D-Brane Charge Groups and Fusion Rings in Wess-Zumino-Witten Models

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Abstract

This thesis presents the computation and investigation of the charges and the corresponding charge groups for untwisted symmetry-preserving D-branes in a Wess-Zumino-Witten model over a compact, connected, simply-connected, simple Lie group. First, some general ideas from conformal field theory are reviewed and applied to Wess-Zumino-Witten models. Boundary conformal field theory is then introduced with the aim of deriving the Cardy constraint relating the consistent boundary conditions to fusion. This is used to justify certain dynamical processes for branes, called condensation, which lead to a conserved charge and constraints on the corresponding charge group (following Fredenhagen and Schomerus). These constraints are then used to determine the charge groups for untwisted symmetry-preserving branes over all compact, connected, simply-connected, simple Lie groups. Rigorous proofs are detailed for the Lie groups $\mathsf{SU}(r+1)$ and $\mathsf{Sp}(2r)$ for all ranks r, and the relevance of these results to K-theory is discussed. These proofs rely on an explicit presentation of the corresponding fusion rings (over $\mathbb{Z}$), which are also rigorously derived for the first time.

This computation is followed by a careful treatment of the Wess-Zumino-Witten model actions; the point being that the consistent quantisation paradigm developed can also be applied to brane charges to determine the charge groups. The usual (string-theoretic) D-brane charges are introduced, and are proved to exactly reproduce the charges of Fredenhagen and Schomerus when certain quantisation effects are brought into play. This is followed by a detailed investigation of the constraints induced on the corresponding charge groups by insisting that the string-theoretic charges be well-defined. These constraints are demonstrated to imply those of Fredenhagen and Schomerus except when the Wess-Zumino-Witten model is over a symplectic Lie group, $\mathsf{Sp}(2r)$. In the symplectic case, numerical computation shows that these constraints can be strictly stronger than those of Fredenhagen and Schomerus. A possible resolution is offered indicating why this need not contradict the K-theoretic interpretation.

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Contents

CHAPTER 1

Introduction

1.1. Background and Motivation

Wess-Zumino-Witten models are examples of string theories, although they need not necessarily satisfy all the consistency conditions that one expects [**90**, **131**, **132**]. The string field is given by a map g from the string worldsheet Σ to the target space. This is illustrated in Figure 1.1. What distinguishes Wess-Zumino-Witten models, and makes them particularly tractable, is that the target space is the underlying manifold of a Lie group G. This tractability allows for a more detailed study of various aspects of string theory than is usually possible on a topologically non-trivial target space. These models also provide convenient examples for testing new ideas in string theory. The action governing Wess-Zumino-Witten models will be constructed in Chapter 6 (where the reasons for their appellation will also become evident).

As string theories, Wess-Zumino-Witten models should be conformally invariant on Σ. That is, these models should define a conformal field theory on the string worldsheet, and this is also referred to as a Wess-Zumino-Witten model. These theories have a special place in the study of (two-dimensional) conformal field theories. They provide relatively accessible examples, with an extended symmetry algebra that is very well understood. Moreover, it is widely believed (though no proof has been offered) that Wess-Zumino-Witten models are the fundamental building blocks out of which all *rational* conformal field theories may be constructed (using the coset construction and "orbifolding" [**87**,**88**]). Wess-Zumino-Witten models will be introduced as conformal field theories in Chapter 3.

Both closed and open strings propogating on the Lie group G may be described by Wess-Zumino-Witten models. In the case of open strings, Σ has a (non-trivial) boundary, and it is necessary to impose boundary conditions at the endpoints in order to define the action consistently. The allowed boundary conditions may be associated with subspaces of G on which the open string endpoints are constrained to lie, and these subspaces are known as Dirichlet-branes (D-branes, or just branes, for short). The consistency of the open string action endows these branes with an interesting geometric structure, but this becomes trivial when pulled back to the worldsheet. In the conformal field theory, branes are only evidenced by the corresponding boundary conditions at $\partial\Sigma$.

Branes, as extended objects, were first considered in the late eighties. It was quickly realised [**50**] that these extended objects should be treated as dynamical objects in their own right (in some more elaborate theory). With the advent of the (still mysterious) M-theory, a setting was established to consider brane processes through various dualities,

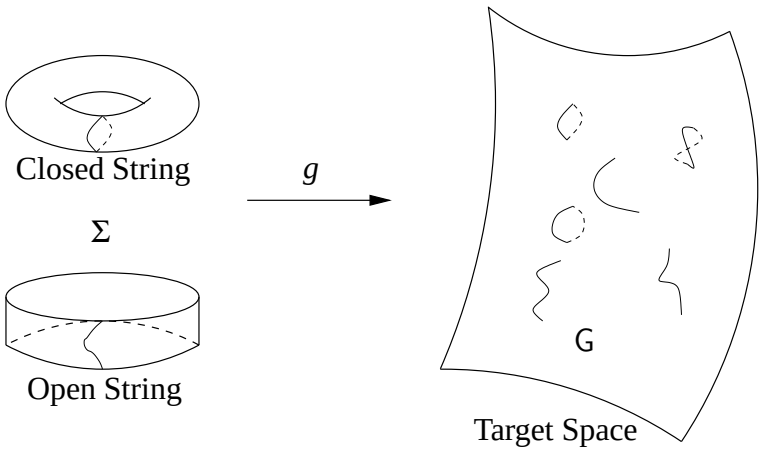

FIGURE 1.1. A schematic illustration of the string field g from the string worldsheet Σ to the target space (Lie group) G. An example of a closed string worldsheet ($\partial\Sigma = \emptyset$) and an open string worldsheet ($\partial\Sigma \neq \emptyset$) is given.

and thence conserved charges. In particular, Polchinski [**129**] used T-duality to identify D-branes as the missing sources of (Ramond-Ramond-)charge in type II string theories on a *flat space*. The brane charge then took the form [**130**]

$$Q = \int_{\text{brane}} e^F,$$

where F is a "$\mathsf{U}(1)$ field strength" (closed 2-form) on the brane. This suggests that brane charges should be classified by some cohomology group.

This conclusion was extended to curved spaces in [**89**, **117**]. On the basis of the cancellation of a certain quantum anomaly, it was suggested that the form of the brane charges must be generalised to

$$Q = \int_{\text{brane}} e^{F-\frac{1}{2}\mathrm{c}_1(N(\text{brane}))} \frac{\widehat{\mathrm{A}}\left(T(\text{brane})\right)}{i^*\sqrt{\widehat{\mathrm{A}}\left(T(\text{space})\right)}},$$

where $T(\cdot)$ and $N(\cdot)$ denote the tangent and normal bundles (respectively), i is the inclusion of the brane into the space, and $\mathrm{c}_1(\cdot)$ and $\widehat{\mathrm{A}}(\cdot)$ denote the first Chern class and the A-roof genus respectively (Appendix C.2). When space is a Lie group, this may be substantially simplified (see Section 7.1.3).

This expression for the brane charges may look rather imposing in all its generality, but it turns out (as remarked by Kontsevich and Segal) that it has a natural interpretation in terms of K-theory. Namely, that the integrand is the image under a modified Chern isomorphism of the class $i_!(E)$, where $i_!$ is the (K-theoretic) Gysin map associated with i, and E is the vector bundle defining the geometric structure of the brane. In other words, this result suggests that brane charges are more naturally classified by some K-group rather than a cohomology group.

Of course, the charge definition given above is written in terms of forms and therefore involves (real) deRham cohomology. The above K-theoretic observation then holds over

the real numbers (and K-theory is isomorphic to deRham cohomology in this case). However, quantisation is generally expected to lead to a corresponding quantisation of charges, in particular, to integral charges. In the fully quantised theory, it is therefore reasonable to suppose that brane charges are valued in some (integral) K-group. The question is now to determine which one.

In [**163**], Witten proposed various K-groups for the various string theories, and in particular, proposed that in the presence of a "Neveu-Schwarz B-field", the brane charges should take values in a *twisted* K-theory. Specifically, such a B-field has a globally defined field strength H which represents a degree-3 class in cohomology. When this class is pure torsion, Witten describes a twisted K-theory which is then proposed to classify brane charge. In a Wess-Zumino-Witten model on a simple Lie group, such a field strength H naturally arises (Section 6.2.2), but the corresponding cohomology class is not torsion. Generalising Witten's proposal to arbitrary field strengths, Bouwknegt and Mathai [**34**] proposed that brane charges should actually be classified by the K-theory of a certain C^*-algebra. This K-theory was first constructed by Rosenberg, and is also known as twisted K-theory. Their proposal reduces to that of Witten when the cohomology class of H is pure torsion.

Subsequently, a dynamical process for branes called condensation was investigated by Alekseev, Recknagel and Schomerus [**5**]. Applying this process to certain branes in a Wess-Zumino-Witten model, Fredenhagen and Schomerus [**63**] were able to derive constraints that any brane charge must satisfy. These constraints yielded a remarkably simple expression for the charge, and for the models based on the Lie group $\mathsf{SU}(r+1)$, were used to predict the form of the group that the charges take values in. In other words, the analysis of this charge and constraints led to highly non-trivial predictions for the twisted K-groups of $\mathsf{SU}(r+1)$ (only the twisted K-group for $r=1$ was at that time known), assuming the proposed classification of brane charges is correct.

A primary motivation for the work of this thesis is to extend the analysis of the constraints of Fredenhagen and Schomerus to other Wess-Zumino-Witten models, and therefore derive predictions for the corresponding twisted K-groups. These predictions are not only of interest in themselves, but provide non-trivial checks of the proposal of Bouwknegt and Mathai when the K-groups are rigorously computed. It is also worth pointing out that the conserved brane charges constructed by Fredenhagen and Schomerus bear little resemblance to the string-theoretic charges discussed above. Another motivation for this thesis is then to investigate the relationship between these charges, and thus link the charges of Fredenhagen and Schomerus with the proposed twisted K-theory classification.

1.2. Overview

This thesis consists of a detailed study of conserved charges for certain branes in Wess-Zumino-Witten models. To be specific, the branes that will be studied are the *untwisted symmetry-preserving* Dirichlet-branes, defined in Sections 3.2.4 and 6.3.2, and the Wess-Zumino-Witten models will describe oriented strings propagating on a compact, connected, simply-connected, simple Lie group G. The following chapters give an account of my original research, published in [**33**, **35**], as well as unpublished material which will be described below. For motivation, personal reasons, and some measure of completeness, this thesis also contains reviews of certain areas. I hope that the approach taken in these reviews will not detract from the thesis as a whole. Indeed, I would like to think that I have attempted to introduce the necessary concepts and theory in a manner which complements, rather than reiterates, the treatments in the literature, at least when not inconvenient.

My publications are divided in their different approaches to the study of brane charges in Wess-Zumino-Witten models. Accordingly, this thesis is divided (perhaps artifically) into an account of the approach via conformal field theory and Lie algebras, and an account of the approach via the global topology of Lie groups. The former (which I have designated the *algebraic* approach) constitutes Chapters 2, 3, and 4, with the exception of Sections 4.1.1 and 4.3.2, and the latter (*geometric* approach) constitutes Chapters 6 and 7. The interlude, Chapter 5, is dedicated to the study of fusion in Wess-Zumino-Witten models. The reader could treat this as an appendix to the calculations of Section 4.2, but I prefer to believe that its results are of significant interest in themselves, and therefore warrant their inclusion as a separate chapter. Of course, by making the divisions outlined above, the occasional forward reference becomes unavoidable. Hopefully, this will not prove troublesome — this thesis is not intended to be read in a strictly sequential manner.

Chapter 2 gives a brief account of the principles of conformal field theory which will be used in the chapter following. It serves to establish the notation (and conventions) which will be used, and most importantly, motivates the fusion process which proves to be of fundamental importance to this thesis. Instead of following the example-oriented, field-theoretic approach to conformal field theory that is ubiquitous in the literature, this chapter introduces the relevant theory in a representation-theoretic manner. In particular, fields are introduced as an auxiliary (though highly useful) construct through the state-field correspondence of the theory of vertex operator algebras. I believe there is some advantage in such a description, in that by concentrating the readers attention on representations from the start, the idea of null fields (and the correlation function constraints they impose) becomes transparent. Of course, this is merely a matter of taste.

In any case, Chapter 3 introduces the Wess-Zumino-Witten models on G as conformal field theories. As a detailed discussion of the defining action of these models requires topological notions, these models are defined by *postulating* conserved (chiral) currents,

which take values in $\mathfrak{g}$, the Lie algebra of G. From these, the symmetry algebra is derived and an energy-momentum field is constructed, which verifies the conformal nature of Wess-Zumino-Witten models (these computations are standard). After establishing the representations comprising the quantum state space, the notion of fusion for these models is examined as in Chapter 2. In particular, the concept of a fusion multiplicity is thoroughly scrutinised.

The second half of this chapter deals with boundary conformal field theory. The conformal boundary condition is introduced and the consistent boundary conditions on the Wess-Zumino-Witten conserved currents are derived. These boundary conditions correspond to branes as will be explained in Chapter 6. The focus of this thesis is on those boundary conditions which preserve the maximal amount of symmetry, and these are shown to be locally constant boundary conditions associated with automorphisms of $\mathfrak{g}$. Quantum states satisfying these boundary conditions are then sought, following the seminal work of Ishibashi and Cardy. However, I have chosen to explicitly search for such states in the space dual to the quantum state space. Rather than give a solution which may be then be verified, it is quite easy to derive the solutions as distributions (functionals) in all generality. The chapter concludes with an account of Cardy's constraint, linking the consistent functionals with the fusion multiplicities (or generalisations thereof).

Chapter 4 begins the study of brane charges. I start by briefly sketching some of the work of Alekseev, Recknagel and Schomerus on brane dynamics, in particular, the process known as condensation. This process may be related to the fusion multiplicities through Cardy's constraint. This is followed by a more detailed account of the charges conserved under brane condensation, and leads to what I have called the fusion constraints on the charge groups. These constraints are due to Fredenhagen and Schomerus, and I give their result for the charge group of a particular class of branes on the Lie group $\mathsf{SU}(r+1)$.

Sections 4.2 and 4.3 then describe the research published in [**33**]. After abstracting the above approach to charge groups, I show how an explicit knowledge of the fusion ring of the Wess-Zumino-Witten model may be used to compute the brane charge group. Using the fusion potential of Gepner, I detail this computation for the Wess-Zumino-Witten models over $\mathsf{SU}(r+1)$, easily reproducing the result of Fredenhagen and Schomerus. I also show how an omitted step in their derivation can be proved.

A fusion potential for the Wess-Zumino-Witten models over the symplectic groups $\mathsf{Sp}(2r)$ is also well-known. I then use this to compute the corresponding charge groups in this case. Unfortunately, no analogous fusion potential appears to be known for the other compact, connected, simply-connected, simple Lie groups. For these cases, I therefore present the charge groups which were suggested by extensive numerical computation. This is followed by an investigation of the symmetries observed in these brane charges, and a brief discussion of the implications of these charge groups for twisted K-theory.

Chapter 5 is devoted to a detailed study of the fusion process in Wess-Zumino-Witten models over compact, connected, simply-connected, simple Lie groups. In particular, to

studying the fusion potentials which were used in Chapter 4 to compute the brane charge groups. I begin with a simple discussion of the fusion multiplicities, and prove the Kac-Walton formula and a result of Gepner in a manner I think is quite transparent. I then discuss fusion potentials and some related, rather unfortunate, shortcomings in the literature. In particular, I point out that the frequent assertion that Gepner's result characterises the fusion ring is incorrect, even when the scalar ring is the complex numbers[1].

Section 5.3 consists of my solutions to these shortcomings. I use a little commutative algebra to write down a quite general presentation for the fusion ring (over the integers). For $\mathsf{SU}(r+1)$, I then show how the theory of symmetric polynomials lets me rewrite this presentation in a form equivalent to that given by the fusion potential. For $\mathsf{Sp}(2r)$, the same technique works using a well-known generalisation of the theory of symmetric polynomials. It follows that in both cases, I have a rigorous *derivation* of the fusion potentials (over $\mathbb{Z}$) from general principles. I am not aware of any similar (complete) rigorous derivation, or even verification, in the literature. These derivations are followed by logically independent verifications that these fusion potentials correctly describe the fusion process over $\mathbb{C}$. Of course, this is already implied by the result over $\mathbb{Z}$. However, these (complete) verifications extend the usual procedure found in the literature and so should be of interest in their own right.

Section 5.4 then discusses how these rigorous results might generalise to the other compact, connected, simply-connected, simple Lie groups. This grew out of my attempts to find justifications for the numerically found brane charge groups of Chapter 4. However, for the derivations over $\mathbb{Z}$, a (seemingly undiscovered) generalisation of the theory of symmetric polynomials appears to be needed, and over $\mathbb{C}$, I show that the type of fusion potential which worked before *cannot* work generally for any other case. Sections 5.3 and 5.4 therefore constitute original research, but this has not been published as yet.

In Chapter 6, I finally begin the geometric approach to Wess-Zumino-Witten models (as oriented strings). Here the closed string action is constructed and the terms of this action balanced so as to get the conserved currents postulated in Chapter 3. In keeping with the spirit that this approach must take matters of global topology into account, I develop a "global" variational method. Again, this is purely a matter of taste, but I find that this method emphasises the mathematical structure and leads to more concise derivations. For later purposes, I spend some time on the topological intricacies in defining the action, and in particular evaluating its ambiguities and their consequences.

[1] I should point out, however, that Gepner's original characterisation in [**83**] is completely correct when the scalar ring is $\mathbb{C}$. One typically uses this characterisation to verify a description of the fusion process (by a fusion potential for example). However, in every case that I am aware of (including the original), this verification does *not* seem to have been properly completed. What is generally called a characterisation of the fusion ring in the literature, and attributed to Gepner, is only a partial characterisation, and this is what I am saying is incorrect. Here, Theorem 5.4 is equivalent to Gepner's original result and does characterise fusion over $\mathbb{C}$. For application to brane charges, a characterisation of fusion over the integers is required, and I indicate immediately afterwards how Gepner's result is extended in this case.

This action is then generalised to open strings which correspond to the boundary conformal field theory description of Chapter 3. The form of the boundary term that must be added to the action is motivated by a careful consideration of how the original action can be defined in this case. The boundary conditions of Chapter 3 then lead to the geometric interpretation of (Dirichlet)-branes, and it is shown how each boundary condition specifies the boundary term of the action (and conversely). Intriguingly, I show in Proposition 6.3 that when the boundary condition is locally constant and associated with an automorphism of $\mathfrak{g}$ (as was required to preserve the maximal amount of symmetry), the construction of the open string action is internally consistent. For a general boundary condition, this construction seems[2] to be inconsistent, and I give an example which demonstrates this. As before, I finish by carefully examining the ambiguities inherent in defining this action, and their consequences.

Chapter 7 consists of a discussion of brane charges and charge groups in this geometric approach. Except where explicitly indicated (descriptions of other's work), this entire chapter constitutes original research, much of which has appeared in [**35**]. I begin by noting that the results of the previous chapter are semiclassical, and that in the quantised theory, one expects certain *quantum shifts*. One of these is standard and well-known, but there is a second which seems to receive far less attention than it should. This is puzzling as the two naturally go together from a Lie-theoretic point of view. I follow this with a detailed study of flux stabilisation. The physical interpretation of this is not dwelt upon, being adequately described in the literature, but I go to some trouble to define what this means mathematically. In particular, it turns out that the $\mathsf{U}(1)$-flux describing this can only be defined up to an easily characterised ambiguity.

The brane charge obtained from this $\mathsf{U}(1)$-flux (*a la* Polchinski) is then evaluated for $\mathsf{SU}(2)$. Including *both* quantum shifts yields the same charges and charge groups that were obtained in Chapter 4. The same is true if only the standard quantum shift is used, but the modified brane charge of Minasian and Moore is used instead. I extend this coincidence of charges by computing them for $\mathsf{SU}(3)$ using Schubert theory, again finding that both geometric charges (incorporating the appropriate quantum shifts) coincide with the algebraic charge. In Theorems 7.2 and 7.4, I prove this rigorously for all compact, connected, simply-connected, simple Lie groups G. Demonstrating that the algebraic charge of Fredenhagen and Schomerus coincides with these geometric charges fills what I believe is an important gap in the literature.

This demonstration is followed by a detailed examination of the ambiguities inherent in the geometric charge definitions. In particular, I show that there are ambiguities which induce constraints on the brane charges which are equivalent to the fusion constraints of Fredenhagen and Schomerus, but only when G is *not* symplectic. In addition, the ambiguity in the $\mathsf{U}(1)$-flux yields constraints on the geometric brane charges which are

[2]Boundary conditions that are not associated with automorphisms do not necessarily lead to inconsistent actions, but it appears that most boundary conditions lead to an inconsistency. Unfortunately, I have not had the time or opportunity to pursue this intriguing result.

stronger than the fusion constraints when G is symplectic. I finish by evaluating these stronger constraints and the charge groups they suggest.

The thesis concludes with three appendices on Lie theory. Appendix A summarises some aspects of the theory of finite-dimensional simple Lie algebras which are used throughout the thesis. Appendix B does the same for the corresponding infinite-dimensional affine Lie algebras, as their theory is fundamental to Chapters 3 and 4. I have taken some trouble to make clear the structure of the group of affine outer automorphisms, its action on the weight space of the horizontal subalgebra, and its relation to the affine Weyl group. Unfortunately, it is not hard to find misleading and even erroneous statements in the literature concerning these points, so I thought it best to include my own understanding, as I use these results in Section 4.3.

Appendix C gives an account of some useful aspects of the theory of compact Lie groups. In particular, I give a detailed account of the centralisers of points in a compact, connected, simply-connected, simple Lie group G. The corresponding quotient spaces are the conjugacy classes of G (which are the worldvolumes of the untwisted symmetry-preserving branes), and I show that there exist conjugacy classes with torsion homology (refuting another claim commonly found in the literature). It is also convenient to include a few standard results from topology here, and collect a few useful results which I have used in the main body of the text.

CHAPTER 2

Conformal Field Theory

This chapter presents a brief introduction to some of the principles of conformal quantum field theory. It is not meant to constitute a comprehensive review, but is meant to refresh the readers memory and to establish notation used elsewhere in this thesis. In particular, one goal is to introduce the fusion process which will form a basic tool in what follows. There are several reviews of this subject available in varying degrees of sophistication [**68**, **74**, **86**, **141**]. The text [**61**] is an excellent source for much of the field, and other treatments which emphasise different ideas may be found in [**102**, **103**, **106**, **143**].

2.1. Conformal Invariance

2.1.1. Conformal Symmetry. A relativistic quantum field theory is generally expected to be invariant under the isometry group of the metric [**128**]. When the metric is that of Minkowski space, the group of such transformations is known as the Poincaré group, the (semidirect) product of the spacetime translations with the group of Lorentz transformations. There are, however, field theories admitting more general symmetries. In particular, free massless theories are (in certain dimensions) invariant under transformations which preserve the metric up to a non-zero scaling factor (which is generally a function of spacetime). It follows that such transformations preserve the "infinitesimal" angle between two curves; accordingly, they are termed *conformal transformations*, and constitute the group of conformal isometries of the spacetime (in general, of a riemannian manifold), hereafter referred to as the conformal group.

As with the Poincaré group, the conformal group of a given riemannian manifold may often be described explicitly. A first step towards such a description is to determine the infinitesimal conformal transformations, that is, the elements of the *conformal algebra*. For euclidean or Minkowski space of dimension $d > 2$, one finds that there are exactly $\frac{1}{2}(d+1)(d+2)$ such linearly independent infinitesimal transformations [**141**, **143**]. Amongst them, the infinitesimal generators of the Poincaré group may be distinguished, as well as a generator corresponding to a dilation. The other generators are termed "special". When $d = 2$, the situation changes unexpectedly, and one finds an infinite number of independent generators. In particular, when the metric has euclidean signature, the condition for an infinitesimal transformation to be conformal may be interpreted (with the aid of a complex structure) as the Cauchy-Riemann equations familiar from complex

function theory. As is well known [**4**], the solutions to these equations consist of holomorphic and antiholomorphic functions[1] (whence the infinite-dimensionality of the conformal algebra).

It should be mentioned that the dimension of the conformal group need not coincide with that of the conformal algebra. Which infinitesimal transformations lift to well-defined group elements is generally determined by the global topology of the riemannian manifold. For example [**45**], on the riemannian manifold $\mathbb{C}$, invertibility restricts the conformal transformations to those of the form $z \longmapsto az+b$, where $a,b \in \mathbb{C}$ and $a \neq 0$ (and the analogous antiholomorphic transformations). On the Riemann sphere, the conformal transformations are more numerous, taking the form

$$z \longmapsto \frac{az+b}{cz+d}, \qquad \text{where } a,b,c,d \in \mathbb{C} \text{ and } ad-bc \neq 0$$

(again with the analogous antiholomorphic transformations). However, the formalism of Noether symmetries only makes use of infinitesimal symmetries, so for field-theoretic purposes, one may safely ignore these topological restrictions (at least to a large extent), and work with the conformal algebra directly.

As such, it is prudent to determine the Lie algebra structure of the conformal algebra. For $\mathbb{R}^d$ with $d > 2$ and signature (p,q), the conformal algebra turns out to be $\mathfrak{so}(p+1,q+1)$. In the more interesting two-dimensional (euclidean) case, the infinitesimal elements are naturally expressed as vector fields $f\partial/\partial z + \overline{f}\partial/\partial\overline{z}$, where f and $\overline{f}$ are, respectively, holomorphic and antiholomorphic. Defining $\ell_n = -z^{n+1}\partial/\partial z$ and $\overline{\ell}_n = -\overline{z}^{n+1}\partial/\partial\overline{z}$ for $n \in \mathbb{Z}$, one finds that the (complexified) conformal algebra is (a topological completion of) two copies of the *Witt algebra*:

$$[\ell_n,\ell_m] = (n-m)\,\ell_{n+m}, \qquad \left[\ell_n,\overline{\ell}_m\right] = 0, \qquad \left[\overline{\ell}_n,\overline{\ell}_m\right] = (n-m)\,\overline{\ell}_{n+m}.$$

The real conformal algebra is spanned by the combinations $\ell_n + \overline{\ell}_n$ and $\mathrm{i}\left(\ell_n - \overline{\ell}_n\right)$. However, it is convenient to work with the complexification in what follows.

In a conformally invariant *quantum* field theory, symmetry generators get promoted to operators on the quantum state space. One might therefore expect that the quantum state space $\mathcal{S}$ admits a representation of (two copies of) the Witt algebra. However, quantum states are only defined up to a non-zero complex multiple. That is, the quantum state space proper is projective in nature. Technically then, it is this projective space $\mathcal{PS}$ that admits a representation of the Witt algebra. Of course, one usually works with the quantum state space as a vector space, rather than as a projective space, so it is convenient to lift this "projective" representation from $\mathcal{PS}$ to $\mathcal{S}$. One finds however [**16**, **143**], that a projective representation may be lifted to a representation of a *central extension* of the Witt algebra

[1] To be precise, the solutions consist of functions which may be *formally* viewed as depending only on the complex variable z or its conjugate $\overline{z}$ (here given the status of independent variables), or linear combinations thereof. In the physics literature, such functions are referred to as being holomorphic or antiholomorphic respectively. This conflicts with the usual mathematical definition wherein holomorphic equates to analytic — physicists certainly allow their holomorphic functions to have poles (and monodromy). This thesis will follow the convention of the physics literature.

on the vector space of quantum states. This leads to a representation on $\mathcal{S}$ of (two copies of) the unique [**16**, **101**] (up to isomorphism) non-trivial central extension of the Witt algebra, the *Virasoro algebra* (denoted by $\mathfrak{Vir}$):

$$[\![L_n, L_m]\!] = (n-m)\,L_{n+m} + \frac{1}{12}\left(n^3 - n\right)\delta_{n+m,0}C, \qquad [\![L_n, C]\!] = 0. \tag{2.1.1}$$

Here, C is the central element of $\mathfrak{Vir}$, and the bracket $[\![\cdot,\cdot]\!]$ is the commutator in $\mathsf{End}\,\mathcal{S}$ (to be distinguished from an abstract Lie bracket). If C were represented by the zero operator, then this would reduce to a representation of the Witt algebra.

2.1.2. The Quantum State Space. The quantum state space $\mathcal{S}$ is therefore a (complex) vector space admitting a representation of two copies of the Virasoro algebra. One now imposes the physical expectation that this representation is unitary, and that there is a unique quantum state of minimal energy. Here, unitary means that $\mathcal{S}$ admits an inner-product[2], $(\cdot,\cdot)$, and that $\mathfrak{Vir}\otimes\mathfrak{Vir}$ admits an antiautomorphism which is represented in $\mathsf{End}\,\mathcal{S}$ by the hermitian adjoint, denoted by †. The energy operator in a conformal field theory is given by the combination $L_0+\overline{L}_0$ (by which is meant $L_0\otimes\mathrm{id}+\mathrm{id}\otimes L_0$). Implicitly, this combination must be a self-adjoint operator on $\mathcal{S}$, whose spectrum is bounded below by a simple (unit multiplicity) eigenvalue.

It is standard practice in conformal field theory to completely separate the holomorphic and antiholomorphic components of the theory. Each half is then referred to as a chiral theory. In particular, one supposes that the above considerations apply to each chiral half separately, so that the quantum state space decomposes (*assuming complete reducibility*[3]) into

$$\mathcal{S} \cong \bigoplus_{i,j} \left(V_i \otimes V_j\right),$$

where V_i and V_j are unitary $\mathfrak{Vir}$-modules. It is also standard practice to now work with quantities from a single chiral theory (for notational convenience), trusting that the extension to the full theory will be clear from the knowledge of which V_i is tensored with which V_j (this knowledge is the content of the *modular invariant* — see Section 3.1.3).

The adjoint in the chiral theory is given by $L_n^\dagger = L_{-n}$ and $C^\dagger = C$, so the chiral energy operator may be taken as the (self-adjoint) L_0. Requiring the spectrum to be bounded from below by a simple eigenvalue is therefore equivalent to requiring that the V_i are *highest*

[2]Of course, the quantum state space may be completed with respect to the metric induced by this inner-product to get a bona-fide Hilbert space. As questions of topological completion have thus far been ignored (as in the relation of the Witt algebras and the conformal algebra), it is not clear that one must define the quantum state space to be a Hilbert space. Indeed, it seems reasonable to suppose that there is a nuclear topology [**18**, **81**, **82**] with respect to which one should complete (to get a *rigged* Hilbert space). Such a topology allows for desirable continuity properties of the unbounded operators one typically deals with in quantum theories. As such, questions regarding the continuity of quantum fields will be ignored in this thesis.

[3]Relaxing this assumption somewhat leads to the consideration of so-called *logarithmic* conformal field theories [**60**].

weight $\mathfrak{Vir}$-modules, and that over all i, there is a highest weight vector of minimal[4] energy, the *vacuum*. This minimal energy must be at least 0 (for unitarity), so the vacuum is usually assumed to have zero energy. It is traditionally denoted[5] by $|0\rangle$. One usually demands that the vacuum is unique in the full theory, meaning that $|0\rangle \otimes |0\rangle$ is the unique (up to constant multiple) eigenvector of the total energy operator, $L_0 + \overline{L}_0$, with eigenvalue 0.

The restriction to unitary highest weight modules is of some mathematical interest. Highest weight $\mathfrak{Vir}$-modules[6] are (partially) characterised by the eigenvalues of C and L_0 on the highest weight vector. These eigenvalues are respectively denoted by c and h; c is the *central charge* of the module, and h is its minimal energy. In fact, these parameters fully characterise *unitary* highest weight $\mathfrak{Vir}$-modules in the following sense [**101**]: If there exists a unitary highest weight $\mathfrak{Vir}$-module of given c and h, then it is unique (among such modules) and irreducible. It is necessary for unitarity that c and h be non-negative real numbers, though this is not sufficient. Indeed, for $c = 0$, there is only one h for which the irreducible highest weight module is unitary, $h = 0$, and this module is trivial (which is why it is the non-trivial central extension of the Witt algebra which plays the starring rôle in conformal field theory).

In summary then, the quantum state space of a (two-dimensional) conformal field theory is expected to have the form

$$\mathcal{S} \cong \bigoplus_{c,h,\overline{c},\overline{h}} \left(V_{c,h} \otimes V_{\overline{c},\overline{h}} \right), \tag{2.1.2}$$

where $V_{c,h}$ is the unitary (hence irreducible) highest weight $\mathfrak{Vir}$-module of central charge $c > 0$ and minimal energy $h \geqslant 0$. The combination $V_{c,0} \otimes V_{\overline{c},0}$ appears exactly once in Equation (2.1.2), and the corresponding highest weight vector is the vacuum of the theory, $|0\rangle \otimes |0\rangle$.

Of course, it is likely that any given example of a conformal field theory will have symmetries more general than the conformal symmetries studied in this section. This will be the case for the theories introduced in Section 3.1 and studied in the remainder of this thesis. In this case, one expects that these symmetries define an infinite-dimensional Lie algebra "extending" the Virasoro algebra, and that the quantum state space decomposition, Equation (2.1.2), may be reorganised into unitary irreducible highest weight representations of this *symmetry algebra*. Of course, these representations must also admit an

[4]This matching of highest weight vectors with minimal energies is another unfortunate result of the standard conventions employed in the literature. Lowest weight modules would be more consistent with this physical intuition, but Lie theorists (arbitrarily) have traditionally worked with highest weights. In any case, it is not difficult to remember, but it is annoying.

[5]In this thesis, a ket $|\cdot\rangle$ will be used to denote vectors in the quantum state space. The bra $\langle\cdot|$ then denotes the linear functional taking value 1 on the corresponding vector and 0 on any vector orthogonal to it (with respect to the appropriate inner product).

[6]A highest weight module is defined to be a quotient module of a Verma module [**99**]. The direct sum of two such modules is therefore not considered to be a highest weight module, even though it may possess a highest weight.

action of the Virasoro algebra, and may in turn be decomposed into (generally infinitely many) irreducible highest weight representations of $\mathfrak{Vir}$.

2.1.3. Conformal Fields. Thus far a conformal field theory has been seen to possess two sets of linear operators, each forming a representation of the Virasoro algebra $\mathfrak{Vir}$ on the state space $\mathcal{S}$, which is composed of unitary highest weight $\mathfrak{Vir}$-modules. The field content of the theory is not *a priori* evident in this abstract construction. In examples however, one generally encounters the fields first through quantisation of (for example) a classical field theory defined by a lagrangian. Fields are therefore operator-valued functions (more generally, distributions) on the classical spacetime. One usually then defines the symmetry algebra by considering the Noether charges corresponding to conserved current fields, and the state space is determined by asking (or choosing) which other fields are allowed in the theory.

This last point suggests that there should be some kind of correspondence between the states and the fields of the theory. This is unimaginatively termed the *state-field correspondence* (and has been elevated to the status of an axiom in the theory of vertex algebras [**100**]). Without being too precise[7], a chiral state $|\psi\rangle \in \mathcal{S}$ of definite energy h corresponds to a chiral field $\psi(z)$ of the form

$$\psi(z) = \sum_n \psi_n \, z^{-n-h}, \qquad \psi_n \in \operatorname{End}\mathcal{S}$$

(in the full theory, one would include an antiholomorphic dependence). The field $\psi(z)$ is said to have *conformal weight h*. This correspondence is then extended to general states in $\mathcal{S}$ by linearity. Generally, the field $\psi(z)$ determines the state $|\psi\rangle$ through the relation [**100**]

$$\psi(z)\,|0\rangle = e^{zL_{-1}}\,|\psi\rangle \qquad \Rightarrow \qquad \lim_{z\to 0}\psi(z)\,|0\rangle = |\psi\rangle\,. \tag{2.1.3}$$

Fields are therefore regular at the origin *when applied to the vacuum*. Conversely, a state $|\psi\rangle$ determines a unique field $\psi(z)$ if it is agreed that all fields corresponding to $0 \in \mathcal{S}$ be considered equivalent (to the zero field). Such fields are called *null fields*, and in this way are quotiented out of the theory. Note that the vacuum corresponds to the identity field.

To illustrate this correspondence, consider the (holomorphic) field constructed from the Virasoro operators L_n, $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-h}$. In simple examples (which are derived from actions), this field may be identified with the holomorphic component of a quantised version of the classical energy-momentum tensor (hence the customary labelling $T(z)$ rather than the more logical $L(z)$). To determine which quantum state this field corresponds to, consider $\lim_{z\to 0} T(z)\,|0\rangle = \lim_{z\to 0}\sum_{n\in\mathbb{Z}} z^{-n-h}L_n\,|0\rangle$. As the vacuum vector is a highest weight vector of minimal energy 0, $L_n\,|0\rangle$ automatically vanishes for $n \geqslant 0$. Additionally, $\|L_{-1}\,|0\rangle\|^2 = 0$, so (by unitarity), $L_n\,|0\rangle$ in fact vanishes for $n \geqslant -1$. Now,

[7] To be somewhat more precise, a field $\psi(z)$ is a formal power series in z whose coefficients, ψ_n, are linear operators on $\mathcal{S}$ which satisfy the following condition: For every $|\phi\rangle \in \mathcal{S}$, there is an N such that $\psi_n\,|\phi\rangle = 0$ whenever $n > N$. This last condition is necessary in order for various products of fields to make sense. In fact, one also allows monodromy around the origin, so the powers of z which appear need not be integral. Such fields should therefore be properly defined on an appropriate Riemann sheet.

$\|L_{-2}|0\rangle\|^2 = \frac{1}{2}c > 0$, so

$$\lim_{z\to 0} T(z)|0\rangle = \lim_{z\to 0}\left[z^{2-h}L_{-2}|0\rangle + z^{3-h}L_{-3}|0\rangle + \ldots\right].$$

It follows that the limit will only exist, and be non-zero, if $h = 2$. Therefore[8], the field

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \quad \text{corresponds to} \quad L_{-2}|0\rangle \in \mathcal{S}. \tag{2.1.4}$$

In this case, the field corresponds to an element of the $\mathfrak{Vir}$-module whose highest weight is the vacuum. Generally, this need not be the case. The decomposition of the state space into highest weight modules distinguishes certain fields, namely those that correspond to the highest weight vectors. Such fields are called *primary fields*; all others are *secondary*. Thus, the identity field is primary whereas the energy-momentum field is secondary.

2.1.4. Radial Ordering and Operator Product Expansions. Consider a holomorphic conformal field $\psi(z)$ of conformal weight h. Its expansion as a series in z suggests a formal expression for the modes ψ_n as a contour integral:

$$\psi(z) = \sum_n \psi_n z^{-n-h} \qquad \Rightarrow \qquad \psi_n = \oint_0 \psi(z) z^{n+h-1}\frac{dz}{2\pi i}.$$

The label 0 on the contour integral indicates that it is taken over a circle around the origin (on the appropriate Riemann sheet). If, in a classical lagrangian theory, $\psi(z)$ corresponds to a conserved current, then one expects to be able to form conserved charges by integrating the current over space. The modes ψ_n would seem to present themselves as obvious candidates for such (quantised) charges. Indeed, the modes may be so interpreted in the following framework. Here, the (compactified) space dimension is defined[9] to run in the angular direction of z, and time runs radially with the origin corresponding to the infinite past. The quantisation of the original classical theory is often performed in this framework, and is then known as *radial quantisation*.

This framework also provides a nice interpretation of the problem of ordering products of operators in the quantised theory. The time-ordering familiar from quantum field theory is transformed into *radial ordering*[10]:

$$\mathcal{R}\Big\{\psi(z)\phi(w)\Big\} = \begin{cases} \psi(z)\phi(w) & \text{if } |z| > |w|, \\ \phi(w)\psi(z) & \text{if } |z| < |w|. \end{cases}$$

An important application of this notion is to the commutators of the modes of conformal fields. One writes, with h_ψ and h_ϕ denoting the conformal weights of $\psi(z)$ and $\phi(w)$

[8]The astute reader will note that this conclusion is actually equivalent to the assumption made in Section 2.1.2 that the vacuum has zero energy.

[9]If, in applications, space and time are already predefined, it is usually possible to make a conformal transformation to end up in this radial framework.

[10]All fields considered in this thesis will be bosonic. Fermionic fields will incur an additional sign change if the order is switched.

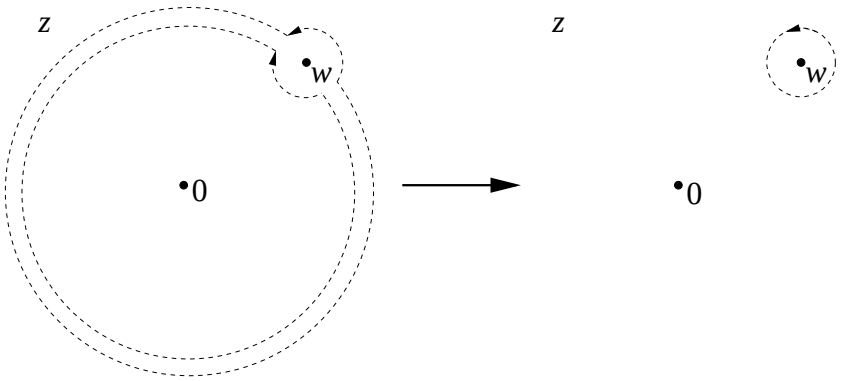

FIGURE 2.1. Contour subtraction: Subtract the inner contour from the outer contour (and take limits as $|z| \to |w|$) to get the contour around w.

(respectively),

$$\left[\!\left[\psi_n, \phi_m\right]\!\right] = \oint_0 \oint_0 \psi(z)\,\phi(w)\, z^{n+h_\psi-1} w^{m+h_\phi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} - \oint_0 \oint_0 \phi(w)\,\psi(z)\, z^{n+h_\psi-1} w^{m+h_\phi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}}.$$

To make contact with radial ordering, suppose that the z-contour in the first integral is taken with $|z| > |w|$ and in the second, $|z| < |w|$. Then, letting the contours approach $|z| = |w|$ appropriately (see Figure 2.1), one finds that

$$\left[\!\left[\psi_n, \phi_m\right]\!\right] = \oint_0 \oint_w \mathcal{R}\Big\{\psi(z)\,\phi(w)\Big\} z^{n+h_\psi-1} w^{m+h_\phi-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}}. \tag{2.1.5}$$

As z may be taken arbitrarily close to w in this integral, it is reasonable to suppose that the radially ordered product may be expanded as a (formal) Laurent series about w. This simple observation leads to the concept of the *operator product expansion*:

$$\mathcal{R}\Big\{\psi(z)\,\phi(w)\Big\} = \sum_{j=-\infty}^{\infty} \frac{A_j(w)}{(z-w)^{j+1}}. \tag{2.1.6}$$

Consider therefore the commutator for the Virasoro algebra. Expanding the corresponding radial product in the right hand side of Equation (2.1.5) gives

$$\oint_0 \oint_w \mathcal{R}\Big\{T(z)\,T(w)\Big\} z^{n+1} w^{m+1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} = \oint_0 \sum_{j=-\infty}^{\infty} \oint_w \frac{A_j(w)}{(z-w)^{j+1}} z^{n+1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} w^{m+1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} = \sum_{j=0}^{n+1} \binom{n+1}{j} \oint_0 A_j(w)\, w^{n+m+2-j} \frac{\mathrm{d}w}{2\pi\mathrm{i}}.$$

The modes on the left-hand-side may also be represented in contour-integral form, giving

$$(n-m) L_{n+m} + \frac{n^3-n}{12} \delta_{n+m,0} C = \oint_0 \left[(n-m)\, T(w)\, w^{n+m+1} + \binom{n+1}{3} \frac{C}{2}\, w^{n+m-1} \right] \frac{\mathrm{d}w}{2\pi\mathrm{i}}.$$

Writing $n-m=2(n+1)-(n+m+2)$, $(n+m+2)w^{n+m+1}=\partial w^{n+m+2}$, and then integrating by parts, the left-hand-side becomes

$$\oint_0\left[\partial T(w)\,w^{n+m+2}+2(n+1)T(w)\,w^{n+m+1}+\binom{n+1}{3}\frac{C}{2}\,w^{n+m-1}\right]\frac{\mathrm{d}w}{2\pi\mathrm{i}}.$$

Equating, one deduces the relation

$$\oint_0\Bigg[(A_0(w)-\partial T(w))\,w^{n+m+2}+(n+1)(A_1(w)-2T(w))\,w^{n+m+1}$$
$$+\binom{n+1}{2}A_2(w)\,w^{n+m}+\binom{n+1}{3}\left(A_2(w)-\frac{C}{2}\right)w^{n+m-1}$$
$$+\binom{n+1}{4}A_4(w)\,w^{n+m-2}+\ldots+A_{n+1}(w)\,w^{m+1}\Bigg]\frac{\mathrm{d}w}{2\pi\mathrm{i}}=0.$$

Now, this relation holds for all m and n in $\mathbb{Z}$. In particular, it has the form

$$\oint_0 f_n(w)\,w^m\frac{\mathrm{d}w}{2\pi\mathrm{i}}=0\qquad\text{for all } m\in\mathbb{Z},$$

where f_n is independent of m; f_n is therefore identically zero (for each $n\in\mathbb{Z}$). But, for $n=-1$, the vanishing of f_n just states that $A_0(w)=\partial T(w)$. Successively taking $n=0,1,2,\ldots$ then gives

$$A_1(w)=2T(w),\qquad A_2(w)=0,\qquad A_3(w)=\frac{C}{2},\qquad\text{and } A_j(w)=0 \text{ for } j>3.$$

It follows (finally) that the Virasoro algebra commutation relations force the operator product expansion

$$\mathcal{R}\Big\{T(z)\,T(w)\Big\}=\frac{\frac{1}{2}C}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}+\ldots \tag{2.1.7}$$

By reversing this argument, it is easy to see that this expansion is actually equivalent to the commutation relations. Note that the regular terms in this expansion (the "…") are not determined, as they contribute nothing to the contour integrals. It is common practice to ignore these terms completely when writing operator product expansions. Accordingly, equality up to regular terms will henceforth be denoted by $\sim$.

As a somewhat different example, recall that a field $\psi(w)$ was called primary if the corresponding state was a highest weight vector, $|\psi\rangle$. Operator product expansions can also be used to characterise this concept in a purely field-theoretic manner. Apply the field $T(z)$ to both sides of the defining relation $\lim_{w\to0}\psi(w)|0\rangle=|\psi\rangle$. On the left-hand-side, $T(z)\,\psi(w)$ can be replaced by its radially ordered version (because $w\to0$). Expanding this radially ordered product as an operator product expansion gives

$$\lim_{w\to0}\sum_{j=-\infty}^{\infty}\frac{A_j(w)}{(z-w)^{j+1}}|0\rangle=T(z)\,|\psi\rangle$$
$$\Rightarrow\qquad\sum_{j=-\infty}^{\infty}z^{-j-1}\lim_{w\to0}A_j(w)\,|0\rangle=\sum_{n=-\infty}^{\infty}z^{-n-2}L_n\,|\psi\rangle\,.$$

Comparing powers of z gives $\lim_{w\to 0} A_{n+1}(w)|0\rangle = L_n|\psi\rangle$. If $|\psi\rangle$ has energy h, the singular coefficients of the operator product expansion therefore satisfy

$$\lim_{w\to 0} A_0(w)|0\rangle = L_{-1}|\psi\rangle, \qquad \lim_{w\to 0} A_1(w)|0\rangle = L_0|\psi\rangle = h|\psi\rangle,$$
$$\text{and} \qquad \lim_{w\to 0} A_j(w)|0\rangle = L_{j-1}|\psi\rangle = 0 \qquad \text{for all } j \geqslant 2.$$

Therefore, $A_1(w) = h\psi(w)$ and $A_j(w) = 0$ for all $j \geqslant 2$. To determine $A_0(w)$, note that Equation (2.1.3) implies that

$$\partial\psi(w)|0\rangle = e^{wL_{-1}}L_{-1}|\psi\rangle \qquad \Rightarrow \qquad \lim_{w\to 0}\partial\psi(w)|0\rangle = L_{-1}|\psi\rangle.$$

Thus, $A_0(w) = \partial\psi(w)$. Conversely, these singular coefficients guarantee that $|\psi\rangle$ is a highest weight vector, provided that $|\psi\rangle$ is also an eigenvector[11] of C. It follows that $\psi(w)$ is a primary field if and only if its operator product expansion with $T(z)$ takes the form

$$\mathcal{R}\Big\{T(z)\,\psi(w)\Big\} \sim \frac{h\,\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}. \tag{2.1.8}$$

Note that $\psi(w)$ then has conformal weight h.

In contrast to the previous example, where the singular terms of the operator product expansion of $T(z)$ with itself were determined by the mode commutator, the operator product expansion of $T(z)$ with a primary field $\psi(z)$ is completely determined (that is, the regular terms are also fixed). Indeed, the coefficient of $(z-w)^n$ is just the field that corresponds to the state $L_{-n-2}|\psi\rangle$. What is not determined is an explicit form for these fields. They will therefore be denoted by $(L_{-n-2}\psi)(z)$.

It is worthwhile pointing out that in the operator product expansion of two arbitrary fields, $\psi(z)$ (of conformal weight h), and $\phi(w)$, the coefficient of $(z-w)^{-(n+h)}$ is the field corresponding to the state $\psi_n|\phi\rangle$. This follows from

$$\lim_{w\to 0}(\psi_n\phi)(w)|0\rangle = \psi_n|\phi\rangle = \lim_{w\to 0}\oint_0 z^{n+h-1}\psi(z)\,\phi(w)\,\frac{\mathrm{d}z}{2\pi\mathrm{i}}|0\rangle,$$

changing the subscript on the integral to w, replacing the power of z by the corresponding power of $z-w$, and noting that the field product is actually radially ordered. Therefore,

$$(\psi_n\phi)(w) = \oint_w (z-w)^{n+h-1}\,\mathcal{R}\Big\{\psi(z)\,\phi(w)\Big\}\frac{\mathrm{d}z}{2\pi\mathrm{i}}, \tag{2.1.9}$$

and the standard operator product expansion gives the required result.

2.1.5. Normal Ordering and Wick's Theorem. The singular terms of an operator product expansion indicate that radially ordered products of quantum fields are not defined when two of the field arguments coincide. However, products of fields are perfectly well defined classically, so it follows that any quantisation procedure must explain what the

[11] Using Equation (2.2.1) and $C = 2[L_2, L_{-2}] - 8L_0$, one can show that C actually commutes with every primary field. Therefore, every representation has the same central charge, and each $|\psi\rangle$ is an eigenvector of C. This is in fact implicitly assumed in most treatments of conformal field theory. If representations of differing central charge are to be allowed, then it is necessary to introduce additional vacua, one for each allowed value.

quantised version of these classical products is. The usual prescription is to note that classically, the ordering of the fields is irrelevant, whereas upon quantisation, the modes of the fields become non-commuting operators. It is therefore plausible that one might be able to find an ordering of these modes, called a *normal ordering*, with which the quantised product is well-defined. Such a normally ordered product will be denoted by

$$:\psi(z)\,\phi(z):.$$

Normal ordering prescriptions are not unique. From the point of view of the operator product expansion, the most natural way to define a non-singular product of two quantum fields is simply to ignore the singular terms. That is, to define

$$\mathcal{R}\Big\{\psi(z)\,\phi(w)\Big\} = \text{singular terms } + :\psi(z)\,\phi(w):.$$

The normally ordered product at w is then well-defined by

$$:\psi(w)\,\phi(w): = \lim_{z\to w} :\psi(z)\,\phi(w):,$$

which is just the first regular term (coefficient of $(z-w)^0$) in the operator product expansion of $\mathcal{R}\Big\{\psi(z)\,\phi(w)\Big\}$. Therefore,

$$:\psi(w)\,\phi(w): = \oint_w \mathcal{R}\Big\{\psi(z)\,\phi(w)\Big\}(z-w)^{-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{2.1.10}$$

It is not hard to check that this does imply an ordering on the modes, hence qualifies to be called a normal ordering. First, reverse the contour manipulation of Figure 2.1 to rewrite Equation (2.1.10) as

$$:\psi(w)\,\phi(w): = \oint_{\substack{0\\|z|>|w|}} \frac{\psi(z)\,\phi(w)}{z-w}\frac{\mathrm{d}z}{2\pi\mathrm{i}} - \oint_{\substack{0\\|z|<|w|}} \frac{\phi(w)\,\psi(z)}{z-w}\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{2.1.11}$$

These integrals can be evaluated by expanding the fields in modes, and $(z-w)^{-1}$ as a (convergent) geometric progression. In the first integral, $|z|>|w|$, so $(z-w)^{-1} = \sum_{p=0}^{\infty} w^p/z^{p+1}$, and in the second, $(z-w)^{-1} = -\sum_{p=0}^{\infty} z^p/w^{p+1}$. The final result is

$$:\psi(w)\,\phi(w): = \left[\sum_{\substack{m+h_\psi\leqslant 0\\ n}} \psi_m\phi_n + \sum_{\substack{m+h_\psi>0\\ n}} \phi_n\psi_m\right] w^{-m-n-h_\psi-h_\phi},$$

where h_ψ and h_ϕ are the conformal weights of $\psi(w)$ and $\phi(w)$, respectively. The mode ordering is therefore

$$:\psi_m\phi_n: = \begin{cases} \psi_m\phi_n & \text{if } m+h_\psi \leqslant 0,\\ \phi_n\psi_m & \text{if } m+h_\psi > 0. \end{cases} \tag{2.1.12}$$

Of course, once one starts introducing normally ordered products of fields, it becomes necessary to enquire as to how one computes the operator product expansion of these

products with other fields. This is the content of Wick's theorem, familiar for free[12] fields from quantum field theory [**128**].

THEOREM 2.1 (Wick). *Let $\psi(z)$, $\phi(z)$, and $\chi(z)$ be mutually free fields. Then,*

$$\mathcal{R}\Big\{\psi(z):\phi(w)\chi(w):\Big\} = :\psi(z):\phi(w)\chi(w): + \overbrace{\psi(z)\phi(w)}\chi(w) + \phi(w)\overbrace{\psi(z)\chi(w)},$$

where $\overbrace{\psi(z)\phi(w)}$ is called the contraction *of $\psi(z)$ and $\phi(w)$, and denotes the singular part of the operator product expansion of $\psi(z)$ with $\phi(w)$.*

This result can be extended inductively to arbitrary numbers of normally ordered free fields, and this extension is also known as Wick's theorem. When the fields are not mutually free (as will be the case in Section 3.1), this result must be modified slightly [**14**,**100**] giving a generalised Wick theorem.

THEOREM 2.2 (Generalised Wick). *For arbitrary fields $\psi(z)$, $\phi(z)$, and $\chi(z)$,*

$$\mathcal{R}\Big\{\psi(z):\phi(w)\chi(w):\Big\} \sim \oint_w \mathcal{R}\Big\{\overbrace{\psi(z)\phi(z')}\chi(w) + \phi(z')\overbrace{\psi(z)\chi(w)}\Big\}(z'-w)^{-1}\frac{dz'}{2\pi i}.$$

2.2. Fusion

2.2.1. Correlation Functions. As in any quantum field theory, the goal of a conformal field theory is to compute the *correlation functions*. Recalling that time-ordering becomes radial ordering in the formalism of Section 2.1.4, the correlation functions have the form

$$\langle 0|\mathcal{R}\Big\{\psi_1(z_1)\psi_2(z_2)\cdots\psi_m(z_m)\Big\}|0\rangle.$$

Not surprisingly, the symmetry algebra constrains the form of these functions. For suppose $n=-1$, 0, or 1, and that $\psi_1(z_1),\ldots,\psi_m(z_m)$ are primary fields of (respective) conformal weights $h_1,\ldots,h_m$. Then, $L_n|0\rangle = L_n^\dagger|0\rangle = 0$, so

$$\begin{aligned}0 &= \langle 0|L_n\mathcal{R}\Big\{\psi_1(z_1)\psi_2(z_2)\cdots\psi_m(z_m)\Big\}|0\rangle \\ &= \sum_{i=1}^m \langle 0|\mathcal{R}\Big\{\psi_1(z_1)\cdots\big[\!\big[L_n,\psi_i(z_i)\big]\!\big]\cdots\psi_m(z_m)\Big\}|0\rangle.\end{aligned}$$

As in Section 2.1.4, the commutator may be replaced by

$$\begin{aligned}\big[\!\big[L_n,\psi_i(z_i)\big]\!\big] &= \oint_{z_i} z^{n+1}\mathcal{R}\Big\{T(z)\psi_i(z_i)\Big\}\frac{dz}{2\pi i} = \oint_{z_i} z^{n+1}\left[\frac{h_i\psi_i(z_i)}{(z-z_i)^2}+\frac{\partial\psi_i(z_i)}{z-z_i}\right]\frac{dz}{2\pi i} \\ &= (n+1)h_i z_i^n\psi_i(z_i) + z_i^{n+1}\partial\psi_i(z_i). \qquad (2.2.1)\end{aligned}$$

[12]For the purposes of Wick's theorem, a collection of fields may be considered to be *mutually free* if the singular coefficients of all the operator product expansions of fields from this collection are just multiples of the identity field [**100**]. $T(z)$ is therefore never free (except in trivial theories).

