota(1) \in V_{L_0}, \tag{5.52}$$

where $\{u^k\}_{k=1}^d$ is any orthonormal basis of $\mathfrak{h}$ and $d = \text{rank}(L_0)$. Note that out of all the $V(i)$, with $i \in (L_0)^\circ/L_0$, only the subspace $V(0) = V_{L_0}$ contains the vacuum and the conformal vector.

The conformal vector gives the Virasoro mode

$$L(0) = \sum_{m=1}^{\infty}\sum_{k=1}^{d} u^k(-m)u^k(m) + \frac{1}{2}\sum_{k=1}^{d} u^k(0)u^k(0),$$

which grades $V(i)$, for $i \in (L_0)^\circ/L_0$, by its eigenvalues, denoted by wt. Recall that in (5.33), the modes of $\widehat{\mathfrak{h}}_*$ are defined to act on $M(1)$, with positive modes annihilating $\iota(a)$, and the zero-modes $\mathfrak{h} = \mathfrak{h} \otimes \mathbb{C}t^0$ are defined to act on $\mathbb{C}\{\widehat{(L_0)^\circ}\}$ by evaluating a in $\iota(a)$. Hence, the $L(0)$-eigenvalues of the basis vectors are

$$\begin{aligned}\text{wt}(\alpha_1(-n_1)\cdots\alpha_\ell(-n_\ell) \otimes \iota(a)) &= \text{wt}(\alpha_1(-n_1)\cdots\alpha_\ell(-n_\ell)) + \text{wt}(\iota(a)) \\ &= \sum_{j=1}^{\ell} n_j + \frac{1}{2}\langle \overline{a}, \overline{a}\rangle,\end{aligned} \tag{5.53}$$

for all $a \in \widehat{L_0 + \lambda_i}$, $\alpha_1, \dots, \alpha_\ell \in L_0$, $n_1, \dots, n_\ell \in \mathbb{Z}_{>0}$, $\ell \in \mathbb{Z}_{\geq 0}$.

REMARK 5.25. Equation (5.53) shows that if $(L_0, \langle \cdot, \cdot \rangle)$ was not positive-definite, then $\bar{a} \in L_0$ could be chosen so that $\iota(a) \in V_{L_0}$ has arbitrarily negative conformal weight. Hence, the positive-definiteness guarantees the grading restriction (2.18). Since V_{L_0} can be written in terms of a PBW-basis, V_{L_0} satisfies the finite-dimensional weight space condition (2.17). △

THEOREM 5.26. (From Section 6.5 of [LL04]). Let $(L_0, \langle \cdot, \cdot \rangle)$ be a positive definite even lattice and let $(\widehat{(L_0)^\circ}, \bar{\cdot})$ be a central extension as in Subsection 5.3.1. Then,

(i) $(V_{L_0}, Y, \mathbf{1}, \omega)$ is a vertex operator algebra.
(ii) The $V(i)$, for $i \in (L_0)^\circ / L_0$, form a complete list of the irreducible V_{L_0}-modules, up to isomorphism.
(iii) Any V_{L_0}-module is completely reducible.

REMARK 5.27. Recall that by "$V(0)$-module", we mean a vertex operator algebra module with finite-dimensional weight spaces and zero-dimensional weight spaces for weights with a sufficiently negative real component. The positive definiteness of $(L_0, \langle \cdot, \cdot \rangle)$ and the PBW-bases from induction provide $V(i)$ with these properties. △

REMARK 5.28. The vertex operator algebra V_{L_0} satisfies Conditions **(B1)**-**(B3)**, needed to guarantee that the category of V_{L_0}-modules naturally has a modular tensor category structure. Since the vacuum module of V_{L_0} is simple, V_{L_0} is a simple vertex operator algebra, that is, **(B1)** holds. Consider any irreducible V_{L_0}-module $V(i)$, $i \in (L_0)^\circ / L_0$. Then, all non-zero $L(0)$-eigenspaces have weight $n + \frac{1}{2}\langle \alpha, \alpha \rangle \geq 0$, for $n \in \mathbb{Z}_{\geq 0}$ and $\alpha \in L_0 + \lambda_i$. Moreover, equality holds only when $n = 0$ and $\alpha = 0 \in L_0$, so **(B2)** holds. Furthermore, **(B3)** is proven in Theorem 3.16 of [DLM97]. △

EXAMPLE 5.29. We will continue Example 5.21, using the $\mathfrak{sl}_2$ root lattice. An orthonormal basis of $\mathfrak{h}$ is $\{\frac{1}{\sqrt{2}}\alpha\}$, so we have

$$L(0) = \frac{1}{4}\alpha(0)\alpha(0) + \frac{1}{2}\sum_{k=1}^{\infty} \alpha(-k)\alpha(k). \tag{5.54}$$

Moreover, $L(0)$ acts on the basis vectors of the V_{L_0}-module $V(i)$, for $i \in (L_0)^\circ / L_0$, as

$$L(0)\alpha(-n_1)\cdots\alpha(-n_\ell)\iota(m\alpha) = \left(m^2 + \sum_{k=1}^{\ell} n_k\right)\alpha(-n_1)\cdots\alpha(-n_\ell)\iota(m\alpha), \tag{5.55}$$

for all $m \in \frac{1}{2}\mathbb{Z}$, $n_1 \geq \cdots n_\ell \geq 1$, $\ell \in \mathbb{Z}_{\geq 0}$.

We can renormalise the modes of $\alpha(-1)\iota(0)$ to be $\alpha_n := \frac{1}{\sqrt{2}}\alpha(n)$, for each $n \in \mathbb{Z}$. Then, $\{\alpha_n : n \in \mathbb{Z}\} \cup \{\mathbf{c}\}$ satisfies the same commutation relations as the mode algebra for H, the rank-1 Heisenberg vertex operator algebra, from Section 2.4. The subspace generated by the modes $\{\alpha_{-n} : n \in \mathbb{Z}_{>0}\}$ from $\iota(0)$, together with $Y(\cdot, z)$ restricted to this subspace, is vertex operator algebra isomorphic to H. We can see that $V_{(L_0)^\circ} = V(0) \oplus V(1)$ is a

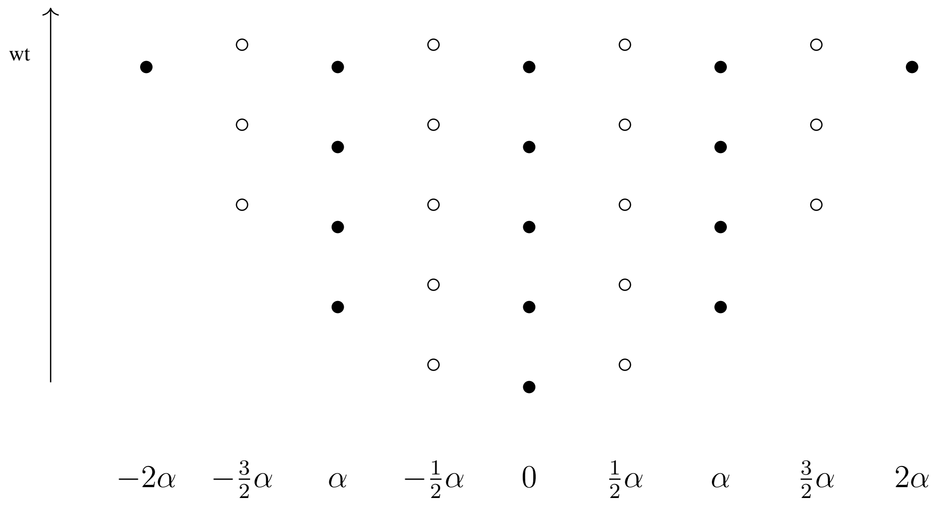

FIGURE 5.1: The solid dots correspond to basis vectors in $V(0)$ and the open dots correspond to basis vectors in $V(1)$. The diagram is organised with conformal weights increasing vertically. The lowest weight in the column labelled by $m\alpha$, for $m \in \frac{1}{2}\mathbb{Z}$, is $\mathrm{wt}\,\iota(m\alpha) = m^2$, and then the weights increase by 1. The dot corresponding to weight $n + \mathrm{wt}\,\iota(m\alpha)$, for $n \in \mathbb{Z}_{\geq 0}$, has a multiplicity of $P(n)$, the number of partitions of n.

direct sum of infinitely many H-modules $F_{\sqrt{2}m}$, for $m \in \frac{1}{2}\mathbb{Z}$. That is, for each $m \in \frac{1}{2}\mathbb{Z}$, the modes $\{\alpha_n : n \in \mathbb{Z}\}$ generate $F_{\sqrt{2}m}$ from the highest weight vector $\iota(m\alpha)$, meaning its α_0-eigenvalue is $\sqrt{2}m$ and $\alpha_n\iota(m\alpha) = 0$, for all $n > 0$. Furthermore, $V(0)$ contains all of the H-modules $F_{\sqrt{2}m}$ with integral m and $V(1)$ contains all of the H-modules $F_{\sqrt{2}m}$ with half-integral m. In Figure 5.1, is a diagram of the $V_{(L_0)^\circ}$ basis vectors organised by conformal weight. $\diamondsuit$

REMARK 5.30. We saw in Example 5.15 that H has an infinitely many isomorphism classes of simple objects and is, hence, not pre-modular. We also saw in the previous example that H is a sub-vertex operator algebra of V_{L_0}, where L_0 is the root lattice of $\mathfrak{sl}_2$. By adding more generators and relations to H, we reduce the set of isomorphism classes of simple modules. In this case, we obtain a finite number, which is needed for pre-modular tensor categories. $\triangle$

EXAMPLE 5.31. Recall that a root system is embedded in a finite-dimensional real vector space equipped with an inner-product. (Note that in our notation we denote this inner-product by $\langle \cdot, \cdot \rangle$.) Hence the corresponding root lattice has a symmetric, non-degenerate and positive-definite $\mathbb{Z}$-linear form. Furthermore, every simply laced (i.e. ADE-type) root system Φ has roots of all the same length. Hence, after normalisation, every simple root $\alpha \in \Delta$ satisfies $\langle \alpha, \alpha \rangle = 2$. Thus, $(\bigoplus_{\alpha\in\Delta} \mathbb{Z}\alpha, \langle \cdot, \cdot \rangle)$ is a positive definite even lattice, which provides the data needed to define a lattice vertex operator. $\diamondsuit$

REMARK 5.32. From a conformal-field-theoretic perspective, the vertex operator algebras in Example 5.31 correspond to certain *affine vertex operator algebras* (to be defined in Chapter 6) at level 1, that is, chiral symmetry algebras for certain Wess-Zumino-Witten models in [Wit84]. As discussed in Section 15.6.3 of [DMS97], lattice vertex operators can be motivated as generalisations of these examples. Physically, each simple root is associated to a free boson on a compactified space time. Here, a "twisted multiplication" $\iota(\alpha)\iota(\beta) = \epsilon(\alpha,\beta)\iota(\alpha+\beta)$, for a phase $\epsilon(\alpha,\beta) \in U(1)$, is introduced to preserve field locality. Then, ϵ can be shown to satisfy a 2-cocyle condition, which manifests itself as the central extension in our definition but restricted from the dual lattice to the original lattice. Sections 5.4-5.6 of [Kac98] similarly motivate the central extension in this manner, but in a mathematical language closer to what we use in this chapter. △

REMARK 5.33. Given a positive definite even lattice L_0, the vertex operator algebra V_{L_0} only depends on the choice of central extension up to isomorphism (Proposition 6.5.5 of [LL04]). By Remark 5.23, a central extension exists, so there is exactly one isomorphism class of lattice vertex operator algebras per each positive definite even lattice. Hence, the only data required for constructing a lattice vertex operator is a positive definite even lattice if we choose the canonical central extension to be that as in Remark 5.23. △

5.3.3 Intertwining operators

In order to find the fusion product of a lattice vertex operator algebra, we want to classify the intertwining maps of type $\binom{V(k)}{V(i)\,V(j)}$, for all $i,j,k \in (L_0)^\circ/L_0$. There is already a vertex operator map $Y(\cdot,z) : V_{(L_0)^\circ} \to (\operatorname{End} V_{(L_0)^\circ})\{z\}$, defined in (5.40), that produces a series in $V_{(L_0)^\circ}\{z\}$ from two vectors in $V_{(L_0)^\circ}$. Since $V_{(L_0)^\circ} = \bigoplus_{i\in(L_0)^\circ/L_0} V(i)$ contains all irreducible V_{L_0}-modules, it appears that $Y(\cdot,z)$ could be used to construct all intertwining *operators* of type $\binom{V(k)}{V(i)\,V(j)}$ for all $i,j,k \in (L_0)^\circ/L_0$. In Chapter 11 and 12 of [DL93], the intertwining operators of the irreducible V_{L_0}-modules are constructed by twisting the vertex operator map by a phase using the bilinear form of $(L_0)^\circ$ and the commutator map (5.28).

DEFINITION 5.34. For each $\alpha \in (L_0)^\circ$, define the operators $e^{i\pi\alpha}$ and $\mathsf{c}(\cdot,\alpha)$ on $V_{(L_0)^\circ}$ by

$$e^{i\pi\alpha}\cdot m\otimes\iota(b) = e^{i\pi\langle\alpha,\bar{b}\rangle} m\otimes\iota(b) \quad \text{and} \quad \mathsf{c}(\cdot,\alpha)\cdot m\otimes\iota(b) = \mathsf{c}(\bar{b},\alpha) m\otimes\iota(b), \tag{5.56}$$

for all $m \in M(1)$ and $b \in \widehat{(L_0)^\circ}$. For all $u \in V(i)$, $i \in (L_0)^\circ/L_0$, define the operator on $V_{(L_0)^\circ}$ by

$$\mathcal{Y}_{\lambda_i}(u,z) = Y(u,z)e^{i\pi\lambda_i}\mathsf{c}(\cdot,\lambda_i). \tag{5.57}$$

THEOREM 5.35. (Propositions 12.2, 12.5 and 12.9 of [DL93])
Let $(L_0,\langle\cdot,\cdot\rangle)$ be a positive definite even lattice, and let $(\widehat{(L_0)^\circ},\bar{\cdot})$ be a central extension as in Subsection 5.3.1. Let $i,j,k \in (L_0)^\circ/L_0$. Then, we have the following.

(i) The map

$$\mathcal{Y}_{i,j}(\cdot,z)\cdot : V(i)\otimes V(j) \to V(i+j)\{z\}, \quad u\otimes v \mapsto \mathcal{Y}_{\lambda_i}(u,z)v \tag{5.58}$$

is a nonzero intertwining operator of type $\binom{V(i+j)}{V(i)\,V(j)}$.

(ii) Any intertwining operator of type $\binom{V(k)}{V(i)\ V(j)}$ is a scalar multiple of $\mathcal{Y}_{i,j}(\cdot, z)\cdot$, when $i + j = k$.

(iii) The only intertwining operator of type $\binom{V(k)}{V(i)\ V(j)}$ is zero, when $i + j \neq k$. $\square$

Note that the maps $\mathcal{Y}_{\lambda_i}$ depend on the choice of coset representatives $\lambda_i, i \in (L_0)^\circ/L_0$. But a different choice of representatives will only change $\mathcal{Y}_{i,j}$ by a non-zero constant. So, the vector space of intertwining operators of type $\binom{V(k)}{V(i)\ V(j)}$, and hence the fusion rules, do not change.

The space of intertwining operators of type $\binom{V(k)}{V(i)\ V(j)}$ is in one-to-one correspondence with the space of $P(w)$-intertwining maps of type $\binom{V(k)}{V(i)\ V(j)}$, by the linear isomorphism (5.5), with $p = 0$. That is,

$$\mathcal{Y}(\cdot, z)\cdot \mapsto I_{\mathcal{Y},0} : m_{(1)} \otimes m_{(2)} \mapsto \mathcal{Y}(m_{(1)}, e^{\log w})m_{(2)}. \tag{5.59}$$

Recall that (5.59) is defined using the substitution convention as defined in Section A.3. When performing calculations below we will often write w^n instead of $e^{n \log w}$, for $n \in \mathbb{C}$. It should be understood what it means to "raise w to an arbitrary complex power" in this sense.

5.4 Fusion product

Let $(L_0, \langle\cdot,\cdot\rangle)$ be a positive definite even lattice together with a central extension $(\widehat{(L_0)^\circ}, \bar{\cdot})$ as in Subsection 5.3.1. Let V_{L_0} be the corresponding vertex operator algebra with the semisimple category $V_{L_0}-\mathsf{Mod}$ of V_{L_0}-modules.

We seek a model for the fusion product in $V_{L_0}-\mathsf{Mod}$, which we can find since we know all the intertwining maps. For simplicity, we will be working with a skeletal monoidal subcategory of $V_{L_0}-\mathsf{Mod}$.

DEFINITION 5.36. Let $V_{L_0}-\mathsf{Mod}'$ be the full subcategory of $V_{L_0}-\mathsf{Mod}$ with the objects

$$\bigoplus_{n=1}^{N} V(i_n) \qquad \text{for } i_1, \ldots, i_N \in (L_0)^\circ/L_0,\ N \in \mathbb{Z}_{\geq 0}, \tag{5.60}$$

which includes a zero object 0. The model for the direct sum in (5.60) is chosen to be such that there is only one object of the form $\bigoplus_{n=1}^{N} V(i_n)$, for any ordering of $(i_1, \ldots, i_N)$. (This can be done by choosing an order for $(L_0)^\circ/L_0$, then defining $\bigoplus_{n=1}^{N} V(i_n)$ inductively.)

Note that $V_{L_0}-\mathsf{Mod}'$ is skeletal and $\mathbb{C}$-linearly equivalent to $V_{L_0}-\mathsf{Mod}$. Furthermore, it is still concrete, so we can perform computations using V_{L_0}-modules and their homomorphisms.

Now we can define a model for the fusion product of $V_{L_0}-\mathsf{Mod}'$. We start by finding the fusion product on the simple modules. Since the tensor product of a tensor category distributes over direct sums, we then know that the fusion product on the simple modules extends to all objects in $V_{L_0}-\mathsf{Mod}'$. We will define $P(w)$-tensor products for all $w \in \mathbb{C}^\times$.

DEFINITION/PROPOSITION 5.37. Let $w \in \mathbb{C}^\times$ and let $i,j \in (L_0)^\circ/L_0$. Define the $P(w)$-tensor product $(V(i) \boxtimes_{P(w)} V(j), \boxtimes_{P(w)})$ by

$$V(i) \boxtimes_{P(w)} V(j) = V(i+j) \quad \text{and} \quad \boxtimes_{P(w)} = \mathcal{Y}_{i,j}(\cdot, w)\cdot\,. \tag{5.61}$$

Proof. Let $i,j,k,k_1,\dots k_n \in (L_0)^\circ/L_0$. First, let $I : V(i)\otimes V(j) \to \overline{V(k)}$ be a $P(w)$-intertwining map. Consider the two cases:

(i) If $k = i+j$, then $I = a\mathcal{Y}_{i,j}$ for some $a \in \mathbb{C}$. So, $\eta = a\,\mathrm{id}_{V(i+j)} : V(i+j) \to V(k)$ is the unique module map such that $\overline{\eta} \circ \boxtimes_{P(w)} = I$.

(ii) If $k \neq i+j$, then $I = 0$ and $\eta = 0 : V(i+j) \to V(k)$ is the unique module map such that $\overline{\eta} \circ \boxtimes_{P(w)} = I$.

Second, let $I : V(i)\otimes V(j) \to \overline{\bigoplus_{n=1}^N V(k_n)}$ be a $P(w)$-intertwining map. Then, I decomposes into the sum of the $P(w)$-intertwining maps $I_n : V(i)\otimes V(j) \to \overline{V(k_n)}$, for $n = 1,\dots,N$. For each n, there is a unique module map $\eta_n : V(i+j) \to V(k_n)$ such that $\overline{\eta_n} \circ \boxtimes = I_n$. Then, $\eta = \sum_{n=1}^N \eta_n : V(i+j) \to \overline{\bigoplus_{n=1}^N V(k_n)}$ is the unique module map such that $\overline{\eta} \circ \boxtimes_{P(w)} = I$. Hence, $(V(i+j), \mathcal{Y}_{i,j})$ is a $P(w)$-tensor product of $V(i)$ and $V(j)$. $\square$

We define the fusion product to be the assignment of $P(1)$-tensor products on objects and morphisms. Hence, the *fusion product bifunctor* of $V_{L_0}-\mathsf{Mod}$ is

$$\begin{gathered} -\boxtimes- : V_{L_0}-\mathsf{Mod}' \times V_{L_0}-\mathsf{Mod}' \to V_{L_0}-\mathsf{Mod}' \\ (V(i),V(j)) \mapsto V(i+j) \qquad (0,V(i)),(V(i),0) \mapsto 0 \\ (a\,\mathrm{id}_{V(i)}, b\,\mathrm{id}_{V(j)}) \mapsto ab\,\mathrm{id}_{V(i+j)} \qquad (0,f),(f,0) \mapsto 0 \end{gathered} \tag{5.62}$$

extended to direct sums of simple modules by the equalities:

$$\left(\bigoplus_{m=1}^M V(i_m)\right) \boxtimes \left(\bigoplus_{n=1}^N V(j_n)\right) = \bigoplus_{m,n=1}^{M,N} (V(i_m)\boxtimes V(j_n)) = \bigoplus_{m,n=1}^{M,N} V(i_m+j_n). \tag{5.63}$$

5.5 Braided monoidal category data

Using HLZ, we have explicitly computed the canonical braided monoidal data for $V_{L_0}-\mathsf{Mod}'$.

THEOREM 5.38. The category $V_{L_0}-\mathsf{Mod}'$ can be naturally given the structure of a braided monoidal category $(V_{L_0}-\mathsf{Mod}', \boxtimes, V_{L_0}, \mathcal{A}, l, r, \mathcal{R})$ consisting of:

(i) the category $V_{L_0}-\mathsf{Mod}'$,

(ii) the bifunctor $-\boxtimes- : V_{L_0}-\mathsf{Mod}' \times V_{L_0}-\mathsf{Mod}' \to V_{L_0}-\mathsf{Mod}'$, as defined by the fusion product (5.62),

(iii) the unit object $V(0) = V_{L_0}$,

(iv) the associator $\mathcal{A} : -\boxtimes(-\boxtimes-) \Rightarrow (-\boxtimes-)\boxtimes-$ with components at triples of simple objects:

$$\mathcal{A}_{V(i),V(j),V(k)} = e^{i\pi\langle\lambda_{i+j}-\lambda_i-\lambda_j,\lambda_k\rangle}\mathsf{c}(\lambda_k, \lambda_{i+j}-\lambda_i-\lambda_j)\,\mathrm{id}_{V(i+j+k)}, \tag{5.64}$$

(v) the left unitor $l : V(0)\boxtimes- \Rightarrow \mathrm{id}_{\mathscr{C}}$ and right unitor $r : -\boxtimes V(0) \Rightarrow \mathrm{id}_{\mathscr{C}}$ as identities,

(vi) the braiding $\mathcal{R} : -\boxtimes\cdot \Rightarrow \cdot\boxtimes-$ with components at pairs of simple objects:

$$\mathcal{R}_{V(i),V(j)} = e^{i\pi\langle\lambda_i,\lambda_j\rangle}\mathsf{c}(\lambda_i,\lambda_j)\,\mathrm{id}_{V(i+j)}\,. \tag{5.65}$$

Moreover, this is a braided tensor category.

We will now compute this data for simple objects by exploiting the method in [HLZg], which we outline for convenience:

- Since V_{L_0} satisfies **(B1)**-**(B3)**, we know that the braided monoidal structure exists.
- Since $V_{L_0}-\mathsf{Mod}'$ is semisimple and the fusion product distributes over direct sums, it suffices to compute the data for simple objects.[2]
- Any endomorphism of a simple object is a scalar multiple of the identity, so such an endomorphism is determined by the image of any non-zero vector.
- To compute the completion of such an endomorphism, it suffices to compute the action on only the lowest weight terms, recalling that the grading is given by the $L(0)$-eigenvalues. That is, we can choose to work with the highest weight vectors $\iota(a)$, for any $a \in \widehat{(L_0)}^\circ$.

5.5.1 Left and right unitors

Let $i \in (L_0)^\circ/L_0$. The left unitor $l_{V(i)}$ is determined by

$$\overline{l_{V(i)}}(\mathbb{1}\boxtimes v) = v \quad \text{for all } v \in V(i). \tag{5.66}$$

Let $v \in V(i)$. Then,

$$\mathbb{1}\boxtimes v = \mathcal{Y}_0(\mathbb{1},1)v = Y(\mathbb{1},1)e^{i\pi 0}\mathsf{c}(\cdot,0)v = v,$$

so $l_{V(i)} = \mathrm{id}_{V(i)}$. The right unitor $r_{V(i)}$ is determined by

$$\overline{r_{V(i)}}(v\boxtimes\mathbb{1}) = e^{L(-1)}v. \tag{5.67}$$

[2]One can verify that the associator, unitors and braiding indeed distribute over direct sums. This is routine using techniques we have already shown, but involves many commutative diagrams, so we will not demonstrate it here.

Let $a \in \widehat{L_0 + \lambda_i}$. Then,

$$\begin{aligned} v \boxtimes \mathbb{1} &= \mathcal{Y}_{\lambda_i}(v,1)\mathbb{1} = Y(v,1)e^{i\pi\lambda_i}\mathsf{c}(\cdot,\lambda_i)\mathbb{1} = Y(\iota(a),1)\iota(1) \\ &= e^{\sum_{n>0}\overline{a}(-n)\frac{1}{n}}\iota(a) = \iota(a) + \text{h.w.t}, \\ e^{L(-1)}\iota(a) &= \sum_{n=0}^{\infty}\frac{1}{n!}(L(-1))^n\iota(a) = \iota(a) + \text{h.w.t}, \end{aligned}$$

where h.w.t denotes the higher weight terms as graded by $L(0)$. So, $r_{V(i)} = \mathrm{id}_{V(i)}$.

5.5.2 Parallel transport isomorphism

Let $w_1, w_2 \in \mathbb{C}^\times$, let γ be a path in $\mathbb{C}^\times$ from w_1 to w_2, and let $l(w_1)$ be the logarithm of w_1 determined by $\log w_2$ and γ. Let $i, j \in (L_0)^\circ/L_0$ and let $\mathcal{Y}$ be the logarithmic intertwining operator associated to the $P(w_2)$-tensor product $V(i) \boxtimes_{P(w_2)} V(j)$. Then, the parallel transport isomorphism

$$\mathcal{T}_\gamma : V(i) \boxtimes_{P(w_1)} V(j) = V(i+j) \to V(i) \boxtimes_{P(w_2)} V(j) = V(i+j)$$

is determined by

$$\begin{aligned} \overline{\mathcal{T}_\gamma}(v_1 \boxtimes_{P(w_1)} v_2) &= \mathcal{Y}(v_1,z)v_2\big|_{z=e^{l(w_1)}} \\ &= y^{L(0)}z^{L(0)}\left(y^{-L(0)}z^{-L(0)}v_1 \boxtimes_{P(w_2)} y^{-L(0)}z^{-L(0)}v_2\right)\Big|_{y=e^{-\log w_2},\ z=e^{l(w_1)}}, \end{aligned}$$

for all $v_1 \in V(i), v_2 \in V(j)$. Let $a \in \widehat{L_0 + \lambda_i}$, $b \in \widehat{L_0 + \lambda_j}$, and $v_1 = \iota(a)$, $v_2 = \iota(b)$. Then, we compute that

$$\begin{aligned} \iota(a) \boxtimes_{P(w_1)} \iota(b) &= \mathcal{Y}_{\lambda_i}(\iota(a),w_1)\iota(b) = Y(\iota(a),w_1)e^{i\pi\lambda_i}\mathsf{c}(\cdot,\lambda_i)\iota(b) \\ &= e^{i\pi\langle\lambda_i,\overline{b}\rangle}\mathsf{c}(\overline{b},\lambda_i)e^{\sum_{n>0}\overline{a}(-n)\frac{w_1^n}{n}}aw_1^{\overline{a}}\iota(b) \\ &= e^{i\pi\langle\lambda_i,\overline{b}\rangle}\mathsf{c}(\overline{b},\lambda_i)w_1^{\langle\overline{a},\overline{b}\rangle}\iota(ab) + \text{h.w.t} \end{aligned}$$

and

$$\begin{aligned} & e^{(l(w_1)-\log w_2)L(0)}\left(e^{(\log w_2 - l(w_1))L(0)}\iota(a) \boxtimes_{P(w_2)} e^{(\log w_2 - l(w_1))L(0)}\iota(b)\right) \\ &= e^{(l(w_1)-\log w_2)L(0)}\left(e^{(\log w_2 - l(w_1))(\frac{1}{2}\langle\overline{a},\overline{a}\rangle + \frac{1}{2}\langle\overline{b},\overline{b}\rangle)}\iota(a) \boxtimes_{P(w_2)} \iota(b)\right) \\ &= e^{(l(w_1)-\log w_2)L(0)}\left(e^{(\log w_2 - l(w_1))(\frac{1}{2}\langle\overline{a},\overline{a}\rangle + \frac{1}{2}\langle\overline{b},\overline{b}\rangle)}e^{i\pi\langle\lambda_i,\overline{b}\rangle}\mathsf{c}(\overline{b},\lambda_i)w_2^{\langle\overline{a},\overline{b}\rangle}\iota(ab) + \text{h.w.t}\right) \\ &= e^{(l(w_1)-\log w_2)(\frac{1}{2}\langle\overline{ab},\overline{ab}\rangle - \frac{1}{2}\langle\overline{a},\overline{a}\rangle - \frac{1}{2}\langle\overline{b},\overline{b}\rangle)}e^{i\pi\langle\lambda_i,\overline{b}\rangle}\mathsf{c}(\overline{b},\lambda_i)w_2^{\langle\overline{a},\overline{b}\rangle}\iota(ab) + \text{h.w.t} \\ &= e^{(l(w_1)-\log w_2)\langle\overline{a},\overline{b}\rangle}e^{i\pi\langle\lambda_i,\overline{b}\rangle}\mathsf{c}(\overline{b},\lambda_i)w_2^{\langle\overline{a},\overline{b}\rangle}\iota(ab) + \text{h.w.t}, \end{aligned}$$

so

$$\mathcal{T}_\gamma = e^{(l(w_1)-\log w_2)\langle\overline{a},\overline{b}\rangle}w_2^{\langle\overline{a},\overline{b}\rangle}w_1^{-\langle\overline{a},\overline{b}\rangle}\,\mathrm{id}_{V(i+j)}\,. \tag{5.68}$$

To simplify this, we choose a and b to be such that $\overline{a} = \lambda_i$ and $\overline{b} = \lambda_j$. Then, we have

$$\begin{aligned} e^{(l(w_1)-\log w_2)\langle \overline{a},\overline{b}\rangle} w_2^{\langle \overline{a},\overline{b}\rangle} w_1^{-\langle \overline{a},\overline{b}\rangle} &= e^{(l(w_1)-\log w_2)\langle \lambda_i,\lambda_j\rangle} e^{\log w_2 \langle \lambda_i,\lambda_j\rangle} e^{-\log w_1 \langle \lambda_i,\lambda_j\rangle} \\ &= e^{(l(w_1)-\log w_1)\langle \lambda_i,\lambda_j\rangle} \end{aligned}$$

Hence, the parallel transport isomorphism associated to γ is

$$\mathcal{T}_\gamma = e^{(l(w_1)-\log w_1)\langle \lambda_i,\lambda_j\rangle}\, \mathrm{id}_{V(i+j)}\,. \tag{5.69}$$

5.5.3 Associator

Let r_1, r_2 be real numbers satisfying $r_1 > r_2 > r_1 - r_2 > 0$. Let γ_1, γ_2, γ_3, γ_4 be paths in $(0,\infty)$ from 1 to r_1, from 1 to r_2, from r_2 to $r_1 - r_2$, from r_2 to 1, respectively. Then, by substituting all principal real logarithms into (5.69) we have that $\mathcal{T}_{\gamma_1}$, $\mathcal{T}_{\gamma_2}$, $\mathcal{T}_{\gamma_3}$ and $\mathcal{T}_{\gamma_4}$ are all identities.

We now calculate the isomorphism

$$\begin{aligned} \mathcal{A}^{P(r_1-r_2)P(r_2)}_{P(r_1)P(r_2)} \colon V(i) \boxtimes_{P(r_1)} (V(j) \boxtimes_{P(r_2)} V(k)) &= V(i+j+k) \\ \longrightarrow (V(i) \boxtimes_{P(r_1-r_2)} V(j)) \boxtimes_{P(r_2)} V(k) &= V(i+j+k) \end{aligned} \tag{5.70}$$

determined by

$$\overline{\mathcal{A}^{P(r_1-r_2)P(r_2)}_{P(r_1)P(r_2)}} : v_1 \boxtimes_{P(r_1)} (v_2 \boxtimes_{P(r_2)} v_3) \mapsto (v_1 \boxtimes_{P(r_1-r_2)} v_2) \boxtimes_{P(r_2)} v_3, \tag{5.71}$$

for all $v_1 \in V(i)$, $v_2 \in V(j)$, $v_3 \in V(k)$.

Let $a \in \widehat{L_0 + \lambda_i}, b \in \widehat{L_0 + \lambda_j}, c \in \widehat{L_0 + \lambda_k}$ and $v_1 = \iota(a), v_2 = \iota(b), v_3 = \iota(c)$. We will compute both sides of (5.71). In what follows, note that a lot of the manipulation is possible since $r_1 > r_2 > r_1 - r_2 > 0$.

We first calculate the left-hand side of (5.71):

$$\begin{aligned} &\iota(a) \boxtimes_{P(r_1)} (\iota(b) \boxtimes_{P(r_2)} \iota(c)) \\ &= \iota(a) \boxtimes_{P(r_1)} e^{\sum_{n>0} \overline{b}(-n) \frac{r_2^n}{n}} r_2^{\langle \overline{b},\overline{c}\rangle} e^{i\pi\langle \lambda_j,\overline{c}\rangle} \mathsf{c}(\overline{c},\lambda_j) \iota(bc) \\ &= \iota(a) \boxtimes_{P(r_1)} r_2^{\langle \overline{b},\overline{c}\rangle} e^{i\pi\langle \lambda_j,\overline{c}\rangle} \mathsf{c}(\overline{c},\lambda_j) \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{\overline{b}(-n_1)\cdots\overline{b}(-n_k)}{n_1\cdots n_k} r_2^{n_1+\cdots+n_k} \iota(bc). \end{aligned}$$

Now, for any $m \in \mathbb{Z}_{>0}$,

$$\begin{aligned} \left[\sum_{n>0} \overline{a}(n) \frac{r_1^{-n}}{-n}, \overline{b}(-m)\right] &= \sum_{n>0} \left[\overline{a}(n), \overline{b}(-m)\right] \frac{r_1^{-n}}{-n} = \sum_{n>0} \left[\overline{a}(n), \overline{b}(-m)\right] \frac{r_1^{-n}}{-n} \\ &= \sum_{n>0} \langle \overline{a}, \overline{b}\rangle n \delta_{n-m,0} \frac{r_1^{-n}}{-n} = -\langle \overline{a}, \overline{b}\rangle r_1^{-m}. \end{aligned}$$

(Recall that these sums are finite when acting on V_{L_0}-modules.) Hence, by use of the fact that $[A,B]=C$ with C central, implies $[A^k,B]=kCA^{k-1}$, for all $k\in\mathbb{Z}_{\geq 0}$, we get

$$\left[e^{\sum_{n>0}\overline{a}(n)\frac{r_1^{-n}}{-n}},\overline{b}(-m)\right]=\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n>0}\overline{a}(n)\frac{r_1^{-n}}{-n}\right)^k,\overline{b}(-m)\right]$$
$$=\sum_{k=1}^{\infty}\frac{1}{k!}\left[\left(\sum_{n>0}\overline{a}(n)\frac{r_1^{-n}}{-n}\right)^k,\overline{b}(-m)\right]=\sum_{k=1}^{\infty}\frac{1}{k!}k(-\langle\overline{a},\overline{b}\rangle r_1^{-m})\left(\sum_{n>0}\overline{a}(n)\frac{r_1^{-n}}{-n}\right)^{k-1}$$
$$=-\langle\overline{a},\overline{b}\rangle r_1^{-m}\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left(\sum_{n>0}\overline{a}(n)\frac{r_1^{-n}}{-n}\right)^{k-1}=-\langle\overline{a},\overline{b}\rangle r_1^{-m}e^{\sum_{n>0}\overline{a}(n)\frac{r_1^{-n}}{-n}}.$$

Hence,

$$\begin{aligned}&\iota(a)\boxtimes_{P(r_1)}\overline{b}(-n_1)\cdots\overline{b}(-n_k)\iota(bc)\\&=e^{\sum_{n>0}\overline{a}(-n)\frac{r_1^{n}}{n}}e^{\sum_{n>0}\overline{a}(n)\frac{r_1^{-n}}{-n}}ar_1^{\overline{a}}e^{i\pi\lambda_i}\mathsf{c}(\cdot,\lambda_i)\overline{b}(-n_1)\cdots\overline{b}(-n_k)\iota(bc)\\&=\left(\overline{b}(-n_1)-\langle\overline{a},\overline{b}\rangle r_1^{-n_1}\right)\cdots\left(\overline{b}(-n_k)-\langle\overline{a},\overline{b}\rangle r_1^{-n_k}\right)\iota(a)\boxtimes_{P(r_1)}\iota(bc)\\&=(-\langle\overline{a},\overline{b}\rangle)^k r_1^{-n_1-\cdots-n_k}r_1^{\langle\overline{a},\overline{bc}\rangle}e^{i\pi\langle\lambda_i,\overline{bc}\rangle}\mathsf{c}(\overline{bc},\lambda_i)\iota(abc)+\text{ h.w.t,}\end{aligned}$$

where we have noted that $e^{\sum_{n>0}\overline{a}(-n)}$, a, r_1^a, $e^{i\pi\lambda_i}$ and $\mathsf{c}(\cdot,\lambda_i)$ commute with $\overline{b}(-m)$, for $m>0$. Hence,

$$\begin{aligned}&\iota(a)\boxtimes_{P(r_1)}(\iota(b)\boxtimes_{P(r_2)}\iota(c))\\&=r_1^{\langle\overline{a},\overline{bc}\rangle}e^{i\pi\langle\lambda_i,\overline{bc}\rangle}\mathsf{c}(\overline{bc},\lambda_i)r_2^{\langle\overline{b},\overline{c}\rangle}e^{i\pi\langle\lambda_j,\overline{c}\rangle}\mathsf{c}(\overline{c},\lambda_j)\\&\quad\cdot\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n_1,\dots,n_k=1}^{\infty}\frac{(-\langle\overline{a},\overline{b}\rangle)^k}{n_1\cdots n_k}\left(\frac{r_2}{r_1}\right)^{n_1+\cdots+n_k}\iota(abc)+\text{ h.w.t}\\&=r_1^{\langle\overline{a},\overline{bc}\rangle}e^{i\pi\langle\lambda_i,\overline{bc}\rangle}\mathsf{c}(\overline{bc},\lambda_i)r_2^{\langle\overline{b},\overline{c}\rangle}e^{i\pi\langle\lambda_j,\overline{c}\rangle}\mathsf{c}(\overline{c},\lambda_j)\\&\quad\cdot\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n>0}\frac{-\langle\overline{a},\overline{b}\rangle}{n}\left(\frac{r_2}{r_1}\right)^n\right)^k\iota(abc)+\text{ h.w.t}\\&=r_1^{\langle\overline{a},\overline{bc}\rangle}e^{i\pi\langle\lambda_i,\overline{bc}\rangle}\mathsf{c}(\overline{bc},\lambda_i)r_2^{\langle\overline{b},\overline{c}\rangle}e^{i\pi\langle\lambda_j,\overline{c}\rangle}\mathsf{c}(\overline{c},\lambda_j)e^{\langle\overline{a},\overline{b}\rangle\log\left(1-\frac{r_2}{r_1}\right)}\iota(abc)+\text{ h.w.t}\\&=r_1^{\langle\overline{a},\overline{bc}\rangle}e^{i\pi\langle\lambda_i,\overline{bc}\rangle}\mathsf{c}(\overline{bc},\lambda_i)r_2^{\langle\overline{b},\overline{c}\rangle}e^{i\pi\langle\lambda_j,\overline{c}\rangle}\mathsf{c}(\overline{c},\lambda_j)\left(1-\frac{r_2}{r_1}\right)^{\langle\overline{a},\overline{b}\rangle}\iota(abc)+\text{ h.w.t}\end{aligned}$$

$$=r_1^{\langle\overline{a},\overline{c}\rangle}r_2^{\langle\overline{b},\overline{c}\rangle}(r_1-r_2)^{\langle\overline{a},\overline{b}\rangle}e^{i\pi\langle\lambda_i,\overline{bc}\rangle}e^{i\pi\langle\lambda_j,\overline{c}\rangle}\mathsf{c}(\overline{bc},\lambda_i)\mathsf{c}(\overline{c},\lambda_j)\iota(abc)+\text{ h.w.t.}\tag{5.72}$$