Let ∂_i denote the partial derivative operator with respect to z_i. It follows that a correlation function of primary fields must satisfy the differential equations (for $n = -1$, 0, or 1):

$$\sum_{i=1}^{m} \left[(n+1) h_i z_i^n + z_i^{n+1} \partial_i \right] \langle 0 | \mathcal{R}\Big\{ \psi_1(z_1) \psi_2(z_2) \cdots \psi_m(z_m) \Big\} |0\rangle = 0. \tag{2.2.2}$$

A correlation function involving a single primary field therefore vanishes unless the field is a multiple of the identity field, in which case the correlation function is just a constant function. Similarly, when there are two and three primary fields, these equations also determine the form of the correlation function as

$$\langle 0 | \mathcal{R}\Big\{ \psi_1(z_1) \psi_2(z_2) \Big\} |0\rangle = \frac{C_{12} \delta_{h_1 h_2}}{(z_1 - z_2)^{h_1 + h_2}}, \tag{2.2.3}$$

and

$$\begin{aligned} &\langle 0 | \mathcal{R}\Big\{ \psi_1(z_1) \psi_2(z_2) \psi_3(z_3) \Big\} |0\rangle \\ &\qquad = \frac{C_{123}}{(z_1 - z_2)^{h_1 + h_2 - h_3} (z_2 - z_3)^{h_2 + h_3 - h_1} (z_3 - z_1)^{h_3 + h_1 - h_2}}, \end{aligned} \tag{2.2.4}$$

where C_{12} and C_{123} are constants depending on the primary fields. When there are more than three primary fields, the functional form of the correlation function is not fixed by these differential equations (although it is still constrained).

It is useful to digress briefly and fix a convenient basis for future purposes. Suppose now (and forever) that for each $h \geqslant 0$, there are only a finite number of linearly independent primary fields of conformal weight h. This finiteness condition ensures that the constants C_{ij} occurring in the correlation function, Equation (2.2.3), of two such fields may be interpreted as entries of a symmetric matrix. There are thus linear combinations of the primary fields of conformal weight h, which diagonalise this matrix. The matrix is non-singular (as a zero eigenvalue would correspond to a null field), hence the linear combinations may be rescaled so that the corresponding constants are $C_{ij} = \delta_{ij}$.

The symmetries corresponding to the modes L_n, $n > 1$ may also constrain the correlation functions, albeit in a more subtle manner. Before discussing this, it is necessary to consider correlation functions involving secondary fields. For simplicity, suppose that there is only one secondary field in the correlation function, and that it has the form $(L_{-n}\psi)(w)$ ($n > 0$), where $\psi(w)$ is primary. According to the discussion at the end of Section 2.1.4, this secondary field is the coefficient of $(z-w)^{n-2}$ in the operator product expansion of $T(z)$ and $\psi(w)$. It follows that

$$\begin{aligned} &\langle 0 | \mathcal{R}\Big\{ (L_{-n}\psi)(w) \psi_1(w_1) \cdots \psi_m(w_m) \Big\} |0\rangle \\ &\qquad = \oint_w (z-w)^{1-n} \langle 0 | \mathcal{R}\Big\{ T(z) \psi(w) \psi_1(w_1) \cdots \psi_m(w_m) \Big\} |0\rangle \frac{\mathrm{d}z}{2\pi \mathrm{i}}. \end{aligned}$$

The radially ordered product will only have singularities in z at $w, w_1, \ldots, w_m$, so the contour integral $\oint_w$ may be replaced by $\oint_* - \sum_i \oint_{w_i}$, where the subscript $*$ indicates a contour that is sufficiently large to enclose all the singularities.

If this $*$-contour is taken to be so large that $|z| > |w|, |w_1|, \ldots, |w_m|$, then $T(z)$ can be taken out of the radial ordering (to the left) and expanded in modes. The integral over the $*$-contour then becomes

$$\sum_{j=-\infty}^{\infty} \oint_* (z-w)^{1-n} z^{-j-2} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \langle 0| L_j \mathcal{R}\Big\{ \psi(w)\, \psi_1(w_1) \cdots \psi_m(w_m) \Big\} |0\rangle .$$

The terms in the sum with $j \leqslant 1$ clearly vanish, as L_j annihilates $\langle 0|$. Furthermore, if $j > 1$, it is easy to see that the integral evaluates to zero[13]. Therefore, the $*$-contour is in fact identically zero.

This therefore establishes the relation

$$\begin{aligned} &\langle 0| \mathcal{R}\Big\{ (L_{-n}\psi)(w)\, \psi_1(w_1) \cdots \psi_m(w_m) \Big\} |0\rangle \\ &\qquad = -\sum_{i=1}^{m} \oint_{w_i} (z-w)^{1-n} \langle 0| \mathcal{R}\Big\{ T(z)\, \psi(w)\, \psi_1(w_1) \cdots \psi_m(w_m) \Big\} |0\rangle \frac{\mathrm{d}z}{2\pi\mathrm{i}}. \end{aligned}$$

Expand the radially ordered product of $T(z)$ and $\psi_i(w_i)$ as usual. As $\psi_i(w_i)$ is a primary field of conformal weight h_i, one finds (after performing the integration) that

$$\begin{aligned} &\langle 0| \mathcal{R}\Big\{ (L_{-n}\psi)(w)\, \psi_1(w_1) \cdots \psi_m(w_m) \Big\} |0\rangle \\ &\quad = \sum_{i=1}^{m} \left[\frac{(n-1)\, h_i}{(w_i - w)^n} - \frac{1}{(w_i - w)^{n-1}} \partial_i \right] \langle 0| \mathcal{R}\Big\{ \psi(w)\, \psi_1(w_1) \cdots \psi_m(w_m) \Big\} |0\rangle . \end{aligned} \quad (2.2.5)$$

Therefore, a correlation function involving a secondary field $(L_{-n}\psi)(w)$ may be obtained from the corresponding correlation function involving only primary fields, by applying a suitable differential operator. This conclusion generalises appropriately to more general secondary fields (and to more than one secondary field in the correlation function). In principle then, all correlation functions can be obtained from those involving primary fields.

Returning to the question of further constraints on the correlation functions, it is obvious that if a correlation function involves a null field (Section 2.1.3), then it must vanish identically. However, in the formalism developed above, such a null field would be descended from a primary field (or a linear combination of them), so the correlation function would be related to the corresponding primary correlation function by a differential operator. Therefore, null fields yield further (differential) constraints on the primary correlation functions. Clearly constraints of this type can only be derived when there are formally independent secondary fields — thus under the state-field correspondence, quantum states — that are in fact linearly dependent. This is precisely the condition that the given primary field corresponds to a highest weight $\mathfrak{Vir}$-module which is not isomorphic to its associated Verma module (Appendix B.3). Demanding unitarity as well, it follows that null field constraints can only be derived for primary fields with $h = 0$ or $c \leqslant 1$ (for certain h) [**101**]. This may seem like a strong restriction, but in fact many important conformal

[13] Actually, this is true when the conformal field theory is defined on the Riemann sphere.

field theories (for example, the minimal models) are composed of such modules. In this way, the representation theory of the symmetry algebra constrains the form of correlation functions involving fields associated with these modules.

As a trivial example, recall that $L_{-1}|0\rangle$ vanishes, so (formally) the corresponding field is null. The corresponding constraint is (by Equation (2.2.5))

$$-\sum_{i=1}^{m}\partial_i\langle 0|\mathcal{R}\Big\{1(w)\,\psi_1(w_1)\cdots\psi_m(w_m)\Big\}|0\rangle=0$$

(where $1(w)$ is the identity field), which is identical to Equation (2.2.2) with $n=-1$. As a less trivial example, it is easy to check that if $c=2h(5-8h)/(1+2h)$, then the vector

$$\left[L_{-2}-\frac{3}{2(1+2h)}L_{-1}^2\right]\big|\psi_{c,h}\big\rangle$$

is null (has zero norm), where $\big|\psi_{c,h}\big\rangle$ is the highest weight vector of central charge c and energy h. This null vector corresponds to a null field which induces the constraint

$$\sum_{i=1}^{m}\left[\frac{h_i}{(w_i-w)^2}-\frac{1}{w_i-w}\partial_i-\sum_{j=1}^{m}\frac{3\partial_i\partial_j}{2(1+2h)}\right]\langle 0|\mathcal{R}\Big\{\psi_{c,h}(w)\,\psi_1(w_1)\cdots\psi_m(w_m)\Big\}|0\rangle=0.$$

This derivation requires an easy generalisation of Equation (2.2.5).

2.2.2. Fusion. It follows from the analysis of Section 2.2.1 that any correlation function is computable, once the primary correlation functions are known. Unfortunately, there is still the problem of having to somehow compute correlation functions involving arbitrarily large numbers of primary fields. However, if the operator product expansions of all the primary fields are known (including regular terms), then one can reduce the evaluation of a correlation function of m primary fields to a (generally infinite) sum of correlation functions involving $m-1$ fields. Using Equation (2.2.5), these may be computed from the correlation functions involving $m-1$ primary fields, and so on.

This is a somewhat daunting task. However, knowing which fields appear in an operator product expansion may simplify the problem. For example, if a correlation function involving primary fields $\psi_1(z_1),\ldots,\psi_{m-1}(z_{m-1})$ vanishes, then it follows that all the correlation functions involving the corresponding secondary fields must also vanish. In determining the function of m primary fields,

$$\langle 0|\mathcal{R}\Big\{\psi_1(z_1),\ldots,\psi_{m-2}(z_{m-2})\,\phi_1(w_1)\,\phi_2(w_2)\Big\}|0\rangle,$$

through the operator product expansion of $\phi_1(w_1)$ and $\phi_2(w_2)$, it follows that any field in the expansion which is descended from $\psi_{m-1}(z_{m-1})$ contributes nothing to the correlation function. Under these circumstances, one only needs to know the *families* that the fields in the operator product expansion belong to, in order to verify if they contribute nothing to the correlation function.

This leads to the concept of *fusion*, which may be defined as an operation on the set of primary fields of the theory. Two primary fields fuse to give a formal sum of primary

fields (or rather the corresponding families), these being the primary fields which appear, or have descendants appearing, in the operator product expansion of the original two primary fields. This operation may be neatly characterised using the correlation function of three primary fields:

$$\langle 0|\mathcal{R}\Big\{\psi_1(z_1)\,\psi_2(z_2)\,\psi_3(z_3)\Big\}|0\rangle = \sum_{j=-\infty}^{\infty} \frac{\langle 0|\mathcal{R}\Big\{A_j(z_2)\,\psi_3(z_3)\Big\}|0\rangle}{(z_1-z_2)^{j+1}}.$$

Here, the $A_j(z_2)$ are the operator product coefficients in the expansion of $\psi_1(z_1)$ and $\psi_2(z_2)$. From the normalisation of the correlation function of two primary fields (Section 2.2.1), it follows that the right-hand-side can only be non-zero if at least one coefficient $A_j(z_2)$ has a component which is a descendant of $\psi_3(z_3)$. That is, if and only if a member of the family headed by $\psi_3(z_3)$ appears in the operator product expansion of $\psi_1(z_1)$ and $\psi_2(z_2)$. That is, this correlation function is non-zero if and only if $\psi_3(z_3)$ appears in the fusion of $\psi_1(z_1)$ and $\psi_2(z_2)$.

Equation (2.2.5) now shows that if $\psi_3(z_3)$ and its descendants do not appear in the operator product expansion of $\psi_1(z_1)$ and $\psi_2(z_2)$, then they do not appear in the operator product expansion of any descendant of $\psi_1(z_1)$ with any descendant of $\psi_2(z_2)$. What this means is that it makes sense (and is most natural) to talk about the fusion of families of fields. In this setting, fusion is customarily denoted thus:

$$[\psi_1]\times[\psi_2] = [\psi_3] + \ldots,$$

where $[\psi_i]$ denotes the family of fields headed by the primary field $\psi_i(z_i)$. This equation signifies that there is a member of the family $[\psi_1]$ and a member of the family $[\psi_2]$, whose operator product expansion contains a member of the family $[\psi_3]$.

A generic fusion rule, with respect to the basis $\{\psi_i(z)\}$ of primary fields, takes the form

$$[\psi_i]\times[\psi_j] = \sum_k \mathcal{N}_{ij}{}^k\,[\psi_k], \tag{2.2.6}$$

where $\mathcal{N}_{ij}{}^k \in \{0,1\}$ is called a *fusion coefficient*. The associativity of the operators on the state space and the commutativity within radial ordering ensure that fusion is an associative, commutative operation. The identity field (denoted now by $\psi_0(z)$) supplies a unit, as the correlation function involving $\psi_0(z)$, $\psi_i(z)$, and $\psi_j(z)$, vanishes unless $i=j$. That is,

$$[\psi_0]\times[\psi_i] = [\psi_i] \qquad \Rightarrow \qquad \mathcal{N}_{0i}{}^j = \delta_{ij}. \tag{2.2.7}$$

It is also easy to see that commutativity ($\mathcal{N}_{ij}{}^k = \mathcal{N}_{ji}{}^k$) is in fact subsumed by the complete symmetry of the indices:

$$\mathcal{N}_{ij}{}^k = \mathcal{N}_{ji}{}^k = \mathcal{N}_{jk}{}^i = \mathcal{N}_{kj}{}^i = \mathcal{N}_{ki}{}^j = \mathcal{N}_{ik}{}^j. \tag{2.2.8}$$

In particular, $\mathcal{N}_{ij}{}^0 = \delta_{ij}$. However, it should be noted that these relations are basis-dependent. These nice properties follow from the fact that the primary fields were chosen

to be (roughly speaking) orthonormal with respect to the correlation function of two fields (Section 2.2.1).

It should be clear from this discussion that a knowledge of the fusion operation associated to a conformal field theory will help determine correlation functions. However, it should also be clear that this knowledge alone is not sufficient. For example, Equation (2.2.2) determines the form of a correlation function of three primary fields up to a multiplicative constant. Fusion determines whether this constant vanishes, but otherwise says nothing about its value[14]. To determine this value, other (dynamical) considerations are required. For example, in the minimal models, a mechanism known as the conformal bootstrap seems to suffice [**61**].

[14]This constant is of non-trivial interest, unlike the constants appearing in the correlation functions of two primary fields which were normalised by choosing an appropriate basis of primary fields.

CHAPTER 3

Wess-Zumino-Witten Branes I: Algebraic Considerations

3.1. Wess-Zumino-Witten Models

The conformal field theories that will be studied in this thesis are known as Wess-Zumino-Witten models (also Wess-Zumino-Novikov-Witten models) for reasons that will be discussed in Chapter 6. These models are especially interesting for a variety of reasons, one being that they describe the dynamics of strings propagating in the underlying manifold of a suitable[1] Lie group. As such, the basic dynamical field in these theories is a map g from the two-dimensional string worldsheet Σ into the Lie group G. However, the action constructed from g which describes these theories is rather intricate, relying heavily on topological notions. As a conformal field theory may be defined (and studied) without giving an action, it is convenient to avoid these intricacies for the time being, and define the Wess-Zumino-Witten models through their conserved currents. In Chapter 6, the appropriate actions will be constructed and analysed; in particular, the conserved currents postulated here will be derived from these actions.

In this chapter, these Wess-Zumino-Witten models will be shown to be conformal field theories with a symmetry algebra extending the Virasoro algebra, and the notion of fusion in these models is carefully defined. The concept of a *brane* is then introduced as a boundary condition (on string endpoints), and the relation between quantised branes and fusion is derived using results from boundary conformal field theory.

3.1.1. Conserved Currents and Symmetry Algebras. Let z be a (local) complex coordinate on the string worldsheet. The string field g is then a map (formally depending on z and $\overline{z}$) which takes values in the Lie group G. In terms of this field, Wess-Zumino-Witten models admit a holomorphic conserved current given by (see Equation (6.2.6))

$$J(z) = kg^{-1}\partial g, \tag{3.1.1}$$

where the scaling factor k is a positive integer called the level that is *a priori* specified (it arises as a coupling constant in the theory, see Equation (6.2.13)). In addition, there is a corresponding antiholomorphic conserved current,

$$\overline{J}(\overline{z}) = -k\overline{\partial} g \cdot g^{-1}. \tag{3.1.2}$$

Unlike the string field g, the holomorphic and antiholomorphic currents both take values in the Lie algebra of G, denoted by $\mathfrak{g}$.

[1]The groups that will be studied in this thesis are the compact, connected, simply-connected groups, but most of the theory can be developed more generally.

The (anti)holomorphicity of these currents is clearly invariant under the transformation $g \longmapsto \overline{f} g f$, provided that f is a holomorphic map and $\overline{f}$ is an antiholomorphic map (taking values in G). Such transformations therefore constitute classical symmetries of the Wess-Zumino-Witten model. The corresponding infinitesimal transformations may be obtained by setting $f = \exp(t\varepsilon(z))$, $\overline{f} = \exp(t\overline{\varepsilon}(\overline{z}))$, and differentiating at $t = 0$. ε and $\overline{\varepsilon}$ then take values in $\mathfrak{g}$. In the formalism of radial quantisation (Section 2.1.4), each infinitesimal symmetry defines a Noether charge which may be taken to have the form

$$Q_\varepsilon = \oint_0 \kappa(\varepsilon(z), J(z)) \frac{\mathrm{d}z}{2\pi\mathrm{i}} \qquad \text{and} \qquad Q_{\overline{\varepsilon}} = -\oint_0 \kappa\left(\overline{\varepsilon}(\overline{z}), \overline{J}(\overline{z})\right) \frac{\mathrm{d}\overline{z}}{2\pi\mathrm{i}}, \tag{3.1.3}$$

where $\kappa(\cdot,\cdot)$ denotes the Killing form of $\mathfrak{g}$ (Appendix A.1). This large collection of conserved charges will define the symmetry algebra of the theory. As expected for a conformal field theory, there are holomorphic and antiholomorphic charges.

In the quantised theory, these conserved charges become operators whose commutation relations may be obtained from the variation of a field under an infinitesimal symmetry [**86**, **141**]. For the holomorphic current, this gives

$$\delta_\varepsilon J(w) = \left[\!\left[Q_\varepsilon, J(w)\right]\!\right], \tag{3.1.4}$$

and the corresponding antiholomorphic relation is analogous. The left-hand-side of this equation is easily found to be $[J(w), \varepsilon(w)] + k\partial\varepsilon(w)$. To avoid confusion between the operator commutator and the Lie bracket of $\mathfrak{g}$ (here denoted by $[\cdot,\cdot]$), it is convenient to decompose $\mathfrak{g}$-valued fields into scalar fields, with respect to a basis $\{t_a\}$ of $\mathfrak{g}$. This basis will be chosen to be orthonormal with respect to the Killing form (more correctly, with respect to the negative of the Killing form). If f_{abc} denotes the structure constants of $\mathfrak{g}$ with respect to this orthonormal basis, then Equation (3.1.4) becomes (summation convention[2] implied)

$$f_{cab} J^c(w)\varepsilon^a(w) + k\partial\varepsilon^b(w) = \oint_0 \varepsilon^a(z) \left[\!\left[J^a(z), J^b(w)\right]\!\right] \frac{\mathrm{d}z}{2\pi\mathrm{i}},$$

where J^a and ε^a denote the components of J and ε, respectively, with respect to $\{t_a\}$. Expanding the commutator, the contour may be chosen so that $|z| > |w|$ for the positively signed term, and $|z| < |w|$ for the negatively signed term. As in Section 2.1.4 (and Figure 2.1), this now becomes

$$f_{cab} J^c(w)\varepsilon^a(w) + k\partial\varepsilon^b(w) = \oint_w \varepsilon^a(z)\,\mathcal{R}\Big\{J^a(z)J^b(w)\Big\}\frac{\mathrm{d}z}{2\pi\mathrm{i}}.$$

Expand $\varepsilon^a(z)$ in a Taylor series about w (supposing that $\varepsilon^a(z)$ has no pole at $z = w$), and the radially ordered product in the usual Laurent series. Then,

$$f_{cab} J^c(w)\varepsilon^a(w) + k\delta_{ab}\partial\varepsilon^a(w) = \sum_{n=0}^{\infty} \frac{A_n^{ab}(w)}{n!}\partial^n\varepsilon^a(w),$$

[2] In this thesis, the usual summation convention involving repeated indices will be employed where it does not cause confusion. However, the convention regarding the pairing of a raised index with a lowered index will not be abided by.

so it follows that $A_0^{ab}(w) = f_{cab}J^c(w) = f_{abc}J^c(w)$, as the structure constants of $\mathfrak{g}$ are completely antisymmetric with respect to an orthonormal basis (Appendix A.1), $A_1^{ab}(w) = k\delta_{ab}$, and $A_m^{ab}(w) = 0$ for every $m \geqslant 2$. The operator product expansions between the conserved currents $J^a(z)$ are therefore given by

$$\mathcal{R}\Big\{J^a(z)J^b(w)\Big\} \sim \frac{k\delta_{ab}}{(z-w)^2} + \frac{f_{abc}J^c(w)}{z-w}. \tag{3.1.5}$$

The corresponding expansions for the antiholomorphic currents are entirely analogous:

$$\mathcal{R}\Big\{\overline{J}^a(\overline{z})\overline{J}^b(\overline{w})\Big\} \sim \frac{k\delta_{ab}}{(\overline{z}-\overline{w})^2} + \frac{f_{abc}\overline{J}^c(\overline{w})}{\overline{z}-\overline{w}}. \tag{3.1.6}$$

Finally, it is easy to show that $\delta_{\overline{\varepsilon}}J(w) = 0$, from which follows

$$\mathcal{R}\Big\{J^a(z)\overline{J}^b(\overline{w})\Big\} \sim 0. \tag{3.1.7}$$

It follows from these expansions that the symmetry algebra of the Wess-Zumino-Witten models is not the Virasoro algebra. It also follows that when G (and hence $\mathfrak{g}$) is non-abelian, the currents $\{J^a(z)\}$ (and $\{\overline{J}^a(\overline{z})\}$) do not form a collection of mutually free fields (Section 2.1.5).

To determine the structure of the symmetry algebra, decompose $J^a(z)$ as[3]

$$J^a(z) = \sum_{n=-\infty}^{\infty} J_n^a z^{-n-1} \qquad \Rightarrow \qquad J_n^a = \oint_0 J^a(z)z^n\frac{\mathrm{d}z}{2\pi\mathrm{i}}.$$

The operator product expansion of Equation (3.1.5) now determines the commutators of these modes through Equation (2.1.5):

$$\begin{aligned}
\Big[\!\Big[J_n^a, J_m^b\Big]\!\Big] &= \oint_0\oint_w \mathcal{R}\Big\{J^a(z)J^b(w)\Big\}z^n w^m\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{d}w}{2\pi\mathrm{i}} \\
&= \oint_0\oint_w \left\{\frac{k\delta_{ab}}{(z-w)^2} + \frac{f_{abc}J^c(w)}{z-w}\right\}z^n\frac{\mathrm{d}z}{2\pi\mathrm{i}}w^m\frac{\mathrm{d}w}{2\pi\mathrm{i}} \\
&= \oint_0 \left\{k\delta_{ab}\,nw^{m+n-1} + f_{abc}J^c(w)\,w^{m+n}\right\}\frac{\mathrm{d}w}{2\pi\mathrm{i}} \\
&= f_{abc}J_{m+n}^c + n\delta_{ab}\,\delta_{m+n,0}\,k.
\end{aligned} \tag{3.1.8}$$

The commutators corresponding to the operator product expansions, Equations (3.1.6) and (3.1.7), are given by

$$\Big[\!\Big[\overline{J}_n^a, \overline{J}_m^b\Big]\!\Big] = f_{abc}\overline{J}_{m+n}^c + n\delta_{ab}\,\delta_{m+n,0}\,k, \tag{3.1.9}$$

and

$$\Big[\!\Big[J_n^a, \overline{J}_m^b\Big]\!\Big] = 0, \tag{3.1.10}$$

[3]The energy of the states corresponding to $J^a(z)$ under the state-field correspondence is 1. This will be obvious from the form of the energy-momentum field (Equation (3.1.11)), and the fact that the energy-momentum field must correspond to a state of energy 2.

respectively. It follows that these holomorphic and antiholomorphic modes together realise a level k representation of $\widehat{\mathfrak{g}} \otimes \widehat{\mathfrak{g}}$ on the quantum state space, where $\widehat{\mathfrak{g}}$ is the *untwisted affine Lie algebra* associated with $\mathfrak{g}$ (and G). This algebra is discussed in Appendix B (see Equation (B.1.1)).

3.1.2. The Energy-Momentum Tensor. Wess-Zumino-Witten models are conformal field theories. To justify this statement, an energy-momentum tensor needs to be identified. Of course, the standard method of identification is to consider the Noether current associated with spacetime translations. However, the fundamental requirement in conformal field theory is that the energy-momentum tensor decomposes into a holomorphic field $T(z)$ and an antiholomorphic field $\overline{T}(\overline{z})$ whose modes furnish representations of the Virasoro algebra. This section is devoted to establishing this requirement.

When G is the abelian group $\mathbb{R}^n$, the classical energy-momentum field is found to be the product $J^a(z)J^a(z)$ (this is the free boson [**61**]). In the general case then, it is reasonable (in the quantised theory) to make the ansatz

$$T(z) = \gamma : J^a(z) J^a(z) :, \tag{3.1.11}$$

and similarly for the antiholomorphic component (γ and $\overline{\gamma}$ are constants to be determined). Because the currents $J^a(z)$ are not free fields, operator product expansions involving $T(z)$ must be computed using the non-commutative Wick formula, Theorem 2.2. For example, using Equation (3.1.5),

$$\begin{aligned}
\mathcal{R}\Big\{J^a(z)\,T(w)\Big\} &= \gamma\,\mathcal{R}\Big\{J^a(z) : J^b(w)\,J^b(w) :\Big\} \\
&\sim \gamma \oint_w \mathcal{R}\Big\{\overbrace{J^a(z)\,J^b(z')}J^b(w) + J^b(z')\overbrace{J^a(z)\,J^b(w)}\Big\}(z'-w)^{-1}\frac{\mathrm{d}z'}{2\pi\mathrm{i}} \\
&= \gamma \oint_w \left[\frac{kJ^a(w)}{(z-z')^2} + \frac{kJ^a(z')}{(z-w)^2} + \frac{f_{abc}}{z-z'}\mathcal{R}\Big\{J^c(z')\,J^b(w)\Big\}\right. \\
&\qquad \left. + \frac{f_{abc}}{z-w}\mathcal{R}\Big\{J^b(z')\,J^c(w)\Big\}\right](z'-w)^{-1}\frac{\mathrm{d}z'}{2\pi\mathrm{i}}.
\end{aligned} \tag{3.1.12}$$

The radially ordered products can now be expanded using Equation (3.1.5) again, but now the regular terms must be included. Indeed, in the term containing the first radially ordered product, the singular terms of the operator product expansion and the first regular term may contribute to the integral (because of the factor $(z-z')$). The term containing the second radially ordered product is somewhat simpler, and it is easy to see that in this case only the first regular term may contribute. The terms involving the radially ordered

products therefore become

$$\gamma f_{abc} \oint_w \left[\frac{k\delta_{cb}}{(z-z')(z'-w)^3} + \frac{f_{cbd} J^d(w)}{(z-z')(z'-w)^2} + \frac{:J^c(w)J^b(w):}{(z-z')(z'-w)} + \frac{:J^b(w)J^c(w):}{(z-w)(z'-w)} \right] \frac{\mathrm{d}z'}{2\pi\mathrm{i}}.$$

Using the antisymmetry of the structure coefficients (Appendix A.1), the first term in this expression immediately vanishes, as do the normally ordered terms after integrating. Therefore, only one term in fact contributes, so Equation (3.1.12) reduces (after performing the integration) to

$$\begin{aligned} \mathcal{R}\Big\{J^a(z)T(w)\Big\} &= \frac{2\gamma k J^a(w)}{(z-w)^2} + \gamma f_{abc} f_{cbd} \frac{J^d(w)}{(z-w)^2} \\ &= \frac{2\gamma(k+\mathrm{h}^\vee)J^a(w)}{(z-w)^2}, \end{aligned}$$

where $\mathrm{h}^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (Equations (A.1.2) and (A.1.3) have been used). It follows that

$$\mathcal{R}\Big\{T(z)J^a(w)\Big\} \sim \frac{2\gamma(k+\mathrm{h}^\vee)J^a(z)}{(w-z)^2} = 2\gamma\big(k+\mathrm{h}^\vee\big)\left[\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\right]. \tag{3.1.13}$$

It is now straight-forward, if somewhat messy, to determine the operator product expansion of $T(z)$ with itself. Using Equation (3.1.13) and Theorem 2.2,

$$\begin{aligned} \mathcal{R}\Big\{T(z)T(w)\Big\} &\sim \gamma \oint_w \mathcal{R}\Big\{\overbrace{T(z)J^a(z')}J^a(w) + J^a(z')\overbrace{T(z)J^a(w)}\Big\}(z'-w)^{-1}\frac{\mathrm{d}z'}{2\pi\mathrm{i}} \\ &= 2\gamma^2\big(k+\mathrm{h}^\vee\big)\oint_w \left[\frac{\mathcal{R}\Big\{J^a(z')J^a(w)\Big\}}{(z-z')^2} + \frac{\mathcal{R}\Big\{\partial J^a(z')J^a(w)\Big\}}{z-z'} \right. \\ &\qquad \left. + \frac{\mathcal{R}\Big\{J^a(z')J^a(w)\Big\}}{(z-w)^2} + \frac{\mathcal{R}\Big\{J^a(z')\partial J^a(w)\Big\}}{z-w}\right](z'-w)^{-1}\frac{\mathrm{d}z'}{2\pi\mathrm{i}}. \end{aligned}$$

In this integrand, the singular terms and the first regular term of the operator product expansions contribute when integrating the first two radially ordered products, whereas only the first regular term contributes when integrating the last two. The singular terms of the operator product expansion of $\partial J^a(z')$ and $J^a(w)$ are computed from

$$\mathcal{R}\Big\{J^a(z')J^a(w)\Big\} \sim \frac{k\dim\mathfrak{g}}{(z'-w)^2} \qquad \Rightarrow \qquad \mathcal{R}\Big\{\partial J^a(z')J^a(w)\Big\} \sim \frac{-2k\dim\mathfrak{g}}{(z'-w)^3},$$

where the $\dim \mathfrak{g}$ factors come from summing over a. It follows that the contributing part of the integrand is

$$\frac{k\dim\mathfrak{g}}{(z-z')^2(z'-w)^3}-\frac{2k\dim\mathfrak{g}}{(z-z')(z'-w)^4}+\frac{:J^a(w)J^a(w):}{(z-z')^2(z'-w)}$$
$$+\frac{:J^a(w)J^a(w):}{(z-w)^2(z'-w)}+\frac{:\partial J^a(w)J^a(w):}{(z-z')(z'-w)}+\frac{:J^a(w)\partial J^a(w):}{(z-w)(z'-w)},$$

and so integration gives

$$\mathcal{R}\Big\{T(z)T(w)\Big\}\sim\frac{2\gamma^2(k+\mathsf{h}^\vee)k\dim\mathfrak{g}}{(z-w)^4}+\frac{4\gamma(k+\mathsf{h}^\vee)T(w)}{(z-w)^2}+\frac{2\gamma(k+\mathsf{h}^\vee)\partial T(w)}{z-w}.$$

By comparing with Equation (2.1.7), it follows that the modes, L_n, of $T(z)$ will form a representation of the Virasoro algebra if and only if $\gamma=1/2(k+\mathsf{h}^\vee)$. In this case,

$$\mathcal{R}\Big\{T(z)T(w)\Big\}\sim\frac{k\dim\mathfrak{g}}{2(k+\mathsf{h}^\vee)(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w},\tag{3.1.14}$$

so the central charge of the (chiral) theory is

$$c=\frac{k\dim\mathfrak{g}}{k+\mathsf{h}^\vee}.\tag{3.1.15}$$

The corresponding result for the antiholomorphic component of the energy-momentum tensor is analogous (in particular, $\overline{c}=c$), so this establishes the conformal nature of the Wess-Zumino-Witten models.

The commutation relations between the Virasoro and affine modes may be deduced from Equation (3.1.13) (and Equation (2.1.5)). The result is

$$\Big[\!\Big[L_n,J^a_m\Big]\!\Big]=-mJ^a_{m+n},\tag{3.1.16}$$

which together with Equations (3.1.8) and (2.1.1) define the chiral symmetry algebra of the Wess-Zumino-Witten model. In fact, Equation (3.1.16) shows that the chiral symmetry algebra is the semidirect sum $\mathfrak{Vir}\oplus\widehat{\mathfrak{g}}$. However, Equation (3.1.11) may be expanded in modes to give (sum over a implied)

$$L_n=\frac{1}{2(k+\mathsf{h}^\vee)}\sum_{m=-\infty}^{\infty}:J^a_mJ^a_{n-m}:,\tag{3.1.17}$$

so $\mathfrak{Vir}$ may be identified with a Lie subalgebra of the universal enveloping algebra of $\widehat{\mathfrak{g}}$ (this is the *Sugawara construction*). Therefore, one can instead regard this enveloping algebra, $\mathfrak{U}(\widehat{\mathfrak{g}})$, as the chiral symmetry algebra of the theory.

3.1.3. The State Space. The space of quantum states $\mathcal{S}$ for the Wess-Zumino-Witten model has not yet been discussed. From the general principles of Section 2.1.2 (and the results of Section 3.1.1), $\mathcal{S}$ decomposes into unitary, irreducible highest weight representations of $\mathfrak{U}(\widehat{\mathfrak{g}})$. The central element K of $\widehat{\mathfrak{g}}$ is represented by $k\,\mathrm{id}_{\mathcal{S}}$ (where $k\in\mathbb{Z}_+$ is the level of the theory), and the central charge C of $\mathfrak{Vir}$ is represented by $c\,\mathrm{id}_{\mathcal{S}}$ (where c is given by Equation (3.1.15)).

The representation theory of untwisted affine Lie algebras is discussed in Appendix B.3. There it is shown that an irreducible highest weight representation is determined by the level k of the representation and the eigenvalues of the zero-grade Cartan subalgebra on the highest weight vector (this defines a weight of the horizontal subalgebra $\mathfrak{g}$). For the representation to be unitary, it is necessary and sufficient that k be a non-negative integer, and that the induced weight λ of $\mathfrak{g}$ belongs to the fundamental affine alcove $\widehat{\mathsf{P}}_k$. This is the set of dominant weights satisfying $(\lambda,\theta)\leqslant k$, where θ is the highest root of $\mathfrak{g}$. These unitary representations (of $\widehat{\mathfrak{g}}$) are also known as integrable highest weight representations. It is important to note that at any given level k, there are only a finite number of such representations.

Whilst the requirement of unitarity severely restricts the allowed representations, it is not an easy matter to elucidate a more detailed picture of the quantum state space. In particular, it is not clear which of the integrable highest weight representations actually occur in the theory, and with what multiplicity. Nor is it clear how the (chiral) representations of $\widehat{\mathfrak{g}}$ should be paired, holomorphic with antiholomorphic, to define the quantum state space of the full theory, as in Section 2.1.2. The simplistic (though traditional) quantisation scheme used in Section 3.1.1 has the profound disadvantage that it does not construct the quantum state space directly.

It should therefore be of comfort to know that the formalism of *geometric quantisation* [**108**, **164**] admits a direct construction of the quantum state space. For Wess-Zumino-Witten models (on simple groups), this construction was analysed in [**57**, **58**] with the result that $\mathcal{S}$ does indeed decompose into integrable highest weight representations, and the particular decomposition depends upon the topology of the group[4]. Specifically,

$$\mathcal{S}\cong\bigoplus_{\lambda,\mu}M_{\lambda\mu}\left(V_\lambda\otimes V_\mu\right), \tag{3.1.18}$$

where V_λ denotes the integrable highest weight $\widehat{\mathfrak{g}}$-module characterised by the weight λ of $\mathfrak{g}$, and M is a matrix of multiplicities, called the *modular invariant*, which depends upon G. As a familiar example, when G is simply-connected, M is the diagonal invariant[5], $M_{\lambda\mu}=\delta_{\lambda\mu}$.

The modular invariant is so-named because of a constraint imposed on the multiplicities by the requirement that the theory be well-defined when the string worldsheet is a torus (generally, a string theory must be well-defined when the worldsheet is any Riemann surface). It is well known that complex structures on the torus are parametrised by an element τ of the upper half-plane [**91**], and that equivalent structures correspond to

[4]In this formalism, the wavefunctions are represented by analytic sections of a line bundle. The highest weight states corresponding to the integrable representations are distinguished from the non-integrable ones by having globally defined wavefunctions. In this way, matters of global topology affect the structure of the conformal field theory.

[5]Some papers refer to the *charge-conjugate* invariant instead, where each representation V_λ appears once, paired with its conjugate representation V_{λ^+} (Appendix A.2). As conjugation is induced by a symmetry of the Dynkin diagram of $\mathfrak{g}$, this invariant is equivalent to the diagonal invariant under a change of labelling of the weights in the antiholomorphic sector.

parameters that are related by the action of the modular group,

$$\tau \longmapsto \frac{a\tau+b}{c\tau+d}, \qquad a,b,c,d \in \mathbb{Z}, \quad ad-bc=1.$$

The constraint imposed on the theory is that the *partition function*,

$$Z(\tau) = \mathrm{tr}_{\mathcal{S}}\, e^{2\pi \mathrm{i}\tau(L_0-c/24)} e^{-2\pi\mathrm{i}\tau^*\left(\overline{L}_0-c/24\right)}, \tag{3.1.19}$$

is invariant under this action [**61**] (τ^* is the complex-conjugate of τ). This may be interpreted as the trace of the exponential of the hamiltonian operator, hence corresponds to the imposition of periodicity in time. As such, the theory relating to the partition function is effectively defined on a torus, whence the relation to the modular group. Note that the partition function may clearly be expressed in terms of the characters of the representations in $\mathcal{S}$ (whose behaviour under modular transformations is noted in Appendix B.3). This leads to a constraint on the matrix of multiplicities M, although this constraint is not generally sufficient to select physically well-defined theories [**79**]. (The direct construction of the state space given in [**57**] does, of course, yield modular invariant partition functions.)

3.1.4. Primary Fields and Correlation Functions. Comparing Equation (3.1.13) with Equation (2.1.8) shows that $J^a(z)$ is a (Virasoro) primary field of weight 1. That is, the conserved current fields correspond to states which are highest weight vectors under the $\mathfrak{Vir}$-action. However, this does not guarantee that these fields are primary with respect to the action of the full symmetry algebra. Such fields, which correspond to highest weight vectors under the $\widehat{\mathfrak{g}}$-action, are termed *affine* primary fields. A similar development to that leading to Equation (2.1.8) describes such a field $\psi(z)$ through its operator product expansions with the currents $J^a(z)$. Specifically, this yields

$$\lim_{w\to 0} A^a_n(w)\,|0\rangle = J^a_n\,|\psi\rangle\,,$$

where $A^a_n(w)$ denotes the appropriate operator product coefficients. Since $|\psi\rangle$ is a highest weight vector, $J^a_n\,|\psi\rangle = 0$ for all $n>0$, so it follows that there is only one non-vanishing singular coefficient, $A^a_0(w)$. This coefficient is precisely the field that corresponds to $J^a_0\,|\psi\rangle$.

The subalgebra of $\widehat{\mathfrak{g}}$ spanned by the zero-grade elements J^a_0 is just a copy of the finite-dimensional Lie algebra $\mathfrak{g}$. Indeed, an explicit isomorphism is given by $J^a_0 \mapsto t_a$, where $\{t_a\}$ is the (orthonormal) basis of $\mathfrak{g}$ with respect to which $J(z)$ was decomposed (Section 3.1.1). It is traditional to use this isomorphism to substitute the quantity $t_a\,|\psi\rangle$ for $J^a_0\,|\psi\rangle$ in the above considerations (here t_a also denotes the endomorphism representing $t_a \in \mathfrak{g}$, and hence $J^a_0 \in \widehat{\mathfrak{g}}$, in the module of highest weight vector $|\psi\rangle$). Whilst this can be rather confusing, it does emphasise the rôle played by the finite-dimensional Lie algebra $\mathfrak{g}$ in this setting. The operator product expansion of $J^a(z)$ with an affine primary field now takes the form

$$J^a(z)\,\psi(w) \sim \frac{(t_a\psi)(w)}{z-w}. \tag{3.1.20}$$

More importantly, such a field is only primary if the singular coefficient vanishes whenever $t_a \in \mathfrak{g}_+$. An affine primary field, associated with a highest weight of $\widehat{\mathfrak{g}}$, may therefore also be associated with a highest weight of $\mathfrak{g}$ — this is just the projection of the highest weight of $\widehat{\mathfrak{g}}$ onto the weight space of the horizontal subalgebra. It should be clear now, from Equation (3.1.5), that $J^a(z)$ is *not* an affine primary field.

It follows from Equations (3.1.17) and (2.1.12) that if $|\psi\rangle$ is an affine highest weight vector, then for $n > 0$ (sum over a),

$$L_n|\psi\rangle = \frac{1}{2(k+\mathsf{h}^\vee)}\left[\sum_{m<0} J^a_m J^a_{n-m} + \sum_{m\geqslant 0} J^a_{n-m}J^a_m\right]|\psi\rangle = \frac{1}{2(k+\mathsf{h}^\vee)} J^a_n J^a_0|\psi\rangle = 0,$$

since J^a_n and J^a_0 commute. Furthermore, by Equation (A.1.5),

$$L_0|\psi\rangle = \frac{1}{2(k+\mathsf{h}^\vee)} J^a_0 J^a_0|\psi\rangle = \frac{1}{2(k+\mathsf{h}^\vee)} t_a t_a|\psi\rangle = \frac{(\psi,\psi+2\rho)}{2(k+\mathsf{h}^\vee)}|\psi\rangle, \tag{3.1.21}$$

where ψ is (also) denoting the weight of $\mathfrak{g}$ corresponding to the highest weight vector $|\psi\rangle$, and ρ is the Weyl vector (weight) of $\mathfrak{g}$. Thus, an affine highest weight vector is also a Virasoro highest weight vector; correspondingly, an affine primary field is a Virasoro primary field. Indeed, if the primary field is associated with the weight ψ of $\mathfrak{g}$, then its conformal weight is $h_\psi = (\psi,\psi+2\rho)/2(k+\mathsf{h}^\vee)$. The converse is not true — a Virasoro primary field is not necessarily an affine primary field, as $J^a(z)$ shows.

It follows immediately that a correlation function of (affine and/or Virasoro) primary fields must satisfy the differential equations that were derived in Section 2.2.1. In particular, Equation (2.2.2) fixes the form of correlation functions involving up to three primary fields (up to a multiplicative constant). The constraints involving (Virasoro) null fields also apply, although the restriction that such null fields may only exist when $h = 0$ or $c \leqslant 1$ [**101**] limits their usefulness[6].

However, one can repeat the derivations of Section 2.2.1 using affine modes and affine primary fields rather than their Virasoro counterparts. The analogue of Equation (2.2.2) is obtained by noting that the zero-grade modes ($J^a_0 \mapsto t_a$) annihilate both $|0\rangle$ and $\langle 0|$. An easy computation using Equation (3.1.20) then yields the following equation:

$$\sum_{i=1}^{m} t^i_a \langle 0|\,\mathcal{R}\Big\{\psi_1(z_1)\cdots\psi_m(z_m)\Big\}|0\rangle = 0. \tag{3.1.22}$$

Here, t^i_a is supposed to indicate that the element $t_a \in \mathfrak{g}$ is acting on $\psi_i(z_i)$ to give $(t_a\psi_i)(z_i)$ inside the correlation function. This equation holds for any element, t_a, of the horizontal algebra $\mathfrak{g}$. Of course, if $t_a \in \mathfrak{g}_+$ then each term vanishes separately (by definition of primary), so the interesting constraints are generated by the other elements.

[6] Among the Wess-Zumino-Witten theories based on simple Lie algebras (with a single exception), the only primary field with null fields as descendants is the identity field. The single exception is the theory with $\mathfrak{g} = \mathfrak{su}(2)$ and level $k = 1$, which has two affine primary fields, both with null fields as descendants.

In particular, when $t_a \in \mathfrak{t}$, the Cartan subalgebra of $\mathfrak{g}$, Equation (3.1.22) becomes

$$\left\langle \sum_{i=1}^{m} \psi_i, t_a \right\rangle \langle 0| \mathcal{R}\Big\{ \psi_1(z_1) \cdots \psi_m(z_m) \Big\} |0\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing of the weights in $\mathfrak{t}^*$ with elements of $\mathfrak{t}$. It follows that the correlation function must vanish unless the sum of the weights ψ_i is zero. Since the weights correspond to highest weights of integrable representations, this can only happen if each weight is zero. In other words, the *only* non-vanishing correlation function of affine primary fields is the trivial one, containing only the identity field.

Note however that the operator product expansion for the field $\psi(w)$ given by Equation (3.1.20) is derived from the requirement that $J^a_n \psi = 0$ for each $n > 0$. This requirement is clearly still satisfied if ψ is a zero-grade descendant of a highest weight vector, so this operator product expansion is still valid for the corresponding fields. It follows that Equation (3.1.22) holds for correlation functions where the $\psi_i(z_i)$ are all such zero-grade fields. Clearly it is possible for the sums of the weights corresponding to such fields to vanish.

As generic correlation functions involving affine primary fields vanish, one cannot expect that all other correlation functions are determined by these primary correlation functions, as was the case for the Virasoro algebra (Section 2.2.1). Instead, one expects that arbitrary correlation functions will be determined by those containing (the non-vanishing) zero-grade fields. Indeed, the analogue of Equation (2.2.5) is

$$\begin{aligned} &\langle 0| \mathcal{R}\Big\{ \left(J^a_{-n}\psi\right)(w)\, \psi_1(w_1) \cdots \psi_m(w_m) \Big\} |0\rangle \\ &\qquad = -\sum_{i=1}^{m} \frac{t^i_a}{(w_i - w)^n} \langle 0| \mathcal{R}\Big\{ \psi(w)\, \psi_1(w_1) \cdots \psi_m(w_m) \Big\} |0\rangle , \end{aligned} \quad (3.1.23)$$

whose derivation is exactly the same, except that Equation (2.1.9) is used as a starting point[7]. This derivation presumes that each $\psi_i(w_i)$ is a zero-grade field, so Equation (3.1.23) reproduces Equation (3.1.22) when $n = 0$ (and $\psi(w)$ is also a zero-grade field).

Of course, Equation (3.1.23) may be generalised to correlation functions involving more general descendant fields. Descendant fields which are null therefore imply constraints on the zero-grade correlation functions. Since $\mathfrak{Vir}$ is acting as a subalgebra of $\mathfrak{U}(\widehat{\mathfrak{g}})$, and zero-grade fields are Virasoro primary fields, Equation (3.1.17) yields an infinite number of null fields. For example, if $|\psi\rangle$ denotes a zero-grade state,

$$L_{-1}|\psi\rangle = \frac{1}{k+\mathrm{h}^\vee} J^a_{-1} J^a_0 |\psi\rangle \qquad \Rightarrow \qquad L_{-1}\psi(z) - \frac{1}{k+\mathrm{h}^\vee} J^a_{-1} t_a \psi(z) \text{ is null.}$$

[7] Again, this derivation assumes that z takes values in the Riemann sphere. That is, that the string worldsheet Σ is topologically a sphere.

From Equation (2.2.5) and (a generalisation of) Equation (3.1.23), the corresponding constraints on the zero-grade correlation functions are the *Knizhnik-Zamolodchikov equations*:

$$\left[\partial_i + \frac{1}{k+\mathrm{h}^\vee}\sum_{j\neq i}\frac{t_a^i t_a^j}{w_i - w_j}\right]\langle 0|\,\mathcal{R}\Big\{\psi_1(w_1)\cdots\psi_m(w_m)\Big\}\,|0\rangle = 0. \tag{3.1.24}$$

The index a is still implicitly summed over.

One last class of constraints is provided by purely affine null fields. These correspond to null vectors in the Verma modules corresponding to the integrable highest weight representations that appear in the (chiral) theory. These null vectors are known in all generality [**99**]. If the highest weight state of this Verma module is $|\lambda\rangle$, then the null vectors are generated by $r+1$ *primitive* states. There are $r = \operatorname{rank}\mathfrak{g}$ such states of the form

$$e_{-\alpha_i}^{\lambda_i+1}|\lambda\rangle\,, \qquad (i=1,2,\ldots,r),$$

where α_i denotes a simple root of $\mathfrak{g}$, and $e_{-\alpha_i}$ is the copy of the corresponding root vector acting in $\widehat{\mathfrak{g}}$ at zero grade. These null vectors arise from the finite-dimensionality of the $\mathfrak{g}$-module defined by the zero-grade states. They reflect the representation theory of $\mathfrak{g}$. The last null vector is somewhat more exotic, and has the form (Appendix B.3)

$$\left(J_{-1}^{\theta}\right)^{k+1-(\lambda,\theta)}|\lambda\rangle\,,$$

where θ is the highest root of $\mathfrak{g}$, and J_{-1}^{θ} is the copy of $e_\theta \in \mathfrak{g}$ acting in $\widehat{\mathfrak{g}}$ at grade -1.

The corresponding constraint is obtained through the repeated application of Equation (3.1.23). This is the *Gepner-Witten equation*:

$$\begin{aligned}
0 &= \langle 0|\,\mathcal{R}\Big\{\left(\left(J_{-1}^{\theta}\right)^p\lambda\right)(w)\,\psi_1(w_1)\cdots\psi_m(w_m)\Big\}\,|0\rangle \\
&= \sum_{i_1=1}^{m}\cdots\sum_{i_p=1}^{m}\frac{e_\theta^{i_1}\cdots e_\theta^{i_p}}{(w-w_{i_1})\cdots(w-w_{i_p})}\,\langle 0|\,\mathcal{R}\Big\{\lambda(w)\,\psi_1(w_1)\cdots\psi_m(w_m)\Big\}\,|0\rangle \\
&= \sum_{\substack{\ell_1,\ldots,\ell_m\geqslant 0\\ \ell_1+\ldots+\ell_m=p}}\frac{p!}{\ell_1!\cdots\ell_m!}\,\frac{\left(e_\theta^1\right)^{\ell_1}\cdots\left(e_\theta^m\right)^{\ell_m}}{(w-w_1)^{\ell_1}\cdots(w-w_m)^{\ell_m}}\,\langle 0|\,\mathcal{R}\Big\{\lambda(w)\,\psi_1(w_1)\cdots\psi_m(w_m)\Big\}\,|0\rangle\,,
\end{aligned} \tag{3.1.25}$$

where $p\geqslant k+1-(\lambda,\theta)$, and $\psi_1(w_1),\ldots,\psi_m(w_m)$ are zero-grade fields. Note that $\lambda(w)$ is assumed to be an affine primary field in this equation.

3.1.5. Fusion. Consider the correlation function of an affine primary field $\lambda(z)$ and a zero-grade field $\mu'(w)$, descended from the primary field $\mu(w)$. By Equations (2.2.3) and (3.1.22), this function vanishes unless the conformal weights match ($h_\lambda = h_\mu$), and $\mu' = -\lambda$ as weights of $\mathfrak{g}$. In fact, Equation (3.1.22) constrains such functions even more severely. Recalling the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{t} \oplus \mathfrak{g}_+$ of Appendix A.1, if

$\mu'(w)$ has a non-vanishing descendant (so $|\mu'\rangle = t\,|\mu''\rangle$ for some $t \in \mathfrak{g}_+$), then

$$\langle 0|\,\mathcal{R}\Big\{\lambda(z)\,\mu'(w)\Big\}\,|0\rangle = -\,\langle 0|\,\mathcal{R}\Big\{(t\lambda)(z)\,\mu''(w)\Big\}\,|0\rangle = 0.$$

This correlation function therefore vanishes unless $|\mu'\rangle$ is a *lowest-weight* state with respect to $\mathfrak{g}$ (is annihilated by $\mathfrak{g}_-$). This, together with $\mu' = -\lambda$, uniquely determines μ to be the highest weight of the representation conjugate (see Appendix A.2) to the representation of highest weight λ.

More generally, in a correlation function involving the field $(t\lambda)(z)$ ($\lambda(z)$ primary, $t \in \mathfrak{g}_-$) and a zero-grade field $\mu'(w)$, it follows from Equation (3.1.22) and the above argument that the correlation function vanishes unless $t\,|\mu'\rangle$ is the lowest weight state in the representation conjugate to that of highest weight λ. As $\mu' = -t\lambda$, the weight μ' is therefore completely determined. However, the multiplicity of μ' in the representation conjugate to that of highest weight λ may be greater than one, so there may be several linearly independent candidates for $|\mu'\rangle$ in this (finite-dimensional) weight space. Of course, t acts linearly on this space, and sends it to the one-dimensional subspace spanned by the lowest weight vector. Therefore, one may choose $|\mu'\rangle$ to be orthogonal to the kernel of this transformation, and in this way, $|\mu'\rangle$ and thus $\mu'(w)$, are uniquely specified (up to an unimportant normalisation).

The above conclusion obviously extends inductively to more general descendant fields of $\lambda(z)$, so it follows that every zero-grade field is uniquely paired with another zero-grade field (hereafter referred to as the *conjugate field*), such that the correlation function involving these two fields is non-vanishing. In this way, the normalisation of affine correlation functions involving two zero-grade fields is established. This should be contrasted with the normalisation chosen for the analogous Virasoro correlation functions in Section 2.2.1. Here, the pairing involves a state in one representation and a corresponding state in the conjugate representation (hence the name), instead of pairing a state with itself.

With the normalisation of these correlation functions determined (from which all others may be derived through the operator product expansion), one can now discuss the process of fusion in Wess-Zumino-Witten models. As in Section 2.2.2, fusion may be characterised through the correlation functions involving three zero-grade fields. However, the analysis is further complicated by the fact that there are, in general, many zero-grade fields corresponding to a given primary field, and that this plethora of zero-grade correlation functions is constrained by Equation (3.1.22).

More specifically, the zero-grade correlation function involving $\lambda'(z_1)$, $\mu'(z_2)$, and $\nu'(z_3)$ will vanish unless a descendant of the field conjugate to $\nu'(z_3)$ appears in the operator product expansion of $\lambda'(z_1)$ and $\mu'(z_2)$. Because of the aforementioned plethora of such (related) correlation functions, the corresponding fusion process traditionally involves the entire families headed by the primary fields. That is, if the above correlation

function is non-vanishing, then the corresponding fusion rule is

$$[\lambda] \times [\mu] = \left[\nu^+\right] + \ldots,$$

where ν^+ denotes the weight conjugate to ν.

It will generally happen that there is more than one non-vanishing correlation function involving zero-grade descendants of fixed primaries. For example, the operator product expansion of two zero-grade fields from the (respective) families, $[\lambda]$ and $[\mu]$, might contain descendants of more than one zero-grade field in the family $[\nu]$ (as the multiplicity of the appropriate weight need not be one). The same applies to other combinations of zero-grade fields in these families, leading to more non-vanishing correlation functions. Of course, many of these non-vanishing correlation functions will be related by Equation (3.1.22), so it is only the *independent couplings* of the zero-grade fields in given families which is of interest.

This number of independent couplings — that is, the number of non-vanishing correlation functions of three zero-grade fields from given families, *modulo* the relations between them implied by Equation (3.1.22) — defines the *fusion multiplicity*. This fusion multiplicity (fusion coefficient) associated with the families $[\lambda]$, $[\mu]$, and $[\nu^+]$ is denoted by $\mathcal{N}_{\lambda\mu}{}^{\nu}$. The conjugation of ν in this definition reflects the normalisation of correlation function of two zero-grade fields. It leads to the general form for a fusion rule:

$$[\lambda] \times [\mu] = \sum_{\nu} \mathcal{N}_{\lambda\mu}{}^{\nu} [\nu]. \tag{3.1.26}$$

In contrast with Equation (2.2.6), where the fusion coefficients take the value 0 or 1, the fusion coefficients in the Wess-Zumino-Witten models may take more general (non-negative integer) values.

To better understand these fusion multiplicities, consider the set of correlation functions involving the zero-grade descendants of $\lambda(z_1)$, $\mu(z_2)$, and $\nu(z_3)$. To describe a non-vanishing correlation function, the corresponding weights λ', μ', and ν' of $\mathfrak{g}$ must be related by $-\nu' = \lambda' + \mu'$. That is, $-\nu'$ must appear as a weight of the *tensor product* of the representations of $\mathfrak{g}$ of highest weight λ and μ (Appendix A.3). Of course, this tensor product decomposes into irreducible highest weight representations, and the representation of highest weight ν^+ (to which $-\nu'$ belongs) appears $N_{\lambda\mu}{}^{\nu^+}$ times. It follows that these $N_{\lambda\mu}{}^{\nu^+}$ independent representations correspond to $N_{\lambda\mu}{}^{\nu^+}$ independent couplings for the set of zero-grade correlation functions being considered.

The preceding discussion appears to suggest that the fusion multiplicities $\mathcal{N}_{\lambda\mu}{}^{\nu}$ are in fact just the tensor product coefficients $N_{\lambda\mu}{}^{\nu}$ of $\mathfrak{g}$. However, this discussion only takes into account the constraints given by Equation (3.1.22). There are further constraints, in particular the Gepner-Witten equation, Equation (3.1.25), which force additional zero-grade correlation functions to vanish (and hence all the functions related to these by Equation (3.1.22) to vanish also). That is, the preceding discussion only demonstrates

that

$$\mathcal{N}_{\lambda\mu}{}^{\nu} \leqslant N_{\lambda\mu}{}^{\nu},$$

a result of not negligible importance. Indeed, a detailed study of the Gepner-Witten equation leads to the so called *depth rule*, the original method for computing fusion multiplicities in Wess-Zumino-Witten models [**85**].

The properties of the fusion multiplicities are similar to those of the tensor product coefficients of $\mathfrak{g}$. As in Equation (2.2.7), the identity field (primary of weight 0) supplies the unit for the fusion process, with

$$\mathcal{N}_{0\lambda}{}^{\mu} = \delta_{\lambda\mu}. \tag{3.1.27}$$

Equation (2.2.8) generalises slightly as $\mathcal{N}_{\lambda\mu}{}^{\nu}$ corresponds to a non-vanishing correlation function involving fields from families $[\lambda]$, $[\mu]$, and $[\nu^+]$. Thus,

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \mathcal{N}_{\mu\lambda}{}^{\nu} = \mathcal{N}_{\mu\nu^+}{}^{\lambda^+} = \mathcal{N}_{\nu^+\mu}{}^{\lambda^+} = \mathcal{N}_{\nu^+\lambda}{}^{\mu^+} = \mathcal{N}_{\lambda\nu^+}{}^{\mu^+}. \tag{3.1.28}$$

Suppose now that ω is an automorphism of $\mathfrak{g}$ corresponding to a Dynkin diagram symmetry (Appendix A.2). Since ω merely corresponds to a certain arbitrariness in the numbering of the fundamental weights of $\mathfrak{g}$ (which can have no bearing on the theory), it follows[8] that the correlation functions must be invariant upon replacing every weight by its image under ω. In particular, applying this logic to correlation functions of three fields gives an additional symmetry of the fusion coefficients, namely

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \mathcal{N}_{\omega(\lambda)\omega(\mu)}{}^{\omega(\nu^+)^+}.$$

Conjugation is such an automorphism, hence it follows from Equation (3.1.28) that

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \mathcal{N}_{\lambda^+\mu^+}{}^{\nu^+} = \mathcal{N}_{\lambda^+\nu}{}^{\mu}. \tag{3.1.29}$$

3.2. Boundary Conformal Field Theory and Branes

Recall that in the string picture of Wess-Zumino-Witten models, the conformal field theory is defined on the string worldsheet, a two-dimensional manifold. This theory describes closed strings, little loops, tracing out the two-dimensional worldsheet in time. This picture can be generalised to a theory of open strings, where the loops are replaced by finite lengths with endpoints. The string worldsheet therefore becomes a two-dimensional manifold *with boundary*, and the theory becomes a *boundary* conformal field theory. This geometric picture of open string Wess-Zumino-Witten models will be studied in much detail in Section 6.3. For the present purposes, however, it will suffice to enquire as to the boundary conditions that should be imposed upon the open string endpoints.

[8] It is tempting to conclude that this will also follow for the automorphisms of $\widehat{\mathfrak{g}}$. However, these more general transformations need not preserve the conformal weights (hence the correlation functions of three fields are not invariant), nor need they preserve the vanishing of the sum of the weights appearing in a non-trivial zero-grade correlation function. It cannot be stressed too much that even though the symmetry algebra is $\mathfrak{U}(\widehat{\mathfrak{g}})$, Wess-Zumino-Witten models are defined on a *finite-dimensional* Lie group G so the fundamental symmetries (of the type used here) are those of G (and hence $\mathfrak{g}$).

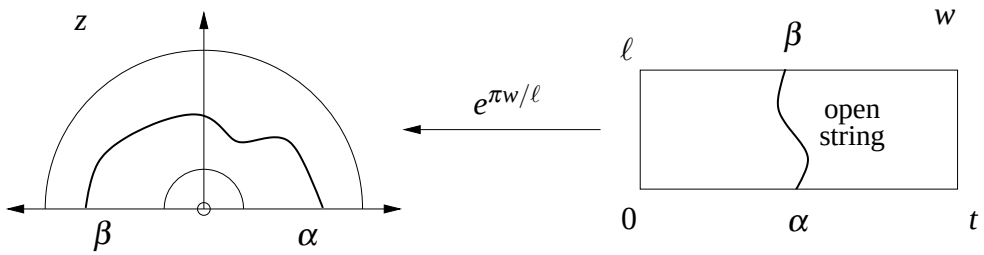

FIGURE 3.1. Changing coordinates from the strip (w) to the half plane (z). Note that the strip boundaries (upper and lower) are mapped to (disconnected pieces of) the half plane boundary, $z = \overline{z}$. The boundary conditions are (generically) labelled by α and β.

3.2.1. Gluing Conditions on the Half Plane. Away from the boundary of the string worldsheet, one can choose (local) complex coordinates as in the previous section, and so most of the theory derived there applies. In this context, this is usually referred to as the *bulk theory*. At the boundary, however, this is no longer the case, and it is convenient to choose the local coordinate chart so that z belongs to the upper half plane, $\mathfrak{Im}\, z \geqslant 0$. Since $\overline{z}$ is (while formally independent of z) to be eventually identified with z^*, the conjugate of z, this coordinate may be taken to belong to the lower half plane, $\mathfrak{Im}\, \overline{z} \leqslant 0$. The most suitable local chart to keep in mind is actually one which describes the (open) string worldsheet as a "strip" with two boundaries. This chart may be conformally mapped to the half plane (minus the origin) as illustrated in Figure 3.1.

As in Section 2.1.1, one now asks what the infinitesimal conformal transformations of the half plane are. It should be clear that these correspond to vector fields of the form $\varepsilon(z)\,\partial/\partial z + \overline{\varepsilon}(\overline{z})\,\partial/\partial\overline{z}$ with the added constraint that these vector fields must preserve the boundary. Decomposing z as $\tau + \mathrm{i}\sigma$ and $\overline{z} = z^*$ as $\tau - \mathrm{i}\sigma$ (so τ parametrises the boundary), these vector fields may be rewritten in the form

$$\frac{1}{2}\left(\varepsilon(z) + \overline{\varepsilon}(\overline{z})\right)\frac{\partial}{\partial\tau} + \frac{\mathrm{i}}{2}\left(\overline{\varepsilon}(\overline{z}) - \varepsilon(z)\right)\frac{\partial}{\partial\sigma}.$$

To preserve the boundary $z = \overline{z}$ ($\sigma = 0$), the coefficient of $\partial/\partial\sigma$ must vanish there, hence the constraint amounts to $\varepsilon(z) = \overline{\varepsilon}(\overline{z})$ at $z = \overline{z}$. In other words, ε and $\overline{\varepsilon}$ are no longer independent. They are both completely determined by by their values on the boundary, where they coincide[9].

It follows that on the boundary, the Lie algebra formed by the infinitesimal conformal transformations is therefore (a completion of) a *single* copy of the Witt algebra. After quantising, one gets a representation of a *single* copy of the Virasoro algebra on the quantum state space. In comparison with the bulk theory, where two copies act, one writes $L_n = \overline{L}_n$ at $z = \overline{z}$ (for each $n \in \mathbb{Z}$). In terms of the energy-momentum field, this is called the *conformal boundary condition*:

$$T(z) = \overline{T}(\overline{z}) \qquad \text{at } z = \overline{z}. \tag{3.2.1}$$

[9]Of course this does not imply that ε and $\overline{\varepsilon}$ coincide as functions. One is holomorphic and the other is antiholomorphic, and more to the point, they are defined on different domains.

This boundary condition has a nice interpretation as requiring that no momentum flows across the boundary, and is therefore often taken as the fundamental requirement of boundary conformal field theory.

However, in a Wess-Zumino-Witten model the energy-momentum field is constructed from the currents $J^a(z)$ (and $\overline{J}^a(\overline{z})$). Therefore, it is of considerable interest to determine the possible boundary conditions on these fields. Making the ansatz, $J^a(z) = \Omega_{ab}\overline{J}^b(\overline{z})$ at $z = \overline{z}$, where Ω_{ab} may vary along the boundary, the conformal boundary condition becomes (see Equation (3.1.11))

$$:J^a(z)J^a(z): = \Omega_{ab}\Omega_{ac}:\overline{J}^b(\overline{z})\overline{J}^c(\overline{z}): = :\overline{J}^b(\overline{z})\overline{J}^b(\overline{z}): \qquad \text{at } z=\overline{z},$$

which is satisfied if and only if $\Omega_{ab}\Omega_{ac} = \delta_{bc}$. If $\Omega\colon \mathfrak{g} \to \mathfrak{g}$ is the matrix with entries Ω_{ab} (with respect to the orthonormal basis $\{t_a\}$), then this just requires that Ω be orthogonal with respect to the Killing form of $\mathfrak{g}$. Note that orthogonal transformations preserve angles.

The consistent boundary conditions that can be imposed on the currents therefore take the form

$$J(z) = \Omega\left(\overline{J}(\overline{z})\right) \qquad \text{at } z=\overline{z}, \tag{3.2.2}$$

where Ω is an orthogonal transformation of $\mathfrak{g}$ (possibly varying along the boundary)[10]. Such boundary conditions are sometimes called *gluing conditions*, as they "glue" the bulk holomorphic and antiholomorphic currents at the boundary. The conformal boundary condition, Equation (3.2.1), breaks exactly half of the conformal symmetry, in that one ends up with a single copy of the Virasoro algebra rather than two. It is therefore interesting to investigate how much of the affine symmetry is broken by a given gluing condition on the currents (clearly at least half the symmetries must be broken).

One way to do this is to see how this affects the operator product expansions at the boundary. Equations (3.1.5) and (3.1.6) give (assuming they are valid)

$$\mathcal{R}\left\{J^a(z)J^b(w)\right\} = \Omega_{aa'}(z)\Omega_{bb'}(w)\mathcal{R}\left\{\overline{J}^a(z)\overline{J}^b(w)\right\},$$

whose singular terms are

$$\begin{aligned}\frac{k\delta_{ab}}{(z-w)^2} + \frac{f_{abc}\Omega_{cc'}(w)\overline{J}^{c'}(w)}{z-w} &= \frac{k\Omega_{aa'}(z)\Omega_{ba'}(w)}{(z-w)^2} + \frac{\Omega_{aa'}(z)\Omega_{ba'}(w)f_{a'b'c'}\overline{J}^{c'}(w)}{z-w} \\ &= \frac{k\Omega_{aa'}(w)\Omega_{ba'}(w)}{(z-w)^2} + \frac{\Omega_{aa'}(w)\Omega_{ba'}(w)f_{a'b'c'}\overline{J}^{c'}(w) + k\partial\Omega_{aa'}(w)\Omega_{ba'}(w)}{z-w}.\end{aligned}$$

Comparing coefficients of $(z-w)^{-2}$ recovers the condition that Ω must be orthogonal. Noting that the operators J^a_n are linearly independent of the identity operator, the $(z-w)^{-1}$

[10]It should be pointed out that more general boundary conditions can be imposed on the currents which are also consistent with the conformal boundary condition, Equation (3.2.1). Examples may be found in [**134**, **135**] which seem to correspond to imposing conditions on a *subalgebra* of $\mathfrak{g}$. These more general boundary conditions will not, however, be considered in this thesis.

coefficients give $\partial\Omega \cdot \Omega^{-1} = 0$ and $f_{abc}\Omega_{cc'} = \Omega_{aa'}\Omega_{bb'}f_{a'b'c'}$. This first constraint requires Ω to be locally constant, and the second constraint is precisely the condition that Ω is an automorphism of $\mathfrak{g}$. Automorphisms of $\mathfrak{g}$ are always orthogonal (with respect to the Killing form), so locally constant automorphisms exhaust the possible Ω which preserve the affine operator product expansions. The regular terms of the operator product expansion can be checked to give no further constraints on Ω.

It follows that the gluing (boundary) condition, Equation (3.2.2), preserves the maximal number (half) of affine symmetries if and only if Ω is constant on each connected component of the boundary, and takes values in $\operatorname{Aut}\mathfrak{g}$. Such boundary conditions will be referred to as *symmetry-preserving*. The boundary conditions considered in the remainder of this chapter will always be assumed to be symmetry-preserving[11]. In the case illustrated in Figure 3.1, the boundary conditions imposed on the two connected components of the boundary, corresponding to the two endpoints of the open string, will be (generically) labelled by α and β. These labels are meant to specify the automorphism as well as any other degrees of freedom. The broken affine symmetries are evidenced by Equation (3.1.7), which is not preserved by these gluing conditions. In fact, this operator product expansion can only be preserved if $\Omega = 0$, which contradicts the conformal boundary condition.

When $\Omega \in \operatorname{Aut}\mathfrak{g}$ then, the gluing condition identifies the holomorphic and antiholomorphic currents up to a "twist" (given by Ω) which preserves half the affine symmetries. The boundary conformal field theory therefore has only one set of conserved currents. Because the operator product expansion of these currents is preserved, the modes $J^a_n = \Omega_{ab}\overline{J}^b_n$ form a representation of $\widehat{\mathfrak{g}}$. It follows that the state space $\mathcal{S}'$ of the quantised half plane theory decomposes as

$$\mathcal{S}' \cong \bigoplus_\lambda {n_{\alpha\beta}}^\lambda V_\lambda ,$$

where V_λ denotes an integrable highest weight module of $\widehat{\mathfrak{g}}$ and ${n_{\alpha\beta}}^\lambda$ is the multiplicity with which it occurs. The subscripts α and β refer to the dependence of these multiplicities on the boundary conditions α and β imposed. The partition function of the theory on the half plane (with these boundary conditions) is therefore

$$Z_{\alpha\beta}(q) = \operatorname{tr}_{\mathcal{S}'} q^{L_0 - c/24} = \sum_\lambda {n_{\alpha\beta}}^\lambda \chi^{\mathfrak{Vir}}_\lambda(q) , \tag{3.2.3}$$

where $\chi^{\mathfrak{Vir}}_\lambda(q)$ denotes the (normalised) character of V_λ as a $\mathfrak{Vir}$-module (see Equation (B.3.1)).

3.2.2. Gluing Conditions on the Annulus. For the purposes of conformal field theory, one can define a brane as the particular boundary condition imposed on the open string endpoints. The symmetry-preserving boundary conditions therefore determine

[11] In fact, it will be shown in Proposition 6.3 that the geometric definition of branes is consistent if $\Omega \in \operatorname{Aut}\mathfrak{g}$, but seems to be inconsistent in most other cases.

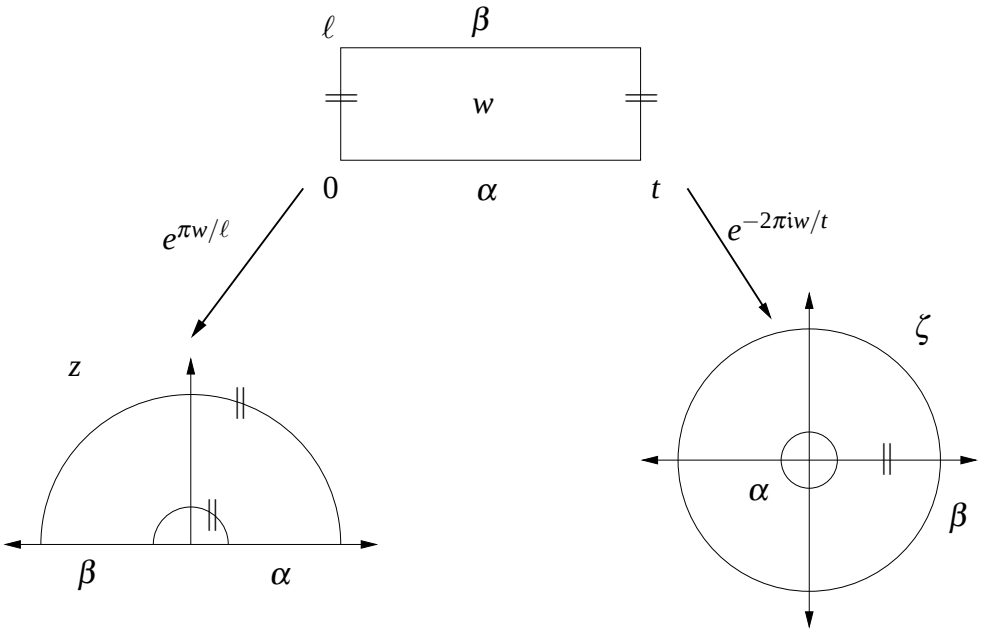

FIGURE 3.2. Changing coordinates from the half plane (z) to the annulus (ζ) via the strip (w). Periodic boundary conditions are labelled with a $\|$, and symmetry-preserving gluing conditions are labelled α and β.

symmetry-preserving branes. To define the concept of a brane properly, one must consider the target space G where the strings live. There, branes are defined geometrically and possess a rich structure which encodes this boundary condition (and will be investigated in detail in Section 6.3). However, this structure does not pull back to the string worldsheet where the conformal field theory is defined. Nevertheless, branes may be profitably studied using the techniques of boundary conformal field theory, in particular using those techniques pioneered by Cardy [**42**–**44**].

First, one imposes periodic boundary conditions in time to facilitate the physical interpretation of the partition function, Equation (3.2.3). With the conventions of Figure 3.2, this sets the variable q of the half plane partition function to $e^{-\pi t/\ell}$. Then one makes a transformation of variables from the half plane to the annulus, again illustrated in Figure 3.2. The currents $J^a(z)$ transform as a 1-form under coordinate changes (see Section 6.2.2), so

$$\frac{\pi}{\ell} z\, J^a(z) = J^a_{\text{strip}}(w) = \frac{-2\pi \mathrm{i}}{t} \zeta\, J^a_{\text{ann.}}(\zeta) \qquad \Rightarrow \qquad \zeta\, J^a_{\text{ann.}}(\zeta) = \frac{\mathrm{i}t}{2\ell} z\, J^a(z)\,.$$

Similarly, $\overline{\zeta}\, \overline{J}^a_{\text{ann.}}\left(\overline{\zeta}\right) = (-\mathrm{i}t/2\ell)\, \overline{z}\, \overline{J}^a(\overline{z})$. Under this change of coordinates, the gluing condition, Equation (3.2.2), becomes

$$\zeta\, J^a_{\text{ann.}}(\zeta) = -\overline{\zeta}\, \Omega_{ab} \overline{J}^b_{\text{ann.}}\left(\overline{\zeta}\right) \qquad \text{at } |\zeta| = 1,\, e^{2\pi\ell/t}.$$

Dropping the subscripts "ann.", the gluing condition of the annulus modes at the boundaries $|\zeta| = r$ ($r = 1$, $e^{2\pi\ell/t}$) is

$$\sum_n J^a_n \zeta^{-n} = -\Omega_{ab} \sum_n \overline{J}^b_n \overline{\zeta}^{-n} = -\sum_n \Omega_{ab} \overline{J}^b_n r^{-2n} \zeta^n = -\sum_n \Omega_{ab} \overline{J}^b_{-n} r^{2n} \zeta^{-n}$$
$$\Rightarrow \qquad J^a_n = -r^{2n} \Omega_{ab} \overline{J}^b_{-n}.$$

Of course, this condition holds with respect to any basis $\{t_a\}$ of $\mathfrak{g}$, not just an orthonormal one. This basis-independent condition is obtained by tensoring with t_a to get

$$J^a_n \otimes t_a = -r^{2n} \overline{J}^a_{-n} \otimes \Omega^{-1}(t_a) \qquad \Rightarrow \qquad J_n = -r^{2n} \Omega^{-1}\left(\overline{J}_{-n}\right),$$

since Ω is orthogonal. It is very convenient to decompose this condition with respect to *different* bases of $\mathfrak{g}$ for the holomorphic and antiholomorphic sector. A particularly nice choice would then be to choose the antiholomorphic basis, $\{\bar{t}_a\}$, to be related to the holomorphic one, $\{t_a\}$, by $\bar{t}_a = \Omega^{-1}(t_a)$. Decomposing the antiholomorphic sector with respect to this choice of $\{\bar{t}_a\}$ would effectively replace Ω_{ab} by δ_{ab}.

However, the structure of $\mathfrak{g}$ (and $\widehat{\mathfrak{g}}$) suggests an even more convenient choice. Recall that the gluing conditions relate holomorphic modes of grade n with antiholomorphic modes of grade $-n$. This suggests that holomorphic raising and lowering operators of $\widehat{\mathfrak{g}}$ should be related to antiholomorphic lowering and raising operators, respectively. That is, that the annulus gluing conditions should swap the subalgebras $\widehat{\mathfrak{g}}_-$ and $\widehat{\mathfrak{g}}_+$ of the triangular decomposition of $\widehat{\mathfrak{g}}$. By considering the zero grade modes, one finds that this will indeed be the case if and only if Ω swaps the corresponding subalgebras $\mathfrak{g}_-$ and $\mathfrak{g}_+$ of $\mathfrak{g}$.

Given a choice of triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{t} \oplus \mathfrak{g}_+$, there is an almost unique[12] automorphism of $\mathfrak{g}$ preserving $\mathfrak{t}$ and swapping $\mathfrak{g}_-$ and $\mathfrak{g}_+$. This is the *Chevalley automorphism* ω_{C} (Appendix A.2). It is convenient at this point to fix the (complexified) holomorphic basis of $\mathfrak{g}$ to consist of the root vectors e_α and the simple coroots $\alpha_i^\vee$. ω_{C} acts on these basis elements to give

$$\omega_{\mathrm{C}}(e_\alpha) = -e_{-\alpha} \qquad \text{and} \qquad \omega_{\mathrm{C}}\left(\alpha_i^\vee\right) = -\alpha_i^\vee.$$

The best choice for the antiholomorphic basis elements is to take $\bar{t}_a = (\omega_{\mathrm{C}} \circ \Omega)^{-1}(t_a) = \left(\Omega^{-1} \circ \omega_{\mathrm{C}}\right)(t_a)$. One finds that[13]

$$J^a_n \otimes t_a = -r^{2n} \overline{J}^{-a}_{-n} \otimes \Omega^{-1}(t_{-a}) = r^{2n} \overline{J}^{-a}_{-n} \otimes \left(\Omega^{-1} \circ \omega_{\mathrm{C}}\right)(t_a) = r^{2n} \overline{J}^{-a}_{-n} \otimes \bar{t}_a.$$

In what follows, this choice of basis will be made, so the annulus mode gluing conditions take the convenient form

$$J^a_n = r^{2n} \overline{J}^{-a}_{-n} \qquad \text{at } |\zeta| = r. \tag{3.2.4}$$

[12] To be precise, unique up to multiplication by a Dynkin symmetry of $\mathfrak{g}$. It will be seen shortly why taking this Dynkin symmetry to be trivial is the best choice.

[13] The label a in J^a_n may correspond to a root vector ($a = \alpha$) or a simple coroot ($a = i$). In the former case, $-a$ refers to the root $-\alpha$, but in the latter case, $-a$ should be understood to refer to i.

With this (relative) choice of basis for the antiholomorphic sector, the gluing conditions relate holomorphic raising and lowering operators with antiholomorphic lowering and raising operators, as promised.

However, this cavalier "removal" of the automorphism Ω does not come without a price. By changing the basis of $\mathfrak{g}$ with respect to which the antiholomorphic sector is decomposed (as above), one changes the triangular decomposition of the antiholomorphic copy of $\mathfrak{g}$, hence $\widehat{\mathfrak{g}}$. That is, the triangular decompositions used in the holomorphic and antiholomorphic sectors now differ (in a relative sense) by the automorphism $\omega_{\mathrm{C}} \circ \Omega$, so the action of the antiholomorphic copy of $\widehat{\mathfrak{g}}$ on the representation space $\mathcal{S}$ has also been changed. (If this basis change was not induced by an automorphism, then the structure of $\mathfrak{g}$ would not be preserved, as the Cartan subalgebra and root spaces would not be mapped to equivalent subspaces, and one would destroy the affine symmetry between the holomorphic and antiholomorphic sectors.)

The set of weights of any integrable highest weight representations will be left invariant if the automorphism is inner (Appendix A.2) though the antiholomorphic Cartan subalgebra will have changed if this inner automorphism is not in the Weyl group. If the automorphism is outer, the Dynkin labels of the weights are just permuted by the corresponding symmetry of the Dynkin diagram of $\mathfrak{g}$. The set of weights of any *given* integrable highest weight representation may not be preserved by such an automorphism, but this set of permuted weights will form the set of weights of *another* integrable highest weight representation, related to the first by the Dynkin symmetry. It is shown in Appendix A.2 that $\omega_{\mathrm{C}} \circ \Omega$ may be written as the product of a unique Dynkin symmetry $\breve{\omega}$ and an inner automorphism. The effect of the above choice of (antiholomorphic) basis is therefore to permute the integrable highest weight representations in the antiholomorphic sector by $\breve{\omega}$. The explicit form of the *modular invariant* (Section 3.1.3), $M_{\lambda\mu}$, is thereby altered to $M_{\lambda\,\breve{\omega}^{-1}(\mu)}$.

In any case, the point of this change of coordinates from the half plane (z) to the annulus (ζ) is to note that the annulus in Figure 3.2 may be embedded in the full complex plane in a manner which may be naturally interpreted in the formalism of radial quantisation. Indeed, if the half plane theory was radially quantised, it is easy to check that the same is true of the annulus theory *except that time and space get interchanged*. That is, upon embedding the annulus in the complex plane, one finds that the spatial direction on the half plane becomes the radial direction on the annulus and the temporal direction on the half plane becomes the (negative) angular direction on the annulus. Since the distinction between space and time is largely a matter of convenience in a euclidean theory, this suggests that the annulus theory may be recovered from an appropriate theory on the full plane by imposing the gluing conditions at the annulus boundaries. It follows that one now has two realisations of the boundary conformal field theory to study, both related to an appropriate theory on the full plane.

3.2.3. Branes as Ishibashi States. It is natural now to ask when a quantum state from the full plane theory satisfies the gluing conditions given by Equation (3.2.4) (meaning they become identities when applied to the state). It is therefore mildly[14] irritating to learn that there are in general no such (non-zero) states. However, one can identify infinite (possibly divergent) linear combinations of states which satisfy the gluing conditions. These formal solutions are identified as representing branes in the quantised Wess-Zumino-Witten theory. Mathematically, it is clear that these formal infinite linear combinations of states are best treated as distributions, specifically as elements of the dual space $\mathcal{S}^*$ of the full plane state space[15]. This distributional interpretation accords well with the intuition that branes should correspond to states localised on the boundary.

A solution to the annulus mode gluing conditions, Equation (3.2.4), will be denoted by $\langle \mathbf{b}| \in \mathcal{S}^*$. The "**b**" denotes "brane" and serves to distinguish these linear functionals from the $\langle \psi| \in \mathcal{S}^*$ used previously to denote the functional that took value 1 on the state $|\psi\rangle \in \mathcal{S}$ and vanished on its orthogonal complement. These brane solutions are characterised by

$$\langle \mathbf{b}| J_n^a |\psi\rangle = \langle \mathbf{b}| r^{2n} \overline{J}_{-n}^{-a} |\psi\rangle ,$$

for all $n \in \mathbb{Z}$, a, and $|\psi\rangle \in \mathcal{S}$. It is convenient to switch to a tensor product notation, where J_n^a becomes $J_n^a \otimes \mathrm{id}$ acting on $\mathcal{S} = \bigoplus_{\lambda\mu} M_{\lambda\mu} \left(V_\lambda \otimes V_\mu\right)$, and $\overline{J}_n^a$ becomes $\mathrm{id} \otimes J_n^a$. The equation characterising brane solutions in this notation is

$$\langle \mathbf{b}| \left\{ J_n^a |\psi\rangle \otimes |\phi\rangle \right\} = r^{2n} \langle \mathbf{b}| \left\{ |\psi\rangle \otimes J_{-n}^{-a} |\phi\rangle \right\} . \tag{3.2.5}$$

It is also convenient to take care regarding the possible presence of non-trivial multiplicities in the modular invariant of the full plane theory. In the remainder of this chapter (unless specified otherwise), the labels λ will *implicitly* include information about this multiplicity. In particular, λ and μ may now be equal as weights, but the highest weight states $|\lambda\rangle$ and $|\mu\rangle$ can still be different.

PROPOSITION 3.1. *Given $r > 0$, the solutions $\langle \mathbf{b}|$ of Equation (3.2.5) act on the states $|\psi\rangle \otimes |\phi\rangle \in V_\lambda \times V_\mu$ by*

$$\langle \mathbf{b}| \left\{ |\psi\rangle \otimes |\phi\rangle \right\} = \sum_{\substack{\lambda=\mu \\ \textit{as weights}}} \left(\Xi_{\lambda\mu} r^{-L_0} |\psi\rangle , r^{-L_0} |\phi\rangle \right)_\mu \langle \mathbf{b}| \left\{ r^{L_0} |\lambda\rangle \otimes r^{L_0} |\mu\rangle \right\} , \tag{3.2.6}$$

where $(\cdot,\cdot)_\mu$ denotes the inner product on V_μ, and $\Xi_{\lambda\mu} \colon \mathcal{S} \to \mathcal{S}$ is the antilinear (chiral) operator which maps $|\lambda\rangle$ to $|\mu\rangle$, annihilates every other highest weight state, and commutes with each lowering operator of $\widehat{\mathfrak{g}}$.

[14] Only mildly, because one really should not expect a closed string to satisfy these boundary conditions!

[15] Recall the footnote in Section 2.1.2 discussing reasons not to complete $\mathcal{S}$, with respect to the Hilbert space topology, to get a bona-fide Hilbert space $\mathcal{H}$. This is another. It seems likely that one can find a natural nuclear topology giving rise to a rigged Hilbert space $(\mathcal{S} \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{S}^*)$, wherein the branes are realised as elements of $\mathcal{S}^*$..

PROOF. Let $|\psi\rangle$ and $|\phi\rangle$ be elements of V_λ and V_μ (respectively) of the form

$$|\psi\rangle = J^{a_1}_{-n_1}\cdots J^{a_p}_{-n_p}|\lambda\rangle \qquad \text{and} \qquad |\psi\rangle = J^{b_1}_{-m_1}\cdots J^{b_q}_{-m_q}|\mu\rangle\,, \tag{3.2.7}$$

where all the modes are lowering operators of $\widehat{\mathfrak{g}}$ (such elements form spanning sets). Then, if $\langle\mathbf{b}|$ is a solution of Equation (3.2.5),

$$\langle\mathbf{b}|\{|\psi\rangle\otimes|\phi\rangle\} = r^{-2\left(n_1+\ldots+n_p\right)}\langle\mathbf{b}|\left\{|\lambda\rangle\otimes J^{-a_p}_{n_p}\cdots J^{-a_1}_{n_1}J^{b_1}_{-m_1}\cdots J^{b_q}_{-m_q}|\mu\rangle\right\}.$$

From the choice of antiholomorphic basis made (relative to the holomorphic one) in Section 3.2.2, it follows that each $J^{-a_i}_{n_i}$ is a raising operator. These can be commuted through the lowering operators $J^{b_j}_{-m_j}$ to annihilate the highest weight vector $|\mu\rangle$, leaving behind terms involving commutators. These commutators can be expanded and any raising operators commuted to the right, and so on. If, after this process terminates, there are any lowering operators remaining, they can be used to annihilate $|\lambda\rangle$. It follows that the only contributing terms are those consisting of modes from the Cartan subalgebra (and the central term). $|\mu\rangle$ is an eigenvector for these, so it follows that the net effect of these modes is just a multiplicative constant.

This constant may be evaluated by noting that[16] $(J^a_n)^\dagger = J^{-a}_{-n}$, so that

$$(|\psi\rangle,|\phi\rangle) = \left(|\lambda\rangle, J^{-a_p}_{n_p}\cdots J^{-a_1}_{n_1}J^{b_1}_{-m_1}\cdots J^{b_q}_{-m_q}|\mu\rangle\right).$$

This is therefore that same multiplicative constant times the inner product of the highest weight vectors $|\lambda\rangle$ and $|\mu\rangle$. Unfortunately, the latter is $\delta_{\lambda\mu}$, so it is necessary to introduce an operator $\Xi_{\lambda\mu}$ on $\mathcal{S}$ which maps $|\lambda\rangle$ to $|\mu\rangle$, annihilates all other highest weight states, and commutes with every lowering operator. Then, one can write

$$\left(\Xi_{\lambda\mu}|\psi\rangle,|\phi\rangle\right) = \left(|\mu\rangle, J^{-a_p}_{n_p}\cdots J^{-a_1}_{n_1}J^{b_1}_{-m_1}\cdots J^{b_q}_{-m_q}|\mu\rangle\right)_\mu,$$

which is exactly the multiplicative constant required.

It follows now that

$$\begin{aligned}\langle\mathbf{b}|\{|\psi\rangle\otimes|\phi\rangle\} &= r^{-2\left(n_1+\ldots+n_p\right)}\left(\Xi_{\lambda\mu}|\psi\rangle,|\phi\rangle\right)\langle\mathbf{b}|\{|\lambda\rangle\otimes|\mu\rangle\}\\ &= \left(\Xi_{\lambda\mu}r^{-2L_0}|\psi\rangle,|\phi\rangle\right)\langle\mathbf{b}|\left\{r^{2L_0}|\lambda\rangle\otimes|\mu\rangle\right\},\end{aligned}$$

by noting that the energy (L_0 eigenvalue) of $|\psi\rangle$ is that of $|\lambda\rangle$ plus the grades $n_1+\ldots+n_p$. This holds whenever $|\psi\rangle$ and $|\phi\rangle$ are elements of the form given in Equation (3.2.7). However, the left hand side is linear in $|\psi\rangle$ whereas the right hand side is (seemingly) antilinear in $|\psi\rangle$. This mismatch is resolved by requiring $\Xi_{\lambda\mu}$ to be an antilinear operator on $\mathcal{S}$ (which completely determines it).

It remains to consider Equation (3.2.5) for $J^a_0 = t_a$ where t_a is an element of the Cartan subalgebra. Remembering that $-a$ should now be interpreted as a, this gives

$$\langle\lambda,t_a\rangle\langle\mathbf{b}|\{|\lambda\rangle\otimes|\mu\rangle\} = \langle\mathbf{b}|\{J^a_0|\lambda\rangle\otimes|\mu\rangle\} = \langle\mathbf{b}|\{|\lambda\rangle\otimes J^a_0|\mu\rangle\} = \langle\mu,t_a\rangle\langle\mathbf{b}|\{|\lambda\rangle\otimes|\mu\rangle\}.$$

[16] The adjoint in an integrable highest weight representation is defined in Appendix B.2 through the action of the Chevalley *anti*automorphism. This choice of adjoint explains why the choice of antiholomorphic basis made earlier is the best.

Since the implicit multiplicity labels are irrelevant inside the pairing $\langle\cdot,\cdot\rangle$, $\langle\mathbf{b}|\{|\lambda\rangle\otimes|\mu\rangle\} = 0$ unless $\lambda=\mu$ as weights. It follows that $h_\lambda = h_\mu$, so one can write (more democratically),

$$\langle\mathbf{b}|\{|\psi\rangle\otimes|\phi\rangle\} = \sum_{\substack{\lambda=\mu\\ \text{as weights}}} \left(\Xi_{\lambda\mu} r^{-L_0}|\psi\rangle, r^{-L_0}|\phi\rangle\right)\langle\mathbf{b}|\left\{r^{L_0}|\lambda\rangle\otimes r^{L_0}|\mu\rangle\right\}. \qquad \blacksquare$$

It follows from this result (and bilinearity) that a solution to Equation (3.2.5) is completely determined by its action on the highest weight states $|\lambda\rangle\otimes|\mu\rangle$, and that it annihilates this state unless λ and μ coincide as weights of $\mathfrak{g}$. Since the multiplicity of the highest weight states $|\lambda\rangle\otimes|\mu\rangle$, with $\lambda=\mu$ as weights, is just $M_{\lambda\lambda}$ (in the modular invariant, λ is just a weight), the following corollary is obvious.

COROLLARY 3.2. *There are precisely* $\operatorname{tr} M$ *linearly independent brane solutions of Equation (3.2.5), where M is the modular invariant matrix.*

There is therefore at least one solution to Equation (3.2.5) as $M_{00}=1$. The obvious basis of independent brane solutions consists of those $\langle\mathbf{b}|$ which take the value 1 on some given highest weight state $|\lambda\rangle\otimes|\mu\rangle$, and vanish on every other highest weight state. These solutions are known as *Ishibashi states* [**97**, **127**], and will be denoted by $\langle\mathbf{b}_{\lambda\mu}|$ (where $\lambda=\mu$ as weights). As an example, when the modular invariant is diagonal ($M_{\lambda\mu}=\delta_{\lambda\mu}$), there is a bijective correspondence between the integrable highest weight representations of $\widehat{\mathfrak{g}}$ and the Ishibashi states. But, if the modular invariant is the charge-conjugate invariant ($M_{\lambda\mu}=\delta_{\lambda\mu^+}$), then the Ishibashi states are in bijection with the integrable highest weights which are self-conjugate.

However, it is very important to recall the choice of bases made in Section 3.2.2, and the consequences of these choices. The form of the modular invariant depends upon the (relative) choice of the bases of $\mathfrak{g}$ used to decompose the holomorphic and antiholomorphic sectors, and the choice used in the above computation depended upon the automorphism $\omega_{\mathrm{C}}\circ\Omega$. Therefore, if one *originally* had the diagonal invariant and $\Omega=\mathrm{id}$, then $\omega_{\mathrm{C}}\circ\Omega=\omega_{\mathrm{C}}$ decomposes into the product of the conjugation automorphism and the longest element of the Weyl group (Appendix A.2), so after making the relative basis choice as above, the modular invariant becomes the charge-conjugate invariant. It follows that the Ishibashi states would then be in bijection with the integrable highest weights which are self-conjugate. Similarly, if the original modular invariant were diagonal but Ω was the conjugation automorphism, the Ishibashi states would correspond to the set of all integrable highest weights. This behaviour, relative to the modular invariant, is how the automorphism allowed in the gluing condition, Equation (3.2.2), affects the *spectrum* of branes (Ishibashi states).

It remains to verify the opening comments of this section and relate these Ishibashi states to infinite linear combinations of elements of $\mathcal{S}$. The value of an Ishibashi state $\langle\mathbf{b}_{\lambda\mu}|$ is given by an inner product on the $\widehat{\mathfrak{g}}$-module V_μ. Let $\{|e_i^\mu\rangle\}$ be an orthonormal basis of (the Hilbert space completion of) this module, which are eigenvectors of L_0 of

eigenvalue $h_\mu + n_i$ (say). Then,

$$\begin{aligned}\left\langle \mathbf{b}_{\lambda\mu}\right| \{|\psi\rangle \otimes |\phi\rangle\} &= r^{2h_\mu}\left(\Xi_{\lambda\mu}|\psi\rangle, r^{-2L_0}|\phi\rangle\right) = r^{2h_\mu}\sum_i \left(\Xi_{\lambda\mu}|\psi\rangle, \left|e_i^\mu\right\rangle\right)\left(\left|e_i^\mu\right\rangle, r^{-2L_0}|\phi\rangle\right)\\ &= r^{2h_\mu}\sum_i \left(\Xi_{\lambda\mu}^\dagger \left|e_i^\mu\right\rangle, |\psi\rangle\right)\left(r^{-2L_0}\left|e_i^\mu\right\rangle, |\phi\rangle\right)\\ &= \sum_i r^{-2n_i}\left(\Xi_{\lambda\mu}^\dagger \left|e_i^\mu\right\rangle \otimes \left|e_i^\mu\right\rangle, |\psi\rangle\otimes|\phi\rangle\right),\end{aligned}$$

where the adjoint of an antilinear operator A is defined by $\left(A^\dagger|\psi\rangle, |\phi\rangle\right) = \left(A|\phi\rangle, |\psi\rangle\right)$.

Since $\Xi_{\lambda\mu}$ commutes with all the modes J_n^a, so does $\Xi_{\lambda\mu}^\dagger$. This adjoint is therefore determined by its action on highest weight vectors, which is easily shown to coincide with the action of $\Xi_{\mu\lambda}$. The Ishibashi state $\left\langle \mathbf{b}_{\lambda\mu}\right|$ may therefore be associated with the formal infinite linear combination of elements of $\mathcal{S}$,

$$\sum_i r^{-2n_i}\Xi_{\lambda\mu}^\dagger \left|e_i^\mu\right\rangle \otimes \left|e_i^\mu\right\rangle = \sum_i r^{-2n_i}\Xi_{\mu\lambda}\left|e_i^\mu\right\rangle \otimes \left|e_i^\mu\right\rangle = \sum_i r^{-2n_i}\left|e_i^\lambda\right\rangle \otimes \left|e_i^\mu\right\rangle. \qquad (3.2.8)$$

The last equality follows as $V_\lambda \cong V_\mu$ (recall $\lambda = \mu$ as weights) so $\Xi_{\mu\lambda}$ takes an orthonormal basis of V_μ to an orthonormal basis of V_λ. Interestingly, if $r > 1$ then this sum converges in the Hilbert space completion of $\mathcal{S}$, whereas if $r = 1$, the sum is divergent[17]. That is, the Ishibashi states may be interpreted as elements of the Hilbert space completion of $\mathcal{S}$ if and only if $r > 1$.

3.2.4. The Annulus Partition Function and Cardy States. Consider the dependence of these Ishibashi states on the annulus boundary radius r. Denoting these states, temporarily, by $\left\langle \mathbf{b}_{\lambda\mu}\right|_r$, one has the relation

$$\begin{aligned}\left\langle \mathbf{b}_{\lambda\mu}\right|_r \{|\psi\rangle\otimes|\phi\rangle\} &= \left(\Xi_{\lambda\mu}r^{-L_0}|\psi\rangle, r^{-L_0}|\phi\rangle\right) r^{h_\lambda+h_\mu}\\ &= r^{h_\lambda+h_\mu}\left\langle \mathbf{b}_{\lambda\mu}\right|_1 \left\{r^{-L_0}|\psi\rangle\otimes r^{-L_0}|\phi\rangle\right\}\\ &= r^{h_\lambda - c/24} r^{h_\mu - c/24}\left\langle \mathbf{b}_{\lambda\mu}\right|_1 \left\{r^{-(L_0-c/24)}|\psi\rangle\otimes r^{-(L_0-c/24)}|\phi\rangle\right\}.\end{aligned}$$

One recognises the action of the (chiral) Hamiltonian, $L_0 - c/24$, and the modular anomaly $m_\lambda = h_\lambda - c/24$ (Appendix B.3. This relation therefore warrants a most important interpretation. The Ishibashi state acting on a closed string state, at the boundary $|\zeta| = r$, is equivalent (up to the modular anomaly) to the corresponding Ishibashi state, at $|\zeta| = 1$, acting on the closed string state *translated back* to the boundary $|\zeta| = 1$. In other words, the Ishibashi states at the boundary $|\zeta| = r$ may be interpreted as having evolved under the Hamiltonian from the corresponding Ishibashi states at $|\zeta| = 1$:

$$\left\langle \mathbf{b}_{\lambda\mu}\right|_1 \xrightarrow{\text{evolution}} r^{-2m_\lambda}\left\langle \mathbf{b}_{\lambda\mu}\right|_r.$$

The partition function for the annulus theory may now be computed as follows. The labels α and β will not generally correspond to periodic boundary conditions, so the

[17] The norm squared of this formal infinite linear combination is just $\sum_i r^{-4n_i}$ which turns out to be a constant times the Virasoro character of V_μ evaluated at $q = r^{-4}$. Virasoro characters converge when $|q| < 1$. This can be elucidated fairly easily from [**99**], Proposition 11.10.

partition function will not be expressed as a trace over the state space. Instead, it takes the form

$$\tilde{Z}_{\alpha\beta}\left(e^{2\pi\ell/t}\right) = \langle \mathbf{b}_\alpha|_1 \left\{ e^{-2\pi\ell\left(L_0+\overline{L}_0-c/12\right)/t} \left\langle \mathbf{b}_\beta \right|_1 \right\},$$

reflecting the evolution of the boundary state $\left\langle \mathbf{b}_\beta \right|_1$ from $r=1$ to $r=e^{2\pi\ell/t}$. These boundary states are linear combinations of Ishibashi states, $\left\langle \mathbf{b}_\beta \right|_r = \sum_{(\lambda\mu)} U_{\beta(\lambda\mu)} \left\langle \mathbf{b}_{\lambda\mu} \right|_r$, where the sum is over labels with $\lambda=\mu$ as weights. Evoking the evolution interpretation for branes, this partition function may be rewritten as

$$\tilde{Z}_{\alpha\beta}\left(e^{2\pi t/\ell}\right) = \sum_{\substack{\lambda=\mu \\ \text{as weights}}} \langle \mathbf{b}_\alpha|_1 \left\{ e^{-4\pi\ell m_\lambda/t} U_{\beta(\lambda\mu)} \left\langle \mathbf{b}_{\lambda\mu} \right|_r \right\}.$$

Now, it is not *a priori* evident that this expression for the partition function makes sense, consisting of a functional acting on another functional. However, the argument of the functional $\langle \mathbf{b}_\alpha|_1$ is a functional at $r=e^{2\pi\ell/t}$, which may be associated with a genuine element of the Hilbert space completion of the state space by Equation (3.2.8). It is therefore possible for the above partition function to make sense by making this replacement[18]. Note that $\left\langle \mathbf{b}_\beta \right|_r = \sum_{(\lambda\mu)} U_{\beta(\lambda\mu)} \left\langle \mathbf{b}_{\lambda\mu} \right|_r$ is associated with the Hilbert space element

$$\sum_{(\lambda\mu)} U^*_{\beta(\lambda\mu)} \sum_i r^{-2n_i} \left| e_i^\lambda \right\rangle \otimes \left| e_i^\mu \right\rangle,$$

by Equation (3.2.8). In particular, note the conjugation that follows from the derivation of this equation.

The evaluation of this partition function is now straight-forward. One finds that

$$\begin{aligned}
\tilde{Z}_{\alpha\beta}\left(e^{2\pi t/\ell}\right) &= \sum_{\substack{\lambda=\mu \\ \text{as weights}}} U^*_{\beta(\lambda\mu)} \sum_i e^{-4\pi\ell(m_\lambda+n_i)/t} \langle \mathbf{b}_\alpha|_1 \left\{ \left| e_i^\lambda \right\rangle \otimes \left| e_i^\mu \right\rangle \right\} \\
&= \sum_{\substack{\lambda=\mu \\ \text{as weights}}} \sum_{\substack{\lambda'=\mu' \\ \text{as weights}}} U_{\alpha(\lambda'\mu')} U^*_{\beta(\lambda\mu)} \sum_i e^{-4\pi\ell(m_\lambda+n_i)/t} \left\langle \mathbf{b}_{\lambda'\mu'} \right|_1 \left\{ \left| e_i^\lambda \right\rangle \otimes \left| e_i^\mu \right\rangle \right\} \\
&= \sum_{\substack{\lambda=\mu \\ \text{as weights}}} U_{\alpha(\lambda\mu)} U^*_{\beta(\lambda\mu)} e^{-4\pi\ell(h_\lambda+n_i-c/24)/t} \\
&= \sum_{\substack{\lambda=\mu \\ \text{as weights}}} U_{\alpha(\lambda\mu)} U^*_{\beta(\lambda\mu)} \operatorname{tr}_{V_\lambda} e^{-4\pi\ell(L_0-c/24)/t}.
\end{aligned}$$

Recognising the character of V_λ as a $\mathfrak{Vir}$-module, this expression for the partition function on the annulus may be compared with the expression, Equation (3.2.3), for the corresponding partition function on the half plane. Insisting that these coincide gives Cardy's

[18] Indeed, Equation (3.2.8) suggests that the boundary states at $r>1$ correspond to infinite linear combinations of orthonormal basis elements whose coefficients are of *rapid decrease*. That is, these boundary states belong to an abstract Schwarz space. In the rigged Hilbert space formalism, the state space should be such an abstract Schwarz space, and the branes should belong to its dual. Hence, one can have some confidence that the partition function described here will not be arrant nonsense.

constraint [**44**]:

$$\sum_\nu n_{\alpha\beta}{}^\nu \chi_\nu^{\mathfrak{Vir}}\left(e^{-\pi t/\ell}\right) = \sum_{\substack{\lambda=\mu \\ \text{as weights}}} U_{\alpha(\lambda\mu)} U^*_{\beta(\lambda\mu)} \chi_\lambda^{\mathfrak{Vir}}\left(e^{-4\pi\ell/t}\right)$$
$$= \sum_{\substack{\lambda=\mu \\ \text{as weights}}} \sum_\nu U_{\alpha(\lambda\mu)} S_{\lambda\nu} U^*_{\beta(\lambda\mu)} \chi_\nu^{\mathfrak{Vir}}\left(e^{-\pi t/\ell}\right).$$

The last equality comes from the modular properties of the Virasoro characters and noticing that $q = e^{2\pi i\tau} = e^{-\pi t/\ell}$ transforms under the modular transformation $S\colon \tau \mapsto -1/\tau$, into $\tilde{q} = e^{-4\pi\ell/t}$ (and vice-versa).

One would like to conclude now that

$$n_{\alpha\beta}{}^\nu = \sum_{\substack{\lambda=\mu \\ \text{as weights}}} U_{\alpha(\lambda\mu)} S_{\lambda\nu} U^*_{\beta(\lambda\mu)}. \tag{3.2.9}$$

This conclusion would be justified if the Virasoro characters $\chi_\nu^{\mathfrak{Vir}}(q)$ were linearly independent. However, they are not, as these characters are left invariant by $\nu \mapsto \omega(\nu)$, where ω is a symmetry of the Dynkin diagram of $\mathfrak{g}$. Nevertheless, it is usually argued, not entirely convincingly, that this conclusion is true [**17**, **44**], so Equation (3.2.9) will be assumed in what follows. One might anticipate that this relation may be derived properly by considering an *extended* partition function involving the full set of commuting observables (thus the full affine characters), not just the energy.

In any case, Equation (3.2.9) may be interpreted as a non-linear constraint on the coefficients $U_{\alpha(\lambda\mu)}$. It is, however, remarkably difficult to solve because the only information known about the coefficients $n_{\alpha\beta}{}^\nu$ is that they represent the multiplicities of the V_ν in the half plane state space $\mathcal{S}'$ (Section 3.2.1). As such, they are non-negative integers.

There is, however, a fundamentally interesting class of solutions when the modular invariant is diagonal (meaning the Ishibashi states are in bijection with the set of integrable highest weights). There are no non-trivial multiplicities, so the labelling $(\lambda\mu)$ can be replaced by a simple λ, and the solution is given by

$$U_{\alpha\lambda} = \frac{S_{\alpha\lambda}}{\sqrt{S_{0\lambda}}} \qquad \Rightarrow \qquad \langle \mathbf{b}_\alpha | = \sum_\lambda \frac{S_{\alpha\lambda}}{\sqrt{S_{0\lambda}}} \langle \mathbf{b}_{\lambda\lambda} |, \tag{3.2.10}$$

which expresses the *Cardy state* $\langle \mathbf{b}_\alpha|$ in terms of the Ishibashi states $\langle \mathbf{b}_{\lambda\lambda}|$. Note that the boundary conditions labels, α, are associated with weights in this solution. It is easy to verify using the symmetry of the S-matrix and the Verlinde formula (Proposition 5.1) that

$$n_{\alpha\beta}{}^\nu = \sum_\lambda \frac{S_{\alpha\lambda} S_{\nu\lambda} S^*_{\beta\lambda}}{S_{0\lambda}} = \mathcal{N}_{\alpha\nu}{}^\beta = \mathcal{N}_{\alpha^+\beta}{}^\nu \tag{3.2.11}$$

is indeed a non-negative integer. It follows that for the diagonal invariant, there are consistent boundary states, called Cardy states, in bijection with the integrable highest weights of $\widehat{\mathfrak{g}}$. The Cardy states correspond to the so-called *untwisted* symmetry-preserving branes, because the "twisting" automorphism $\omega_C \circ \Omega$ is trivial (relative to the modular invariant).

Note however, that this analysis does not rule out the existence of other consistent solutions that correspond to this automorphism and modular invariant.

It does, however, suggest a method to get consistent solutions for more general modular invariants. Given an invariant M, let $\mathcal{E}_M$ denote the (multi)set of weights (with multiplicity) which are paired with themselves in the modular invariant. Thus, $\lambda \equiv (\lambda\mu) \in \mathcal{E}_M$ and $|\mathcal{E}_M| = \mathrm{tr}M$.

PROPOSITION 3.3. *Define matrices n^ν by $(n^\nu)_{\alpha\beta} = n_{\alpha\beta}{}^\nu$, where α and β label boundary states and $\nu \in \mathcal{E}_M$. Then, if the n^ν constitute a (normal) non-negative integer matrix representation (NIMrep) of the fusion ring (see Section 5.1), there exist corresponding solutions to Equation (3.2.9).*

PROOF. Define a matrix V by $V_{\alpha\lambda} = \sqrt{S_{0\lambda}}U_{\alpha\lambda}$. Then, Equation (3.2.9) takes the form

$$n_{\alpha\beta}{}^\nu = \sum_{\lambda\in\mathcal{E}_M} V_{\alpha\lambda}\frac{S_{\nu\lambda}}{S_{0\lambda}}V^\dagger_{\lambda\beta}.$$

As the n^ν constitute a normal representation of the fusion ring (meaning the representing matrices are normal), their eigenvalues have the form $S_{\nu\lambda}/S_{0\lambda}$ (Section 5.1.1). The n^ν must all commute, so V may therefore be taken as the unitary matrix which simultaneously diagonalises each n^ν. ∎

Note that as V is chosen to be unitary in this proof, the boundary states end up being in bijection with $\mathcal{E}_M$. It follows that if one can find a NIMrep of the fusion ring consisting of m by m normal matrices ($m = \mathrm{tr}M$), then one automatically knows of m linearly independent consistent boundary states solving Equation (3.2.9). When M is the diagonal invariant, m is the number of integrable highest weights, and V may be taken to be the modular S-matrix. The NIMrep is then the regular representation furnished by the fusion matrices (Section 5.1.1), and the boundary states are the Cardy states as described above. Of course, nothing precludes the possibility of there being solutions to Equation (3.2.9) which do not correspond to a NIMrep of the fusion ring. However, in view of the difficulty of actually obtaining solutions, it is not surprising that NIMreps have become a fundamental object of study (see for example [**79**]).

However they are obtained, the boundary states which solve Equation (3.2.9) may be interpreted as consistent (or quantised) *branes*, which specify the boundary conditions on the open string theory. As exemplified by the Cardy states, one generally finds more than one consistent brane for each modular invariant. In Section 6.3, branes will be studied *globally* from a geometric point of view, and it will be shown there that these different branes correspond to geometric objects at different positions in the target space (the Lie group G).

CHAPTER 4

Brane Charge Groups

In this chapter, a conserved charge for the quantised branes of a Wess-Zumino-Witten model is introduced and studied. The main goal is to compute the abelian group which these brane charges take values in, and for the untwisted symmetry-preserving branes, this goal is achieved in Section 4.2. These computations require a detailed knowledge of the fusion process in Wess-Zumino-Witten models, a comprehensive study of which follows in Chapter 5.