Next we calculate the right-hand side:

$$
\begin{aligned}
&(\iota(a) \boxtimes_{P(r_1-r_2)} \iota(b)) \boxtimes_{P(r_2)} \iota(c)\\
&\quad = \left(e^{\sum_{n>0} \overline{a}(-n)\frac{(r_1-r_2)^n}{n}} (r_1 - r_1)^{\langle \overline{a},\overline{b}\rangle} e^{i\pi\langle \lambda_i,\overline{b}\rangle} \mathsf{c}(\overline{b}, \lambda_i)\iota(ab)\right) \boxtimes_{P(r_2)} \iota(c)\\
&\quad = \Bigg((r_1 - r_2)^{\langle \overline{a},\overline{b}\rangle} e^{i\pi\langle \lambda_i,\overline{b}\rangle} \mathsf{c}(\overline{b}, \lambda_i)\\
&\qquad \cdot \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n_1,\dots,n_k=1}^{\infty} \frac{\overline{a}(-n_1)\cdots\overline{a}(-n_k)}{n_1\cdots n_k} (r_1 - r_2)^{n_1+\cdots+n_k} \iota(ab)\Bigg) \boxtimes_{P(r_2)} \iota(c).
\end{aligned}
$$

Now,

$$
\begin{aligned}
&(\overline{a}(-n_1)\cdots\overline{a}(-n_k)\iota(ab)) \boxtimes_{P(r_2)} \iota(c)\\
&\quad = {\circ\atop\circ}\left(\frac{1}{(n_1-1)!}\left(\frac{d}{dr_2}\right)^{n_1-1}\overline{a}(r_2)\right)\cdots\left(\frac{1}{(n_k-1)!}\left(\frac{d}{dr_2}\right)^{n_k-1}\overline{a}(r_2)\right) Y(\iota(ab), r_2){\circ\atop\circ}\\
&\qquad \cdot e^{i\pi\langle \lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{c}, \lambda_{i+j})\iota(c)\\
&\quad = {\circ\atop\circ} \sum_{m_1\in\mathbb{Z}} \binom{-m_1-1}{n_1-1} \overline{a}(m_1) r_2^{-m_1-n_1} \cdots \sum_{m_k\in\mathbb{Z}} \binom{-m_k-1}{n_k-1} \overline{a}(m_k) r_2^{-m_k-n_k}\\
&\qquad \cdot e^{\sum_{n>0} \overline{a}(-n)\frac{r_2^n}{n}} e^{\sum_{n>0} \overline{a}(n)\frac{r_2^{-n}}{-n}} (ab) r_2^{\overline{ab}} {\circ\atop\circ}\; e^{i\pi\langle \lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{c}, \lambda_{i+j})\iota(c)\\
&\quad = (ab) r_2^{\overline{ab}} \binom{-1}{n_1-1} \overline{a}(0) r_2^{-n_1} \cdots \binom{-1}{n_k-1} \overline{a}(0) r_2^{-n_k} e^{i\pi\langle \lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{c}, \lambda_{i+j})\iota(c) + \text{h.w.t}\\
&\quad = (ab) r_2^{\overline{ab}} (-1)^{n_1+\cdots n_k-k} \langle \overline{a},\overline{c}\rangle^k r_2^{-n_1-\cdots-n_k} e^{i\pi\langle \lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{c}, \lambda_{i+j})\iota(c) + \text{h.w.t}\\
&\quad = r_2^{\langle \overline{ab},\overline{c}\rangle} (-1)^{n_1+\cdots n_k-k} \langle \overline{a},\overline{c}\rangle^k r_2^{-n_1-\cdots-n_k} e^{i\pi\langle \lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{c}, \lambda_{i+j})\iota(abc) + \text{h.w.t.}
\end{aligned}
$$

(Note that $ab \in \widehat{L_0 + \lambda_i + \lambda_j} = \widehat{L_0 + \lambda_{i+j}}$, but $\lambda_i + \lambda_j \neq \lambda_{i+j}$ in general.) So,

(5.73)

$$
\begin{aligned}
&(\iota(a) \boxtimes_{P(r_1-r_2)} \iota(b)) \boxtimes_{P(r_2)} \iota(c)\\
&\quad = (r_1 - r_2)^{\langle \overline{a},\overline{b}\rangle} e^{i\pi\langle \lambda_i,\overline{b}\rangle} \mathsf{c}(\overline{b}, \lambda_i) r_2^{\langle \overline{ab},\overline{c}\rangle} e^{i\pi\langle \lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{c}, \lambda_{i+j})\\
&\qquad \cdot \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n_1,\dots,n_k=1}^{\infty} \frac{(-\langle \overline{a},\overline{c}\rangle)^k}{n_1\cdots n_k} \left(-\frac{r_1-r_2}{r_2}\right)^{n_1+\cdots+n_k} \iota(abc) + \text{h.w.t}\\
&\quad = (r_1 - r_2)^{\langle \overline{a},\overline{b}\rangle} e^{i\pi\langle \lambda_i,\overline{b}\rangle} \mathsf{c}(\overline{b}, \lambda_i) r_2^{\langle \overline{ab},\overline{c}\rangle} e^{i\pi\langle \lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{c}, \lambda_{i+j}) e^{\langle \overline{a},\overline{c}\rangle \log\left(1+\frac{r_1-r_2}{r_2}\right)} \iota(abc) + \text{h.w.t}\\
&\quad = (r_1 - r_2)^{\langle \overline{a},\overline{b}\rangle} e^{i\pi\langle \lambda_i,\overline{b}\rangle} \mathsf{c}(\overline{b}, \lambda_i) r_2^{\langle \overline{ab},\overline{c}\rangle} e^{i\pi\langle \lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{c}, \lambda_{i+j}) \left(1+\frac{r_1-r_2}{r_2}\right)^{\langle \overline{a},\overline{c}\rangle} \iota(abc) + \text{h.w.t}\\
&\quad = r_1^{\langle \overline{a},\overline{c}\rangle} r_2^{\langle \overline{b},\overline{c}\rangle} (r_1 - r_2)^{\langle \overline{a},\overline{b}\rangle} e^{i\pi\langle \lambda_i,\overline{b}\rangle} e^{i\pi\langle \lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{b}, \lambda_i) \mathsf{c}(\overline{c}, \lambda_{i+j})\iota(abc) + \text{h.w.t.}
\end{aligned}
$$

Hence, by comparing the left-hand and right-hand sides of (5.71), that is, comparing (5.72) to (5.73), we get

$$
\begin{aligned}
\mathcal{A}_{P(r_1)P(r_2)}^{P(r_1-r_2)P(r_2)} &= r_1^{\langle\overline{a},\overline{c}\rangle} r_2^{\langle\overline{b},\overline{c}\rangle} (r_1-r_2)^{\langle\overline{a},\overline{b}\rangle} e^{i\pi\langle\lambda_i,\overline{b}\rangle} e^{i\pi\langle\lambda_{i+j},\overline{c}\rangle} \mathsf{c}(\overline{b},\lambda_i)\mathsf{c}(\overline{c},\lambda_{i+j}) \\
&\quad \cdot \left(r_1^{\langle\overline{a},\overline{c}\rangle} r_2^{\langle\overline{b},\overline{c}\rangle} (r_1-r_2)^{\langle\overline{a},\overline{b}\rangle} e^{i\pi\langle\lambda_i,\overline{bc}\rangle} e^{i\pi\langle\lambda_j,\overline{c}\rangle} \mathsf{c}(\overline{bc},\lambda_i)\mathsf{c}(\overline{c},\lambda_j) \right)^{-1} \mathrm{id}_{V(i+j+k)} \\
&= e^{i\pi\langle\lambda_{i+j}-\lambda_i-\lambda_j,\overline{c}\rangle} \mathsf{c}(\overline{c},\lambda_{i+j}-\lambda_i-\lambda_j)\,\mathrm{id}_{V(i+j+k)},
\end{aligned}
\tag{5.74}
$$

where we have used the following fact: for all $a, b, c \in \widehat{(L_0)^\circ}$,

$$
\begin{aligned}
\mathsf{c}(\overline{a}+\overline{b},\overline{c}) = \mathsf{c}(\overline{ab},\overline{c}) &= (ab)c(ab)^{-1}c^{-1} = abcb^{-1}a^{-1}c^{-1} = abcb^{-1}c^{-1}ca^{-1}c^{-1} \\
&= a\mathsf{c}(\overline{b},\overline{c})ca^{-1}c^{-1} = aca^{-1}c^{-1}\mathsf{c}(\overline{b},\overline{c}) = \mathsf{c}(\overline{a},\overline{c})\mathsf{c}(\overline{b},\overline{c}).
\end{aligned}
$$

By choosing c such that $\overline{c} = \lambda_k$, we finally arrive at

$$
\mathcal{A}_{P(r_1)P(r_2)}^{P(r_1-r_2)P(r_2)} = e^{i\pi\langle\lambda_{i+j}-\lambda_i-\lambda_j,\lambda_k\rangle} \mathsf{c}(\lambda_k,\lambda_{i+j}-\lambda_i-\lambda_j)\,\mathrm{id}_{V(i+j+k)}\,.
$$

Thus, the associator

$$
\mathcal{A}_{V(i),V(j),V(k)} : V(i) \boxtimes (V(j) \boxtimes V(k)) \to (V(i) \boxtimes V(j)) \boxtimes V(k)
\tag{5.75}
$$

is the composition

$$
\begin{aligned}
\mathcal{A}_{V(i),V(j),V(k)} &= \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(z_2)} \mathrm{id}_{V(k)}) \circ \mathcal{A}_{P(r_1)P(r_2)}^{P(r_1-r_2)P(r_2)} \circ (\mathrm{id}_{V(i)} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1} \\
&= e^{i\pi\langle\lambda_{i+j}-\lambda_i-\lambda_j,\lambda_k\rangle} \mathsf{c}(\lambda_k,\lambda_{i+j}-\lambda_i-\lambda_j)\,\mathrm{id}_{V(i+j+k)}\,.
\end{aligned}
\tag{5.76}
$$

REMARK 5.39. We can now see that fixing the choice of representatives λ_i, for $i \in (L_0)^\circ/L_0$, does indeed matter, since in general $\lambda_i + \lambda_j \neq \lambda_{i+j}$. However, it does not matter which set of representatives we choose to fix, as seen in Proposition 5.42. △

REMARK 5.40. For verification and illumination, we will check that the monoidal category axioms are satisfied. Let $i, j, k, \ell \in (L_0)^\circ/L_0$. We show that the pentagon identity,

$$
\mathcal{A}_{i+j,k,\ell} \circ \mathcal{A}_{i,j,k+\ell} = (\mathcal{A}_{i,j,k} \boxtimes \mathrm{id}_\ell) \circ \mathcal{A}_{i,j+k,\ell} \circ (\mathrm{id}_i \boxtimes \mathcal{A}_{j,k,\ell}),
\tag{5.77}
$$

is satisfied. Since $\lambda_{k+\ell} = \lambda_k + \lambda_\ell + \alpha$ for some $\alpha \in L_0$ and $\lambda_{i+j} - \lambda_i - \lambda_j \in L_0$, then $\langle \lambda_{i+j} - \lambda_i - \lambda_j, \alpha\rangle \in \mathbb{Z}$ and we have

$$
e^{i\pi\langle\lambda_{i+j}-\lambda_i-\lambda_j,\alpha\rangle} \mathsf{c}(\alpha,\lambda_{i+j}-\lambda_i-\lambda_j) = (-1)^{\langle\lambda_{i+j}-\lambda_i-\lambda_j,\alpha\rangle}(-1)^{\langle\alpha,\lambda_{i+j}-\lambda_i-\lambda_j\rangle} = 1.
\tag{5.78}
$$

Hence,

$$\begin{aligned}
\mathcal{A}_{i+j,k,\ell} \circ \mathcal{A}_{i,j,k+\ell} &= e^{i\pi\langle \lambda_{i+j+k}-\lambda_{i+j}-\lambda_k,\lambda_\ell\rangle}\mathsf{c}(\lambda_\ell, \lambda_{i+j+k}-\lambda_{i+j}-\lambda_k) \\
&\quad \cdot e^{i\pi\langle \lambda_{i+j}-\lambda_i-\lambda_j,\lambda_{k+\ell}\rangle}\mathsf{c}(\lambda_{k+\ell}, \lambda_{i+j}-\lambda_i-\lambda_j)\,\mathrm{id}_{V(i+j+k+\ell)} \\
&= e^{i\pi\langle \lambda_{i+j+k}-\lambda_{i+j}-\lambda_k,\lambda_\ell\rangle}\mathsf{c}(\lambda_\ell, \lambda_{i+j+k}-\lambda_{i+j}-\lambda_k) \\
&\quad \cdot e^{i\pi\langle \lambda_{i+j}-\lambda_i-\lambda_j,\lambda_k+\lambda_\ell\rangle}\mathsf{c}(\lambda_k+\lambda_\ell, \lambda_{i+j}-\lambda_i-\lambda_j)\,\mathrm{id}_{V(i+j+k+\ell)} \\
&= e^{i\pi\langle \lambda_{i+j+k}-\lambda_{i+j}-\lambda_k,\lambda_\ell\rangle}\mathsf{c}(\lambda_\ell, \lambda_{i+j+k}-\lambda_{i+j}-\lambda_k) \\
&\quad \cdot e^{i\pi\langle \lambda_{i+j}-\lambda_i-\lambda_j,\lambda_\ell\rangle}\mathsf{c}(\lambda_\ell, \lambda_{i+j}-\lambda_i-\lambda_j) \\
&\quad \cdot e^{i\pi\langle \lambda_{i+j}-\lambda_i-\lambda_j,\lambda_k\rangle}\mathsf{c}(\lambda_k, \lambda_{i+j}-\lambda_i-\lambda_j)\,\mathrm{id}_{V(i+j+k+\ell)} \\
&= e^{i\pi\langle \lambda_{i+j+k}-\lambda_i-\lambda_j-\lambda_k,\lambda_\ell\rangle}\mathsf{c}(\lambda_\ell, \lambda_{i+j+k}-\lambda_i-\lambda_j-\lambda_k) \\
&\quad \cdot e^{i\pi\langle \lambda_{i+j}-\lambda_i-\lambda_j,\lambda_k\rangle}\mathsf{c}(\lambda_k, \lambda_{i+j}-\lambda_i-\lambda_j)\,\mathrm{id}_{V(i+j+k+\ell)} \\
&= e^{i\pi\langle \lambda_{i+j}-\lambda_i-\lambda_j,\lambda_k\rangle}\mathsf{c}(\lambda_k, \lambda_{i+j}-\lambda_i-\lambda_j) \\
&\quad \cdot e^{i\pi\langle \lambda_{i+j+k}-\lambda_i-\lambda_{j+k},\lambda_\ell\rangle}\mathsf{c}(\lambda_\ell, \lambda_{i+j+k}-\lambda_i-\lambda_{j+k}) \\
&\quad \cdot e^{i\pi\langle \lambda_{j+k}-\lambda_j-\lambda_k,\lambda_\ell\rangle}\mathsf{c}(\lambda_\ell, \lambda_{j+k}-\lambda_j-\lambda_k)\,\mathrm{id}_{V(i+j+k+\ell)} \\
&= (\mathcal{A}_{i,j,k}\boxtimes \mathrm{id}_\ell)\circ \mathcal{A}_{i,j+k,\ell}\circ(\mathrm{id}_i \boxtimes \mathcal{A}_{j,k,\ell}).
\end{aligned}$$

The triangle identity,

$$(r_i \boxtimes \mathrm{id}_j)\circ \mathcal{A}_{i,L_0,j} = \mathrm{id}_i \boxtimes l_j, \tag{5.79}$$

is also satisfied. For this, we note that the identity L_0 in $(L_0)^\circ/L_0$ corresponds to the unit object $V(0)$ and has representative $\lambda_{L_0} = 0 \in (L_0)^\circ$. Hence,

$$(r_i \boxtimes \mathrm{id}_j)\circ \mathcal{A}_{i,L_0,j} = e^{i\pi\langle \lambda_{i+L_0}-\lambda_i-\lambda_{L_0},\lambda_j\rangle}\,\mathrm{id}_{i+j} = e^{i\pi\langle \lambda_i-\lambda_i,\lambda_j\rangle}\,\mathrm{id}_{i+j} = \mathrm{id}_{i+j} = \mathrm{id}_i \boxtimes l_j.$$

Thus, we have verified that $(V\text{--}\mathsf{Mod}', \boxtimes, V(0), \mathcal{A}, l, r)$ is a monoidal category. △

5.5.4 Braiding

Let γ^- be a path from -1 to 1 in the closed upper half plane with 0 deleted. Hence, $l(-1) = \log(-1) = i\pi$. Let $i, j \in (L_0)^\circ/L_0$. Then, by (5.69), we have

$$\mathcal{T}_{\gamma^-} = \mathrm{id}_{V(i+j)}, \tag{5.80}$$

since γ^{-1} is in the principal branch sheet. We now find $\mathcal{R}_{P(1)} : V(i)\boxtimes_{P(1)} V(j) \to V(j)\boxtimes_{P(-1)} V(i)$ which is determined by

$$\overline{\mathcal{R}_{P(1)}}(v_1 \boxtimes_{P(1)} v_2) = e^{L(-1)}(v_2 \boxtimes_{P(-1)} v_1) \quad \text{for } v_1 \in V(i), v_2 \in V(j). \tag{5.81}$$

Let $a \in \widehat{L+\lambda_i}$, $b \in \widehat{L+\lambda_j}$ and $v_1 = \iota(a)$, $v_2 = \iota(b)$. Then,

$$\iota(a)\boxtimes_{P(1)} \iota(b) = e^{i\pi\langle \lambda_i, \bar{b}\rangle}\mathsf{c}(\bar{b}, \lambda_i)\iota(ab) + \text{h.w.t}$$

and

$$e^{L(-1)}(\iota(b)\boxtimes_{P(-1)}\iota(a)) = e^{L(-1)}(e^{i\pi\langle\overline{b},\overline{a}\rangle}e^{i\pi\langle\lambda_j,\overline{a}\rangle}\mathsf{c}(\overline{a},\lambda_j)\iota(ba) + \text{h.w.t })$$
$$= e^{L(-1)}(e^{i\pi\langle\overline{b},\overline{a}\rangle}e^{i\pi\langle\lambda_j,\overline{a}\rangle}\mathsf{c}(\overline{a},\lambda_j)\mathsf{c}(\overline{b},\overline{a})\iota(ab) + \text{h.w.t })$$
$$= e^{i\pi\langle\overline{b},\overline{a}\rangle}e^{i\pi\langle\lambda_j,\overline{a}\rangle}\mathsf{c}(\overline{a},\lambda_j)\mathsf{c}(\overline{b},\overline{a})\iota(ab) + \text{h.w.t }.$$

Hence,

$$R_{P(1)} = e^{i\pi\langle\overline{b},\overline{a}\rangle}e^{i\pi\langle\lambda_j,\overline{a}\rangle}\mathsf{c}(\overline{a},\lambda_j)\mathsf{c}(\overline{b},\overline{a})e^{-i\pi\langle\lambda_i,\overline{b}\rangle}\mathsf{c}(\overline{b},\lambda_i)^{-1}\,\mathrm{id}_{V(i+j)}$$
$$= e^{i\pi(\langle\lambda_j,\overline{a}\rangle-\langle\lambda_i,\overline{b}\rangle+\langle\overline{b},\overline{a}\rangle)}\mathsf{c}(\overline{a},\lambda_j)\mathsf{c}(\overline{b},\overline{a})\mathsf{c}(\lambda_i,\overline{b}).$$

Choose a and b such that $\overline{a}=\lambda_i$ and $\overline{b}=\lambda_j$. So, we have

$$R_{P(1)} = e^{i\pi(\langle\lambda_j,\lambda_i\rangle-\langle\lambda_i,\lambda_j\rangle+\langle\lambda_j,\lambda_i\rangle)}\mathsf{c}(\lambda_i,\lambda_j)\mathsf{c}(\lambda_j,\lambda_i)\mathsf{c}(\lambda_i,\lambda_j)$$
$$= e^{i\pi\langle\lambda_i,\lambda_j\rangle}\mathsf{c}(\lambda_i,\lambda_j)\,\mathrm{id}_{V(i+j)}\,. \tag{5.82}$$

Thus, the braiding is

$$\mathcal{R} = \mathcal{T}_{\gamma^-}\circ\mathcal{R}_{P(1)} = \mathrm{id}_{V(i+j)}\circ e^{i\pi\langle\lambda_i,\lambda_j\rangle}\mathsf{c}(\lambda_i,\lambda_j)\,\mathrm{id}_{V(i+j)}$$
$$= e^{i\pi\langle\lambda_i,\lambda_j\rangle}\mathsf{c}(\lambda_i,\lambda_j)\,\mathrm{id}_{V(i+j)}\,. \tag{5.83}$$

REMARK 5.41. Let $i,j,k\in(L_0)^\circ/L_0$. For illumination, we will show one of the hexagon identities, namely

$$\mathcal{R}_{i,j+k} = \mathcal{A}_{j,k,i}\circ(\mathrm{id}_j\boxtimes\mathcal{R}_{i,k})\circ\mathcal{A}_{j,i,k}^{-1}\circ(\mathcal{R}_{i,j}\boxtimes\mathrm{id}_k)\circ\mathcal{A}_{i,j,k}. \tag{5.84}$$

Computing the right-hand side explicitly, we get

$$\begin{aligned}
&\mathcal{A}_{j,k,i}\circ(\mathrm{id}_j\boxtimes\mathcal{R}_{i,k})\circ\mathcal{A}_{j,i,k}^{-1}\circ(\mathcal{R}_{i,j}\boxtimes\mathrm{id}_k)\circ\mathcal{A}_{i,j,k}\\
&\quad= e^{i\pi\langle\lambda_{j+k}-\lambda_j-\lambda_k,\lambda_i\rangle}\mathsf{c}(\lambda_i,\lambda_{j+k}-\lambda_j-\lambda_k)e^{i\pi\langle\lambda_i,\lambda_k\rangle}\mathsf{c}(\lambda_i,\lambda_k)\\
&\quad\quad\cdot(e^{i\pi\langle\lambda_{j+i}-\lambda_j-\lambda_i,\lambda_k\rangle}\mathsf{c}(\lambda_k,\lambda_{j+i}-\lambda_j-\lambda_i))^{-1}e^{i\pi\langle\lambda_i,\lambda_j\rangle}\mathsf{c}(\lambda_i,\lambda_j)\\
&\quad\quad\cdot e^{i\pi\langle\lambda_{i+j}-\lambda_i-\lambda_j,\lambda_k\rangle}\mathsf{c}(\lambda_k,\lambda_{i+j}-\lambda_i-\lambda_j)\,\mathrm{id}_{i+j+k}\\
&\quad= e^{i\pi\langle\lambda_i,\lambda_{j+k}\rangle}\mathsf{c}(\lambda_i,\lambda_{j+k})\,\mathrm{id}_{i+j+k} = \mathcal{R}_{i,j+k}.
\end{aligned}$$

The remaining hexagon identity can be verified similarly. Hence, we have verified that $(V_{L_0}\text{−}\mathsf{Mod}',\boxtimes,V(0),\mathcal{A},l,r,\mathcal{R})$ is a braided monoidal category. △

PROPOSITION 5.42. The braided monoidal category $(V_{L_0}\text{−}\mathsf{Mod}',\boxtimes,V(0),\mathcal{A},l,r,\mathcal{R})$ is independent of the choice of representatives $\{\lambda_i\in L_0+\lambda_i : i\in(L_0)^\circ/L_0\}$, up to braided monoidal equivalence.

Proof. Let $\{\lambda_i\}_{i\in(L_0)^\circ/L_0}$ and $\{\widetilde{\lambda}_i\}_{i\in(L_0)^\circ/L_0}$ be sets of coset representatives and let $\alpha_i=\widetilde{\lambda}_i-\lambda_i$. Let $(V_{L_0}\text{−}\mathsf{Mod}',\boxtimes,V(0),\mathcal{A},l,r,\mathcal{R})$ and $(V_{L_0}\text{−}\mathsf{Mod}',\boxtimes,V(0),\widetilde{\mathcal{A}},\widetilde{l},\widetilde{r},\widetilde{\mathcal{R}})$ be the respective braided monoidal categories as per the construction above. Endow the identity functor $\mathrm{id}_{V_{L_0}-\mathsf{Mod}'}$ with the natural isomorphism

$$J_{V(i),V(j)} = e^{i\pi\langle\alpha_i,\lambda_j\rangle}\mathsf{c}(\lambda_j,\alpha_i)\,\mathrm{id}_{V(i+j)} : V(i)\boxtimes V(j)\to V(i)\boxtimes V(j), \tag{5.85}$$

for all $i,j\in(L_0)^\circ/L_0$ and the isomorphism $\varphi=\mathrm{id}_{V(0)}$. To show that

$$(\mathrm{id}_{V_{L_0}-\mathsf{Mod}'},J,\varphi) : (V_{L_0}\text{−}\mathsf{Mod}',\boxtimes,V(0),\widetilde{\mathcal{A}},\widetilde{l},\widetilde{r},\widetilde{\mathcal{R}})\to(V_{L_0}\text{−}\mathsf{Mod}',\boxtimes,V(0),\mathcal{A},l,r,\mathcal{R})$$

is braided-monoidal, it suffices to check the compatibility of associators, unitors and braiding for the simple objects. These diagrams are compositions of multiples of identities and can be directly checked to commute. Since $\mathrm{id}_{V_{L_0}-\mathsf{Mod}'}$ is an equivalence of categories, we have that $(\mathrm{id}_{V_{L_0}-\mathsf{Mod}'}, J, \varphi)$ is a braided monoidal equivalence. $\square$

5.6 Modularity

Finally, we will use [Hua08] to compute the canonical pre-modular structure of $V_{L_0}-\mathsf{Mod}'$.

THEOREM 5.43. The category $V_{L_0}-\mathsf{Mod}'$ can be naturally given the structure of a pre-modular category $(V_{L_0}-\mathsf{Mod}', \boxtimes, V(0), \mathcal{A}, l, r, \mathcal{R}, \theta)$ consisting of:

(i) the braided monoidal category $(V_{L_0}-\mathsf{Mod}', \boxtimes, V(0), \mathcal{A}, l, r, \mathcal{R})$ from Theorem 5.38,

(ii) the ribbon structure (twist) $\theta : \mathrm{id}_{V_{L_0}-\mathsf{Mod}'} \Rightarrow \mathrm{id}_{V_{L_0}-\mathsf{Mod}'}$ with components

$$\theta_X = e^{2\pi i L(0)} : X \to X. \tag{5.86}$$

Here, the duals are $V(i)^* = V(-i)$, for all $i \in (L_0)^\circ/L_0$. Moreover, this pre-modular category has an invertible S-matrix

$$S = (s_{V(i),V(j)})_{i,j\in(L_0)^\circ/L_0} = \left(e^{2\pi i\langle\lambda_i,\lambda_j\rangle}\right)_{i,j\in(L_0)^\circ/L_0}. \tag{5.87}$$

Thus, $V_{L_0}-\mathsf{Mod}'$ is modular.

5.6.1 Rigidity

The rigidity of $V_{L_0}-\mathsf{Mod}$ is a result of Section 3 of [Hua08], but we are working with $V_{L_0}-\mathsf{Mod}'$. Recall from Proposition 4.4 that dual objects are unique up to isomorphism, provided they exist. So, even though we do not have contragredient modules in $V_{L_0}-\mathsf{Mod}'$ (as a restricted dual vector space), we can still verify that the duals exist by using modules that are isomorphic to the contragredient modules whilst still being objects in $V_{L_0}-\mathsf{Mod}'$.

Let $i \in (L_0)^\circ/L_0$. For $j \in (L_0)^\circ/L_0$, we have $V(i) \boxtimes V(j) = V(j) \boxtimes V(i) = V(i+j)$. If we want non-zero module maps $V(i+j) \to V(0)$ and $V(0) \to V(i+j)$, then we need $i + j = 0$, since $V(0)$ and $V(i+j)$ are simple. So, we define the dual object to be

$$V(i)^* = V(-i). \tag{5.88}$$

Recalling (4.1) and (4.2), for left rigidity, we must impose the following conditions:

$$r_{V(i)} \circ (\mathrm{id}_{V(i)} \boxtimes \mathrm{ev}_{V(i)}) \circ \mathcal{A}^{-1}_{V(i),V(-i),V(i)} \circ (\mathrm{coev}_{V(i)} \boxtimes \mathrm{id}_{V(i)}) \circ l^{-1}_{V(i)} = \mathrm{id}_{V(i)},$$
$$l_{V(-i)} \circ (\mathrm{ev}_{V(i)} \boxtimes \mathrm{id}_{V(-i)}) \circ \mathcal{A}_{V(-i),V(i),V(-i)} \circ (\mathrm{id}_{V(-i)} \boxtimes \mathrm{coev}_{V(i)}) \circ r^{-1}_{V(-i)} = \mathrm{id}_{V(-i)}.$$

Since

$$\begin{aligned}\mathcal{A}_{V(i),V(-i),V(i)} &= e^{i\pi\langle \lambda_{i-i}-\lambda_i-\lambda_{-i},\lambda_i\rangle}\mathsf{c}(\lambda_i, \lambda_{i-i}-\lambda_i-\lambda_{-i})\,\mathrm{id}_{V(i-i+i)}\\ &= e^{i\pi\langle -\lambda_i-\lambda_{-i},\lambda_i\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)\,\mathrm{id}_{V(i)},\end{aligned}$$

we may choose evaluation and coevaluation to be given by

$$\mathrm{ev}_{V(i)} = e^{i\pi\langle -\lambda_i-\lambda_{-i},\lambda_i\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)\,\mathrm{id}_{V(0)} : V(i)^* \boxtimes V(i) = V(0) \to V(0), \tag{5.89}$$

$$\mathrm{coev}_{V(i)} = \mathrm{id}_{V(0)} : V(0) \to V(i) \boxtimes V(i)^* = V(0). \tag{5.90}$$

We now verify the evaluation and coevaluation conditions. First,

$$\begin{aligned}&r_{V(i)} \circ (\mathrm{id}_{V(i)} \boxtimes \mathrm{ev}_{V(i)}) \circ \mathcal{A}^{-1}_{V(i),V(-i),V(i)} \circ (\mathrm{coev}_{V(i)} \boxtimes \mathrm{id}_{V(i)}) \circ l^{-1}_{V(i)}\\ &= e^{i\pi\langle -\lambda_i-\lambda_{-i},\lambda_i\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)\big(e^{i\pi\langle -\lambda_i-\lambda_{-i},\lambda_i\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)\big)^{-1}\,\mathrm{id}_{V(i)} = \mathrm{id}_{V(i)}\,.\end{aligned}$$

Second, we observe that

$$\begin{aligned}\mathcal{A}_{V(-i),V(i),V(-i)} &= e^{i\pi\langle \lambda_{-i+i}-\lambda_{-i}-\lambda_i,\lambda_{-i}\rangle}\mathsf{c}(\lambda_{-i}, \lambda_{-i+i}-\lambda_{-i}-\lambda_i)\,\mathrm{id}_{V(-i+i-i)}\\ &= e^{i\pi\langle -\lambda_{-i}-\lambda_i,\lambda_{-i}\rangle}\mathsf{c}(\lambda_i,\lambda_{-i})\,\mathrm{id}_{V(-i)}\,.\end{aligned}$$

Hence, by using $\lambda_i + \lambda_{-i} \in L_0$, we have

$$\begin{aligned}&l_{V(-i)} \circ (\mathrm{ev}_{V(i)} \boxtimes \mathrm{id}_{V(-i)}) \circ \mathcal{A}_{V(-i),V(i),V(-i)} \circ (\mathrm{id}_{V(-i)} \boxtimes \mathrm{coev}_{V(i)}) \circ r^{-1}_{V(-i)}\\ &= e^{i\pi\langle -\lambda_i-\lambda_{-i},\lambda_i\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)e^{i\pi\langle -\lambda_{-i}-\lambda_i,\lambda_{-i}\rangle}\mathsf{c}(\lambda_i,\lambda_{-i})\,\mathrm{id}_{V(-i)}\\ &= e^{-i\pi\langle \lambda_i+\lambda_{-i},\lambda_i+\lambda_{-i}\rangle}\,\mathrm{id}_{V(-i)} = \mathrm{id}_{V(-i)}\,.\end{aligned}$$

So, $(V(i)^*, \mathrm{ev}_{V(i)}, \mathrm{coev}_{V(i)})$ is a left dual of $V(i)$. A similar procedure can be used to show that $V(i)^*$ is a right dual object of $V(i)$, together with

$$\mathrm{ev}'_{V(i)} = e^{i\pi\langle \lambda_i+\lambda_{-i},\lambda_i\rangle}\mathsf{c}(\lambda_i,\lambda_{-i})\,\mathrm{id}_{V(0)} \quad \text{and} \quad \mathrm{coev}'_{V(i)} = \mathrm{id}_{V(0)}\,. \tag{5.91}$$

These duals then extend over direct sums, hence $V_{L_0}{-}\mathsf{Mod}'$ is rigid.

This shows that $V_{L_0}{-}\mathsf{Mod}'$ is a finite semisimple $\mathbb{C}$-linear abelian rigid monoidal category with $\boxtimes$ bilinear on morphisms and $\mathrm{End}_{V_{L_0}-\mathsf{Mod}'}(V(0)) \cong \mathbb{C}$. That is, $V_{L_0}{-}\mathsf{Mod}'$ is a fusion category. We still need to give $V_{L_0}{-}\mathsf{Mod}'$ the canonical ribbon structure and show that the S-matrix on $V_{L_0}{-}\mathsf{Mod}'$ is invertible.

REMARK 5.44. We could have used the fact that $V_{L_0}{-}\mathsf{Mod}$ is rigid and monoidally equivalent to $V_{L_0}{-}\mathsf{Mod}'$ to obtain the rigidity of $V_{L_0}{-}\mathsf{Mod}'$ as a result of Proposition 4.42. Nonetheless, we have chosen to use the above method for its explicitness. △

From now on, fix the dual objects, and (co)evaluation morphisms as above.

5.6.2 Twist

As discussed in [Hua08], the twist is given by

$$\theta : \mathrm{id}_{V_{L_0}-\mathsf{Mod}'} \Rightarrow \mathrm{id}_{V_{L_0}-\mathsf{Mod}'}\,, \qquad \theta_X = e^{2\pi i L(0)} : X \to X. \tag{5.92}$$

We verify that this is indeed a twist on $V_{L_0}-\mathsf{Mod}'$ by checking (4.30). Consider a simple V_{L_0}-module $V(i)$. Then, $\theta_{V(i)} = e^{2\pi i L(0)}$ is a module endomorphism since for all $m\iota(a) \in V(i)$, $m \in M(1)$, $a \in \widehat{L+\lambda_i}$, with $\overline{a} = \alpha + \lambda_i$, we have

$$\begin{aligned}\theta_{V(i)}(m\iota(a)) &= e^{2\pi i L(0)} m\iota(a) = e^{2\pi i(\mathrm{wt}\, m + \mathrm{wt}\, \iota(a))} m\iota(a) = e^{2\pi i(\mathrm{wt}\, m + \frac{1}{2}\langle \alpha+\lambda_i, \alpha+\lambda_i\rangle)} m\iota(a)\\ &= e^{2\pi i(\mathrm{wt}\, m + \frac{1}{2}\langle \alpha,\alpha\rangle + \langle \alpha,\lambda_i\rangle + \frac{1}{2}\langle \lambda_i,\lambda_i\rangle)} m\iota(a) = e^{\pi i\langle \lambda_i,\lambda_i\rangle} m\iota(a),\end{aligned}$$

since $\mathrm{wt}\, m$, $\langle \alpha, \lambda_i\rangle \in \mathbb{Z}$ and $\langle \alpha, \alpha\rangle \in 2\mathbb{Z}$. That is,

$$\theta_{V(i)} = e^{\pi i\langle \lambda_i,\lambda_i\rangle}\, \mathrm{id}_{V(i)}\,. \tag{5.93}$$

Let $i, j \in (L_0)^\circ/L_0$. Then, using that $\lambda_{i+j} = \lambda_i + \lambda_j + \alpha$, for some $\alpha \in L_0$, we get

$$\begin{aligned}\theta_{V(i)\boxtimes V(j)} &= \theta_{V(i+j)} = e^{\pi i\langle \lambda_{i+j},\lambda_{i+j}\rangle}\, \mathrm{id}_{V(i+j)} = e^{\pi i\langle \lambda_i+\lambda_j+\alpha, \lambda_i+\lambda_j+\alpha\rangle}\, \mathrm{id}_{V(i+j)}\\ &= e^{\pi i(\langle \lambda_i+\lambda_j,\lambda_i+\lambda_j\rangle + 2\langle \lambda_i+\lambda_j,\alpha\rangle + \langle \alpha,\alpha\rangle)}\, \mathrm{id}_{V(i+j)} = e^{\pi i\langle \lambda_i+\lambda_j,\lambda_i+\lambda_j\rangle}\, \mathrm{id}_{V(i+j)}\\ &= e^{\pi i\langle \lambda_i,\lambda_i\rangle} e^{\pi i\langle \lambda_j,\lambda_j\rangle} e^{\pi i\langle \lambda_i,\lambda_j\rangle} e^{\pi i\langle \lambda_j,\lambda_i\rangle}\, \mathrm{id}_{V(i+j)}\\ &= (\theta_{V(i)} \boxtimes \theta_{V(j)}) \circ \mathcal{R}_{V(j),V(i)} \circ \mathcal{R}_{V(i),V(j)}.\end{aligned}$$

So, θ is indeed a twist. Recall the definition of the left dual of a morphism from Definition 4.5. The left dual of the twist is

$$\begin{aligned}(\theta_{V(i)})^* &= l_{V(-i)} \circ (\mathrm{ev}_{V(i)} \boxtimes \mathrm{id}_{V(-i)}) \circ ((\mathrm{id}_{V(-i)} \boxtimes \theta_{V(i)}) \boxtimes \mathrm{id}_{V(-i)})\\ &\qquad \circ \mathcal{A}_{V(-i),V(i),V(-i)} \circ (\mathrm{id}_{V(-i)} \boxtimes \mathrm{coev}_{V(i)}) \circ r^{-1}_{V(-i)}\\ &= e^{i\pi\langle -\lambda_i-\lambda_{-i},\lambda_i\rangle} \mathsf{c}(\lambda_{-i},\lambda_i) e^{\pi i\langle \lambda_i,\lambda_i\rangle} e^{i\pi\langle -\lambda_{-i}-\lambda_i,\lambda_{-i}\rangle} \mathsf{c}(\lambda_i,\lambda_{-i})\, \mathrm{id}_{V(-i)}\\ &= e^{\pi i\langle \lambda_i,\lambda_i\rangle}\, \mathrm{id}_{V(-i)} = e^{\pi i\langle -\lambda_{-i}+\alpha, -\lambda_{-i}+\alpha\rangle}\, \mathrm{id}_{V(-i)}\\ &= e^{\pi i(\langle \lambda_{-i},\lambda_{-i}\rangle - 2\langle \lambda_{-i},\alpha\rangle + \langle \alpha,\alpha\rangle)}\, \mathrm{id}_{V(-i)} = e^{\pi i\langle \lambda_{-i},\lambda_{-i}\rangle}\, \mathrm{id}_{V(-i)} = \theta_{V(i)^*},\end{aligned}$$

where $\lambda_i = -\lambda_{-i} + \alpha$ for some $\alpha \in L_0$. Thus, θ is a ribbon structure on $V_{L_0}-\mathsf{Mod}'$, hence $V_{L_0}-\mathsf{Mod}'$ is a ribbon fusion category (i.e. a pre-modular category).

REMARK 5.45. The braided monoidal functor $(\mathrm{id}_{V_{L_0}-\mathsf{Mod}'}, J, \varphi)$, from the proof of Proposition 5.42, is $\mathbb{C}$-linear and preserves twists, since its underlying functor is the identity functor. That is, it is a pre-modular functor. Hence, the pre-modular structure of $V_{L_0}-\mathsf{Mod}'$ is independent of the choice of coset representatives λ_i, for $i \in (L_0)^\circ/L_0$, up to pre-modular equivalence. $\triangle$

5.6.3 The S-matrix

We first find the canonical pivotal (spherical) structure $a : \mathrm{id}_{\mathscr{C}} \Rightarrow (\cdot)^{**}$ of $V_{L_0}-\mathsf{Mod}'$ with respect to the twist θ. Recall, from Remark 4.37, that there is a natural transformation $u_X : X \to X^{**}$, for $X \in V_{L_0}-\mathsf{Mod}'$, given by the composition

$$\begin{aligned}X &\xrightarrow{r_X^{-1}} X \boxtimes V(0) \xrightarrow{\mathrm{id}_X \boxtimes \mathrm{coev}_{X^*}} X \boxtimes (X^* \boxtimes X^{**}) \xrightarrow{\mathcal{A}_{X,X^*,X^{**}}} (X \boxtimes X^*) \boxtimes X^{**}\\ &\xrightarrow{\mathcal{R}_{X,X^*} \boxtimes \mathrm{id}_{X^{**}}} (X^* \boxtimes X) \boxtimes X^{**} \xrightarrow{\mathrm{ev}_X \boxtimes \mathrm{id}_{X^{**}}} V(0) \boxtimes X^{**} \xrightarrow{l_{X^{**}}} X^{**}.\end{aligned}$$

Then, the canonical pivotal structure a is given by $a_X = u_X \circ \theta_X$. Specifically, let $i \in (L_0)^\circ/L_0$. Then,

$$\begin{aligned} u_{V(i)} &= l_{V(i)} \circ (\mathrm{ev}_{V(i)} \boxtimes \mathrm{id}_{V(i)}) \circ (\mathcal{R}_{V(i),V(-i)} \boxtimes \mathrm{id}_{X^{**}}) \\ &\qquad \circ \mathcal{A}_{V(i),V(-i),V(i)} \circ (\mathrm{id}_{V(i)} \boxtimes \mathrm{coev}_{V(-i)}) \circ r^{-1}_{V(i)} \\ &= e^{i\pi\langle -\lambda_i-\lambda_{-i},\lambda_i\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)e^{i\pi\langle\lambda_i,\lambda_{-i}\rangle}\mathsf{c}(\lambda_i,\lambda_{-i})e^{i\pi\langle -\lambda_i-\lambda_{-i},\lambda_i\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)\,\mathrm{id}_{V(i)} \\ &= e^{i\pi\langle\lambda_i,-2\lambda_i-\lambda_{-i}\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)\,\mathrm{id}_{V(i)} \end{aligned}$$

and so, the pivotal structure is

$$\begin{aligned} a_{V(i)} = u_{V(i)} \circ \theta_{V(i)} &= \left(e^{i\pi\langle\lambda_i,-2\lambda_i-\lambda_{-i}\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)\,\mathrm{id}_{V(i)}\right) \circ \left(e^{i\pi\langle\lambda_i,\lambda_i\rangle}\,\mathrm{id}_{V(i)}\right) \\ &= e^{i\pi\langle\lambda_i,-\lambda_i-\lambda_{-i}\rangle}\mathsf{c}(\lambda_{-i},\lambda_i)\,\mathrm{id}_{V(i)}\,. \end{aligned}$$

We can now find the S-matrix. Let $i, j \in (L_0)^\circ/L_0$. Recall that the S-matrix has components given by

$$s_{V(i),V(j)} = \mathrm{Tr}_a(\mathcal{R}_{V(j),V(i)} \circ \mathcal{R}_{V(i),V(j)}). \tag{5.94}$$

Hence, we calculate

$$\begin{aligned} s_{V(i),V(j)} &= \mathrm{ev}_{V(-i-j)} \circ \left((a_{V(i+j)} \circ \mathcal{R}_{V(j),V(i)} \circ \mathcal{R}_{V(i),V(j)}) \boxtimes \mathrm{id}_{V(-i-j)}\right) \circ \mathrm{coev}_{V(i+j)} \\ &= e^{i\pi\langle -\lambda_{-i-j}-\lambda_{i+j},\lambda_{-i-j}\rangle}\mathsf{c}(\lambda_{i+j},\lambda_{-i-j})e^{i\pi\langle\lambda_{i+j},-\lambda_{i+j}-\lambda_{-i-j}\rangle}\mathsf{c}(\lambda_{-i-j},\lambda_{i+j}) \\ &\quad \cdot e^{i\pi\langle\lambda_i,\lambda_j\rangle}\mathsf{c}(\lambda_i,\lambda_j)e^{i\pi\langle\lambda_j\lambda_i\rangle}\mathsf{c}(\lambda_j,\lambda_i)\,\mathrm{id}_{V(0)} \\ &= e^{-i\pi\langle\lambda_{i+j}+\lambda_{-i-j},\lambda_{i+j}+\lambda_{-i-j}\rangle}\mathsf{c}(\lambda_{i+j},\lambda_{-i-j})\mathsf{c}(\lambda_{-i-j},\lambda_{i+j}) \\ &\quad \cdot e^{2i\pi\langle\lambda_i,\lambda_j\rangle}\mathsf{c}(\lambda_i,\lambda_j)\mathsf{c}(\lambda_j,\lambda_i)\,\mathrm{id}_{V(0)} \\ &= e^{2i\pi\langle\lambda_i,\lambda_j\rangle}\,\mathrm{id}_{V(0)}\,. \end{aligned}$$

Thus, after the identification $\mathrm{End}\,V(0) = \mathbb{C}$, by $\mathrm{id}_{V(0)} \mapsto 1$, we have

$$s_{V(i),V(j)} = e^{2i\pi\langle\lambda_i,\lambda_j\rangle} \quad \text{for } i, j \in (L_0)^\circ/L_0. \tag{5.95}$$

5.6.4 Invertibility of the S-matrix

PROPOSITION 5.46. The S-matrix of $V_{L_0}-\mathsf{Mod}'$ is invertible.

Proof. Since L_0 is equipped with a symmetric bilinear form, it has an orthogonal basis, say $\{\alpha_s\}_{s=1}^d$. Then, $(L_0)^\circ$ has a basis $\left\{\beta_s = \frac{1}{\langle\alpha_s,\alpha_s\rangle}\alpha_s\right\}_{s=1}^d$ with the set isomorphisms

$$\begin{aligned} (L_0)^\circ/L_0 &\leftrightarrow \{\lambda_i : i \in (L_0)^\circ/L_0\} \\ &\qquad \leftrightarrow \left\{\sum_{s=1}^d m_s\beta_s : m_s = 0, \ldots, \langle\alpha_s,\alpha_s\rangle - 1 \text{ and } s = 1, \ldots, d\right\}, \end{aligned}$$

where each λ_i is congruent to some $\sum_{s=1}^d m_s\beta_s$, modulo L_0.

We will show that $S = \left(e^{2i\pi\langle\lambda_i,\lambda_j\rangle}\right)_{i,j\in(L_0)^\circ/L_0}$ is invertible by showing that $S(S^t)^* = SS^* = (|(L_0)^\circ/L_0|\delta_{i,j})$, which is invertible (and S can even be renormalised by $1/\sqrt{|(L_0)^\circ/L_0|}$

to be unitary). For $i, j \in (L_0)^\circ/L_0$, we have

$$\left(SS^*\right)_{i,j} = \sum_{k\in(L_0)^\circ/L_0} e^{2i\pi\langle\lambda_i,\lambda_k\rangle}e^{-2i\pi\langle\lambda_k,\lambda_j\rangle} = \sum_{k\in(L_0)^\circ/L_0} e^{2i\pi\langle\lambda_i-\lambda_j,\lambda_k\rangle}.$$

In the case that $i = j$, then $\lambda_i = \lambda_j$ and hence

$$\left(SS^*\right)_{i,i} = \sum_{k\in(L_0)^\circ/L_0} e^{2i\pi\langle 0,\lambda_k\rangle} = \left|(L_0)^\circ/L_0\right| .$$

In the case that $i \neq j$, let $\ell = i - j$ and write

$$\lambda_i - \lambda_j = \lambda_\ell = \sum_{s=1}^{d} n_s\beta_s = \sum_{s=1}^{d} \frac{n_s}{\langle\alpha_s,\alpha_s\rangle}\alpha_s \text{ (mod } L_0), \tag{5.96}$$

for some $n_s \in \{0, \ldots, \langle\alpha_s, \alpha_s\rangle - 1\}$ with $s \in \{1, \ldots, d\}$. Since $\lambda_\ell \notin L_0$, we can choose some $t \in \{1, \ldots, d\}$ such that $\frac{n_t}{\langle\alpha_t,\alpha_t\rangle} \notin \mathbb{Z}$. So, we can write

$$\frac{n_t}{\langle\alpha_t,\alpha_t\rangle} = \frac{p}{q}, \quad \text{where } p \text{ and } q \text{ are coprime and } q \mid \langle\alpha_t,\alpha_t\rangle. \tag{5.97}$$

Now, consider the list

$$\langle n_t\beta_t, m\beta_t\rangle = \left\langle \frac{n_t}{\langle\alpha_t,\alpha_t\rangle}\alpha_t, \frac{m}{\langle\alpha_t,\alpha_t\rangle}\alpha_t \right\rangle = m\frac{n_t}{\langle\alpha_t,\alpha_t\rangle} = m\frac{p}{q}, \tag{5.98}$$

for $m \in \{0, \ldots, \langle\alpha_t, \alpha_t\rangle - 1\}$. This list is can also be written as

$$\frac{0}{q}, \frac{1}{q} \ldots, \frac{q-1}{q} \text{ (mod } \mathbb{Z}), \qquad \text{repeated } \frac{\langle\alpha_t,\alpha_t\rangle}{q} \text{ times.} \tag{5.99}$$

Hence,

$$\begin{aligned}
\sum_{k\in L/L_0} e^{2i\pi\langle\lambda_\ell,\lambda_k\rangle} &= \sum_{m_1=0}^{\langle\alpha_1,\alpha_1\rangle-1} \cdots \sum_{m_d=0}^{\langle\alpha_d,\alpha_d\rangle-1} \exp\left(2\pi i \left\langle \lambda_\ell, \sum_{s=1}^{d} m_s\beta_s \right\rangle\right) \\
&= \sum_{m_1=0}^{\langle\alpha_1,\alpha_1\rangle-1} \cdots \sum_{m_d=0}^{\langle\alpha_d,\alpha_d\rangle-1} \exp\left(2\pi i \sum_{s=1}^{d} \langle n_s\beta_s, m_s\beta_s\rangle\right) \\
&= \sum_{m_1,\ldots,m_d} \exp\left(2\pi i \left(\langle n_t\beta_t, m_t\beta_t\rangle + \sum_{\substack{s=1\\ s\neq t}}^{d} \langle n_s\beta_s, m_s\beta_s\rangle\right)\right) \\
&= \sum_{m_t} \exp\left(2\pi i \langle n_t\gamma_t, m_t\gamma_t\rangle\right) \sum_{m_1,\ldots,\widehat{m_t},\ldots,m_d} \exp\left(2\pi i \sum_{\substack{s=1\\ s\neq t}}^{d} \langle n_s\beta_s, m_s\beta_s\rangle\right) \\
&= 0,
\end{aligned}$$

where we have used

$$\sum_{m=0}^{\langle\alpha_t,\alpha_t\rangle-1} \exp\left(2\pi i \langle n_t\beta_t, m\beta_t\rangle\right) = \frac{\langle\alpha_t,\alpha_t\rangle}{q}\sum_{m=0}^{q-1}\exp\left(2\pi i\frac{m}{q}\right) = 0. \tag{5.100}$$

Hence, $SS^* = (|(L_0)^\circ/L_0|\delta_{i,j})$. Since SS^* is invertible, then so is the S-matrix. Thus, $V_{L_0}-\mathsf{Mod}'$ is a modular tensor category. □

REMARK 5.47. From $V_{L_0}-\mathsf{Mod}'$, we immediately obtain many other examples of pre-modular categories, which we briefly discuss to stress the fact that modularity is rarer than pre-modularity. Let L be a sublattice of $(L_0)^\circ$ containing L_0:

$$L_0 \subseteq L \subseteq (L_0)^\circ. \tag{5.101}$$

The subspace $V_L = \bigoplus_{i\in L/L_0} V(i)$ gives the restriction $Y(\cdot,z) : V_L \to (\text{End } V_L)\{z\}$ since L/L_0 is a subgroup of $(L_0)^\circ/L_0$. Define $V_{L_0}-\mathsf{Mod}'_L$ to be the full subcategory of $V_{L_0}-\mathsf{Mod}'$ with the objects

$$\bigoplus_{n=1}^{N} V(i_n) \qquad \text{for } i_1,\ldots,i_N \in L/L_0,\ N\in\mathbb{Z}_{\geq 0}. \tag{5.102}$$

Consider the full subcategory of $V_{L_0}-\mathsf{Mod}'_L$ that contains only the simple objects. This monoidal category is of the form as in Example B.9, with the group $G = L/L_0$ and the abelian group $A = \mathbb{C}^\times$. Together with the left duals, this forms a *2-group* (or *categorical group*) since the tensor product of an object with its dual is isomorphic to the identity. In this sense, $V_{L_0}-\mathsf{Mod}'_L$ can be thought of as a categorification of the group ring $\mathbb{Z}[L/L_0]$. (Recall that $(V_{L_0}-\mathsf{Mod}'_L,\oplus)$ actually decategorifies to an abelian monoid, but nonetheless, the analogy works for its Grothendieck group).

We observe that $V_{L_0}-\mathsf{Mod}'_L$ is closed under the fusion product since it is a "direct sum extension" of the group product in $(L_0)^\circ/L_0$. So, $V_{L_0}-\mathsf{Mod}'_L$ has the structure of a monoidal subcategory and, hence, the structure of a ribbon subcategory. Furthermore, the dual functor $(\cdot)^* : V(i)\mapsto V(-i)$, is a "direct sum extension" of the inversion in $(L_0)^\circ/L_0$. Hence, $V_{L_0}-\mathsf{Mod}'_L$ is also a ribbon fusion category.

Finally, we give a counter-example illustrating that $V_{L_0}-\mathsf{Mod}'_L$, for $L\neq(L_0)^\circ$, need not be a modular category. Let $L_0 = \mathbb{Z}\alpha$, with $\langle\alpha,\alpha\rangle = 4$, so $(L_0)^\circ = \frac{1}{4}\mathbb{Z}\alpha$. Let $L = \frac{1}{2}\mathbb{Z}\alpha$, so that $L_0 \subsetneq L \subsetneq (L_0)^\circ = \frac{1}{4}\mathbb{Z}\alpha$. Then, $L/L_0 = \{L_0, L_0+\frac{1}{2}\alpha\}$, with $\langle\frac{1}{2}\alpha,\frac{1}{2}\alpha\rangle = 1$. So, the S-matrix of $V_{L_0}-\mathsf{Mod}'_L$ is the singular matrix

$$S = \begin{bmatrix} e^{2\pi i\langle 0,0\rangle} & e^{2\pi i\langle 0,\frac{1}{2}\alpha\rangle} \\ e^{2\pi i\langle\frac{1}{2}\alpha,0\rangle} & e^{2\pi i\langle\frac{1}{2}\alpha,\frac{1}{2}\alpha\rangle}\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}. \tag{5.103}$$

△

EXAMPLE 5.48. Continuing Example 5.29, where L_0 is the root lattice of $\mathfrak{sl}_2$, the modular tensor category $V_{L_0}-\mathsf{Mod}'$ consists of:

(i) simple objects $V(i)$, $i\in\mathbb{Z}_2 = \{0,1\} \cong_{\mathsf{Groups}} \{L_0, L_0+\frac{1}{2}\alpha\}$,

(ii) the tensor product $V(i)\boxtimes V(j) = V(i+j)$,

(iii) the unit object $V(0) = V_{L_0}$,

(iv) the associator $\mathcal{A}_{V(i),V(j),V(k)} = \begin{cases} -\operatorname{id}_{V(1)} & \text{if } i=j=k=1, \\ \operatorname{id}_{V(i+j+k)}, & \text{otherwise} \end{cases}$

(v) left and right unitors as identities,

(vi) the braiding $\mathcal{R}_{V(i),V(j)} = \begin{cases} i\operatorname{id}_{V(0)} & \text{if } i=j=1, \\ \operatorname{id}_{V(i+j)} & \text{otherwise,} \end{cases}$

(vii) dual objects $V(0)^* = V(0)$ and $V(1)^* = V(1)$,

(viii) the ribbon structure $\theta_{V(0)} = \operatorname{id}_{V(0)}$ and $\theta_{V(1)} = i\operatorname{id}_{V(1)}$.

The S-matrix is

$$S = \begin{bmatrix} e^{2\pi i\langle\lambda_0,\lambda_0\rangle} & e^{2\pi i\langle\lambda_0,\lambda_1\rangle} \\ e^{2\pi i\langle\lambda_1,\lambda_0\rangle} & e^{2\pi i\langle\lambda_1,\lambda_1\rangle} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{5.104}$$ $\Diamond$

EXAMPLE 5.49. Continuing Example 5.22, where L_0 is the root lattice of $\mathfrak{sl}_3$, the modular tensor category $V_{L_0}-\mathsf{Mod}'$ consists of:

(i) simple objects $V(n)$, $n \in \mathbb{Z}_3 = \{0,1,2\} \cong_{\mathsf{Groups}} \{L_0, L_0+\beta_2, L_0+2\beta_2\}$,

(ii) the tensor product $V(m) \boxtimes V(n) = V(m+n)$,

(iii) the unit object $V(0) = V_{L_0}$,

(iv) the associator $\mathcal{A}_{V(\ell),V(m),V(n)} = \begin{cases} e^{\frac{4\pi i}{3}n}\operatorname{id}_{V(\ell+m+n)} & \text{if } \ell=m=2, \\ \operatorname{id}_{V(\ell+m+n)} & \text{otherwise,} \end{cases}$

(v) left and right unitors as identities,

(vi) the braiding $\mathcal{R}_{V(m),V(n)} = e^{\frac{2\pi i}{3}mn}$,

(vii) dual objects $V(0)^* = V(0)$, $V(1)^* = V(2)$ and $V(2)^* = V(1)$,

(viii) the ribbon structure (twist) $\theta_{V(0)} = \operatorname{id}_{V(0)}$, $\theta_{V(n)} = e^{\frac{2\pi i}{3}}\operatorname{id}_{V(n)}$, for $n=1,2$.

The S-matrix is

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ 1 & e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} \end{bmatrix}. \tag{5.105}$$ $\Diamond$

In the next chapter we will compare the explicit modular data computed in Example 5.48 to that of a modular tensor category constructed from a quantum group associated to $\mathfrak{sl}_2$.

Chapter 6

An explicit Kazhdan-Lusztig correspondence at a non-negative integral level

Moore and Seiberg remarked in [MS88] and [MS89] that rational vertex operator algebras have canonically braided tensor categories of modules and, in modern language, modular tensor structure. In [Dri90], Drinfeld showed that for any ADE-type simple Lie algebra $\mathfrak{g}$, the quantised universal enveloping algebra $U_h(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})[[h]]$ has a non-strict braided tensor category of modules. The associator is given by a system of differential equations from conformal field theory, namely, the Knizhnik–Zamolodchikov equations from [KZ84].

Motivated by [MS88], [MS89] and [Dri90], Kazhdan and Lusztig in [KL91] used $U_h(\mathfrak{g})$ and the KZ equations to define a monoidal structure on a certain category of modules for the affine Lie algebra $\widehat{\mathfrak{g}}$. In [KL93a; KL93b; KL94a; KL94b], they constructed a braided monoidal equivalence between two rigid braided tensor categories, one constructed from certain $\widehat{\mathfrak{g}}$-modules at level k, and the other constructed from a certain category of modules for a $\mathfrak{g}$-quantum group specialised at $q = e^{i\pi/(k+h^\vee)}$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. This *Kazhdan-Lusztig correspondence* was originally proven for levels $k \in \mathbb{C}$ such that $k + h^\vee < 0$ (and later extended to $k + h^\vee \notin \mathbb{Q}_{\geq 0}$), despite the fact that Moore and Seiberg's work was for non-negative integral levels.

In this chapter, we will detail a correspondence for $\mathfrak{sl}_2$ at level 1. Our braided tensor equivalence will involve an explicit construction of a functor which does not factor through a category of U_h-modules. We will show also that the equivalence holds on the level of modular tensor categories. In the final section, we will explain some details of our construction.

6.1 Affine vertex operator algebras

Since this thesis has a vertex-operator-algebraic perspective, we will use *affine vertex operator algebras* instead of using affine Lie algebras directly. Here, we give the definition of an affine vertex operator algebra, following [FZ92]. The construction is similar to that of the Heisenberg vertex operator algebra discussed in Chapter 2, but with $\mathfrak{gl}_1$ replaced by a simple Lie algebra.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra.[1] Assume that a Cartan subalgebra, root system and system of positive roots are fixed. Let

$$\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}, \tag{6.1}$$

be the Killing form normalised so that the highest root θ has norm-squared $\langle\theta,\theta\rangle = 2$.

DEFINITION 6.1. The *affine Lie algebra* $\widehat{\mathfrak{g}}$ associated with the pair $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ consists of the vector space

$$\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\mathbf{k}, \tag{6.2}$$

with the Lie bracket relations

$$[a_m,b_n] = [a,b]_{m+n} + m\langle a,b\rangle\delta_{m+n,0}\mathbf{k} \qquad \text{for all } a,b\in\mathfrak{g},\ m,n\in\mathbb{Z}, \tag{6.3}$$

where we write $a_m = a\otimes t^m$, and $\mathbf{k}$ a central element.

DEFINITION 6.2. A $\widehat{\mathfrak{g}}$-module is said to be of *level* k if $\mathbf{k}$ acts as multiplication by the complex scalar k.

Decompose $\widehat{\mathfrak{g}}$ into the subalgebras

$$\widehat{\mathfrak{g}}_{\geq 0} := \mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}\mathbf{k} \qquad \text{and} \qquad \widehat{\mathfrak{g}}_{-} := \mathfrak{g}\otimes\mathbb{C}[t^{-1}]t^{-1}, \tag{6.4}$$

so that $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_{\geq 0}\oplus\widehat{\mathfrak{g}}_{-}$. Let $k\in\mathbb{C}$ and let $\mathbb{C}_k = \mathbb{C}$ be the $\widehat{\mathfrak{g}}_{\geq 0}$-module with $\mathfrak{g}\otimes\mathbb{C}[t]$ acting as zero and $\mathbf{k}$ acting as multiplication by k. Define a $\mathcal{U}(\widehat{\mathfrak{g}})$ module by the induction

$$V_k(\mathfrak{g}) = \mathcal{U}(\widehat{\mathfrak{g}})\otimes_{\mathcal{U}(\widehat{\mathfrak{g}}_{\geq 0})}\mathbb{C}_k. \tag{6.5}$$

Let $\mathbf{1} = 1\otimes 1\in V_k(\mathfrak{g})$. Note that $V_k(\mathfrak{g})\cong\mathcal{U}(\widehat{\mathfrak{g}}_{-})$ possesses a PBW-basis with respect to an ordered basis $(a^{(i)})_{i=1}^{\dim\mathfrak{g}}$ of $\mathfrak{g}$,

$$\{a^{(i_1)}_{-n_1}\cdots a^{(i_\ell)}_{-n_\ell}\mathbf{1} : n_1\geq\cdots\geq n_\ell\geq 1,\ \text{and}\ \ i_j\leq i_{j+1} \text{ if } n_j = n_{j+1}\}. \tag{6.6}$$

For convenience, in what follows, we assume that the basis $(a^{(i)})_{i=1}^{\dim\mathfrak{g}}$ is orthonormal with respect to $\langle\cdot,\cdot\rangle$. Now assume that $k\neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. A Construction Theorem (Proposition 2.36) argument shows that $V_k(\mathfrak{g})$ has the following vertex algebra structure. The *Sugawara construction*[2] can be used to produce a conformal vector for $V_k(\mathfrak{g})$, hence giving $V_k(\mathfrak{g})$ the structure of a vertex operator algebra.

---

[1]For the Kazhdan-Lusztig correspsondence, we use the construction from [FZ92] where $\mathfrak{g}$ is simple. However, in general $\mathfrak{g}$ can be non-simple, as long as it is complex and equipped with a symmetric invariant non-generate bilinear form $\langle\cdot,\cdot\rangle$. See the construction in Section 6.2 of [LL04], for example.

[2]This construction has its physical origin in [Sug68], but these days it takes on a fairly unrecognisable form. The conformal vector (6.8) is now "well-known" and is commonly referred to as the *Sugawara construction* or the *Sugawara vector*.

DEFINITION 6.3. The *universal affine vertex operator algebra associated to* $\mathfrak{g}$ *at level* k is the 4-tuple $(V_k(\mathfrak{g}), Y, \mathbf{1}, \omega)$ consisting of the following data:

(i) the underlying vector space $V_k(\mathfrak{g})$,

(ii) the vertex operator map defined by

$$Y(a^{(i_1)}_{-n_1}\cdots a^{(i_\ell)}_{-n_\ell}\mathbf{1}) = {}^\circ_\circ \frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1} a^{(i_1)}(z)\cdots \frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1} a^{(i_1)}(z){}^\circ_\circ, \tag{6.7}$$

for all $a^{(i_1)}_{-n_1}\cdots a^{(i_\ell)}_{-n_\ell}\mathbf{1}$ in the PBW-basis of $V_k(\mathfrak{g})$,

(iii) the vacuum vector $\mathbf{1}$,

(iv) the conformal vector

$$\omega = \frac{1}{2(k+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}} a^{(i)}_{-1}a^{(i)}_{-1}\mathbf{1}. \tag{6.8}$$

REMARK 6.4. The $\mathbb{Z}$-grading of $V_k(\mathfrak{g})$ is given by its $L(0)$-eigenvalues, given by

$$L(0)a^{(i_1)}_{-n_1}\cdots a^{(i_\ell)}_{-n_\ell}\mathbf{1} = \left(\sum_{j=1}^{\ell} n_j\right) a^{(i_1)}_{-n_1}\cdots a^{(i_\ell)}_{-n_\ell}\mathbf{1}. \tag{6.9}$$

The vertex operator algebra $(V_k(\mathfrak{g}), Y, \mathbf{1}, \omega)$ has central charge $c = \dfrac{k\dim\mathfrak{g}}{k+h^\vee}$. △

We are interested in affine vertex operator algebras that are simple (recall Definition/Proposition 2.33). Since $L(0) = \omega_1$ grades a vertex operator algebra module, ideals are graded subspaces. In the case of $V_k(\mathfrak{g})$, we have $V_k(\mathfrak{g})_{(0)} = \mathbb{C}\mathbf{1}$. So, any proper ideal of $V_k(\mathfrak{g})$ does not contain any weight zero vectors, since the vacuum vector generates $V_k(\mathfrak{g})$. Hence, the *unique maximal ideal* J is the sum of all proper ideals and we thus have the following simple vertex operator algebra.

DEFINITION 6.5. The *simple affine vertex operator algebra associated to* $\mathfrak{g}$ *at level* k ($k \neq h^\vee$) is the quotient vertex operator algebra

$$L_k(\mathfrak{g}) = V_k(\mathfrak{g})/J. \tag{6.10}$$

REMARK 6.6. In [GK06], a necessary and sufficient condition for the irreducibility of $V_k(\mathfrak{g})$ is given. It follows that $V_k(\mathfrak{g})$ is simple when $k+h^\vee \notin \mathbb{Q}_{\geq 0}$, the case in the original Kazhdan-Lusztig correspondence. △

6.2 The Kazhdan-Lusztig correspondence

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and consider $L_k(\mathfrak{g})$, the simple affine vertex operator algebra associated to $\mathfrak{g}$ at level k. We first state the result by Kazhdan-Lusztig, as quoted in [Hua].

DEFINITION 6.7. Let λ be a weight of $\mathfrak{g}$ (i.e. a dual vector of a fixed Cartan subalgebra of $\mathfrak{g}$) and let $L(\lambda)$ be the irreducible highest weight module of highest weight λ. Promote $L(\lambda)$ to a $\widehat{\mathfrak{g}}_{\geq 0}$-module by identifying $\mathfrak{g} = \mathfrak{g} \otimes t^0$, letting $\mathfrak{g} \otimes \mathbb{C}[t]t$ act trivially and letting $\mathbf{k}$ act as the complex scalar k.

Define $L(k, \lambda)$ to be the unique irreducible quotient of the following $\widehat{\mathfrak{g}}$-module at level k:

$$\mathcal{U}(\widehat{\mathfrak{g}}) \otimes_{\mathcal{U}(\widehat{\mathfrak{g}}_{\geq 0})} L(\lambda). \tag{6.11}$$

Let $k \in \mathbb{C}$ be such that $k + h^\vee \notin \mathbb{Q}_{\geq 0}$. Denote by $\mathcal{O}_k$ the category $\widehat{\mathfrak{g}}$-modules at level k of finite Jordan-Hölder length whose irreducible subquotients are of the form $L(k, \lambda)$ with λ a dominant integral weight of $\mathfrak{g}$ (i.e. a non-negative integral sum of the fundamental weights of $\mathfrak{g}$).

THEOREM 6.8 (KAZHDAN-LUSZTIG). Let $k \in \mathbb{C}$ be such that $k + h^\vee \notin \mathbb{Q}_{\geq 0}$. Then, $\mathcal{O}_k$ has a natural rigid braided tensor category structure. Moreover, this is equivalent (as rigid braided tensor categories) to the rigid braided tensor category of finite-dimensional weight modules for a quantum group constructed from $\mathfrak{g}$ at $q = e^{\frac{i\pi}{k+h^\vee}}$.

REMARK 6.9. As remarked in Section 4.2 of [Hua], $\mathcal{O}_k$ is equivalent (as rigid braided tensor categories) to the rigid braided tensor category $L_k(\mathfrak{g})-\mathsf{Mod}$. Recall that we define vertex operator algebra modules to have finite-dimensional $L(0)$-eigenspaces. It follows that $\mathcal{O}_k$ is not the usual level k is *not* the usual level k BGG category for $\widehat{\mathfrak{g}}$. △

In [Fin96] and [Fin13], Finkelberg used Kazhdan and Lusztig's original work to extend the correspondence to nearly all non-negative integral levels and simple Lie algebras. Work towards a braided tensor structure for categories of modules of affine vertex operator algebras at *admissible levels* has been made in [CHY18]. See Section 4 of [Hua] for the various problems and conjectures arising from extending Theorem 6.8 to different levels. Our aim is not to solve one of these general problems, but instead, to explore an example of a Kazhdan-Lusztig correspondence at a level k such that $k + h^\vee \in \mathbb{Q}_{\geq 0}$. Our problem is stated as follows.

PROBLEM 6.10. Find a rigid braided tensor category constructed from a category of finite-dimensional modules for a quantum group associated to $\mathfrak{sl}_2$ that is equivalent (as rigid braided tensor categories) to the rigid braided tensor category $L_k(\mathfrak{sl}_2)-\mathsf{Mod}$ at level $k = 1$. Can this equivalence be strengthened to a modular equivalence? ▷

By "quantum group", we mean one of the Drinfeld-Jimbo-type quantum groups with underlying Hopf algebra structure discussed in Appendix D. As suggested by Conjecture 4.10 of [Hua], the quantum group will be specialised at $q = e^{\frac{i\pi}{k+h^\vee}}$ and the associated rigid braided tensor category will be constructed from a semisimplification (see Appendix E) of a full subcategory of finite-dimensional modules. We will explicitly construct an equivalence and show that it can be strengthened to an equivalence of modular tensor categories.

6.3 The vertex operator algebra $L_1(\mathfrak{sl}_2)$

We want to use $L_1(\mathfrak{sl}_2)$, however, we have not computed the modular tensor category data for $L_1(\mathfrak{sl}_2)$, let alone discussed its representation theory. Fortunately, $L_1(\mathfrak{sl}_2)$ is isomorphic as a vertex operator algebra to V_{L_0}, where L_0 is the root lattice of $\mathfrak{sl}_2$, so we can use the modular tensor category data for V_{L_0} computed in the last chapter in Example 5.48. (From here on, the notation for L_0 is fixed for this meaning.)

We will show that $L_1(\mathfrak{sl}_2) \cong V_{L_0}$ as vertex operator algebras. Denote by

$$h = h_{-1}\mathbf{1}, \qquad e = e_{-1}\mathbf{1}, \qquad f = f_{-1}\mathbf{1}, \tag{6.12}$$

the generators of $V_1(\mathfrak{sl}_2)$, so that

$$Y(a, z) = a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \qquad \text{for } a = h, e, f. \tag{6.13}$$

Define the map $\widetilde{\varphi}$ on the span of $\{\mathbf{1}, h, e, f\}$,

$$\begin{gathered}\widetilde{\varphi} : \text{span}\{\mathbf{1}, h, e, f\} \to V_{L_0}\\ \mathbf{1} \mapsto \iota(0), \quad h \mapsto \alpha_{-1}\iota(0), \quad e \mapsto \iota(\alpha), \quad f \mapsto \iota(-\alpha),\end{gathered} \tag{6.14}$$

recalling the notation $\iota(a) = a \otimes 1$ from Subsection 5.3.1 and Example 5.21.

We claim that we can extend $\widetilde{\varphi}$ to the linear map, defined on the PBW-basis of $V_1(\mathfrak{sl}_2)$, by

$$\varphi : V_1(\mathfrak{sl}_2) \to V_{L_0}, \quad \varphi(a^{(i_1)}_{-n_1} \cdots a^{(i_\ell)}_{-n_\ell}\mathbf{1}) = \widetilde{\varphi}(a^{(i_1)})_{-n_1} \cdots \widetilde{\varphi}(a^{(i_\ell)})_{-n_\ell}\iota(0), \tag{6.15}$$

with respect to the ordered basis $(a^{(1)}, a^{(2)}, a^{(3)}) := (h, e, f)$ of $\mathfrak{sl}_2$. For φ to be well-defined, we need it to respect the commutation relations of the generating modes. The commutation relations of the vertex operators can be computed from the definitions of the vertex operator maps for both lattice and affine vertex operator algebras. One can check the following:

$$\begin{gathered}[h(x), e(y)] = 2y^{-1}\delta\left(\frac{x}{y}\right) e(y), \qquad [h(x), f(y)] = -2y^{-1}\delta\left(\frac{x}{y}\right) f(y),\\ [e(x), f(y)] = y^{-1}\delta\left(\frac{x}{y}\right) h(y) - y^{-1}\frac{\partial}{\partial x}\delta\left(\frac{x}{y}\right),\\ [h(x), h(y)] = -2y^{-1}\frac{\partial}{\partial x}\delta\left(\frac{x}{y}\right), \qquad [e(x), e(y)] = [f(x), f(y)] = 0,\end{gathered} \tag{6.16}$$

and

$$\begin{gathered}[Y(\alpha_{-1}\iota(0), x), Y(\iota(\pm\alpha), y)] = \pm 2y^{-1}\delta\left(\frac{x}{y}\right) Y(\iota(\pm\alpha), y),\\ [Y(\iota(\alpha), x), Y(\iota(-\alpha), y)] = y^{-1}\delta\left(\frac{x}{y}\right) Y(\alpha_{-1}\iota(0), y) - y^{-1}\frac{\partial}{\partial x}\delta\left(\frac{x}{y}\right),\\ [Y(\alpha_{-1}\iota(0), x), Y(\alpha_{-1}\iota(0), y)] = -2y^{-1}\frac{\partial}{\partial x}\delta\left(\frac{x}{y}\right),\\ [Y(\iota(\pm\alpha), x), Y(\iota(\pm\alpha), y)] = 0.\end{gathered} \tag{6.17}$$

So, by induction we have

$$\varphi([a_m, b_n]v) = [\widetilde{\varphi}(a)_m, \widetilde{\varphi}(b)_n]\varphi(v), \tag{6.18}$$

for all $v \in V_1(\mathfrak{sl}_2)$, $a, b = h, e, f$, and $m, n \in \mathbb{Z}$. Thus, φ is a well-defined linear map, hence a vertex algebra homomorphism. Moreover, a basis for V_{L_0} is contained in the image of φ, so φ is surjective.

For the conformal vector, we can find an orthonormal basis so that

$$\varphi(\omega_{V_1(\mathfrak{sl}_2)}) = \varphi\left(\frac{1}{12}(h_{-1}h_{-1}\mathbf{1} + 2e_{-1}f_{-1}\mathbf{1} + 2f_{-1}e_{-1}\mathbf{1})\right) = \frac{1}{4}\alpha_{-1}^2\iota(0) = \omega_{V_{L_0}}. \tag{6.19}$$

Hence, $\varphi : V_1(\mathfrak{sl}_2) \to V_{L_0}$ is a vertex operator algebra homomorphism.

Since $L_1(\mathfrak{sl}_2)$ is the *unique* simple quotient of $V_1(\mathfrak{sl}_2)$, V_{L_0} is simple and $\mathrm{im}(\varphi) \neq 0$, we have an isomorphism of vertex operator algebras:

$$L_1(\mathfrak{sl}_2) = V_1(\mathfrak{sl}_2)/J = V_1(\mathfrak{sl}_2)/\ker(\varphi) \cong \mathrm{im}(\varphi) = V_{L_0}. \tag{6.20}$$

We have already given an explicit description of the modular structure of $V_{L_0}-\mathsf{Mod}$ in Example 5.48. Thus, it remains to seek a quantum group of $\mathfrak{sl}_2$ with a category of its modules that can be semisimplified to a modular tensor category that is modular equivalent to $V_{L_0}-\mathsf{Mod}$.

6.4 An explicit correspondence

The quantum group we work with is the Lusztig (restricted specialisation) $\mathfrak{sl}_2$-quantum group, with q specialised to ϵ, the primitive $2(k+h^\vee)^{\text{th}}$ root of unity, for $k = 1$ and $h^\vee = 2$. That is, U_ϵ^{res} at $\epsilon = e^{i\pi/3}$. (See Subsection 6.5.3 below for an explanation why we do not use the small quantum group.) Consider the ribbon tensor category $U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ from Proposition D.35.

We will focus our attention on the Weyl modules $W_\epsilon^{\text{res}}(n)$, for $n = 0, 1, 2$. These modules are in fact irreducible (one can simply check this directly). The E, F, K actions are summarised by the diagrams

$$\begin{tikzcd} \underset{1}{v_0^{(0)}}, \end{tikzcd} \qquad \begin{tikzcd} \underset{\epsilon^1}{v_0^{(1)}} \arrow[r, bend left, dashed] & \underset{\epsilon^{-1}}{v_1^{(1)}} \arrow[l, bend left] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \underset{\epsilon^2}{v_0^{(2)}} \arrow[r, bend left, dashed] & \underset{1}{v_1^{(2)}} \arrow[l, bend left] \arrow[r, bend left, dashed] & \underset{\epsilon^{-2}}{v_2^{(2)}}, \arrow[l, bend left] \end{tikzcd} \tag{6.21}$$

where the nodes denote the basis vectors, the labels below the nodes are the K-eigenvalues, the dashed arrows denote the actions by F, the solid arrows denote the actions by E, and each action has a factor of 1.

For brevity, we will now write $W(i) = W_\epsilon^{\text{res}}(i)$ for $i = 0, 1, 2$, $u = v_0^{(0)}$, $v_0 = v_0^{(1)}$ and $v_1 = v_1^{(1)}$.