4.1. Brane Dynamics and Conserved Charges

Recall that branes have thus far been introduced as boundary (gluing) conditions imposed on the currents of the conformal field theory. It has also been mentioned that branes have a geometric interpretation in the corresponding string theory as extended objects in the target space on which open string endpoints are constrained. In this setting, there is no notion of brane dynamics whatsoever. Nevertheless, in the proposed extension of string theory, M-theory, the status of the branes of the theory is expected to be raised to that of dynamical objects, on a par with strings.

Unfortunately, M-theory is not a well-defined theory at present, so the study of brane dynamics is not a straight-forward matter. Indeed, it is hoped that knowledge of brane dynamics will help to determine the form of M-theory. One popular approach is to consider an effective field-theoretic description, valid at low energies, of the open string theory associated with a given brane. Such a quantum field theory serves as an approximation of this string theory. More importantly, it should also serve as a low-energy approximation to the corresponding M-theory describing the brane. That is, to the expected brane dynamics.

If some well-defined dynamical processes for branes can be identified from this approximation, then one can attempt to determine corresponding conserved charges. Even though such processes and charges would be derived in a low-energy approximation, it is reasonable to hope that they might still be valid (at least in part) in the full theory. It is in this way that such approximations are supposed to provide guidelines for constructing M-theory.

These ideas were applied to Wess-Zumino-Witten models by Alekseev, Recknagel and Schomerus [**5**], for brane dynamics, and Fredenhagen and Schomerus [**63**], for conserved charges. In this section, their results are presented. Whilst these results are of fundamental importance for the rest of this chapter (and motivate much of this thesis), the corresponding derivations will not be detailed. They rely on advanced ideas from string

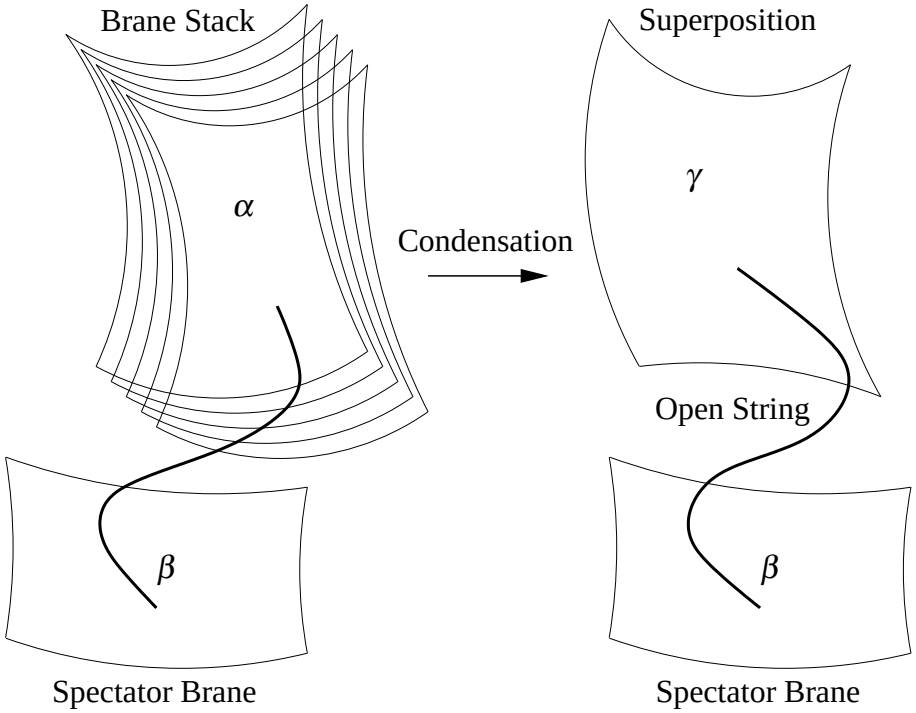

FIGURE 4.1. A prototypical condensation process for branes: A stack of coincident branes, all of the same type α, decays into a superposition of other branes (labelled by γ).

theory, condensed matter physics, and non-commutative geometry which are largely irrelevant to the remainder of this thesis.

4.1.1. Dynamical Studies. Recall from Section 3.2.4 that the consistent branes in a quantised Wess-Zumino-Witten model (with the diagonal modular invariant) are in bijective correspondence with (a subset of) the set of integrable highest weight representations of the associated affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. The brane configuration considered in [**5**] consists of a "stack" of m coincident branes, all of the same type, labelled by the weight (boundary condition) α of $\mathfrak{g}$. The idea is that this configuration is unstable and if perturbed, *condenses* into another (more stable) configuration or superposition of branes. This is illustrated in Figure 4.1.

In a low-energy effective field theory approximation, the field content will consist of fields A which (classically) take values in the Lie algebra $\mathfrak{g}$ (compare with the conserved currents of Equations (3.1.1) and (3.1.2)). Decomposing with respect to a basis of $\mathfrak{g}$, the components A^a are then just real-valued functions. This is the situation for a single brane. When m coincident branes (of the same type) are being considered, the components A^a must be promoted to functions taking values in the set of m by m matrices. Components such as these are sometimes known as Chan-Paton factors.

An action for the effective field theory was computed in [**5**] to first order. It turns out to be independent of the specific type of branes involved, and yields a classical equation

of motion which specialises to (summation convention assumed)

$$\left[A^a, \left[A^a, A^b\right] - f_{abc}A^c\right] = 0,$$

when the matrix-valued functions A^a are assumed constant (the f_{abc} denote the structure constants of $\mathfrak{g}$). There are two obvious solutions. The first is obtained by setting $\left[A^a, A^b\right] = 0$ for all a, b. This solution is expected based on the analogous study with flat target spaces [**162**], and corresponds to translations of each of the m branes (without changing their type). The second solution is obtained if m is the dimension of some representation of $\mathfrak{g}$ (assumed to be irreducible for convenience), and the A^a are taken to realise this representation. This solution coincides with the first when $\mathfrak{g}$ is abelian (which corresponds to a flat target space) but is otherwise quite different. The interpretation proposed in [**5**] for this solution is that it corresponds to the condensation process of Figure 4.1.

More specifically, suppose that m is the dimension of the irreducible $\mathfrak{g}$-module of highest weight λ. Then, the proposal is that the stack of m branes labelled by α condenses into a superposition of branes, where the number of branes of type γ is given by $n_{\alpha\gamma}{}^{\lambda}$. The coefficients $n_{\alpha\gamma}{}^{\lambda}$ are precisely those which appeared in the half plane partition function, Equation (3.2.3). (Recall that the labels α and γ there described the different boundary conditions imposed on the string endpoints.) As an important example, for the untwisted symmetry-preserving branes (Section 3.2.4), these coefficients are just the fusion coefficients,

$$n_{\alpha\gamma}{}^{\lambda} = \mathcal{N}_{\alpha\lambda}{}^{\gamma} = \mathcal{N}_{\lambda\alpha}{}^{\gamma},$$

by Equations (3.1.28) and (3.2.11).

Direct evidence for this proposal is given in [**5**] for these untwisted symmetry-preserving branes, when the level k is sent to infinity. Under these circumstances, the coefficients $n_{\alpha\gamma}{}^{\lambda}$ become the tensor product coefficients $N_{\lambda\alpha}{}^{\gamma}$, and the evidence given is largely representation-theoretic. To justify this proposal when the level is finite, recourse is made to a formal analogy with the much studied *Kondo model* of condensed matter physics. This analogy is (tersely) described in [**63**] (some more detail may be found in [**111**]), and the condensation interpretation is then derived from the "absorption of boundary spin principle" developed by Affleck and Ludwig in [**2**].

In the context of brane dynamics, this principle of Affleck and Ludwig is interpreted to imply the following rule expressing the effect of perturbing the stack of m branes:

$$m\chi_{\mu}^{\mathfrak{Vir}}(q) \longrightarrow \sum_{\nu} \mathcal{N}_{\lambda^+\mu}{}^{\nu}\, \chi_{\nu}^{\mathfrak{Vir}}(q),$$

where $\chi_{\mu}^{\mathfrak{Vir}}(q)$ is the character of the irreducible representation of $\mathfrak{g}$ of highest weight μ, treated as a $\mathfrak{Vir}$-module. Multiplying both sides of this rule by $n_{\alpha\beta}{}^{\mu}$ and summing over μ, one finds that

$$mZ_{\alpha\beta}(q) \longrightarrow \sum_{\mu} n_{\alpha\beta}{}^{\mu} \sum_{\nu} \mathcal{N}_{\lambda^+\mu}{}^{\nu}\, \chi_{\nu}^{\mathfrak{Vir}}(q),$$

where the left hand side is recognised as m times the half plane partition function for an open string with boundary conditions (branes) labelled by α and β (Equation (3.2.3)). The factor m accounts for the fact that there is a stack of m coincident branes of type α rather than just one. Assuming that the ${n_{\alpha\beta}}^{\mu}$ define a (normal) NIMrep of the fusion ring (see Proposition 3.3), the right hand side becomes (using Equation (3.1.29))

$$\sum_{\nu}\left[\sum_{\mu}{\mathcal{N}_{\lambda\nu}}^{\mu}{n_{\alpha\beta}}^{\mu}\right]\chi_{\nu}^{\mathfrak{Vir}}(q)=\sum_{\nu}\left[\sum_{\gamma}{n_{\alpha\gamma}}^{\lambda}{n_{\gamma\beta}}^{\nu}\right]\chi_{\nu}^{\mathfrak{Vir}}(q).$$

The perturbation rule can therefore be expressed as

$$mZ_{\alpha\beta}(q)\longrightarrow\sum_{\gamma}{n_{\alpha\gamma}}^{\lambda}Z_{\gamma\beta}(q),$$

which is precisely the proposal for brane condensation (Figure 4.1), at the level of the Virasoro-specialised characters.

It should however be noted that this brane condensation process as described above is not at all well-defined. The problem is that a stack of m branes does not automatically distinguish a representation of dimension m. Even if one restricts to irreducible representations (which one is not obliged to), those whose highest weights are related by a Dynkin symmetry of $\mathfrak{g}$ will have the same dimension. Moreover, two irreducible representations of $\mathfrak{g}$ may have the same dimension whilst being completely unrelated by any symmetry. The solution is to recall that it is the Chan-Paton factors A^a which carry the information about which representation is involved. Indeed, the perturbation used in [**5**, **63**] to initiate the condensation process involves coupling these Chan-Paton factors on the brane to the currents $J^a(z)$. Nevertheless, the use of Virasoro-specialised characters means that one cannot distinguish a quantity (character) labelled by λ from the quantity corresponding to the image of λ under a Dynkin symmetry. The above argument therefore contains such ambiguities. As with the derivation of the Cardy constraint, Equation (3.2.9), one should really be working at the level of the extended partition function (involving a complete set of commuting observables) and thus the full affine characters.

4.1.2. Brane Charges. Having developed the dynamical process of brane condensation, one can study charges conserved by this process. Given a brane labelled by an integrable highest weight λ, the conserved charge of this brane will be denoted by $Q_{\mathrm{alg}}(\lambda)$ (the subscript "alg" stands for *algebraic* and distinguishes this charge from the geometric charges to be introduced in Chapter 7). Brane condensation processes are now translated into equalities that must be satisfied by the brane charges. Assuming that charges are additive (hence valued in an abelian group), these take the form

$$\dim(\lambda)\,Q_{\mathrm{alg}}(\alpha)=\sum_{\gamma}{n_{\alpha\gamma}}^{\lambda}\,Q_{\mathrm{alg}}(\gamma),\tag{4.1.1}$$

where $\dim(\lambda)$ denotes the dimension of the irreducible $\mathfrak{g}$-module of highest weight λ.

The coefficients ${n_{\alpha\gamma}}^{\lambda}$ seem to be rather difficult to study in general, so for the remainder of this chapter, it will prove convenient to specialise to the untwisted symmetry-preserving branes. These coefficients now become the fusion coefficients ${\mathcal{N}_{\lambda\alpha}}^{\gamma}$ which are significantly more tractable. In particular, taking $\alpha = 0$ and applying Equations (3.1.27) and (3.1.28) gives

$$\dim(\lambda)\, Q_{\text{alg}}(0) = \sum_{\gamma} {\mathcal{N}_{\lambda 0}}^{\gamma} Q_{\text{alg}}(\gamma) = Q_{\text{alg}}(\lambda).$$

Normalising the charge by setting $Q_{\text{alg}}(0) = 1$ (if this charge vanished, then all charges would vanish) gives the conserved charge associated with an untwisted symmetry-preserving brane as the dimension of the corresponding irreducible representation:

$$Q_{\text{alg}}(\lambda) = \dim(\lambda). \tag{4.1.2}$$

Note, however, that this definition does not (naïvely) satisfy Equation (4.1.1) (with the fusion coefficients). Whilst the form of the brane charges is determined by Equation (4.1.2), it remains to determine *which* abelian group the charges take values in. Since the charges of the untwisted symmetry-preserving branes are all multiples of $Q_{\text{alg}}(0)$, the charge group has the form $\mathbb{Z}_x$. The parameter x is then *constrained* by brane condensation, giving

$$\dim(\lambda)\dim(\mu) = \sum_{\nu} {\mathcal{N}_{\lambda\mu}}^{\nu} \dim(\nu) \qquad (\text{mod } x). \tag{4.1.3}$$

These constraints will be referred to as *fusion constraints* in what follows. Note that they would become trivial (be satisfied for all x) if the fusion coefficients were replaced by the corresponding tensor product coefficients.

The fusion constraints were analysed in detail by Fredenhagen and Schomerus for the algebras $\mathfrak{su}(r+1)$ [**63**]. By making use of some fairly complicated induction arguments and a modified Littlewood-Richardson rule for fusion products [**61**], they were able to reduce this set of constraints to those of the form

$$\dim(k\Lambda_1)\dim(\Lambda_i) = \sum_{\nu} {\mathcal{N}_{k\Lambda_1,\Lambda_i}}^{\nu} \dim(\nu) \qquad (\text{mod } x),$$

for $i = 1, \ldots, r$. Here k is the level of the Wess-Zumino-Witten model and Λ_i denotes the fundamental weights of $\mathfrak{g}$. The fusion coefficients may be computed from the corresponding tensor product computation,

$$(k\Lambda_1) \otimes (\Lambda_i) = (k\Lambda_1 + \Lambda_i) \oplus ((k-1)\Lambda_1 + \Lambda_{i+1}),$$

with Λ_{r+1} understood to vanish. By the Kac-Walton formula (Proposition 5.2), only the weight $(k-1)\Lambda_1 + \Lambda_{i+1}$ survives in the corresponding fusion product. The fusion constraints are therefore equivalent (for $\mathfrak{g} = \mathfrak{su}(r+1)$) to

$$\dim(k\Lambda_1 + \Lambda_i) = \frac{i}{k+i}\binom{k+r+1}{r+1}\binom{r+1}{i} = 0 \qquad (\text{mod } x), \tag{4.1.4}$$

for $i = 1, \ldots, r$. The first equality follows from Weyl's dimension formula, Equation (A.3.4). They then claim (but only sketch a proof) that the largest x satisfying these constraints is[1]

$$x = \frac{k+r+1}{\gcd\{k+r+1, \operatorname{lcm}\{1,2,\ldots r\}\}}. \tag{4.1.5}$$

A proof of this claim will be given in Section 4.2.2.

This result was subsequently rederived by Maldacena, Moore and Seiberg in [**111**]. Their derivation is based on a different set of constraints, given by

$$\dim(\widehat{\omega}(\lambda)) = \det\widehat{\omega}\ \dim(\lambda) \qquad (\mathrm{mod}\ x),$$

where $\widehat{\omega}$ is a Dynkin symmetry of the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ acting on the weight space. The motivation for these constraints is geometric, and will be discussed in Section 4.3.1. It should be noted however, that while these constraints end up predicting the same charge group as those of Fredenhagen and Schomerus, it is not *a priori* clear that these constraints are as strong. Indeed, one could not expect the corresponding constraints to be exhaustive for more general Lie algebras as the group of Dynkin symmetries need not be non-trivial.

4.2. Charge Group Computations

In this section, the charge groups of the untwisted symmetry-preserving branes are computed for a Wess-Zumino-Witten model based on a general simple Lie algebra. These results have appeared in [**33**].

Consider an arbitrary fusion rule, expressed (as in Chapter 5) as an operation on the integral weights in the fundamental alcove:

$$\lambda \times \mu = \sum_{\nu} \mathcal{N}_{\lambda\mu}{}^{\nu}\, \nu.$$

Such a fusion rule is sent to the corresponding fusion constraint, Equation (4.1.3), by the replacement $\lambda \mapsto \dim(\lambda) \pmod{x}$. The strategy adopted in this section to analyse the fusion constraints is to instead consider the fusion rules, recovering the required results through this replacement.

Algebraically, the fusion rules of a Wess-Zumino-Witten model at level k define a commutative ring with unity, $\mathcal{F}_k^{\mathbb{Z}}$, called the *fusion ring*. This ring is studied in Section 5.1.2 where it is represented as a quotient of the character ring (Theorem 5.4):

$$\mathcal{F}_k^{\mathbb{Z}} \cong \frac{\mathbb{Z}[\chi_1, \ldots, \chi_r]}{\mathcal{I}_k^{\mathbb{Z}}}.$$

Here χ_i denotes the fundamental characters, those describing the irreducible representations of highest weight Λ_i, $i = 1, \ldots, r$, and $\mathcal{I}_k^{\mathbb{Z}}$ is the *fusion ideal*. The dimension map

$$\dim\colon \mathbb{Z}[\chi_1, \ldots, \chi_r] \longrightarrow \mathbb{Z}, \qquad \chi_\lambda \longmapsto \dim(\lambda),$$

[1] It can indeed happen that the largest such x is $x = 1$. This first occurs for $\mathfrak{su}(4)$ at level 2, but is not uncommon when the level is small. The interpretation is then that the corresponding brane charges always vanish and therefore give no dynamical information whatsoever.

extends to a ring homomorphism (in fact, it is just evaluation at the weight 0), so the problem may be rephrased as the determination of the maximal value of x such that dim induces a homomorphism between the quotient spaces

$$\dim' \colon \mathcal{F}_k^{\mathbb{Z}} \cong \frac{\mathbb{Z}[\chi_1,\ldots,\chi_r]}{\mathcal{J}_k^{\mathbb{Z}}} \longrightarrow \frac{\mathbb{Z}}{x\mathbb{Z}} = \mathbb{Z}_x.$$

A necessary and sufficient condition for this to happen is that the image of the fusion ideal under dim must be contained in $x\mathbb{Z}$. Since every ideal of $\mathbb{Z}$ has the form $m\mathbb{Z}$ for some m, it follows that x will be maximised by taking $x\mathbb{Z} = \dim \mathcal{J}_k^{\mathbb{Z}}$. If $\{p_i\}$ is a set of generators of the fusion ideal, it follows that the maximum value x may take is $\gcd\{\dim p_i\}$.

4.2.1. The Charge Group of $\mathfrak{su}(2)$. To illustrate this strategy, consider first the simplest case, $\mathfrak{su}(2)$. The character ring $\mathbb{Z}[\chi_1]$ is a principal ideal domain [**12**], so the fusion ideal is generated by a single polynomial in χ_1. It should be clear that to find this generator, it is sufficient to restrict to fusion rules involving the fundamental weight Λ_1. The tensor product rules involving Λ_1 are well known:

$$\begin{aligned} 0 \otimes \Lambda_1 = \Lambda_1 \qquad &\Rightarrow \qquad \chi_0 = 1, \\ [i\Lambda_1] \otimes \Lambda_1 = [(i+1)\Lambda_1] \oplus [(i-1)\Lambda_1] \quad &\Rightarrow \quad \chi_{(i+1)\Lambda_1} = \chi_1 \chi_{i\Lambda_1} - \chi_{(i-1)\Lambda_1} \quad \text{for all } i \geqslant 1. \end{aligned}$$

This gives a recursion relation for the characters $\chi_{i\Lambda_1}$ as polynomials in χ_1 which may be solved to yield Chebyshev polynomials of the second kind [**61**].

It follows from the Kac-Walton formula, Proposition 5.2, that the corresponding fusion rules at level k are just

$$\begin{aligned} 0 \times \Lambda_1 &= \Lambda_1, \\ [i\Lambda_1] \times \Lambda_1 &= [(i+1)\Lambda_1] + [(i-1)\Lambda_1] \qquad \text{for } 1 \leqslant i \leqslant k-1, \\ [k\Lambda_1] \times \Lambda_1 &= [(k-1)\Lambda_1]. \end{aligned}$$

Comparing with the tensor product rules therefore shows that the only difference is that $[(k+1)\Lambda_1]$ no longer appears. Equivalently, $\chi_{(k+1)\Lambda_1}$ must be set to zero in the fusion ring, hence the fusion ideal of $\mathfrak{su}(2)$ is

$$\mathcal{J}_k^{\mathbb{Z}} = \left\langle \chi_{(k+1)\Lambda_1} \right\rangle.$$

Following the strategy outlined above, the charge group for the untwisted symmetry-preserving branes of $\mathfrak{su}(2)$ at level k is obtained by applying dim to the generator of the fusion ideal. This is trivially seen to yield the charge group $\mathbb{Z}_{k+2}$.

4.2.2. The Charge Group of $\mathfrak{su}(r+1)$. The fusion ideals $\mathcal{J}_k^{\mathbb{Z}}$ of $\mathfrak{su}(r+1)$ have been explicitly described [**83**]. A convenient set of generators for each $\mathcal{J}_k^{\mathbb{Z}}$ is given by the partial derivatives (with respect to the fundamental characters) of a single function, called the *fusion potential*. This will be shown rigorously[2] in Section 5.3 (Theorem 5.8). At

[2]The two arguments given in [**83**] to show that the fusion ring is described by this fusion potential are not quite complete. The author indicates that completing the first requires "some rather lengthy manipulations

level k, this potential has the form

$$V_{k+r+1} = \frac{1}{k+r+1} \sum_{i=1}^{r+1} q_i^{k+r+1}, \tag{4.2.1}$$

where the variables q_i denote the formal exponentials of the weights of the irreducible $\mathfrak{su}(r+1)$-module of highest weight Λ_1 (thus $q_1 q_2 \cdots q_{r+1} = 1$). It will be seen in Section 5.2.1 (or Proposition 5.15) that the derivatives of V_{k+r+1} with respect to the fundamental characters may indeed be expressed as polynomials in the fundamental characters with integer coefficients (that is, as elements of the character ring). To compute these polynomials, it is convenient to form a generating function (Section 5.2.1):

$$V_{\mathfrak{su}(r+1)}(t) = \sum_{m=1}^{\infty} (-1)^{m-1} V_m t^m = \log\left[1 + \chi_1 t + \chi_2 t^2 + \ldots + \chi_r t^r + t^{r+1}\right]. \tag{4.2.2}$$

Given this information, it is simple to compute the (maximal) charge group parameter x. The homomorphism dim may be extended trivially to $\mathbb{Z}[\chi_1, \ldots, \chi_r][\![t]\!]$ (which just means formal power series in t with coefficients from the character ring). Differentiate the generating function and apply dim, noting that $\dim(\Lambda_j) = \binom{r+1}{j}$, to get

$$\frac{\partial V_{\mathfrak{su}(r+1)}(t)}{\partial \chi_i} = \frac{t^i}{1 + \chi_1 t + \ldots + \chi_r t^r + t^{r+1}} \xrightarrow{\dim} \frac{t^i}{(1+t)^{r+1}}.$$

According to Equation (4.2.2), the result of applying dim to the corresponding generator of the fusion ideal is given (up to a sign) by the coefficient of t^{k+r+1} in this formal power series. Since

$$\frac{t^i}{(1+t)^{r+1}} = \sum_{n=0}^{\infty} (-1)^n \binom{n+r}{r} t^{n+i} = \sum_{m=1}^{\infty} (-1)^{m-i} \binom{m-i+r}{r} t^m,$$

it follows that

$$x = \gcd\left\{\binom{k+2r+1-i}{r} : i = 1, \ldots, r\right\} = \gcd\left\{\binom{k+r+i}{r} : i = 1, \ldots, r\right\}. \tag{4.2.3}$$

Expressions like these may often be simplified by using the simple identity

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1} = \binom{n+1}{i} - \binom{n}{i-1}. \tag{4.2.4}$$

whose derivation is trivial. It is also convenient to have a graphical representation to visualise the simplifications[3]. Applying this identity to the binomial $\binom{k+r+2}{r}$ gives $\binom{k+r+1}{r} + \binom{k+r+1}{r-1}$. This is illustrated in Figure 4.2 by the arrows south and south-west from $\binom{k+r+2}{r}$.

of determinants", the details of which were omitted. The second argument falls significantly shorter, and is in fact not even sufficient to prove that the fusion potential describes the complexification of the fusion ring (where torsion is irrelevant). Note that the application to brane charges developed here requires a description of the fusion ring (over $\mathbb{Z}$). Unfortunately, it is this second argument which seems to have been the focus of attention in the literature.

[3]Such a visualisation serves to make the *structure* of the simplification obvious. Formal inductive procedures will therefore be omitted. The simplifications needed in this section are easily understood, so the graphical representation adds little perhaps. Those required in the next section are more complicated however.

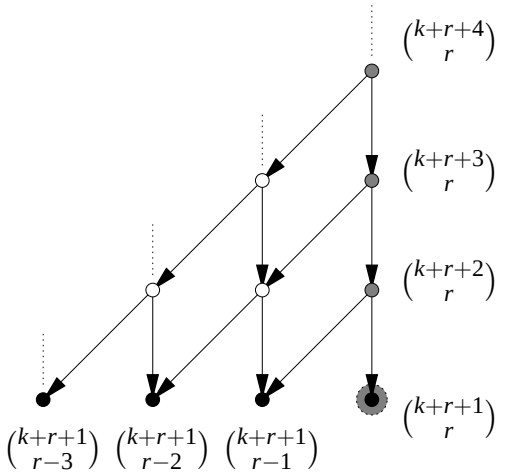

FIGURE 4.2. This diagram depicts the manipulation of the binomials in the gcd of Equation (4.2.3). The points represent binomials $\binom{m}{n}$ where m increases upwards and n increases to the right. The grey circles represent the original binomials in the gcd, and the black circles what they are replaced by. The arrows indicate the application of Equation (4.2.4) to the given binomial.

Since $\binom{k+r+1}{r}$ is already present in the gcd, it follows that $\binom{k+r+2}{r}$ may be replaced (in the gcd) by $\binom{k+r+1}{r-1}$. Similarly, applying Equation (4.2.4) iteratively to $\binom{k+r+3}{r}$ gives (see Figure 4.2) $\binom{k+r+1}{r-2}$ plus binomials already in the gcd. Thus, $\binom{k+r+3}{r}$ may be replaced by $\binom{k+r+1}{r-2}$. Continuing this replacement gives

$$x = \gcd\left\{\binom{k+r+1}{i} : i = 1,\ldots,r\right\}. \tag{4.2.5}$$

This can be compared with the result of Fredenhagen and Schomerus, Equation (4.1.5), with the help of a lemma from [**111**], itself a special case of a result of [**98**].

LEMMA 4.1.

$$\gcd\left\{\binom{k}{1},\binom{k}{2},\ldots,\binom{k}{r}\right\} = \frac{k}{\gcd\{k,\operatorname{lcm}\{1,2,\ldots r\}\}}.$$

It follows from Lemma 4.1 that the $\mathfrak{su}(r+1)$ fusion potential, Equation (4.2.1), easily reproduces the (untwisted symmetry-preserving) brane charge group $\mathbb{Z}_x$ given by Fredenhagen and Schomerus, Equation (4.1.5). In fact, as shown in Section 5.3.1, this fusion potential may be derived from the fact (Equation (5.3.2), see also [**83**]) that the fusion ideal $\mathcal{I}_k^{\mathbb{Z}}$ is generated by the characters $\chi_{(k+i)\Lambda_1}$, $i = 1,2,\ldots,r$. Since $\dim((k+i)\Lambda_1) = \binom{k+r+i}{r}$, this leads directly to Equation (4.2.3), giving a slightly easier derivation of Equation (4.1.5).

One can also make contact with the method used by Fredenhagen and Schomerus. Recall from Equation (4.1.4) that they had shown that

$$x = \gcd\left\{\frac{i}{k+i}\binom{k+r+1}{r+1}\binom{r+1}{i} : i = 1,2,\ldots,r\right\}.$$

Whilst these numbers are a little cumbersome to manipulate, it is easy to check that

$$\frac{i}{k+i}\binom{k+r+1}{r+1}\binom{r+1}{i} = \binom{k+i-1}{i-1}\binom{k+r+1}{r+1-i}.$$

Rewrite Equation (4.2.4) as

$$\binom{n}{i} = \binom{n+1}{i} - \binom{n}{i-1}, \tag{4.2.6}$$

and iterate to get $\binom{n}{i} = \sum_{j=0}^{i}(-1)^j\binom{n+1}{i-j}$. Replacing i with $i-1$ and n with $k+i-1$ now gives

$$\binom{k+i-1}{i-1} = \sum_{j=0}^{i-1}(-1)^j\binom{k+1}{i-j-1}.$$

Multiplying both sides by $\binom{k+r+1}{r+1-i}$ and simplifying then yields

$$\binom{k+i-1}{i-1}\binom{k+r+1}{r+1-i} = \sum_{j=0}^{i-1}(-1)^j\binom{r-j}{i-j-1}\binom{k+r+1}{r-j}.$$

It follows easily from this identity that

$$\begin{aligned}
x &= \gcd\left\{\binom{k+i-1}{i-1}\binom{k+r+1}{r+1-i} : i = 1,2,\ldots,r\right\}\\
&= \gcd\left\{\binom{r}{i-1}\binom{k+r+1}{r} - \binom{r-1}{i-2}\binom{k+r+1}{r-1} + \ldots\right.\\
&\qquad \left. + (-1)^{i-2}\binom{r+2-i}{1}\binom{k+r+1}{r+2-i} + (-1)^{i-1}\binom{k+r+1}{r+1-i} : i = 1,2,\ldots,r\right\}\\
&= \gcd\left\{\binom{k+r+1}{r+1-i} : i = 1,2,\ldots,r\right\}\\
&= \gcd\left\{\binom{k+r+1}{i} : i = 1,2,\ldots,r\right\},
\end{aligned}$$

which is just Equation (4.2.5).

4.2.3. The Charge Group of $\mathfrak{sp}(2r)$. The fusion ideals $\mathcal{I}_k^{\mathbb{Z}}$ of the symplectic algebras $\mathfrak{sp}(2r)$ have also been described in terms of fusion potentials. These potentials were first proposed in [**31**], and then subsequently in [**84**]. This description of the symplectic fusion rings will also be rigorously[4] derived in Section 5.3 (Theorem 5.8). At level k, the fusion potential takes the form

$$V_{k+r+1} = \frac{1}{k+r+1}\sum_{i=1}^{r}\left(q_i^{k+r+1} + q_i^{-(k+r+1)}\right), \tag{4.2.7}$$

where the variables q_i and q_i^{-1} denote the formal exponentials of the weights of the irreducible $\mathfrak{su}(r+1)$-module of highest weight Λ_1 (these weights have the form $\pm\varepsilon_i$,

[4]Alas, both [**31**] and [**84**] justify their fusion potential through the second argument of [**83**], which (as noted before) does not even establish that the potential correctly describes the complexification of the fusion ring. A set of generators of the fusion ring is proposed in [**31**], in analogy with the first argument of [**83**], but the proof that these are indeed generators is not given (it is stated that this proof "will appear elsewhere").

$i = 1,2,\ldots,r$). Again (Section 5.2.1 or Proposition 5.15), the derivatives of V_{k+r+1} with respect to the fundamental characters are elements of the character ring, and the generating function corresponding to Equation (4.2.2) is computed in Section 5.2.1 to be

$$V_{\mathfrak{sp}(2r)}(t) = \log\left[1 + E_1 t + \ldots + E_{r-1}t^{r-1} + E_r t^r + E_{r-1}t^{r+1} + \ldots + E_1 t^{2r-1} + t^{2r}\right], \tag{4.2.8}$$

where $E_n = \chi_n + \chi_{n-2} + \chi_{n-4} + \ldots$, and it is understood that $\chi_0 = 1$ and $\chi_n = 0$ whenever $n < 0$.

As in Section 4.2.2, computing the (maximal) charge group parameter x is now straightforward. Differentiate the potential and apply dim to get

$$\begin{aligned}\frac{\partial V_{\mathfrak{sp}(2r)}(t)}{\partial \chi_i} &= \frac{t^i + t^{i+2} + \ldots + t^{2r-i-2} + t^{2r-i}}{1 + E_1 t + \ldots + E_{r-1}t^{r-1} + E_r t^r + E_{r-1}t^{r+1} + \ldots + E_1 t^{2r-1} + t^{2r}} \\ &\xrightarrow{\dim} \frac{t^i + t^{i+2} + \ldots + t^{2r-i-2} + t^{2r-i}}{(1+t)^{2r}},\end{aligned}$$

where use has been made of $\dim(\Lambda_j) = \binom{2r}{j} - \binom{2r}{j-2}$, thus $E_j \xrightarrow{\dim} \binom{2r}{j}$. Expanding as a formal power series in t,

$$\frac{t^j}{(1+t)^{2r}} = \sum_{m=1}^{\infty} (-1)^{m-j} \binom{m-j+2r-1}{2r-1} t^m,$$

gives

$$\begin{aligned} x &= \gcd\left\{ \binom{k+3r-i}{2r-1} + \binom{k+3r-i-2}{2r-1} + \ldots \right. \\ &\qquad \left. + \binom{k+r+i+2}{2r-1} + \binom{k+r+i}{2r-1} : i = 1,2,\ldots,r \right\} \\ &= \gcd\left\{ \binom{k+3r-1}{2r-1} + \binom{k+r+1}{2r-1}, \binom{k+3r-2}{2r-1} + \binom{k+r+2}{2r-1}, \ldots \right. \\ &\qquad \left. , \binom{k+2r+1}{2r-1} + \binom{k+2r-1}{2r-1}, \binom{k+2r}{2r-1} \right\}. \end{aligned} \tag{4.2.9}$$

To simplify this expression, consider the element $\binom{k+2r+1}{2r-1} + \binom{k+2r-1}{2r-1}$ of the gcd. Applying Equation (4.2.4) to the first binomial and Equation (4.2.6) to the second, one obtains $\binom{k+2r}{2r-2} - \binom{k+2r-1}{2r-2}$ plus twice $\binom{k+2r}{2r-1}$ (which is already in the gcd). Applying Equation (4.2.6) once more, one finds that the negative sign in this identity leads to a fortuitous cancellation and that $\binom{k+2r+1}{2r-1} + \binom{k+2r-1}{2r-1}$ may be replaced by $\binom{k+2r-1}{2r-3}$. This is illustrated in Figure 4.3.

This cancellation persists in general. This is most easily seen by considering the diagrams of Figure 4.3. By removing a small number of (dashed) arrows, each diagram naturally splits into two isomorphic graphs, which meet at the points corresponding to the binomials $\binom{k+r+j}{2j-1}$. The multiplicity of this binomial in $\binom{k+3r-i}{2r-1} + \binom{k+r+i}{2r-1}$ is then the sum of the number of paths from $\binom{k+3r-i}{2r-1}$ to $\binom{k+r+j}{2j-1}$ (including the dashed arrows) and from $\binom{k+r+i}{2r-1}$ to $\binom{k+r+j}{2j-1}$. It is clear that the multiplicity of $\binom{k+r+i}{2i-1}$ is 1 as the only path

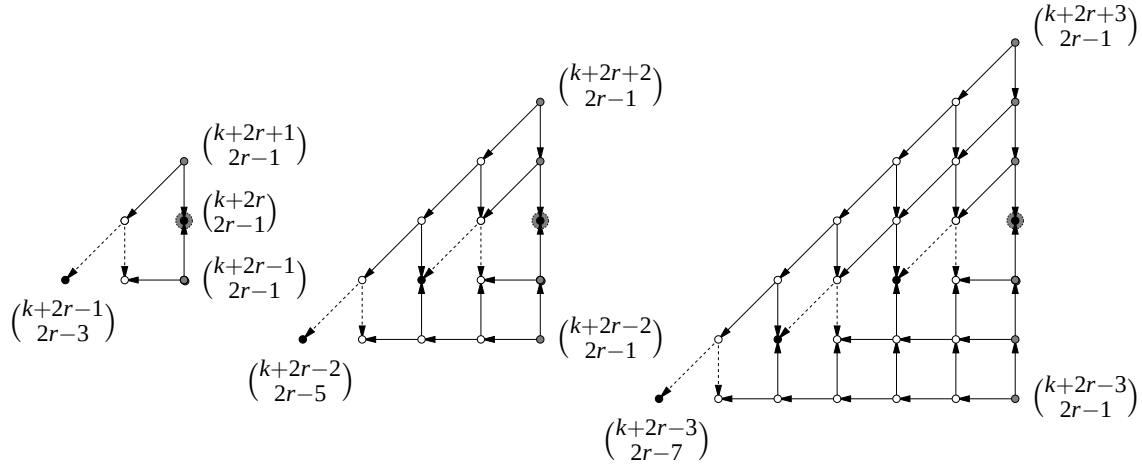

FIGURE 4.3. These diagrams depict the manipulation of the binomials in the gcd of Equation (4.2.9), as in Figure 4.2. The arrows correspond to Equation (4.2.4) (south and south-west) and Equation (4.2.6) (north and west). Arrows pointing west therefore carry a relative sign. Observe that by removing the dashed arrows, each diagram is split into two isomorphic graphs by the black circles. Observe also that there are white circles from which no arrows leave. These mark where "cancellation" occurs.

is the straight south-west one. The two isomorphic components of the diagram are also connected (by a dashed arrow) at points corresponding to the binomials $\binom{k+r+j}{2j}$. The isomorphic nature of the components means that the number of paths to $\binom{k+r+j}{2j}$ from $\binom{k+3r-i}{2r-1}$ and $\binom{k+r+i}{2r-1}$ is exactly the same. However, the multiplicity of $\binom{k+r+j}{2j}$ is the *difference* between these numbers of paths (because of the negative sign in Equation (4.2.6)), and so vanishes.

It follows that each $\binom{k+3r-i}{2r-1}+\binom{k+r+i}{2r-1}$ is equal to $\binom{k+r+i}{2i-1}$ plus integral multiples of binomials already in the gcd. Therefore,

$$x=\gcd\left\{\binom{k+r+1}{1},\binom{k+r+2}{3},\ldots,\binom{k+2r}{2r-1}\right\}.$$

This is now in a form amenable to a generalisation of Lemma 4.1 (which is proven in the same manner).

LEMMA 4.2.

$$\gcd\left\{\binom{k}{1},\binom{k+1}{3},\ldots,\binom{k+r-1}{2r-1}\right\}=\frac{k}{\gcd\{k,\operatorname{lcm}\{1,2,\ldots,r,1,3,5,\ldots,2r-1\}\}}.$$

Therefore the (untwisted symmetry-preserving) brane charge group for the $\mathfrak{sp}(2r)$ Wess-Zumino-Witten model at level k is

$$\mathbb{Z}_x \qquad \text{where} \qquad x=\frac{k+r+1}{\gcd\{k+r+1,\operatorname{lcm}\{1,2,\ldots,r,1,3,5,\ldots,2r-1\}\}}. \tag{4.2.10}$$

As in Section 4.2.2, a small saving in effort can be made by not starting with the fusion potential, Equation (4.2.7), but rather with the generators that it is derived from. These generators are given in Equation (5.3.3), and give Equation (4.2.9) directly.

4.2.4. The Charge Group in General. It would be nice to generalise the result for $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ to general simple Lie algebras $\mathfrak{g}$. Unfortunately, fusion potentials for the corresponding Wess-Zumino-Witten models are not known, at least not in a form analogous[5] to Equations (4.2.1) and (4.2.7). The reason why such analogous fusion potentials have not been found will be described in detail in Section 5.4. Generating sets for the fusion ideal $\mathcal{I}_k^{\mathbb{Z}}$ are also described there (explicitly for $\mathfrak{g}_2$), but they are not found to be suited to the above (exact) charge group computations (meaning that they yield complicated expressions which seem difficult to simplify). Instead, one is forced to resort to numerical computation, the results of which will now be described.

The fusion rules of a Wess-Zumino-Witten model may be computed exhaustively when the level k and the rank r of the simple Lie algebra are fairly small (10 was generally found to be too large). A relatively efficient algorithm for doing this is implemented in the program KAC [**142**]. The output of this program was processed through a simple python script which computed the (maximal) charge group parameter x. The low rank and level results thus obtained suggest (with a few low level exceptions that will be addressed in Section 4.3.3) the general formula

$$x = \frac{k+\mathsf{h}^\vee}{\gcd\{k+\mathsf{h}^\vee, y\}}, \tag{4.2.11}$$

where $\mathsf{h}^\vee$ is the dual Coxeter number (listed along with other useful data in Table A.1), and $y = y(\mathfrak{g})$ is an integer independent of the level. The suggested value of the integer y is given in Table 4.1. This value always has the form $y = \operatorname{lcm}\{y_\alpha\}$ for some y_α. The set $\{y_\alpha\}$ does not coincide with the set of exponents of $\mathfrak{g}$ as one might have naïvely hoped (though it does contain it). However, it should be noted that with the exception of the symplectic algebras, the y_α run from 1 to $\mathsf{h}-1$ consecutively (where h is the Coxeter number of $\mathfrak{g}$). This observation will be returned to in Section 4.3.1.

Further evidence for these parameters $y(\mathfrak{g})$, $\mathfrak{g} \neq \mathfrak{su}(r+1), \mathfrak{sp}(2r)$, was detailed in [**33**]. There, certain characters χ_λ with $(\lambda,\theta) = k+1$ (where θ is the highest root of $\mathfrak{g}$) were listed which were believed to generate the fusion ideal $\mathcal{I}_k^{\mathbb{Z}}$ (they belong to $\mathcal{I}_k^{\mathbb{Z}}$ by Proposition 5.5). They were found, numerically, to reproduce the values of y listed in Table 4.1, for all $\mathfrak{g}$ with $r < 10$ and levels $k < 5000$. It will be seen in Section 5.4.1 that this belief is unwarranted — the elements given do not (usually) generate the fusion ring. However, a correct set of generators is given there, and these have been numerically checked to also reproduce the listed values of y (though for less ambitious ranges of ranks and levels).

Some intuitive feeling for where this formula for x, Equation (4.2.11), comes from may be obtained from Weyl's dimension formula, Equation (A.3.4):

$$\dim(\lambda) = \prod_{\alpha\in\Delta_+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)} = \prod_{\alpha\in\Delta_+} \frac{(\lambda+\rho,\alpha^\vee)}{(\rho,\alpha^\vee)}.$$

[5] See Section 5.2.2 for a discussion of general, if not particularly useful, forms of the fusion potentials.

$\mathfrak{g}$	$y(\mathfrak{g})$
$\mathfrak{su}(r+1)$	$\operatorname{lcm}\{1,2,\ldots,r\}$
$\mathfrak{so}(2r+1)$	$\operatorname{lcm}\{1,2,\ldots,2r-1\}$
$\mathfrak{sp}(2r)$	$\operatorname{lcm}\{1,2,\ldots,r,1,3,5,\ldots,2r-1\}$
$\mathfrak{so}(2r)$	$\operatorname{lcm}\{1,2,\ldots,2r-3\}$
$\mathfrak{e}_6$	$\operatorname{lcm}\{1,2,\ldots,11\}$
$\mathfrak{e}_7$	$\operatorname{lcm}\{1,2,\ldots,17\}$
$\mathfrak{e}_8$	$\operatorname{lcm}\{1,2,\ldots,29\}$
$\mathfrak{f}_4$	$\operatorname{lcm}\{1,2,\ldots,11\}$
$\mathfrak{g}_2$	$\operatorname{lcm}\{1,2,\ldots,5\}$

TABLE 4.1. The integers y appearing in Equation (4.2.11) for each simple Lie algebra $\mathfrak{g}$.

Here Δ_+ denotes the positive roots of $\mathfrak{g}$. Consider the characters χ_λ with $(\lambda,\theta)=k+1$. As remarked above, these are elements of $\mathcal{I}_k^{\mathbb{Z}}$. Indeed, this set of characters can be augmented to form a generating set for $\mathcal{I}_k^{\mathbb{Z}}$ (Section 5.4.1). Under dim, these characters map to $\dim(\lambda)$ which always contains the factor $(\lambda+\rho,\theta)=k+\mathsf{h}^\vee$. Similarly, characters χ_λ with λ on a shifted affine alcove boundary will have $(\lambda,\alpha)\in(k+\mathsf{h}^\vee)\mathbb{Z}$ for some $\alpha\in\Delta_+$, so the factor $k+\mathsf{h}^\vee$ may again be identified in the corresponding dimension.

The other generators of $\mathcal{I}_k^{\mathbb{Z}}$ may be taken to have the form $\chi_\lambda-\det\widehat{w}\,\chi_{\widehat{w}\cdot\lambda}$, where $\widehat{w}$ is an element of the affine Weyl group $\widehat{\mathsf{W}}_k$ at level k. Write $\widehat{w}\cdot\lambda=w\cdot\lambda+(k+\mathsf{h}^\vee)\,q^\vee$, where $w\in\mathsf{W}$ (the Weyl group of $\mathfrak{g}$) and $q^\vee\in\mathsf{Q}^\vee$ (the coroot lattice of $\mathfrak{g}$). Since Weyl's dimension formula is anti-invariant under the shifted action of W, it is sufficient to restrict to pure translations. One finds that

$$\begin{aligned}\dim\left(\lambda+\left(k+\mathsf{h}^\vee\right)q^\vee\right)-\dim(\lambda)&\\ &=\frac{\prod_{\alpha\in\Delta_+}\left[(\lambda+\rho,\alpha)+(k+\mathsf{h}^\vee)\,(q^\vee,\alpha)\right]-\prod_{\alpha\in\Delta_+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Delta_+}(\rho,\alpha)}\\ &=\sum_{R\subsetneq\Delta_+}\left(k+\mathsf{h}^\vee\right)^{|\Delta_+\setminus R|}\prod_{\alpha\in R}\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}\prod_{\beta\in\Delta_+\setminus R}\frac{(q^\vee,\beta)}{(\rho,\beta)},\end{aligned}$$

so the factor $k+\mathsf{h}^\vee$ may again be identified ($|\Delta_+\setminus R|\geqslant 1$). It is therefore not surprising that the formula for x gives a divisor of this factor.

Consider now the denominator $\prod_{\alpha\in\Delta_+}(\rho,\alpha)$. When $\mathfrak{g}$ is simply-laced, the factors (ρ,α) are all integers and run (with repetitions) from 1 up to $(\rho,\theta)=\mathsf{h}^\vee-1=\mathsf{h}-1$. In other words, these are the integers y_α appearing in the lcm of Table 4.1. When $\mathfrak{g}$ is not simply-laced, (ρ,α) need not be an integer. One finds however that defining

$$y_\alpha=\begin{cases}(\rho,\alpha) & \text{if } (\rho,\alpha)\in\mathbb{Z},\\ (\rho,\alpha^\vee) & \text{if } (\rho,\alpha)\notin\mathbb{Z},\end{cases}$$

and setting $y = \operatorname{lcm}\{y_\alpha \colon \alpha \in \Delta_+\}$ reproduces all the results[6] in Table 4.1. It should be mentioned that generally $(\rho,\alpha) \in \mathbb{Z}$ need not imply that $(\lambda+\rho,\alpha) \in \mathbb{Z}$, so the above prescription need not guarantee integer factors in the numerator when applied to Weyl's dimension formula.

4.3. Addenda: Symmetries and K-Theory

In this last section some interesting observations are made regarding the brane charges computed in the previous section. The chapter then concludes with some remarks concerning the relevance of the results of this chapter to certain K-theories, and a brief discussion of a seeming ambiguity in the presentation of the fusion ring.

4.3.1. Charge Symmetries. Consider the brane charges $Q_{\text{alg}}(\lambda) = \dim(\lambda) \pmod{x}$, where x is given by Equation (4.2.11) (and Table 4.1). For convenience, extend the domain of the charges to all integral weights $\lambda \in \mathsf{P}$ through Weyl's dimension formula, Equation (A.3.4).

By Proposition 5.5, the combination $\chi_\lambda - \det\widehat{w}\ \chi_{\widehat{w}\cdot\lambda}$ always belongs to the fusion ideal $\mathcal{I}_k^{\mathbb{Z}}$ for $\widehat{w} \in \widehat{\mathsf{W}}_k$, the affine Weyl group. The brane charges are therefore invariant (up to a sign) under the shifted action of $\widehat{\mathsf{W}}_k = \mathsf{W} \ltimes \mathsf{Q}^\vee$, so in particular, under the shifted action of W (this also follows from Weyl's dimension formula) and under translations by $(k+\mathsf{h}^\vee)\,\mathsf{Q}^\vee$:

$$\text{For all } q^\vee \in \mathsf{Q}^\vee, \qquad Q_{\text{alg}}(\lambda) = Q_{\text{alg}}\left(\lambda + \left(k+\mathsf{h}^\vee\right) q^\vee\right) \qquad \pmod{x}. \tag{4.3.1}$$

Weyl's dimension formula is also invariant under the action (shifted or otherwise) of the Dynkin symmetries (outer automorphisms) of $\mathfrak{g}$, $\mathsf{Out}\,\mathfrak{g}$. With W, this gives a charge symmetry under the shifted action of the group of automorphisms of $\mathfrak{g}$ which preserve the Cartan subalgebra $\mathfrak{t}$, $\mathsf{Aut}_\mathfrak{t}\,\mathfrak{g}$. Each $\omega \in \mathsf{Aut}_\mathfrak{t}\,\mathfrak{g}$ may be uniquely decomposed as $\omega = w_\omega \eta_\omega$ (Appendix A.2), where $w_\omega \in \mathsf{W}$ and $\eta_\omega \in \mathsf{Out}\,\mathfrak{g}$. Since the Dynkin symmetries of $\mathfrak{g}$ preserve the set of positive roots,

$$\dim(\omega\cdot\lambda) = \det w_\omega\ \dim(\lambda) \qquad \Rightarrow \qquad Q_{\text{alg}}(\omega\cdot\lambda) = \det w_\omega\ Q_{\text{alg}}(\lambda). \tag{4.3.2}$$

Note that $\det\omega$ and $\det w_\omega$ need not coincide in general. Together with Equation (4.3.1), this extends the symmetry group of the brane charges to $\mathsf{Aut}_\mathfrak{t}\,\mathfrak{g} \ltimes \mathsf{Q}^\vee$.

This (rather trivial) observation explains a symmetry in the $\mathfrak{su}(3)$ brane charges, depicted in Figure 4.4 by reflection about the dashed lines. The reflection lines correspond to the non-trivial element ω of $\mathsf{Out}\,\mathfrak{su}(3)$ (note that $w_\omega = \mathrm{id}$, so the charges are left invariant despite the symmetry being a reflection). There are also rather obvious additional symmetries under reflection about the other two bisection axes and rotation by $2\pi/3$. These transformations correspond to the action of the Dynkin symmetries of $\widehat{\mathfrak{g}}$, $\mathsf{Out}\,\widehat{\mathfrak{g}}$.

[6] For $\mathfrak{f}_4$, one finds that $\{y_\alpha \colon \alpha \in \Delta_+\} = \{1,2,\ldots,9,11\}$ (that is, 10 is missing). Obviously, this does not affect the lcm.

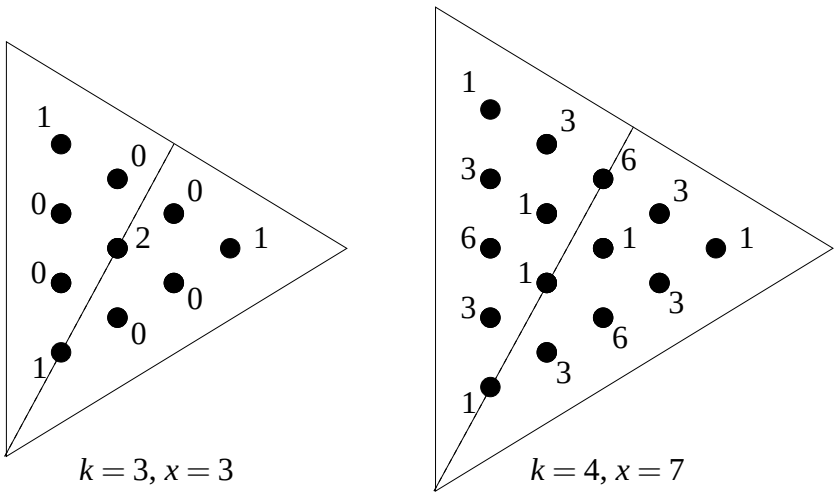

FIGURE 4.4. Brane charges for $\mathfrak{su}(3)$ at levels $k = 3$ (left) and $k = 4$ (right). The charge is given adjacent to the weight in the fundamental alcove that labels the brane. The dashed line represents the reflection plane corresponding to the action of the non-trivial Dynkin symmetry.

A Dynkin symmetry $\widehat{\omega} \in \mathsf{Out}\,\widehat{\mathfrak{g}}$ has a shifted action on the weight space of $\mathfrak{g}$ given by (compare Equation (B.2.2))

$$\widehat{\omega} \cdot \lambda = \omega \cdot \lambda + \left(k + \mathsf{h}^\vee\right) \Lambda_i = \omega \cdot \left(\lambda + \left(k + \mathsf{h}^\vee\right) \omega^{-1}(\Lambda_i)\right),$$

where $\omega \in \mathsf{Aut}_\mathfrak{t}\,\mathfrak{g}$ and the Dynkin symmetry $\widehat{\omega}$ takes node 0 to i in the Dynkin diagram of $\widehat{\mathfrak{g}}$. Decomposing ω as above, it follows that

$$\dim(\widehat{\omega} \cdot \lambda) = \det w_\omega \, \dim\left(\lambda + \left(k + \mathsf{h}^\vee\right) \omega^{-1}(\Lambda_i)\right). \tag{4.3.3}$$

By calculating brane charges for more general Lie algebras, one is led to hypothesise that a Dynkin symmetry $\widehat{\omega} \in \mathsf{Out}\,\widehat{\mathfrak{g}}$ induces a brane charge symmetry of the form

$$Q_{\text{alg}}(\widehat{\omega} \cdot \lambda) = \det w_\omega \, Q_{\text{alg}}(\lambda) \qquad (\text{mod } x). \tag{4.3.4}$$

For $\mathfrak{su}(3)$, Equation (B.2.3) gives $\det w_\omega = (-1)^{2(\Lambda_i, \rho)} = 1$ for all $\widehat{\omega} \in \mathsf{Out}\,\widehat{\mathfrak{su}}(3)$, in accordance with the observed symmetries of Figure 4.4. As with Equation (4.3.1), these $\mathsf{Out}\,\widehat{\mathfrak{g}}$ symmetries may be considered symmetries under translations (without any loss of generality). Indeed, it is shown in Appendix B.2 that extending $\mathsf{Aut}_\mathfrak{t}\,\mathfrak{g} \ltimes \mathsf{Q}^\vee$ by the affine outer automophisms leads to the symmetry group $\mathsf{Aut}_\mathfrak{t}\,\mathfrak{g} \ltimes \mathsf{Q}^*$, where Q^* is the dual root lattice. With Equation (4.3.1), the hypothesised $\mathsf{Out}\,\widehat{\mathfrak{g}}$ symmetries may therefore be reduced to:

$$\text{For all } q^\circ \in \mathsf{Q}^*, \qquad Q_{\text{alg}}(\lambda) = Q_{\text{alg}}\left(\lambda + \left(k + \mathsf{h}^\vee\right) q^\circ\right) \qquad (\text{mod } x). \tag{4.3.5}$$

It should be noted that such symmetries are certainly not derived from the fusion ideal $\mathcal{I}_k^{\mathbb{Z}}$. As noted before, the elements of the fusion ideal give rise to symmetries under translation by elements of $(k + \mathsf{h}^\vee)\,\mathsf{Q}^\vee$, and $\mathsf{Q}^\vee \subseteq \mathsf{Q}^*$ (with equality if and only if $\mathfrak{g} \in \{\mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2\}$). To illustrate this, consider $\mathfrak{g} = \mathfrak{su}(2)$, for which $\mathsf{Q}^* = \mathsf{P}$ and $\mathsf{Q}^\vee = 2\mathsf{P}$.