The unit object in $U_\epsilon^{\text{res}}\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ is $W(0)$. So, we have

$$W(0) \otimes W(0) \cong W(0), \qquad W(0) \otimes W(1) \cong W(1) \quad \text{and} \quad W(1) \otimes W(0) \cong W(1).$$

We can also decompose $W(1) \otimes W(1)$ into a direct sum of indecomposable modules. The coproduct for U_ϵ^{res} is used to compute the U_ϵ^{res}-action on $W(1) \otimes W(1)$, giving the decomposition:

$$\begin{array}{c} -\epsilon v_0 \otimes v_1 + v_1 \otimes v_0 \\ 1 \\ \bigoplus \\ \begin{tikzcd} v_0 \otimes v_0 \arrow[r, dashed, bend left] & \epsilon^{-1} v_0 \otimes v_1 + v_1 \otimes v_0 \arrow[l, bend left] \arrow[r, dashed, bend left] & v_1 \otimes v_1 \arrow[l, bend left] \end{tikzcd} \\ \epsilon^2 \qquad\qquad\qquad 1 \qquad\qquad\qquad \epsilon^{-2} \end{array} \cong W(0) \oplus W(2). \tag{6.22}$$

This monoidal structure (multiplication table) on $W(0)$ and $W(1)$ resembles the monoidal structure on the simple objects of $V_{L_0}\text{−}\mathsf{Mod}$, provided we disregard $W(2)$ (recall that $V(i) \boxtimes V(j) = V(i+j)$, for $i, j \in \mathbb{Z}_2 = \{0, 1\}$, from Example 5.48). The process of semisimplification, as discussed in Appendix E, can produce a monoidal category with this desired monoidal structure if it recontextualises $W(2)$ as a zero object. As we will explain in Section 6.5 below, we must first choose a subcategory before we perform the semisimiplification.

Definition 6.11. Let $\mathsf{Gen}_{0,1}$ be the "subcategory generated by $W_\epsilon^{\text{res}}(0)$ and $W_\epsilon^{\text{res}}(1)$". Put precisely, let $\mathsf{Gen}_{0,1}$ be the smallest (by inclusion of object classes) full subcategory of $U_\epsilon^{\text{res}}\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ satisfying the following conditions:

(i) all modules isomorphic to $W_\epsilon^{\text{res}}(0)$ and $W_\epsilon^{\text{res}}(1)$ are objects,

(ii) closure under tensor products,

(iii) closure under duals,

(iv) closure under direct sums,

(v) closure under direct summands.

Remark 6.12. The category $\mathsf{Gen}_{0,1}$ exists because $U_\epsilon^{\text{res}}\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ satisfies conditions (i)-(v). In fact, we will not need to know all of the objects in $\mathsf{Gen}_{0,1}$ explicitly because we will eventually semisimplify it. As discussed in Remark E.12, it will be sufficient to know just the indecomposable objects of non-zero dimension.

We could have instead used the construction of the category of tilting modules given in Section 11.3 of [CP95] for odd roots of unity, or in our case, Section 3 of [Saw06] for general roots of unity, but this construction would take too long to present here. We are able to bypass the tilting module construction because we expect only two simple objects after semisimplification for the case of $U_\epsilon^{\text{res}}(\mathfrak{sl}_2)$ at $\epsilon = e^{i\pi/3}$, hence this example is small enough to compute all necessary data explicitly. However, if we were to generalise this procedure to other simple Lie algebras or higher roots of unity, then our procedure would not be computationally viable. △

By definition, $\mathsf{Gen}_{0,1}$ is a rigid monoidal subcategory and closed under direct sums and direct summands. Recall from Remark E.12 that $\mathsf{Gen}_{0,1}$ can be semisimplified to produce a semisimple pivotal tensor category. The canonical pivotal structure was described in Remark 4.37. In the case of quasi-triangular Hopf algebras, the natural isomorphism u in (4.38) is exactly the action by the element $u = \nabla((S \otimes \mathrm{id})R_{21})$ in (D.7). Since the canonical twist for $U_\epsilon^{\mathrm{res}}$ is given by the action of $(e^{-hH}u)^{-1}$ (the inverse of the ribbon element (D.7) in U_h), the pivotal structure is given by the action of $u(e^{-hH}u)^{-1} = e^{hH} = K$. Note that this is equivalent to giving $U_\epsilon^{\mathrm{res}}$ the structure of a pivotal Hopf algebra (see Example 4.24) with pivot K, hence to giving $U_\epsilon^{\mathrm{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ the pivotal structure a^K, as defined in (4.27).

Given a module X in $U_\epsilon^{\mathrm{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$, K acting semisimply implies that we can choose a basis $\{x_i\}_{i=1}^{\dim_{\mathbb{C}} X}$ of K-eigenvectors with dual basis $\{x^i\}_{i=1}^{\dim_{\mathbb{C}} X}$. The (left categorical) dimension of the module is:

$$\dim_{a^K}(X) = \sum_{i=1}^{\dim_{\mathbb{C}} X} x^i(Kx_i) = \sum_{i=1}^{\dim_{\mathbb{C}} X} \lambda_i, \quad \text{where } \lambda_i \in \mathbb{C} \text{ satisfies } Kx_i = \lambda_i x_i. \tag{6.23}$$

From (6.21), we can see that the dimensions of the irreducible modules are

$$\dim_{a^K}(W_\epsilon^{\mathrm{res}}(0)) = 1, \quad \dim_{a^K}(W_\epsilon^{\mathrm{res}}(1)) = 1, \quad \text{and} \quad \dim_{a^K}(W_\epsilon^{\mathrm{res}}(2)) = 0. \tag{6.24}$$

(Note the right categorical dimension is the sum of the reciprocals of the K-eigenvalues. Since the eigenvalues are roots of unity, and the dimensions are real, the left and right categorical dimensions of the Weyl modules coincide. For this reason, we can use $\dim_{a^K}$ to denote either the left of right categorical dimensions in what follows.)

After using the pivotal structure of $U_\epsilon^{\mathrm{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ to semisimplify $\mathsf{Gen}_{0,1}$, we can now understand the abelian structure of $\overline{\mathsf{Gen}_{0,1}}$, the semisimplification of $\mathsf{Gen}_{0,1}$.

PROPOSITION 6.13. The ribbon tensor category $\overline{\mathsf{Gen}_{0,1}}$ has two isomorphism classes of simple objects. It follows that $\overline{\mathsf{Gen}_{0,1}}$ is a pre-modular category.

Proof. Since $W(0)$ and $W(1)$ are non-isomorphic, non-zero dimensional indecomposable objects in $\mathsf{Gen}_{0,1}$, there are at least two simple objects in $\overline{\mathsf{Gen}_{0,1}}$, up to isomorphism. Any simple object M in $\overline{\mathsf{Gen}_{0,1}}$ must be isomorphic to an indecomposable object in $\mathsf{Gen}_{0,1}$ of non-zero dimension. Since $\mathsf{Gen}_{0,1}$ was defined to be the smallest category with its defining properties, there is a finite sequence of tensor products, duals, direct sums and direct summands used to obtain M from $W(0)$ and $W(1)$.

Assume the case where M is obtained only from modules isomorphic to $W(0)$ or $W(1)$ (i.e. never tensor with or take the dual of an object containing $W(2)$ as a direct summand). Since $W(0)$ or $W(1)$ are simple and self dual in $\mathsf{Gen}_{0,1}$, then M is isomorphic to either $W(0)$ or $W(1)$.

Suppose M, at some point in the sequence, is generated from $W(2)$ via a finite sequence of tensor products, duals and direct summands. But, $W(2)$ is a zero object in $\overline{\mathsf{Gen}_{0,1}}$, and any tensor product, dual or direct summand (in $\overline{\mathsf{Gen}_{0,1}}$) of a zero object is zero. So, M is a zero in $\overline{\mathsf{Gen}_{0,1}}$, contradicting its simplicity.

Hence, M is isomorphic to $W(0)$ or $W(1)$. Thus, $\overline{\mathsf{Gen}_{0,1}}$ has exactly two simple objects, up to isomorphism, and is a hence a finite semisimple ribbon tensor category. That is, a pre-modular category. $\square$

We now need to produce a $\mathbb{C}$-linear braided monoidal equivalence between $\overline{\mathsf{Gen}_{0,1}}$ and $V_{L_0}-\mathsf{Mod}$. Since the skeletal category $V_{L_0}-\mathsf{Mod}'$ is modular equivalent to $\underline{V_{L_0}-\mathsf{Mod}}$, it will suffice to construct a $\mathbb{C}$-linear braided monoidal equivalence between $\overline{\mathsf{Gen}_{0,1}}$ and $V_{L_0}-\mathsf{Mod}'$. We work with $V_{L_0}-\mathsf{Mod}'$ because it is easier to write down a functor and check the compatibility conditions when the source category is skeletal.

We have effectively already computed the tensor product in $\overline{\mathsf{Gen}_{0,1}}$ for all simple objects: The trivial module $W(0)$ is the unit object in $U_\epsilon^{\mathrm{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$, hence is the unit object in $\mathsf{Gen}_{0,1}$ and $\overline{\mathsf{Gen}_{0,1}}$. Since $W(2)$ is a zero object in $\overline{\mathsf{Gen}_{0,1}}$, it follows from (6.22) that

$$W(1) \otimes W(1) \cong_{\overline{\mathsf{Gen}_{0,1}}} W(0). \tag{6.25}$$

We define a monoidal functor $F : V_{L_0}-\mathsf{Mod}' \to \overline{\mathsf{Gen}_{0,1}}$, on simple objects, by

$$F(V(0)) = W(0) \qquad \text{and} \qquad F(V(1)) = W(1), \tag{6.26}$$

so that tensor product structure is preserved. The images of the morphisms are, of course,

$$F(\mathrm{id}_{V(0)}) = \mathrm{id}_{W(0)} \quad \text{and} \quad F(\mathrm{id}_{V(1)}) = \mathrm{id}_{W(1)}, \tag{6.27}$$

which can be extended $\mathbb{C}$-linearly. Extending F over direct sums gives a $\mathbb{C}$-linear additive functor that respects tensor products (up to isomorphism).

For a monoidal functor, we also need to define the data

$$J : F(-) \otimes F(-) \Rightarrow F(- \boxtimes -) \quad \text{and} \quad \varphi : W(0) \to F(V(0)).$$

Define $\varphi = \mathrm{id}_{W(0)}$ and define the following morphisms in $\mathsf{Gen}_{0,1}$:

$$\mu_{0,0} \colon W(0) \otimes W(0) \to W(0), \qquad u \otimes u \mapsto u, \tag{6.28}$$

$$\mu_{0,1} \colon W(0) \otimes W(1) \to W(1), \qquad u \otimes v_i \mapsto v_i, \tag{6.29}$$

$$\mu_{1,0} \colon W(1) \otimes W(0) \to W(1), \qquad v_i \otimes u \mapsto v_i, \tag{6.30}$$

$$\mu_{1,1} \colon W(1) \otimes W(1) \to W(0), \qquad \begin{aligned} -\epsilon v_0 \otimes v_1 + v_1 \otimes v_0 &\mapsto u, \\ v_0 \otimes v_0 &\mapsto 0, \\ \epsilon^{-1} v_0 \otimes v_1 + v_1 \otimes v_0 &\mapsto 0, \\ v_1 \otimes v_1 &\mapsto 0, \end{aligned} \tag{6.31}$$

$$\widetilde{\mu}_{1,1} \colon W(0) \to W(1) \otimes W(1), \qquad u \mapsto -\epsilon v_0 \otimes v_1 + v_1 \otimes v_0. \tag{6.32}$$

Then, define the following morphisms in $\overline{\mathsf{Gen}_{0,1}}$, for $i, j \in \mathbb{Z}_2 = \{0, 1\}$,

$$J_{V(i),V(j)} = [\mu_{i,j}] : F(V(i)) \otimes F(V(j)) \Rightarrow F(V(i) \boxtimes V(j)) = F(V(i+j)). \tag{6.33}$$

Note that $\mu_{0,0}$, $\mu_{0,1}$, $\mu_{1,0}$ are isomorphisms in $\mathsf{Gen}_{0,1}$, so their semisimplifications are isomorphisms in $\overline{\mathsf{Gen}_{0,1}}$. Moreover, $\mu_{1,1} \circ \widetilde{\mu}_{1,1} = \mathrm{id}_{W(0)}$ while $\widetilde{\mu}_{1,1} \circ \mu_{1,1}$ is the identity on the non-zero dimension summand. So, $[\mu_{1,1}] \circ [\widetilde{\mu}_{1,1}] = [\mathrm{id}_{W(0)}]$ and $[\widetilde{\mu}_{1,1}] \circ [\mu_{1,1}] =$

$[\mathrm{id}_{W(1)\otimes W(1)}]$, hence $J_{V(1),V(1)}$ is an isomorphism. Since the hom-spaces between the simple objects are either zero or one $\mathbb{C}$-dimensional, J is natural on simple objects and hence its extension to direct sums is natural.

We now have the data (F, J, φ) needed for a monoidal functor, so we are left to show the compatibility of the associators and unitors. Recall that the semisimplification functor $S : \mathsf{Gen}_{0,1} \to \overline{\mathsf{Gen}_{0,1}}$ acts as the identity on objects. To show the commutativity of a diagram $D : \mathscr{D} \to \overline{\mathsf{Gen}_{0,1}}$, it therefore suffices to show the commutativity of a diagram $D' : \mathscr{D} \to \mathsf{Gen}_{0,1}$, with the same objects as in the image of D, such that $S \circ D' = D$. This enables us to perform concrete computations in $\mathsf{Gen}_{0,1}$, to be used in the non-concrete category of $\overline{\mathsf{Gen}_{0,1}}$.

The compatibility of associators that we wish to show can be written as

$$\begin{tikzcd} FV(i)\otimes(FV(j)\otimes FV(k)) \arrow[r, "{[\alpha_{FV(i),FV(j),FV(k)}]}"] & (FV(i)\otimes FV(j))\otimes FV(k) \arrow[d, "J_{V(i),V(j)}\otimes \mathrm{id}"] \\ FV(i)\otimes FV(j+k) \arrow[u, "\mathrm{id}\otimes J^{-1}_{V(i),V(j)}"] & FV(i+j)\otimes FV(k) \arrow[d, "J_{V(i+j),V(k)}"] \\ FV(i+j+k) \arrow[u, "J^{-1}_{V(i),V(j+k)}"] \arrow[r, "F(\mathcal{A}_{V(i),V(j),V(k)})"'] & FV(i+j+k) \end{tikzcd} \quad . \tag{6.34}$$

Hence, it suffices to show the commutativity of

$$\begin{tikzcd} W(i)\otimes(W(j)\otimes W(k)) \arrow[r, "\alpha_{W(i),W(j),W(k)}"] & (W(i)\otimes W(j))\otimes W(k) \arrow[d, "\mu_{i,j}\otimes \mathrm{id}"] \\ W(i)\otimes W(j+k) \arrow[u, "\mathrm{id}\otimes\widetilde{\mu}_{i,j}"] & W(i+j)\otimes W(k) \arrow[d, "\mu_{i+j,k}"] \\ W(i+j+k) \arrow[u, "\widetilde{\mu}_{i,j+k}"] \arrow[r, "A_{i,j,k}\,\mathrm{id}_{W(i+j+k)}"'] & W(i+j+k) \end{tikzcd} \quad , \tag{6.35}$$

where we use $i, j, k \in \mathbb{Z}_2 = \{0, 1\}$, write $\widetilde{\mu}_{i,j} = \mu_{i,j}^{-1}$ for $(i, j) = (0, 0), (0, 1), (1, 0)$, recall $\widetilde{\mu}_{1,1}$ from (6.32), and define the complex numbers

$$A_{i,j,k} = \begin{cases} -1 & \text{if } i = j = k = 1, \\ 1 & \text{otherwise.} \end{cases} \tag{6.36}$$

To explicitly show that (6.35) commutes, we must show that mapping the basis vectors of $W(i + j + k)$ by the composition of linear maps, going "up and around" the diagram, results in a factor of $A_{i,j,k}$. It will help to first compute the following:

$$\mu_{1,1}(v_0 \otimes v_1) = \mu_{1,1}(-(-\epsilon v_0 \otimes v_1 + v_1 \otimes v_0) + \epsilon^{-1} v_0 \otimes v_1 + v_1 \otimes v_0) = -u,$$
$$\mu_{1,1}(v_1 \otimes v_0) = \mu_{1,1}(\epsilon^{-1}(-\epsilon v_0 \otimes v_1 + v_1 \otimes v_0) + \epsilon(\epsilon^{-1} v_0 \otimes v_1 + v_1 \otimes v_0)) = \epsilon^{-1} u.$$

For brevity we omit the tensor products on elements in what follows. Recall that the associator in a category of finite dimensional modules, for some Hopf algebra over $\mathbb{C}$, is the associator from $\mathbb{C}-\mathsf{Vect}$. The computations are as follows.

$$
\begin{aligned}
(i,j,k) = (0,0,0), \quad & u \xmapsto{\widetilde{\mu}_{0,0}} uu \xmapsto{\mathrm{id}\otimes\widetilde{\mu}_{0,0}} u(uu) \xmapsto{\alpha} (uu)u \xmapsto{\mu_{0,0}\otimes\mathrm{id}} uu \xmapsto{\mu_{0,0}} u \\
(i,j,k) = (0,0,1), \quad & v_i \xmapsto{\widetilde{\mu}_{0,1}} uv_i \xmapsto{\mathrm{id}\otimes\widetilde{\mu}_{0,1}} u(uv_i) \xmapsto{\alpha} (uu)v_i \xmapsto{\mu_{0,0}\otimes\mathrm{id}} uv_i \xmapsto{\mu_{0,1}} v_i \\
(i,j,k) = (0,1,0), \quad & v_i \xmapsto{\widetilde{\mu}_{0,1}} uv_i \xmapsto{\mathrm{id}\otimes\widetilde{\mu}_{1,0}} u(v_iu) \xmapsto{\alpha} (uv_i)u \xmapsto{\mu_{0,1}\otimes\mathrm{id}} v_iu \xmapsto{\mu_{1,0}} v_i \\
(i,j,k) = (0,1,1), \quad & u \xmapsto{\widetilde{\mu}_{0,0}} uu \xmapsto{\mathrm{id}\otimes\widetilde{\mu}_{1,1}} u(-\epsilon v_0v_1 + v_1v_0) \xmapsto{\alpha} -\epsilon(uv_0)v_1 + (uv_1)v_0 \\
& \xmapsto{\mu_{0,1}\otimes\mathrm{id}} -\epsilon v_0v_1 + v_1v_0 \xmapsto{\mu_{0,0}} u \\
(i,j,k) = (1,0,0), \quad & v_i \xmapsto{\widetilde{\mu}_{1,0}} v_iu \xmapsto{\mathrm{id}\otimes\widetilde{\mu}_{0,0}} v_i(uu) \xmapsto{\alpha} (v_iu)u \xmapsto{\mu_{1,0}\otimes\mathrm{id}} v_iu \xmapsto{\mu_{1,0}} v_i \\
(i,j,k) = (1,0,1), \quad & u \xmapsto{\widetilde{\mu}_{1,1}} -\epsilon v_0v_1 + v_1v_0 \xmapsto{\mathrm{id}\otimes\widetilde{\mu}_{0,1}} -\epsilon v_0(uv_1) + v_1(uv_0) \\
& \xmapsto{\alpha} -\epsilon(v_0u)v_1 + (v_1u)v_0 \xmapsto{\mu_{1,0}\otimes\mathrm{id}} -\epsilon v_0v_1 + v_1v_0 \xmapsto{\mu_{1,1}} u \\
(i,j,k) = (1,1,0), \quad & u \xmapsto{\widetilde{\mu}_{1,1}} -\epsilon v_0v_1 + v_1v_0 \xmapsto{\mathrm{id}\otimes\widetilde{\mu}_{1,0}} -\epsilon v_0(v_1u) + v_1(v_0u) \\
& \xmapsto{\alpha} -\epsilon(v_0v_1)u + (v_1v_0)u \xmapsto{\mu_{1,1}\otimes\mathrm{id}} uu \xmapsto{\mu_{0,0}} u \\
(i,j,k) = (1,1,1), \quad & v_0 \xmapsto{\widetilde{\mu}_{0,1}} v_0u \xmapsto{\mathrm{id}\otimes\widetilde{\mu}_{1,1}} v_0(-\epsilon v_0v_1 + v_1v_0) \\
& \xmapsto{\alpha} -\epsilon(v_0v_0)v_1 + (v_0v_1)v_0 \xmapsto{\mu_{1,1}\otimes\mathrm{id}} -uv_0 \xmapsto{\mu_{0,1}} -v_0 \\
(i,j,k) = (1,1,1), \quad & v_1 \xmapsto{\widetilde{\mu}_{0,1}} v_1u \xmapsto{\mathrm{id}\otimes\widetilde{\mu}_{1,1}} v_1(-\epsilon v_0v_1 + v_1v_0) \\
& \xmapsto{\alpha} -\epsilon(v_1v_0)v_1 + (v_1v_1)v_0 \xmapsto{\mu_{1,1}\otimes\mathrm{id}} -uv_1 \xmapsto{\mu_{0,1}} -v_1
\end{aligned}
$$

From this, we see that the compatibility of associators is satisfied.

The compatibility of unitors is also satisfied, which comes from the fact that we chose $\mu_{0,i} = l_{W(i)}$ and $\mu_{i,0} = r_{W(i)}$, for $i = 0, 1$, and $\varphi = \mathrm{id}_{W(0)}$. Recall that the unitors in a category of finite dimensional modules, for some Hopf algebra over $\mathbb{C}$, are the unitors from $\mathbb{C}-\mathsf{Vect}$. That is, the commutativity of

(6.37)
$$
\begin{tikzcd}
W(0)\otimes W(i) \arrow[r, "l_{W(i)}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}"'] & W(i) \\
W(0)\otimes W(i) \arrow[r, "\mu_{0,i}"'] & W(i) \arrow[u, "\mathrm{id}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
W(i)\otimes W(0) \arrow[r, "r_{W(i)}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}"'] & W(i) \\
W(i)\otimes W(0) \arrow[r, "\mu_{i,0}"'] & W(i) \arrow[u, "\mathrm{id}"']
\end{tikzcd}
,
$$

and the application of the semisimplification functor gives the commutativity of

(6.38)
$$
\begin{tikzcd}
W(0)\otimes W(i) \arrow[r, "{[l_{W(i)}]}"] \arrow[d, "\varphi\otimes\mathrm{id}"'] & W(i) \\
W(0)\otimes W(i) \arrow[r, "J_{V(0),V(i)}"'] & W(i) \arrow[u, "F(\lambda_{V(i)})"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
W(i)\otimes W(0) \arrow[r, "{[r_{W(i)}]}"] \arrow[d, "\mathrm{id}\otimes\varphi"'] & W(i) \\
W(i)\otimes W(0) \arrow[r, "J_{V(i),V(0)}"'] & W(i) \arrow[u, "F(\rho_{V(i)})"']
\end{tikzcd}
.
$$

Thus, (F, J, φ) is a monoidal functor from $V_{L_0}-\mathsf{Mod}'$ to $\overline{\mathsf{Gen}_{0,1}}$.

We have defined F to be bijective on the end-spaces of each simple object $V(i)$. The only morphism between non-isomorphic simple objects is zero, so F is fully faithful. We have defined F to be a one-to-one correspondence on the simple objects, up to isomorphism, of $V_{L_0}-\mathsf{Mod}'$ and $\overline{\mathsf{Gen}_{0,1}}$. Since $\overline{\mathsf{Gen}_{0,1}}$ is semisimple, we have that F is essentially surjective. Thus, F is an equivalence of categories.

Now we have to show that F preserves the braiding and ribbon structures. Recall that the braiding and ribbon structure of $U_\epsilon^{\mathrm{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ gets transferred to $\overline{\mathsf{Gen}_{0,1}}$ in the semisimplification process. We have defined F so that $W(0)$ and $W(1)$ are the simple objects in its image category. Hence, to verify the braiding of F, it suffices to know the braiding on $W(0)$ and $W(1)$ in $\overline{\mathsf{Gen}_{0,1}}$.

We first explicitly find the braiding on the simple objects $W(0)$ and $W(1)$ in $U_\epsilon^{\mathrm{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$. After we find the braiding concretely we can transfer it via semisimplification. Recall from Proposition D.33, that the braiding on $U_\epsilon^{\mathrm{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ is defined by $c_{V,W} = \tau \circ R$, where $\tau : v \otimes w \mapsto w \otimes v$,

$$R = \epsilon^{\frac{1}{2}H\otimes H} \sum_{n\geq 0} \epsilon^{\frac{n(n-1)}{2}} \frac{(\epsilon - \epsilon^{-1})^n}{[n]_\epsilon!} E^n \otimes F^n \tag{6.39}$$

and $\epsilon^{\frac{1}{2}H\otimes H} v\otimes w = \epsilon^{\langle\lambda,\mu\rangle} v\otimes w$ for weight vectors v and w of weights λ and μ, respectively. (Recall that $\epsilon^{n/2} = e^{i\pi n/6}$.) The action of R on the basis elements of $W(i) \otimes W(j)$, for $i, j = 0, 1$, are as follows:

$$\begin{aligned}
Ru \otimes u &= \epsilon^0 u \otimes u + 0 + \cdots = u \otimes u,\\
Ru \otimes v_i &= \epsilon^0 u \otimes v_i + 0 + \cdots = u \otimes v_i,\\
Rv_i \otimes u &= \epsilon^0 v_i \otimes u + 0 + \cdots = v_i \otimes u,\\
Rv_0 \otimes v_0 &= \epsilon^{1/2} v_0 \otimes v_0 + 0 + \cdots = \epsilon^{1/2} v_0 \otimes v_0,\\
Rv_0 \otimes v_1 &= \epsilon^{-1/2} v_0 \otimes v_1 + 0 + \cdots = \epsilon^{-1/2} v_0 \otimes v_1,\\
Rv_1 \otimes v_0 &= \epsilon^{\frac{1}{2}H\otimes H}(v_1 \otimes v_0 + (\epsilon - \epsilon^{-1}) v_0 \otimes v_1 + 0 \cdots)\\
&= \epsilon^{-1/2} v_1 \otimes v_0 + \epsilon^{-1/2}(\epsilon - \epsilon^{-1}) v_0 \otimes v_1,\\
Rv_1 \otimes v_1 &= \epsilon^{1/2} v_1 \otimes v_1 + 0 + \cdots = \epsilon^{1/2} v_1 \otimes v_1.
\end{aligned} \tag{6.40}$$

We now show that the monoidal functor (F, J, φ) satisfies the braiding compatibility conditions. Since $\overline{\mathsf{Gen}_{0,1}}$ and $V_{L_0}-\mathsf{Mod}$ are semisimple, it suffices to show that

$$\begin{tikzcd}
F(V(i)) \otimes F(V(j)) \arrow[r, "{[\tau\circ R]}"] & F(V(j)) \otimes F(V(i)) \arrow[d, "J_{j,i}"] \\
F(V(i) \boxtimes V(j)) \arrow[u, "J_{i,j}^{-1}"] \arrow[r, "F(\mathcal{R}_{V(i),V(j)})"'] & F(V(j) \boxtimes V(i))
\end{tikzcd} \tag{6.41}$$

commutes in $\overline{\mathsf{Gen}_{0,1}}$, for $i, j = 0, 1$. We first perform the concrete calculations for

$$\begin{tikzcd} W(i) \otimes W(j) \arrow[r, "\tau\circ R"] & W(j) \otimes W(i) \arrow[d, "\mu_{j,i}"] \\ W(i+j) \arrow[u, "\widetilde{\mu}_{i,j}"] \arrow[r, "r_{i,j}\,\mathrm{id}_{W(i+j)}"'] & W(j+i) \end{tikzcd}, \tag{6.42}$$

where we define the complex numbers $r_{i,j} = \begin{cases} 1 & \text{if } (i,j) = (0,0), (0,1), (1,0), \\ i & \text{if } (i,j) = (1,1). \end{cases}$

As before, we show that the composition "going up and around" (6.42) results in the factor $r_{i,j}$. The computations are as follows.

$$\begin{aligned}
(i,j) = (0,0), &\quad u \xmapsto{\widetilde{\mu}_{0,0}} u \otimes u \xmapsto{\tau\circ R} u \otimes u \xmapsto{\mu_{0,0}} u \\
(i,j) = (0,1), &\quad v_i \xmapsto{\widetilde{\mu}_{0,1}} u \otimes v_i \xmapsto{\tau\circ R} v_i \otimes u \xmapsto{\mu_{1,0}} v_i \\
(i,j) = (1,0), &\quad v_i \xmapsto{\widetilde{\mu}_{1,0}} v_i \otimes u \xmapsto{\tau\circ R} u \otimes v_i \xmapsto{\mu_{0,1}} v_i \\
(i,j) = (1,1), &\quad u \xmapsto{\widetilde{\mu}_{1,1}} -\epsilon v_0 \otimes v_1 + v_1 \otimes v_0 \\
&\quad \xmapsto{\tau\circ R} -\epsilon(\epsilon^{-1/2} v_1 \otimes v_0) + \epsilon^{-1/2}(\epsilon - \epsilon^{-1}) v_1 \otimes v_0 + \epsilon^{-1/2} v_0 \otimes v_1 \\
&\quad = -\epsilon^{-3/2} v_1 \otimes v_0 + \epsilon^{-1/2} v_0 \otimes v_1 \xmapsto{\mu_{1,1}} -\epsilon^{-3/2}(\epsilon^{-1} u) + \epsilon^{-1/2}(-u) \\
&\quad = (-\epsilon^{-5/2} - \epsilon^{-1/2}) u = iu
\end{aligned}$$

Hence, (6.42) commutes. So, after semisimplification, (6.41) also commutes and we conclude that $(F, J, \varphi) : V_{L_0}\text{–}\mathsf{Mod} \to \overline{\mathsf{Gen}_{0,1}}$ is a braided monoidal functor.

Finally, we show that (F, J, φ) preserves twists. Recall from Proposition D.35 that

$$\nu = K^{-1} \sum_{n \geq 0} \epsilon^{\frac{n(n-1)}{2}} \frac{(\epsilon - \epsilon^{-1})^n}{[n]_\epsilon!} (-KF)^n \epsilon^{-\frac{1}{2}H^2} E^n, \tag{6.43}$$

where $\epsilon^{-\frac{1}{2}H^2} v = \epsilon^{-\frac{1}{2}\langle \lambda, \alpha \rangle^2} v$, where v has weight λ. The twist is the inverse of the action of ν, and the action of ν on $W(i)$, $i = 0, 1$, is computed by:

$$\begin{aligned}
\nu u &= \epsilon^0 \epsilon^0 u + 0 + \cdots = u, \\
\nu v_0 &= \epsilon^{-1} \epsilon^{-1/2} v_0 + 0 + \cdots = -i v_0, \\
\nu v_1 &= \epsilon^1 \epsilon^{-1/2} v_1 + \epsilon^1 (\epsilon - \epsilon^{-1})(-\epsilon^{-1}) \epsilon^{-1/2} v_1 + 0 + \cdots = -i v_1.
\end{aligned}$$

Hence,

$$\theta_{W(0)} = \mathrm{id}_{W(0)} \quad \text{and} \quad \theta_{W(1)} = i\, \mathrm{id}_{W(1)}. \tag{6.44}$$

So, recalling the twists in $V_{L_0}\text{–}\mathsf{Mod}$ from Example 5.48,

$$F(\theta_{V(0)}^{V_{L_0}\text{–}\mathsf{Mod}}) = F(\mathrm{id}_{V(0)}) = \mathrm{id}_{F(V(0))} = \theta_{F(V(0))}^{\overline{\mathsf{Gen}_{0,1}}}, \tag{6.45}$$

$$F(\theta_{V(1)}^{V_{L_0}\text{–}\mathsf{Mod}}) = F(i\, \mathrm{id}_{V(1)}) = i\, \mathrm{id}_{F(V(1))} = \theta_{F(V(1))}^{\overline{\mathsf{Gen}_{0,1}}}. \tag{6.46}$$

Thus, (F, J, φ) is a ribbon functor. Since F is also a $\mathbb{C}$-linear equivalence, (F, J, φ) is a pre-modular equivalence. Moreover, $V_{L_0}-\mathsf{Mod}'$ is modular, so this an equivalence of modular tensor categories by Proposition 4.55. Recall that $V_{L_0}-\mathsf{Mod}'$ is modular equivalent to $V_{L_0}-\mathsf{Mod}$, which is modular equivalent to $L_1(\mathfrak{sl}_2)-\mathsf{Mod}$ by the functor induced by the isomorphism (6.20). Thus, a Kazhdan-Lusztig correspondence for $\mathfrak{sl}_2$ at level 1 has been constructed and strengthened to an equivalence of modular tensor categories. Our results are summarised in the following theorem.

THEOREM 6.14. The modular tensor category $L_1(\mathfrak{sl}_2)-\mathsf{Mod}$ is modular equivalent to $\overline{\mathsf{Gen}_{0,1}}$, the semisimplification of the full subcategory $\mathsf{Gen}_{0,1}$ of finite-dimensional type 1 $U^{\text{res}}_{e^{i\pi/3}}(\mathfrak{sl}_2)$-modules generated by $W(0)$ and $W(1)$. □

6.5 Discussion

While we were able to construct a modular equivalence as in Theorem 6.14, our functor is opaque in the sense that it is not a concrete algebraic construction of a module in the target category from a module in the source category.

We can shed some light on the vertex-operator-algebraic side. First, $L_1(\mathfrak{sl}_2)-\mathsf{Mod}$ is *isomorphic* to $V_{L_0}-\mathsf{Mod}$, since $L_1(\mathfrak{sl}_2)$ and V_{L_0} are isomorphic vertex operator algebras, so we *are* working with $L_1(\mathfrak{sl}_2)$-modules. Second, the two irreducible $L_1(\mathfrak{sl}_2)$-modules, the vacuum module and the non-vacuum module, correspond to the $\mathfrak{sl}_2$-weights $\lambda = 0, \frac{1}{2}\alpha$, respectively. (The irreducible modules of an affine vertex operator algebra at non-negative integral level can be found in Theorem 1.3 of [HL99], along with their braided monoidal data.) Indeed, the equivalence maps the irreducible $L_1(\mathfrak{sl}_2)$-modules to the corresponding irreducible (Weyl) $U^{\text{res}}_{e^{i\pi/3}}(\mathfrak{sl}_2)$-modules with highest weights $\lambda = 0, \frac{1}{2}\alpha$.

This post hoc analysis reveals some structure behind our functor (6.26), despite it being constructed by inspection of the tensor products. Besides this, it is still unclear to us how to directly construct a finite-dimensional $U^{\text{res}}_{e^{i\pi/3}}(\mathfrak{sl}_2)$-module from an $L_1(\mathfrak{sl}_2)$-module. In [McR16], it is suggested that an equivalence for non-negative levels may be constructed by factoring through a category of U_h-modules, as is originally done by Kazhdan and Lusztig. However, we are trying to avoid this factorisation into the world of quasi-Hopf algebras. Finding an equivalence that is a direct construction, and general for all finite-dimensional complex simple Lie algebras and non-negative integral levels, is stated in [Hua] to be an open problem.

Our process has effectively constructed the equivalence out of a *non-monoidal* functor from $L_1(\mathfrak{sl}_2)-\mathsf{Mod}$ to a subcategory of $U^{\text{res}}_{e^{i\pi/3}}(\mathfrak{sl}_2)-\mathsf{Mod}$ composed with the semisimplification functor. (This just happened to be the first functor we found to work when constructing functors in a trial-and-error process.) Even though it is an equivalence, this functor is opaque for the direction it is going, and if given more time, we would refine this process. Instead, we would have constructed a $\mathbb{C}$-linear monoidal functor

$F : \mathsf{Gen}_{0,1} \to L_1(\mathfrak{sl}_2)-\mathsf{Mod}$ by assigning the isomorphism classes of indecomposable modules as $W(0) \mapsto V(0)$, $W(1) \mapsto V(1)$ and assigning the negligible (i.e. zero quantum dimension) indecomposable modules to the zero object. Recalling our analogy for tensor categories being categorified rings, we can think of the negligible modules as the "kernel" of F. Since semisimplification is a "quotient" process, we can think of $\overline{\mathsf{Gen}_{0,1}}$ as $\mathsf{Gen}_{0,1}/\ker(F)$. We expect there should be an analogue of the first isomorphism theorem in the category of pivotal tensor categories. And since F is essentially surjective, we expect a $\mathbb{C}$-linear monoidal equivalence between $L_1(\mathfrak{sl}_2)-\mathsf{Mod}$ and $\overline{\mathsf{Gen}_{0,1}}$. Perhaps this isomorphism theorem holds in the category of ribbon tensor categories.

We conclude by explaining some of the reasoning behind the semisimplification process and our choice of quantum group.

6.5.1 Reason for choosing a subcategory

We will explain why we needed to choose a subcategory of $U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}$ before semisimplifying. Suppose we semisimplify $U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}$. Then consider the indecomposable module $W_\epsilon^{\text{res}}(3)$ (obtained as a Weyl module as in Definition D.28):

(6.47)
$$\begin{tikzcd}
v_0^{(3)} \arrow[r, dashed, bend left] \arrow[rr, dashed, bend left=40] \arrow[rrr, dashed, bend left=50] & v_1^{(3)} \arrow[r, dashed, bend left] & v_2^{(3)} \arrow[l, bend left] & v_3^{(3)}, \arrow[l, bend left] \arrow[ll, bend left=40] \arrow[lll, bend left=50] \\
\epsilon^3 & \epsilon^1 & \epsilon^{-1} & \epsilon^{-3}
\end{tikzcd}$$

where the nodes denote the basis vectors, the labels below the nodes are the K-eigenvalues, the dashed arrows denote the actions by F, the solid arrows denote the actions by E, and each action has a factor of 1.

This module has categorical dimension

$$\dim_{a^K}(W_\epsilon^{\text{res}}(3)) = \epsilon^3 + \epsilon^1 + \epsilon^{-1} + \epsilon^{-3} = -1. \tag{6.48}$$

Since $W_\epsilon^{\text{res}}(3)$ is an indecomposable object with non-zero dimension, it is a simple object in $\overline{U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}}$. We want a semisimple category with two simple objects, however, we will show that $W_\epsilon^{\text{res}}(3)$ is not isomorphic to either $W_\epsilon^{\text{res}}(0)$ or $W_\epsilon^{\text{res}}(1)$ in $\overline{U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}}$. Since $W_\epsilon^{\text{res}}(3)$ has a basis of K-eigenvectors, with none of them of eigenvalue 1, then $\hom_{U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}}(W_\epsilon^{\text{res}}(3), W_\epsilon^{\text{res}}(0)) = 0$. So, $\hom_{\overline{U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}}}(W_\epsilon^{\text{res}}(3), W_\epsilon^{\text{res}}(0)) = 0$, and hence, $W_\epsilon^{\text{res}}(3)$ and $W_\epsilon^{\text{res}}(0)$ are non-isomorphic objects in $\overline{U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}}$. Suppose then that $f : W_\epsilon^{\text{res}}(3) \to W_\epsilon^{\text{res}}(1)$ is a morphism in $U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}$. Then, $f(v_0^{(3)}) = 0$ since there are no K-eigenvectors of eigenvalue ϵ^3 in $W_\epsilon^{\text{res}}(1)$. Similarly, $f(v_3^{(3)}) = 0$. Then,

$$f(v_1^{(3)}) = f(Fv_0^{(3)}) = Ff(v_0^{(3)}) = 0 \quad \text{and} \quad f(v_2^{(3)}) = f(Fv_1^{(3)}) = Ff(v_1^{(3)}) = 0.$$

So, $\hom_{\overline{U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}}}(W_\epsilon^{\text{res}}(3), W_\epsilon^{\text{res}}(1)) = 0$, and hence, $W_\epsilon^{\text{res}}(3)$ and $W_\epsilon^{\text{res}}(1)$ are non-isomorphic objects in $\overline{U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}}$. Hence, $\overline{U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}}$ has more than two isomorphism classes of simple objects.

Thus, $\overline{U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}}$ cannot be additively equivalent to $V_{L_0}-\mathsf{Mod}$. This is why we must take a subcategory of $U_\epsilon^{\text{res}}-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ before semisimplifying.

6.5.2 Reason for semisimplification

We can see from the direct sum decomposition $W(1) \otimes W(1) \cong W(0) \oplus W(2)$, in (6.22), that semisimplification is used to recontextualise $W(2)$ as a zero object. As we were trying to construct the correspondence, two other reasons for semisimplification became apparent, both stemming from the fact that the tensor product of two finite-dimensional modules for a Hopf algebra is modelled after the tensor product of their underlying vector spaces.

First, the monoidal category of finite-dimensional modules for a (non-quasi-)Hopf algebra has "trivial" associators and unitors. That is, after the canonical identifications for the $\mathbb{C}$-linear tensor product of three modules or for a module with the unit, the associators are identities. Semisimplification can remove this triviality because the associators are not necessarily set-theoretic maps anymore, but instead, elements in a quotiented hom-space. It is this mechanism that enabled us to pick up the non-trivial factors $A_{i,j,k}$ in (6.35) and hence the non-trivial associators in (6.34).

The second reason is due to the dimensions of the modules. The unit in our desired category of U_ϵ^{res}-modules must be the trivial module, which has $\mathbb{C}$-dimension one. If we want a U_ϵ^{res}-module $M = F(V(1))$ such that $M \otimes M = F(V(1)) \otimes F(V(1)) \cong F(V(0)) = \mathbb{1}$, then M must be of $\mathbb{C}$-dimension one. This means M is either $V_\epsilon(1, 0)$ or $V_\epsilon(-1, 0)$, up to isomorphism. For an additive equivalence, we require M to be non-isomorphic to $\mathbb{1} = V_\epsilon(1, 0)$, hence we have $M \cong V_\epsilon(-1, 0)$. Ignoring the fact that this is not a type 1 module (and as result has no canonical braiding or ribbon structure) the semisimple monoidal subcategory consisting of modules isomorphic to $V_\epsilon(\pm 1, 0)$ is additively equivalent to $V_{L_0}-\mathsf{Mod}$. Even though this category has the same fusion rules as $V_{L_0}-\mathsf{Mod}$, it is monoidally equivalent to a skeletal category with trivial associators. Hence, by similar arguments to Proposition 6.16 below, there can be no monoidal equivalence between this subcategory and $V_{L_0}-\mathsf{Mod}$. Since our desired subcategory must take modules of $\mathbb{C}$-dimension greater than one into account, we must also use semisimplification to allow isomorphisms between modules of different $\mathbb{C}$-dimensions.

6.5.3 Reason for not using the small quantum group

Our goal was to find a quantum group associated to $\mathfrak{sl}_2$, with a semisimplified category of modules that is braided tensor equivalent to $V_{L_0}-\mathsf{Mod}$. A promising candidate was the small quantum group $\overline{U}_\epsilon$ at a root of unity, since its irreducible representations are known (see Theorem 6.5.7 of [Kas95]) and, as discussed in Remark D.18, we have a canonical braided and ribbon structure. That is, we might expect a braided (or possibly ribbon) tensor equivalence between $V_{L_0}-\mathsf{Mod}$ and a semisimplified subcategory of $\overline{U}_\epsilon-\mathsf{Mod}_{\mathsf{fd}}$ at $\epsilon = e^{i\pi/3}$. We will show that no such braided tensor equivalence exists, and in fact, no monoidal equivalence exists between the two monoidal categories.

The small quantum group $\overline{U}_\epsilon$ has the irreducible representations $V_\epsilon(1,0)$, $V_\epsilon(-1,1)$ and $V_\epsilon(1,2)$ (which can be verified directly or using Theorem 6.5.7 of [Kas95]). The E, F, K actions are summarised by the following diagrams.

$$\begin{tikzcd} u, & v_0 \arrow[r, dashed, bend left] & v_1 \arrow[l, bend left, "-1"] \\ 1 & -\epsilon^1 & -\epsilon^{-1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} w_0 \arrow[r, dashed, bend left] & w_1 \arrow[l, bend left] \arrow[r, dashed, bend left] & w_2. \arrow[l, bend left] \\ \epsilon^2 & 1 & \epsilon^{-2} \end{tikzcd} \tag{6.49}$$

The dashed arrows represent F, the solid arrows represent E, the numbers below the basis vectors are their K-weights and the arrows with no labels have a factor of 1. Similarly to before, the categorical dimensions can be found be summing over the weights. That is,

$$\dim_{a^K}(V_\epsilon(1,0)) = 1, \quad \dim_{a^K} V_\epsilon(-1,1)) = -\epsilon - \epsilon^{-1} = -1,$$
$$\dim_{a^K}(V_\epsilon(1,2)) = \epsilon^2 + 1 + \epsilon^{-2} = 0.$$

Consider the smallest full subcategory $\mathscr{C}$ of $\overline{U}_\epsilon-\mathsf{Mod}_{\mathsf{fd}}$ that is closed under tensor products, duals, direct sums and direct summands, containing $V_\epsilon(1,0)$ and $V_\epsilon(-1,1)$, up to isomorphism. After semisimplification, $\overline{\mathscr{C}}$ contains the simple objects of $\mathscr{C}$ of non-zero dimension. That is, at least, $V_\epsilon(1,0)$ and $V_\epsilon(-1,1)$, up to isomorphism. Using similar arguments as in Proposition 6.13, $\overline{\mathscr{C}}$ has exactly two simple objects, up to isomorphism.

We need to compute the tensor product on $\overline{\mathscr{C}}$ for its simple objects. The trivial module $V_\epsilon(1,0) \cong \mathbb{C}$ is the unit object in $\overline{U}_\epsilon-\mathsf{Mod}_{\mathsf{fd}}$, hence is the unit object in $\mathscr{C}$ and $\overline{\mathscr{C}}$. So, we are left to compute the tensor product for $V_\epsilon(-1,1) \otimes V_\epsilon(-1,1)$, which decomposes into

$$\begin{array}{c} \begin{tikzcd} v_0 \otimes v_1 + \epsilon^{-1} v_1 \otimes v_0 \\ 1 \end{tikzcd} \\ \bigoplus \\ \begin{tikzcd} v_0 \otimes v_0 \arrow[r, dashed, bend left] & \epsilon^2 v_0 \otimes v_1 + v_1 \otimes v_0 \arrow[l, bend left] \arrow[r, dashed, bend left] & -v_1 \otimes v_1 \arrow[l, bend left] \\ \epsilon^2 & 1 & \epsilon^{-2} \end{tikzcd} \end{array} \cong \quad V_\epsilon(1,0) \oplus V_\epsilon(1,2). \tag{6.50}$$

We will construct a strict monoidal category that is monoidally equivalent to $\overline{\mathscr{C}}$ (this is because it is easier to see when skeletal monoidal categories are not equivalent). Define $\widetilde{\mathscr{C}}$ to be the skeletal $\mathbb{C}$-linear semisimple abelian category with simple objects X_0 and X_1. That is, $\widetilde{\mathscr{C}}$ contains a zero object, denoted by 0, and the finite direct sums of X_0 and X_1, with exactly one object in each isomorphism class.

We endow $\widetilde{\mathscr{C}}$ with the tensor product $\widetilde{\otimes} : \widetilde{\mathscr{C}} \times \widetilde{\mathscr{C}} \to \widetilde{\mathscr{C}}$, defined by

$$X_i \widetilde{\otimes} X_j = X_{i+j}, \quad \text{and} \quad \mathrm{id}_{X_i} \widetilde{\otimes} \mathrm{id}_{X_j} = \mathrm{id}_{X_{i+j}}, \tag{6.51}$$

with subscripts $i, j \in \mathbb{Z}_2 = \{0, 1\}$. This is extended bilinearly and to direct sums. Define the associator and unitors, $\widetilde{\alpha}$, $\widetilde{\lambda}$ and $\widetilde{\rho}$, to be identities. Then, $(\widetilde{\mathscr{C}}, \widetilde{\otimes}, X_0, \widetilde{\alpha}, \widetilde{\lambda}, \widetilde{\rho})$ is a skeletal strict monoidal category.

PROPOSITION 6.15. The monoidal categories $\widetilde{\mathscr{C}}$ and $\overline{\mathscr{C}}$ are monoidally equivalent.

Proof. Define the $\mathbb{C}$-linear fully faithful functor on the objects

$$F : \widetilde{\mathscr{C}} \to \overline{\mathscr{C}}, \qquad 0 \mapsto 0, \quad X_i \mapsto V_\epsilon(i),$$

and extend this to all direct sums. Since $\overline{\mathscr{C}}$ is semisimple with exactly two isomorphism classes of simple objects, then F is essentially surjective. We now endow the categorical equivalence F with monoidal structure.

Define $\varphi = \mathrm{id}_{V_\epsilon(0)} : V_\epsilon(0) \to F(X_0)$ and define the following morphisms in $\mathscr{C}$:

$$\mu_{0,0} \colon V_\epsilon(0) \otimes V_\epsilon(0) \to V_\epsilon(0), \qquad u \otimes u \mapsto u, \tag{6.52}$$

$$\mu_{0,1} \colon V_\epsilon(0) \otimes V_\epsilon(1) \to V_\epsilon(1), \qquad u \otimes v_i \mapsto v_i, \tag{6.53}$$

$$\mu_{1,0} \colon V_\epsilon(1) \otimes V_\epsilon(0) \to V_\epsilon(1), \qquad v_i \otimes u \mapsto v_i, \tag{6.54}$$

$$\mu_{1,1} \colon V_\epsilon(1) \otimes V_\epsilon(1) \to V_\epsilon(0), \qquad \begin{aligned} v_0 \otimes v_1 + \epsilon^{-1} v_1 \otimes v_0 &\mapsto u, \\ v_0 \otimes v_0 &\mapsto 0, \\ \epsilon^2 v_0 \otimes v_1 + v_1 \otimes v_0 &\mapsto 0, \\ -v_1 \otimes v_1 &\mapsto 0, \end{aligned} \tag{6.55}$$

$$\widetilde{\mu}_{1,1} \colon V_\epsilon(0) \to V_\epsilon(1) \otimes V_\epsilon(1), \qquad u \mapsto v_0 \otimes v_1 + \epsilon^{-1} v_1 \otimes v_0. \tag{6.56}$$

Then, define the morphisms

$$J_{X_i, X_j} = [\mu_{i,j}] : F(X_i) \otimes F(X_j) \Rightarrow F(X_i \widetilde{\otimes} X_j) = F(X_{i+j}) \tag{6.57}$$

in $\overline{\mathscr{C}}$, noting that $[\widetilde{\mu}_{1,1}] = J^{-1}_{X_1, X_1}$.

We will not go through the details here, but (F, J, φ) is a monoidal functor and a monoidal equivalence using the same procedure as we have shown in the previous section. ☐

PROPOSITION 6.16. There is no monoidal equivalence between $\mathscr{C}$ and $V_{L_0}-\mathsf{Mod}$.

Proof. We have that $\widetilde{\mathscr{C}}$ is monoidally equivalent to $\mathscr{C}$ and $V_{L_0}-\mathsf{Mod}'$ is monoidally equivalent to $V_{L_0}-\mathsf{Mod}$. So, it suffices to show that there is no monoidal equivalence between $\widetilde{\mathscr{C}}$ and $V_{L_0}-\mathsf{Mod}'$.

Suppose that there is an $\mathbb{C}$-linear monoidal equivalence:

$$(G : \widetilde{\mathscr{C}} \to V_{L_0}\text{−Mod}',\ K : G(-) \otimes G(-) \to G(-\widetilde{\otimes}-),\ \psi : V(0) \to G(X_0)). \tag{6.58}$$

Since G is an additive equivalence, it gives a one-to-one correspondence between isomorphism classes of simple objects. And since G is monoidal we have $G(X_i) = V(i)$, for $i = 0, 1$. So, we have the commutative diagram

$$\begin{tikzcd}
V(1) \boxtimes (V(1) \boxtimes V(1)) \arrow[r, "\mathcal{A}_{1,1,1}=-\,\mathrm{id}"] \arrow[d, "\mathrm{id}\boxtimes K_{X_1,X_1}"'] & (V(1) \boxtimes V(1)) \boxtimes V(1) \arrow[d, "K_{X_1,X_1}\boxtimes \mathrm{id}"] \\
V(1) \boxtimes V(0) \arrow[d, "K_{X_1,X_0}"'] & V(0) \boxtimes V(1) \arrow[d, "K_{X_0,X_1}"] \\
V(1) \arrow[r, "G(\widetilde{\alpha}_{X_1,X_1,X_1})=\mathrm{id}"'] & V(1)
\end{tikzcd} \tag{6.59}$$

Since K_{X_i,X_j} are invertible scalar multiples of $\mathrm{id}_{V(i+j)}$, the commutativity of (6.59) gives $K_{X_0,X_1} = -K_{X_1,X_0}$. But, we also have

(6.60)

$$\begin{tikzcd}
V(0) \otimes V(1) \arrow[r, "\lambda_{V(1)}=\mathrm{id}"] \arrow[d, "\psi\otimes\mathrm{id}"'] & V(1) \\
V(0) \otimes V(1) \arrow[r, "K_{X_0,X_1}"'] & V(1) \arrow[u, "G(\widetilde{\lambda}_{X_1})=\mathrm{id}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
V(1) \otimes V(0) \arrow[r, "\rho_{V(1)}=\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\psi"'] & V(1) \\
V(1) \otimes V(0) \arrow[r, "K_{X_1,X_0}"'] & V(1) \arrow[u, "G(\widetilde{\rho}_{X_1})=\mathrm{id}"']
\end{tikzcd}.$$

Since ψ is an invertible multiple of an identity, we have $K_{X_0,X_1} = K_{X_1,X_0}$. However, we cannot have $-K_{X_1,X_0} = K_{X_1,X_0}$ since it is invertible. This contradiction shows that no such (G, K, ψ) can exist. □

In summary, there is no monoidal equivalence and hence no braided tensor equivalence between the semisimplification of $\mathscr{C}$ and V_{L_0}−Mod. This explains why we attempted to construct an equivalence using the Lusztig quantum group instead.

REMARK 6.17. The failure of the small quantum group equivalence demonstrates the importance of stating all the data in a pre-modular category. Despite $\widetilde{\mathscr{C}}$ and V_{L_0}−Mod being ribbon fusion categories that are additively equivalent with the same fusion rules:

$$V(i) \boxtimes V(j) \cong V(i+j) \quad \text{and} \quad W(i) \otimes W(j) \cong W(i+j), \qquad \text{for } i, j \in \mathbb{Z}_2, \tag{6.61}$$

they are not monoidally equivalent. This is clear if we have an explicit description of the associators and unitors. Given a pre-modular category, there is no reason to assume that the monoidal categories are strict, especially since the associator for vertex operator algebras are in general non-trivial and the semisimplification process removes the triviality of the quantum group associator. △

Chapter 7

Conclusion

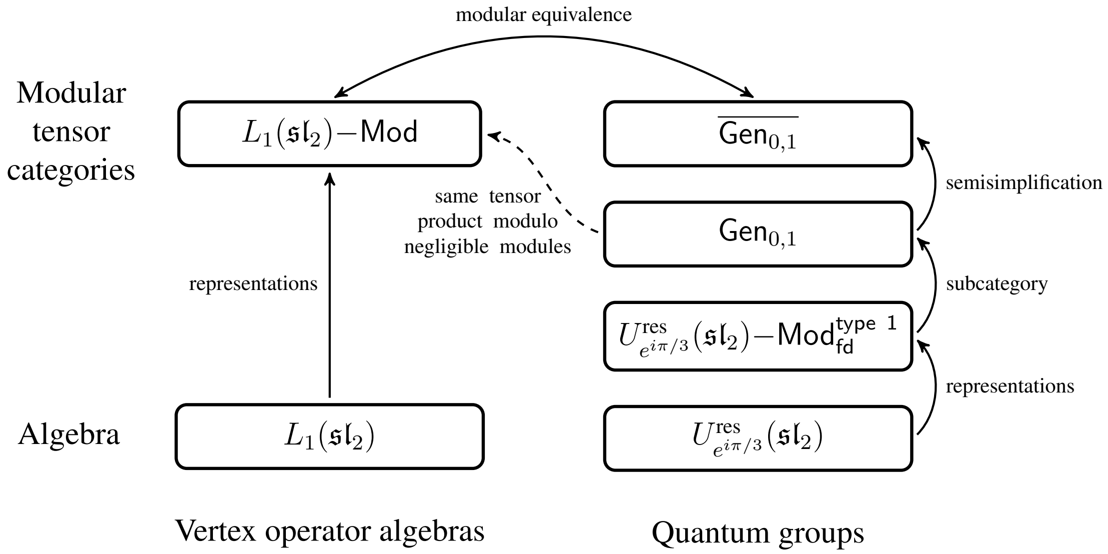

FIGURE 7.1: A schematic summary of our main problem: the Kazhdan-Lusztig correspondence specifically for $\mathfrak{sl}_2$ at level $k = 1$ and root of unity $\epsilon = e^{i\pi/3}$. The column on the left-hand side represents the world of vertex operator algebras and the column on the right-hand side represents the world of quantum groups. The vertical direction represents the progression from algebraic objects, to categories of representations, to increasing levels of categorified algebraic structure.

7.1 Summary

Throughout this thesis, we successfully built up our knowledge to a point where we could affirmatively answer our main problem.

Starting on the algebraic level, we learned about vertex algebras, specifically their conformally symmetric relatives, vertex operator algebras. We initially used the Heisenberg vertex operator algebra to develop an intuition for their structure. We then studied their representation theory using their modules. As expected, the category of modules has an abelian structure, but we progressed further to explore what other categorified structures the categories of modules can canonically obtain. The $P(w)$-tensor product formed the basis for these structures. It provided the fusion product to be used as the tensor product bifunctor in the monoidal categorical structure. Braided monoidal data was then obtained by analytic means using the intertwining operators associated to the $P(w)$-tensor product. Again, the Heisenberg vertex operator algebra was used as a guiding example.

We then built towards the definition of a pre-modular category, using Hopf algebras to provide concrete examples along the way. Notions of equivalences were also discussed in preparation for our correspondence. We arrived at a point where we needed examples of modular tensor categories, but elementary Hopf algebra examples were of no help. Furthermore, the Heisenberg vertex operator algebra was unable to provide any non-trivial modular tensor categories. So, we moved on to the lattice vertex operator algebras, which were able to produce modular tensor categories, and we explicitly computed their modular data by exploiting the HLZ procedure. One of our examples turned out to be equivalent to $L_1(\mathfrak{sl}_2)-\mathsf{Mod}$, the modular tensor category in our Kazhdan-Lusztig correspondence.

On the quantum-group-theoretic side, we studied various forms of quantum groups, but only focusing on those associated with $\mathfrak{sl}_2$. Their representations formed categories with ribbon tensor structure and canonical pivotal structure. After specialising to the sixth root of unity, there were subcategories with tensor structure similar to that of $L_1(\mathfrak{sl}_2)-\mathsf{Mod}$ but with extra modules. Semisimplification offered a way to quotient out such negligible modules and we examined the possible $\mathbb{C}$-linear additive equivalences between the semisimplifications and $L_1(\mathfrak{sl}_2)-\mathsf{Mod}$. We found that the Lusztig form $U^{\text{res}}_{e^{i\pi/3}}$ provided a monoidal equivalence, whereas the small quantum group $\overline{U}_{e^{i\pi/3}}$ did not. This monoidal equivalence extended to a braided equivalence and, finally, to a modular equivalence. Thus, we constructed our desired Kazhdan-Lusztig correspondence, connecting $L_1(\mathfrak{sl}_2)$ to an $\mathfrak{sl}_2$-quantum through an equivalence on the level of modular tensor categories.

7.2 Future directions

We constructed our functor by reducing the modular tensor categories to a state that we could explicitly understand without the use of the algebraic objects from which they originally came. In this sense, the functor we constructed lives strictly in the upper level of Figure 7.1. Unfortunately, this does not reveal "why" a such a functor exists. A future step could be to "pull" the functor back down to a concrete level. That is, a functorial construction of an object in $\overline{\mathsf{Gen}_{0,1}}$ given any $L_1(\mathfrak{sl}_2)$-module, or vice versa.

To develop our intuition, it may help to make another explicit Kazhdan-Lusztig correspondence; we have already computed the modular tensor data for $L_1(\mathfrak{sl}_3)-\mathsf{Mod}$ in Example 5.49. The desired modular tensor category on the right-hand side is expected to be the semisimplification of the category of $U_\epsilon^{\mathrm{res}}(\mathfrak{sl}_3)$-modules, generated by the first three Weyl modules. Here, ϵ is the eighth root of unity, using $2(k+h^\vee) = 2(1+3)$. Following the $\mathfrak{sl}_2$ case, the "first three Weyl" modules are the three Weyl modules corresponding to the dominant integral weights corresponding to the irreducible $L_1(\mathfrak{sl}_3)$-modules, i.e. $0, \omega_1, \omega_2$ (note that this is $0, \alpha_1^*, \alpha_2^*$ in the notation of Example 5.49). This agrees with the *Weyl alcove* used for general affine vertex operators and quantum groups—however, we are yet to understand this general approach and we are simply working with our intuition developed for lattice vertex operator algebras.

We would like to understand the general theory of affine vertex operator algebras and quantum groups, associated to finite-dimensional simple Lie algebras, at non-negative integral levels and roots of unity, respectively. On the vertex-operator-algebraic side, the first place to start would be [HL99], where the braided monoidal data is readily computed. On the quantum-group-theoretic side, [Saw06] lays out the general construction of modular tensor categories at roots of unity. Sawin uses the *tilting module construction*, which we were able to bypass by performing other explicit computations, but will need to adopt for general cases. We expect that the ribbon tensor category of tilting modules has the same tensor product decomposition rules as an affine vertex operator algebra modular tensor category, modulo the negligible modules. These fusion rules should depend heavily on the weight lattice of the corresponding finite-dimensional simple Lie algebra.

We suspect that the construction of the equivalence is similar to as in Figure 7.1. The top triangle in Figure 7.1 is an analogue of the first isomorphism theorem. The negligible modules are exactly those in the "kernel" and the semisimplification "quotients" out the kernel (by recontextualising them as zero objects). Since the functor from the subcategory of quantum group modules to the category of affine vertex operator algebra modules is surjective, we obtain an equivalence of categories. We would like to further investigate this process in the category of pivotal tensor categories. Since semisimplification preserves so much structure, we expect similar processes to hold in other categories of categories with underlying pivotal tensor structure, namely, the category of ribbon tensor categories, which contains all pre-modular and modular tensor categories.

Throughout this thesis, we have discussed vertex operator algebras, modular tensor categories and the Kazhdan-Lusztig correspondence. For the readers knowledgeable in either just vertex operator algebras or quantum groups, we have demonstrated how these algebraic structures are connected on the level of modular tensor categories. This thesis should also have provided some insight to the reader on the Kazhdan-Lusztig correspondence, at non-negative integral levels, by the use of an explicit example. As we were unable to find such detailed examples in the literature, others may also benefit from this thesis in this regard. We also hope that the reader has developed an understanding of the bigger picture, as in Figure 7.1, while still appreciating the finer detailed mathematics at each level along the way.

Appendix A

Formal algebra

Formal algebra and formal calculus provide analogues of complex-analytic notions in an algebraic way. Concepts such as series, differentiation, residues and the delta function have formal analogues. Such notions will be needed for the formulation of vertex algebras introduced in Chapter 2. This appendix breifly summarises the definitions and notation to be used in the body of this thesis. Some examples will be given to illustrate how the formalism works, while simultaneously working towards an identity that will be used to verify our first (but trivial) example of a vertex algebra. We recommend the references [LL04], [FB04] and [Sch08] for learning formal calculus.

A.1 Formal calculus

Let V be a complex vector space. We will use *formal variables*, typically denoted by x, y, z, z_1, z_2, etc., as symbols with powers that index sequences.

DEFINITION A.1. A *(doubly-infinite) formal Laurent series* in z with coefficients in V is a doubly-infinite sequence

$$\sum_{n\in\mathbb{Z}} a_n z^n := (a_n)_{n\in\mathbb{Z}}, \quad \text{with } a_n \in V. \tag{A.1}$$

Here, we use a purely formal sum notation. We will also call this a *formal distribution* or, simply, a *series*. We will commonly denote series by $a(z) = \sum_n a_n z^n$. Note that there is flexibility for the indexing to change, for example, we may write $\sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ instead.

DEFINITION A.2. The *vector space of (doubly-infinite) formal Laurent series* in z with coefficients in V, denoted by $V[[z, z^{-1}]]$ or $V[[z^{\pm 1}]]$, is the complex vector space of all series in z with coefficients in V. The scalar multiplication and vector addition is inherited from the vector space of sequences.

DEFINITION A.3. The following are important subspaces of $V[[z, z^{-1}]]$.

(i) the space of *polynomials*

$$V[z] = \left\{ \sum_{n\in\mathbb{Z}_{\geq 0}} a_n z^n : a_n \in V \text{ with only finitely many non-zero} \right\}, \tag{A.2}$$

(ii) the space of *formal Laurent polynomials*

$$V[z, z^{-1}] = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n : a_n \in V \text{ with only finitely many non-zero} \right\}, \tag{A.3}$$

(iii) the space of *formal power series*

$$V[[z]] = \left\{ \sum_{n\in\mathbb{Z}_{\geq 0}} a_n z^n : a_n \in V \right\}, \tag{A.4}$$

(iv) the space of *truncated formal Laurent series*

$$V((z)) = \left\{ \sum_{n\in\mathbb{Z}_{\geq k}} a_n z^n : a_n \in V,\ k \in \mathbb{Z} \right\}. \tag{A.5}$$

DEFINITION A.4. Let $a(z) \in V[[z, z^{-1}]]$ and $w \in \mathbb{C}^\times$. We define the *substitutions*

$$a(wz) = \sum_{n\in\mathbb{Z}} a_n z^n|_{z\mapsto wz} = \sum_{n\in\mathbb{Z}} w^n a_n z^n \in V[[z, z^{-1}]], \tag{A.6}$$

$$a(w) = a(z)|_{z=w} = \lim_{z\to w} a(z) = \sum_{n\in\mathbb{Z}} w^n a_n \in V \qquad \text{if } a(z) \in V[z, z^{-1}], \tag{A.7}$$

$$a(0) = a(z)|_{z=0} = \lim_{z\to 0} a(z) = a_0 \in V \qquad \text{if } a(z) \in V[[z]]. \tag{A.8}$$

Note that the sum in (A.7) is actually finite since $a(z)$ is a formal Laurent polynomial.

For multiple formal variables $z_1, \ldots, z_k$, we have similar definitions of formal series with multiple indices:

$$a(z_1, \ldots, z_k) = \sum_{(n_1,\ldots,n_k)\in\mathbb{Z}^k} a_{n_1,\ldots,n_k} z_1^{n_1} \cdots z_k^{n_k} \in V[[z_1^{\pm 1}, \ldots, z_k^{\pm 1}]]. \tag{A.9}$$

The formal variables "commute", for example, $x^m y^n = y^n x^m$.

DEFINITION A.5. The notion of differentiation has a formal analogue. The *formal derivative* of a series $\sum_{n\in\mathbb{Z}} a_n$, with respect to z, is defined as

$$\frac{d}{dz} \sum_{n\in\mathbb{Z}} a_n z^n = \sum_{n\in\mathbb{Z}} n a_n z^{n-1}. \tag{A.10}$$

There are similar notions for *formal partial derivatives*, for example,

$$\frac{\partial}{\partial z_1} \sum_{(n_1,\ldots,n_k)\in\mathbb{Z}^k} a_{n_1,\ldots,n_k} z_1^{n_1} \cdots z_k^{n_k} = \sum_{(n_1,\ldots,n_k)\in\mathbb{Z}^k} n_1 a_{n_1,\ldots,n_k} z_1^{n_1-1} z_2^{n_2} \cdots z_k^{n_k}. \tag{A.11}$$

DEFINITION A.6. The notion of a small contour integral around a point has a formal analogue. The *formal residue* of a series $\sum_{n\in\mathbb{Z}} a_n$, with respect to z, is defined to be

$$\operatorname{Res}_z \sum_{n\in\mathbb{Z}} a_n z^n = a_{-1} \in V. \tag{A.12}$$

That is, the residue of a given series is its coefficient of z^{-1}.

DEFINITION A.7. The *binomial expansion* of two formal variables x and y, for $n \in \mathbb{Z}$, is defined as

$$(x+y)^n = \sum_{k\geq 0} \binom{n}{k} x^{n-k} y^k \in \mathbb{C}[[x, y, x^{-1}, y^{-1}]], \tag{A.13}$$

where we use the standard binomial coefficients, for $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$,

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad \text{when } (n,k) \neq (0,0), \quad \text{and} \quad \binom{0}{0} = 1. \tag{A.14}$$

We can also define binomial expansions, such as

$$\begin{aligned}(x-y)^n &= \sum_{k\geq 0} \binom{n}{k} x^{n-k} y^k |_{y=-y} \\ &= \sum_{k\geq 0} (-1)^k \binom{n}{k} x^{n-k} y^k \in \mathbb{C}[[x, y, x^{-1}, y^{-1}]].\end{aligned} \tag{A.15}$$

REMARK A.8. For $n \geq 0$, we have $(x+y)^n = (y+x)^n$, as expected. However, when $n < 0$, we have $(x+y)^n \neq (y+x)^n$ because the expansion has only negative powers of the first formal variable and non-negative powers of the second formal variable. So, we should think of the notation $(x+y)^n$ as a function of two formal variables that cannot be swapped in general. △

EXAMPLE A.9. We can define the *shifting operator*, of y by $-x$, as

$$\exp\left(-x\frac{\partial}{\partial y}\right) = \sum_{n\geq 0} \frac{(-x)^n}{n!} \left(\frac{\partial}{\partial y}\right)^n. \tag{A.16}$$

Since series in $\mathbb{C}[[x, y, x^{-1}, y^{-1}]]$ are doubly indexed, the shifting operator is a well-defined map from $\mathbb{C}[[y, y^{-1}]]$ to $\mathbb{C}[[x, y, x^{-1}, y^{-1}]]$. We can see that shifting y^m,

$$\exp\left(-x\frac{\partial}{\partial y}\right) y^m = \sum_{n\geq 0} \frac{(-x)^n}{n!} \left(\frac{\partial}{\partial y}\right)^n y^m = \sum_{n\geq 0} \binom{m}{n} (-x)^n y^{m-n} = (y-x)^m,$$

introduces a binomial expansion in y and $-x$. ◊

DEFINITION A.10. There is a formal notion of *multiplication*

$$\cdot : V[z, z^{-1}] \times \mathbb{C}[[z, z^{-1}]] \to V[[z, z^{-1}]] \text{ defined by} \tag{A.17}$$

$$\left(\sum_{m\in\mathbb{Z}} a_m z^m\right) \cdot \left(\sum_{n\in\mathbb{Z}} w_n z^n\right) = \sum_{k\in\mathbb{Z}} \left(\sum_{\substack{m,n\in\mathbb{Z} \\ m+n=k}} w_n a_m\right) z^k.$$

There are other variants, for example, polynomials in V with series in $\mathbb{C}$, truncated Laurent series in V and $\mathbb{C}$, multiplication with series in $\mathbb{C}$ on the left, etc.

REMARK A.11. The internal sum, on the right-hand side of (A.17), is addition in V, and the external sum is a formal sum. Notice that in general, a series in $V[[z, z^{-1}]]$ cannot be multiplied with a series in $\mathbb{C}[[z, z^{-1}]]$; this would result in infinite sums in V. A way around this is to use different formal variables. For example,

$$\left(\sum_{m\in\mathbb{Z}} a_m x^m\right) \cdot \left(\sum_{n\in\mathbb{Z}} w_n y^n\right) = \sum_{m,n\in\mathbb{Z}} w_n a_m x^m y^n \tag{A.18}$$

is a well-defined multiplication from $V[[x, x^{-1}]] \times \mathbb{C}[[y, y^{-1}]]$ to $V[[x, y, x^{-1}, y^{-1}]]$. △

REMARK A.12. The space of formal Laurent polynomials with coefficients in $\mathbb{C}$ is a formal analogue of the space of test functions for the formal Laurent series with coefficients in V. We can see this by defining

(A.19)

$$\langle\cdot,\cdot\rangle : V[[z, z^{-1}]] \times \mathbb{C}[z, z^{-1}] \to V[[z, z^{-1}]] \to V, \quad \langle a(z), \varphi(z)\rangle = \mathrm{Res}_z\left(a(z)\varphi(z)\right).$$

Then, for any $a(z), b(z) \in V[[z, z^{-1}]]$, we have that

$$a(z) = b(z) \quad \text{if and only if} \quad \langle a(z), \varphi(z)\rangle = \langle b(z), \varphi(z)\rangle \text{ for all } \varphi(z) \in \mathbb{C}[z, z^{-1}].$$

This justifies the name "formal distribution". △

DEFINITION A.13. The *formal delta function* in z is

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n \in \mathbb{C}[[z, z^{-1}]]. \tag{A.20}$$

REMARK A.14. There is another common definition of the formal delta function used in [Kac98], [FB04], [Sch08], etc. In this thesis we only use Definition A.13, as used in [FLM88], [LL04], [HLZ14], etc. Despite the notation, (A.20) should be thought of as an analogue of the Dirac delta function at $z = 1$. We will see why in the following proposition. △

PROPOSITION A.15. Let $a(z) \in V[z, z^{-1}]$. Then

$$a(z)\delta(z) = a(1)\delta(z). \tag{A.21}$$

Proof. Given $a(z) \in V[z, z^{-1}]$, we have

$$\begin{aligned} a(z)\delta(z) &= \sum_{m\in\mathbb{Z}} a_m z^m \cdot \sum_{n\in\mathbb{Z}} z^n = \sum_{k\in\mathbb{Z}} \sum_{\substack{m,n\in\mathbb{Z} \\ m+n=k}} a_m z^k \\ &= \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_m z^k = \sum_{m\in\mathbb{Z}} a_m \cdot \sum_{k\in\mathbb{Z}} z^k = a(1)\delta(z). \end{aligned}$$

It also follows that $\mathrm{Res}_z\big(a(z)\delta(z)\big) = a(1)$. □

DEFINITION A.16. The *formal delta function of two variables*, x and y, is

$$\delta\left(\frac{x}{y}\right) = \sum_{n\in\mathbb{Z}} x^n y^{-n} \in \mathbb{C}[[x^{\pm1}, y^{\pm1}]]. \tag{A.22}$$

We will now give an identity for a formal delta function of two variables, which will eventually be used in verifying the axioms for our first example of a vertex algebra.

EXAMPLE A.17. We first show that

$$y^{-1}\delta\left(\frac{x}{y}\right) = (x-y)^{-1} + (y-x)^{-1}. \tag{A.23}$$

Here we use $\binom{-1}{k} = (-1)^k$ to obtain

$$\begin{aligned}
(x-y)^{-1} + (y-x)^{-1} &= \sum_{k\geq 0}\binom{-1}{k}x^{-1-k}(-y)^k + \sum_{k\geq 0}\binom{-1}{k}y^{-1-k}(-x)^k \\
&= \sum_{k\geq 0} x^{-1-k}y^k + \sum_{k\geq 0} y^{-1-k}x^k \\
&= \sum_{k\leq -1} x^k y^{-1-k} + \sum_{k\geq 0} y^{-1-k}x^k \\
&= \sum_{k\in\mathbb{Z}} x^k y^{-1-k} = y^{-1}\delta\left(\frac{x}{y}\right).
\end{aligned}$$

It then follows that

$$\frac{(-1)^n}{n!}\left(\frac{\partial}{\partial x}\right)^n y^{-1}\delta\left(\frac{x}{y}\right) = (x-y)^{-n-1} - (-y+x)^{-n-1}. \tag{A.24}$$

For example, the second term is computed as

$$\begin{aligned}
\frac{(-1)^n}{n!}\left(\frac{\partial}{\partial x}\right)^n \sum_{k\geq 0} y^{-1-k}x^k &= \sum_{k\geq n}(-1)^n\binom{k}{n}y^{-1-k}x^{k-n} = \sum_{k\geq 0}(-1)^n\binom{k+n}{n}y^{-1-k-n}x^k \\
&= \sum_{k\geq 0} -\binom{-n-1}{k}(-y)^{-1-k-n}x^k = -(-y+x)^{-n-1},
\end{aligned}$$

where we have used

$$\begin{aligned}
(-1)^n\binom{k+n}{n} &= (-1)^k\binom{k+n}{n}\binom{-1}{k+n} \\
&= (-1)^k\frac{(k+n)\cdots(k+1)}{n!}\frac{(-1)\cdots(-k-n)}{(k+n)!} \\
&= \frac{(-n-1)\cdots(-k-n)}{k!} = \binom{-n-1}{k}.
\end{aligned}$$

The first term can be computed similarly. $\lozenge$

DEFINITION A.18. The *formal delta function of three variables*, x, y and z, is

$$\delta\left(\frac{x+y}{z}\right) = \sum_{n\in\mathbb{Z}}(x+y)^n z^{-n} = \sum_{n\in\mathbb{Z}}\sum_{k\geq 0}\binom{n}{k}x^{n-k}y^k z^{-n} \in \mathbb{C}[[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]]. \tag{A.25}$$

We will also write three variable formal delta functions of the kind

$$\delta\left(\frac{x-y}{z}\right) = \delta\left(\frac{x+y}{z}\right)\Big|_{y=-y} \tag{A.26}$$

and other similar variations.

REMARK A.19. The three variable formal delta function is needed to state a version of the Jacobi identity, which is an axiom of vertex algebras. Observe that the order of x and y *does* matter, since the formal delta function uses binomial expansions of negative powers. △

EXAMPLE A.20. To demonstrate how formal calculus and the formal delta function work, we will show that

$$x^{-1}\delta\left(\frac{y-z}{x}\right) - x^{-1}\delta\left(\frac{z-y}{-x}\right) = z^{-1}\delta\left(\frac{y-x}{z}\right). \tag{A.27}$$

First consider

$$\begin{aligned}\exp\left(-x\frac{\partial}{\partial y}\right) z^{-1}\delta\left(\frac{y}{z}\right) &= \exp\left(-x\frac{\partial}{\partial y}\right) z^{-1}\sum_{n\in\mathbb{Z}} y^n z^{-n} = z^{-1}\sum_{n\in\mathbb{Z}} \exp\left(-x\frac{\partial}{\partial y}\right) y^n z^{-n} \\ &= z^{-1}\sum_{n\in\mathbb{Z}}(y-x)^n z^{-n} = z^{-1}\delta\left(\frac{y-x}{z}\right).\end{aligned}$$

Then we use Example A.17 to obtain

$$\begin{aligned}\exp\left(-x\frac{\partial}{\partial y}\right) z^{-1}\delta\left(\frac{y}{z}\right) &= \sum_{n\geq 0}\frac{(-x)^n}{n!}\left(\frac{\partial}{\partial z}\right)^n z^{-1}\delta\left(\frac{y}{z}\right) \\ &= \sum_{n\geq 0} x^n\left((y-z)^{-n-1} - (-z+y)^{-n-1}\right) \\ &= \sum_{n\in\mathbb{Z}} x^n\left((y-z)^{-n-1} - (-z+y)^{-n-1}\right) \\ &= \sum_{n\in\mathbb{Z}} x^{-n-1}(y-z)^n - \sum_{n\in\mathbb{Z}} -(-x)^{-n-1}(z-y)^n \\ &= x^{-1}\delta\left(\frac{y-z}{x}\right) - x^{-1}\delta\left(\frac{z-y}{-x}\right).\end{aligned}$$ ◊

REMARK A.21. The previous example is used to verify the Jacobi identity for the simplest family of examples of vertex (operator) algebras, namely, commutative associative unital algebras in Example 2.3. △

A.2 Normal ordering

Let V be an associative algebra over $\mathbb{C}$.