Taking $\lambda = 0$ in Equation (4.3.5) and $q^\circ = \Lambda_1$ gives

$$1 = Q_{\text{alg}}(0) = Q_{\text{alg}}((k+2)\Lambda_1) = k+3 \qquad (\text{mod } k+2),$$

which is obviously true. However, if $\chi_{(k+2)\Lambda_1} - \chi_0 = \chi_{(k+2)\Lambda_1} - 1$ were in the fusion ideal $\mathcal{I}_k^{\mathbb{Z}}$, then $1 + \chi_{k\Lambda_1} \in \mathcal{I}_k^{\mathbb{Z}}$ as $\chi_{(k+2)\Lambda_1} + \chi_{k\Lambda_1} \in \mathcal{I}_k^{\mathbb{Z}}$ by Proposition 5.5. But this contradicts the fact that $\mathcal{I}_k^{\mathbb{Z}}$ is generated by $\chi_{(k+1)\Lambda_1}$ (Section 4.2.2 or Section 5.3.1) as $1 + \chi_{k\Lambda_1} \in \mathcal{I}_k^{\mathbb{Z}}$ is a degree k polynomial in χ_{Λ_1} whereas $\chi_{(k+1)\Lambda_1}$ is degree $k+1$.

It would be quite interesting to find a mathematical proof showing that by satisfying the fusion constraints, Equation (4.1.3), one automatically satisfies the symmetries Equation (4.3.4) (and hence its equivalent, Equation (4.3.5)). Such a proof has remained elusive unfortunately. From a physical point of view, however, these symmetries are quite unremarkable, and indeed are expected. This is due to the fact that these additional symmetries are naturally parametrised by the *centre* $\mathsf{Z}(\mathsf{G})$ of the (compact, connected, simply-connected) Lie group G corresponding to $\mathfrak{g}$ (for precise statements, see Appendices B.2 and C.1). In the geometric picture of branes (Section 6.3.2), the action of $\mathsf{Out}\,\widehat{\mathfrak{g}}$ corresponds to translating the brane on G by the corresponding central element. This translation does not affect the geometric structure of the brane except possibly to change its orientation, and so *on physical grounds* one does not expect its charge to change, except possibly by a sign. This expectation is precisely that employed by Maldacena, Moore and Seiberg [**111**] (this was remarked upon in Section 4.1.2) to compute the brane charge groups of $\mathfrak{su}(r+1)$ (see also [**145**, **148**] for $\mathfrak{su}(3)$ where these symmetries are referred to as the "multiplet structure").

It seems reasonable to search for any additional symmetries. Consider the brane charges of $\mathfrak{sp}(4)$ at level $k = 4$ depicted in Figure 4.5. There is one non-trivial element of $\mathsf{Out}\,\widehat{\mathfrak{sp}}(4)$ which takes node 0 of the Dynkin diagram to node 2. The corresponding charge symmetry is exhibited by reflection about the dashed line. Note that in this case, the charges are negated $(\text{mod } 7)$ under this reflection in accordance with the sign $(-1)^{2(\Lambda_2,\rho)} = (-1)^3$.

What is even more interesting is the observation that there are two further reflections in the fundamental alcove of $\widehat{\mathfrak{sp}}(4)$ (marked by dotted lines in Figure 4.5) about which the charges are negated. Similar symmetries are also evident when investigating the brane charges of $\mathfrak{g}_2$ (for which $\mathsf{Out}\,\widehat{\mathfrak{g}}_2$ is trivial). That these symmetries are unrelated to the elements of $\mathsf{Out}\,\widehat{\mathfrak{g}}$ is evident from the fact that the reflections do *not* preserve the fundamental alcove. Nevertheless, the examples shown in Figure 4.5 indicate that these mysterious symmetries divide the fundamental alcove into several congruent subalcoves (fundamental domains), just as the $\mathsf{Out}\,\widehat{\mathfrak{g}}$ symmetries do. This seems to be indicative of an interesting structure generalising the outer automorphisms.

To investigate further, one can try to decompose the reflections corresponding to these mysterious symmetries into reflections through the origin and translations. The $\mathsf{Out}\,\widehat{\mathfrak{g}}$ symmetry bisecting the fundamental alcove of $\mathfrak{sp}(4)$ decomposes into a Weyl reflection and a translation by $(k+3)\Lambda_2$ ($\Lambda_2 \in \mathsf{Q}^*$) as expected. The two mysterious symmetries

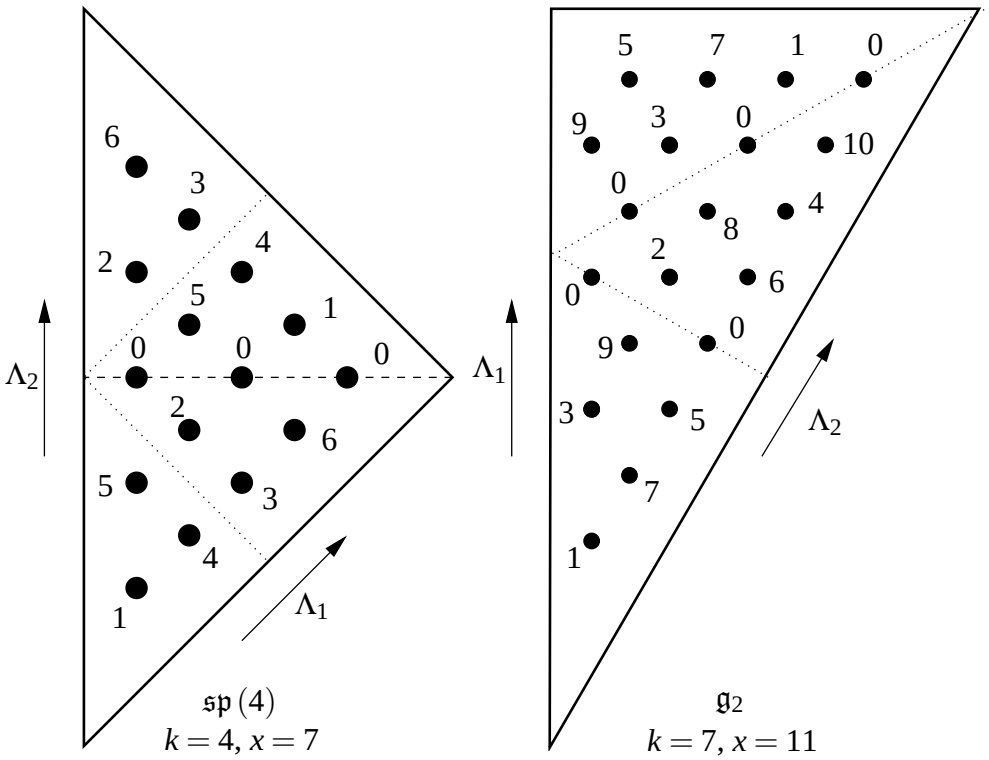

FIGURE 4.5. Brane charges for $\mathfrak{sp}(4)$ at level $k = 4$ (left) and $\mathfrak{g}_2$ at level $k = 7$ (right). The charge is given adjacent to the weight in the fundamental alcove that labels the brane. The dashed line represents the reflection plane corresponding to the action of the non-trivial Dynkin symmetry. The dotted lines represent additional symmetries.

can be checked to correspond to Weyl reflections (accounting for the sign change) and translations by $(k+3)\Lambda_1$ and $(k+3)(\Lambda_2 - \Lambda_1)$. Similarly, the mysterious symmetries observed in the fundamental alcove of $\mathfrak{g}_2$ correspond to Weyl reflections and translations by $(k+4)\Lambda_2$ and $(k+4)(\Lambda_1 - \Lambda_2)$. In each case, the translation is by an element of the weight lattice P which is not in the dual root lattice.

In fact, one can check (Figure 4.6) that in both cases, the weights corresponding to the mysterious symmetries generate the non-trivial cosets of P/Q^*. Similarly, the weight Λ_2 of $\mathfrak{sp}(4)$ generates the non-trivial element of $\mathsf{Q}^*/\mathsf{Q}^\vee$. It follows that, assuming these symmetries are real and not coincidental, *every* element of P generates a charge symmetry (those not seen in Figure 4.5 will be found in other alcoves):

$$\text{For all } \mu \in \mathsf{P}, \qquad Q_{\text{alg}}(\lambda) = Q_{\text{alg}}\left(\lambda + \left(k + \mathsf{h}^\vee\right)\mu\right) \qquad (\text{mod } x). \tag{4.3.6}$$

These hypothesised symmetries further enhance the symmetry group of the brane charges to $\mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g} \ltimes \mathsf{P}$.

When $\mathfrak{g}$ is simply laced, $\mathsf{Q}^* = \mathsf{P}$, so the weight lattice symmetries of Equation (4.3.6) reduce to the $\mathsf{Out}\,\widehat{\mathfrak{g}}$ symmetries of Equation (4.3.5). Therefore, when investigating if the weight lattice symmetries observed for $\mathfrak{sp}(4)$ and $\mathfrak{g}_2$ are present more generally, one must restrict attention to the algebras $\mathfrak{so}(2r+1)$, $\mathfrak{sp}(2r)$ and $\mathfrak{f}_4$ (this explains why no additional symmetries were seen in Figure 4.4 for $\mathfrak{su}(3)$). Numerically investigating brane charges

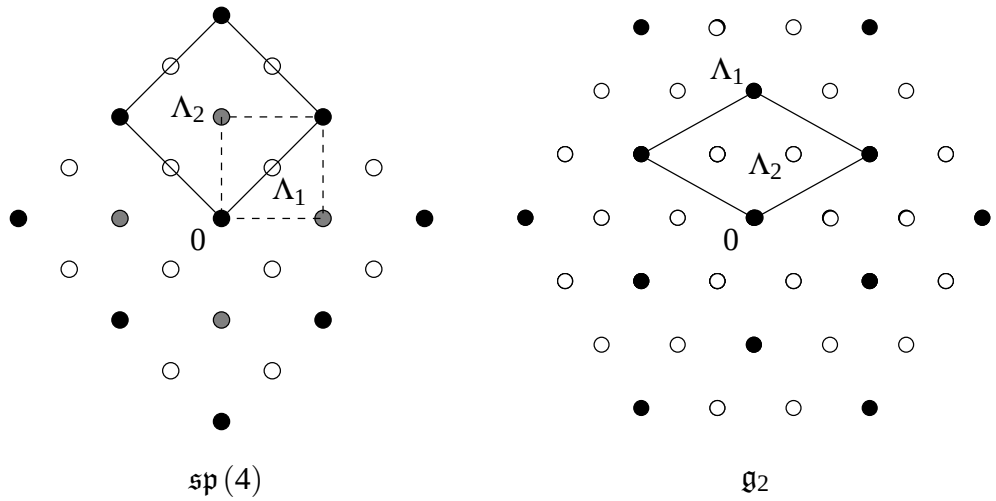

FIGURE 4.6. The lattices P (white), Q^* (grey), and $\mathsf{Q}^\vee$ (black) for $\mathfrak{sp}(4)$ (left) and the lattices P (white) and $\mathsf{Q}^* = \mathsf{Q}^\vee$ (black) for $\mathfrak{g}_2$ (right). Fundamental domains representing P/Q^* (dashed) and $\mathsf{P}/\mathsf{Q}^\vee$ (solid) are also indicated.

(where x is given by Equation (4.2.11) and Table 4.1) at low rank and level, one observes these weight lattice symmetries for the algebras $\mathfrak{so}(2r+1)$ and $\mathfrak{f}_4$. However, symmetries corresponding to Equation (4.3.6) are *not* always observed for $\mathfrak{sp}(2r)$. Specifically, the expected weight lattice symmetries are absent when r is not a power of 2 and $k+\mathsf{h}^\vee$ is divisible by the smallest power of 2 greater than r (that is, $2^{\lceil \log_2 r \rceil} \mid k+r+1$).

Equivalently, one can investigate (numerically) what charge group $\mathbb{Z}_\xi$ is implied by including the weight lattice symmetries of Equation (4.3.6). In agreement with the above observations, one finds that $\xi \neq x$ only for $\mathfrak{sp}(2r)$ where r is not a power of 2 (and the k given previously). In every such case where the weight lattice symmetries imply a deviant charge group, one finds that x is even and $\xi = x/2$. Indeed, for $\mathfrak{sp}(2r)$, one finds that ξ is generally given by[7] (compare with Table 4.1)

$$\xi = \frac{k+\mathsf{h}^\vee}{\gcd\{k+\mathsf{h}^\vee, \operatorname{lcm}\{1,2,\ldots,2r-1\}\}}.$$

Note that this implies (refer to Section 4.2.4) that ξ has the aesthetically pleasing form, common to *all* simple Lie algebras, given by

$$\xi = \frac{k+\mathsf{h}^\vee}{\gcd\{k+\mathsf{h}^\vee, \operatorname{lcm}\{1,2,\ldots,h-1\}\}}. \tag{4.3.7}$$

The observed presence of these weight lattice symmetries for non-symplectic algebras, coupled with the fact that imposing these symmetries on the symplectic algebras leads to a universal formula for the charge group, suggests that perhaps the fusion constraints of Fredenhagen and Schomerus do not account for all the physical constraints. After all, it has never been claimed that the fusion constraints are exhaustive, hence the

[7] Indeed, the condition that r not be a power of 2 guarantees that there is a power of 2 in $\{1,2,\ldots,2r-1\}$ which is missing from $\{1,2,\ldots,r,1,3,5,\ldots,2r-1\}$. This power of 2 is $2^{\lceil \log_2 r \rceil}$ which was noted to divide $k+r+1$ when the weight lattice prediction deviates from that of the fusion constraints.

true charge group parameter could be a proper divisor of the x given in Section 4.2.4. Of course, no evidence (or interpretation) has been given to indicate that the weight lattice symmetries are physically relevant. Indeed, one could argue based on the orders of the quotient groups[8] P/Q^* that the observation of these symmetries is coincidental, and the reason why these symmetries are not observed in the symplectic case is that there would have to be so many of them. This argument is not particularly satisfying however, as it fails to address the regular structure of these symmetries. In any case, it seems reasonable to keep in mind that there might be physically (or mathematically) relevant constraints, independent of the fusion constraints of Fredenhagen and Schomerus, which lead to the imposition of these weight lattice symmetries. This idea will be revisited in Section 7.3.3.

4.3.2. The K-Theory Connection. It has been shown above, quite explicitly, that the brane charges for untwisted symmetry-preserving branes take values in an abelian group $\mathbb{Z}_x$. Generally, one demands that the charge group for more general branes will also take values in some abelian group. This abelian group should obviously depend (in some nice way) on the string theory being investigated, in particular, on the target space that the strings (and branes) live in. For the Wess-Zumino-Witten models that are studied in this thesis, the target space is then (the underlying manifold of) the Lie group G.

It is a favourite pastime among mathematicians to associate abelian groups to (suitably nice) spaces, and many such associations have been developed. One of the oldest and best understood is the familiar example of the cohomology groups. The geometric approach to brane charges (outlined in more detail in Section 7.1) initially led to the suggestion that brane charges for a target space X should take values in the cohomology groups $\mathsf{H}^*(\mathsf{G};\mathbb{Z})$. This suggestion proved to be too naïve however, as evidenced by the above results and the well known fact (apparently due to Pontrjagin [**133**]) that the groups $\mathsf{SU}(r+1)$ have torsion-free cohomology. Instead, a more realistic class of candidates for the brane charge group is given by the various *K-theories* $\mathsf{K}^*(\mathsf{G})$.

K-theory is a type of generalised cohomology theory [**10**] introduced by Atiyah and Hirzebruch, and Grothendieck (amongst others). Its relation to the cohomology groups is perhaps most easily seen in the topological theory, wherein the elements of the K-theory of X are formal differences of isomorphism classes of vector bundles over X. This should be compared with the characteristic classes of such vector bundles which live in (various) cohomology groups. In fact, one has an isomorphism over the rationals given by the Chern character [**11**]:

$$\mathsf{K}^*_{\text{top}}(X)\otimes\mathbb{Q}\cong\mathsf{H}^*(X;\mathbb{Q})\,.$$

Topological K-theory therefore may differ from cohomology only through its torsion.

The K-theory that is currently accepted to classify brane charges on X is not a topological theory, but rather the K-theory of a C*-algebra of sections of a particular infinite-dimensional bundle over X. This brane charge classification was first proposed by Bouwknegt

[8] These orders may be extracted from Table B.1, and are 1 for simply laced algebras, 2 for $\mathfrak{so}(2r+1)$, 3 for $\mathfrak{g}_2$, 4 for $\mathfrak{f}_4$, and 2^{r-1} for $\mathfrak{sp}(2r)$.

and Mathai [**34**] (see also [**32**]), and the appropriate K-theory is called *twisted K-theory*. This K-theory was first defined by Rosenberg [**138**], and will be denoted by ${}^{H}\mathsf{K}^{*}(X)$. H here denotes an element of $\mathsf{H}^{3}(X;\mathbb{Z})$ which describes the "twisting". For the case of interest in this thesis, the class $H \in \mathsf{H}^{3}(\mathsf{G};\mathbb{Z}) \cong \mathbb{Z}$ is fundamentally involved in the (geometric) definition of the Wess-Zumino-Witten model, and (in $\mathbb{Z}$) is represented by $k+\mathsf{h}^{\vee}$ (see Sections 6.2.5 and 7.1.1). The class H will be studied in detail in Section 6.2.

The computation of this K-theory is not, however, straight-forward. When Fredenhagen and Schomerus first published their results on the brane charges of $\mathfrak{su}(r+1)$ (described in Section 4.1.2), only the twisted K-theory of $\mathsf{SU}(2)$ was known. This K-group agreed with their result for $\mathfrak{su}(2)$. The subsequent paper of Maldacena, Moore and Seiberg contained a physical (that is, not mathematically rigorous) computation of ${}^{H}\mathsf{K}^{*}(\mathsf{SU}(3))$ which also agreed with the result of Fredenhagen and Schomerus. However, it also announced a (previously unpublished) result of Hopkins concerning these K-groups:

$$ {}^{k+r+1}\mathsf{K}^{*}(\mathsf{SU}(r+1)) \cong \mathbb{Z}_{x} \otimes \wedge_{\mathbb{Z}}[w_5, w_7, \ldots, w_{2r+1}], $$

where $\wedge_{\mathbb{Z}}[w_i]$ denotes the (graded) exterior algebra over $\mathbb{Z}$ generated by the degree i elements w_i, and x is given by Equation (4.1.5). The result of Fredenhagen and Schomerus for the untwisted symmetry-preserving branes of $\mathfrak{su}(r+1)$ therefore agrees nicely with ${}^{k+r+1}\mathsf{K}^{*}(\mathsf{SU}(r+1))$, meaning that the torsion orders match. Note that by ignoring the ring structure (hence treating ${}^{k+r+1}\mathsf{K}^{*}(\mathsf{G})$ as an abelian group), this reduces to $\mathbb{Z}_{x}^{\oplus 2^{r-1}}$. Hopkins' computation therefore also suggests that for $r>1$ there are other generating branes (in a K-theoretic sense) with the same charge group $\mathbb{Z}_x$.

When the results of Sections 4.2.3 and 4.2.4 were first published, the twisted K-theories corresponding to these cases had also not been computed. All that was known[9] was that the torsion order had to be a divisor of $k+\mathsf{h}^{\vee}$. The results of Equation (4.2.11) and Table 4.1 therefore made highly non-trivial predictions regarding the torsion orders of these K-groups. These predictions were first checked by Braun [**36**] who showed how an assumption regarding the existence of a fusion potential (for the fusion ring) and an announced result of Freed, Hopkins and Teleman (Theorem 4.3 below) allows one to reduce the general K-theory computation to the strategy adopted in Section 4.2. Combining his result with Equation (4.2.11) then gives ${}^{k+\mathsf{h}^{\vee}}\mathsf{K}^{*}(\mathsf{G}) \cong \mathbb{Z}_{x}^{\oplus 2^{r-1}}$ as an abelian group (the ring structure was not determined), when G is compact, connected, simply-connected, and simple. Subsequently, the twisted K-theory of the groups $\mathsf{Sp}(2r)$, $\mathsf{Spin}(r)$, and G_2 were directly (and independently) computed by Douglas [**51**] with the result that

$$ {}^{k+\mathsf{h}^{\vee}}\mathsf{K}^{*}(\mathsf{G}) \cong \mathbb{Z}_{x} \otimes \wedge_{\mathbb{Z}}[w_1, w_2, \ldots, w_{r-1}], $$

[9] This follows from a generalisation of the Atiyah-Hirzebruch spectral sequence to twisted K-theory [**138**]. Only one differential is explicitly known from this sequence, and this guarantees the statement about torsion orders. For the case of $\mathsf{SU}(2)$ (and only in this case), one can be sure that the other (undetermined) differentials are trivial for dimensional reasons.

where x seems to be in agreement[10] with Equation (4.2.11) and Table 4.1 (here the w_i need not denote elements of degree i). This generalises Hopkins' unpublished result for $\mathsf{G} = \mathsf{SU}(r+1)$.

It should be noted at this point that the speculation in Section 4.3.1 regarding the charge groups of the symplectic algebras is not borne out by these K-theory computations. Specifically, the charge groups computed for $\mathfrak{sp}(2r)$ in Section 4.2.3 from the fusion constraints are in perfect agreement with ${}^{k+\mathsf{h}^\vee}\mathsf{K}^*(\mathsf{Sp}(2r))$, as computed by Braun and Douglas. It follows that if the weight lattice symmetries of Equation (4.3.6) (observed for other algebras) are imposed upon the $\mathfrak{sp}(2r)$ (untwisted symmetry-preserving) brane charges, then the order of the charge group implied by these symmetries will *not* match the torsion order of the classifying K-group.

It would be quite surprising if imposing weight lattice symmetries did lead to a match in torsion with K-theory. This is because of a remarkable result of Freed, Hopkins and Teleman, announced[11] in [**64**]:

THEOREM 4.3. *Let* G *be a compact, connected, simply-connected, simple Lie group with Lie algebra* $\mathfrak{g}$. *If* $\mathcal{F}_k^{\mathbb{Z}}$ *denotes the fusion ring of the Wess-Zumino-Witten model associated with* $\mathfrak{g}$ *(and the diagonal modular invariant), then*

$$\mathcal{F}_k^{\mathbb{Z}} \cong {}^{k+\mathsf{h}^\vee}\mathsf{K}_{\mathsf{G}}^*(\mathsf{G}),$$

where the right hand side is the twisted G-equivariant *(under conjugation) K-theory of* G.

A partial proof of this result, showing that the fusion ring and the twisted equivariant K-theory coincide as abelian groups, has appeared in [**65**]. This theorem gives a very precise statement of the relationship between fusion and K-theory, hence it would be very surprising if the brane charge groups derived from the fusion constraints, Equation (4.1.3), did not match the corresponding K-groups.

4.3.3. A Seeming Ambiguity. It remains to address the remark made in Section 4.2.4 concerning low level exceptions to the charge group result, Equation (4.2.11). An example will make this exceptional behaviour clear. Consider the fusion ring corresponding to the algebra $\mathfrak{g}_2$ at level $k = 1$. There are only two weights in the corresponding fundamental alcove, 0 and Λ_2, and the only non-trivial fusion rule is given by KAC (or by hand) as

$$\Lambda_2 \times \Lambda_2 = 0 + \Lambda_2. \tag{4.3.8}$$

Applying dim to this fusion rule therefore gives $49 = 8$ which is satisfied (mod 41). However, Equation (4.2.11) and Table 4.1 give $x = 1$ for $\mathfrak{g}_2$, level 1. Furthermore, 41 is certainly not a divisor of $k + \mathsf{h}^\vee = 5$, so this result must also be in direct conflict with the K-group ${}^5\mathsf{K}^*(\mathsf{G}_2)$.

[10] Douglas gives the torsion orders of the twisted K-theories in terms of some rather complicated expressions involving binomials. Numerically, they have been checked to match the results given above over a large range of ranks and levels.

[11] The announced result is in fact much more general.

The resolution of this discrepancy comes from considering the abstract presentation of the fusion ring of $\mathfrak{g}_2$, level 1. According to the above considerations, the fusion ring is generated by Λ_2 modulo the non-trivial fusion rule, Equation (4.3.8). That is,

$$\mathcal{F}_1^{\mathbb{Z}} \cong \frac{\mathbb{Z}[\chi_2]}{\langle \chi_2^2 - \chi_2 - 1 \rangle}.$$

This ideal then predicts $x = 41$. However, this is *not* expressed as a quotient of the character ring (as was generally assumed throughout Section 4.2. To do this, one needs to include the redundant weight Λ_1 to get

$$\mathcal{F}_1^{\mathbb{Z}} \cong \frac{\mathbb{Z}[\chi_1, \chi_2]}{\langle \chi_1, \chi_2^2 - \chi_2 - 1 \rangle}.$$

Applying dim now gives $x = \gcd\{14, 41\} = 1$ as expected. Similar redundancies, and therefore possible discrepancies, will occur whenever a fundamental weight does not belong to the fundamental alcove (that is, when the level is less than the maximal comark, $k < \max_i \{a_i^\vee\}$).

It follows that there are two presentations of the fusion ring which, despite being isomorphic as rings, yield different predictions for the charge group parameter x. Naïvely, one might prefer the former description as the latter has an obvious redundancy in its description. Indeed, one could add arbitrarily many additional redundant variables[12] to this presentation of the fusion ring in the same manner. However, the comparison with K-theory noted above indicates that it is this latter description which is relevant (at least mathematically).

Further insight may be gleaned from consideration of the tensor product decomposition corresponding to Equation (4.3.8). This is

$$\Lambda_2 \otimes \Lambda_2 = 0 \oplus \Lambda_2 \oplus \Lambda_1 \oplus (2\Lambda_2).$$

Noting that Λ_1 and $2\Lambda_2$ are both on the boundary of the shifted fundamental alcove, it follows from Proposition 5.5 that $\chi_{\Lambda_1} \equiv \chi_1$ and $\chi_{2\Lambda_2}$ should both (separately) be set to zero in the character ring $\mathbb{Z}[\chi_1, \chi_2]$. However, truncating this tensor product rule to get Equation (4.3.8) shows that it is in fact the combination $\chi_1 + \chi_{2\Lambda_2}$ which is set to zero. This latter truncation corresponds to the presentation of $\mathcal{F}_1^{\mathbb{Z}}$ as a quotient of $\mathbb{Z}[\chi_2]$. In other words, whilst these two presentations are isomorphic as rings, their treatment of boundary weights differs, and this makes all the difference to the corresponding charge group predictions.

To decide which presentation of the fusion ring gives the physically relevant charge groups, one must recall the condensation process described in Section 4.1.1. There, a stack of m identical branes, described in the low energy effective field theory by $m \times m$ Chan-Paton matrices A^a, was seen to be unstable with respect to condensation when the A^a spanned a representation of $\mathfrak{g}$. *Any* representation was admissible, and choosing

[12] These variables, however, would need to be naturally acted upon by dim (evaluation at 0).

irreducible ones led to the fusion constraints, Equation (4.1.3), which are in turn, the images under dim of the fusion rules $\lambda \times \mu = \sum_\nu \mathcal{N}_{\lambda\mu}{}^\nu \nu$.

The point[13] is that any irreducible representation is admissible, so the weight λ appearing in the fusion constraints (and hence the fusion rules) that determine the brane charge groups may be taken to be *any* dominant integral weight (not just those in the fundamental alcove). In particular, when λ is on the boundary of a shifted affine alcove, the fusion rule

$$\lambda \times 0 = \emptyset \qquad \Rightarrow \qquad \chi_\lambda = 0$$

must be included in the constraints (here $\emptyset$ is denoting the additive identity of the fusion ring). It follows that the presentation of the fusion ring describing this physics must set the character of *every* boundary weight to zero. The (smallest) presentation which does this is clearly that which describes the fusion ring as a quotient of the full character ring.

To summarise, the physical description of the condensation process giving rise to constraints on the brane charge groups *forces* the use of the presentation of the fusion ring as a quotient of the character ring. This in turn, leads to charge groups given by Equation (4.2.11) and Table 4.1 (with *no* exceptions), and these charge groups correctly predict the torsion orders of the corresponding classifying K-theories.

[13]This point was made transparent during private communication with Volker Braun.

CHAPTER 5

Fusion Rings

In this chapter, a detailed investigation of the fusion process is undertaken for the Wess-Zumino-Witten models based on a simple Lie algebra $\mathfrak{g}$ with the diagonal modular invariant. The aim is to prove rigorously that the fusion process for $\mathfrak{g} = \mathfrak{su}(r+1), \mathfrak{sp}(2r)$ may be described by the fusion potentials introduced in Section 4.2. The main ideas behind these proofs, and indeed many of the components, will be developed for general simple algebras whenever possible, so these investigations also shed light on why it is that analogous fusion potentials for the other algebras have not been found. Indeed, it is shown that strictly analogous potentials do not exist.

5.1. Fusion Rings and Algebras

5.1.1. Fusion Coefficients. As discussed in Sections 2.2.2 and 3.1.5, the process of fusion may be viewed as an algebraic operation on the set of primary fields in the conformal field theory. In the case of the Wess-Zumino-Witten model associated with the simple Lie algebra $\mathfrak{g}$ (with the diagonal modular invariant), the primary fields are in bijection with the integrable highest weight representations of $\widehat{\mathfrak{g}}$. These in turn are uniquely determined (Appendix A.2) by the projection of their highest weight onto the corresponding weight of the horizontal subalgebra. It follows that fusion in Wess-Zumino-Witten models may be viewed as an algebraic operation on the set, $\widehat{\mathsf{P}}_k$, of integral weights (of $\mathfrak{g}$) in the affine fundamental alcove at level k. This operation is denoted by $\times$:

$$\lambda \times \mu = \sum_{\nu} \mathcal{N}_{\lambda\mu}{}^{\nu}\, \nu. \tag{5.1.1}$$

The summation here is formal, and should not be confused with the usual sum of weights.

The important properties of fusion derived in Sections 2.2.2 and 3.1.5 will be briefly reviewed. First, the weight 0 defines a unit:

$$\mathcal{N}_{0\mu}{}^{\nu} = \delta_{\mu\nu} \qquad \text{(identity)}.$$

Second, the commutativity and associativity of the operator product expansion translates into

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \mathcal{N}_{\mu\lambda}{}^{\nu} \qquad \text{(commutativity)},$$

$$\text{and} \qquad \sum_{\sigma} \mathcal{N}_{\lambda\mu}{}^{\sigma}\mathcal{N}_{\sigma\nu}{}^{\tau} = \sum_{\sigma} \mathcal{N}_{\mu\nu}{}^{\sigma}\mathcal{N}_{\lambda\sigma}{}^{\tau} \qquad \text{(associativity)}.$$

Finally, Equation (3.1.29) gives

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \mathcal{N}_{\lambda^+\nu}{}^{\mu} \qquad \text{(symmetry)},$$

where λ^+ is the weight conjugate to λ.

Defining *fusion matrices* by $[\mathcal{N}_\lambda]_{\mu\nu} = \mathcal{N}_{\lambda\mu}{}^{\nu}$, these conditions imply

$$\mathcal{N}_0 = \mathrm{id}, \qquad \mathcal{N}_{\lambda^+} = \mathcal{N}_\lambda^{\mathsf{T}}$$

$$\mathcal{N}_\lambda\mathcal{N}_\mu = \mathcal{N}_\mu\mathcal{N}_\lambda \qquad \mathcal{N}_\lambda\mathcal{N}_\mu = \mathcal{N}_{\lambda\mu}{}^{\nu}\mathcal{N}_\nu,$$

where $^{\mathsf{T}}$ denotes transposition. The fusion matrices are therefore a commuting set of normal matrices furnishing a representation of the fusion rules, called the regular representation [**66**]. As such, the fusion matrices may be simultaneously diagonalised by a unitary matrix U, and this eigenspace decomposition corresponds to the decomposition of the regular representation into its irreducible components (which are of course one-dimensional over $\mathbb{C}$). The diagonalisation $\mathcal{N}_\lambda U = UD_\lambda$ (D_λ is diagonal) is equivalent to $\sum_\nu \mathcal{N}_{\lambda\mu}{}^{\nu} U_{\nu\sigma} = U_{\mu\sigma} d_{\lambda\sigma}$ where $d_{\lambda\sigma}$ are the eigenvalues of $\mathcal{N}_\lambda$ (and no summation over σ is implied). Putting $\mu = 0$ then gives $U_{\lambda\sigma} = U_{0\sigma} d_{\lambda\sigma}$, which determines the eigenvalues of the fusion matrices when $U_{0\sigma}$ is non-vanishing.

PROPOSITION 5.1 (The Verlinde Formula). *The fusion matrices of a Wess-Zumino-Witten model (associated to a simple Lie algebra $\mathfrak{g}$ with the diagonal modular invariant) are diagonalised by the modular S-matrix of $\widehat{\mathfrak{g}}$. Since $S_{0\sigma} > 0$ (Appendix B.3), the fusion coefficients may be expressed in the form*

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \sum_\sigma \frac{S_{\lambda\sigma} S_{\mu\sigma} S^*_{\nu\sigma}}{S_{0\sigma}}.$$

The Verlinde formula was first conjectured, rather more generally, in [**154**], and was demonstrated there for $\mathfrak{g} = \mathfrak{su}(2)$. This surprising, deep, yet utterly fundamental result generated much excitement in the mathematical literature, and has now been proven in various levels of generality [**55**, **153**]. However, these levels still fall short of the original expectation that this formula should apply to all rational conformal field theories.

There is another well-known formula for the fusion coefficients which provides a useful geometric algorithm for studying the fusion process. This is the Kac-Walton formula, originally set as an exercise in [**99**] and independently conjectured in [**157**], then proved in [**70**, **156**].

PROPOSITION 5.2 (The Kac-Walton Formula). *The fusion coefficients of a Wess-Zumino-Witten model (associated to a simple Lie algebra $\mathfrak{g}$ and with the diagonal modular invariant) are given by*

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \sum_{\widehat{w}\in\widehat{W}_k} \det \widehat{w}\, N_{\lambda\mu}{}^{\widehat{w}\cdot\nu},$$

where $\widehat{\mathsf{W}}_k$ is the affine Weyl group (at level k), and $N_{\lambda\mu}{}^{\nu}$ denotes the multiplicity of the representation of highest weight ν in the tensor product of the representations of highest weight λ and μ.

PROOF. Let k be the level of the Wess-Zumino-Witten model, and let

$$\xi_\sigma = -2\pi\mathrm{i}\,(\sigma+\rho)\,/\,\left(k+\mathsf{h}^\vee\right).$$

From the shifted affine action, Equation (B.2.1), and its effect on the characters, Equation (A.3.6), it follows that

$$\chi_{\widehat{w}\cdot\lambda}\,(\xi_\sigma) = \det\widehat{w}\;\chi_\lambda\,(\xi_\sigma).$$

Therefore, if λ is on the boundary of any shifted affine alcove, it is fixed by an affine reflection, so $\chi_\lambda\,(\xi_\sigma)=0$ for every $\sigma\in\widehat{\mathsf{P}}_k$.

Recall from Proposition 5.1 that the eigenvalues of the fusion matrices are given by

$$\frac{S_{\lambda\sigma}}{S_{0\sigma}} = \chi_\lambda\,(\xi_\sigma) \tag{5.1.2}$$

(see Equation (B.3.2)). It follows that

$$\sum_{\nu'\in\mathsf{P}} N_{\lambda\mu}{}^{\nu'}\,\chi_{\nu'}\,(\xi_\sigma) = \chi_\lambda\,(\xi_\sigma)\,\chi_\mu\,(\xi_\sigma) = \sum_{\nu\in\widehat{\mathsf{P}}_k} \mathcal{N}_{\lambda\mu}{}^{\nu}\,\chi_\nu\,(\xi_\sigma).$$

To compare the sum over P (the weight lattice) with the sum over $\widehat{\mathsf{P}}_k$ (the integrable highest weights at level k), note that every $\nu'\in\mathsf{P}$ is either on a shifted alcove boundary or there is a unique $\widehat{w}\in\widehat{\mathsf{W}}_k$ whose shifted action maps ν' into $\widehat{\mathsf{P}}_k$. As the boundary weights do not contribute to the sums, it follows that

$$\begin{aligned}\sum_{\nu\in\widehat{\mathsf{P}}_k} \mathcal{N}_{\lambda\mu}{}^{\nu}\,\chi_\nu\,(\xi_\sigma) &= \sum_{\nu\in\widehat{\mathsf{P}}_k}\sum_{\widehat{w}\in\widehat{\mathsf{W}}_k} N_{\lambda\mu}{}^{\widehat{w}\cdot\nu}\,\chi_{\widehat{w}\cdot\nu}\,(\xi_\sigma)\\ &= \sum_{\nu\in\widehat{\mathsf{P}}_k}\left[\sum_{\widehat{w}\in\widehat{\mathsf{W}}_k}\det\widehat{w}\;N_{\lambda\mu}{}^{\widehat{w}\cdot\nu}\right]\chi_\nu\,(\xi_\sigma).\end{aligned}$$

Using Equation (5.1.2) once more, cancelling $S_{0\sigma}$, and noting the invertibility of the S-matrix now gives the required result. ∎

The Kac-Walton formula suggests an algorithm for computing the fusion of two weights. This consists of computing the weights (with multiplicity) in the tensor product, discarding any weights that lie on a shifted affine boundary, and using affine Weyl group transformations to map any weight outside the shifted fundamental alcove to its interior (remembering to include the determinant of the transformation used). This algorithm, or rather the idea behind it, will prove very useful throughout this chapter.

5.1.2. Abstract Structure. Mathematically, the set of weights $\widehat{\mathsf{P}}_k$ with the fusion product defines a finitely-generated, associative, commutative, unital ring. Moreover, this

fusion ring is freely generated as a $\mathbb{Z}$-module (abelian group), and possesses a distinguished basis in which the structure constants are all non-negative integers. It is often convenient to generalise this structure to a fusion algebra by allowing coefficients in an algebraically closed field, $\mathbb{C}$ say. The fusion ring (at level k) will be denoted by $\mathcal{F}_k^{\mathbb{Z}}$, and the corresponding fusion algebra (over $\mathbb{C}$) by $\mathcal{F}_k^{\mathbb{C}} = \mathcal{F}_k^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$.

PROPOSITION 5.3. *The fusion ring/algebra has no (non-zero) nilpotent elements.*

PROOF. Consider first the fusion algebra, and define the elements $\pi_\lambda = S_{0\lambda} \sum_\mu S^*_{\lambda\mu} \mu$. Since S is unitary, the π_λ constitute a (vector space) basis for the fusion algebra. They also form a set of orthogonal idempotents:

$$\pi_\lambda \times \pi_\mu = S_{0\lambda} S_{0\mu} \sum_{\sigma,\tau,\nu} S^*_{\lambda\sigma} S^*_{\mu\tau} \mathcal{N}_{\sigma\tau}{}^{\nu}\, \nu = S_{0\lambda} S_{0\mu} \sum_{\sigma,\tau,\nu,\zeta} S^*_{\lambda\sigma} S^*_{\mu\tau} \frac{S_{\sigma\zeta} S_{\tau\zeta} S^*_{\nu\zeta}}{S_{0\zeta}} \nu$$

$$= S_{0\lambda} S_{0\mu} \sum_{\nu,\zeta} \delta_{\lambda\zeta} \delta_{\mu\zeta} \frac{S^*_{\nu\zeta}}{S_{0\zeta}} \nu = \delta_{\lambda\mu} \pi_\mu .$$

Here, the Verlinde formula (Proposition 5.1) has been used, as well as the symmetry of the S-matrix.

Suppose now that $\Phi = \sum_\lambda \Phi_\lambda \pi_\lambda$ is a nilpotent element of the fusion algebra. Then $\Phi^n = \sum_\lambda \Phi_\lambda^n \pi_\lambda = 0$ for some positive integer n, so $\Phi_\lambda^n = 0$ for each λ, hence $\Phi_\lambda = 0$ and so $\Phi = 0$. It follows that there are no non-zero nilpotent elements in the fusion algebra. Finally, a nilpotent element of the fusion ring will have a nilpotent image in the fusion algebra (under the obvious inclusion), hence the image must be zero. But the fusion ring has no torsion as a $\mathbb{Z}$-module, so the inclusion is injective and the original nilpotent element must also be zero. ∎

Since the fusion algebra is finitely-generated, associative, and commutative, it may be presented as a free polynomial ring (over $\mathbb{C}$) in its generators, modulo an ideal $\mathcal{I}_k^{\mathbb{C}}$. The lack of non-trivial nilpotent elements implies that this ideal is radical[1], hence completely determined by the affine variety consisting of points at which every polynomial in the ideal vanishes [**47**]. This variety will be referred to as the fusion variety.

The fusion algebra is a finite-dimensional vector space over $\mathbb{C}$ so it follows from this radicality that the fusion variety consists of a finite number of points, one for each basis element [**47**]. Since the π_λ of the proof of Proposition 5.3 form a basis of idempotents, they correspond to polynomials which take the values 0 and 1 on the fusion variety. Furthermore, their orthogonality ensures that polynomials corresponding to distinct idempotents must take value 1 on disjoint subsets of the fusion variety. But the π_λ form a basis, so the corresponding polynomials must take value 1 somewhere on the variety, hence they take value 1 at exactly one point of the variety. The point of the fusion variety at which the polynomial corresponding to π_λ takes value 1 will be denoted by ν^λ. It now follows from

[1] An ideal is radical if whenever some positive power of a ring element belongs to the ideal, so does the ring element itself [**12**].

$\lambda = \sum_\mu \left(S_{\lambda\mu}/S_{0\mu}\right)\pi_\mu$ that the polynomial representing λ takes the value $S_{\lambda\mu}/S_{0\mu}$ at v^μ. This polynomial will be denoted by p_λ.

The Kac-Walton formula (Proposition 5.2), which relates the fusion coefficients to the tensor product coefficients, suggests that it may be advantageous to fix the free polynomial ring as the representation ring (character ring) of $\mathfrak{g}$ (this was also suggested by Section 4.3.3). This ring is generated by the (characters of the) fundamental weights (Appendix A.3), and polynomial multiplication corresponds to the tensor product of representations. That is, the polynomial corresponding to the weight Λ_i is just its character $\chi_{\Lambda_i} \equiv \chi_i$. As the variables of the character ring are these fundamental characters, it follows that the coordinates of the points of the fusion variety are just

$$v_i^\lambda = p_{\Lambda_i}\left(v^\lambda\right) = \frac{S_{\Lambda_i\lambda}}{S_{0\lambda}}.$$

This proves the following fundamental result of Gepner [**83**]:

THEOREM 5.4. *The fusion algebra $\mathcal{F}_k^{\mathbb{C}}$ of the Wess-Zumino-Witten model associated with a simple Lie algebra $\mathfrak{g}$ of rank r (and diagonal modular invariant), is isomorphic to $\mathbb{C}[\chi_1,\ldots,\chi_r]/\mathcal{I}_k^{\mathbb{C}}$, where $\mathcal{I}_k^{\mathbb{C}}$ is the (radical) ideal of polynomials vanishing on the points*

$$\left\{\left(\frac{S_{\Lambda_1\lambda}}{S_{0\lambda}},\ldots,\frac{S_{\Lambda_r\lambda}}{S_{0\lambda}}\right) \in \mathbb{C}^r \colon \lambda \in \widehat{\mathsf{P}}_k\right\}.$$

Note that the points of the fusion variety have coordinates which are the eigenvalues of the fusion generators in the regular representation.

The fusion ring may likewise be represented as a quotient of $\mathbb{Z}[\chi_1,\ldots,\chi_r]$. The fusion ideal in this case is then $\mathcal{I}_k^{\mathbb{Z}} = \mathcal{I}_k^{\mathbb{C}} \cap \mathbb{Z}[\chi_1,\ldots,\chi_r]$. This ideal has the property that if any integral multiple of an integer polynomial is in $\mathcal{I}_k^{\mathbb{Z}}$, then so is the polynomial, thus ensuring that the quotient is a free $\mathbb{Z}$-module. Ideals with this property will be referred to as being *dividing*.

5.2. Fusion Potentials

5.2.1. $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$. The fusion rings have been explicitly described for the Wess-Zumino-Witten models over the algebras $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$. In both cases, the fusion ideal is generated by the partial derivatives of a single function referred to as a fusion potential. For $\mathfrak{su}(r+1)$, the fusion potential for the level k theory is given by Equation (4.2.1),

$$V_{k+r+1} = \frac{1}{k+r+1}\sum_{i=1}^{r+1} e^{(k+r+1)\varepsilon_i}.$$

Here, the ε_i, $i = 1,\ldots,r+1$ denote the projections of the standard orthonormal elements[2] onto the weight space (thus $\varepsilon_1 + \ldots \varepsilon_{r+1} = 0$). Since the ε_i are permuted by the Weyl group, W, of $\mathfrak{su}(r+1)$, V_{k+r+1} is a W-invariant function of (formal) exponentials of

[2]These orthonormal elements form a basis of $\mathbb{R}^{r+1}$ wherein the weight space is the hyperplane orthogonal to the sum of the basis elements. These basis elements are useful as they are permuted by the Weyl group $\mathsf{W} = \mathsf{S}_{r+1}$ of $\mathfrak{su}(r+1)$, and the roots are simply differences of these elements.

weights, hence may be expressed as a polynomial in the characters of the fundamental representations [**30**]. It is perhaps simpler to note that $\varepsilon_1 = \Lambda_1$ and so the ε_i coincide with the weights of the fundamental representation of $\mathfrak{su}(r+1)$. The fusion potential may therefore be written in the forms

$$V_{k+r+1} = \frac{1}{k+r+1} \sum_{\mu \in \mathsf{W}(\Lambda_1)} e^{(k+r+1)\mu} = \frac{1}{k+r+1} \sum_{\mu \in \Delta^{\Lambda_1}} e^{(k+r+1)\mu}, \tag{5.2.1}$$

where $\mathsf{W}(\lambda)$ is the Weyl orbit of λ and Δ^{λ} is the set of weights of the representation with highest weight λ.

For $\mathfrak{sp}(2r)$, the fusion potential for the level k theory is [**31**, **84**]

$$V_{k+r+1} = \frac{1}{k+r+1} \sum_{i=1}^{r} \left[e^{(k+r+1)\varepsilon_i} + e^{-(k+r+1)\varepsilon_i} \right],$$

where the ε_i, $i = 1, \ldots, r$ now constitute an orthonormal basis of the weight space on which the Weyl group of $\mathfrak{sp}(2r)$ acts by permutation and negation. Again, $\varepsilon_1 = \Lambda_1$ and so the fusion potential may again be written in exactly the same forms given in Equation (5.2.1).

This loudly demands the consideration of a generating function for these fusion potentials. For later purposes, define the "character" of Φ, where Φ is an arbitrary set of weights, by $\chi_\Phi = \sum_{\mu \in \Phi} e^{\mu}$. This corresponds to a genuine character when Φ is the set of weights of a representation. Let

$$V_m = \frac{1}{m} \sum_{\mu \in \Phi} e^{m\mu} = \frac{1}{m} \chi_{m\Phi},$$

where $m\Phi$ means the set consisting of multiples by m of elements of Φ, and define $V(t) = \sum_{m=1}^{\infty} (-1)^{m-1} V_m t^m$. Then,

$$V(t) = \sum_{\mu \in \Phi} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} e^{m\mu} t^m}{m} = \log \left[\prod_{\mu \in \Phi} (1 + e^{\mu} t) \right] = \log \left[\sum_{n=0}^{|\Phi|} \chi_{\wedge^n \Phi} t^n \right], \tag{5.2.2}$$

where $\wedge^n \Phi$ is the n^{th} exterior power[3] of Φ. In the fusion potentials for $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$, $\Phi = \Delta^{\Lambda_1}$, so one requires knowledge of the exterior powers of the fundamental representations (for each algebra).

These are well-known [**73**]. For $\mathfrak{su}(r+1)$, the n^{th} exterior power of the fundamental representation is just the representation of highest weight Λ_n (and the trivial representation if $n = 0, r+1$). For $\mathfrak{sp}(2r)$, the exterior powers are not always irreducible. When $n \leqslant r$, the n^{th} exterior power is the direct sum of the representations of highest weight Λ_{n-2i}, $i = 0, 1, 2, \ldots$ When $n \geqslant r$, the n^{th} exterior power coincides with the $(2r-n)^{\text{th}}$ exterior

[3]The exterior power $\wedge^n \Phi$ of a set of weights Φ is defined to be the set $\{0\}$ if $n = 0$, and otherwise, the set

$$\{\phi_1 + \phi_2 + \ldots + \phi_n \colon \phi_1, \phi_2, \ldots, \phi_n \in \Phi \text{ are distinct.}\}$$

power. The generating functions for $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ therefore have the forms

$$V_{\mathfrak{su}(r+1)}(t) = \log\left[\sum_{n=0}^{r+1} \chi_n t^n\right]$$

with $\chi_0 = \chi_{r+1} = 1$, and

$$V_{\mathfrak{sp}(2r)}(t) = \log\left[\sum_{n=0}^{r-1} E_n\left(t^n + t^{2r-n}\right) + E_r t^r\right],$$

where $E_n = \chi_n + \chi_{n-2} + \chi_{n-4} + \ldots$, $\chi_0 = 1$, and $\chi_n = 0$ whenever $n < 0$. This demonstrates Equations (4.2.2) and (4.2.8).

5.2.2. The General Case. An ideal which is generated by the partial derivatives of a single function is sometimes called a *jacobian* ideal [**54**]. One should expect such ideals to be comparatively rare, so it somewhat surprising to find that the fusion ideals for the $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ Wess-Zumino-Witten models are of this type. Nevertheless, it seems to be accepted that *every* fusion ideal is jacobian (at least over $\mathbb{C}$) — this was in fact originally conjectured by Gepner [**83**].

The first claim to have established the jacobian nature of the fusion ideal for general Wess-Zumino-Witten models appears in [**49**]. There, explicit formulae for the fusion potentials are presented for each simple algebra (implicitly, with the diagonal modular invariant). However, this claim suffers from the problem that it does not seem to make clear how the variables used in these explicit fusion potentials relate to the variables of the polynomial ring. To clarify, in the $\mathfrak{su}(r+1)$ fusion potential given in Equation (5.2.1), the variables are the weights of the fundamental representation. The polynomial ring, however, is the character ring, so the variables are the characters χ_i. The relation between these two sets of variables is therefore evident. It is the corresponding relation between the sets of variables used in [**49**] which is missing, making the explicit fusion potentials impossible to use (and impossible to check).

In any case, Aharony [**3**] later gave an explicit fusion potential for an *arbitrary* rational conformal field theory. The variables used to construct this potential form a generating set of the fusion ring, whose choice is constrained only by the requirement that at least one element of this set must have distinct eigenvalues in the regular representation (it is argued in [**66**] that every fusion ring contains such an element). The explicit form of the potential is rather unwieldy and will not be reproduced here. Suffice to say, it is constructed in such a way that its partial derivatives vanish precisely at the points whose coordinates are eigenvalues of the generating set in the regular representation. It should be noted however, that one requires explicit knowledge of these eigenvalues in order to write down Aharony's potential.

Strictly speaking, this construction establishes that the potential given reproduces the correct fusion *variety* (Section 5.1.2). To conclude that the corresponding ideal is in fact the fusion ideal (over $\mathbb{C}$), one also needs to verify that the potential generates a radical

ideal. This point does not appear to have been addressed in [**3**] (or anywhere else it seems). Assuming this property, one may then conclude that this potential correctly describes the fusion *algebra*. To extend this conclusion to a description of the fusion ring, further information is required regarding integrality properties of the derivatives of the potential. Whilst the rationality of the coefficients of these potentials are discussed, this alone is not nearly sufficient to prove that the fusion ring is correctly described by these potentials[4].

In any case, these fusion potentials do have their shortcomings. First, they are inelegant descriptions, involving the explicit forms of S-matrix elements (Appendix B.3), which one would not generally wish to compute. Second, they are not unique — there are enormous amounts of ambiguity in their construction, particularly in choosing a generating set and then choosing a distinguished element whose eigenvalues are distinct. Third, they do not seem to have any obvious interpretation involving other areas of mathematics. This last point is perhaps not valid for an arbitrary rational conformal field theory, but for the specific case of Wess-Zumino-Witten models, the examples of Section 5.2.1 suggest that one should be able to interpret a fusion potential in Lie-theoretic terms. For the purposes of this thesis, in particular the computations of Section 4.2, it should be clear that these inelegant fusion potentials are simply not suitable.

5.3. Proofs

In this section, the potentials given by Equation (5.2.1) for $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ are proven, rigorously, to describe the fusion *rings* of the corresponding Wess-Zumino-Witten models (with the diagonal modular invariant). This is precisely what was used in the computations of Section 4.2. There are three results that one can try to prove:

- The given potential describes the fusion *variety* of Theorem 5.4.
- The given potential defines a *radical* ideal of the complexified character ring.
- The given potential defines a *dividing* ideal of the integral character ring.

The first two establish that the potential is indeed the fusion potential for the fusion algebra, and the third is what is required to extend that conclusion to the fusion ring. In what follows, an elegant proof is first presented which directly addresses none of these results. It uses ideas from commutative algebra to directly construct a presentation of the fusion ideal (which is determined by the above three properties it must satisfy). The fusion potentials are then *derived* from this presentation through some simple symmetric group theory. It should be mentioned that such a derivation seems not to have appeared in the literature, where fusion potentials have been postulated without any satisfactory underlying motivation.

Unfortunately, the symmetric group theory used in this proof does not seem to have been extended to general simple Lie algebras. It therefore seems reasonable to study potentials of a similar form to those that successfully described the fusion rings of $\mathfrak{su}(r+1)$

[4]Specifically, one needs to show that the ideal generated by the derivatives in $\mathbb{Z}[\chi_1,\ldots,\chi_r]$ is dividing (as defined at the end of Section 5.1.2).

and $\mathfrak{sp}(2r)$. Before undertaking such a study, it is useful to build up enough theory to prove directly that the fusion potentials given for $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ define radical ideals corresponding to the (respective) fusion varieties[5]. Section 5.3.2 is devoted to this, and the theory developed in the course of these proofs will be used to undertake a more general study of potentials in Section 5.4.2.

5.3.1. Proofs over $\mathbb{Z}$. Let $\xi_\sigma = -2\pi\mathrm{i}(\sigma+\rho)/(k+\mathsf{h}^\vee)$. Recall from the proof of the Kac-Walton formula, Proposition 5.2, that if λ is a weight on the boundary of a shifted affine alcove (hence fixed by the shifted action of some reflection of $\widehat{\mathsf{W}}_k$), then $\chi_\lambda(\xi_\sigma) = 0$ for each integrable highest weight $\sigma \in \widehat{\mathsf{P}}_k$. Characters may always be expressed as polynomials with integer coefficients in the fundamental characters, $\chi_\lambda = p_\lambda(\chi_1,\ldots,\chi_r)$, so by Equation (5.1.2),

$$0 = \chi_\lambda(\xi_\sigma) = p_\lambda(\chi_1(\xi_\sigma),\ldots,\chi_r(\xi_\sigma)) = p_\lambda\left(\frac{S_{\Lambda_1\sigma}}{S_{0\sigma}},\ldots,\frac{S_{\Lambda_r\sigma}}{S_{0\sigma}}\right).$$

The polynomial p_λ therefore vanishes on the fusion variety. As the fusion ideal is radical, it follows that $p_\lambda \in \mathcal{I}_k^{\mathbb{C}}$, and as p_λ has integer coefficients, $p_\lambda \in \mathcal{I}_k^{\mathbb{Z}}$. More generally[6], the same argument proves the following proposition.

PROPOSITION 5.5. *Given any $\lambda \in \mathsf{P}$ and $\widehat{w} \in \widehat{\mathsf{W}}_k$, it follows that $p_\lambda - \det\widehat{w}\, p_{\widehat{w}\cdot\lambda} \in \mathcal{I}_k^{\mathbb{Z}}$.*

This argument shows, among other things, that $p_\lambda \in \mathcal{I}_k^{\mathbb{Z}}$ whenever $(\lambda,\theta) = k+1$ (where θ denotes the highest root of $\mathfrak{g}$ as usual). Let L_μ denote the irreducible representation of $\mathfrak{g}$ of highest weight μ. Writing $\lambda = \sum_i \lambda_i\Lambda_i$, it follows from the familiar properties of the representation ring that λ is the (unique) *highest* weight in the representation $L_{\Lambda_1}^{\otimes\lambda_1}\otimes\cdots\otimes L_{\Lambda_r}^{\otimes\lambda_r}$. Therefore,

$$p_\lambda(\chi_1,\ldots,\chi_r) = \chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r} - \ldots,$$

and in some sense, the omitted terms (the "…") correspond to lower weights and should be regarded as less important. To make this lack of importance precise, one introduces a monomial ordering on the character ring. This monomial ordering will be designed to have the property that the leading term, $\mathrm{LT}(p_\lambda)$, of p_λ is the monomial $\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}$. However, it will also prove very useful to have an ordering which respects the boundary $(\lambda,\theta) = k+1$ of the shifted fundamental affine alcove.

LEMMA 5.6. *Define a monomial ordering $\prec$ on $\mathbb{Z}[\chi_1,\ldots,\chi_r]$ by*

$$\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r} \prec \chi_1^{\mu_1}\cdots\chi_r^{\mu_r} \qquad \textit{if and only if}$$

[5]This proves the first two results listed above, hence that the fusion potentials correctly describe the fusion *algebra*. It seems to be difficult to *directly* prove the third result, that the ideal is dividing.

[6]The following proposition is indeed a generalisation, for applied to a boundary weight λ fixed by $\widehat{w}$, one gets $2p_\lambda \in \mathcal{I}_k^{\mathbb{Z}}$. But, $\mathcal{I}_k^{\mathbb{Z}}$ is dividing (Section 5.1.2), so $p_\lambda \in \mathcal{I}_k^{\mathbb{Z}}$ is recovered.

$$(\lambda,\theta)<(\mu,\theta), \quad \textit{or}$$
$$(\lambda,\theta)=(\mu,\theta) \quad \textit{and} \quad (\lambda,\rho)<(\mu,\rho), \quad \textit{or}$$
$$(\lambda,\theta)=(\mu,\theta) \quad \textit{and} \quad (\lambda,\rho)=(\mu,\rho) \quad \textit{and} \quad \chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r} <' \chi_1^{\mu_1}\cdots\chi_r^{\mu_r},$$

where $<'$ is some other monomial ordering (lexicographic for definiteness). Then,

$$\mathrm{LT}\left(p_\lambda\right)=\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r},$$

and if λ is a dominant integral weight on the boundary of the shifted fundamental alcove, $\mathrm{LT}\left(p_\mu\right)\prec\mathrm{LT}\left(p_\lambda\right)$ for all $\mu\in\widehat{\mathsf{P}}_k$.

PROOF. First, this is indeed a monomial ordering, as follows from the properties of the bilinear form $(\cdot,\cdot)$ and the fact that $<'$ is a monomial ordering. The statement concerning boundary weights is trivially checked as it implies $(\lambda,\theta)=k+1$ whereas $(\mu,\theta)\leqslant k$ for all $\mu\in\widehat{\mathsf{P}}_k$. It remains to verify that $\mathrm{LT}\left(p_\lambda\right)=\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}$.

This may be proved inductively (on the height), for it is trivially checked when λ is a fundamental weight. Generally, decompose $L_{\Lambda_1}^{\otimes\lambda_1}\otimes\cdots\otimes L_{\Lambda_r}^{\otimes\lambda_r}$ into irreducible representations, so that

$$\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}=\chi_\lambda+\sum_\mu c_\mu\chi_\mu,$$

where the μ are all of lower height than λ. By induction, $\mathrm{LT}\left(p_\lambda\right)$ is the greatest (under $\prec$) of $\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}$ and the monomials $-c_\mu\chi_1^{\mu_1}\cdots\chi_r^{\mu_r}$. Now, since each μ is a weight of $L_{\Lambda_1}^{\otimes\lambda_1}\otimes\cdots\otimes L_{\Lambda_r}^{\otimes\lambda_r}$, $\mu=\lambda-\sum_i m_i\alpha_i$, where the m_i are non-negative integers and α_i are the simple roots of $\mathfrak{g}$. It follows that $(\lambda,\theta)\geqslant(\mu,\theta)$ since the Dynkin labels of θ are never negative. But, $(\lambda,\rho)>(\mu,\rho)$ since ρ has positive Dynkin labels (indeed, this is what it means for λ to be higher than μ), hence $\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}$ is the greatest of the monomials (under $\prec$), completing the proof. ∎

Consider now the ideal $\left\langle\mathrm{LT}\left(\mathcal{I}_k^{\mathbb{Z}}\right)\right\rangle$ generated by the leading terms (with respect to $\prec$) of the polynomials in the fusion ideal. Since the fusion ring, $\mathcal{F}_k^{\mathbb{Z}}=\mathbb{Z}\left[\chi_1,\ldots,\chi_r\right]/\mathcal{I}_k^{\mathbb{Z}}$, is freely generated as an abelian group by the (cosets of the) characters of the weights in $\widehat{\mathsf{P}}_k$, the leading terms $\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}$, with $(\lambda,\theta)\leqslant k$ are the only monomials not in $\left\langle\mathrm{LT}\left(\mathcal{I}_k^{\mathbb{Z}}\right)\right\rangle$. That is, $\left\langle\mathrm{LT}\left(\mathcal{I}_k^{\mathbb{Z}}\right)\right\rangle$ is freely generated as an abelian group by the monomials $\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}$ with $(\lambda,\theta)>k$. It is easy now to determine a set of generators, as an ideal, for $\left\langle\mathrm{LT}\left(\mathcal{I}_k^{\mathbb{Z}}\right)\right\rangle$. For $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$, the monomials $\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}$ with $(\lambda,\theta)=k+1$ suffice[7] (as the comarks are all unity). That is, for these algebras,

$$\left\langle\mathrm{LT}\left(\mathcal{I}_k^{\mathbb{Z}}\right)\right\rangle=\left\langle\mathrm{LT}\left(p_\lambda\right):\ (\lambda,\theta)=k+1\right\rangle,$$

by Lemma 5.6. (Note that the leading coefficient of these characters is unity.) But, Proposition 5.5 states that $p_\lambda\in\mathcal{I}_k^{\mathbb{Z}}$ when $(\lambda,\theta)=k+1$, so this equation proclaims in fact that the p_λ form a *Gröbner basis* of $\mathcal{I}_k^{\mathbb{Z}}$ [**47**, **48**].

[7] If this is not obvious, see Section 5.4.1.

THEOREM 5.7. *For $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$, the characters $\chi_\lambda = p_\lambda(\chi_1,\ldots,\chi_r)$ with $(\lambda,\theta) = k+1$ form a Gröbner basis for the fusion ideal $\mathcal{I}_k^{\mathbb{Z}}$, with respect to the monomial ordering $\prec$.*

Note the crucial, but subtle, rôle played by the monomial ordering $\prec$. The extension of this result to the other simple Lie algebras will be discussed in Section 5.4.

Note also that because the Gröbner basis given has elements whose leading coefficient is unity, this presentation shows explicitly that the fusion ideal is dividing in the sense of Section 5.1.2. Radicality now follows explicitly from the fact that the rank of the fusion ring is equal to the number of points in the fusion variety (generally the rank is not less than this number, with equality if and only if the ideal is radical [**48**]).

Theorem 5.7 gives a very nice generating set for the fusion ideal of $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$, nice because of the fact that the set constitutes a Gröbner basis, and also because of the obvious Lie-theoretic interpretation of the set. However, this set does have disadvantages when compared with the fusion potential description of Section 5.2, most notably that the number of generators increases polynomially with the level k. It has also proven difficult to use these generating sets in the computations of Section 4.2, as the dimensions of general (irreducible) representations can be rather cumbersome to manipulate. The aim is therefore to reduce this generating set to the generators defined by the fusion potential.

For simplicity, consider the fusion ring associated with $\mathfrak{su}(r+1)$, specifically the characters $\chi_\lambda = p_\lambda(\chi_1,\ldots,\chi_r) \in \mathcal{I}_k^{\mathbb{Z}}$, with $(\lambda,\theta) = k+1$. Recall that the characters of $\mathfrak{su}(r+1)$ may be expressed as *Schur polynomials* in the variables $q_i = e^{\varepsilon_i}$, $i = 1,\ldots,r+1$ (ε_i the weights of the fundamental representation, so $q_1\cdots q_{r+1} = 1$). By the *Jacobi-Trudy identity* [**73**], Schur polynomials may be expressed in terms of complete symmetric polynomials H_i:

$$\chi_\lambda = \det\left(H_{\lambda^i+j-i}\right) = \begin{vmatrix} H_{\lambda^1} & H_{\lambda^1+1} & \cdots & H_{\lambda^1+r-1} \\ H_{\lambda^2-1} & H_{\lambda^2} & \cdots & H_{\lambda^2+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\lambda^r+1-r} & H_{\lambda^r+2-r} & \cdots & H_{\lambda^r} \end{vmatrix},$$

where $\lambda = \sum_i \lambda^i \varepsilon_i$ and λ^{r+1} is chosen to vanish (fixing the other λ^i). Noting that $\theta = \varepsilon_1 - \varepsilon_{r+1}$ [**30**], one finds that $k+1 = (\lambda,\theta) = \lambda^1 - \lambda^{r+1} = \lambda^1$, so the top row of this determinant has entries H_{k+i}, $i = 1,\ldots,r$. But, H_j is clearly the character of the representation of highest weight $j\varepsilon_1 = j\Lambda_1$ (again by the Jacobi-Trudy identity), so it follows by expanding the determinant along the top row that each χ_λ with $(\lambda,\theta) = k+1$ may be expressed as a linear combination of the $\chi_{(k+i)\Lambda_1}$, $i = 1,\ldots,r$, with polynomial coefficients. These polynomial coefficients are products of complete symmetric polynomials, hence can be expressed as integer polynomials in the fundamental characters. Therefore,

$$\mathcal{I}_k^{\mathbb{Z}} = \langle p_\lambda :\ (\lambda,\theta) = k+1\rangle \subseteq \left\langle p_{(k+i)\Lambda_1} : i = 1,\ldots,r\right\rangle. \tag{5.3.1}$$

To show that the inclusion in Equation (5.3.1) is in fact an equality, one only needs to show that each $p_{(k+i)\Lambda_1}$ is in the fusion ideal. By Proposition 5.5, this will be the case if $(k+i)\Lambda_1$ is fixed by some shifted affine Weyl reflection, which in turn will be the case if $((k+i)\Lambda_1+\rho,\alpha)\in(k+\mathsf{h}^\vee)\mathbb{Z}$ for some root α. It may easily be verified that there is such a root[8], so it follows that

$$\mathcal{I}_k^{\mathbb{Z}}=\left\langle p_{(k+i)\Lambda_1}:\, i=1,\ldots,r\right\rangle. \tag{5.3.2}$$

The story for the fusion ring of $\mathfrak{sp}(2r)$ is similar. There is a generalisation of the Jacobi-Trudy identity due to Weyl [**160**] which expresses the characters in terms of the determinant of a matrix whose entries involve complete symmetric polynomials (in the formal exponentials of the weights of the fundamental representation). This expression is also reproduced in [**73**]. Exactly as before, this generalised Jacobi-Trudy identity may be used to show that

$$\mathcal{I}_k^{\mathbb{Z}}\subseteq\left\langle p_{(k+1)\Lambda_1},p_{(k+1+i)\Lambda_1}+p_{(k+1-i)\Lambda_1}:\, i=1,\ldots,r-1\right\rangle.$$

The opposite inclusion is simplicity itself. $(k+1)\Lambda_1$ is obviously fixed by the (shifted) affine reflection $\widehat{w}_\theta$ about the hyperplane $(\lambda,\theta)=k+1$, and $p_{(k+1+i)\Lambda_1}+p_{(k+1-i)\Lambda_1}=p_{(k+1+i)\Lambda_1}-\det\widehat{w}_\theta\ p_{\widehat{w}_\theta\cdot(k+1+i)\Lambda_1}$. Proposition 5.5 again shows that these polynomials are elements of the fusion ideal, so

$$\mathcal{I}_k^{\mathbb{Z}}=\left\langle p_{(k+1)\Lambda_1},p_{(k+1+i)\Lambda_1}+p_{(k+1-i)\Lambda_1}:\, i=1,\ldots,r-1\right\rangle. \tag{5.3.3}$$

Note that the number of generators in these presentations is r, independent of the level k. Note also that instead of involving general characters, these generators consist of (the relatively simple) characters corresponding to multiples of the first fundamental weight. This set of generators for the fusion ideal is therefore perfectly suited for the computations of Section 4.2.

Whilst these generators are in fact all that is needed for these computations, it is not difficult to show how to derive from these the fusion potentials of Section 5.2. This will be detailed for $\mathfrak{su}(r+1)$, as the symplectic case is analogous if slightly more cumbersome. First, recall that the characters $\chi_{(k+i)\Lambda_1}$ are the complete symmetric functions $H_{k+i}(q_1,\ldots,q_{r+1})$, $i=1,\ldots,r$. To facilitate their manipulation, the standard generating function is introduced. H_{k+i} is then the coefficient of t^{k+i} in $\prod_{j=1}^{r+1}\left(1-q_jt\right)^{-1}$. Recalling that $\mathsf{h}^\vee=r+1$, this is equivalent to saying that $H_{k+\mathsf{h}^\vee-i}$ is the coefficient of $t^{k+\mathsf{h}^\vee}$ in $t^i\prod_{j=1}^{r+1}\left(1-q_jt\right)^{-1}$. Replacing t by minus $-t$, and absorbing any overall sign in the generators, one finds that the generators may be taken as the coefficients of $t^{k+\mathsf{h}^\vee}$ in

$$t^i\prod_{j=1}^{r+1}\left(1+q_jt\right)^{-1}=t^i\left(\sum_{n=0}^{r+1}E_nt^n\right)^{-1}=\frac{t^i}{1+\chi_1t+\ldots+\chi_rt^r+t^{r+1}},$$

[8]This verification is most conveniently performed in the orthonormal basis of $\mathbb{R}^{r+1}$ introduced in Section 5.2.1. Writing $\lambda=\sum_i\lambda^i\varepsilon_i$, one may take $\Lambda_1^i=\delta_{1i}$ and $\rho^i=r+1-i$. The roots have the form $\varepsilon_i-\varepsilon_j$, so one may choose $\alpha=\varepsilon_1-\varepsilon_{r+2-i}$. This gives $((k+i)\Lambda_1+\rho,\alpha)=k+r+1=k+\mathsf{h}^\vee$.

where the E_n are the elementary symmetric polynomials, and one recognises that $\chi_n = E_n(q_1,\ldots,q_{r+1})$ [**73**] (and $E_{r+1} = q_1\cdots q_{r+1} = 1$). This is obviously just the partial derivative with respect to χ_i of

$$V_{\mathfrak{su}(r+1)}(t) = \log\left[1+\chi_1 t+\ldots+\chi_r t^r + t^{r+1}\right],$$

which is exactly the generating function for the fusion potentials, Equation (4.2.2). With the analogous symplectic case, this now proves the main result of this chapter.

THEOREM 5.8. *The fusion ring $\mathcal{F}_k^{\mathbb{Z}}$ of the Wess-Zumino-Witten model associated with $\mathfrak{su}(r+1)$ or $\mathfrak{sp}(2r)$ at level k is described by the fusion potential $V_{k+\mathsf{h}^\vee}$ given by Equation (5.2.1):*

$$\mathcal{F}_k^{\mathbb{Z}} \cong \frac{\mathbb{Z}[\chi_1,\ldots,\chi_r]}{\left\langle \frac{\partial V_{k+\mathsf{h}^\vee}}{\partial\chi_1},\ldots,\frac{\partial V_{k+\mathsf{h}^\vee}}{\partial\chi_r}\right\rangle}.$$

5.3.2. Proofs over $\mathbb{C}$. In Theorem 5.8, the fusion ideals $\mathcal{I}_k^{\mathbb{Z}}$ for $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ were shown to be generated by the partial derivatives of the (respective) fusion potentials. That is, these fusion potentials correctly describe the (respective) fusion rings. As mentioned earlier, it will be useful to directly verify the analogous statements for the fusion algebras (which is of course implied by the theorem). This verification amounts to a logically independent proof (over $\mathbb{C}$), whose demonstration is certainly of independent interest. It consists of showing that the fusion potentials given define a radical ideal that vanishes at precisely the points of the fusion variety.

The first aim is therefore to establish that the derivatives of the fusion potentials given by Equation (5.2.1) vanish on each point $v^\lambda \in \mathbb{C}^r$ of the fusion variety. Recall from Theorem 5.4 and Equation (B.3.2) that

$$v_i^\lambda = \frac{S_{\Lambda_i\lambda}}{S_{0\lambda}} = \chi_i\left(-2\pi\mathrm{i}\frac{\lambda+\rho}{k+\mathsf{h}^\vee}\right).$$

The fusion potentials should therefore have critical points when the characters are evaluated at $\xi_\lambda = -2\pi\mathrm{i}(\lambda+\rho)/(k+\mathsf{h}^\vee)$, for $\lambda\in\widehat{\mathsf{P}}_k$. Indeed, the functions

$$\varkappa_i\colon \lambda\mapsto\chi_i\left(-2\pi\mathrm{i}\frac{\lambda+\rho}{k+\mathsf{h}^\vee}\right) = \sum_{\mu\in\Delta^{\Lambda_i}} e^{-2\pi\mathrm{i}(\mu,\lambda+\rho)/(k+\mathsf{h}^\vee)}$$

on the weight space may be easily checked to be invariant under the shifted action of the affine Weyl group $\widehat{\mathsf{W}}_k$. Therefore, the aim reduces to establishing that the potentials have critical points when the characters are evaluated at $\chi_i = \varkappa_i(\lambda)$, for any $\lambda\in\mathsf{P}$ which is not on a shifted alcove boundary.

It is obviously convenient to work with the potentials as functions on the weight space rather than as polynomials on the fundamental characters. Evaluating the potentials, Equation (5.2.1), as above gives

$$V_{k+\mathsf{h}^\vee}(\varkappa_1(\lambda),\ldots,\varkappa_r(\lambda)) = \frac{1}{k+\mathsf{h}^\vee}\sum_{\mu\in\Delta^{\Lambda_1}} e^{-2\pi\mathrm{i}(\mu,\lambda+\rho)} = \frac{1}{k+\mathsf{h}^\vee}\chi_1(-2\pi\mathrm{i}(\lambda+\rho)). \tag{5.3.4}$$

Notice that the level dependence of the fusion potentials becomes quite trivial. This drastic simplification encourages the investigation of the critical points of these potentials with respect to λ. Denoting the gradient operations with respect to the fundamental characters χ_i and the Dynkin labels λ_j by ∇_χ and ∇_λ respectively, one has

$$\nabla_\lambda V_m = \left(\nabla_\chi V_m\right)\, J, \tag{5.3.5}$$

where J is the jacobian of the functions $\varkappa_i$ with respect to the λ_j. It follows that if the fusion potential has an critical point with respect to λ, and J is non-singular there, then there is a corresponding critical point with respect to the fundamental characters. It is clearly useful now to determine when J becomes singular.

LEMMA 5.9. *The determinant of the jacobian J is an anti-invariant function under the induced action of* W.

PROOF. Differentiating the $\varkappa_i$ with respect to the Dynkin indices gives

$$\frac{\partial \varkappa_i}{\partial \lambda_j} = \frac{-2\pi \mathrm{i}}{k+\mathsf{h}^\vee} \sum_{\mu \in \Delta^{\Lambda_i}} \left(\mu, \Lambda_j\right) e^{-2\pi \mathrm{i}(\mu, \lambda+\rho)/(k+\mathsf{h}^\vee)}. \tag{5.3.6}$$

Letting $\nu = -2\pi \mathrm{i}\,(\lambda+\rho)\,/\,(k+\mathsf{h}^\vee)$, the entries of the jacobian, as functions on the weight space, satisfy

$$\begin{aligned} J_{ij}\left(w(\nu)\right) &= \frac{-2\pi \mathrm{i}}{k+\mathsf{h}^\vee} \sum_{\mu \in \Delta^{\Lambda_i}} \left(\mu, \Lambda_j\right) e^{(\mu, w(\nu))} = \frac{-2\pi \mathrm{i}}{k+\mathsf{h}^\vee} \sum_{\mu \in \Delta^{\Lambda_i}} \left(w^{-1}(\mu), w^{-1}\left(\Lambda_j\right)\right) e^{\left(w^{-1}(\mu), \nu\right)} \\ &= \frac{-2\pi \mathrm{i}}{k+\mathsf{h}^\vee} \sum_{\mu \in \Delta^{\Lambda_i}} \left(\mu, w^{-1}\left(\Lambda_j\right)\right) e^{(\mu, \nu)} = \frac{-2\pi \mathrm{i}}{k+\mathsf{h}^\vee} \sum_{k=1}^{r} \left(w^{-1}\right)_{jk} \sum_{\mu \in \Delta^{\Lambda_i}} (\mu, \Lambda_k)\, e^{(\mu, \nu)} \\ &= \sum_{k=1}^{r} J_{ik}(\nu)\, w_{kj}, \end{aligned}$$

where w_{kj} is the matrix representation of w in the basis of fundamental weights (this matrix is orthogonal). Hence,

$$J\left(w(\nu)\right) = J(\nu)\, w \qquad \Rightarrow \qquad \det J\left(w(\nu)\right) = \det w \det J(\nu). \qquad \blacksquare$$

PROPOSITION 5.10.

$$\det J = \left(\frac{-2\pi i}{k+\mathsf{h}^\vee}\right)^r \frac{1}{|\mathsf{P}/\mathsf{Q}^\vee|} \prod_{\alpha \in \Delta_+} \left(e^{\alpha/2} - e^{-\alpha/2}\right),$$

where Δ_+ denotes the set of positive roots (and the orders $|\mathsf{P}/\mathsf{Q}^\vee|$ may be extracted from Table B.1).

PROOF. By Lemma 5.9, $\det J$ is anti-invariant under W, so it factors [**30**] into an invariant element multiplied by the primitive anti-invariant element (see Appendix A.3), $\prod_{\alpha \in \Delta_+} \left(e^{\alpha/2} - e^{-\alpha/2}\right)$. From Equation (5.3.6), the term of highest weight in J_{ij} is easily verified to be

$$\frac{-2\pi \mathrm{i}}{k+\mathsf{h}^\vee} \left(\Lambda_i, \Lambda_j\right) e^{\Lambda_i},$$

so the term of maximal height in $\det J$ is

$$\left(\frac{-2\pi i}{k+\mathsf{h}^\vee}\right)^r \det\left(\Lambda_i,\Lambda_j\right) e^\rho.$$

Since $\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)$ has highest term $e^{\sum_{\alpha\in\Delta_+}\alpha/2}=e^\rho$, the invariant factor must be a constant. The result now follows from the fact that the matrix with entries $\left(\Lambda_i,\Lambda_j\right)$ is inverse to the symmetrised Cartan matrix (Appendix A.1). ∎

Evaluating $\det J$ at $-2\pi\mathrm{i}\left(\lambda+\rho\right)/\left(k+\mathsf{h}^\vee\right)$, it follows from Proposition 5.10 that the jacobian is singular precisely when

$$\prod_{\alpha\in\Delta_+}\left(e^{-\mathrm{i}\pi(\alpha,\lambda+\rho)/(k+\mathsf{h}^\vee)}-e^{\mathrm{i}\pi(\alpha,\lambda+\rho)/(k+\mathsf{h}^\vee)}\right)=0$$

$$\Rightarrow\qquad \prod_{\alpha\in\Delta_+}\sin\left[\pi\frac{(\alpha,\lambda+\rho)}{k+\mathsf{h}^\vee}\right]=0.$$

That is, when λ is on the boundary of a shifted affine alcove. It follows that the only place that a potential may have critical points with respect to λ which are not critical points with respect to the χ_i is on these boundaries.

PROPOSITION 5.11. *For the fusion potentials given by* (5.2.1)*, the critical points with respect to λ occur precisely at the weight lattice* P.

PROOF. First consider the potential for $\mathfrak{sp}\left(2r\right)$. From Equation (5.3.4), it follows that

$$V_{k+\mathsf{h}^\vee}\left(-2\pi\mathrm{i}\frac{\lambda+\rho}{k+\mathsf{h}^\vee}\right)=\frac{1}{k+\mathsf{h}^\vee}\sum_{\mu\in\Delta^{\Lambda_1}}e^{-2\pi\mathrm{i}(\mu,\lambda+\rho)}=\frac{2}{k+\mathsf{h}^\vee}\sum_{\mu\in\Delta_+^{\Lambda_1}}\cos\left[2\pi\left(\mu,\lambda+\rho\right)\right],$$

where $\Delta_+^{\Lambda_1}$ denotes the set of "positive" weights of the fundamental representation (the full set of weights consists of these weights and their negatives). Critical points therefore occur when

$$\sum_{\mu\in\Delta_+^{\Lambda_1}}\left(\mu,\Lambda_i\right)\sin\left[2\pi\left(\mu,\lambda+\rho\right)\right]=0,$$

for each $i=1,\ldots,r$. Consider the matrix A with entries $A_{i\mu}=\left(\mu,\Lambda_i\right)$. Since $\left|\Delta_+^{\Lambda_1}\right|=r$, this matrix is square. The positive weights of the fundamental representation can be verified to have the form (identifying the weight space with the Cartan subalgebra)

$$\Lambda_1-\alpha_1-\ldots-\alpha_{j-1}=\frac{1}{2}\left(\alpha_j^\vee+\ldots+\alpha_r^\vee\right)\qquad(j=1,\ldots,r),$$

so the corresponding entry of the matrix is $1/2$ if $i\geqslant j$ and 0 otherwise. The matrix A is thus lower-triangular, therefore invertible, so the critical points occur precisely when

$$\sin\left[2\pi\left(\mu,\lambda+\rho\right)\right]=\sin\left[\pi\left(\lambda_j+\rho_j+\ldots+\lambda_r+\rho_r\right)\right]=0\qquad\text{for all } j=1,\ldots,r.$$

It follows that $\lambda_j+\ldots+\lambda_r\in\mathbb{Z}$ for each $j=1,\ldots,r$, hence $\lambda\in\mathsf{P}$.

For $\mathfrak{su}\left(r+1\right)$, the number of weights in the fundamental representation is $r+1$, and this set need not be closed under negation. Proceeding as in the symplectic case would

therefore lead to the consideration of an $r\times(r+1)$-dimensional matrix. This suggests changing variables to the orthonormal basis $\{\varepsilon_1,\ldots,\varepsilon_{r+1}\}$ of $\mathbb{C}^{r+1}$. The weights of the fundamental representation are then the projection of the ε_i onto the hyperplane of $\mathbb{C}^{r+1}$ orthogonal to $\varepsilon_1+\ldots\varepsilon_{r+1}$. Writing $\lambda=\sum_i\lambda^i\varepsilon_i$, the λ^i are related to the Dynkin labels by $\lambda_i=\lambda^i-\lambda^{i+1}$. Finding the critical points of a potential on the weight space becomes a constrained optimisation problem in this setting, so the fusion potential should be altered by a Lagrange multiplier Ω to

$$\widetilde{V}_{k+\mathsf{h}^\vee}\left(-2\pi\mathrm{i}\frac{\lambda+\rho}{k+\mathsf{h}^\vee}\right)=\frac{1}{k+\mathsf{h}^\vee}\sum_{\mu\in\Delta^{\Lambda_1}}e^{-2\pi\mathrm{i}(\mu,\lambda+\rho)}+\Omega\left(\lambda,\varepsilon_1+\ldots+\varepsilon_{r+1}\right).$$

The weights of Δ^{Λ_1} are related to the ε_j by $\mu=\varepsilon_j-\left(\varepsilon_1+\ldots+\varepsilon_{r+1}\right)/(r+1)$. Differentiating with respect to the λ^i, and solving for the critical points therefore gives

$$e^{-2\pi\mathrm{i}\left(\lambda^i+\rho^i\right)}=\frac{k+\mathsf{h}^\vee}{2\pi\mathrm{i}}\Omega+\frac{1}{r+1}\sum_{j=1}^{r+1}e^{-2\pi\mathrm{i}\left(\lambda^j+\rho^j\right)}.$$

That is, $\exp\left[-2\pi i\left(\lambda^i+\rho^i\right)\right]$ is independent of i, so $\lambda_i+\rho_i=\lambda^i+\rho^i-\lambda^{i+1}-\rho^{i+1}\in\mathbb{Z}$. It follows that $\lambda\in\mathsf{P}$, as required. ∎

Recall that the fusion variety corresponds to integral weights λ which do *not* lie on an affine alcove boundary. Proposition 5.11 indicates that the potentials for $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ given by Equation (5.2.1), have critical points with respect to λ at these weights. Finally, Proposition 5.10 shows that the jacobian is non-singular away from the shifted alcove boundaries, so the fusion potentials also have critical points with respect to the fundamental characters at these weights. It therefore remains to show that there are no additional critical points (with respect to the fundamental characters) on the shifted alcove boundaries, and that the jacobian ideals of $\mathbb{C}[\chi_1,\ldots,\chi_r]$ described by the potentials are radical.

Turning to the question of radicality first, it follows from standard multiplicity theory, specifically the theory of *Milnor numbers* [**48**, **115**], that a jacobian ideal described by a potential will be radical if the hessian matrix of the potential is non-singular at each point of the corresponding (zero-dimensional) variety[9]. The hessian matrices for the fusion potentials of $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ are therefore worth investigating.

PROPOSITION 5.12. *For $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$, the hessian matrix H_λ of the fusion potential $V_{k+\mathsf{h}^\vee}(\xi_\lambda)$, with respect to λ, is non-singular on* P.

PROOF. This may be shown uniformly for all simply-laced groups, as in this case $\mathsf{P}=\mathsf{Q}^*$, the dual of the root lattice. Taking $\lambda+\rho\in\mathsf{Q}^*$ then implies that

$$(\mu,\lambda+\rho)=(\Lambda_1,\lambda+\rho)\qquad\qquad\mod\mathbb{Z},$$

[9]Indeed, if the hessian matrix is non-singular, then the *multiplicity* associated with each point of the variety is just 1. The sum of the multiplicities is the dimension of the corresponding (finite-dimensional) factor ring. Thus, when the multiplicities are all unity, this dimension equals the number of points in the variety. This equality can occur (for finite varieties) if and only if the corresponding ideal is radical (this was also noted in Section 5.3.1).

for all weights μ in Δ^{Λ_1}. It follows that

$$\begin{aligned}(H_\lambda)_{ij} &= \frac{-4\pi^2}{k+\mathsf{h}^\vee}\sum_{\mu\in\Delta^{\Lambda_1}}(\mu,\Lambda_i)\left(\mu,\Lambda_j\right)e^{-2\pi\mathrm{i}(\mu,\lambda+\rho)}\\ &= \frac{-4\pi^2}{k+\mathsf{h}^\vee}e^{-2\pi\mathrm{i}(\Lambda_1,\lambda+\rho)}\sum_{\mu\in\Delta^{\Lambda_1}}(\mu,\Lambda_i)\left(\mu,\Lambda_j\right)\\ &= \frac{-4\pi^2}{k+\mathsf{h}^\vee}e^{-2\pi\mathrm{i}(\Lambda_1,\lambda+\rho)}I_{\Lambda_1}\left(\Lambda_i,\Lambda_j\right),\end{aligned}$$

by Equation (A.1.7). Here, $I_\Lambda = (\Lambda,\Lambda+2\rho)\dim(\Lambda)/\dim\mathfrak{g}$ denotes the Dynkin index of the representation of highest weight Λ. It follows that

$$\det H_\lambda = \left(\frac{-4\pi^2 I_\Lambda}{k+\mathsf{h}^\vee}\right)^r e^{-2\pi\mathrm{i}r(\Lambda_1,\lambda+\rho)}\left|\mathsf{P}/\mathsf{Q}^\vee\right|^{-1} \qquad \text{when } \lambda+\rho\in\mathsf{Q}^*.$$

This is clearly non-zero, so the proposition is verified for $\mathfrak{su}(r+1)$.

For $\mathfrak{sp}(2r)$, a more specific calculation is required. Recall from the proof of Proposition 5.11 that the weights of Δ^{Λ_1} take the form $\mu_\ell = \pm\frac{1}{2}\left(\alpha_\ell^\vee+\ldots+\alpha_r^\vee\right)$, for $\ell = 1,2,\ldots,r$. It follows that $(\mu_\ell,\Lambda_i)\left(\mu_\ell,\Lambda_j\right) = \frac{1}{4}$ if $i\geqslant\ell$ and $j\geqslant\ell$, and 0 otherwise. Computing the hessian as before gives

$$(H_\lambda)_{ij} = \frac{-2\pi^2}{k+\mathsf{h}^\vee}\sum_{\ell=1}^{\min\{i,j\}}\cos\left[\pi\left(\lambda_\ell+\ldots+\lambda_r+r-\ell+1\right)\right].$$

Let $h_\ell = \cos\left[\pi\left(\lambda_\ell+\ldots+\lambda_r+r-\ell+1\right)\right]$. Then, elementary row operations give

$$\begin{aligned}\det H_\lambda &= \left(\frac{-2\pi^2}{k+\mathsf{h}^\vee}\right)^r\det\begin{pmatrix}h_1 & h_1 & h_1 & \cdots & h_1\\ h_1 & h_1+h_2 & h_1+h_2 & \cdots & h_1+h_2\\ h_1 & h_1+h_2 & h_1+h_2+h_3 & \cdots & h_1+h_2+h_3\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ h_1 & h_1+h_2 & h_1+h_2+h_3 & \cdots & h_1+\ldots+h_r\end{pmatrix}\\ &= \left(\frac{-2\pi^2}{k+\mathsf{h}^\vee}\right)^r\det\begin{pmatrix}h_1 & h_1 & h_1 & \cdots & h_1\\ 0 & h_2 & h_2 & \cdots & h_2\\ 0 & 0 & h_3 & \cdots & h_3\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & h_r\end{pmatrix}\\ &= \left(\frac{-2\pi^2}{k+\mathsf{h}^\vee}\right)^r\prod_{\ell=1}^{r}\cos\left[\pi\left(\lambda_\ell+\ldots+\lambda_r+r-\ell+1\right)\right].\end{aligned}$$

Therefore, when $\lambda\in\mathsf{P}$, $\det H_\lambda = \pm\left(2\pi^2\right)^r/\left(k+\mathsf{h}^\vee\right)^r\neq 0$, so the hessian is again non-singular. ∎

Of course, it is the hessian of $V_{k+\mathsf{h}^\vee}$ with respect to the fundamental characters that is required to be non-singular at each point of the fusion variety. Denoting this hessian by

H_χ, one has the relationship

$$\frac{\partial^2 V_{k+\mathsf{h}^\vee}}{\partial\lambda_i\partial\lambda_j} = \sum_{s,t}\frac{\partial\chi_s}{\partial\lambda_i}\frac{\partial^2 V_{k+\mathsf{h}^\vee}}{\partial\chi_s\partial\chi_t}\frac{\partial\chi_t}{\partial\lambda_j} + \sum_\ell \frac{\partial V_{k+\mathsf{h}^\vee}}{\partial\chi_\ell}\frac{\partial^2\chi_\ell}{\partial\lambda_i\partial\lambda_j}$$

$$\Rightarrow \qquad H_\lambda = J^{\mathsf{T}} H_\chi J \qquad \text{when } \nabla_\chi V_{k+\mathsf{h}^\vee} = 0. \tag{5.3.7}$$

When $\lambda \in \mathsf{P}$ is not on the boundary of a shifted alcove, the hessian H_λ is non-singular by Proposition 5.12, and the jacobian J is non-singular by Proposition 5.10. The remark after Proposition 5.11 shows that $\nabla_\chi V_{k+\mathsf{h}^\vee} = 0$ at such points, so Equation (5.3.7) forces the hessian H_χ to be non-singular there too. The multiplicity of *these* points is therefore 1 as is required. It remains only to consider the possibility that $\nabla_\chi V_{k+\mathsf{h}^\vee}$ might vanish at some $\lambda \in \mathsf{P}$ on the boundary of a shifted alcove.

COROLLARY 5.13. *The integral weights on the shifted alcove boundaries do not correspond to zeroes of* $\nabla_\chi V_{k+\mathsf{h}^\vee}$.

PROOF. If an integral weight on a shifted alcove boundary did correspond to such a zero, Equation (5.3.7) would give $\det H_\lambda = (\det J)^2 \det H_\chi = 0$, since J is singular on the boundaries, and $\det H_\chi$ is a polynomial in the characters (hence finite at any given point). But this contradicts Proposition 5.12. ∎

It follows immediately that the potentials $V_{k+\mathsf{h}^\vee}$ do indeed describe the fusion variety correctly. Furthermore, since the multiplicity at *every* point of the variety is now known to be 1, the ideal generated by the derivatives of the potentials is radical. This demonstrates that the fusion potentials correctly describe the fusion algebra as claimed.

THEOREM 5.14. *The fusion algebra of the Wess-Zumino-Witten model associated with* $\mathfrak{su}(r+1)$ *or* $\mathfrak{sp}(2r)$ *at level* k *is described by the fusion potential* $V_{k+\mathsf{h}^\vee}$ *given by Equation (5.2.1):*

$$\mathcal{F}_k^{\mathbb{C}} \cong \frac{\mathbb{C}[\chi_1,\ldots,\chi_r]}{\left\langle \frac{\partial V_{k+\mathsf{h}^\vee}}{\partial\chi_1},\ldots,\frac{\partial V_{k+\mathsf{h}^\vee}}{\partial\chi_r}\right\rangle}.$$

5.4. Generalisations

The theory developed in Section 5.3 is specific to the algebras $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$, but only in a rather limited way. For example, Propositions 5.11 and 5.12 both merely verify that a certain fact is true for these algebras. The other results, and more importantly the logic behind them, hold for general simple Lie algebras. It seems plausible then that convenient descriptions might similarly be found for the fusion rings of these general algebras. This section is devoted to some attempts to investigate these fusion rings, illustrated by the (easily visualised) example of the simple Lie algebra $\mathfrak{g}_2$, and highlights the various obstacles that such investigations need to overcome.

5.4.1. Obstructions over $\mathbb{Z}$. Recall from Section 5.3.1 that there exists a monomial ordering $\prec$ with respect to which the ideal $\left\langle \mathrm{LT}\left(\mathcal{J}_k^{\mathbb{Z}}\right)\right\rangle$ of $\mathbb{Z}[\chi_1,\ldots,\chi_r]$ is freely generated (as an abelian group) by the monomials $\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}$ with $(\lambda,\theta) > k$. A set of ideal

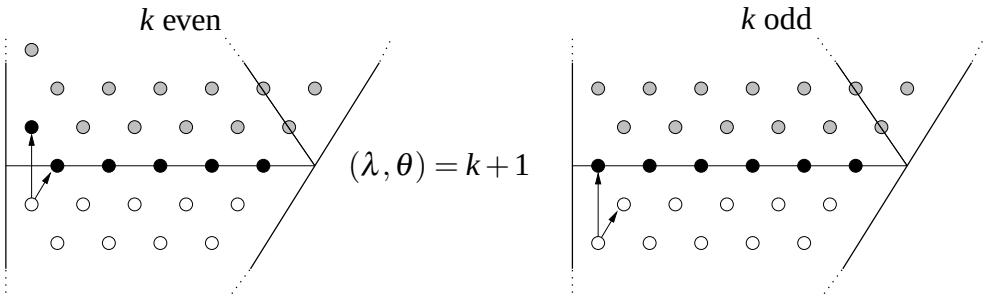

FIGURE 5.1. The atomic monomials for the ideal $\langle \mathrm{LT}(\mathcal{J}_k^{\mathbb{Z}})\rangle$ associated with the Lie algebra $\mathfrak{g}_2$ at even and odd level. Weights corresponding to monomials in the ideal are grey or black, the latter corresponding to atomic monomials. The arrows indicate the effect of multiplying by χ_1 and χ_2.

generators for $\langle \mathrm{LT}(\mathcal{J}_k^{\mathbb{Z}})\rangle$ is therefore obtained by selecting the *atomic* monomials from the set of group generators given above. The atomic monomials of a set of monomials $\mathcal{M}=\left\{\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}\right\}$ are defined to be those which *cannot* be expressed as the product of a fundamental character and a monomial from $\mathcal{M}$. Equivalently, atomic monomials are those corresponding to weights from which one cannot subtract any fundamental weight and still remain in the set of weights corresponding to $\mathcal{M}$.

It should be clear that for the set $\mathcal{M}=\left\{\chi_1^{\lambda_1}\cdots\chi_r^{\lambda_r}:\ (\lambda,\theta)>k\right\}$, every weight λ with $(\lambda,\theta)=k+1$ corresponds to an atomic monomial. For $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$, these are all the atomic monomials, as $a_i^\vee=(\Lambda_i,\theta)=1$ (so if $(\mu,\theta)>k+1$, one can always subtract a fundamental weight from μ yet remain in $\mathcal{M}$). For other algebras, it will generally be necessary to include other monomials. For example, $(\Lambda_1,\theta)=2$ for the algebra $\mathfrak{g}_2$, so it follows that when the level k is even, the monomial $\chi_1^{(k+2)/2}$ is also atomic (see Figure 5.1).

Given the atomic monomials, one now has to find polynomials in the fusion ideal $\mathcal{J}_k^{\mathbb{Z}}$ whose leading terms with respect to $\prec$ are these monomials. This is straightforward for the monomials corresponding to weights with $(\lambda,\theta)=k+1$ — as in Section 5.3.1, one may take the characters χ_λ. For the remaining atomic monomial of $\mathfrak{g}_2$ (when k is even), Proposition 5.5 gives $\chi_{(k+2)\Lambda_1/2}+\chi_{k\Lambda_1/2}\in\mathcal{J}_k^{\mathbb{Z}}$, and this combination has the correct leading term. Therefore, a Gröbner basis for the (integral) fusion ideal of $\mathfrak{g}_2$ is given by

$$\begin{aligned} &\{\chi_\lambda:\ (\lambda,\theta)=k+1\} && \text{if } k \text{ is odd}\\ \text{and}\qquad &\left\{\chi_{(k+2)\Lambda_1/2}+\chi_{k\Lambda_1/2}\right\}\cup\{\chi_\lambda:\ (\lambda,\theta)=k+1\} && \text{if } k \text{ is even.}\end{aligned}$$

This example disproves the belief expressed in [**33**] that the fusion ring (at level k) was always generated by the characters χ_λ with $(\lambda,\theta)=k+1$. Clearly it is straight-forward to compute corresponding Gröbner bases for the other simple Lie algebras[10]. These give a complete description of the fusion ring, at least in principle.

[10]Note that every simple Lie algebra except $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ has at least one comark greater than unity. It follows that at certain levels, the fusion ring of these algebras will also not be generated by the χ_λ with $(\lambda,\theta)=k+1$. The fusion rings of the Lie algebras $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ are therefore distinguished in this regard.

As in Section 5.3.1, these bases suffer from the disadvantage that the number of generators increases polynomially with the level. However, the means to bypass this disadvantage, the Jacobi-Trudy identity (and its generalisation to $\mathfrak{sp}(2r)$), are no longer available. It is quite disappointing to find that no further generalisations of this identity (to the other simple Lie algebras) seem to have been discovered[11]. Without an appropriate generalisation, and indeed a generalisation of the theory of symmetric polynomials, the argument given to derive the fusion potentials breaks down. At this stage then, the obvious alternative is to study the behaviour of suitably general potentials, hoping that the results obtained from this study will suggest how to proceed in describing these general fusion rings.

5.4.2. Obstructions over $\mathbb{C}$. Consider therefore, for a general Lie algebra, the potentials

$$V_m^\Phi = \frac{1}{m}\sum_{\mu\in\Phi} e^{m\mu}, \tag{5.4.1}$$

where m is a positive integer and Φ is a finite W-invariant set of integral weights. The conditions on Φ ensure that these potentials may be written as polynomials in the fundamental characters with rational coefficients. However, if these potentials are to describe a fusion ring, their derivatives with respect to the fundamental characters should be polynomials *with integer coefficients*. In Section 5.3, this property was built in to the construction of the fusion potentials. Here, this integrality is perhaps not entirely evident, and requires checking.

PROPOSITION 5.15. *The derivatives with respect to the fundamental characters of the potentials given by Equation (5.4.1) may be expressed as polynomials in the fundamental characters with integer coefficients.*

PROOF. Consider the formal generating function

$$V^\Phi(t) = \sum_{m=1}^{\infty}(-1)^{m-1}V_m^\Phi t^m = \log\left[\prod_{\mu\in\Phi}(1+e^\mu t)\right] = \log\left[\sum_n \chi_{\wedge^n\Phi}t^n\right].$$

The formal generating functions for the derivatives then take the form

$$\frac{\partial V^\Phi(t)}{\partial\chi_i} = \frac{\sum_n(\partial\chi_{\wedge^n\Phi}/\partial\chi_i)t^n}{\prod_{\mu\in\Phi}(1+e^\mu t)}.$$

Now observe that the inverse of the denominator is a finite product of geometric series (in t), each of whose coefficients are (up to a sign) integer powers of the e^μ. As the denominator is W-invariant, the coefficents of the powers of t in the product of the geometric series must also be W-invariant. It follows that these coefficients may be expressed as integer polynomials in the fundamental characters.

[11]There is a generalisation to $\mathfrak{so}(n)$, also due to Weyl [**160**], but it only applies to tensor representations. The fusion rings considered in this thesis involve spinor representations as well, so a further generalisation is required. Nevertheless, a postulated fusion potential based on these tensor representations may be found in [**118**]. It is not clear if this potential describes any physically relevant fusion process.

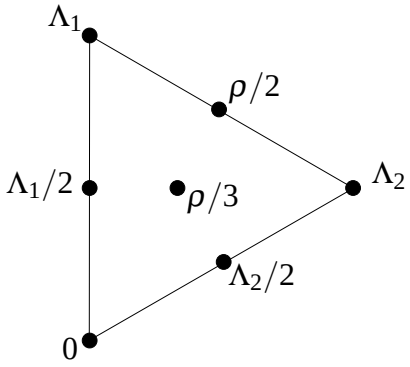

FIGURE 5.2. The critical points $\lambda+\rho$ of the potential $V_m^{\mathsf{W}(\theta)}$ for $\mathfrak{su}(3)$ as a function of the weight space. Only the points in the fundamental alcove are shown — the rest may be obtained through $\widehat{\mathsf{W}}$-invariance.

Finally, note that $\chi_{\wedge^n\Phi}$ is also W-invariant, hence an integer polynomial in the fundamental characters, so its derivatives are integer polynomials too. Multiplying the numerator by the product of geometric series therefore gives a formal power series in t whose coefficients are integer polynomials in the fundamental characters. ■

The aim is now to determine the variety described by these general potentials, and compare with the appropriate fusion variety. As in Section 5.3.2, it is extremely useful to view these potentials as functions on the weight space by evaluating each fundamental character at $-2\pi\mathrm{i}\,(\lambda+\rho)\,/\,(k+\mathsf{h}^\vee)$. Identifying m with $k+\mathsf{h}^\vee$, the potentials given by Equation (5.4.1) become (compare Equation (5.3.4))

$$V^{\Phi}_{k+\mathsf{h}^\vee}\left(-2\pi\mathrm{i}\frac{\lambda+\rho}{k+\mathsf{h}^\vee}\right)=\frac{1}{k+\mathsf{h}^\vee}\sum_{\mu\in\Phi}e^{-2\pi\mathrm{i}(\mu,\lambda+\rho)}=\frac{1}{k+\mathsf{h}^\vee}\chi_\Phi\left(-2\pi\mathrm{i}\,(\lambda+\rho)\right). \quad (5.4.2)$$

Observe that this expression as a function of λ is not only invariant under the shifted action of W (because Φ is W-invariant), but also under translations by the coroot lattice $\mathsf{Q}^\vee$ (because $\Phi\subset\mathsf{P}$). Therefore, as functions on the weight space, these potentials are invariant under the shifted action of the affine Weyl groups $\widehat{\mathsf{W}}_k$, *for all* k. It follows from this simple observation that the set of critical points of the potentials will be invariant under the shifted action of these affine Weyl groups.

The behaviour of these sets of critical points is perhaps best illustrated through examples. The set of critical points for $\Phi=\Delta^{\Lambda_1}$ was shown to coincide with P in Proposition 5.11 for the algebras $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$ (note that P is indeed invariant under the shifted action of the affine Weyl groups). The most easily visualised examples remaining are the potentials associated with $\mathfrak{g}_2$. It turns out to be computationally convenient to consider a related example first.

LEMMA 5.16. *The critical points of* $V_m^{\mathsf{W}(\theta)}$ *for* $\mathfrak{su}(3)$ *as a function of* λ *are given by the union of two (shifted) lattices:*

$$\lambda+\rho\in\mathrm{span}_{\mathbb{Z}}\left\{\frac{1}{2}\Lambda_1,\frac{1}{2}\Lambda_2\right\}\cup\mathrm{span}_{\mathbb{Z}}\left\{\frac{1}{3}\left(2\Lambda_1-\Lambda_2\right),\frac{1}{3}\left(2\Lambda_2-\Lambda_1\right)\right\}$$

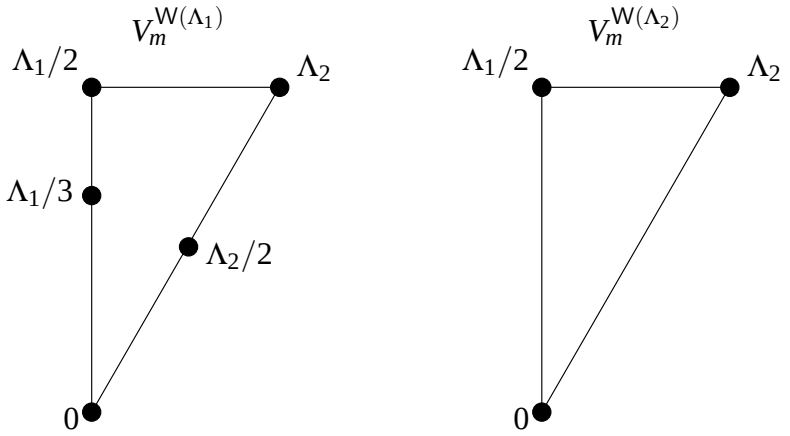

FIGURE 5.3. The critical points $\lambda+\rho$ of the potentials $V_m^{\mathsf{W}(\Lambda_1)}$ and $V_m^{\mathsf{W}(\Lambda_2)}$ for $\mathfrak{g}_2$ as a function of the weight space. Again, only the points in the fundamental alcove are shown.

(see Figure 5.2).

PROOF. Evaluating this potential at $-2\pi i\lambda'/m$ (where $\lambda'=\lambda+\rho$ for convenience) gives

$$mV_m^{\mathsf{W}(\theta)} = 2\cos\left[2\pi\lambda_1'\right]+2\cos\left[2\pi\lambda_2'\right]+2\cos\left[2\pi\left(\lambda_1'+\lambda_2'\right)\right],$$

so the critical points occur when

$$\sin\left[2\pi\lambda_1'\right] = -\sin\left[2\pi\left(\lambda_1'+\lambda_2'\right)\right] = \sin\left[2\pi\lambda_2'\right].$$

Since $\sin(2\pi A)=\sin(2\pi B)$ if and only if $A-B\in\mathbb{Z}$ or $A+B\in\mathbb{Z}+\frac{1}{2}$, the conditions for critical points become

$$\left(2\lambda_1'+\lambda_2'\in\mathbb{Z} \quad \text{or} \quad \lambda_2'\in\mathbb{Z}+\frac{1}{2}\right) \quad \text{and} \quad \left(\lambda_1'+2\lambda_2'\in\mathbb{Z} \quad \text{or} \quad \lambda_1'\in\mathbb{Z}+\frac{1}{2}\right).$$

Carefully solving these conditions gives

$$(\lambda_1',\lambda_2')\in\left[\left(\frac{1}{2}\mathbb{Z}\right)\times\left(\frac{1}{2}\mathbb{Z}\right)\right]\cup\left[\left(\mathbb{Z}+\frac{1}{3}\right)\times\left(\mathbb{Z}+\frac{1}{3}\right)\right]\cup\left[\left(\mathbb{Z}+\frac{2}{3}\right)\times\left(\mathbb{Z}+\frac{2}{3}\right)\right]. \quad \blacksquare$$

The weights of $\mathsf{W}(\theta)$ are just the roots of $\mathfrak{su}(3)$. This set therefore consists of the simple roots α_1 and α_2, their sum θ, and the negatives of all three. The relevance to $\mathfrak{g}_2$ is that the sets $\mathsf{W}(\Lambda_1)$ (the long roots of $\mathfrak{g}_2$) and $\mathsf{W}(\Lambda_2)$ (the short roots) have precisely the same structure. Indeed, the corresponding potentials have the form

$$mV_m^{\mathsf{W}(\Lambda_1)} = 2\cos\left[2\pi\lambda_1'\right]+2\cos\left[2\pi\left(\lambda_1'+\lambda_2'\right)\right]+2\cos\left[2\pi\left(2\lambda_1'+\lambda_2'\right)\right],$$

$$\text{and} \quad mV_m^{\mathsf{W}(\Lambda_2)} = 2\cos\left[2\pi\frac{\lambda_2'}{3}\right]+2\cos\left[2\pi\frac{3\lambda_1'+\lambda_2'}{3}\right]+2\cos\left[2\pi\frac{3\lambda_1'+2\lambda_2'}{3}\right],$$

so setting $\lambda_1''=\lambda_1'$, $\lambda_2''=\lambda_1'+\lambda_2'$, and $\lambda_1''=\lambda_1'+\lambda_2'/3$, $\lambda_2''=\lambda_2'/3$, respectively, brings these into the form of the $\mathfrak{su}(3)$ potential. Lemma 5.16 determines the critical points of this potential, so it is a simple matter to substitute back to get the critical points of the $\mathfrak{g}_2$ potentials.

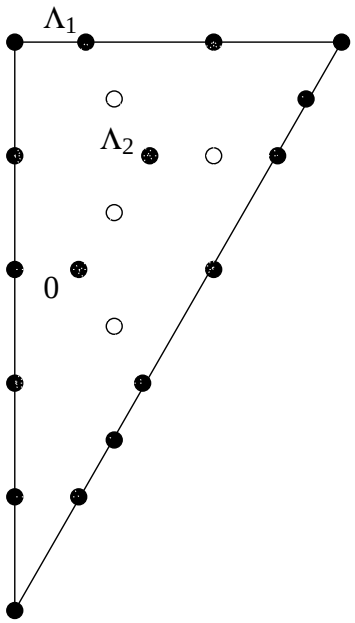

FIGURE 5.4. The critical points λ of the potential $V_m^{\mathsf{W}(\Lambda_2)}$ for $\mathfrak{g}_2$ in the shifted fundamental alcove at level $k = 1$. The white points denote those in the interior which do not belong to the weight lattice.

PROPOSITION 5.17. *The critical points of the potentials* $V_m^{\mathsf{W}(\Lambda_1)}$ *and* $V_m^{\mathsf{W}(\Lambda_2)}$ *for* $\mathfrak{g}_2$ *as a function of* λ *are given by the shifted lattices*

$$\lambda+\rho \in \operatorname{span}_{\mathbb{Z}}\left\{\frac{\Lambda_1}{2},\frac{\Lambda_2}{2}\right\} \cup \operatorname{span}_{\mathbb{Z}}\left\{\frac{\Lambda_1}{3},\Lambda_2\right\},$$

$$\textit{and} \quad \lambda+\rho \in \mathsf{P} \cup \operatorname{span}_{\mathbb{Z}}\left\{\frac{\Lambda_1}{2},\frac{3\Lambda_2}{2}\right\},$$

respectively (see Figure 5.3).

The set of critical points of the $\mathfrak{g}_2$ potentials with respect to λ therefore includes, but is not limited to, the weight lattice P. The additional points demonstrate that these potentials *do not* describe the fusion variety of $\mathfrak{g}_2$. For at any given level $k > 0$, some of these additional critical points will be found in the *interior* of the (shifted) affine alcoves of this level (this is illustrated in Figure 5.4). At such points, the jacobian matrix J is non-singular (Proposition 5.10), so these points are critical points with respect to the fundamental characters as well. Thus the variety generated by these potentials strictly contains the fusion variety. Generalising slightly, this argument shows that if the set of critical points (with respect to λ) of a potential is not the weight lattice, then this potential cannot describe the fusion variety.