DEFINITION A.22. Let $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in V[[z, z^{-1}]]$ (note the indexing; the coefficient of z^{-n-1} is a_n). Define, respectively, the *singular* and *regular* parts of $a(z)$ as

$$a(z)_- = \sum_{n\geq 0} a_n z^{-n-1} \text{ and } a(z)_+ = \sum_{n<0} a_n z^{-n-1}. \tag{A.28}$$

DEFINITION A.23. Let $a(x) \in V[[x, x^{-1}]]$ and $b(y) \in V[[y, y^{-1}]]$. Define the *normal ordered product* of $a(x)$ and $b(y)$ as the series

$$\overset{\circ}{\circ} a(x)b(y) \overset{\circ}{\circ} = a(x)_+ b(y) + b(y)a(x)_- \in V[[x^{\pm 1}, y^{\pm 1}]]. \tag{A.29}$$

Equivalently,

$$\overset{\circ}{\circ} a(x)b(y) \overset{\circ}{\circ} = \sum_{n\in\mathbb{Z}} \left(\sum_{m<0} a_m b_n x^{-m-1} + \sum_{m\geq 0} b_n a_m x^{-m-1} \right) y^{-n-1}. \tag{A.30}$$

Normal ordered products can be iteratively defined for any finite number of formal series $a^{(1)}(z_1), \dots, a^{(n)}(z_n)$ in distinct formal variables $z_1, \dots, z_n$ with coefficients in V. A single series is defined to have the normal ordered product

$$\overset{\circ}{\circ} a^{(1)}(z_1) \overset{\circ}{\circ} = a^{(1)}(z_1). \tag{A.31}$$

We then inductively define the *(right-nested) normal ordered product* as

(A.32)

$$\overset{\circ}{\circ} a^{(n)}(z_n) \cdots a^{(1)}(z_1) \overset{\circ}{\circ} = \overset{\circ}{\circ} a^{(n)}(z_n) \overset{\circ}{\circ} a^{(n-1)}(z_{n-1}) \overset{\circ}{\circ} \cdots \overset{\circ}{\circ} a^{(2)}(z_2) a^{(1)}(z_1) \overset{\circ}{\circ} \cdots \overset{\circ}{\circ}\overset{\circ}{\circ}\overset{\circ}{\circ}.$$

The normal ordered product of two series can be interpreted as being a product with $\overset{\circ}{\circ} \cdot \overset{\circ}{\circ}$ acting "formally linearly" (i.e. distributing over formal sums). That is,

$$\overset{\circ}{\circ} a(x)b(y) \overset{\circ}{\circ} = \overset{\circ}{\circ} \sum_{m\in\mathbb{Z}} a_m x^{-m-1} \sum_{n\in\mathbb{Z}} b_n y^{-n-1} \overset{\circ}{\circ} =: \sum_{m,n\in\mathbb{Z}} \overset{\circ}{\circ} a_m b_n \overset{\circ}{\circ} x^{-m-1} y^{-n-1}. \tag{A.33}$$

Comparing this with (A.30), we have the following definition.

DEFINITION A.24. Let $a(x)$ and $b(y)$ be series with coefficients in V. For $m, n \in \mathbb{Z}$, the *normal ordered product* of a_m and b_n is defined to be

$$\overset{\circ}{\circ} a_m b_n \overset{\circ}{\circ} = \begin{cases} a_m b_n & m < 0, \\ b_n a_m & m \geq 0. \end{cases} \tag{A.34}$$

Note that this definition depends on the series $a(x)$ and $b(y)$, and their choice of indexing. It is not necessarily a general prescription for multiplication in the algebra V.

REMARK A.25. Note that, in (A.30), the formal variables cannot be equal in general, as this could result in infinite sums in V. In Chapter 2, the definition of a vertex algebra allows for normal ordered products between certain series in the same variable. $\triangle$

A.3 Complex indexing

We will state some definitions and conventions to be used in Chapter 5.

Let V be a vector space over $\mathbb{C}$.

DEFINITION A.26. For a function $a : \mathbb{C} \to V$, $n \mapsto a_n$, we will use the formal series notation

$$a(z) = \sum_{n\in\mathbb{C}} a_n z^n := a. \tag{A.35}$$

Denote by

$$V\{z\} = \left\{ \sum_{n \in \mathbb{C}} a_n z^n : a_n \in V, \text{for } n \in \mathbb{C} \right\} \tag{A.36}$$

the vector space of V-valued functions on $\mathbb{C}$. We will call elements in $V\{z\}$ *formal series indexed by* $\mathbb{C}$ *with coefficients in* V, or again, simply *series*. There are analogous definitions for multiple formal variables

$$\sum_{(n_1,\ldots,n_k) \in \mathbb{C}^k} a_{(n_1,\ldots,n_k)} z_1^{n_1} \cdots z_k^{n_k}. \tag{A.37}$$

We will now set our convention and notation for logarithms. We will typically use w for numbers in the punctured complex plane $\mathbb{C}^\times$, and reserve x, y, z, etc. for formal variables. We will use the branch cut for $\log w$ and $\arg w$ such that the imaginary component of $\log w$ is $\arg w$ with

$$0 \leq \arg w < 2\pi. \tag{A.38}$$

For logarithms not within the principal branch sheet, we will use the notation

$$l_p(w) = \log w + 2\pi i p, \quad \text{for } p \in \mathbb{Z}. \tag{A.39}$$

DEFINITION A.27. Let $a(z) = \sum_{n \in \mathbb{C}} a_n z^n \in V\{z\}$ and let $\zeta \in \mathbb{C}$. If the substitution $a(z)|_{z^n = e^{\zeta n}}$ exists in V, then we write the *substitution* as

$$a(e^\zeta) = a(z)|_{z = e^\zeta} := a(z)|_{z^n = e^{\zeta n}} \in V. \tag{A.40}$$

If $w \in \mathbb{C}^\times$ and $a(e^{\log w})$ exists, then we also simply denote $a(e^{\log w})$ by $a(w)$.

Appendix B

Braided monoidal categories

Braided monoidal categories with non-symmetric braiding were first introduced in [JS86]. Shortly after, in [MS88], these structures where suggested to emerge from conformal field theory. By [KL91], it was understood that braided monoidal structures can arise from quantum groups and affine Lie algebras. In [KL93a]–[KL94b], certain equivalences between these two sources of braided monoidal categories where shown to exist, forming, what we now call today, the Kazhdan-Lusztig correspondence. It was shown in [HLZg] that certain categories of vertex operator algebra modules can also be naturally endowed with the structure of braided monoidal categories. Explicit examples of braided monoidal categories constructed from vertex operator algebras will be computed Chapter 5. Braided monoidal categories also serve as the underlying structure for (pre)-modular categories as defined in Chapter 4.

This appendix summarises the necessary definitions and results used throughout this thesis. Some elementary proofs are included to highlight the key features of (braided) monoidal categories. The proofs that we have selected to show also helped us to form the insight used to write the proofs in Section 4.5 and Chapter 6.

B.1 Monoidal categories

Monoidal categories naturally arise from the tensor product-like structure in categories of representations of "typical" algebraic objects. For example, two representations of a fixed group can be combined via the tensor product to produce a third representation, and tensoring any representation with the one dimensional trivial representation leaves it unchanged, up to isomorphism. This tensor product structure can be thought of as a categorification of a monoid. Recall the following set-theoretic definition, which we include for analogy.

DEFINITION B.1. A *monoid* $(M, \cdot, e)$ consists of the following data:

(i) a set M,

(ii) a binary operation $- \cdot - : M \times M \to M$,

(iii) an element $e \in M$, called the *identity element*,

satisfying the following conditions:

(i) (*associativity*) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, for all $x, y, z \in M$,

(ii) (*identity*) $e \cdot x = x$ and $x \cdot e = x$, for all $x \in M$.

The following definition models the previous definition and categorifies it by promoting: the set to a category, the binary operation to a bifunctor, and the equations to natural isomorphisms. An additional identity will be added to ensure that the isomorphism relating nested products is independent of the choice of order used to associate them. Another identity will be added to ensure that the isomorphism relating nested products with the unit object is independent of the choice to use the unit from the left or the right.

DEFINITION B.2. A *monoidal category* $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ consists of the following data:

(i) a category $\mathscr{C}$,

(ii) a bifunctor $- \otimes - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the *tensor product*,

(iii) an object $\mathbb{1} \in \mathrm{ob}(\mathscr{C})$ called the *unit object*,

(iv) a natural isomorphism $\alpha : - \otimes (- \otimes -) \Rightarrow (- \otimes -) \otimes -$ called the *associator*,

(v) a natural isomorphism $\lambda : \mathbb{1} \otimes - \Rightarrow \mathrm{id}_{\mathscr{C}}$ called the *left unitor*,

(vi) a natural isomorphism $\rho : - \otimes \mathbb{1} \Rightarrow \mathrm{id}_{\mathscr{C}}$ called the *right unitor*,

satisfying the following conditions:

(i) (*pentagon identity*) for all $W, X, Y, Z \in \mathrm{ob}(\mathscr{C})$, the following diagram commutes

$$
\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W \otimes X, Y, Z}"] & \\
W \otimes (X \otimes (Y \otimes Z)) \arrow[ur, "\alpha_{W, X, Y \otimes Z}"] \arrow[d, "\mathrm{id}_W \otimes \alpha_{X,Y,Z}"'] & & ((W \otimes X) \otimes Y) \otimes Z \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\alpha_{W, X \otimes Y, Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[u, "\alpha_{W,X,Y} \otimes \mathrm{id}_Z"']
\end{tikzcd}
\tag{B.1}
$$

(ii) (*triangle identity*) for all $X, Y \in \mathrm{ob}(\mathscr{C})$, the following diagram commutes

$$
\begin{tikzcd}
X \otimes (\mathbb{1} \otimes Y) \arrow[rr, "\alpha_{X, \mathbb{1}, Y}"] \arrow[dr, "\mathrm{id}_X \otimes \lambda_Y"'] & & (X \otimes \mathbb{1}) \otimes Y \arrow[dl, "\rho_X \otimes \mathrm{id}_Y"] \\
 & X \otimes Y &
\end{tikzcd}
\tag{B.2}
$$
.

REMARK B.3. It is common to also call our definition of a monoidal category a *tensor category*. We instead reserve this name for the notion of the tensor category defined in Chapter 4.

Another (possibly more) common version of this definition has the associator natural isomorphism $\alpha : (-\otimes-)\otimes- \Rightarrow -\otimes(-\otimes-)$ being the inverse of what we have in Definition B.2. This reverses the direction of the associator arrows in the previous definition.

There is another definition with slightly different data, replacing the left and right unitors for a single isomorphism $\iota : \mathbb{1}\otimes\mathbb{1} \to \mathbb{1}$ in $\mathscr{C}$. The triangle axiom is replaced with another axiom involving the unit object. These definitions are equivalent as seen in Section 2.1 of [EGNO16]. We have chosen our definition to match the definition and associator direction convention used in [HLZg]. $\triangle$

REMARK B.4. Similarly to the pentagon and triangle identities, there could be other nested combinations of tensor products and unit objects. We would like all possible ways of relating these products with compositions of associators and unitors to be *coherent*, that is, equal. Mac Lane's coherence theorem shows that the pentagon and triangle identities are sufficient to enforce coherence. See Theorem B.19 below for the exact statement. $\triangle$

EXAMPLE B.5. The motivating example for a monoidal category consists of:

(i) the category $\mathbb{k}-\mathsf{Vect}$ of $\mathbb{k}$-vector spaces for a fixed field $\mathbb{k}$,

(ii) the tensor product given by $\otimes_{\mathbb{k}}$, the standard tensor product over $\mathbb{k}$,

(iii) the unit object is $\mathbb{k}$ as a $\mathbb{k}$-vector space,

(iv) the associator, on $\mathbb{k}$-vector spaces X, Y and Z, given by

$$\alpha_{X,Y,Z} : X \otimes_{\mathbb{k}} (Y \otimes_{\mathbb{k}} Z) \to (X \otimes_{\mathbb{k}} Y) \otimes_{\mathbb{k}} Z, \quad x \otimes_{\mathbb{k}} (y \otimes_{\mathbb{k}} z) \mapsto (x \otimes_{\mathbb{k}} y) \otimes_{\mathbb{k}} z,$$

(v) the left and right unitors, on a $\mathbb{k}$-vector space X, given by

$$\lambda_X : \mathbb{1} \otimes_{\mathbb{k}} X \to X, \quad 1 \otimes_{\mathbb{k}} x \mapsto x \qquad \text{and} \qquad \rho_X : X \otimes_{\mathbb{k}} \mathbb{1} \to X, \quad x \otimes_{\mathbb{k}} 1 \mapsto x.$$

The data $(\mathbb{k}-\mathsf{Vect}, \otimes_{\mathbb{k}}, \mathbb{k}, \alpha, \lambda, \rho)$ satisfies the triangle and pentagon identities. This example explains the origin of the name and notation of the tensor product in a monoidal category. A similar family of examples can be made by using a commutative ring R and the category of R-modules. $\lozenge$

EXAMPLE B.6. Let G be a group and let $\mathbb{k}[G]$ be the group algebra over a field $\mathbb{k}$. Given two $\mathbb{k}[G]$-modules X and Y, the tensor product $X \otimes_{\mathbb{k}} Y$ can be given a $\mathbb{k}[G]$-module structure by defining

$$g \cdot (x \otimes_{\mathbb{k}} y) = (g \cdot x) \otimes_{\mathbb{k}} (g \cdot y) \quad \text{for all } g \in G,\ x \in X,\ y \in Y.$$

This G-module tensor product construction together with the trivial representation on $\mathbb{k}$ as the unit object, and associator and unitors adopted from $\mathbb{k}-\mathsf{Vect}$, give a monoidal category structure to $\mathbb{k}[G]-\mathsf{Mod}$. $\lozenge$

EXAMPLE B.7. Let $\mathfrak{g}$ be a Lie algebra over a field $\Bbbk$. Given two $\mathfrak{g}$-modules X and Y, the tensor product $X \otimes_{\Bbbk} Y$ can be given a $\mathfrak{g}$-module structure by defining

$$g \cdot (x \otimes_{\Bbbk} y) = (g \cdot x) \otimes_{\Bbbk} y + x \otimes_{\Bbbk} (g \cdot y) \quad \text{for all } g \in \mathfrak{g},\ x \in X,\ y \in Y.$$

This $\mathfrak{g}$-module tensor product construction together with the trivial representation on $\Bbbk$ as the unit object, and associator, left and right unitors adopted from $\Bbbk-\mathsf{Vect}$, give a monoidal category structure to $\mathfrak{g}-\mathsf{Mod}$, the category of $\mathfrak{g}$-modules. $\Diamond$

REMARK B.8. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Recall that there is a canonical isomorphism of categories between $\mathfrak{g}-\mathsf{Mod}$ and the category $\mathcal{U}(\mathfrak{g})-\mathsf{Mod}$ of $\mathcal{U}(\mathfrak{g})$-modules. Then, $\mathcal{U}(\mathfrak{g})-\mathsf{Mod}$ can be given "the same" monoidal structure as $\mathfrak{g}-\mathsf{Mod}$; as seen in Example C.8. (To make this more precise we need the notion of a monoidal structure preserving functor from Definition B.10 below.) Examples B.6 and B.7 (or more precisely, $\mathcal{U}(\mathfrak{g})-\mathsf{Mod}$) are special cases of the monoidal structure that arises from bialgebras. See Appendix C for the definition. $\triangle$

EXAMPLE B.9. There are monoidal categories that are not defined as concrete categories of modules or vector spaces. For example, let G be a group and let A be an abelian group. Let ω be a 3-cocycle of G with values in A for the trivial group action. That is, a function $\omega : G \times G \times G \to A$ satisfying, for all $w, x, y, z \in G$,

$$\omega(wx, y, z)\omega(w, x, yz) = \omega(w, x, y)\omega(w, xy, z)\omega(x, y, z). \tag{B.3}$$

Define the following:

(i) the category $\mathscr{C}$ with objects and morphisms

$$\mathrm{ob}(\mathscr{C}) = G \qquad \text{and} \qquad \hom(x, y) = \begin{cases} A & x = y, \\ \emptyset & x \neq y, \end{cases} \quad \text{for all } x, y \in G.$$

The identity morphisms are the identity element in A, and the composition of morphisms is given by the group product in A,

(ii) the tensor product $- \otimes - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ defined by

$$x \otimes y = xy \quad \text{for all } x, y \in G \qquad \text{and} \qquad a \otimes b = ab \quad \text{for all } a, b \in A,$$

(iii) the unit object is the identity $\mathbb{1}_G \in G$,

(iv) the associator on $x, y, z \in G$ is $\alpha_{x,y,z} = \omega(x, y, z)$,

(v) the left and right unitors on $x \in G$ are $\lambda_x = \mathbb{1}_A = \rho_x$ the identity in A.

The triangle identity is satisfied since $\omega(x, \mathbb{1}_G, y) = \mathbb{1}_A$ for all $x, y \in G$. The pentagon identity is satisfied by the 3-cocycle condition (B.3). $\Diamond$

In Chapter 5 and Chapter 6 there will be monoidal categories with monoidal subcategories equivalent to those in Example B.9. We will now introduce a notion for comparing the structure of monoidal categories.

DEFINITION B.10. Let $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and $(\mathscr{D}, \boxtimes, \mathbf{1}, a, l, r)$ be monoidal categories. A *monoidal functor* (F, J, φ) from $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ to $(\mathscr{D}, \boxtimes, \mathbf{1}, a, l, r)$ consists of the following data:

(i) a functor $F : \mathscr{C} \to \mathscr{D}$,

(ii) a natural isomorphism $J : F(-) \boxtimes F(-) \Rightarrow F(- \otimes -)$,

(iii) an isomorphism $\varphi : \mathbf{1} \to F(\mathbb{1})$ in $\mathscr{D}$,

satisfying the following conditions:

(i) (*compatibility of associators*) for all objects $X, Y, Z \in \mathrm{ob}(\mathscr{C})$, the following diagram commutes

$$\begin{tikzcd}
F(X) \boxtimes (F(Y) \boxtimes F(Z)) \arrow[rr, "a_{F(X),F(Y),F(Z)}"] \arrow[d, "\mathrm{id}_{F(X)} \boxtimes J_{Y,Z}"'] & & (F(X) \boxtimes F(Y)) \boxtimes F(Z) \arrow[d, "J_{X,Y} \boxtimes \mathrm{id}_{F(Z)}"] \\
F(X) \boxtimes F(Y \otimes Z) \arrow[d, "J_{X,Y \otimes Z}"'] & & F(X \otimes Y) \boxtimes F(Z) \arrow[d, "J_{X \otimes Y, Z}"] \\
F(X \otimes (Y \otimes Z)) \arrow[rr, "F(\alpha_{X,Y,Z})"] & & F((X \otimes Y) \otimes Z)
\end{tikzcd} \tag{B.4}$$

(ii) (*compatibility of unitors*) for each object $X \in \mathrm{ob}(\mathscr{C})$, the following diagrams commute

$$\begin{tikzcd}
\mathbf{1} \boxtimes F(X) \arrow[r, "l_{F(X)}"] \arrow[d, "\varphi \boxtimes \mathrm{id}_{F(X)}"'] & F(X) \\
F(\mathbb{1}) \boxtimes F(X) \arrow[r, "J_{\mathbb{1},X}"] & F(\mathbb{1} \otimes X) \arrow[u, "F(\lambda_X)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \boxtimes \mathbf{1} \arrow[r, "r_{F(X)}"] \arrow[d, "\mathrm{id}_{F(X)} \boxtimes \varphi"'] & F(X) \\
F(X) \boxtimes F(\mathbb{1}) \arrow[r, "J_{X,\mathbb{1}}"] & F(X \otimes \mathbb{1}) \arrow[u, "F(\rho_X)"']
\end{tikzcd} . \tag{B.5}$$

REMARK B.11. The functor F imposes compatibility of the underlying categories, while the natural isomorphism J imposes compatibility of the tensor products, and the isomorphism φ imposes compatibility of the unit objects. If we replace the arrows on the right-hand side of the compatibility of associator condition with their inverses, the condition reads that "the associator of the image is the image of the associator". And similarly the compatibility of the unitors condition reads that "the unitor of the image is the image of the unitor". Hence, a monoidal functor imposes compatibility of all the structure in monoidal categories. △

EXAMPLE B.12. Consider the monoidal category structure on $\mathbb{k}[G]-\mathsf{Mod}$ from Example B.6. Consider the functor $F : \mathbb{k}[G]-\mathsf{Mod} \to \mathbb{k}-\mathsf{Vect}$ that forgets the $\mathbb{k}[G]$-module structure leaving a $\mathbb{k}$-vector space. Let $J_{X,Y} = \mathrm{id}_{X \otimes_{\mathbb{k}} Y}$ for all $\mathbb{k}[G]$-modules X and Y. Let $\varphi = \mathrm{id}_{\mathbb{k}}$. Then, (F, J, φ) is a monoidal functor. This can be interpreted as $\mathbb{k}[G]-\mathsf{Mod}$ having a compatible monoidal structure with $\mathbb{k}-\mathsf{Vect}$. In fact, all bialgebras will have a category of modules with a similar monoidal forgetful functor.

For comparison, consider the category of vertex operator algebra modules endowed with the fusion product as in Chapter 3. There is no such monoidal forgetful functor because the fusion product does not have the same underlying vector space as the vector space tensor product. ◊

DEFINITION B.13. Let $(\mathscr{C}^i, \otimes^i, \mathbb{1}^i, \alpha^i, \lambda^i, \rho^i)$, for $i = 1, 2, 3$, be monoidal categories. Let (F, J, φ) and (G, K, ψ) be monoidal functors from monoidal categories with $i = 1$ to $i = 2$ and $i = 2$ to $i = 3$, respectively. We define the *composition* of (G, K, ψ) after (F, J, φ) to be the triple consisting of:

(i) the functor composition $G \circ F = GF : \mathscr{C}^1 \to \mathscr{C}^3$,

(ii) the natural isomorphism

$$\begin{aligned} K \bullet J : GF(-) \otimes^3 GF(-) \Rightarrow GF(- \otimes^1 -) \quad &\text{defined by} \\ (K \bullet J)_{(X,Y)} = G(J_{X,Y}) \circ K_{F(X),F(Y)} \end{aligned} \tag{B.6}$$

(we will show naturality in Proposition B.14 below),

(iii) the isomorphism in $\mathscr{C}^3$, $\psi \bullet \varphi = G(\varphi) \circ \psi : \mathbb{1}^3 \to GF(\mathbb{1}^1)$.

We denote this composition by $(G \circ F, K \bullet J, \psi \bullet \varphi) = (G, K, \psi) \circ (F, J, \varphi)$.

PROPOSITION B.14. The composition of two monoidal functors is a monoidal functor.

Proof. The assignment $K \bullet J$ is natural since the following diagram commutes for all morphisms $f : X \to X'$ and $g : Y \to Y'$ in $\mathscr{C}^1$.

$$\begin{tikzcd} GF(X) \otimes^3 GF(Y) \arrow[r, "GF(f)\otimes^3 GF(g)"] \arrow[d, "K_{F(X),F(Y)}"'] & GF(X') \otimes^3 GF(Y') \arrow[d, "K_{F(X'),F(Y')}"] \\ G(F(X) \otimes^2 F(Y)) \arrow[r, "G(F(f)\otimes^2 F(g))"] \arrow[d, "G(J_{X,Y})"'] & G(F(X') \otimes^2 F(Y')) \arrow[d, "G(J_{X',Y'})"] \\ GF(X \otimes^1 Y) \arrow[r, "GF(f\otimes^1 g)"] & GF(X' \otimes^1 Y') \end{tikzcd} \tag{B.7}$$

The commutativity of the top square is given by the naturality of K using $F(f)$ and $F(g)$. Commutativity of the bottom square is given by the naturality of J using f and g to which the functor G is applied. The morphism $J_{X,Y}$ is an isomorphism so its image $G(J_{X,Y})$ is an isomorphism; and $K_{F(X),F(Y)}$ is an isomorphism so the composition $(K \bullet J)_{X,Y}$ is an isomorphism.

The compatibility of associators is satisfied for all objects X, Y and Z in $\mathscr{C}^1$ by the commutative diagram below.
(B.8)

$$\begin{tikzcd}
 & GFX(GFYGFZ) \arrow[r, "\alpha_{GFX,GFY,GFZ}"] \arrow[d, "\mathrm{id}\otimes K_{FY,FZ}"'] & (GFXGFY)GFZ \arrow[d, "K_{FX,FY}\otimes\mathrm{id}"] & \\
 & GFXG(FYFZ) \arrow[dl, bend right, "\mathrm{id}\otimes GJ_{Y,Z}"'] \arrow[d, "K_{FX,FYFZ}"'] & G(FXFY)GFZ \arrow[dr, bend left, "GJ_{X,Y}\otimes\mathrm{id}"] \arrow[d, "K_{FXFY,FZ}"] & \\
GFXGF(YZ) \arrow[dr, bend right, "K_{FX,F(YZ)}"'] & G(FX(FYFZ)) \arrow[r, "G\alpha_{FX,FY,FZ}"] \arrow[d, "G(\mathrm{id}\otimes J_{Y,Z})"'] & G((FXFY)FZ) \arrow[d, "G(J_{X,Y}\otimes\mathrm{id})"] & GF(XY)GFZ \arrow[dl, bend left, "K_{F(XY),FZ}"] \\
 & G(FXF(YZ)) \arrow[d, "GJ_{X,YZ}"'] & G(F(XY)FZ) \arrow[d, "GJ_{XY,Z}"] & \\
 & GF(X(YZ)) \arrow[r, "GF\alpha_{X,Y,Z}"] & GF((XY)Z) &
\end{tikzcd}$$

Note that all brackets, tensor products and superscripts that can be understood from context have been omitted. The top and bottom hexagons commute due to compatibility of associators. The quadrilaterals on the sides commute by the naturality of K. The outer morphisms compose to give the compatibility of associators α^1 and α^3. The composition of the upper outer morphisms use the fact that the tensor product is a bifunctor.

The compatibility of left unitors is satisfied for all objects X in $\mathscr{C}^1$ by the commutative diagram below.

$$\begin{tikzcd}
\mathbb{1}^3 \otimes^3 GF(X) \arrow[rr, "\lambda^3_{GF(X)}"] \arrow[d, "\psi\otimes^3\mathrm{id}_{GF(X)}"'] & & GF(X) \\
G(\mathbb{1}^2) \otimes^3 GF(X) \arrow[r, "K_{\mathbb{1}^2,F(X)}"] \arrow[d, "G(\varphi)\otimes^3\mathrm{id}_{GF(X)}"'] & G(\mathbb{1}^2 \otimes^2 F(X)) \arrow[ur, "G(\lambda^2_{F(X)})"] & \\
GF(\mathbb{1}^1) \otimes^3 GF(X) \arrow[r, "K_{F(\mathbb{1}^1),F(X)}"'] & G(F(\mathbb{1}^1) \otimes^2 F(X)) \arrow[r, "G(J_{\mathbb{1}^1,X})"'] \arrow[u, "G(\varphi\otimes^2\mathrm{id}_{F(X)})^{-1}"'] & GF(\mathbb{1}^1 \otimes^1 X) \arrow[uu, "GF(\lambda^1_X)"']
\end{tikzcd} \tag{B.9}$$

The top quadrilateral commutes by the compatibility of the left unitors λ^3 and λ^2. The right quadrilateral commutes by the compatibility of the left unitors λ^2 and λ^1 to which the functor G is applied. The bottom square commutes by naturality of K with morphisms φ and $\mathrm{id}_{F(X)}$. We have used the fact that φ has the inverse $G(\varphi^{-1} \otimes^2 \mathrm{id}_{F(X)})$ to reverse the middle vertical arrow. The right unitors are compatible in a similar way. $\square$

This proof demonstrates common techniques used in proofs throughout this thesis, namely, the application of naturality and functoriality to produce commutative diagrams.

DEFINITION/PROPOSITION B.15. The collection of monoidal categories, the collection of monoidal functors and monoidal composition form a category MonCat.

Proof. The monoidal functor $(\mathrm{id}_{\mathscr{C}}, J_{X,Y} = \mathrm{id}_{X \otimes Y}, \varphi = \mathrm{id}_{\mathbb{1}})$ gives the identity morphism for each monoidal category. Let (F^i, J^i, φ^i), for $i = 1, 2, 3$, be monoidal functors. Then, their monoidal composition is associative. This can be seen from the associativity of functors and the following equations:

$$\begin{aligned}
(J^3 \bullet (J^2 \bullet J^1))_{X,Y} &= F^3(F_2(J^1_{X,Y}) \circ J^2_{F^1(X),F^1(Y)}) \circ J^3_{(F^2 \circ F^1)(X),(F^2 \circ F^1)(Y)} \\
&= (F^3 \circ F^2)(J^1_{X,Y}) \circ (F^3)J^2_{F^1(X),F^2(Y)}) \circ J^3_{(F^2 \circ F^1)(X),(F^2 \circ F^1)(Y)}) \\
&= ((J^3 \bullet J^2) \bullet J^1)_{X,Y}, \\
\varphi^3 \bullet (\varphi^2 \bullet \varphi^1) &= F^3(F^2(\varphi^1) \circ \varphi^2) \circ \varphi^3 \\
&= (F^3 \circ F^2)(\varphi^1) \circ (F^3(\varphi^2) \circ \varphi^3) = (\varphi^3 \bullet \varphi^2) \bullet \varphi^1. \qquad \square
\end{aligned}$$

In Chapter 4, a property called rigidity will be shown to be an invariant under *monoidal equivalence* in MonCat. But first, we need a notion of equivalent monoidal categories.

PROPOSITION B.16. Let $(F, J, \varphi) : (\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho) \to (\mathscr{D}, \boxtimes, \mathbf{1}, a, l, r)$ be a monoidal functor. Assume that F is an equivalence of categories. Let

$$F : \mathscr{C} \to \mathscr{D}, \quad G : \mathscr{D} \to \mathscr{C}, \quad \varepsilon : FG \to \mathrm{id}_{\mathscr{D}}, \quad \eta : \mathrm{id}_{\mathscr{C}} \to GF$$

be an adjoint equivalence. Then, G has a canonical structure of a monoidal functor.

Proof. Define the natural isomorphism $K : G(-) \otimes G(-) \Rightarrow G(- \boxtimes -)$ with components

$$\begin{aligned}
&K_{X,Y} = G(\varepsilon_X \boxtimes \varepsilon_Y) \circ G(J^{-1}_{X,Y}) \circ \eta_{GX \otimes GY} : \\
&GX \otimes GY \to GF(GX \otimes GY) \to G(FGX \boxtimes FGY) \to G(X \boxtimes Y).
\end{aligned} \tag{B.10}$$

Define the isomorphism

$$\psi = G(\varphi^{-1}) \circ \eta_{\mathbb{1}} : \mathbb{1} \to GF\mathbb{1} \to G\mathbf{1}. \tag{B.11}$$

Then, (G, K, ψ) is a monoidal functor. The compatibility of associators can be found from the following commutative diagrams.

$$\begin{tikzcd}
GXG(YZ) \arrow[d, "\eta"'] & GX(GYGZ) \arrow[l, "\mathrm{id} \otimes K"'] \arrow[r, "\alpha"] \arrow[d, "\eta"] & (GXGY)GZ \arrow[d, "\eta"] \\
GF(GXG(YZ)) \arrow[d, "GJ^{-1}"'] & GF(GX(GYGZ)) \arrow[l, "GF(\mathrm{id} \otimes K)"'] \arrow[r, "GF\alpha"] \arrow[d, "GJ^{-1}"] & GF((GXGY)GZ) \arrow[d, "GJ^{-1}"] \\
G(FGXFG(YZ)) \arrow[ddr, "G(\varepsilon \otimes \varepsilon)"'] & G(FGXF(GYGZ)) \arrow[l, "G(\mathrm{id} \otimes FK)"'] \arrow[d, "G(\mathrm{id} \otimes J^{-1})"] & G(F(GXGY)FGZ) \arrow[d, "G(J^{-1} \otimes \mathrm{id})"] \\
 & G(FGX(FGYFGZ)) \arrow[r, "Ga"] \arrow[d, "G(\varepsilon \otimes (\varepsilon \otimes \varepsilon))"] & G((FGXFGY)FGZ) \arrow[d, "G((\varepsilon \otimes \varepsilon) \otimes \varepsilon)"] \\
 & G(X(YZ)) \arrow[r, "Ga"] & G((XY)Z)
\end{tikzcd} \tag{B.12}$$

(B.13)

$$\begin{tikzcd}
G(FGXFGF(GYGZ)) \arrow[d, "G(\mathrm{id}\otimes FGJ^{-1})"'] & G(FGXF(GYGZ)) \arrow[l, "G(\mathrm{id}\otimes F\eta)"'] \arrow[d, "G(\mathrm{id}\otimes J^{-1})"] \\
G(FGXFG(FGYFGZ)) \arrow[r, shift right, "G(\mathrm{id}\otimes\varepsilon)"'] \arrow[d, "G(\mathrm{id}\otimes FG(\varepsilon\otimes\varepsilon))"'] & G(FGX(FGYFGZ)) \arrow[l, shift right, "G(\mathrm{id}\otimes F\eta)"'] \arrow[d, "G(\varepsilon\otimes(\varepsilon\otimes\varepsilon))"] \\
G(FGXFG(YZ)) \arrow[r, "G(\varepsilon\otimes\varepsilon)"'] & G(X(YZ))]
\end{tikzcd}$$

(Tensor products and subscripts can be found from context.) The idea is to build outwards from the middle hexagon of (B.12) making use of the compatibility of associators in F. All squares commute by naturality. The bottom-left quadrilateral in (B.12) is given by (B.13). The middle of (B.13) makes use of the counit-unit adjunction and its zig-zag equations. The right side of diagram (B.12) can be completed by mirroring the left side. Finally, the compatibility of associators is then given by composing the outside morphisms.

The compatibility of the left unitors is give by the commutative diagram:

(B.14)

$$\begin{tikzcd}
G(\mathbf{1}X) \arrow[r, "G\lambda"] & GX & \\
G(\mathbf{1}FGX) \arrow[u, "G(\mathrm{id}\,\varepsilon)"] \arrow[r, "G\lambda"] \arrow[d, shift left, "G(\varphi\,\mathrm{id})"] & GFGX \arrow[u, "G\varepsilon"'] & GX \arrow[l, "\eta"'] \arrow[ul, "\mathrm{id}"'] \\
G(F\mathbb{1}FGX) \arrow[u, shift left, "G(\varphi^{-1}\,\mathrm{id})"] \arrow[r, shift left, "GJ"] \arrow[d, shift left, "G(F\eta\,\mathrm{id})"] & GF(\mathbb{1}GX) \arrow[l, shift left, "GJ^{-1}"] \arrow[u, "GFl"'] \arrow[d, "GF(\eta\,\mathrm{id})"] & \mathbb{1}GX \arrow[l, "\eta"'] \arrow[u, "l"'] \arrow[d, "\eta\,\mathrm{id}"] \\
G(FGF\mathbb{1}FGX) \arrow[u, shift left, "G(\varepsilon\,\mathrm{id})"] \arrow[d, "G(FG\varphi^{-1}\,\mathrm{id})"] & GF(GF\mathbb{1}GX) \arrow[l, "GJ^{-1}"] \arrow[d, "GF(G\varphi^{-1}\,\mathrm{id})"] & GF\mathbb{1}GX \arrow[l, "\eta"'] \arrow[d, "G\varphi^{-1}\,\mathrm{id}"] \\
G(FG\mathbf{1}FGX) \arrow[uuu, bend left=60, "G(\varepsilon\,\mathrm{id})"] & GF(G\mathbf{1}GX) \arrow[l, "GJ^{-1}"'] & G\mathbf{1}GX \arrow[l, "\eta"']
\end{tikzcd} .$$

This diagram is built outwards from the compatibility of left unitors of F. The remaining squares commute by naturality and the triangle is a zig-zag equation. □

The previous proofs highlight the importance that monoidal categories and their morphisms are built from category-theoretic notions such as functors and natural isomorphisms. This category-theoretic perspective provides the intuition used in Chapter 6.

DEFINITION B.17. An *equivalence of monoidal categories* or a *monoidal equivalence* is a monoidal functor (F, J, φ) such that F is an equivalence of categories.

We will now state three important theorems.

THEOREM B.18. *Mac Lane's Strictness Theorem.* Any monoidal category is monoidally equivalent to a strict monoidal category, that is, one with associators and unitors as identities.

THEOREM B.19. *Mac Lane's Coherence Theorem.* Let $X_1, \ldots, X_n$ be objects in a monoidal category $\mathscr{C}$. Let Y and Z be tensor products of $X_1, \ldots, X_n$, retaining this order of objects, with any arbitrary insertions of the unit object and parentheses in any order. Let $f, g : Y \to Z$ be morphisms in $\mathscr{C}$ made from the composition of associators, unitors and identities. Then, $f = g$.

THEOREM B.20. Any monoidal category is monoidally equivalent to a skeletal monoidal category, that is, one with exactly one object in each isomorphism class.

Proofs can be found in Sections 2.8 and 2.9 of [EGNO16]. Mac Lane's coherence theorem follows from Mac Lane's strictness theorem.

REMARK B.21. Working with a strict category comes at the cost of a larger (by inclusion) object class and a loss of concreteness (see how the object class is constructed in Section 6.6 of [Kas95]). In general, a monoidally equivalent skeletal category cannot be made to be strict; this fact is important for Chapter 6.

Vertex operator algebras often produce monoidal categories with non-trivial associators. In Chapter 5, we will prefer skeletal categories over strict ones, and these categories will still be concrete. This preference comes at the cost of non-trivial associators, but this makes the explicit descriptions interesting from a morphism perspective. △

B.2 Braided monoidal categories

A monoid $(M, \cdot, e)$ is *commutative* if it satisfies

(B.15) $$x \cdot y = y \cdot x \qquad \text{for all } x, y \in M.$$

To categorify this condition, we promote the equality to a natural isomorphism.

DEFINITION B.22. A *braided monoidal category* $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, c)$ consists of the following data:

(i) a monoidal category $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$,

(ii) a natural isomorphism $c : \cdot \otimes - \Rightarrow - \otimes \cdot$, called the *braiding*, with the components

(B.16) $$c_{X,Y} : X \otimes Y \to Y \otimes X \qquad \text{for } X, Y \in \mathrm{ob}(\mathscr{C}),$$

for which the following diagrams commute, for all $X, Y, Z \in \mathrm{ob}(\mathscr{C})$:

(i) (*hexagon identity 1*)

$$
\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] \arrow[dl, "\alpha_{X,Y,Z}"'] & (Y \otimes Z) \otimes X & \\
(X \otimes Y) \otimes Z \arrow[dr, "c_{X,Y}\otimes \mathrm{id}_Z"'] & & & Y \otimes (Z \otimes X) \arrow[ul, "\alpha_{Y,Z,X}"'] \\
& (Y \otimes X) \otimes Z \arrow[r, "\alpha^{-1}_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes c_{X,Y}"'] &
\end{tikzcd} \tag{B.17}
$$

(ii) (*hexagon identity 2*)

$$
\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[r, "c_{X\otimes Y,Z}"] \arrow[dl, "\alpha^{-1}_{X,Y,Z}"'] & Z \otimes (X \otimes Y) & \\
X \otimes (Y \otimes Z) \arrow[dr, "\mathrm{id}_X \otimes c_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \arrow[ul, "\alpha^{-1}_{Z,X,Y}"'] \\
& X \otimes (Z \otimes Y) \arrow[r, "\alpha_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z}\otimes \mathrm{id}_Y"'] &
\end{tikzcd} . \tag{B.18}
$$

The hexagon identities enforce that braiding with a tensor product is the same as braiding individually.

EXAMPLE B.23. We can endow the monoidal category of $\mathbb{k}$-vector spaces, in Example B.5, with a braiding

$$c_{X,Y} : X \otimes_{\mathbb{k}} Y \to Y \otimes_{\mathbb{k}} X, \quad x \otimes_{\mathbb{k}} y = y \otimes_{\mathbb{k}} x. \tag{B.19}$$

This braiding is also inherited by $\mathbb{k}[G]-$Mod and $\mathfrak{g}-$Mod from Examples B.6 and B.7, respectively. In these cases, we have the property that $c_{X,Y} = c^{-1}_{Y,X}$, for all $X, Y \in \mathbb{k}-$Vect. Note that the definition of braiding does not require this condition. $\Diamond$

DEFINITION B.24. A braided monoidal category $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, c)$ is called *symmetric* if it satisfies the following condition

$$c_{X,Y} = c^{-1}_{Y,X} \qquad \text{for all } X, Y \in \mathrm{ob}(\mathscr{C}). \tag{B.20}$$

REMARK B.25. In some literature, braided monoidal categories are called “quasitensor categories” and symmetric braided monoidal categories are called “tensor categories”. We will not be using this terminology here. $\triangle$

In Chapter 5, we study examples of non-symmetric braided monoidal categories.