Of course, the potentials considered for $\mathfrak{g}_2$ are only the two simplest. However, their critical points (with respect to λ) are, in a sense, characteristic of all possible potentials. Given any finite W-invariant set $\Phi \subset \mathsf{P}$, $mV_m^{\Phi} = \chi_{\Phi}(-2\pi\mathrm{i}(\lambda+\rho))$ (see Equation (5.4.2)) may be expressed as an integer polynomial in the fundamental characters, or equivalently,

in the "characters" of the $\mathsf{W}(\Lambda_j)$. That is,

$$mV_m^{\Phi} = p_{\Phi}\left(mV_m^{\mathsf{W}(\Lambda_1)}, \ldots, mV_m^{\mathsf{W}(\Lambda_r)}\right),$$

for some integer polynomial p_{Φ}. It follows now from the chain rule for differentiation that if $\lambda+\rho$ is a common critical point of all the $V_m^{\mathsf{W}(\Lambda_i)}$, then it is also a critical point of $V_m^{\mathsf{W}(\sigma)}$. This proves the following result:

PROPOSITION 5.18. *Let Φ be any (finite) W-invariant set of integral weights. Then, the critical points of the potential V_m^{Φ} as a function on the weight space include any critical points which are common to all the potentials $V_m^{\mathsf{W}(\Lambda_i)}$ (or $V_m^{\Delta^{\Lambda_i}}$), $i=1,\ldots,r$.*

For $\mathfrak{g}_2$, the common critical points are given by Proposition 5.17 as the (shifted) lattices

$$\lambda+\rho \in \mathsf{P} \cup \mathrm{span}_{\mathbb{Z}}\left\{\frac{\Lambda_1}{2}, \frac{3\Lambda_2}{2}\right\},$$

which coincides geometrically with the vertices of the affine alcoves. Proposition 5.18 states that *any* (allowable) potential V_m^{Φ} has these critical points (usually it will have many others), and therefore, as argued above, does *not* describe the fusion variety.

This result shows that in constrast with the cases of $\mathfrak{su}(r+1)$ and $\mathfrak{sp}(2r)$, there is no fusion potential of the form V_m^{Φ} for $\mathfrak{g}_2$. The situation for the other simple algebras is similarly bleak, because of the following result.

PROPOSITION 5.19. *The potentials $V_m^{\Delta^{\Lambda_i}} = \frac{1}{m}\chi_i(-2\pi\mathrm{i}(\lambda+\rho))$, $i=1,\ldots,r$, always have critical points (with respect to λ) when $\lambda+\rho$ is the vertex of an affine alcove.*

PROOF. Identifying m with $k+\mathsf{h}^{\vee}$, the condition for $V_m^{\Delta^{\Lambda_i}}$ to have a critical point is just that $J_{ij}(-2\pi\mathrm{i}(\lambda+\rho))=0$ for each j, where J is the jacobian matrix of Equation (5.3.5). Proving the proposition therefore amounts to showing that $J(-2\pi\mathrm{i}\nu)=0$ whenever ν is an alcove vertex.

Recall from the proof of Lemma 5.9 that $J(w(\nu))=J(\nu)w$, where the w on the right hand side denotes the matrix representing w on the weight space (with respect to the basis of fundamental weights). Denoting the i^{th} row of J by $\nabla_{\lambda}\chi_i$, it follows that

$$\nabla_{\lambda}\chi_i(-2\pi\mathrm{i}w(\nu)) = \nabla_{\lambda}\chi_i(-2\pi\mathrm{i}\nu)w.$$

$\nabla_{\lambda}\chi_i(-2\pi\mathrm{i}\nu)$ is therefore a row vector which may be associated with an element of the dual of the weight space (the Cartan subalgebra).

Consider the fundamental alcove vertices (the general case will follow from the $\widehat{\mathsf{W}}$-invariance of the characters). If $\nu=0$, then ν is fixed by every $w\in\mathsf{W}$, so $\nabla_{\lambda}\chi_i(-2\pi\mathrm{i}\nu)$ is a row vector fixed by every $w\in\mathsf{W}$. Thus, $\nabla_{\lambda}\chi_i(0)$ is the zero vector (for each i), verifying the proposition for this vertex.

The other fundamental alcove vertices have the form (Appendix B.2) $\nu=\Lambda_j/a_j^{\vee}$, where $a_j^{\vee}$ denotes the comarks of the Lie algebra. Then, ν is invariant under all the simple Weyl reflections except w_j, so $\nabla_{\lambda}\chi_i(-2\pi\mathrm{i}\nu)$ is invariant under all these simple reflections, hence $\nabla_{\lambda}\chi_i(-2\pi\mathrm{i}\nu)$ is orthogonal to every simple root except α_j. But, ν is

fixed by the affine reflection about the hyperplane $(\mu,\theta)=1$. This reflection has the form $\widehat{w}(\mu)=w_\theta(\mu)+\theta$, where $w_\theta\in\mathsf{W}$ is the Weyl reflection associated with the highest root θ. Hence, using the invariance of the characters under translations in $\mathsf{Q}^\vee$,

$$\nabla_\lambda\chi_i(-2\pi\mathrm{i}\nu)=\nabla_\lambda\chi_i(-2\pi\mathrm{i}(w_\theta(\nu)+\theta))=\nabla_\lambda\chi_i(-2\pi\mathrm{i}w_\theta(\nu))=\nabla_\lambda\chi_i(-2\pi\mathrm{i}\nu)\,w_\theta.$$

It follows now that $\nabla_\lambda\chi_i(-2\pi\mathrm{i}\nu)$ is also orthogonal to θ. But, θ and the simple roots, excepting α_j, together constitute a basis of the weight space (as the mark a_j never vanishes). Thus, $\nabla_\lambda\chi_i(-2\pi\mathrm{i}\nu)$ is again the zero vector, verifying the proposition for all the vertices of the fundamental alcove. ∎

It follows from Propositions 5.18 and 5.19 that every potential of the form V_m^Φ will have critical points at every affine alcove vertex. Unless the set of affine alcove vertices coincides with the weight lattice, there is no hope that a potential of this form will correctly describe the fusion variety. But, this coincidence can only occur when the comarks of the algebra are all unity, so it follows that these potentials cannot describe the fusion variety unless the algebra is $\mathfrak{su}(r+1)$ or $\mathfrak{sp}(2r)$.

5.4.3. A Partial Result. The result of the previous section forces the search for a "nice" presentation of the fusion ring to move beyond the consideration of the potentials V_m^Φ. It is not clear what should replace these potentials in this consideration (neither is it clear that a "nice" presentation exists). Indeed, one would hope that a successful generalisation of the Jacobi-Trudy identity to all simple Lie algebras would help to decide exactly this. In any case, this chapter will conclude with an observation regarding a conjectured presentation of the fusion ring of $\mathfrak{g}_2$ when the level is *even*.

The observation is simply this: Stare at Figure 5.3 and note that if the critical points of $V_{k+\mathsf{h}^\vee}^{\mathsf{W}(\Lambda_1)}$ are dilated by a factor of two, their intersection with the critical points of $V_{k+\mathsf{h}^\vee}^{\mathsf{W}(\Lambda_2)}$ will precisely be the weight lattice. Such a dilation might be achieved by noting that from

$$V_{k+\mathsf{h}^\vee}^\Phi=\frac{1}{k+\mathsf{h}^\vee}\sum_{\mu\in\Phi}e^{(k+\mathsf{h}^\vee)\mu}\quad\Rightarrow\quad V_{k+\mathsf{h}^\vee}^\Phi\left(-2\pi\mathrm{i}\frac{\lambda+\rho}{k+\mathsf{h}^\vee}\right)=\frac{1}{k+\mathsf{h}^\vee}\sum_{\mu\in\Phi}e^{-2\pi\mathrm{i}(\mu,\lambda+\rho)},$$

halving each $\mu\in\Phi$ effectively doubles each critical point $\lambda+\rho$. One would like to therefore consider the potentials $V_{k+\mathsf{h}^\vee}^{\mathsf{W}(\Lambda_2)}$ and $V_{k+\mathsf{h}^\vee}^{\mathsf{W}(\Lambda_1/2)}$. Unfortunately, the latter potential is (of course) not expressible as a polynomial in the fundamental characters as it involves non-integral weights.

However, the same effect can be achieved by halving not μ but $k+\mathsf{h}^\vee$. More precisely, by considering $V_{k+\mathsf{h}^\vee}^{\mathsf{W}(\Lambda_2)}$ and $V_{(k+\mathsf{h}^\vee)/2}^{\mathsf{W}(\Lambda_1)}$ (both potentials are still to be evaluated at $-2\pi\mathrm{i}(\lambda+\rho)/(k+\mathsf{h}^\vee)$). It follows immediately that when k is even ($\mathsf{h}^\vee=4$), the latter potential involves integral weights only, so therefore defines a polynomial in the fundamental characters (it is not clear how to proceed when the level is odd). The ideal generated by the derivatives of *both* these potentials therefore vanishes on the fusion variety of $\mathfrak{g}_2$. Moreover, the hessians of both potentials with respect to λ may be computed (by

brute force) and are non-singular at each $\lambda \in \mathsf{P}$. Corollary 5.13 then shows that this ideal vanishes *precisely* on the fusion variety.

Time constraints have not allowed the consideration of whether this ideal is radical (and therefore correctly describes the fusion algebra of $\mathfrak{g}_2$). Because the ideal is not jacobian, the computation of the multiplicities appears to be more complicated than checking the determinant of the hessian. Similarly, an investigation of the behaviour of the corresponding ideal of $\mathbb{Z}[\chi_1,\ldots,\chi_r]$ has not been undertaken (although the generators are integral polynomials by Proposition 5.15). However, numerical simulations of the brane charge groups obtained using this ideal are not in conflict with the results of Section 4.2.4. It is therefore reasonable to conclude with a conjecture:

CONJECTURE 5.1. *The fusion ring of* $\mathfrak{g}_2$ *at* even *level k may be represented by*

$$\mathcal{F}_k^{\mathbb{Z}} \cong \frac{\mathbb{Z}[\chi_1,\ldots,\chi_r]}{\left\langle \frac{\partial V_{(k+4)/2}^{\mathsf{W}(\Lambda_1)}}{\partial \chi_1},\ldots,\frac{\partial V_{(k+4)/2}^{\mathsf{W}(\Lambda_1)}}{\partial \chi_r},\frac{\partial V_{k+4}^{\mathsf{W}(\Lambda_2)}}{\partial \chi_1},\ldots,\frac{\partial V_{k+4}^{\mathsf{W}(\Lambda_2)}}{\partial \chi_r}\right\rangle}.$$

For completeness, the generating functions of these potentials are

$$\begin{aligned} V^{\mathsf{W}(\Lambda_1)}(t) = \log\Big[1 &+ (\chi_1-\chi_2-1)t + \left(\chi_2^3-3\chi_1\chi_2-2\chi_1-\chi_2+1\right)t^2 \\ &+ \left(\chi_1^2-\chi_2^3+4\chi_1\chi_2+\chi_2^2+4\chi_1+2\chi_2-1\right)t^3 \\ &+ \left(\chi_2^3-3\chi_1\chi_2-2\chi_1-\chi_2+1\right)t^4 + (\chi_1-\chi_2-1)t^5+t^6\Big] \end{aligned}$$

and

$$\begin{aligned} V^{\mathsf{W}(\Lambda_2)}(t) = \log\Big[1 &+ (\chi_2-1)t + (\chi_1+1)t^2 \\ &+ \left(\chi_2^2-2\chi_1-1\right)t^3 + (\chi_1+1)t^4 + (\chi_2-1)t^5+t^6\Big]. \end{aligned}$$

CHAPTER 6

Wess-Zumino-Witten Branes II: Geometric Considerations

Classically, strings in a Wess-Zumino-Witten model are described by a map $g\colon \Sigma \to \mathsf{G}$, where Σ is a two-dimensional manifold, the *string worldsheet*, and G is a real (finite-dimensional) Lie group, the *target space* (with Lie algebra $\mathfrak{g}$). It may be assumed that G is connected, as different connected components are physically isolated from one another. In what follows, several additional technical simplifications will be made: Σ will be compact and orientable (hence only *oriented* string theories will be realised), and G will *always* be compact, simply-connected, and semisimple, in fact simple (for simplicity). It turns out [**57**, **58**] that these conditions on G imply that the modular invariant (Section 3.1.3) of the corresponding Wess-Zumino-Witten theory is the diagonal invariant. Thus the Wess-Zumino-Witten models associated with the Lie algebra $\mathfrak{g}$ and the diagonal invariant may be more succinctly described as the Wess-Zumino-Witten models on the simply-connected target space G. Other modular invariants correspond to non-simply-connected groups, and perhaps more general orbifolds [**112**].

6.1. Some Algebraic Preliminaries

6.1.1. $\mathfrak{g}$-valued Forms. Consider $\mathrm{d}g\colon T(\Sigma) \to T(\mathsf{G})$. Clearly $g^{-1}\mathrm{d}g$ takes values in $T_{\mathrm{id}}(\mathsf{G}) = \mathfrak{g}$, so therefore it is a $\mathfrak{g}$-valued 1-form on Σ,

$$g^{-1}\mathrm{d}g \in \Omega^1(\Sigma) \otimes \mathfrak{g}.$$

In fact, it is the pullback by g of ϑ, the canonical, left-invariant 1-form on G. This form is the basic building block of the string actions that will be introduced in this chapter. ϑ may be defined by

$$\vartheta = \vartheta_i \otimes t_i \tag{6.1.1}$$

(summation convention implied), where $\{t_i\}$ is a basis for $\mathfrak{g}$, considered as left-invariant vector fields on G, and $\{\vartheta_i\}$ is the dual basis of 1-forms.

The space $\Omega^1(\Sigma) \otimes \mathfrak{g}$ inherits a natural structure as a commutative algebra via

$$\left[\omega_i \otimes t_i \mathbin{\mathring{\wedge}} \eta_j \otimes t_j\right] = \omega_i \wedge \eta_j \otimes \left[t_i, t_j\right].$$

This extends to $\Omega^\bullet(\Sigma) \otimes \mathfrak{g}$ with graded commutativity,

$$\left[\omega \mathbin{\mathring{\wedge}} \eta\right] = (-1)^{pq+1}\left[\eta \mathbin{\mathring{\wedge}} \omega\right],$$

where ω is a $\mathfrak{g}$-valued p-form and η is a $\mathfrak{g}$-valued q-form. As an important example, the derivative of the canonical 1-form may now be expressed as

$$\mathrm{d}\vartheta = -\vartheta \wedge \vartheta = \frac{-1}{2}\left[\vartheta \owedge \vartheta\right]. \tag{6.1.2}$$

The Jacobi identity also generalises to $\Omega^\bullet(\Sigma)\otimes\mathfrak{g}$ in a graded form:

$$[\omega \owedge [\eta \owedge \zeta]] + (-1)^{p(q+r)}[\eta \owedge [\zeta \owedge \omega]] + (-1)^{(p+q)r}[\zeta \owedge [\omega \owedge \eta]] = 0, \tag{6.1.3}$$

where $\omega \in \Omega^p(\Sigma)\otimes\mathfrak{g}$, $\eta \in \Omega^q(\Sigma)\otimes\mathfrak{g}$, and $\zeta \in \Omega^r(\Sigma)\otimes\mathfrak{g}$.

Both $\Omega^\bullet(\Sigma)$ and $\mathfrak{g}$ have a natural Hilbert space structure, that of $\mathfrak{g}$ coming from the Killing form $\kappa(\cdot,\cdot)$ (or rather its negative, as $\mathfrak{g}$ is the compact real form of a complex Lie algebra — see Appendix A.1), and that of $\Omega^\bullet(\Sigma)$ defined by

$$(\omega,\eta) = \int_\Sigma \omega \wedge *\eta,$$

where $*$ is the Hodge star operator [**46**, **124**, **155**]. The inner product on $\Omega^\bullet(\Sigma)\otimes\mathfrak{g}$ is the tensor product of these, hence takes the form

$$\left(\omega_i \otimes t_i \owedge \eta_j \otimes t_j\right) = \kappa\left(t_i,t_j\right)\int_\Sigma \omega_i \wedge *\eta_j = \int_\Sigma \kappa\left(\omega_i \otimes t_i \owedge *\eta_j \otimes t_j\right),$$

where $\kappa(\cdot \owedge \cdot)$ denotes the obvious extension of the Killing form to $\Omega^\bullet(\Sigma)\otimes\mathfrak{g}$.

Note that this extended Killing form converts a $\mathfrak{g}$-valued p-form and a $\mathfrak{g}$-valued q-form to a genuine $(p+q)$-form. It can quickly be checked that this extended Killing form is graded-symmetric and associative:

$$\kappa(\omega \owedge \eta) = (-1)^{pq}\kappa(\eta \owedge \omega) \qquad \text{and} \qquad \kappa([\omega \owedge \eta] \owedge \zeta) = \kappa(\omega \owedge [\eta \owedge \zeta]), \tag{6.1.4}$$

where $\omega \in \Omega^p(\Sigma)\otimes\mathfrak{g}$, $\eta \in \Omega^q(\Sigma)\otimes\mathfrak{g}$, and $\zeta \in \Omega^r(\Sigma)\otimes\mathfrak{g}$.

6.1.2. Action Variation. It will be useful to collect a few facts here regarding a global definition of the usual notion of the variation of a functional, specifically an action. Consider then an action $\mathrm{S}[g]$ depending on a map g and its derivatives. The variation is then supposed to be given by

$$\delta\mathrm{S}[g] = \mathrm{S}[g+\delta g] - \mathrm{S}[g],$$

expanded to first order in δg. Of course, when the map g takes values in a non-linear space, in particular a simple Lie group, this definition makes little sense when viewed literally, and therefore requires an appropriate interpretation.

Thinking in terms of global data, it seems reasonable to replace this first-order expression by the infinitesimal action of some flow (locally deforming g) derived from a vector field X on the group G. Define then (locally) the variation of g on this domain by

$$\delta_X g = X^\mu \partial_\mu g,$$

the derivative of g in the direction given by X. Multiplying on the left by g^{-1} suggests the global definition[1],

$$g^{-1}\delta_X g = \iota_X\left(g^{-1}\mathrm{d}g\right), \tag{6.1.5}$$

where ι_X is the interior product with respect to X (recall that $g^{-1}\mathrm{d}g$ is a $\mathfrak{g}$-valued 1-form). If G were a linear space, then g could be viewed as a 0-form, and the variation would reduce to $\delta_X g = \iota_X \mathrm{d}g = \mathcal{L}_X g$, where $\mathcal{L}_X$ is the Lie derivative with respect to X.

Consider now the variation of $g^{-1}\mathrm{d}g$. Using the standard first-order expansion,

$$\begin{aligned}
\delta_X\left(g^{-1}\mathrm{d}g\right) &= g^{-1}\mathrm{d}\left(\delta_X g\right) - g^{-1}\delta_X g \cdot g^{-1}\mathrm{d}g \\
&= \mathrm{d}\left(g^{-1}\delta_X g\right) + g^{-1}\mathrm{d}g \cdot g^{-1}\delta_X g - g^{-1}\delta_X g \cdot g^{-1}\mathrm{d}g \\
&= \mathrm{d}\iota_X\left(g^{-1}\mathrm{d}g\right) - \frac{1}{2}\iota_X\left[g^{-1}\mathrm{d}g \owedge g^{-1}\mathrm{d}g\right] \\
&= \mathcal{L}_X\left(g^{-1}\mathrm{d}g\right),
\end{aligned} \tag{6.1.6}$$

by Equation (6.1.2). If δ_X was a derivation on $\Omega^\bullet(\Sigma)$, one might think from these two calculations that δ_X and $\mathcal{L}_X$ were identical, as implied in [**80**]. This conclusion is absurd however, as it would imply that the variation of any action defined over a compact orientable manifold is identically zero:

$$\int_\Sigma \mathcal{L}_X \eta = \int_\Sigma \left[\mathrm{d}\left(\iota_X \eta\right) + \iota_X\left(\mathrm{d}\eta\right)\right] = 0,$$

(by Stokes' theorem and the fact that η must be a form of maximal degree).

Of course, δ_X is not a derivation on $\Omega^\bullet(\Sigma)$, and only appears to act as one on the algebra generated by $g^{-1}\mathrm{d}g$. Indeed, δ_X acts on the map g, not on $\Omega^\bullet(\mathsf{G})$, and this confusion arises solely from the fact that the forms considered in the study of Wess-Zumino-Witten actions are all pullbacks by g of forms on $\Omega^\bullet(\mathsf{G})$. However, it can be quite useful to make the identification $\delta_X = \mathcal{L}_X$, provided that one is aware of its limitations. The advantage of this identification is to provide a global framework for the variation of actions defined over non-trivial manifolds, and as a side benefit, it can simplify the computations significantly.

As an important example highlighting the difference between δ_X and $\mathcal{L}_X$, let $f \in \Omega^0\left(\mathbb{R}^2\right) \otimes \mathfrak{g}$ and compare $\delta_X * f = \delta_X(f) * (1) = *\delta_X f$ with the computation

$$\begin{aligned}
\mathcal{L}_X * f &= \mathcal{L}_X\left(f\mathrm{d}x^1 \wedge \mathrm{d}x^2\right) = \mathrm{d}\left(fX^1\mathrm{d}x^2 - fX^2\mathrm{d}x^1\right) \\
&= \partial_\mu\left(fX^\mu\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 = *\left(\partial_\mu f X^\mu + f\partial_\mu X^\mu\right) \\
&= *\mathcal{L}_X f + *\left(f\partial_\mu X^\mu\right).
\end{aligned}$$

(Here the volume form was taken to be $\mathrm{d}x^1 \wedge \mathrm{d}x^2$.) The variational operator δ_X therefore commutes with the Hodge star whereas the Lie derivative $\mathcal{L}_X$ does not.

A somewhat more common limitation is that alluded to before — the Lie derivative of a form η of maximal degree is exact, as the term $\iota_X \mathrm{d}\eta$ vanishes, whereas the variation of such a (pulled back) form generally does not vanish. This mismatch appears to be an

[1] Of course, one might like to allow more general variations. However, these prove to be sufficient for the purposes of this chapter.

unavoidable consequence of the fact that δ_X and $\mathcal{L}_X$ act on different algebraic structures. However, from a purely algebraic point of view, this mismatch may be circumvented by the simple expedient of ignoring the fact that the exterior derivative of a form of maximal degree must vanish. That is, both d and then $\imath_X$ must be allowed to act before the degree of the form is brought into question.

6.2. Closed String Wess-Zumino-Witten Models

6.2.1. A Non-Linear Sigma Model. Recall the string field $g\colon \Sigma \to \mathsf{G}$, and the canonical left-invariant 1-form ϑ. The most natural action that can be constructed from these ingredients is obtained from the "length" (Section 6.1.1) of the pullback form,

$$\mathrm{S}_0[g] = a\left(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\right) = a\int_\Sigma \kappa\left(g^{-1}\mathrm{d}g \wedge *\left(g^{-1}\mathrm{d}g\right)\right), \tag{6.2.1}$$

where a is some arbitrary constant. As Σ is homeomorphic to the unit disc in $\mathbb{R}^2$ with various parts of the boundary identified [**71**], there is a chart covering Σ up to a set of measure zero. Taking coordinates $\left(x^1, x^2\right)$ in such a chart gives

$$\begin{aligned}\mathrm{S}_0[g] &= a\int_\Sigma \kappa\left(g^{-1}\partial_\mu g, g^{-1}\partial_\nu g\right)\mathrm{d}x^\mu \wedge *\mathrm{d}x^\nu \\ &= \frac{a}{I}\int_\Sigma \mathrm{tr}\left[g^{-1}\partial_1 g\cdot g^{-1}\partial_1 g + g^{-1}\partial_2 g\cdot g^{-1}\partial_2 g\right]\mathrm{d}x^1\wedge\mathrm{d}x^2 \\ &= \frac{a}{I}\int_\Sigma \mathrm{tr}\left[g^{-1}\partial_\mu g\cdot g^{-1}\partial^\mu g\right].\end{aligned}$$

Here the metric was chosen so that $*\left(\mathrm{d}x^1\right) = \mathrm{d}x^2$ and $*\left(\mathrm{d}x^2\right) = -\mathrm{d}x^1$, and I represents the Dynkin index of the representation of $\mathfrak{g}$ used (implicitly) in the Killing form trace (Appendix A.1). $\mathrm{S}_0[g]$ may therefore be recognised as a *non-linear sigma model* action. With $z = x^1 + \mathrm{i}x^2$, it can also be expressed in the form

$$\mathrm{S}_0[g] = 2\mathrm{i}a\int_\Sigma \kappa\left(g^{-1}\partial g, g^{-1}\overline{\partial} g\right)\mathrm{d}z\wedge\mathrm{d}\overline{z}.$$

Because of the existence of such a chart, this sigma model action may be varied in local coordinates. However, to facilitate comparison with what follows, the global formalism of Section 6.1.2 will be used. Varying Equation (6.2.1) in this manner gives

$$\begin{aligned}\delta_X \mathrm{S}_0[g] &= a\int_\Sigma \left\{\kappa\left(\delta_X\left(g^{-1}\mathrm{d}g\right)\wedge *\left(g^{-1}\mathrm{d}g\right)\right) + \kappa\left(\left(g^{-1}\mathrm{d}g\right)\wedge *\delta_X\left(g^{-1}\mathrm{d}g\right)\right)\right\} \\ &= 2a\int_\Sigma \kappa\left(\delta_X\left(g^{-1}\mathrm{d}g\right)\wedge *\left(g^{-1}\mathrm{d}g\right)\right) \\ &= a\int_\Sigma \left\{2\kappa\left(\mathrm{d}\left(g^{-1}\delta_X g\right)\wedge *\left(g^{-1}\mathrm{d}g\right)\right) - \kappa\left(\imath_X\left[g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\right]\wedge *\left(g^{-1}\mathrm{d}g\right)\right)\right\},\end{aligned}$$

using the symmetry of the inner product and Equation (6.1.6). The second term vanishes identically via

$$\imath_X\left[g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\right] = 2\left[g^{-1}\delta_X g\wedge g^{-1}\mathrm{d}g\right],$$

and the computation ($A = g^{-1}\delta_X g$ and $B = g^{-1}\mathrm{d}g$)

$$\begin{aligned}\int_\Sigma \kappa([A \wedge B] \wedge *B) &= \int_\Sigma \kappa(B \wedge *[A \wedge B]) = \int_\Sigma \kappa(B \wedge [A \wedge *B]) \\ &= \int_\Sigma \kappa([B \wedge A] \wedge *B) = -\int_\Sigma \kappa([A \wedge B] \wedge *B)\end{aligned}$$

(which uses graded symmetry and associativity, Equation (6.1.4), and the symmetry of the inner product). The variation now becomes

$$\begin{aligned}\delta_X \mathrm{S}_0[g] &= 2a\int_\Sigma \left\{\mathrm{d}\kappa\left(g^{-1}\delta_X g \wedge *\left(g^{-1}\mathrm{d}g\right)\right) - \kappa\left(g^{-1}\delta_X g \wedge \mathrm{d}*\left(g^{-1}\mathrm{d}g\right)\right)\right\} \qquad (6.2.2)\\ &= -2a\int_\Sigma \kappa\left(g^{-1}\delta_X g \wedge \mathrm{d}*\left(g^{-1}\mathrm{d}g\right)\right),\end{aligned}$$

and so the equations of motion are just

$$\mathrm{d}*\left(g^{-1}\mathrm{d}g\right) = 0.$$

(Note that extracting the equations of motion required that the Killing form be non-degenerate, hence the restriction that G be (semi)simple.)

In local coordinates, $\left(x^1, x^2\right)$, these equations of motion take the form

$$\partial_\mu\left(g^{-1}\partial_\mu g\right) = 0,$$

and in complex coordinates, $z = x^1 + \mathrm{i}x^2$, they become

$$\partial\left(g^{-1}\overline{\partial}g\right) + \overline{\partial}\left(g^{-1}\partial g\right) = 0, \qquad (6.2.3)$$

where ∂ and $\overline{\partial}$ are the derivatives with respect to z and $\bar{z}$ respectively. The equations of motion do not, therefore, define holomorphic and antiholomorphic fields, as is characteristic of a two-dimensional conformal field theory. It follows that the non-linear sigma model action requires modifying.

6.2.2. The Wess-Zumino Term. It is not obvious how to modify the non-linear sigma model action to achieve conformal invariance. The solution lies in constructing a so-called *Wess-Zumino* term [**159**], a topological quantity whose definition requires some care [**125**], and adding it to the original sigma model action. This solution was found by Witten [**161**], and as the name implies, this leads to the conformal field theories known as Wess-Zumino-Witten models. These describe strings propagating in the topologically non-trivial background of the group manifold, and as Σ is assumed to have no boundary, the strings are closed.

The Wess-Zumino term is given by

$$\mathrm{S}_{\mathrm{WZ}}[g] = b\int_\Gamma \kappa\left(\tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \mathrm{d}\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}\right)\right) = b\int_\Gamma \tilde{g}^*\kappa\left(\vartheta \wedge \mathrm{d}\vartheta\right), \qquad (6.2.4)$$

where b is another constant, Γ is a *three*-dimensional manifold whose boundary is Σ, and $\tilde{g}\colon \Gamma \to \mathsf{G}$ is an extension of the field g from Σ to Γ. Whilst there are intricacies in this definition to deal with, it should first be checked that it does achieve its aim of restoring conformal invariance to the theory (at least formally).

First, it is convenient to define a form $H \in \Omega^3(\mathsf{G})$ by

$$H = \frac{b}{2\pi\mathrm{i}}\,\kappa(\vartheta \wedge\!\!\!\wedge \mathrm{d}\vartheta),$$

so that the Wess-Zumino action, Equation (6.2.4), takes the form

$$\mathrm{S}_{\mathrm{WZ}}[g] = 2\pi\mathrm{i}\int_\Gamma \tilde{g}^* H.$$

PROPOSITION 6.1. *$H \in \Omega^3(\mathsf{G})$ is closed, hence defines a class in $\mathsf{H}^3(\mathsf{G};\mathbb{C})$.*

PROOF. It follows immediately from the graded Jacobi identity (Equation (6.1.3)) that

$$[\vartheta \wedge\!\!\!\wedge [\vartheta \wedge\!\!\!\wedge \vartheta]] = 0.$$

Thus, Equation (6.1.2) and associativity give

$$\mathrm{d}H = \frac{b}{2\pi\mathrm{i}}\,\kappa(\mathrm{d}\vartheta \wedge\!\!\!\wedge \mathrm{d}\vartheta) = \frac{b}{8\pi\mathrm{i}}\,\kappa([\vartheta \wedge\!\!\!\wedge \vartheta] \wedge\!\!\!\wedge [\vartheta \wedge\!\!\!\wedge \vartheta]) = \frac{b}{8\pi\mathrm{i}}\,\kappa(\vartheta \wedge\!\!\!\wedge [\vartheta \wedge\!\!\!\wedge [\vartheta \wedge\!\!\!\wedge \vartheta]]) = 0. \quad \blacksquare$$

To compute the variation in the action given by Equation (6.2.4), one should take a global approach. As the integrand is a simple function of $\tilde{g}$ and its derivatives (there are no Hodge stars present), and a variation δ_X of g may be extended to a corresponding variation $\delta_{\tilde{X}}$ of $\tilde{g}$, it follows from Section 6.1.2 that the variation is given by the Lie derivative. From Proposition 6.1, the only contributing term is a boundary term, so

$$\begin{aligned}\delta_X \mathrm{S}_{\mathrm{WZ}}[g] &= b\int_\Gamma \mathrm{d}\iota_{\tilde{X}}\,\kappa\left(\tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge\!\!\!\wedge \mathrm{d}\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}\right)\right)\\ &= \frac{-b}{2}\int_\Sigma \iota_X\,\kappa\left(g^{-1}\mathrm{d}g \wedge\!\!\!\wedge \left[g^{-1}\mathrm{d}g \wedge\!\!\!\wedge g^{-1}\mathrm{d}g\right]\right).\end{aligned}$$

Note that the variation is independent of the extension $\tilde{g}$.

Using graded commutativity and associativity, it is easy to show that

$$\iota_X\,\kappa\left(g^{-1}\mathrm{d}g \wedge\!\!\!\wedge \left[g^{-1}\mathrm{d}g \wedge\!\!\!\wedge g^{-1}\mathrm{d}g\right]\right) = 3\kappa\left(g^{-1}\delta_X g \wedge\!\!\!\wedge \left[g^{-1}\mathrm{d}g \wedge\!\!\!\wedge g^{-1}\mathrm{d}g\right]\right),$$

and hence that the variation of the Wess-Zumino term is just

$$\delta\mathrm{S}_{\mathrm{WZ}}[g] = 3b\int_\Sigma \kappa\left(g^{-1}\delta_X g \wedge\!\!\!\wedge \mathrm{d}\left(g^{-1}\mathrm{d}g\right)\right). \tag{6.2.5}$$

Adding this to the sigma model variation implies that the equations of motion for the total action take the form

$$\mathrm{d}\left[3bg^{-1}\mathrm{d}g - 2a*\left(g^{-1}\mathrm{d}g\right)\right] = 0.$$

Set $b = 2\mathrm{i}a/3$. In complex coordinates this now becomes

$$-4\mathrm{i}a\overline{\partial}\left(g^{-1}\partial g\right)\mathrm{d}z\wedge\mathrm{d}\overline{z} = 0,$$

so the equations of motion define a holomorphic field (scaled appropriately) by[2]

$$J(z) = 8\pi a g^{-1}\partial g. \tag{6.2.6}$$

[2]In fact, the equations of motion define a holomorphic 1-form which is locally represented by $J(z)\,\mathrm{d}z$.

The existence of the corresponding antiholomorphic field is established by noting that

$$\begin{aligned}\overline{\partial}\left(g^{-1}\partial g\right) &= -g^{-1}\overline{\partial}g\cdot g^{-1}\partial g + g^{-1}\overline{\partial}\partial g \\ &= g^{-1}\left(\partial\overline{\partial}g\cdot g^{-1} - \overline{\partial}g\cdot g^{-1}\partial g\cdot g^{-1}\right)g = g^{-1}\partial\left(\overline{\partial}g\cdot g^{-1}\right)g,\end{aligned} \tag{6.2.7}$$

so the equations of motion also give the (scaled) antiholomorphic field

$$\overline{J}(\overline{z}) = -8\pi a\overline{\partial}g\cdot g^{-1}. \tag{6.2.8}$$

It is interesting to note that if b is instead set to $-2\mathrm{i}a/3$, the equations of motion define (scaled) holomorphic and antiholomorphic fields, this time given by $-8\pi a\partial g\cdot g^{-1}$ and $8\pi a g^{-1}\overline{\partial}g$ respectively. Changing the sign of the Wess-Zumino term therefore leads to an entirely equivalent theory where the relationship between the holomorphic and antiholomorphic fields, and the string field g, is also reversed.

The Wess-Zumino-Witten action

$$\mathrm{S}_{\mathrm{WZW}}[g] = a\int_{\Sigma}\kappa\left(g^{-1}\mathrm{d}g\mathbin{\hat{\wedge}}*\left(g^{-1}\mathrm{d}g\right)\right) + \frac{2\mathrm{i}a}{3}\int_{\Gamma}\kappa\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}\mathbin{\hat{\wedge}}\mathrm{d}\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}\right)\right), \tag{6.2.9}$$

therefore does give rise to holomorphic and antiholomorphic fields, and the results of Chapter 3 then show that the theory generated by these fields is a conformal field theory. However, there are topological intricacies present in the definition of the Wess-Zumino term which have so far been ignored. Their consideration is not only vital to the standard Wess-Zumino-Witten theory, but also provides a paradigm that guides the study of brane charges in these models through topological and geometric means.

6.2.3. Ambiguities and Quantisation. Recall that in defining the Wess-Zumino term (Equation (6.2.4)), a three-dimensional manifold Γ with boundary Σ was introduced, and the field g was extended from Σ to a field $\tilde{g}$ on Γ in some way. Whilst the existence of the abstract manifold Γ is clear, it is necessary to check that it may be mapped into the target space G in a manner compatible with g, that is, that the extension $\tilde{g}$ makes sense. If this is the case, then it remains to consider the effect of different extensions, for $\tilde{g}$ will not be unique. Indeed, Γ is not even uniquely specified.

The existence of $\tilde{g}$ is easily dealt with. What is to be shown is that the image $g(\Sigma)$ is always the boundary of some submanifold of G, which may then be taken to be $\tilde{g}(\Gamma)$. Translating into the language of homology, any 2-cycle $g(\Sigma)$ must be the boundary of some 3-chain $\tilde{g}(\Gamma)$. So, g may always be extended to $\tilde{g}$ provided $\mathsf{H}_2(\mathsf{G};\mathbb{Z}) = 0$. Happily, this is always the case for a compact Lie group (Appendix C.3).

Consider then, the effect of the lack of uniqueness of Γ and thus $\tilde{g}$. Recall that the variation, Equation (6.2.5), of the Wess-Zumino term involved a total derivative, so it did not actually depend on Γ or $\tilde{g}$. It follows that the equations of motion, and hence the classical physics defined by them, are completely insensitive to this ambiguity in the definition of the action. In the quantised theory, however, the action enters directly through the (euclidean) Feynman amplitudes $\exp(-\mathrm{S}[g])$. Therefore this ambiguity must be carefully examined if the theory is to be quantised.

Suppose now that two choices Γ and Γ' are made for the three-dimensional manifold with boundary Σ. The difference between the Wess-Zumino actions, that is the ambiguity in $S_{WZ}[g]$, obtained from these choices is just

$$2\pi i \int_{\Gamma-\Gamma'} \tilde{g}^* H,$$

where $\Gamma-\Gamma'$ is the oriented difference. As a 3-chain, $\Gamma-\Gamma'$ is the difference of two 3-chains with the same boundary, and so is a genuine 3-cycle. It follows that the ambiguities in the action are therefore just the periods of $2\pi i H$ over the 3-cycles in G. The question now arises as to the nature of the cohomology class represented by H. If it is null, meaning H is exact, then there is no ambiguity and the action is well-defined. However, as G is assumed simple, $H^3(G;\mathbb{R}) = \mathbb{R}$ (Appendix C.3), so H is not necessarily exact.

In any case, suppose the fundamental period of H is p. Then the ambiguities in $S_{WZ}[g]$, and hence the full action $S_{WZW}[g]$, are integral multiples of $2\pi i p$. For the quantum theory to be well-defined, it is only necessary for the Feynman amplitudes to be well-defined, and the ambiguities in these are just the multiplicative phases $\exp(-2\pi i m p)$, where $m \in \mathbb{Z}$ is arbitrary (being the image of $\Gamma-\Gamma'$ in $H_3(G;\mathbb{Z}) = \mathbb{Z}$). It now follows that for the Wess-Zumino-Witten model to define a consistent quantum theory, the closed form H appearing in the Wess-Zumino term must have an integral fundamental period. That is,

$$H \in H^3(G;\mathbb{Z}). \tag{6.2.10}$$

This quantisation condition may clearly be satisfied by suitably fixing a.

6.2.4. $SU(2)$: An Example. Consider the most tractable of the simple Lie groups, $SU(2)$. It is easiest to work in the fundamental (defining) representation for which the Dynkin index is $I_{\Lambda_1} = 1$. The traditional parametrisation in this representation uses Euler angles, and can be used to quickly determine the values of a satisfying Equation (6.2.10). However, to simplify a later calculation (in Section 6.3.5), it is convenient to introduce a different parametrisation. This parametrisation is based on the stereographic projection of the 3-sphere S^3 onto $\mathbb{R}^3$, and the fact that $SU(2)$ is diffeomorphic to S^3. It may be expressed as $\beta : \mathbb{R}^3 \to SU(2)$, where

$$\beta(r,\theta,\phi) = \frac{1}{4+r^2}\begin{pmatrix} 4-r^2+4ir\cos\theta & 4r\sin\theta e^{i\phi} \\ -4r\sin\theta e^{-i\phi} & 4-r^2-4ir\cos\theta \end{pmatrix}, \tag{6.2.11}$$

where $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant \pi$, and $0 \leqslant \phi < 2\pi$ denote the usual polar coordinates in $\mathbb{R}^3$. This parametrisation is injective when $r > 0$ and $0 < \theta < \pi$, forming a coordinate chart covering $SU(2)$ up to a set of measure zero. Obviously the entire group manifold itself generates $H_3(SU(2);\mathbb{Z})$.

One can quickly compute that in this parametrisation,

$$\begin{aligned}\beta^{-1}\mathrm{d}\beta &= \frac{4}{4+r^2}\begin{pmatrix}\mathrm{i}\cos\theta & \sin\theta\, e^{\mathrm{i}\phi}\\ -\sin\theta\, e^{-\mathrm{i}\phi} & -\mathrm{i}\cos\theta\end{pmatrix}\mathrm{d}r\\ &\quad+\frac{4r}{(4+r^2)^2}\begin{pmatrix}-\mathrm{i}\left(4-r^2\right)\sin\theta & \left(\left(4-r^2\right)\cos\theta-4\mathrm{i}r\right)e^{\mathrm{i}\phi}\\ -\left(\left(4-r^2\right)\cos\theta+4\mathrm{i}r\right)e^{-\mathrm{i}\phi} & \mathrm{i}\left(4-r^2\right)\sin\theta\end{pmatrix}\mathrm{d}\theta\\ &\qquad+\frac{4\mathrm{i}r\sin\theta}{(4+r^2)^2}\begin{pmatrix}-4r\sin\theta & \left(4-r^2-4\mathrm{i}r\cos\theta\right)e^{\mathrm{i}\phi}\\ \left(4-r^2+4\mathrm{i}r\cos\theta\right)e^{\mathrm{i}\phi} & 4r\sin\theta\end{pmatrix}\mathrm{d}\phi,\end{aligned}$$

so

$$\begin{aligned}\beta^*H &= \frac{-a}{6\pi}\kappa\left(\beta^{-1}\mathrm{d}\beta\wedge\left[\beta^{-1}\mathrm{d}\beta\wedge\beta^{-1}\mathrm{d}\beta\right]\right)=\frac{-a}{3\pi}\operatorname{tr}\left(\beta^{-1}\mathrm{d}\beta\right)^{\wedge 3}\\ &= \frac{256a}{\pi}\frac{r^2\sin\theta}{\left(4+r^2\right)^3}\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\\ \Rightarrow\quad \int_{\mathsf{SU}(2)}H &= \frac{256a}{\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{r^2\sin\theta}{\left(4+r^2\right)^3}\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi=8\pi a.\end{aligned}$$

It follows that for $\mathsf{G}=\mathsf{SU}(2)$, the quantum Wess-Zumino-Witten model can only be well-defined when $a=k/8\pi$, for $k\in\mathbb{Z}$.

6.2.5. The General Case. Consider now the inclusion $j\colon \mathsf{SU}(2)\hookrightarrow\mathsf{G}$ of the $\mathsf{SU}(2)$-subgroup corresponding to the root α. This is a group homomorphism. The pullback j^*H is a closed 3-form on $\mathsf{SU}(2)$, so it is of interest to compare it to the form $H_{\mathsf{SU}(2)}$ appearing in the $\mathsf{SU}(2)$ Wess-Zumino-Witten action.

PROPOSITION 6.2. *Let j be the group homomorphism including the $\mathsf{SU}(2)$-subgroup corresponding to the root α into G. Then,*

$$j^*H=\frac{2}{\|\alpha\|^2}H_{\mathsf{SU}(2)}.$$

PROOF. Let $\left\{x^1,x^2,x^3\right\}$ be a basis of $\mathfrak{su}(2)$. As j is injective, j_* is a linear injection, so the set $\left\{j_*x^1,j_*x^2,j_*x^3\right\}$ may be extended to a basis $\{y^a\}$ of $\mathfrak{g}$. Let $\{\vartheta_a\}$ be the dual basis (of left-invariant 1-forms), and $\eta_a=j^*\vartheta_a$. Then,

$$\eta_a\left(x^b\right)=\left(j^*\vartheta_a\right)\left(x^b\right)=\vartheta_a\left(j_*x^b\right)=\vartheta_a\left(y^b\right)=\delta_{ab},$$

where $a=1,\ldots,\dim\mathfrak{g}$ and $b=1,2,3$. So, $\{\eta_1,\eta_2,\eta_3\}$ is the dual basis to $\left\{x^1,x^2,x^3\right\}$, and $\eta_a=0$ when $a>3$.

It follows now that

$$\begin{aligned}j^*H &= \frac{-a}{3\pi}j^*\left[\sum_{a,b=1}^{\dim\mathfrak{g}}\vartheta_a\wedge\mathrm{d}\vartheta_b\cdot\kappa\left(y^a,y^b\right)\right]=\frac{-a}{3\pi}\sum_{a,b=1}^{\dim\mathfrak{g}}j^*\vartheta_a\wedge\mathrm{d}j^*\vartheta_b\cdot\kappa\left(y^a,y^b\right)\\ &= \frac{-a}{3\pi}\sum_{a,b=1}^{3}\eta_a\wedge\mathrm{d}\eta_b\cdot\kappa\left(j_*x^a,j_*x^b\right)=\frac{-a}{3\pi}j^*\kappa\left(\vartheta_{\mathsf{SU}(2)}\wedge\mathrm{d}\vartheta_{\mathsf{SU}(2)}\right).\end{aligned}$$

Now $j^*\kappa(\cdot,\cdot)$ is easily checked to be bilinear, symmetric, and associative on $\mathsf{SU}(2)$ (associativity follows from j being a group homomorphism). Hence it must be a multiple of the Killing form on $\mathsf{SU}(2)$ (Appendix A.1). Thus, j^*H is this multiple of $H_{\mathsf{SU}(2)}$.

To determine this multiple, observe that j_* sends the coroot $\alpha^\vee_{\mathsf{SU}(2)} \in \mathfrak{su}(2)^{\mathbb{C}}$ to $\pm\alpha^\vee \in \mathfrak{g}^{\mathbb{C}}$, and therefore (in the complexified algebra),

$$j^*\kappa\left(\alpha^\vee_{\mathsf{SU}(2)}, \alpha^\vee_{\mathsf{SU}(2)}\right) = \kappa\left(\alpha^\vee, \alpha^\vee\right) = \frac{4}{\|\alpha\|^2}.$$

But the Killing form on $\mathsf{SU}(2)$ assigns this coroot a length of 2. ∎

The relevance of Proposition 6.2 is that an $\mathsf{SU}(2)$-subgroup of G corresponding to the root α, denoted by $\mathsf{SU}(2)_\alpha$, is homologically non-trivial:

$$\int_{\mathsf{SU}(2)_\alpha} H = \int_{\mathsf{SU}(2)} j^*H = \int_{\mathsf{SU}(2)} \frac{2}{\|\alpha\|^2} H_{\mathsf{SU}(2)} = \frac{16\pi a}{\|\alpha\|^2},$$

by the result of Section 6.2.4. In fact, it is a result of Bott and Samelson [**28**] that any subgroup $\mathsf{SU}(2)_\alpha$, with α long, *generates* $\mathsf{H}_3(\mathsf{G};\mathbb{Z})$. It now follows that in a general compact, connected, simply-connected, simple Lie group, the 3-form H has fundamental period $8\pi a$. To satisfy the quantisation condition, Equation (6.2.10), a must be fixed to $k/8\pi$ where $k \in \mathbb{Z}_+$ (k is, of course, the level). Note that k could be taken to be a negative integer, but negating k merely amounts to a change of orientation on G, and so is physically irrelevant (however, it does affect the form of the fields $J(z)$ and $\overline{J}(\overline{z})$, as remarked at the end of Section 6.2.2). It is clear that $k = 0$ gives a physically vacuous theory.

This discussion therefore, fixes the normalisation of H to

$$H = \frac{k}{24\pi^2}\kappa(\vartheta \wedge \mathrm{d}\vartheta). \tag{6.2.12}$$

The Wess-Zumino-Witten action for *closed* strings now takes its final form:

$$\begin{aligned} \mathsf{S}_{\mathrm{WZW}}[g] &= \frac{k}{8\pi}\int_\Sigma \kappa\left(g^{-1}\mathrm{d}g \wedge *\left(g^{-1}\mathrm{d}g\right)\right) - \frac{k}{12\pi\mathrm{i}}\int_\Gamma \kappa\left(\tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \mathrm{d}\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}\right)\right) \\ &= \frac{k}{8\pi}\int_\Sigma \kappa(g^*\vartheta \wedge *g^*\vartheta) + 2\pi\mathrm{i}\int_\Gamma \tilde{g}^*H. \end{aligned} \tag{6.2.13}$$

6.3. Open String Wess-Zumino-Witten Models

6.3.1. The Open String Action. The closed string action may be generalised so that it can describe both closed and open strings propagating on the group manifold. To accommodate the open strings, the string worldsheet Σ is allowed to have a boundary. Hence, Σ will now denote a compact, orientable, two-dimensional manifold with boundary $\partial\Sigma$, a compact one-dimensional manifold (that is, a finite collection of circles). For clarity of notation, it is convenient to assume that this boundary has only one connected component (circle) — the general case follows easily from this one (and the no-boundary case discussed in Section 6.2).

It appears that the first detailed study of Wess-Zumino-Witten models with open string worldsheets was conducted in [**107**]. Here, the effects associated to the non-trivial boundary were studied using the theory of T-duality [**8**, **9**, **40**, **41**]. The geometric study of boundary conditions for open strings leads to similar results [**7**, **104**, **147**], and this is the approach which will be detailed below.

The generalisation of the Wess-Zumino-Witten action (Equation (6.2.13)) strikes an immediate problem. As Σ has a boundary, there is no 3-chain Γ with $\partial\Gamma = \Sigma$. The Wess-Zumino term is therefore not defined. To overcome this, it proves necessary to (temporarily) remove the effect of the boundary by writing Σ as $\Sigma' - D$ where Σ' is a compact, orientable, two-dimensional manifold (without boundary) and D is a compact, orientable, two-dimensional manifold with (oriented) boundary $-\partial\Sigma$. As G is simply-connected, $\mathsf{H}_1(\mathsf{G};\mathbb{Z}) = 0$, so the string field can be extended from $g\colon \Sigma \to \mathsf{G}$ to $g'\colon \Sigma' \to \mathsf{G}$. The idea now is that this is the same situation encountered in Section 6.2.2, so the Wess-Zumino action may be constructed as before. However, the contribution from the manifold D needs to be cancelled somehow. That said, let $\tilde{g}'\colon \Gamma' \to \mathsf{G}$ be an extension of g' where $\partial\Gamma' = \Sigma'$. The Wess-Zumino term is therefore

$$2\pi\mathrm{i}\int_{\Gamma'}(\tilde{g}')^{*}H.$$

To motivate the manner in which the contribution from D is cancelled, note that D is two-dimensional, so H is exact when restricted to any tubular neighbourhood[3] T_D of $g'(D)$. That is, $H = \mathrm{d}\omega$ on T_D for some $\omega \in \Omega^2(T_D)$. Suppose, for the moment, that it is possible to find a 3-chain $\Lambda \subseteq T_D$ whose boundary is $g'(D)$. Then H would be exact on Λ, and the contribution of D to the Wess-Zumino term could be cancelled by subtracting

$$2\pi\mathrm{i}\int_{\Lambda}H = 2\pi\mathrm{i}\int_{g'(D)}\omega.$$

Of course, there is no such Λ, as $\partial g'(D) \neq \emptyset$, so this procedure is absurd. However, the right-hand-side of this expression is perfectly well-defined, so it seems plausible that subtracting this quantity will lead to a reasonable theory. The action incorporating this "quasi-cancellation" is then

$$\mathrm{S}_{\mathrm{WZW}}[g] = \frac{k}{8\pi}\int_{\Sigma}\kappa(g^{*}\vartheta \wedge *g^{*}\vartheta) + 2\pi\mathrm{i}\left[\int_{\Gamma'}(\tilde{g}')^{*}H - \int_{D}(g')^{*}\omega\right], \tag{6.3.1}$$

and will serve as the definition for the open string Wess-Zumino-Witten action. In this context, the cancellation term involving ω will be referred to as the *boundary term*, $\mathrm{S}_{\partial\Sigma}[g]$.

[3] A tubular neighbourhood of a submanifold N of a manifold M is an open neighbourhood of N in M which is diffeomorphic to the normal bundle of N in M. Tubular neighbourhoods always exist [**92**]. As the cohomology of any *vector* bundle is the same as that of its base, exactness follows trivially.

Consider now the variation of this action. The sigma model term gives the same variation as in Section 6.2.1, but with an extra term over $\partial\Sigma$ (refer to Equation (6.2.2)),

$$\delta_X \mathrm{S}_0[g] = \frac{k}{4\pi}\left[\int_\Sigma \kappa\left(g^{-1}\delta_X g \stackrel{\wedge}{,} \mathrm{d} * \left(g^{-1}\mathrm{d}g\right)\right) - \int_{\partial\Sigma} \kappa\left(g^{-1}\delta_X g \stackrel{\wedge}{,} * \left(g^{-1}\mathrm{d}g\right)\right)\right],$$

the Wess-Zumino term has variation

$$\delta_X \mathrm{S}_{\mathrm{WZ}}[g] = 2\pi\mathrm{i}\int_{\Sigma'} \iota_X\left(\tilde{g}'\right)^* H = 2\pi\mathrm{i}\int_\Sigma \iota_X\left(\tilde{g}'\right)^* H + 2\pi\mathrm{i}\int_D \iota_X\left(\tilde{g}'\right)^* H,$$

whereas the boundary term gives a contribution of (recall $\partial D = -\partial\Sigma$)

$$\delta_X \mathrm{S}_{\partial\Sigma}[g] = -2\pi\mathrm{i}\int_D \left(\iota_X \mathrm{d}\left(g'\right)^*\omega + \mathrm{d}\iota_X\left(g'\right)^*\omega\right) = -2\pi\mathrm{i}\int_D \iota_X\left(\tilde{g}'\right)^* H + 2\pi\mathrm{i}\int_{\partial\Sigma} \iota_X g^*\omega.$$

Note that the boundary term does introduce the required cancellation, at least in the action variation.

The total variation is thus exactly that of the closed string model, but with a boundary contribution

$$\int_{\partial\Sigma}\left[2\pi\mathrm{i}\,\iota_X g^*\omega + \frac{k}{4\pi}\kappa\left(g^{-1}\delta_X g \stackrel{\wedge}{,} * \left(g^{-1}\mathrm{d}g\right)\right)\right]. \tag{6.3.2}$$

It follows that the equations of motion for the string *away from the boundary* (in the bulk) are identical to those of the closed string of Section 6.2. In particular, the theory is conformal, and the holomorphic and antiholomorphic fields $J(z)$ and $\overline{J}(\overline{z})$ are defined (where the coordinate chart is such that z belongs to the upper half plane, and the real axis corresponds to $\partial\Sigma$).

Note that the boundary contribution to the variation can only determine the dynamics of the open string on the boundary $\partial\Sigma$. Hence, ω determines the equation of motion for the string endpoints, that is, the boundary conditions. However, these boundary conditions clearly depend upon the specific choice of ω, whereas this form has thus far been only constrained by $\mathrm{d}\omega = H$ on T_D. It follows, perhaps unsurprisingly, that the boundary conditions must be chosen first, and these will determine the form ω appearing in the open string action Equation (6.3.1).

6.3.2. Boundary Conditions. Recall from Section 3.2.1 that the open string boundary conditions consistent with conformal invariance take the form

$$J(z) = \Omega\left(\overline{J}(\overline{z})\right) \qquad \text{at } z = \overline{z},$$

where $\Omega\colon \mathfrak{g} \to \mathfrak{g}$ is orthogonal (so angle-preserving) with respect to the Killing form, and z in the upper half plane parametrises a complex chart on Σ with the real axis corresponding to (part of) $\partial\Sigma$. This is the form of the conformal boundary condition at $\mathfrak{g} = T_{\mathrm{id}}(\mathsf{G})$. In order to extract the geometric meaning (on the group manifold), it is necessary to left-translate this condition to $T_g(\mathsf{G})$ [**147**], giving

$$\partial g = -g\Omega\left(\overline{\partial} g \cdot g^{-1}\right) \equiv \Omega_g \overline{\partial} g \qquad \text{at } z = \overline{z}. \tag{6.3.3}$$

Note that $\Omega_g\colon T_g(\mathsf{G}) \to T_g(\mathsf{G})$ is a linear operator, orthogonal with respect to the Killing form (also translated to $T_g(\mathsf{G})$).