We require a notion of morphisms for braided monoidal categories. The structure we must preserve consists of a monoidal category and the braiding. That is, we need a monoidal functor and a condition that "the braiding of the image is the image of the braiding". We do not need to add any additional data to the monoidal functor. So, we have the following definition.

DEFINITION B.26. Let $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, c)$ and $(\mathscr{D}, \boxtimes, \mathbf{1}, a, l, r, d)$ be braided monoidal categories. A braided monoidal functor from $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, c)$ to $(\mathscr{D}, \boxtimes, \mathbf{1}, a, l, r, d)$ is a monoidal functor (F, J, φ) from $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ to $(\mathscr{D}, \boxtimes, \mathbf{1}, a, l, r)$ satisfying the following condition:

(i) (*compatibility of braiding*) for all objects X, Y in $\mathscr{C}$, the following diagram commutes

$$\begin{tikzcd} F(X) \boxtimes F(Y) \arrow[r, "d_{F(X),F(Y)}"] \arrow[d, "J_{X,Y}"'] & F(Y) \boxtimes F(X) \arrow[d, "J_{Y,X}"] \\ F(X \otimes Y) \arrow[r, "F(c_{X,Y})"'] & F(Y \otimes X) \end{tikzcd} \tag{B.21}$$

EXAMPLE B.27. The identity monoidal functor is braided and the monoidal composition of two braided monoidal functors is braided. The proof for the latter claim is summarised in the following commutative diagram.

$$\begin{tikzcd} GFX \otimes^3 GFY \arrow[r, "c^3_{GFX,GFY}"] \arrow[d, "K_{FX,FY}"'] & GFY \otimes^3 GFX \arrow[d, "K_{FY,FX}"] \\ G(FX \otimes^2 FY) \arrow[r, "Gc^2_{FX,FY}"] \arrow[d, "GJ_{X,Y}"'] & G(FY \otimes^2 FX) \arrow[d, "GJ_{Y,X}"] \\ GF(X \otimes^1 Y) \arrow[r, "GFc^1_{X,Y}"] & GF(Y \otimes^1 X) \end{tikzcd} \tag{B.22}$$

$\Diamond$

From Proposition B.15 and Example B.27 we have a category of braided monoidal categories.

DEFINITION B.28. The collection of braided monoidal categories with the collection of braided monoidal functors and monoidal composition forms a category $\mathsf{BraidCat}$.

PROPOSITION B.29. Let $(F, J, \varphi) : (\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, c) \to (\mathscr{D}, \boxtimes, \mathbf{1}, a, l, r, d)$ be a braided monoidal functor with F an equivalence of categories. Let (G, K, ψ) be the canonical quasi-inverse monoidal functor from the proof of Proposition B.16. Then, (G, K, ψ) is braided.

Proof. We use the same notation as in Proposition B.16. For all objects X, Y in $\mathscr{D}$, we have the following commutative diagram.

$$\begin{tikzcd}
GX \otimes GY \arrow[r, "c_{GX,GY}"] \arrow[d, "\eta_{GX\otimes GY}"'] & GY \otimes GX \arrow[d, "\eta_{GY\otimes GX}"] \\
GF(GX \otimes GY) \arrow[r, "GFc_{GX,GY}"] \arrow[d, "GJ^{-1}_{X,Y}"'] & GF(GY \otimes GX) \arrow[d, "GJ^{-1}_{Y,X}"] \\
G(FGX \boxtimes FGY) \arrow[r, "Gd_{FGX,FGY}"] \arrow[d, "G(\varepsilon_X \boxtimes \varepsilon_Y)"'] & G(FGY \boxtimes FGX) \arrow[d, "G(\varepsilon_Y \boxtimes \varepsilon_X)"] \\
G(X \boxtimes Y) \arrow[r, "Gd_{X,Y}"] & G(Y \boxtimes X)
\end{tikzcd} \tag{B.23}$$

The commutativity of the squares, from top to bottom, respectively use the naturality of η, the fact that F is braided and naturality of the braiding d. $\square$

DEFINITION B.30. An *equivalence of braided monoidal categories* or a *braided monoidal equivalence* is a braided monoidal functor (F, J, φ) such that F is an equivalence of categories.

The final chapter of this thesis aims to explicitly compute a braided monoidal equivalences in a Kazhdan-Lusztig correspondence. In Chapter 4, we explore additional structures on categories with underlying monoidal category structures, namely, modular tensor categories. The themes of equivalence used in this appendix will be replicated in Chapter 4 to show that there is a notion of (pre)-modular equivalence built from an equivalence of categories. There, we will utilise the braided monoidal structure that underlies the (pre)-modular structure.

Appendix C

Hopf algebras

We will need Hopf algebras for two reasons: first, to produce guiding examples in Chapter 4, and second, to provide a foundation for quantum groups, which will be used in Chapter 6 to provide an explicit example of a Kazhdan-Lusztig correspondence. We will not give detailed proofs for any statements; this appendix only serves as a self contained collection of definitions to refer to. Our suggested references for this material are [EGNO16] and [Kas95].

In what follows, let $\Bbbk$ be a field (not necessarily $\mathbb{C}$). In this appendix, all vector spaces and linear maps are assumed to be over $\Bbbk$, unless otherwise stated.

C.1 Algebras, coalgebras and bialgebras

We will give two equivalent definitions for an associative unital algebra. The first definition is traditional and defined on the level of elements, whereas the second definition is defined on the level of morphisms.

DEFINITION C.1. An *associative unital algebra* $(A, \cdot)$ over $\Bbbk$ consists of the following data:

(i) a vector space A,

(ii) a bilinear map $\cdot : A \times A \to A$,

satisfying the following conditions:

(i) (*associativity*) if $x, y, z \in A$, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,

(ii) (*unit*) there exists an element $1 \in A$ such that if $x \in A$ then $x \cdot 1 = x = 1 \cdot x$.

DEFINITION C.2. An *associative unital algebra* (A, ∇, η) over $\Bbbk$ consists of the following data:

(i) a vector space A,

(ii) a linear map $\nabla : A \otimes A \to A$, called the *product*,

(iii) a linear map $\eta : \Bbbk \to A$, called the *unit*,

satisfying the following conditions:

(i) (*associativity*) the following diagram commutes

(C.1)
$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\nabla\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\nabla"'] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd},$$

where $A \otimes (A \otimes A) \cong A \otimes A \otimes A \cong (A \otimes A) \otimes A$ are canonically identified,

(ii) (*unit*) the following diagram commutes

(C.2)
$$\begin{tikzcd} A \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\eta"'] \arrow[dr, "\mathrm{id}"] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd},$$

where $A \otimes \Bbbk \cong A \cong \Bbbk \otimes A$ are canonically identified.

Given an associative unital algebra $(A, \cdot)$ we can define the product by $\nabla(x \otimes y) = x \cdot y$, for $x, y \in A$, and the unit $\eta : 1 \in \Bbbk \mapsto 1 \in A$. Given an associative unital algebra (A, ∇, η) we can define $x \cdot y = \nabla(x \otimes y)$, for $x, y \in A$, and the unit $1 \in A$ is given by $\eta(1)$. Then, the associativity and unit conditions in Definition C.1 give the associativity and unit conditions in Definition C.2, respectively, and vice versa.

When using Definition C.2, we will often use the notation of Definition C.1 and write $\nabla(x \otimes y) = x \cdot y = xy$ and $\eta(1) = 1$, for brevity. So, in this sense, we will treat the definitions above as different choices of notation. The purpose of Definition C.2 is to have a definition that we can categorically dualise.

DEFINITION C.3. A *coassociative counital coalgebra* (C, Δ, ε) over $\Bbbk$ consists of the following data:

(i) a vector space C,

(ii) a linear map $\Delta : C \to C \otimes C$, called the *coproduct*, (usally written with *Sweedler notation* as $\Delta(h) = \sum_{(h)} h' \otimes h''$)

(iii) a linear map $\varepsilon : C \to \Bbbk$, called the *counit*,

satisfying the following conditions:

(i) (*coassociativity*) the following diagram commutes

(C.3)
$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd},$$

where $C \otimes (C \otimes C) \cong C \otimes C \otimes C \cong (C \otimes C) \otimes C$ are canonically identified,

(ii) (*counit*) the following diagram commutes

(C.4)
$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & C \otimes C \arrow[d, "\mathrm{id} \otimes \varepsilon"] \\ C \otimes C \arrow[r, "\varepsilon \otimes \mathrm{id}"'] & C \end{tikzcd},$$

where $\Bbbk \otimes C \cong C \cong C \otimes \Bbbk$ are canonically identified.

From now on we will use the terms algebra and coalgebra to refer to associative unital algebras and coassociative counital coalgebras, respectively.

A bialgebra should be a vector space with an algebra structure and a coalgebra structure such that these structures are compatible.

DEFINITION C.4. A *bialgebra* $(B, \nabla, \eta, \Delta, \varepsilon)$ over $\Bbbk$ consists of the following data:

(i) a vector space B,

(ii) four linear maps

$$\nabla : B \otimes B \to B, \quad \eta : \Bbbk \to B, \quad \Delta : B \to B \otimes B \quad \text{and} \quad \varepsilon : B \to \Bbbk,$$

satisfying the following conditions:

(i) (B, ∇, ε) is an algebra,

(ii) (B, Δ, η) is a coalgebra,

(iii) the four compatibility conditions:

(C.5)
$$\begin{tikzcd} B \otimes B \arrow[r, "\nabla"] \arrow[d, "\Delta \otimes \Delta"'] & B \arrow[r, "\Delta"] & B \otimes B \\ B \otimes B \otimes B \otimes B \arrow[rr, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"'] & & B \otimes B \otimes B \otimes B \arrow[u, "\nabla \otimes \nabla"'] \end{tikzcd},$$

where $\tau : x \otimes y \mapsto y \otimes x$, and

(C.6)
$$\begin{tikzcd} B \otimes B \arrow[rr, "\nabla"] \arrow[dr, "\varepsilon \otimes \varepsilon"'] & & B \arrow[dl, "\varepsilon"] \\ & \Bbbk & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "\Delta"] & & B \otimes B \\ & \Bbbk \arrow[ul, "\eta"] \arrow[ur, "\eta \otimes \eta"'] & \end{tikzcd}$$

(C.7)
$$\begin{tikzcd} \Bbbk \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta"'] & & \Bbbk \\ & B \arrow[ur, "\varepsilon"'] & \end{tikzcd}.$$

The bialgebra data contains four linear maps ∇, η, Δ and ε. If we naturally give $\Bbbk$ and $B \otimes B$ algebra or coalgebra structures, then the compatibility conditions ensure that these maps are algebra or coalgebra homomorphisms with respect to the algebra or coalgebra structure of B, respectively.

Given a bialgebra $(B, \nabla, \eta, \Delta, \varepsilon)$, consider the category $B-\mathsf{Mod}$ of B-modules; by B-module, we mean a module of the associative algebra (B, ∇, η). The coproduct Δ and counit ε canonically produce a monoidal structure on $B-\mathsf{Mod}$. This is explained in detail in Example 4.3, but essentially: Δ defines the tensor product bifunctor, ε defines the unit object, and the associator and unitors are inherited from $\Bbbk-\mathsf{Vect}$.

C.2 Hopf algebras

A Hopf algebra is a bialgebra with additional data that canonically equips its category of finite-dimensional modules with the structure of a left rigid (see Section 4.1) monoidal category, that is, every object has left duals.

DEFINITION C.5. A *Hopf algebra* $(H, \nabla, \eta, \Delta, \varepsilon, S)$ over $\Bbbk$ consists of the following data:

(i) a bialgebra $(H, \nabla, \eta, \Delta, \varepsilon)$,

(ii) a linear map $S : H \to H$, called the antipode,

such that the following diagram commutes.

(C.8)
$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes \mathrm{id}"] & & H \otimes H \arrow[dr, "\nabla"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & \Bbbk \arrow[rr, "\eta"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id} \otimes S"'] & & H \otimes H \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

Given a Hopf algebra $(H, \nabla, \eta, \Delta, \varepsilon, S)$, the underlying bialgebra structure gives a monoidal structure to the category $H-\mathsf{Mod}_{\mathsf{fd}}$ of finite-dimensional H-modules. The antipode S provides left duals for each object in $H-\mathsf{Mod}_{\mathsf{fd}}$. If the antipode S is invertible, then S provides left and right duals for each object in $H-\mathsf{Mod}_{\mathsf{fd}}$. This is explained in detail in Examples and 4.3 and 4.10.

PROPOSITION C.6. Let $(H, \nabla, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. Then,

(C.9)
$$S(xy) = S(y)S(x) \qquad \text{and} \qquad S(1) = 1.$$

That is, the antipode is an associative unital algebra antihomomorphism.

We now provide two familiar examples of Hopf algebras.

EXAMPLE C.7. Let G be a group and consider the group algebra $\Bbbk[G]$ with the usual product and unit. Then, $\Bbbk[G]$ can be equipped with the coproduct defined by $\Delta(g) = g\otimes g$, the counit defined by $\varepsilon(g) = 1$, and the antipode $S(g) = g^{-1}$, for all $g \in G$. The category $\Bbbk[G]-\mathsf{Mod}_{\mathsf{fd}}$ has the same monoidal structure as the tensor product given to finite-dimensional representations of G (recall Example B.6). The left and right duals correspond to the contragredient representation. $\Diamond$

EXAMPLE C.8. Let $\mathfrak{g}$ be a Lie algebra and consider the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ with the usual product and unit. Then, $\mathcal{U}(\mathfrak{g})$ can be equipped with the coproduct defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, for all $x \in \mathfrak{g} \subseteq \mathcal{U}(\mathfrak{g})$. As required by the compatibility conditions, we define $\Delta(1) = 1 \otimes 1$ and $\Delta(xy) = \Delta(x)\Delta(y)$, for all $x, y \in \mathfrak{g}$, hence uniquely extending Δ to all of $\mathcal{U}(\mathfrak{g})$ (the commutator of $\mathcal{U}(\mathfrak{g})$ is indeed respected). The counit is defined by $\varepsilon(1) = 1$ and $\varepsilon(x) = 0$, for all $x \in \mathfrak{g}$. As required by the compatibility conditions, we define $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$, for all $x, y \in \mathcal{U}(\mathfrak{g})$, hence uniquely extending ε to all of $\mathcal{U}(\mathfrak{g})$. Finally, the antipode is $S(x) = -x$, for all $x \in \mathfrak{g}$. Requiring that $S(1) = 1$ and $S(xy) = S(y)S(x)$, for all $x, y \in \mathfrak{g}$, is enough to uniquely define an antipode satisfying diagram (C.8). The category $\mathcal{U}(\mathfrak{g})-\mathsf{Mod}_{\mathsf{fd}}$ has a monoidal structure that is monoidally equivalent to the tensor product given to Lie algebra representations of $\mathfrak{g}$ (recall Example B.7). The left and right duals correspond to contragredient representations. $\Diamond$

Some bialgebras can be equipped with additional data that gives the monoidal category of finite-dimensional modules a braiding or ribbon structure.

DEFINITION C.9. A *quasi-triangular bialgebra* $(B, \nabla, \eta, \Delta, \varepsilon, R)$ consists of the following data:

(i) a bialgebra $(B, \nabla, \eta, \Delta, \varepsilon, S)$,

(ii) an invertible element R in $B \otimes B$ called the *universal R-matrix*,

satisfying the following conditions:

$$R\Delta(x)R^{-1} = (\tau \circ \Delta)(x) \qquad \text{for all } x \in B, \tag{C.10}$$

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23} \qquad \text{and} \qquad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}, \tag{C.11}$$

where $\tau : x \otimes y \mapsto y \otimes x$. If $R = \sum_i R'_i \otimes R''_i$, then we use the notation

$$R_{12} = \sum_i R'_i \otimes R''_i \otimes 1, \quad R_{13} = \sum_i R'_i \otimes 1 \otimes R''_i, \quad R_{23} = \sum_i 1 \otimes R'_i \otimes R''_i. \tag{C.12}$$

DEFINITION C.10. A *quasi-triangular Hopf algebra* or a *braided Hopf algebra* is a Hopf algebra together with a universal R-matrix and quasi-triangular bialgebra structure.

Given a quasi-triangular bialgebra $(B, \nabla, \eta, \Delta, \varepsilon, R)$, the universal R-matrix provides a braiding on the monoidal category $B-\mathsf{Mod}$ defined by

$$c_{X,Y}(x \otimes y) = \tau_{X,Y}(R \cdot (x \otimes y)), \tag{C.13}$$

for all $x \in X$, $y \in Y$ and $X, Y \in \mathrm{ob}(H-\mathsf{Mod}_{\mathsf{fd}})$. Here, τ is the braiding on $\Bbbk-\mathsf{Vect}$, that is, $\tau_{X,Y}(x \otimes y) = y \otimes x$. Equation (C.10) ensures that the components of c are B-module homomorphisms. The hexagon identities ((B.17) and (B.18)) are satisfied by (C.11).

In Section 4.2, we define ribbon categories. We now equip a quasi-triangular Hopf algebra with additional data that canonically gives a ribbon structure to its category of finite-dimensional modules.

DEFINITION C.11. A *ribbon Hopf algebra* $(H, \nabla, \eta, \Delta, \varepsilon, S, R, \nu)$ consists of the following data:

(i) a quasi-triangular Hopf algebra $(H, \nabla, \eta, \Delta, \varepsilon, S, R)$ with an invertible antipode,

(ii) an invertible central element ν in H, called the *ribbon element*,

satisfying the following relations:

$$\Delta(\nu) = (R_{21}R)^{-1}(\nu \otimes \nu) \qquad \text{and} \qquad S(\nu) = \nu. \tag{C.14}$$

Given a ribbon Hopf algebra $(H, \nabla, \eta, \Delta, \varepsilon, S, R, \nu)$ with an invertible antipode, the category $H\text{−}\mathsf{Mod}_{\mathsf{fd}}$ is a braided rigid monoidal category equipped with the ribbon structure (twist) defined by

$$\theta_X(x) = \nu^{-1} \cdot x \qquad \text{for all } x \in X \text{ and } X \in \mathrm{ob}(H\text{−}\mathsf{Mod}_{\mathsf{fd}}). \tag{C.15}$$

Since ν is central, the components of θ are H-module homomorphisms. The first relation in (C.14) ensures that the twist condition (4.30) is satisfied, while the second relation in (C.14) ensures that the ribbon condition (4.31) is satisfied. Note that we require the antipode to be invertible since we have defined ribbon categories to be rigid instead of just left rigid.

We will use ribbon Hopf algebras in Chapter 6 to construct ribbon categories from quantum groups.

Now, we highlight a feature of bialgebras. The monoidal categories generated by bialgebras have "trivial" associators and unitors in the sense that

$$\alpha_{X,Y,Z} : x \otimes (y \otimes z) \mapsto (x \otimes y) \otimes z, \tag{C.16}$$

$$\lambda_X : 1 \otimes x \mapsto x \qquad \text{and} \qquad \rho_X : x \otimes 1 \mapsto x. \tag{C.17}$$

If we were to make the canonical identifications

$$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z \qquad \text{and} \qquad \Bbbk \otimes X = X = X \otimes \Bbbk,$$

then the monoidal category becomes strict.

There are notions of *quasi-bialgebras* and *quasi-Hopf algebras* that endow bialgebras and Hopf algebras with data used to define non-trivial associators and unitors for their monoidal categories of (resp. finite-dimensional) modules. These definitions can be found in Chapter 15 of [Kas95] (although the associators are the inverse of what we use here). We will not use these notions in this thesis and, instead, obtain a non-trivial associator from a Hopf algebra using semi-simplification in Chapter 6.

Appendix D

Quantum groups associated to $\mathfrak{sl}_2$

This appendix contains the definitions and results for the quantum groups associated to $\mathfrak{sl}_2$ to be applied in Chapter 6. Our aim is to find $\mathfrak{sl}_2$-quantum groups with canonical ribbon tensor categories of modules. For our application's purposes, we will present the following definitions and results in the special case for $\mathfrak{sl}_2$ only. However, there are general definitions for finite-dimensional complex simple Lie algebras as well, which we would need if we were to generalise the results from Chapter 6.

We will use the terminology and conventions from [CP95], however, other references we have used are [Kas95], [Saw06] and [ES02].

D.1 Algebras

Even though the Hopf algebras of interest in Chapter 6 are over $\mathbb{C}$, we will first take a detour to discuss Hopf algebras over (topological) commutative rings. The definitions from Appendix C can be generalised to commutative rings, as seen in Chapter 4 of [CP95]. That is, replace the field $\Bbbk$ with a commutative unital ring k and the $\Bbbk$-vector spaces with k-modules. In the case that k is equipped with a topology, we equip H and $H \otimes H$ with a k-module topology. We also replace $H \otimes H$ with its completion and require that the unit, multiplication, counit and comultiplication are continuous. Doing so gives *topological Hopf algebras*, *topological quasi-triangular Hopf algebras* etc. We will now introduce the topology that is needed to eventually obtain ribbon tensor categories from certain $\mathfrak{sl}_2$-quantum groups.

Let h be a formal variable. We will define a Hopf algebra over $\mathbb{C}[[h]]$, the ring of formal series in h with coefficients in $\mathbb{C}$.

DEFINITION D.1. Let V be a $\mathbb{C}[[h]]$-module. The *h-adic topology* on V is defined by requiring that $\{h^n V : n \in \mathbb{Z}_{\geq 0}\}$ is a neighbourhood base for $0 \in V$ and that translations and scalar multiplication are continuous.

REMARK D.2. For $V = \mathbb{C}[[h]]$, this defines a topology for $\mathbb{C}[[h]]$ as a topological ring. A general $\mathbb{C}[[h]]$-module V then becomes a topological $\mathbb{C}[[h]]$-module. It follows that every $\mathbb{C}[[h]]$-module homomorphism is continuous. △

REMARK D.3. We need topological algebras over $\mathbb{C}[[h]]$, with the h-adic topology, in order to define a universal R-matrix and ribbon element for the following quantum group. These elements will eventually transfer the braiding and ribbon structure to categories of other non-quasi-triangular $\mathfrak{sl}_2$-quantum group modules, providing the canonical ribbon tensor structures to be used in Chapter 6. △

The following definition is from Definition/Proposition 6.4.3, Lemma 8.3.6 and Corollary 8.3.16 of [CP95].

DEFINITION/PROPOSITION D.4. Write $e^h = \sum_{n\geq 0} \frac{h^n}{n!}$. The *Drinfeld-Jimbo quantum group* $U_h = U_h(\mathfrak{sl}_2)$ is a topological ribbon Hopf algebra over $\mathbb{C}[[h]]$ consisting of:

(i) the generators X, Y and H, and relations

$$[H,X] = 2X, \qquad [H,Y] = -2Y, \qquad [X,Y] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}, \tag{D.1}$$

(ii) the coproduct and counit given by

$$\Delta(X) = X\otimes e^{hH} + 1\otimes X, \quad \Delta(Y) = Y\otimes 1 + e^{-hH}\otimes Y, \quad \Delta(H) = H\otimes 1 + 1\otimes H, \tag{D.2}$$

$$\varepsilon(X) = \varepsilon(Y) = \varepsilon(H) = 0, \tag{D.3}$$

(iii) the antipode given by

$$S(X) = -Xe^{-hH}, \qquad S(Y) = -e^{hH}Y, \qquad S(H) = -H, \tag{D.4}$$

(iv) the universal R-matrix

$$R = e^{\frac{1}{2}hH\otimes H} \sum_{n\geq 0} e^{\frac{n(n-1)}{2}h} \frac{(e^h - e^{-h})^n}{[n]_{e^h}!} X^n \otimes Y^n, \tag{D.5}$$

where we denote

$$[n]_{e^h}! = \frac{\sinh(nh)}{\sinh(h)} = \frac{e^{nh} - e^{-nh}}{e^h - e^{-h}}, \tag{D.6}$$

(v) the ribbon element

$$\nu = e^{-hH}u, \qquad \text{where } u = \nabla((S\otimes \mathrm{id})R_{21}). \tag{D.7}$$

REMARK D.5. The universal R-matrix (D.5) is an element in the completion of $U_h \otimes U_h$ with respect to the h-adic topology. △

REMARK D.6. The Drinfeld-Jimbo quantum group is a *quantised universal enveloping algebra* because U_h is a deformation of the universal enveloping algebra of $\mathfrak{sl}_2$, that is, U_h/hU_h is isomorphic to $\mathcal{U}(\mathfrak{sl}_2)$ as Hopf algebras over $\mathbb{C}$. We can see this by comparing Definition D.4 with Example C.8 for $\mathfrak{g} = \mathfrak{sl}_2$. (The non-immediate part is to check that $[X,Y] = H$ modulo hU_h.) Heuristically, we think of this as $U_h \to \mathcal{U}(\mathfrak{sl}_2)$ as $h \to 0$. In the $h \to 0$ "limit", the universal R-matrix (D.5) and ribbon element (D.7) become trivial, as expected for $\mathcal{U}(\mathfrak{sl}_2)-\mathsf{Mod}_{\mathsf{fd}}$, highlighting the need to consider quantum groups when seeking non-trivial ribbon tensor categories associated to $\mathfrak{sl}_2$. $\triangle$

Let q be a formal variable.

DEFINITION D.7. We define the following *q-analogues* of numbers, factorials and binomial coefficients. Let $n, k \in \mathbb{Z}$. The *q-number* or *q-bracket* of n is

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}. \tag{D.8}$$

Let $0 \leq k \leq n$. The *q-factorial* of n is

$$[n]_q! = [1]_q[2]_q \cdots [n]_q, \quad \text{when } n > 0, \tag{D.9}$$

and $[0]_q! = 1$, when $n = 0$. The *q-binomial coefficient "n choose k"* is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}. \tag{D.10}$$

There are similar definitions with the formal variable q replaced with $\epsilon \in \mathbb{C}\backslash\{0, \pm 1\}$.

REMARK D.8. The following quantum group can be defined as an algebra over $\Bbbk(q)$, the field of rational functions in q, for a field $\Bbbk$. It is most common for $\Bbbk$ to be $\mathbb{Q}$, but because we are interested in constructing a $\mathbb{C}$-linear modular tensor category in Chapter 6, we keep to the case where $\Bbbk = \mathbb{C}$ only. $\triangle$

DEFINITION D.9. The *Drinfeld-Jimbo quantum group* $U_q = U_q(\mathfrak{sl}_2)$ is the associative unital algebra over $\mathbb{C}(q)$ consisting of:

(i) the generators E, F, K and K^{-1} and relations

$$\begin{gathered} KK^{-1} = K^{-1}K = 1, \\ KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2}F, \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \end{gathered} \tag{D.11}$$

(ii) the bialgebra structure given by the coproduct Δ and counit ε defined by

$$\begin{gathered} \Delta(E) = 1 \otimes E + E \otimes K, \qquad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \\ \Delta(K) = K \otimes K, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \\ \varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K) = \varepsilon(K^{-1}) = 1, \end{gathered} \tag{D.12}$$

(iii) the antipode S defined by

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K. \tag{D.13}$$

REMARK D.10. Notice that the relations for U_h in (D.1) resemble the relations for U_q in (D.11) in the following sense. The relations (D.1) imply

$$e^{hH}Xe^{-hH} = e^{2h}X, \qquad e^{hH}Ye^{-hH} = e^{-2h}Y, \qquad [X, Y] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}. \tag{D.14}$$

Hence, U_q injects into U_h by the map

$$E \mapsto X, \qquad F \mapsto Y, \qquad K^{\pm 1} \mapsto e^{\pm hH}, \qquad q \mapsto e^h. \tag{D.15}$$

Note that this is not a $\mathbb{C}(q)$-linear map, since we are identifying $\mathbb{C}(q)$ with the subring $\mathbb{C}(e^h)$ of $\mathbb{C}[[h]]$, but, nonetheless, the algebra relations are still preserved. This "injection" relates U_q with U_h and will be used to transfer the ribbon structure of $U_h-\mathsf{Mod}_{\mathsf{fd}}$ to $U_q-\mathsf{Mod}_{\mathsf{fd}}$; see Propositions D.33 and D.35 below. △

REMARK D.11. Using the algebra antihomomorphism property of S, we have

$$S^2(E) = KEK^{-1}, \quad S^2(F) = KFK^{-1} \quad \text{and} \quad S^2(K^{\pm 1}) = K^{\pm 1}.$$

So, $S^2(u) = KuK^{-1}$, for all $u \in U_q$. It follows that $u \mapsto K^{-1}S(u)K$ is the inverse of S. Since U_q is a Hopf algebra with an invertible antipode, the category $U_q-\mathsf{Mod}_{\mathsf{fd}}$ has a rigid monoidal structure and, hence, the structure of a tensor category. △

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials in q with coefficients in $\mathbb{Z}$. An *integral form* of U_q is an $\mathcal{A}$-subalgebra V of U_q such that U_q can be recovered by an extension of scalars from $\mathcal{A}$ to $\mathbb{C}(q)$. That is,

$$U_q = V \otimes_{\mathcal{A}} \mathbb{C}(q). \tag{D.16}$$

We will see two integral forms of U_q.

DEFINITION/PROPOSITION D.12. (Definition/Proposition 9.3.1 of [CP95]) The *divided powers*, for $n \in \mathbb{Z}_{>0}$, are defined as

$$E^{(n)} = \frac{E^n}{[n]_q!} \qquad \text{and} \qquad F^{(n)} = \frac{F^n}{[n]_q!}. \tag{D.17}$$

The *restricted integral form* $U_{\mathcal{A}}^{\mathrm{res}} = U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{sl}_2)$ is the $\mathcal{A}$-subalgebra of U_q generated by the elements $\{E^{(n)}, F^{(n)}, K, K^{-1} : n \in \mathbb{Z}_{>0}\}$. This is, non-trivially, an integral form of U_q.

REMARK D.13. One should initially be worried that $U_{\mathcal{A}}^{\mathrm{res}}$ may not be an algebra over $\mathcal{A}$ since $[E, F] = \frac{K-K^{-1}}{q-q^{-1}}$. This is resolved in Theorem 9.3.4 of [CP95] with a long list of generators and relations. However, such an explicit theorem will not be needed for our computations in Chapter 6. △

DEFINITION D.14. The *non-restricted integral form* $U_{\mathcal{A}} = U_{\mathcal{A}}(\mathfrak{sl}_2)$ is the $\mathcal{A}$-subalgebra of U_q generated by the elements $\{E, F, K, K^{-1}, [K; 0]\}$, where $[K; 0] := \frac{K-K^{-1}}{q-q^{-1}}$. This is an integral form of U_q.

The integral forms can be used to specialise the formal variable q to a complex number $\epsilon \in \mathbb{C}\backslash\{0, \pm 1\}$.

DEFINITION D.15. The *restricted specialisation* $U_{\epsilon}^{\mathrm{res}} = U_{\epsilon}^{\mathrm{res}}(\mathfrak{sl}_2)$ is the $\mathbb{C}$-algebra

$$U_{\epsilon}^{\mathrm{res}} = U_{\mathcal{A}}^{\mathrm{res}} \otimes_{\mathcal{A}} \mathbb{C}. \tag{D.18}$$

This is also called the *Lusztig quantum group*.

DEFINITION D.16. The *non-restricted specialisation* $U_{\epsilon} = U_{\epsilon}(\mathfrak{sl}_2)$ is the $\mathbb{C}$-algebra

$$U_{\epsilon} = U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}. \tag{D.19}$$

This is also called the *De Concini-Kac quantum group*.

The De Concini-Kac quantum group can be defined similarly to Definition D.9, but with the formal variable q replaced with $\epsilon \in \mathbb{C}\backslash\{0, \pm 1\}$.

When ϵ is not a root of unity, the q-numbers $[n]_\epsilon$ are zero only when $n = 0$. So, the divided powers $E^{(n)}$ and $F^{(n)}$ in U_ϵ^{res} are simply $E^n/[n]_\epsilon$ and $F^n/[n]_\epsilon$, respectively. Hence, the restricted and non-restricted specialisations coincide.

In the case when ϵ is a root of unity, the restricted and non-restricted specialisations do not coincide, and neither do their representation theories. Moreover, there is third version of a quantum group at root of unity arising from the non-restricted specialisation.

Denote by ℓ the smallest positive integer such that $\epsilon^\ell = 1$ and define

$$\ell' = \begin{cases} \ell & \text{if } \ell \text{ is odd} \\ \ell/2 & \text{if } \ell \text{ is even} \end{cases}. \tag{D.20}$$

For simplicity, we will assume that ϵ is the ℓ^{th}-primitive root of unity $\epsilon = e^{2\pi i/\ell}$. Note that, $[n]_\epsilon = 0$ if and only if n is a multiple of ℓ'.

DEFINITION/PROPOSITION D.17. (Definition 6.5.6 and Proposition 6.5.8 of [Kas95]) The *small quantum group* $\overline{U}_\epsilon = \overline{U}_\epsilon(\mathfrak{sl}_2)$ is the quotient

$$\overline{U}_\epsilon = U_\epsilon/\langle E^{\ell'}, F^{\ell'}, K^{\ell'} - 1\rangle, \tag{D.21}$$

where $\langle E^{\ell'}, F^{\ell'}, K^{\ell'} - 1\rangle$ denotes the ideal generated by the central elements $E^{\ell'}$, $F^{\ell'}$ and $K^{\ell'} - 1$. The small quantum group is in fact finite-dimensional with a basis

$$\{E^i F^j K^k : 0 \leq i, j, k \leq \ell' - 1\}. \tag{D.22}$$

REMARK D.18. The small quantum group can be equipped with a universal R-matrix and ribbon element as seen in Theorem 9.7.1 and Proposition 14.6.5 of [Kas95]. We will not need these explicitly, but instead remark that $\overline{U}_\epsilon$ is a ribbon Hopf algebra with an invertible antipode and, hence, $\overline{U}_\epsilon-\mathsf{Mod}_{\mathsf{fd}}$ is canonically a ribbon tensor category. Thus, it initially appears to be a good candidate for the quantum group in a Kazhdan-Lusztig correspondence, however, we will see that this is not the case in Section 6.5. △

D.2 Representation theory

Now that we have seen several different Hopf algebras associated to $\mathfrak{sl}_2$, we can discuss their representation theories. As is typical for Hopf algebras, we will only consider the finite-dimensional representations. Our main goal is to discuss the representations for U_ϵ^{res} and $\overline{U}_\epsilon$ at ϵ a root of unity. However, we will first need to discuss the representations of U_h and U_q as well.

PROPOSITION D.19. (Proposition 6.4.10 of [CP95]) The *indecomposable* U_h-modules of finite-rank (as a free $\mathbb{C}[[h]]$-module) are classified by $\mathbb{Z}_{\geq 0}$. More precisely, for each

$n \in \mathbb{Z}_{\geq 0}$, there is an indecomposable U_h-module $V_h(n)$ with $\mathbb{C}[[h]]$-basis

$$\{v_0^{(n)}, \ldots, v_n^{(n)}\} \tag{D.23}$$

and U_h-action given by

$$Hv_r^{(n)} = (n-2r)v_r^{(n)}, \quad Xv_r^{(n)} = [n-r+1]_{e^h} v_{r-1}^{(n)}, \quad Yv_r^{(n)} = [r+1]_{e^h} v_{r+1}^{(n)}, \tag{D.24}$$

where $v_{-1}^{(n)} = v_{n+1}^{(n)} = 0$. Every finite-rank indecomposable U_h-module is isomorphic to $V_h(n)$, for some $n \in \mathbb{Z}_{\geq 0}$.

REMARK D.20. The U_h-modules $V_h(n)$ become the finite-dimensional irreducible $\mathfrak{sl}_2$-modules when $h \to 0$. Put precisely, $V_h(n)/hV_h(n)$ is the irreducible $\mathfrak{sl}_2$-module of $\mathbb{C}$-dimension $(n+1)$, after we identify U_h/hU_h with $\mathcal{U}(\mathfrak{sl}_2)$. △

The representation theory of U_q is similar to the classical case of $\mathfrak{sl}_2$-modules, in the sense that any finite-dimensional irreducible U_q-module can be shown to be a highest weight module. The highest weights can then be solved for, and the modules can be constructed, thus classifying all finite-dimensional irreducible modules, up to isomorphism.

However, the terminology will differ in the quantum case. Given a U_q-module V, the *weight* of a vector v in V is an element $\lambda \in \mathbb{C}(q)$ such that $Kv = \lambda v$. In this case, we call v a *weight vector*. Furthermore, we call v a *highest weight vector* if, in addition, $Ev = 0$, and in this case, we call λ a *highest weight*. A *highest weight module* is one generated by a highest weight vector.

The representation theory of U_q differs from $\mathfrak{sl}_2$ in the following way. The irreducible U_q-modules of finite dimension are highest weight modules with highest weight vectors of weights $\lambda = \sigma q^n$, where $\sigma = \pm 1$ and $n \in \mathbb{Z}_{\geq 0}$. Note that the highest weights are classified by, not only the non-negative integers, but by $\{(\sigma, n) : \sigma = \pm 1,\ n \in \mathbb{Z}_{\geq 0}\}$.

PROPOSITION D.21. (Example 10.1.3 of [CP95]) Let $\sigma = \pm 1$ and $n \in \mathbb{Z}_{\geq 0}$. Then, there is an irreducible U_q-module $V_q(\sigma, n)$ with $\mathbb{C}(q)$-basis

$$\{v_0^{(\sigma,n)}, \ldots, v_n^{(\sigma,n)}\}, \tag{D.25}$$

and U_q-action given by

$$Kv_r^{(\sigma,n)} = \sigma q^{n-2r} v_r^{(\sigma,n)}, \quad Ev_r^{(\sigma,n)} = \sigma[n-r+1]_q v_{r-1}^{(\sigma,n)}, \quad Fv_r^{(\sigma,n)} = [r+1]_q v_{r+1}^{(\sigma,n)}, \tag{D.26}$$

where $v_{-1}^{(\sigma,n)} = v_{n+1}^{(\sigma,n)} = 0$. Every finite-dimensional irreducible U_q-module is isomorphic to $V_q(\sigma, n)$, for some $\sigma = \pm 1$ and $n \in \mathbb{Z}_{\geq 0}$.

Similarly to the classical case, we have the following propositions.

PROPOSITION D.22. (Proposition 10.1.2) Every finite-dimensional irreducible U_q-module is a *weight module*, that is, decomposes into a direct sum of its weight spaces.

PROPOSITION D.23. (Theorem 10.1.7) Every finite-dimensional U_q-module is completely reducible.

REMARK D.24. The irreducible modules with $\sigma = 1$ are said to be of *type 1* and, for brevity, we denote these by $V_q(n) = V_q(1,n)$ with basis elements $v^n = v^{(1,n)}$. Note that the H-action in (D.24) exponentiates to $e^{hH}v_r^{(n)} = e^{h(n-2r)}v_r^{(n)}$. That is, the type 1 U_q-modules $V_q(n)$ are the analogues of the U_h-modules $V_h(n)$ via the injection from Remark D.10. This will be important when transferring the braiding and ribbon structure from U_h to U_q.

By Proposition D.23, we have that in general, a U_q-module V is of *type 1* if K acts semisimply on V with eigenvalues q^n, for some $n \in \mathbb{Z}$ (importantly, not $-q^n$). In this case, the irreducible modules are classified by the non-negative weight lattice, similarly to the classical case. We can think of the weight q^n as corresponding to the *weight* λ in P, the weight lattice of $\mathfrak{sl}_2$, such that $Kv = q^{\langle\lambda,\alpha\rangle}v = q^n v$. △

PROPOSITION D.25. (Proposition 10.1.16 of [CP95], but with the proof adapted for a formal variable instead of a non-root of unity) Let $\sigma = \pm 1$ and $m \geq n \in \mathbb{Z}_{\geq 0}$. Then,

$$V_q(\sigma,m) \cong V_q(\sigma,0)\otimes V_q(m) \cong V_q(m)\otimes V_q(\sigma,0), \quad V_q(0) \cong V_q(\sigma,0)\otimes V_q(\sigma,0), \tag{D.27}$$

$$V_q(m)\otimes V_q(n) \cong V_q(m+n)\oplus V_q(m+n-2)\oplus\cdots\oplus V_q(m-n). \tag{D.28}$$

REMARK D.26. Propositions D.23 and D.25 tell us that the category $U_q\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ of type 1 finite-dimensional U_q-modules is closed under the tensor product. Since $S(K) = K^{-1}$, the dual of a type 1 U_q-module is also of type 1. So, $U_q\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ is a rigid monoidal subcategory of $U_q\text{−}\mathsf{Mod}_{\mathsf{fd}}$. Finally, every type 1 module decomposes into irreducible type 1 modules, hence $U_q\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ is closed under kernels and cokernels. It is thus also a tensor category. △

The representation theory of U_q can be used to construct De Concini-Kac quantum group modules. In fact, the representation theory is the same when ϵ is not a root of unity and hence is the same for the Lusztig quantum group at a non-root of unity. The same arguments as in Proposition D.21, but with q replaced by ϵ, yield the finite-dimensional irreducible U_ϵ-modules $V_\epsilon(\sigma,n)$, for $\sigma = \pm 1$, $n \in \mathbb{Z}_{\geq 0}$. The U_ϵ-modules $V_\epsilon(\sigma,n)$ are defined when ϵ is a root of unity, however, these modules may not be irreducible, and in fact, no finite-dimensional module can be irreducible if its dimension is greater than ℓ' (see Proposition 6.5.2 of [Kas95]).

The representation theory of U_q can also be used to construct Lusztig quantum group modules. Even though the finite-dimensional irreducible U_q-modules produce highest weight U_ϵ^{res}-modules, these are not necessarily irreducible.

DEFINITION D.27. Let $V_q(n)$ be a type 1 finite-dimensional irreducible module of U_q with a highest weight vector $v_0^{(n)}$. Consider the $U_{\mathcal{A}}^{\text{res}}$-module generated by $v_0^{(n)}$,

$$V_{\mathcal{A}}^{\text{res}}(n) = U_{\mathcal{A}}^{\text{res}}v_0^{(n)}. \tag{D.29}$$

It has a basis

$$\{v_0^{(n)},\ldots,v_n^{(n)}\}, \tag{D.30}$$

$$
\begin{tikzcd}
v_0^{(4)} \arrow[r, bend left, dashed, "{[1]_q}"] & v_1^{(4)} \arrow[l, bend left, "{[4]_q}"] \arrow[r, bend left, dashed, "{[2]_q}"] & v_2^{(4)} \arrow[l, bend left, "{[3]_q}"] \arrow[r, bend left, dashed, "{[3]_q}"] & v_3^{(4)} \arrow[l, bend left, "{[2]_q}"] \arrow[r, bend left, dashed, "{[4]_q}"] & v_4^{(4)} \arrow[l, bend left, "{[1]_q}"] \\
q^4 & q^2 & q^0 & q^{-2} & q^{-4}
\end{tikzcd}
$$

FIGURE D.1: Example of the irreducible U_q module $V_q(1,4)$. The action by F is drawn as dashed arrows, action by E is drawn as solid arrows, and K-eigenvalues are labelled underneath the basis vectors.

with U_q-action given by

(D.31)
$$Kv_r^{(n)} = q^{n-2r}v_r^{(n)}, \quad E^{(s)}v_r^{(n)} = \begin{bmatrix} n-r+s \\ s \end{bmatrix}_q v_{r-s}^{(n)}, \quad F^{(s)}v_r^{(n)} = \begin{bmatrix} r+s \\ s \end{bmatrix}_q v_{r+s}^{(n)}.$$

We now replace q with a root of unity $\epsilon \neq \pm 1$.

DEFINITION D.28. A *Weyl module* is a U_ϵ^{res}-module defined as

$$W_\epsilon^{\text{res}}(n) = V_{\mathcal{A}}^{\text{res}}(n) \otimes_{\mathcal{A}} \mathbb{C}, \tag{D.32}$$

where q acts on $\mathbb{C}$ as multiplication by ϵ.

REMARK D.29. The Weyl modules are *type 1* U_ϵ^{res}-modules. Importantly, each Weyl module $W_\epsilon^{\text{res}}(n)$ has a basis of vectors $\{v_r^{(n)}\}$ with well-defined H-actions, that is, weights λ of $\mathfrak{sl}_2$ such that $Kv = \epsilon^{\langle \lambda, \alpha \rangle}v$. Note that the weights are able to be easily deduced when ϵ is not a root of unity, but not when ϵ is a root of unity. However, in the root of unity case, we can use Weyl modules to deduce the weights. △

The irreducible finite-dimensional U_q-modules can give $\overline{U}_\epsilon$-modules. The modules $V_\epsilon(\sigma, n)$ give modules for the small quantum group if $E^{\ell'}$, $F^{\ell'}$ and $K^{\ell'} - 1$ act as zero. Modules not of type 1 may be taken into account. For example, if $\epsilon = e^{i\pi/3}$, then $V_\epsilon(1,3)$ has K^3 acting as -1 whereas $V_\epsilon(-1,3)$ has K^3 acting as 1. We will see that Lusztig and small quantum groups have "different representation theories", as explained in Section 6.5.

In Figures D.1, D.2 and D.3, we show examples of type 1 U_q-, U_ϵ^{res}- and $\overline{U}_\epsilon$-modules, respectively, of dimension 4 and at $\epsilon = e^{\pi i/3}$, to develop an intuition for their structures by comparing the actions of the generators.

Since the small quantum group comes equipped with a universal R-matrix and ribbon element, $\overline{U}_\epsilon{-}\mathsf{Mod}_{\mathsf{fd}}$ is canonically a ribbon category. However, if we are to obtain a ribbon category from the Lusztig quantum group, we need to restrict the category of modules to those analogous to U_h-modules, that is, type 1 modules. In these modules, there is a basis of weight vectors with well-defined H-actions.

Let $\epsilon \in \mathbb{C}\backslash\{0, \pm 1\}$.

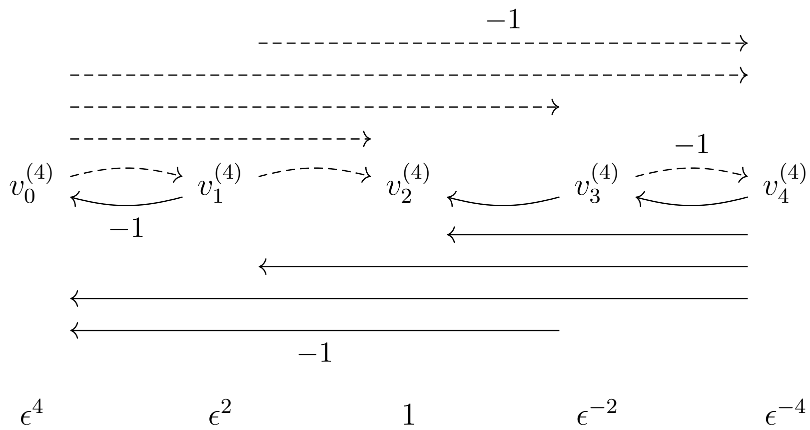

FIGURE D.2: Example of the reducible U_ϵ^{res} module $W_\epsilon^{\text{res}}(n)$ for $\epsilon = e^{\pi i/3}$, the primitive sixth root of unity. The actions by $F^{(s)}$ are drawn as dashed arrows, actions by $E^{(s)}$ are drawn as solid arrows, for $s = 1, \dots, 4$, and K-eigenvalues are labelled underneath the basis vectors. Any arrows that are not drawn, represent an action of zero. The arrows without labels, have a factor of 1.

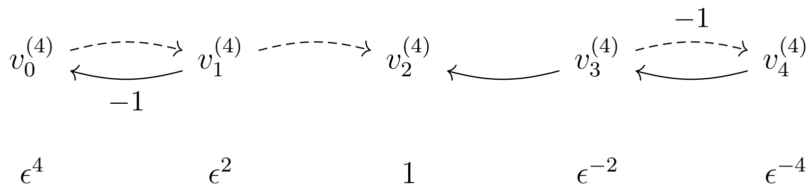

FIGURE D.3: Example of a reducible $\overline{U}_\epsilon$ module for $\epsilon = e^{\pi i/3}$. The actions by F are drawn as dashed arrows, actions by E are drawn as solid arrows, and K-eigenvalues are labelled underneath the basis vectors. Any arrows that are not drawn, represent an action of zero. The arrows without labels, have a factor of 1.

DEFINITION D.30. Let $V \in \text{ob}(U_q\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1})$. Decompose V into the direct sum of irreducible submodules then generate $U_{\mathcal{A}}^{\text{res}}$-modules from their highest weight vectors, as in Definition D.27. Define $V_{\mathcal{A}}^{\text{res}}$ to be the $\mathcal{A}$-linear direct sum of these $U_{\mathcal{A}}^{\text{res}}$-modules and define the U_ϵ^{res}-module $V_\epsilon^{\text{res}} = V_{\mathcal{A}}^{\text{res}} \otimes_{\mathcal{A}} \mathbb{C}$. Denote by $U_{\mathcal{A}}^{\text{res}}\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ the full subcategory of $U_{\mathcal{A}}^{\text{res}}$-modules, with objects comprising of all such $V_{\mathcal{A}}^{\text{res}}$. Define $U_\epsilon^{\text{res}}\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ to be the smallest (by inclusion of object-classes) full subcategory of U_ϵ^{res}-modules satisfying the following conditions:

(i) contains all objects V_ϵ^{res}, for $V \in \text{ob}(U_q\text{−}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1})$,

(ii) closure under tensor products,

(iii) closure under direct sums,

(iv) closure under duals,

(v) closure under subobjects and quotients.

REMARK D.31. Given a module M in $U_\epsilon^{\text{res}}\text{–}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$, there is a finite sequence of tensor products, direct sums, duals, subobjects and quotients (starting with objects isomorphic to the Weyl modules) used to obtain M. Since the weight of a weight vector is preserved under each of these steps, there is a well-defined basis of weight vectors of M. This is necessary for a well-defined H-action, which is needed below to let the universal R-matrix and ribbon element of U_h act on a module in $U_\epsilon^{\text{res}}\text{–}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$.

Since $U_\epsilon^{\text{res}}\text{–}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ is closed under the tensor product and duals, it is a rigid monoidal subcategory of $U_\epsilon^{\text{res}}\text{–}\mathsf{Mod}_{\mathsf{fd}}$. Furthermore, $U_\epsilon^{\text{res}}\text{–}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ is closed under direct sums, kernels and cokernels, so it also inherits the abelian structure from $U_\epsilon^{\text{res}}\text{–}\mathsf{Mod}_{\mathsf{fd}}$. Hence, $U_\epsilon^{\text{res}}\text{–}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ has a tensor category structure. △

REMARK D.32. We will not explicitly construct the category of finite-dimensional type 1 U_ϵ^{res}-modules because it will not be needed in Chapter 6. However, we will direct the reader to the resources. For non-roots of unity, this category is the same of the category of finite-dimensional type 1 U_ϵ^{res}-modules, but with q replaced with ϵ. For odd roots of unity, the characterisation of weight modules with $K^\ell = 1$ is used in Chapter 11 of [CP95]. A general construction for all roots of unity can be found in [Saw06], but the quantum group used by Sawin is slightly different to those we have presented here. △

It remains to give $U_\epsilon^{\text{res}}\text{–}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ a canonical braiding and ribbon structure.

D.3 Ribbon tensor categories

Definition D.4 gave a topological ribbon Hopf structure for the quantum group U_h with universal R-matrix R and ribbon element ν. So, $U_h\text{–}\mathsf{Mod}_{\mathsf{fd}}$ is a rigid monoidal category equipped with braiding

(D.33) $$c_{V,W} : V \otimes W \to W \otimes V, \qquad c_{V,W}(v \otimes w) = \tau(R(w \otimes v))$$

(where $\tau : v \otimes w \mapsto w \otimes v$) and ribbon structure

(D.34) $$\theta_V : V \to V, \qquad \theta_V(v) = \nu^{-1}v.$$

On the other hand, we do not have a universal R-matrix for U, where U is U_q, $U_{\mathcal{A}}^{\text{res}}$ or U_ϵ^{res}. However, by viewing U_q as a "sub-Hopf algebra" of U_h, as in Remark D.10, we are still able to define a braiding and ribbon structure for U.

PROPOSITION D.33. (Corollary 10.1.20 of [CP95]) The monoidal category $U\text{–}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ is a braided monoidal category with the braiding

(D.35) $$c_{V,W} : V \otimes W \to W \otimes V, \qquad c_{V,W}(v \otimes w) = \tau(R(w \otimes v))$$

where $\tau : v \otimes w \mapsto w \otimes v$ and R acts on $V \otimes W$ with only finitely many terms non-zero.

REMARK D.34. For the precise meaning of "R acts on $V \otimes W$", see Section 10.1 D of [CP95]. In essence, we use the injection from Remark D.10 to allow X to act as E, Y act as F and $e^{\frac{1}{2}hH\otimes H}$ to act as the scalar $q^{\langle\lambda,\mu\rangle}$ or $\epsilon^{\langle\lambda,\mu\rangle}$ on $v\otimes w$, where v is of weight λ and w is of weight μ (using "weight" as in Remark D.24). The U-modules are weight modules, so $e^{\frac{1}{2}hH\otimes H}$ is defined to act on all of $V\otimes W$. Also, sufficiently high powers of E and F will annihilate finite-dimensional modules, so R acts as a finite sum and no topology is required on $V\otimes W$.

We point out that $q^{\langle\lambda,\mu\rangle}$ is not in the ground field $\mathbb{C}(q)$ when $\langle\lambda,\mu\rangle$ is half integral. Thus, we must extend the base field from $\mathbb{C}(q)$ to $\mathbb{C}(q^{\frac{1}{2}})$ and the base ring from $\mathbb{Z}[q,q^{-1}]$ to $\mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$; fortunately it makes no difference to have done this from the start. (An alternative approach is to redefine $q=s^2$, for another formal variable s, as done in [Saw06].) For the specialised case, we must choose a branch sheet for ϵ, that is, recognise that $\epsilon^{\frac{n}{2}}=e^{\frac{n}{2}\log\epsilon}$, for all $n\in\mathbb{Z}$, for some fixed logarithm of ϵ. In the root of unity case, we already had chosen the ℓ^{th}-primitive root of unity, $\epsilon=e^{2\pi i/\ell}$, so we will use $\epsilon^{\frac{n}{2}}=\epsilon^{n\pi i/\ell}$. △

We will now extend Proposition D.33 to ribbon structures.

PROPOSITION D.35. The braided monoidal category $U-\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type}\ 1}$ is a ribbon category with the ribbon structure

$$\theta_V : V\to V,\qquad \theta_V(v)=\nu^{-1}v, \tag{D.36}$$

where ν^{-1} acts on V with only finitely many terms non-zero.

Proof. We can rewrite the ribbon element in U_h as

$$\begin{aligned}
\nu &= e^{-hH}\nabla(S\otimes\mathrm{id})R_{21}\\
&= e^{-hH}\nabla(S\otimes\mathrm{id})\sum_{k\geq 0}\sum_{n\geq 0}\frac{(\frac{1}{2}h)^k}{k!}e^{\frac{n(n-1)}{2}h}\frac{(e^h-e^{-h})^n}{[n]_{e^h}!}(H^kX^n)\otimes(H^kY^n)\\
&= e^{-hH}\sum_{k\geq 0}\sum_{n\geq 0}\frac{(\frac{1}{2}h)^k}{k!}e^{\frac{n(n-1)}{2}h}\frac{(e^h-e^{-h})^n}{[n]_{e^h}!}(S(Y))^n(-H)^kH^kX^n\\
&= e^{-hH}\sum_{n\geq 0}e^{\frac{n(n-1)}{2}h}\frac{(e^h-e^{-h})^n}{[n]_{e^h}!}(S(Y))^ne^{-\frac{1}{2}hH^2}X^n\\
&= e^{-hH}\sum_{n\geq 0}e^{\frac{n(n-1)}{2}h}\frac{(e^h-e^{-h})^n}{[n]_{e^h}!}(-e^{hH}Y)^ne^{-\frac{1}{2}hH^2}X^n.
\end{aligned}\tag{D.37}$$

Via the injection from Remark D.10, we let X and Y act on U-modules as E and F, respectively. Since h and H appear only in exponentials, we can replace their actions with $e^{\frac{1}{2}h}\mapsto q^{\frac{1}{2}}$, $e^{\pm hH}\mapsto K^{\pm 1}$ and $e^{-\frac{1}{2}hH^2}\mapsto q^{-\frac{1}{2}H^2}$, where $q^{-\frac{1}{2}H^2}$ acts on the λ-weight space as multiplication by $q^{-\frac{1}{2}\langle\lambda,\alpha\rangle^2}$. That is to say, ν acts as in element in $U_{\mathcal{A}}$, recalling that we have redefined $\mathcal{A}=\mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$. The action of E^n will annihilate all vectors in the module for sufficiently large n, so ν will act with only finitely many non-zero terms. Hence, ν acts on all type 1 finite-dimensional U-modules. Since ν is invertible and central in U, ν and its inverse acts as an automorphism, which is needed for twists. Furthermore, the E, F and K actions and weights are preserved by U-module homomorphisms, so (D.36) is a natural isomorphism. The same arguments hold when q is specialised to ϵ.

Recall from (C.14) that, as a ribbon element on in U_h, we have $\Delta(\nu^{-1}) = (\nu^{-1} \otimes \nu^{-1})(R_{21}R)$ and $S(\nu^{-1}) = \nu^{-1}$. That is, (D.36) and (D.35) satisfy the ribbon conditions:

$$\theta_{V\otimes W} = (\theta_V \otimes \theta_W) \circ c_{W,V} \circ c_{V,W} \qquad \text{and} \qquad (\theta_V)^* = \theta_{V^*}. \tag{D.38}$$

Thus, $(U\text{--}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}, c, \theta)$ is a ribbon tensor category. □

Recall the canonical pivotal structure of a ribbon tensor category from Definition/Proposition 4.37. The natural isomorphism u in (4.38) is exactly the action by the element $u = \nabla((S \otimes \mathrm{id})R_{21})$ in (D.7). Since the ribbon structure of $U_\epsilon^{\mathrm{res}}\text{--}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}$ is given by the action of $(e^{-hH}u)^{-1}$ (the inverse of the ribbon element (D.7) in U_h), the pivotal structure is given by the action of $u(e^{-hH}u)^{-1} = e^{hH} = K$. Note that this is equivalent to giving $U_\epsilon^{\mathrm{res}}$ the structure of a pivotal Hopf algebra (from Example 4.24) with the pivot K, hence giving $U_\epsilon^{\mathrm{res}}\text{--}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}$ the pivotal structure a^K, as defined in (4.27).

Assume we are given a module V in $U_\epsilon^{\mathrm{res}}\text{--}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}$. We can choose a basis $\{v_i\}_{i=1}^{\dim_{\mathbb{C}} V}$ of weight vectors, with the dual basis $\{v^i\}_{i=1}^{\dim_{\mathbb{C}} V}$. Then, the (left categorical) dimension of V is

$$\dim^L_{a^K}(V) = \sum_{i=1}^{\dim_{\mathbb{C}} V} v^i(Kv_i) = \sum_{i=1}^{\dim_{\mathbb{C}} V} \lambda_i, \quad \text{where } \lambda_i \text{ is the } K\text{-eigenvalue of } v_i. \tag{D.39}$$

REMARK D.36. Despite being formulated differently, our notion of trace and dimension (by promoting $(U_\epsilon^{\mathrm{res}}, K)$ to a pivotal Hopf algebra) coincides with the usual definition of quantum trace and quantum dimension (for example, Section 11.3B of [CP95]). △

In Chapter 6, we are interested in constructing a pre-modular category of U-modules. But, $U\text{--}\mathsf{Mod}_{\mathsf{fd}}^{\mathsf{type\ 1}}$ may have an infinite number of isomorphism classes of indecomposable modules. In Appendix E, we discuss a process called *semisimiplification* which can potentially reduce the number of isomorphism classes of indecomposable objects in a tensor category, while still retaining much of its structure.

Appendix E

Semisimplification

Semisimplifications of pivotal and spherical categories were introduced in [BW99] to obtain semisimple categories from representations of Hopf algebras. In Chapter 6, we will need such a notion to produce a pre-modular category from a non-semisimple, non-finite ribbon tensor category. Proofs for any of the following statements that are not proven here can be found in [EO]. Recall that the definitions relating to pre-modular categories can be found in Chapter 4.

E.1 Tensor ideals and quotients

A quotient category can be made from a category by retaining the object class and quotienting the hom-classes by certain equivalence relations. The following definition provides a notion of a quotient that is compatible with $\Bbbk$-linear monoidal structure.

DEFINITION E.1. Let $\Bbbk$ be an field and let $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a $\Bbbk$-linear monoidal category. A *tensor ideal I* in $\mathscr{C}$ is a collection of subspaces

(E.1) $$I(X,Y) \subseteq \hom_{\mathscr{C}}(X,Y) \qquad \text{for all } X,Y \in \mathrm{ob}(\mathscr{C}),$$

satisfying the following conditions, for all $W,X,Y,Z \in \mathrm{ob}(\mathscr{C})$:

(i) for all $f \in I(X,Y)$, $g \in \hom_{\mathscr{C}}(Y,Z)$, $h \in \hom_{\mathscr{C}}(W,X)$,

(E.2) $$g \circ f \in I(X,Z) \qquad \text{and} \qquad f \circ h \in I(W,Y),$$

(ii) for all $f \in I(X,Y)$, $g \in \hom_{\mathscr{C}}(W,Z)$,

(E.3) $$f \otimes g \in I(X \otimes W, Y \otimes Z) \qquad \text{and} \qquad g \otimes f \in I(W \otimes X, Z \otimes Y).$$

That is, I is an ideal for composition and tensors. A tensor ideal allows us to quotient the $\Bbbk$-linear hom-spaces of $\mathscr{C}$ in a way that is well-defined on tensors.

REMARK E.2. Note that in Definition E.1, by $\Bbbk$-linear we only require that $\mathscr{C}$ is *additive* with $\Bbbk$-linear structured hom-spaces and bilinear composition. We do not yet need to require that $\mathscr{C}$ is $\Bbbk$-linear abelian with a bilinear tensor product. $\triangle$

DEFINITION/PROPOSITION E.3. Let I be a tensor ideal in $\mathscr{C}$. The *quotient* of $\mathscr{C}$ by I is the category $\overline{\mathscr{C}} = \mathscr{C}/I$ consisting of the following:

(i) the objects $\mathrm{ob}(\overline{\mathscr{C}}) = \mathrm{ob}(\mathscr{C})$,

(ii) the morphisms $\hom_{\overline{\mathscr{C}}}(X,Y) = \hom_{\mathscr{C}}(X,Y)/I(X,Y)$, for all $X, Y \in \mathrm{ob}(\mathscr{C})$.

(iii) composition $[g] \circ [f] = [g \circ f]$, for all $[f] \in \hom_{\overline{\mathscr{C}}}(X,Y)$, $[g] \in \hom_{\overline{\mathscr{C}}}(Y,Z)$.

(iv) identity morphisms $\mathrm{id}_X = [\mathrm{id}_X]$, for all $X \in \mathrm{ob}(\mathscr{C})$.

The canonical quotient functor $Q : \mathscr{C} \to \overline{\mathscr{C}}$ is the identity on objects and the quotient map on morphisms.

REMARK E.4. The composition is well-defined since I is an ideal on composition. The quotient category $\overline{\mathscr{C}}$ canonically adopts the $\Bbbk-\mathrm{Vect}$-enriched structure since the new hom-spaces are quotient $\Bbbk$-linear spaces. A zero object in $\mathscr{C}$ is a zero object in $\overline{\mathscr{C}}$ since $\hom_{\overline{\mathscr{C}}}(0,X) = 0/I(0,X) = 0$ and $\hom_{\overline{\mathscr{C}}}(X,0) = 0/I(X,0) = 0$ for each object X in $\mathscr{C}$. In fact, in Proposition E.11 we will see that non-zero objects in $\mathscr{C}$ can become zero objects in $\overline{\mathscr{C}}$. Suppose we have $(X_1 \oplus X_2, \pi_i : X_1 \oplus X_2 \to X_i, \iota_i : X_i \to X_1 \oplus X_2)$, a biproduct of objects X_1 and X_2 in $\mathcal{C}$. Let $X[\oplus]Y = X \oplus Y$, $[\pi]_i = [\pi_i]$, $[\iota]_i = [\iota_i]$. Then,

$$[\pi]_i \circ [\iota]_j = [\pi_i \circ \iota_j] = \begin{cases} [\mathrm{id}_{X_i}] & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{E.4}$$

which is sufficient for a biproduct. So, $\overline{\mathscr{C}}$ is $\Bbbk$-linear additive. Since Q is a quotient map on morphisms, it is $\Bbbk$-linear and hence additive. $\triangle$

REMARK E.5. Tensor ideals are defined to ensure the quotient category inherits the monoidal structure $(\overline{\mathscr{C}}, [\otimes], \mathbb{1}, [\alpha], [\lambda], [\rho])$. The tensor product is defined as

$$-[\otimes]- : \overline{\mathscr{C}} \times \overline{\mathscr{C}} \to \overline{\mathscr{C}}, \quad X[\otimes]Y = X \otimes Y \text{ and } [f][\otimes][g] = [f \otimes g]. \tag{E.5}$$

The tensor product is well-defined and bilinear since I is an ideal on the tensor product of morphisms. The unit object of $\overline{\mathscr{C}}$ is the unit object of $\mathscr{C}$. The associator and unitors of $\overline{\mathscr{C}}$ are the quotient of the associator and unitors of $\mathscr{C}$. That is,

$$[\alpha]_{X,Y,Z} = [\alpha_{X,Y,Z}] : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z, \tag{E.6}$$

$$[\lambda]_X = [\lambda_X] : \mathbb{1} \otimes X \to X \quad \text{and} \quad [\rho]_X = [\rho_X] : X \otimes \mathbb{1} \to X. \tag{E.7}$$

These are isomorphisms since Q is a functor. The naturality of $[\alpha]$, $[\lambda]$ and $[\rho]$ comes from applying the quotient to the naturality squares of α, λ and ρ in $\mathscr{C}$, and using the functoriality of Q. Similarly, we can apply the quotient to the pentagon and triangle identities, thus giving $\overline{\mathscr{C}}$ its adopted monoidal structure. This also makes (Q, J, φ) a monoidal functor with J and φ as identities. $\triangle$

PROPOSITION E.6. If $\mathscr{C}$ is rigid, pivotal, spherical, braided or ribbon, then so is $\overline{\mathscr{C}}$.

Proof. The functor (Q, J, φ) is monoidal, so it takes left duals to left duals, and similarly for right duals (see Proposition 4.7). So, the rigidity of $\mathscr{C}$ transfers onto $\overline{\mathscr{C}}$, since Q is surjective on objects.

If $a : \mathrm{id}_{\mathscr{C}} \Rightarrow (-)^{**}$ is a pivotal structure for $\mathscr{C}$, then we can define a pivotal structure for $\overline{\mathscr{C}}$ with components $[a]_X = [a_X] : X \to X^{**}$. Then, for all objects X, Y and morphisms $f : X \to Y$ in $\mathscr{C}$, we have

$$[a]_{X[\otimes]Y} = [a_{X\otimes Y}] = [a_X \otimes a_Y] = [a_X][\otimes][a_Y] = [a]_X[\otimes][a]_Y, \tag{E.8}$$

and, since Q is a monoidal functor, the commutativity of

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a_X"'] & Y \arrow[d, "a_Y"] \\ X^{**} \arrow[r, "f^{**}"'] & Y^{**} \end{tikzcd} \quad \text{gives the commutativity of} \quad \begin{tikzcd} X \arrow[r, "{[f]}"] \arrow[d, "{[a]_X}"'] & Y \arrow[d, "{[a]_Y}"] \\ X^{**} \arrow[r, "{[f^{**}]=[f]^{**}}"'] & Y^{**} \end{tikzcd}. \tag{E.9}$$

Note that $[f^*] = [f]^*$, since Q is monoidal. So, $[a]$ is a pivot for $\overline{\mathscr{C}}$.

The trace of the image is the image of the trace since, for all endomorphisms $[f] : X \to X$ in $\overline{\mathscr{C}}$, we have that

$$\begin{aligned} \mathrm{Tr}^L_{[a]}([f]) &= [\mathrm{ev}_{X^*}] \circ ([a]_x[\otimes][\mathrm{id}_{X^*}]) \circ ([f][\otimes][\mathrm{id}_{X^*}]) \circ [\mathrm{coev}_X] \\ &= [\mathrm{ev}_{X^*} \circ (a_x \otimes \mathrm{id}_{X^*}) \circ (f \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X] = [\mathrm{Tr}^L_a(f)]. \end{aligned}$$

It follows that when $\mathscr{C}$ is spherical we have

$$\dim^L_{[a]}(X) = \mathrm{Tr}^L_{[a]}([\mathrm{id}_X]) = [\mathrm{Tr}^L_a(\mathrm{id}_X)] = [\mathrm{Tr}^L_a(\mathrm{id}_{X^*})] = \mathrm{Tr}^L_{[a]}([\mathrm{id}_{X^*}]) = \dim^L_{[a]}(X^*),$$

for all objects X in $\mathcal{C}$. So, $\overline{\mathscr{C}}$ is spherical as well.

Assume that c is a braiding on $\mathscr{C}$. Then, we can define a braiding on $\overline{\mathscr{C}}$ with components $[c]_{X[\otimes]Y} = [c_{X\otimes Y}] : X \otimes Y \to Y \otimes X$. We can apply the quotient functor to the hexagon identities in $\mathscr{C}$ to obtain the hexagon identities in $\overline{\mathscr{C}}$. Furthermore, we obtain the naturality of $[c]$ by applying the quotient functor to the naturality squares in $\mathscr{C}$.

Assume that θ is a ribbon structure for $\mathscr{C}$. Then, $[\theta]_X = [\theta_X]$ is a ribbon structure for $\overline{\mathscr{C}}$ since it is a natural isomorphism and, for all objects X and Y in $\mathscr{C}$, we have

$$[\theta]_{X\otimes Y} = [\theta_{X\otimes Y}] = [(\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}] = ([\theta]_X[\otimes][\theta]_Y) \circ [c]_{Y,X} \circ [c]_{X,Y}$$

and $[\theta_X]^* = [\theta_X^*] = [\theta_{X^*}] = [\theta]_{X^*}$. So, $\overline{\mathscr{C}}$ naturally adopts the ribbon structure from $\mathscr{C}$. $\square$

REMARK E.7. We have just seen that quotienting by a tensor ideal preserves *a lot* of structure and does so in a very natural way. We will see that a certain type of quotienting can change the abelian structure in such way that it creates a semisimple category. The number of simple objects can decrease as well, potentially constructing a finite semisimple ribbon tensor category (that is, a pre-modular category) out of a non-finite non-semisimple ribbon tensor category. $\triangle$

E.2 Semisimplification of pivotal tensor categories

We will see that the pivotal structure of a pivotal tensor category naturally produces a tensor ideal and the quotient is a semisimple pivotal category.

In what follows, let $\Bbbk$ be an algebraically closed field and $\mathscr{C}$ a pivotal tensor category over $\Bbbk$ with pivotal structure a. We use Tr_a to denote either Tr_a^L or Tr_a^R, and use $\dim_a$ to denote either $\dim_a^L$ or $\dim_a^R$.

DEFINITION E.8. Let $f : X \to Y$ be a morphism in $\mathscr{C}$. We say that f is *negligible* if it satisfies

(E.10) $$\mathrm{Tr}_a(f \circ g) = 0 \qquad \text{for all } g \in \hom_{\mathscr{C}}(Y, X).$$

Let $\mathcal{N}(\mathscr{C})$ denote the collection of subspaces of negligible morphisms in $\mathscr{C}$.

LEMMA E.9. Let $X = \bigoplus_i X_i$ and $Y = \bigoplus_j Y_j$ be objects in $\mathscr{C}$ decomposed into indecomposable objects X_i and Y_j. Let $f = \bigoplus_{i,j} f_{i,j} : X \to Y$ be a morphism decomposed into $f_{i,j} : X_i \to Y_j$. Then, f is negligible if and only if for each i, j we have any one of the following:

$$\dim_a(X_i) = 0, \quad \dim_a(Y_j) = 0, \quad \text{or} \quad f_{i,j} \text{ is not an isomorphism.}$$

LEMMA E.10. The collection $\mathcal{N}(\mathscr{C})$ is a tensor ideal in $\mathscr{C}$.

PROPOSITION E.11. Let $\overline{\mathscr{C}}$ be the quotient of $\mathscr{C}$ by $\mathcal{N}(\mathscr{C})$. Then, $\overline{\mathscr{C}}$ is a semisimple tensor category. Furthermore,

(i) every indecomposable object in $\overline{\mathscr{C}}$ is $\overline{\mathscr{C}}$-isomorphic to some indecomposable object in $\mathscr{C}$,

(ii) if X is an indecomposable object in $\mathscr{C}$ of zero dimension, then X is a zero object in $\overline{\mathscr{C}}$,

(iii) if X is an indecomposable object in $\mathscr{C}$ of non-zero dimension, then X is a simple object in $\overline{\mathscr{C}}$.

Proof. We will show that $\overline{\mathscr{C}}$ is abelian after we first show (i) - (iii). At the moment we mean *indecomposable* (i.e. a non-zero object that is not isomorphic to a direct sum of two non-zero objects) and *simple* (i.e. a non-zero object with its only subobjects isomorphic to zero and itself) in the additive sense only.

(i) Assume that X is an indecomposable object in $\overline{\mathscr{C}}$. Since the direct sum in $\overline{\mathscr{C}}$ is adopted from the direct sum in $\mathscr{C}$, we can first consider a decomposition of $X \cong_{\mathscr{C}} \bigoplus_i X_i$ into indecomposable objects. Since isomorphisms are preserved by functors, we have $X \cong_{\overline{\mathscr{C}}} \underline{\bigoplus_i X_i} = [\bigoplus]_i X_i$. But X is indecomposable in $\overline{\mathscr{C}}$, so all the X_i are zero-objects in $\overline{\mathscr{C}}$, except for exactly one, say $X_j \cong_{\overline{\mathscr{C}}} X$. That is, X is $\overline{\mathscr{C}}$-isomorphic to an indecomposable object in $\mathscr{C}$.

(ii) Let X be an indecomposable object in $\mathscr{C}$ with $\dim_a(X) = 0$. Let $f : X \to Y$, $g : Z \to X$ be morphisms. Then, by Lemma E.9, f and g are negligible. That is to say, $\hom_{\overline{\mathscr{C}}}(X, Y) = 0$ and $\hom_{\overline{\mathscr{C}}}(Z, X) = 0$, making X a zero object in $\overline{\mathscr{C}}$.

(iii) Let X be an indecomposable object in $\mathscr{C}$ with $\dim_a(X) \neq 0$. To find if X is simple in $\overline{\mathscr{C}}$ we need to compare it to every indecomposable object in $\overline{\mathscr{C}}$. By (i) and (ii), we only need to check non-zero dimension indecomposable objects in $\mathscr{C}$. Assume that Y is a non-zero dimension indecomposable object in $\mathscr{C}$ and $[f] : Y \to X$ is a morphism in $\overline{\mathscr{C}}$. By Lemma E.9, $\dim_a(X) \neq 0$ and $\dim_a(Y) \neq 0$, so $[f] \neq 0$ implies f is an isomorphism in $\mathscr{C}$, hence $[f]$ is an isomorphism in $\overline{\mathscr{C}}$. So, the only subobjects of X in $\overline{\mathscr{C}}$ are isomorphic to X or zero objects in $\overline{\mathscr{C}}$.

By (ii) and (iii), any indecomposable object in $\mathscr{C}$ is either a zero object in $\overline{\mathscr{C}}$ or a simple object in $\overline{\mathscr{C}}$. By (i), we have that every indecomposable object in $\overline{\mathscr{C}}$ is simple. Hence, $\overline{\mathscr{C}}$ is semisimple in the additive sense.

We will now show that $\overline{\mathscr{C}}$ is abelian. Let X and Y be non-zero dimensional indecomposable objects in $\mathscr{C}$ and let $f : X \to Y$ be a morphism in $\mathscr{C}$. By Lemma E.9, we have

$$\dim_{\Bbbk}\left(\hom_{\overline{\mathscr{C}}}(X,Y)\right) = \begin{cases} 0 & \text{if } X \ncong_{\mathscr{C}} Y \text{ or equivalently } X \ncong_{\overline{\mathscr{C}}} Y, \\ 1 & \text{if } X \cong_{\mathscr{C}} Y \text{ or equivalently } X \cong_{\overline{\mathscr{C}}} Y. \end{cases} \tag{E.11}$$

So, $\overline{\mathscr{C}}$ is a $\Bbbk$-linear semisimple additive category such that equation (E.11) holds. Consider a simple $\Bbbk$-linear abelian category $\mathscr{D}$ with simple objects in one-to-one correspondence with isomorphism classes of simple objects in $\overline{\mathscr{C}}$. Since $\Bbbk$ is algebraically closed, the hom-spaces in $\mathscr{D}$ between simple objects are either zero or one dimensional, by Schur's lemma. Hence, we can construct an additive functor from $\mathscr{D}$ to $\overline{\mathscr{C}}$ by using the one-to-one correspondence of simple objects. This functor would also be an equivalence of categories, so it preserves limits and colimits, namely, kernels and cokernels. Hence, $\overline{\mathscr{C}}$ is a semisimple abelian category.

By Proposition E.6, $\overline{\mathscr{C}}$ is rigid. Since the tensor product $[\otimes]$ of $\overline{\mathscr{C}}$ is well-defined, it is still $\Bbbk$-bilinear on morphisms. In a tensor category, the unit object is simple (Proposition 4.32) and its endomorphism space is a one-dimensional $\Bbbk$-vector space. So, the only endomorphism that is not an isomorphism is zero. By (E.11), the quotient endomorphism space is a one-dimensional $\Bbbk$-vector space, hence $\overline{\mathscr{C}}$ is a tensor category. □

REMARK E.12. Proposition E.11 says that in order to know the simple objects in $\overline{\mathscr{C}}$, it is sufficient to know all the non-zero dimensional indecomposable objects in $\mathscr{C}$; we exploit this in Chapter 6.

Note that the abelian structure of $\overline{\mathscr{C}}$ does not directly come from the abelian structure of $\mathscr{C}$ in the sense that kernels (or cokernels) in $\overline{\mathscr{C}}$ are not the quotient image of the kernels (or cokernels) in $\mathscr{C}$. In fact, the proof does not require $\mathscr{C}$ to be abelian. Since we just need the additive structure and the pivotal structure, we can instead require $\mathscr{C}$ to be a full rigid monoidal subcategory of a pivotal tensor category such that $\mathscr{C}$ is closed under direct sums and direct summands. This will be useful for the semisimplification in Section 6.4. △

DEFINITION E.13. We call the quotient category $\overline{\mathscr{C}}$ the *semisimplification of* $\mathscr{C}$. The quotient functor is called the *semisimplification functor* $S : \mathscr{C} \to \overline{\mathscr{C}}$.

DEFINITION E.14. Let $S : \mathscr{C} \to \overline{\mathscr{C}}$ be a semisimplification. We say an object X in $\mathscr{C}$ *negligible* if $S(X)$ is a zero object in $\overline{\mathscr{C}}$. Equivalently, X is negligible if it is a zero object or it can decompose into a direct sum of indecomposable objects, each of zero categorical dimension.

REMARK E.15. The pivotal structure is needed for defining categorical traces and dimensions. If the pivotal structure is spherical, then left and right traces, and hence dimensions, coincide. Without this condition, these left and right semisimplifications may not coincide since left and right dimensions do not have to be zero at the same time. One could fix this by assuming that for all objects X in $\mathscr{C}$, $\dim_a^L(X) = 0$ if and only if $\dim_a^R(X) = 0$, but we are not concerned about this here. Instead, we will simply use the left trace. $\triangle$

REMARK E.16. If $\mathscr{C}$ is a ribbon tensor category, then we can define the canonical pivotal structure as in Definition/Proposition 4.37. Hence, every ribbon tensor category has a canonical semisimplification. In the case that $\overline{\mathscr{C}}$ is finite, the quotient image of the canonical pivotal structure is the canonical *spherical* structure of $\overline{\mathscr{C}}$ as a ribbon fusion (i.e. pre-modular) category. $\triangle$

By Proposition E.6, spherical structures, along with braiding and ribbon structures, are naturally carried across by the semisimplification process. In Chapter 6, we will use semisimplification to produce pre-modular categories from subcategories of modules of quantum groups.

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