Suppose that $\overline{\partial} g$ was an eigenvector for Ω_g with eigenvalue -1. Then, Equation (6.3.3) becomes $\partial g = -\overline{\partial} g$, giving $\partial_1 g = 0$ (where $z = x^1 + \mathrm{i}x^2$) at the boundary. This corresponds to a *Dirichlet* boundary condition fixing the endpoint of the string. Similarly, if the eigenvalue is $+1$, $\partial_2 g = 0$ at the boundary (as the string endpoint is *a priori* constrained to stay on the boundary, this equation is redundant). In this case, the endpoint of the string satisfies a *Neumann* boundary condition leaving it free to wander along the boundary. Finally, a pair of complex conjugate eigenvalues corresponds to a two-dimensional subspace of $T_g(\mathsf{G})$ in which ∂g and $\overline{\partial} g$ need only preserve their angle. Hence this also corresponds to a Neumann boundary condition.

The general picture therefore is that the tangent space at g decomposes orthogonally into a Dirichlet subspace in which the string endpoint is fixed (eigenvalue -1) and a Neumann subspace in which the string endpoint is free. Consider now the effect of varying $g \in \mathsf{G}$. Whilst the eigenvalues corresponding to the Neumann subspace stay away from -1, this orthogonal decomposition of the tangent spaces varies smoothly [**105**]. It follows that this decomposition defines two transversal families of submanifolds of G, one corresponding to trajectories (through some point g) along which the string endpoint may continuously move, and one corresponding to trajectories along which the endpoint may not continuously move. The submanifolds along which endpoints may move are (perhaps incorrectly) called Dirichlet branes, abbreviated *D-branes*.

Of course, it would be naïve to think that the Neumann eigenvalues will always stay away from -1 as the string endpoint moves along an allowed trajectory. Generally, there will be points g where two Neumann eigenvalues coalesce at -1, or two -1 eigenvalues split to become genuine complex conjugates. There the dimensions of the Dirichlet and Neumann subspaces will change by 2 and the smoothness of the orthogonal decomposition will be lost. The corresponding objects in the group manifold will therefore not be smooth submanifolds; however they are still referred to as D-branes.

As an important example, consider $\Omega = \mathrm{id}$, the identity transformation. Then, Ω_g acts on $T_g(\mathsf{G})$ as $\Omega_g(xg) = -gx$. The Dirichlet subspace is therefore the space of vectors xg (with $x \in \mathfrak{g}$) satisfying $xg = gx$. It follows that for every $t \in \mathbb{R}$,

$$\exp(2\pi\mathrm{i}tx)\, g = g \exp(2\pi\mathrm{i}tx)\,,$$

and therefore that $\exp(2\pi\mathrm{i}tx)$ belongs to the centraliser of g, $\mathsf{Z}(g)$. Hence x belongs to the Lie algebra of this centraliser, $\mathfrak{z}(g) = T_{\mathrm{id}}(\mathsf{Z}(g))$, and the Dirichlet subspace is the right-translate by g of this space. But as g is obviously in the centre of its centraliser, the Dirichlet subspace is just $T_g(\mathsf{Z}(g))$. It follows that the Neumann subspace is its orthogonal complement, the tangent space to the conjugacy class through g, $T_g(\mathcal{C}(g))$ (Appendix C.3). The D-brane through $g \in \mathsf{G}$ corresponding to the choice $\Omega = \mathrm{id}$ is thus the conjugacy class $\mathcal{C}(g)$. Such branes will be referred to as *untwisted symmetry-preserving*

branes (compare with Section 3.2.4), and will form the main focus of the rest of this thesis.

This may be generalised to the case Ω an inner automorphism $\mathrm{Ad}(h)$ say, $h \in \mathsf{G}$. The same argument shows that the Dirichlet subspace is the right translate by g of $\mathfrak{z}(gh)$, which is $T_{gh}(\mathsf{Z}(gh))h^{-1}$. The Neumann subspace is therefore the right-translate by h^{-1} of the tangent space at gh to the conjugacy class through gh. The corresponding D-brane through $g \in \mathsf{G}$ is thus $\mathcal{C}(gh)h^{-1}$, a translated conjugacy class.

More general cases can be analysed, in particular, the case when Ω is an outer automorphism derived from a symmetry of the Dynkin diagram of $\mathfrak{g}$. This gives rise to D-branes described by the *twined* conjugacy classes of [**56**]. In the language of Section 3.2, these are *twisted* symmetry-preserving branes (the "twisting" is the action of Ω). As with regular conjugacy classes, these D-branes coincide with smooth submanifolds of G, called the *worldvolumes* of the branes. However, when Ω is not an automorphism, more general structures are encountered (see for instance [**147**] for a mild example). These more general structures will be referred to as non-symmetry-preserving.

6.3.3. Fixing ω. Consider now the boundary condition given by Equation (6.3.3), recast in (x^1, x^2)-coordinates. It becomes

$$(\mathrm{id} - \Omega_g)\,\partial_1 g = \mathrm{i}\,(\mathrm{id} + \Omega_g)\,\partial_2 g \text{ at } x^2 = 0.$$

Since the x^1-axis is identified with (part of) the string worldsheet boundary, its image under g must lie on a D-brane. Projecting the boundary condition onto the Dirichlet and Neumann subspaces of the brane gives

$$\text{Dirichlet:} \qquad 2\,(\partial_1 g)_{\mathrm{D}} = 0 \tag{6.3.4}$$

$$\text{Neumann:} \qquad (\mathrm{id} - \Omega_g)\,(\partial_1 g)_{\mathrm{N}} = \mathrm{i}\,(\mathrm{id} + \Omega_g)\,(\partial_2 g)_{\mathrm{N}}$$

$$\Rightarrow \qquad (\partial_2 g)_{\mathrm{N}} = -\mathrm{i}\frac{\mathrm{id} - \Omega_g}{\mathrm{id} + \Omega_g}\,(\partial_1 g)_{\mathrm{N}}, \tag{6.3.5}$$

as $\mathrm{id} + \Omega_g$ is invertible on the Neumann subspace. Obviously the first condition just fixes the string endpoint in the Dirichlet directions, whereas the second details the boundary condition along the brane worldvolume. In effect, this latter condition is the equation of motion for the string endpoint along the brane worldvolume.

Recall the contribution, Equation (6.3.2), of the boundary to the variation of the open string Wess-Zumino-Witten model action. The aim is now to determine ω such that this action variation reproduces the equations of motion of the string endpoint, Equation (6.3.5). The form of this variation suggests the following ansatz:

$$\iota_X g^* \omega = \frac{k}{8\pi^2}\kappa\left(g^{-1}\delta_X g \wedge \eta\right), \qquad \text{for some } \mathfrak{g}\text{-valued 1-form } \eta. \tag{6.3.6}$$

As the boundary contribution to the action variation is an integral over $\partial\Sigma$, which is represented by the x^1-axis in local coordinates, it follows that only the coefficient of $\mathrm{d}x^1$ in the integrand contributes. Furthermore, at the boundary, g must take values on the

brane, hence $g^{-1}\delta_X g = \iota_X\left(g^{-1}\mathrm{d}g\right)$ must take values in the Neumann subspace at $T_g(\mathsf{G})$, left-translated by g^{-1} back to $\mathfrak{g}$. It follows that the equations of motion only determine the coefficient η_1 of $\mathrm{d}x^1$, and even then, only the value that this coefficient takes in the (translated) Neumann subspace:

$$\mathrm{i}\,(\eta_1)_{\mathrm{N}} - g^{-1}\,(\partial_2 g)_{\mathrm{N}} = 0.$$

This obviously suggests that η (and hence ω) should only be defined on the brane, where this condition fixes it uniquely. It is clear that this is consistent with the open string Wess-Zumino-Witten action, Equation (6.3.1), *provided* that the 2-chain $g'(D)$, over which ω is integrated, is contained within the brane.

The boundary condition, Equation (6.3.5), in the Neumann direction, now gives (dropping the Neumann subscripts)

$$\eta_1 = -g^{-1}\frac{\mathrm{id}-\Omega_g}{\mathrm{id}+\Omega_g}\partial_1 g \qquad \Rightarrow \qquad \eta = -g^{-1}\frac{\mathrm{id}-\Omega_g}{\mathrm{id}+\Omega_g}\mathrm{d}g,$$

remembering that η is only being defined on the brane. Recalling the definition of Ω_g, Equation (6.3.3), it follows that

$$\eta = -\frac{\mathrm{id}+\Omega\circ\mathrm{Ad}\,(g)}{\mathrm{id}-\Omega\circ\mathrm{Ad}\,(g)}g^{-1}\mathrm{d}g. \tag{6.3.7}$$

PROPOSITION 6.3. *Each D-brane associated with a conformal boundary condition*

$$J(z) = \Omega\left(\overline{J}(\overline{z})\right) \qquad \text{at } z = \overline{z},$$

supports a 2-form ω such that the variation of the open string Wess-Zumino-Witten action, Equation (6.3.1), reproduces this conformal boundary condition. ω is given by

$$g^*\omega = \frac{-k}{16\pi^2}\kappa\left(g^{-1}\mathrm{d}g \owedge \frac{\mathrm{id}+\Omega\circ\mathrm{Ad}\,(g)}{\mathrm{id}-\Omega\circ\mathrm{Ad}\,(g)}g^{-1}\mathrm{d}g\right), \tag{6.3.8}$$

and, moreover, if $\Omega \in \mathsf{Aut}\,\mathfrak{g}$, then $\mathrm{d}\omega = H$ (where H is restricted to the brane).

PROOF. Consider first the operator $A = (\mathrm{id}-B)^{-1}(\mathrm{id}+B) : \mathfrak{g} \to \mathfrak{g}$, where $B = \Omega\circ\mathrm{Ad}\,(g)$. With respect to the Killing form, Ω is orthogonal as the boundary condition is conformal, as is $\mathrm{Ad}\,(g)$. Their product, B, is thus orthogonal (hence normal) and so has eigenvalues on the unit circle (with 1 removed). By the functional calculus [**137**], A is therefore a normal operator with purely imaginary eigenvalues, hence is skew-symmetric.

It follows now that

$$\begin{aligned}
\iota_X g^*\omega &= \frac{-k}{16\pi^2}\left[\kappa\left(g^{-1}\delta_X g \owedge A\left(g^{-1}\mathrm{d}g\right)\right) - \kappa\left(g^{-1}\mathrm{d}g \owedge A\left(g^{-1}\delta_X g\right)\right)\right] \\
&= \frac{-k}{16\pi^2}\left[\kappa\left(g^{-1}\delta_X g \owedge A\left(g^{-1}\mathrm{d}g\right)\right) + \kappa\left(A\left(g^{-1}\mathrm{d}g\right) \owedge g^{-1}\delta_X g\right)\right] \\
&= \frac{k}{8\pi^2}\kappa\left(g^{-1}\delta_X g \owedge -A\left(g^{-1}\mathrm{d}g\right)\right),
\end{aligned}$$

in agreement with the ansatz, Equation (6.3.6), and Equation (6.3.7). Since X is an arbitrary vector field, this establishes Equation (6.3.8) for ω.

To show that $\mathrm{d}\omega = H$, which establishes that the open string Wess-Zumino-Witten action is self-consistent, some auxiliary calculation is required. As $\mathrm{id} - B$ is invertible on the translated Neumann subspace of $\mathfrak{g} = T_{\mathrm{id}}(\mathsf{G})$, ζ can be uniquely defined by $(\mathrm{id} - B)\, g^* \zeta = g^* \vartheta$ (where ϑ is restricted to the brane). It is easy to verify that

$$\mathrm{d}B = B \circ \mathrm{ad}\left(g^{-1}\mathrm{d}g\right) = B \circ \mathrm{ad}\left((\mathrm{id} - B)\, g^* \zeta\right). \tag{6.3.9}$$

Furthermore, Equation (6.1.2) may be used to show that (the pullbacks will be dropped in the remainder of this proof)

$$(\mathrm{id} - B)\, \mathrm{d}\zeta = (\mathrm{id} - B)\, [B\zeta \wedge \zeta] - \frac{1}{2}(\mathrm{id} - B)\, [\zeta \wedge \zeta] + \frac{1}{2}\tau, \tag{6.3.10}$$

where $\tau = B[\zeta \wedge \zeta] - [B\zeta \wedge B\zeta]$ is clearly zero if Ω is an automorphism[4] of $\mathfrak{g}$.

Consider now

$$\omega = \frac{-k}{16\pi^2}\kappa\left(\vartheta \wedge \frac{\mathrm{id} + B}{\mathrm{id} - B}\vartheta\right) = \frac{-k}{16\pi^2}\kappa\left((\mathrm{id} - B)\, \zeta \wedge (\mathrm{id} + B)\, \zeta\right) = \frac{k}{8\pi^2}\kappa(B\zeta \wedge \zeta).$$

Its derivative may be computed directly, if inelegantly, using the orthogonality of B, and Equations (6.3.9) and (6.3.10). One finds that

$$\begin{aligned}
\mathrm{d}\omega &= \frac{k}{16\pi^2}\left[\kappa(B\zeta \wedge [\zeta \wedge \zeta]) - \kappa(\zeta \wedge [B\zeta \wedge B\zeta]) - \kappa(B\zeta \wedge \tau)\right] \\
&= \frac{k}{16\pi^2}\left[\kappa(B\zeta \wedge [B\zeta \wedge B\zeta]) - \kappa(\zeta \wedge [B\zeta \wedge B\zeta]) + \kappa(B\zeta \wedge [\zeta \wedge \zeta]) - \kappa(\zeta \wedge [\zeta \wedge \zeta])\right].
\end{aligned}$$

However, a similar expansion gives

$$\begin{aligned}
H &= \frac{-k}{48\pi^2}\kappa\left((\mathrm{id} - B)\, \zeta \wedge [(\mathrm{id} - B)\, \zeta \wedge (\mathrm{id} - B)\, \zeta]\right) \\
&= \frac{k}{48\pi^2}\left[\kappa(B\zeta \wedge [B\zeta \wedge B\zeta]) - 3\kappa(\zeta \wedge [B\zeta \wedge B\zeta]) + 3\kappa(B\zeta \wedge [\zeta \wedge \zeta]) - \kappa(\zeta \wedge [\zeta \wedge \zeta])\right].
\end{aligned}$$

Sadly, it follows that

$$H - \mathrm{d}\omega = \frac{k}{24\pi^2}\left[\kappa(\zeta \wedge [\zeta \wedge \zeta]) - \kappa(B\zeta \wedge [B\zeta \wedge B\zeta])\right] = \frac{k}{24\pi^2}\kappa(B\zeta \wedge \tau),$$

which seems to be non-zero in general, but vanishes when $\Omega \in \mathsf{Aut}(\mathfrak{g})$. ∎

This demonstrates that the conformal boundary conditions may be derived from the open string Wess-Zumino-Witten action, Equation (6.3.1), by choosing ω appropriately. Furthermore, when $\Omega \in \mathsf{Aut}(\mathfrak{g})$, this action is self-consistent in that $\mathrm{d}\omega = H$. In the general case, where Ω is orthogonal on $\mathfrak{g}$ but not an automorphism, the corresponding self-consistency has not been shown. Indeed, the proof of Proposition 6.3 suggests that self-consistency seems rather doubtful in this case. For instance, taking $\Omega = -\mathrm{id}$ gives

$$H - \mathrm{d}\omega = \frac{k}{12\pi^2}\kappa(\zeta \wedge [\zeta \wedge \zeta]),$$

[4] As ζ is only allowed to take values in the translated Neumann subspace of $\mathfrak{g}$, τ vanishing need not *require* Ω to be an automorphism.

which looks suspiciously non-zero. In what follows, Ω will therefore be taken to be an automorphism of $\mathfrak{g}$. In fact, Ω will usually be chosen to be the identity map, as these correspond to the untwisted symmetry-preserving branes, which are the main focus of this thesis.

6.3.4. More Ambiguities and Quantisation. The mathematical data for defining the open string Wess-Zumino-Witten action therefore consists of the following. The string worldsheet is a two-dimensional, compact, orientable manifold Σ with boundary, and this worldsheet is mapped by the field g into a compact, connected, simply-connected, simple Lie group G. The boundary, assumed to be connected for simplicity, is mapped into an *a priori* chosen D-brane $\mathcal{C}$, which is determined by the gluing condition, Equation (3.2.2), characterised by an automorphism $\Omega \in \mathsf{Aut}(\mathfrak{g})$. G supports a 3-form H given by Equation (6.2.12), and $\mathcal{C}$ supports a 2-form ω, given by Equation (6.3.8), which satisfies $\mathrm{d}\omega = H$ (restricted to $\mathcal{C}$). To construct the action, Equation (6.3.1), g is extended from Σ to $g' \colon \Sigma' = \Sigma + D \to \mathsf{G}$, where D is a two-dimensional, compact, orientable manifold with boundary satisfying $g'(D) \subseteq \mathcal{C}$. g' is then extended to $\tilde{g}' \colon \Gamma' \to \mathsf{G}$ where $\partial\Gamma' = \Sigma'$.

As in Section 6.2.3, it is now necessary to determine the validity of all these extensions, and the effect of any ambiguities in the action caused by a lack of uniqueness in these extensions. First, this formalism requires that g can be extended from Σ to Σ', hence that $g'(D)$ can be consistently defined in $\mathcal{C} \subset \mathsf{G}$. This requires that the 1-cycle $g(\partial\Sigma)$ is a boundary in $\mathcal{C}$, hence this extension makes sense provided that $\mathsf{H}_1(\mathcal{C};\mathbb{Z}) = 0$. This condition is not easy to check for a general brane, as their worldvolumes have not been elucidated. However, for the symmetry-preserving branes of interest in this thesis, the worldvolumes are conjugacy classes (Section 6.3.2), which are simply-connected (Appendix C.3). This extension is therefore valid.

Similarly, the extension from Σ' to Γ' requires that $\mathsf{H}_2(\mathsf{G};\mathbb{Z}) = 0$ as in Section 6.2.3, hence this extension is also valid. Instead of considering these two extensions separately, it is somewhat more elegant to view them as a single composite extension from Σ to Γ'. Now, $\partial\Gamma' = \Sigma + D \equiv \Sigma \pmod{\mathcal{C}}$, so Σ is required to be a boundary *modulo* $\mathcal{C}$. This fits naturally in the formalism of relative homology (Appendix C.2), leading to the conclusion that this single extension makes sense when $\mathsf{H}_2(\mathsf{G},\mathcal{C};\mathbb{Z}) = 0$. From the long exact sequence in relative homology,

$$0 = \mathsf{H}_2(\mathsf{G};\mathbb{Z}) \longrightarrow \mathsf{H}_2(\mathsf{G},\mathcal{C};\mathbb{Z}) \longrightarrow \mathsf{H}_1(\mathcal{C};\mathbb{Z}) \longrightarrow \mathsf{H}_1(\mathsf{G};\mathbb{Z}) = 0,$$

it follows that $\mathsf{H}_2(\mathsf{G},\mathcal{C};\mathbb{Z}) = \mathsf{H}_1(\mathcal{C};\mathbb{Z}) = 0$ (in a simply-connected Lie group G). This shows (again) that the composite extension is valid.

It remains then to consider the ambiguities in the action due to different choices of extension. Consider two different extensions Γ'_1 and Γ'_2 with respective manifolds D_1 and D_2 in $\mathcal{C}$. The action ambiguity may be expressed as

$$2\pi\mathrm{i}\left[\int_{\Gamma'_1-\Gamma'_2}\left(\tilde{g}'\right)^* H - \int_{D_1-D_2}\left(g'\right)^*\omega\right] = 2\pi\mathrm{i}\left[\int_Z H - \int_S \omega\right],$$

where S is the 2-cycle $g'(D_1) - g'(D_2)$ and Z is the 3-chain $\tilde{g}'(\Gamma'_1) - \tilde{g}'(\Gamma'_2)$. Note that $\partial Z = S$. This ambiguity may now be interpreted through the natural pairing of relative homology with relative cohomology (see Appendix C.2) — the relative 3-cycle $(Z,S) \in \mathsf{H}_3(\mathsf{G},\mathcal{C};\mathbb{Z})$ with the relatively closed 3-form $(H,\omega) \in \mathsf{H}^3(\mathsf{G},\mathcal{C};\mathbb{R})$.

It follows that the Feynman amplitudes will be unambiguous, so the quantum theory will be well-defined, if the relatively closed 3-form (H,ω) has integral periods. The quantisation condition for the open string Wess-Zumino-Witten model is therefore

$$(H,\omega) \in \mathsf{H}^3(\mathsf{G},\mathcal{C};\mathbb{Z}). \tag{6.3.11}$$

This generalises the closed string quantisation condition, Equation (6.2.10), in that when the 2-cycle S is homologically trivial in the brane $\mathcal{C}$, it follows that the periods of (H,ω) over the relative cycle (Z,S) reduce to the periods of H over genuine cycles. H is already fixed by this case to be integral, and ω is fixed by the choice of brane and the normalisation of H. Therefore, the presence of homologically non-trivial 2-cycles can only fix the last remaining degree of freedom: The specific brane which ω is defined upon.

It follows that Equation (6.3.11) selects a subset of the classical D-branes, for which the quantised theory is consistent. Note however, that if $\mathsf{H}_2(\mathcal{C};\mathbb{Z}) = 0$, then the open string quantisation condition reduces to the closed string quantisation condition, so the brane $\mathcal{C}$ is consistent in the quantised theory. In particular, all branes of dimension lower than 2 are consistent.

6.3.5. $\mathsf{SU}(2)$**: Another Example.** It is again useful to consider the most tractable case, that of the symmetry-preserving branes on $\mathsf{SU}(2)$. As shown in Section 6.3.2, the worldvolumes of these branes are just the conjugacy classes. The "stereographic-polar" parametrisation, Equation (6.2.11), of the fundamental representation of $\mathsf{SU}(2)$ admits a useful diagonalisation:

$$\beta(r,\theta,\phi) = \gamma(\theta,\phi)\, h(r)\, \gamma(\theta,\phi)^{-1}$$

where

$$\gamma(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2} & \mathrm{i}\sin\frac{\theta}{2}e^{\mathrm{i}\phi} \\ \mathrm{i}\sin\frac{\theta}{2}e^{-\mathrm{i}\phi} & \cos\frac{\theta}{2} \end{pmatrix} \quad \text{and} \quad h(r) = \frac{1}{4+r^2}\begin{pmatrix} (2+\mathrm{i}r)^2 & 0 \\ 0 & (2-\mathrm{i}r)^2 \end{pmatrix}.$$

As unitary matrices are conjugate if and only if they have the same eigenvalues, it follows that in this parametrisation, the conjugacy classes are the 2-spheres of constant r (with the degenerate cases $r = 0$ and $r = \infty$ corresponding to the point conjugacy classes of the two elements of $\mathsf{Z}(\mathsf{SU}(2))$). This simple portrayal of the conjugacy classes is the reason why this parametrisation was chosen in the first place.

Substituting this diagonalisation into $\beta^{-1}\mathrm{d}\beta$ and fixing h (that is, setting $\mathrm{d}h = 0$) gives

$$\beta^{-1}\mathrm{d}\beta = \left[\mathrm{Ad}\left(\beta^{-1}\right) - \mathrm{id}\right]\mathrm{d}\gamma\cdot\gamma^{-1}.$$

Using Equation (6.3.8) (with $\Omega = \mathrm{id}$), it now follows that in the stereographic-polar parametrisation, ω takes the form ($\zeta = \mathrm{Ad}\left(\beta^{-1}\right)\mathrm{d}\gamma\cdot\gamma^{-1}$)

$$\begin{aligned}\beta^*\omega &= \frac{-k}{16\pi^2}\kappa\left(\left[\mathrm{id}-\mathrm{Ad}\left(\beta\right)\right]\zeta\mathbin{\tilde{\wedge}}\left[\mathrm{id}+\mathrm{Ad}\left(\beta\right)\right]\zeta\right) = \frac{k}{8\pi^2}\kappa\left(\mathrm{Ad}\left(\beta\right)\zeta\mathbin{\tilde{\wedge}}\zeta\right)\\ &= \frac{k}{8\pi^2}\,\mathrm{tr}\left(\gamma^{-1}\mathrm{d}\gamma h^{-1}\gamma^{-1}\mathrm{d}\gamma h\right) = \frac{-k}{\pi^2}\frac{r\left(4-r^2\right)}{\left(4+r^2\right)^2}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi.\end{aligned}$$

From Section 6.2.4,

$$\beta^*H = \frac{32k}{\pi^2}\frac{r^2}{\left(4+r^2\right)^3}\sin\theta\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi.$$

The quantisation condition, Equation (6.3.11), is now easily evaluated. The homologically non-trivial 2-cycle S is the entire conjugacy class (at $r = r_0$, say, in stereographic-polar coordinates), and the 3-chain Z may be chosen as the region $r \leqslant r_0$. Integration gives

$$\begin{aligned}\int_Z H - \int_S \omega &= \frac{128k}{\pi}\int_0^{r_0}\frac{r^2\,\mathrm{d}r}{\left(4+r^2\right)^3} + \frac{4k}{\pi}\frac{r_0\left(4-r_0^2\right)}{\left(4+r_0^2\right)^2}\\ &= \frac{2k}{\pi}\tan^{-1}\frac{r_0}{2} - \frac{4k}{\pi}\frac{r_0\left(4-r_0^2\right)}{\left(4+r_0^2\right)^2} + \frac{4k}{\pi}\frac{r_0\left(4-r_0^2\right)}{\left(4+r_0^2\right)^2}\\ &= \frac{2k}{\pi}\tan^{-1}\frac{r_0}{2}.\end{aligned}$$

It follows that the only symmetry-preserving branes in the $\mathsf{SU}(2)$ Wess-Zumino-Witten model which are consistent in the quantised theory are those corresponding to radii $r_0 = 2\tan\left(m\pi/2k\right)$, where $m \in \mathbb{Z}$. There are exactly $k+1$ such branes, corresponding to $m = 0, 1, \ldots, k$, and both the zero-dimensional branes are included (as expected) as the cases $m = 0$ and $m = k$.

This result can be interpreted in a more Lie-theoretic manner by noting that the fundamental representation of $\mathfrak{su}(2)^{\mathbb{C}}$ has highest weight Λ_1, so the coroot $\alpha_1^\vee$ is represented by the diagonal matrix with entries $\pm\langle\Lambda_1, \alpha_1^\vee\rangle = \pm 1$. Thus, $\exp\left(2\pi\mathrm{i}t\alpha_1^\vee\right)$ is represented by the diagonal matrix with entries $e^{2\pi\mathrm{i}t}$ and $e^{-2\pi\mathrm{i}t}$. Recall that $h(r_0)$ is such a matrix, with entries

$$\frac{\left(2\pm\mathrm{i}r_0\right)^2}{4+r_0^2} = \frac{2\pm 2\mathrm{i}\tan\frac{m\pi}{2k}}{2\mp 2\mathrm{i}\tan\frac{m\pi}{2k}} = e^{\pm\mathrm{i}m\pi/k}.$$

It follows that the allowed symmetry-preserving branes are the conjugacy classes containing one of the elements

$$h(r_0) = \exp\left(\mathrm{i}m\pi\alpha_1^\vee/k\right) = \exp\left(2\pi\mathrm{i}m\Lambda_1/k\right).$$

In this formalism, the quantisation condition on the branes states that the conjugacy class through $\exp\left(2\pi\mathrm{i}\lambda/k\right)$ is an allowed brane after quantisation if $\lambda \in \mathsf{P}$. That is, if $\langle\lambda, \alpha_1^\vee\rangle = m \in \mathbb{Z}$.

It is important to realise that there is still one ambiguity left unaccounted for. This is the choice of the 3-chain Z. In the computation above, this was chosen to be the region $r \leqslant r_0$; however, there is no reason to prefer this choice to the region $r \geqslant r_0$. The associated ambiguity in the computation of the period of (H,ω) will then be just the period of H over the oriented difference of these regions. This oriented difference is the entire group manifold, and H was normalised in Section 6.2.4 so that this period is the level $k \in \mathbb{Z}$. This ambiguity therefore only affects the evaluation of the relative period of (H,ω) by an integer, and therefore plays no part in determining which symmetry-preserving branes satisfy the quantisation condition.

6.3.6. General untwisted symmetry-preserving Branes. Consider now an untwisted symmetry-preserving brane in a general Wess-Zumino-Witten model. Its worldvolume is a conjugacy class $\mathcal{C}(h)$ consisting of elements conjugate to some h in the maximal torus. To analyse the quantisation condition, Equation (6.3.11), it is sufficient to consider relative cycles (Z,S) where S is a generator of $\mathsf{H}_2(\mathcal{C}(h);\mathbb{Z})$, since homologically trivial 2-cycles in $\mathcal{C}(h)$ contribute nothing to the brane quantisation. As the brane is simply-connected, it follows from the Hurewicz isomorphism [**144**] that these generators are 2-spheres; in fact, a convenient set of generators is given by [**28**]

$$S_i = \left\{ \mathrm{Ad}(\gamma)\, h \colon \gamma \in \mathsf{SU}(2)_{\alpha_i} \right\} \qquad i = 1, \ldots, \mathrm{rank}\,\mathsf{G}, \tag{6.3.12}$$

where $\mathsf{SU}(2)_{\alpha_i}$ is the $\mathsf{SU}(2)$-subgroup of G corresponding to the simple root α_i. This set is not homologically independent in general, but it always spans $\mathsf{H}_2(\mathcal{C}(h);\mathbb{Z})$.

PROPOSITION 6.4. *The homology cycle S_i is the translation of a conjugacy class of the subgroup $\mathsf{SU}(2)_{\alpha_i}$.*

PROOF. Let $h = \exp(2\pi i y)$, and project y onto the subspace spanned by $\alpha_i^\vee$:

$$y = y_\perp + y_\parallel, \qquad \text{where } y_\parallel = \frac{\kappa(\alpha_i^\vee, y)}{\kappa(\alpha_i^\vee, \alpha_i^\vee)}\alpha_i^\vee = \frac{\langle \alpha_i, y\rangle}{2}\alpha_i^\vee .$$

Then, $y_\perp$ commutes with the root vectors corresponding to α_i:

$$[y_\perp, e_{\pm\alpha_i}] = \left[y - \frac{\langle \alpha_i, y\rangle}{2}\alpha_i^\vee, e_{\pm\alpha_i} \right] = \left(\langle \pm\alpha_i, y\rangle - \frac{\langle \alpha_i, y\rangle}{2}\langle \pm\alpha_i, \alpha_i^\vee\rangle \right) e_{\pm\alpha_i} = 0.$$

It follows that h decomposes analogously as $h_\perp h_\parallel$, and this decomposition has the property that $h_\perp$ commutes with $\mathsf{SU}(2)_{\alpha_i}$ whereas $h_\parallel \in \mathsf{SU}(2)_{\alpha_i}$. Therefore,

$$\begin{aligned} S_i &= \left\{ \gamma h \gamma^{-1} \colon \gamma \in \mathsf{SU}(2)_{\alpha_i} \right\} = \left\{ h_\perp \gamma h_\parallel \gamma^{-1} \colon \gamma \in \mathsf{SU}(2)_{\alpha_i} \right\} \\ &= h_\perp \mathcal{C}\left(h_\parallel; \mathsf{SU}(2)_{\alpha_i}\right). \end{aligned}$$

■

Using this result, the computation of the periods of (H,ω) over the relative cycles (Z_i, S_i) ($\partial Z_i = S_i$) may be reduced to the $\mathsf{SU}(2)$ case of Section 6.3.5. Let $S_i' = h_\perp^{-1} S_i$ be the conjugacy class of $\mathsf{SU}(2)_{\alpha_i}$ corresponding to S_i, and let Z_i' be the 3-chain bounding S_i' (chosen as in Section 6.3.5). Z_i is then defined to be $h_\perp Z_i'$. As in Section 6.2.5, let

j denote the inclusion of $\mathsf{SU}(2)$ into G as $\mathsf{SU}(2)_{\alpha_i}$. Finally, let Z and S be the inverse images $j^{-1}(Z_i')$ and $j^{-1}(S_i')$ respectively.

As H is left-invariant, it easily follows from Proposition 6.2 that

$$\int_{Z_i} H = \int_{h_\perp^{-1} Z_i'} H = \int_{Z_i'} H = \int_Z j^* H = \frac{2}{\|\alpha_i\|^2} \int_Z H_{\mathsf{SU}(2)}.$$

ω however, is not left-invariant, being only defined on the conjugacy class $\mathcal{C}(h)$. Instead,

$$\int_{S_i} \omega = \int_{S_i'} \ell_{h_\perp}^* \omega,$$

where $\ell_g \colon \mathsf{G} \to \mathsf{G}$ denotes left-translation by g.

However, note that Equation (6.3.8) may be rewritten as

$$\omega = \frac{-k}{16\pi^2} \kappa(\vartheta \wedge \mathcal{A}\vartheta),$$

where $\mathcal{A} \colon \mathcal{C}(h) \to \mathsf{End}\,\mathfrak{g}$ is given by $\mathcal{A}(g) = (\mathrm{id} - \mathrm{Ad}(g))^{-1}(\mathrm{id} + \mathrm{Ad}(g))$. It follows that

$$\ell_{h_\perp}^* \omega = \frac{-k}{16\pi^2} \kappa\left(\vartheta \wedge \left(\ell_{h_\perp}^* \mathcal{A}\right)\vartheta\right), \text{ and } \left(\ell_{h_\perp}^* \mathcal{A}\right)(g) = \frac{\mathrm{id} + \mathrm{Ad}(h_\perp)\,\mathrm{Ad}(g)}{\mathrm{id} - \mathrm{Ad}(h_\perp)\,\mathrm{Ad}(g)}.$$

Now, if $g \in S_i' \subset \mathsf{SU}(2)_{\alpha_i}$, the proof of Proposition 6.4 shows that $h_\perp$ commutes with g, hence $\mathrm{Ad}(h_\perp)$ acts trivially on $g^*\vartheta$. Therefore, *when g is restricted to take values in S_i'*,

$$g^* \ell_{h_\perp}^* \omega = \frac{-k}{16\pi^2} \kappa\left(g^{-1}\mathrm{d}g \wedge \frac{\mathrm{id} + \mathrm{Ad}(g)}{\mathrm{id} - \mathrm{Ad}(g)} g^{-1}\mathrm{d}g\right),$$

which is just the expression defining the 2-form $\omega_{\mathsf{SU}(2)_{\alpha_i}}$ characterising the boundary condition on the conjugacy class $S_i' = \mathcal{C}\left(h_\parallel; \mathsf{SU}(2)_{\alpha_i}\right)$ in the $\mathsf{SU}(2)_{\alpha_i}$ Wess-Zumino-Witten model.

Finally then,

$$\int_{S_i} \omega = \int_{S_i'} \ell_{h_\perp}^* \omega = \int_{S_i'} \omega_{\mathsf{SU}(2)_{\alpha_i}} = \int_S j^* \omega_{\mathsf{SU}(2)_{\alpha_i}} = \frac{2}{\|\alpha_i\|^2} \int_S \omega_{\mathsf{SU}(2)},$$

where the last equality follows from a slight modification of Proposition 6.2 (with the normalisation from $\mathrm{d}\omega = H$).

For the conjugacy class $\mathcal{C}(h)$ where $h = \exp(2\pi \mathrm{i} y)$, the result of Section 6.3.5 therefore gives

$$\int_{Z_i} H - \int_{S_i} \omega = \frac{2}{\|\alpha_i\|^2} \left[\int_Z H_{\mathsf{SU}(2)} - \int_S \omega_{\mathsf{SU}(2)}\right] = \frac{2}{\|\alpha_i\|^2} \langle \alpha_i, ky \rangle = \kappa\left(\alpha_i^\vee, ky\right).$$

The quantisation condition, Equation (6.3.11), demands that this must be an integer for each $i = 1, \ldots, \mathrm{rank}\,\mathsf{G}$, so it follows that this conjugacy class is the worldvolume of a consistent brane in the quantised theory if and only if $ky \in \mathsf{P}$.

As in Section 6.3.5, it should be remarked that this computation involved a special choice of 3-cycles Z_i. Again, the ambiguity in computing the relative period of (H, ω) over different cycles will be a period of H, that is, a multiple of k. This will also not play any part is determining which branes are allowed in the quantised theory. However,

this ambiguity (among others) will be important in Section 7.3 in the context of assigning conserved charges to the consistent quantised branes.

CHAPTER 7

Brane Charge Groups Revisited

As in Chapter 6, G will always denote a compact, connected, simply-connected, simple Lie group, and $\mathfrak{g}$ its Lie algebra. A maximal torus T of G will also be fixed from the outset, and its Lie algebra will be denoted by $\mathfrak{t}$. In this chapter, the geometric definition and computation of a conserved charge for quantised D-branes in the open string Wess-Zumino-Witten model will be discussed. To simplify the analysis, the D-branes will be restricted to untwisted symmetry-preserving branes, which correspond to the trivial automorphism in the gluing condition, Equation (3.2.2). It is expected that more general branes, particularly those corresponding to twined conjugacy classes [**56**], may also be studied using similar techniques as those that are developed in this chapter.

7.1. Geometric Charge Definitions

7.1.1. Another Quantisation Effect. As shown in Sections 6.3.2 and 6.3.6, the untwisted symmetry-preserving brane worldvolumes may be identified with certain conjugacy classes of G. Specifically, if the level of the theory is denoted by k, then the allowed D-branes correspond to the conjugacy classes

$$\mathcal{C}\left(h'_\lambda\right) = \mathsf{G}/\mathsf{Z}\left(h'_\lambda\right), \text{ where } h'_\lambda = \exp\left(2\pi\mathrm{i}\lambda/k\right)$$

and $\lambda \in \mathsf{P}$, the weight lattice. In fact, since $w(h) \in \mathcal{C}(h)$ for every Weyl transformation $w \in \mathsf{W}$, and $\ker\exp = 2\pi\mathrm{i}\mathsf{Q}^\vee$ (where $\mathsf{Q}^\vee$ is the coroot lattice), it follows that λ may be chosen in the affine fundamental alcove at level k. These conjugacy classes are therefore in bijection with the set of integrable highest weight modules of the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$ at level k.

A conjugacy class $\mathcal{C}(h_\lambda)$ — and by association, the corresponding brane — will be called regular (singular) if h_λ is a regular (singular) element of T, respectively (Appendix C.1). Regular branes then have worldvolume diffeomorphic to G/T, are maximal in dimension among the untwisted symmetry-preserving branes, and correspond to weights in the interior of the affine fundamental alcove. Singular branes have strictly lower dimension, and correspond to weights on the boundary of the affine alcove.

This picture is still, however, semiclassical. Whilst removing the ambiguity in the Feynman amplitudes identifies a finite set of allowed untwisted symmetry-preserving branes, this is not the whole story. In particular, this analysis misses the well-known, but still poorly understood, non-perturbative level shift:

$$k \mapsto k + \mathsf{h}^\vee.$$

This "shifted level" is ubiquitous in the conformal field theory description of Wess-Zumino-Witten models (for example Equations (3.1.15) and (3.1.17)). Indeed, this shift has even been identified in a perturbative treatment of these models [**25**] (at least to two loops). It seems likely that in a careful analysis using the theory of geometric quantisation [**108**], this shift would be identified as a consequence of the familiar (though mysterious) *metaplectic shift* [**53**, **164**]. Alas, such an analysis for branes does not seem to appear in the literature, and is beyond the scope of this thesis (but see [**140**]).

There is, however, an analysis [**56**, **112**] which aims to extract geometric information about the branes from their conformal field theory description as boundary states (recall that the Wess-Zumino-Witten model was *quantised* as a conformal field theory in Chapter 3). The results of this analysis suggest that not only does the level undergo the expected shift, but that the weights labelling the branes are also subjected to a corresponding shift:

$$\lambda \mapsto \lambda + \rho .$$

More precisely, the allowed untwisted symmetry-preserving branes in fact correspond to the conjugacy classes

$$\mathcal{C}(h_\lambda), \text{ where } h_\lambda = \exp\left(2\pi \mathrm{i} \frac{\lambda+\rho}{k+\mathsf{h}^\vee}\right),$$

and λ denotes an integral weight in the fundamental alcove at level k. Equivalently, λ is an integral weight in the *interior* of the shifted fundamental alcove at level k (Appendix B.2). The conclusion therefore is that the allowed branes in the fully quantised theory are in fact all regular, with worldvolumes diffeomorphic to G/T. This weight shift has also been observed [**152**] in the context of path-integral quantisation, for a point particle moving along a "quantisable orbit" of $\mathsf{SU}(2)$ (for example, a string endpoint moving along an untwisted symmetry-preserving brane).

On a cautionary note, however, the analysis in [**56**] starts from an expression for the boundary states which appears to disagree with Equation (3.2.10) (and most other sources). The argument in [**112**] is in better agreement, but requires the use of many approximations. These analyses also arrive at the conclusion that the D-branes in the fully quantised theory are "smeared out" in directions transverse to the conjugacy class. This fits in well with intuition about branes as dynamical quantum objects in some extension of string theory, but seems to be at odds with the identification in the conformal field theory description of the brane as a *non-normalisable* state (functional) in the space of closed strings. This identification suggests that brane wavefunctions could not be smooth functions on the group, and rather must be expressed as a distribution, for instance by a delta function on the conjugacy class. What is actually computed in [**56**] is the overlap of the brane with certain closed strings, so it seems plausible that the observed "smearing out" may actually be due to these strings[1]. In any case, geometric quantisation should

[1] In [**112**], the string states are chosen to be (approximate) delta functions. The overlap is then computed to be, approximately, a delta function on the shifted brane. Interestingly, there is a remark in [**111**] which admits that brane positions cannot be resolved using strings, due to finite size effects.

also be able to resolve this issue directly by computing the brane wavefunction (in the Schrödinger representation) corresponding to Equation (3.2.8).

Nevertheless, the conclusion of [**56**], namely the quantum shifts described above and the consequence that all untwisted symmetry-preserving branes are regular, will be used repeatedly in what follows. It is perhaps worth reiterating that these shifts are expected to be consequences of the metaplectic shift in the geometric quantisation formalism.

7.1.2. Geometric Charges I: $\mathsf{U}(1)$-flux. Recall that the Wess-Zumino-Witten action for an open string with untwisted symmetry-preserving boundary conditions, Equation (6.3.1), involves the closed 3-form H given by Equation (6.2.12), and a 2-form ω given by Equation (6.3.8) on the brane worldvolume $\mathcal{C}(h_\lambda)$. Since the branes are in bijection with the integral weights λ of the fundamental alcove at level k (Section 7.1.1), it is not unreasonable to denote this 2-form by ω_λ.

Since a D-brane is entirely characterised by the 2-form ω_λ, it might seem reasonable to define a brane charge by integrating (some suitable function of) this form over the brane. By analogy with the coupling of the Ramond-Ramond fields in string theory [**129**, **130**], this function should just be the simple exponential e^{ω_λ}. This is to be interpreted as a power series in ω_λ (actually a polynomial as the branes have finite dimension).

For $\mathsf{SU}(2)$, this charge may be computed explicitly using the results of Section 6.3.5. Taking into account the quantum shift of Section 7.1.1, it follows that

$$\int_{\mathcal{C}(h_\lambda)} e^{\omega_\lambda} = \int_{\mathcal{C}(h_\lambda)} \omega_\lambda = \frac{-4(k+2)}{\pi} \frac{r_0\left(4-r_0^2\right)}{\left(4+r_0^2\right)^2} = -\frac{k+2}{2\pi} \sin \frac{2\pi\langle\lambda+\rho,\alpha^\vee\rangle}{k+2},$$

since $r_0 = 2\tan\frac{(m+1)\pi}{2(k+2)}$ and $m = \langle\lambda,\alpha^\vee\rangle$. Normalising so that the charge corresponding to $\lambda = 0$ is 1 gives

$$Q_\omega(\lambda) = \frac{\sin\frac{2\pi\langle\lambda+\rho,\alpha^\vee\rangle}{k+2}}{\sin\frac{2\pi}{k+2}},$$

which are not even rational numbers in general, let alone integers[2]. Hence, this definition does not seem to define a charge analogous to the algebraic charge introduced in Equation (4.1.2).

Note that ω_λ is not generally a closed form, hence the integration in $Q_\omega(\lambda)$ does not have an obvious cohomological interpretation. That is, this charge is not obviously conserved under small brane deformations. This can be rectified by a judicious consideration of the closed 3-form H. By Theorem C.1, H is exact on $\mathcal{C}(h_\lambda) \cong \mathsf{G}/\mathsf{T}$, so $H = \mathrm{d}B$, and $F_\lambda = B - \omega_\lambda$ represents a degree-two cohomology class on the brane. This closed 2-form F_λ is sometimes referred to as the $\mathsf{U}(1)$-flux, as it may be interpreted (assuming

[2]Interestingly, the numbers $Q_\omega(\lambda)$ have a form very similar to (but not identical to) the *quantum dimensions* [**61**] given (for $\mathsf{SU}(2)$) by

$$\frac{S_{\lambda 0}}{S_{00}} = \frac{\sin\frac{\pi\langle\lambda+\rho,\alpha^\vee\rangle}{k+2}}{\sin\frac{\pi}{k+2}}.$$

It is indicated in [**121**] (though not demonstrated) that this similarity persists for other groups. However, there appears to be no understanding of why this should be so.

the integrality condition is met) as the first Chern class of a $\mathsf{U}(1)$-bundle. This bundle was originally introduced to explain the stability of the branes against collapse under tension [**13**], this stability generally being referred to as *flux stabilisation*.

Dually, one might suppose that this stability could also be understood by postulating a (related) conserved charge that prevented the brane from collapsing dynamically. The obvious modification of the charge $Q_\omega(\lambda)$ then suggests the candidate

$$Q_F(\lambda) = \int_{\mathcal{C}(h_\lambda)} e^{F_\lambda}. \tag{7.1.1}$$

However, before getting carried away with this definition, it should be remarked that the cohomology class defined by F_λ is, as yet, *completely arbitrary*, because B is only determined up to a closed form.

To rectify this situation, it proves useful to (briefly) adopt the philosophy of [**59**, **146**] regarding flux stabilisation. There it was argued that there is no need to invoke such a $\mathsf{U}(1)$-bundle; rather one should work directly with the global quantities H and ω_λ. Furthermore, it was argued that there was a cohomological obstruction to defining this bundle over the whole brane, so one must work with the global quantities and promote the $\mathsf{U}(1)$-bundle structure to that of a *bundle gerbe* [**122**]. Whilst the first point is philosophically useful, the obstruction argument is flawed due to what seems to be a misidentification of (H,ω_λ) and $(0,F_\lambda)$ in $\mathsf{H}^3(\mathsf{G},\mathcal{C}(h_\lambda);\mathbb{R})$.

The periods of F_λ can be suggestively rewritten as

$$\int_S F_\lambda = \int_S (B-\omega_\lambda) = \int_Z H - \int_S \omega_\lambda,$$

where S is a 2-cycle in $\mathcal{C}(h_\lambda)$ and Z is a 3-chain in G with boundary $\partial Z = S$, and $H = \mathrm{d}B$ on Z. Global philosophy now inspires the proclaimation that when the $\mathsf{U}(1)$-flux is referred to in the literature, what is really meant is the closed 2-form with these periods. Thus, the periods of F_λ are identical to the *relative* periods of (H,ω_λ). Comparison with the brane quantisation of Section 6.3.6 shows that these periods are integers, hence F_λ defines a class in $\mathsf{H}^2(\mathcal{C}(h_\lambda);\mathbb{Z})$. As a consequence, it may indeed be interpreted[3] as the first Chern class of a $\mathsf{U}(1)$-bundle on the brane, as required for flux stabilisation. A more direct construction of F_λ is given in the following proposition.

PROPOSITION 7.1. *There exists a complex $\mathfrak{C}_\lambda$ in G which contains $\mathcal{C}(h_\lambda)$, and on which H is exact. If $H = \mathrm{d}B$ on this complex, the cohomology class of $F_\lambda = B - \omega_\lambda$ is uniquely determined, even though B is not.*

PROOF. Let $\{S_i\}_{i=1}^r$ be a basis of $\mathsf{H}_2(\mathcal{C}(h_\lambda);\mathbb{Z})$ (by Theorem C.1, this homology group is $\mathbb{Z}^r$, where $r = \operatorname{rank}\mathsf{G}$). Since $\mathcal{C}(h_\lambda)$ is simply-connected, this basis can be chosen to consist of 2-spheres and as $\pi_2(\mathsf{G}) = 0$, these spheres are contractible in G. For each i,

[3]One really should distinguish $F_\lambda \in \mathsf{H}^2(\mathcal{C}(h_\lambda);\mathbb{R})$ from the class it represents in $\mathsf{H}^2(\mathcal{C}(h_\lambda);\mathbb{Z})$. However, for convenience of notation, this will not be done explicitly. That is, each integral cohomology class will be labelled by its deRham representative.

let D_i be the image of a homotopy mapping from $[0,1]\times S^2$ into G which demonstrates this contractibility. D_i is then a 3-cell in G which is attached to $\mathcal{C}(h_\lambda)$ along S_i.

If C_j denotes the complex formed by attaching the D_i with $i\leqslant j$ to $\mathcal{C}(h_\lambda)$ (and $C_0=\mathcal{C}(h_\lambda)$), the Mayer-Vietoris exact sequence for attaching D_j, Equation (C.2.2), becomes

$$0\to \mathsf{H}_3\left(C_{j-1};\mathbb{Z}\right)\to \mathsf{H}_3\left(C_j;\mathbb{Z}\right)\to \mathsf{H}_2\left(S^2;\mathbb{Z}\right)\xrightarrow{\imath_*}\mathsf{H}_2\left(C_{j-1};\mathbb{Z}\right)\to \mathsf{H}_2\left(C_j;\mathbb{Z}\right)\to 0,$$

where $\imath\colon S^2\to C_{j-1}$ is the inclusion of the 2-sphere as S_j in $\mathcal{C}(h_\lambda)\subseteq C_{j-1}$. For $j=1$, the attaching converts the homology generator S_1 into a homologically trivial cycle, so $\mathsf{H}_2(C_1;\mathbb{Z})=\mathbb{Z}^{r-1}$. The map $\imath_*$ is an injection by construction, so it follows from exactness that $\mathsf{H}_3(C_1;\mathbb{Z})=\mathsf{H}_3(C_0;\mathbb{Z})=0$. Inductively then, $\mathsf{H}_2\left(C_j;\mathbb{Z}\right)=\mathbb{Z}^{r-j}$ and $\mathsf{H}_3\left(C_j;\mathbb{Z}\right)=0$ for all j. Setting $\mathfrak{C}_\lambda=C_r$ gives $\mathsf{H}_3(\mathfrak{C}_\lambda;\mathbb{Z})=\mathsf{H}_2(\mathfrak{C}_\lambda;\mathbb{Z})=0$. Attaching 3-cells does not affect the degree 1 homology, so by the Universal Coefficient Theorem [**144**],

$$\mathsf{H}^3(\mathfrak{C}_\lambda;\mathbb{Z})=\mathsf{H}^2(\mathfrak{C}_\lambda;\mathbb{Z})=0.$$

It follows that H is exact on $\mathfrak{C}_\lambda$, so $H=\mathrm{d}B$ for some $B\in\Omega^2(\mathfrak{C}_\lambda)$. B is still only defined up to a closed form, but as $\mathsf{H}^2(\mathfrak{C}_\lambda;\mathbb{Z})=0$, the closed forms on $\mathfrak{C}_\lambda$ are all exact. Restricting to $\mathcal{C}(h_\lambda)$, it is clear that the ambiguity in B is *still* just a set of exact forms. Thus, $F_\lambda=B-\omega_\lambda$ is well-defined as a cohomology class on $\mathcal{C}(h_\lambda)$. ∎

It is perhaps tempting to conclude that the $\mathsf{U}(1)$-flux F_λ is now completely specified. However, as noted in Sections 6.3.5 and 6.3.6, the relative periods of (H,ω_λ) are only well-defined *modulo* $k+\mathsf{h}^\vee$, due to an ambiguity in choosing the 3-chain to integrate H over (note the quantum shift). The same is true for the proclaimed periods of F_λ, for the same reason. In the direct construction of Proposition 7.1, this ambiguity is manifested as a choice of (perhaps homotopically distinct) 3-cells to attach to the brane $\mathcal{C}(h_\lambda)$ (recall from Appendix C.3 that $\pi_3(\mathsf{G})=\mathbb{Z}$). The consequences of this ambiguity will be examined in Section 7.3.3.

7.1.3. Geometric Charges II: Quantum Anomalies. The charge $Q_F(\lambda)$ of Equation (7.1.1) can now be trivially evaluated for $\mathsf{SU}(2)$. As

$$Q_F(\lambda)=\int_{\mathcal{C}(h_\lambda)}e^{F_\lambda}=\int_{\mathcal{C}(h_\lambda)}F_\lambda,$$

this charge is therefore just the period of F_λ over the 2-sphere constituting the (regular) conjugacy class, hence is the corresponding relative period of (H,ω_λ). From Section 6.3.5, this is just the integer

$$Q_F(\lambda)=\left\langle\lambda+\rho,\alpha^\vee\right\rangle \tag{7.1.2}$$

(note the quantum shift). As the charge corresponding to $\lambda=0$ is 1, it is not necessary to normalise. The charge of the brane labelled by λ is therefore given by the dimension of the irreducible $\mathfrak{su}(2)$-module with highest weight λ, in exact agreement with the algebraic charge $Q_{\mathrm{alg}}(\lambda)$ postulated in Section 4.1.2. Furthermore, the ambiguity in F_λ leads to the conclusion that the charge $Q_F(\lambda)$ is only defined *modulo* $k+2$. That is, the

charges take values in $\mathbb{Z}_{k+2}$, again in exact agreement with the fusion constraints result of Section 4.2.1.

This brane charge was computed in [**13**, **146**] for $\mathsf{SU}(2)$. These computations did not, however, incorporate the quantum shift — the conjugacy classes $\mathcal{C}\left(h'_\lambda\right)$ were taken to be the consistent quantum branes ($h' = \exp(2\pi \mathrm{i}\lambda/k)$), rather than $\mathcal{C}(h_\lambda)$. Consequently, the charge computed there was $\langle \lambda, \alpha^\vee \rangle$, 1 less than that computed above. There does appear to be an error in their calculation in that if λ is not shifted, taking $\lambda = 0$ should correspond to a degenerate point-brane. Therefore, the charge for $\lambda = 0$ should be the integral of 1 over a point, *which is* 1, not 0. This should also be true for the other point-brane, corresponding to $\lambda = k\Lambda$.

This discrepancy was removed in [**6**], but by using a modified charge rather than incorporating the shift $\lambda \mapsto \lambda + \rho$. This modified charge was originally proposed in [**117**], based on the cancellation of certain quantum anomalies (and [**89**]). It was also shown there that this modified charge had the desirable property that it could be naturally interpreted as an element of some K-group. The modified charge, after specialising to the case of interest in this thesis, amounts to

$$Q_{F'}(\lambda) = \int_{\mathcal{C}\left(h'_\lambda\right)} e^{F'_\lambda} \operatorname{Td}\left(T\left(\mathcal{C}\left(h'_\lambda\right)\right)\right), \tag{7.1.3}$$

where F'_λ denotes the $\mathsf{U}(1)$-flux on the unshifted brane, $T(M)$ denotes the tangent bundle of the manifold M, and $\operatorname{Td}(E)$ denotes the Todd class of the vector bundle E (see Appendix C.2).

When the conjugacy class is a 2-sphere, the $\mathsf{SU}(2)$ computation of [**6**] reduces to noting that the contributing part of the integrand in Equation (7.1.3) is

$$F'_\lambda \operatorname{Td}_0\left(T\left(S^2\right)\right) + \operatorname{Td}_1\left(T\left(S^2\right)\right) = F'_\lambda + \frac{1}{2}\mathrm{c}_1\left(T\left(S^2\right)\right),$$

where $\operatorname{Td}_i(E)$ denotes the i^{th} Todd polynomial, $\mathrm{c}_1(E)$ the first Chern class of E, and Equation (C.2.4) has been used. As $T\left(S^2\right)$ may be viewed as a complex line bundle, its first Chern class is its Euler class [**29**], so its integral is just the Euler characteristic of S^2, which is 2. Thus,

$$Q_{F'}(\lambda) = \left\langle \lambda, \alpha^\vee \right\rangle + 1 = \left\langle \lambda + \rho, \alpha^\vee \right\rangle,$$

in agreement with the unmodified, quantum-shifted, charge result, Equation (7.1.2). If the conjugacy class is a point, corresponding to $\lambda = 0$ or $k\Lambda$, then the modified charge is 1. It is not clear if these singular branes should be considered in the theory. Certainly, the results of [**56**] discussed in Section 7.1.1 seem to indicate that they should not. Nevertheless, the modified charge of these singular branes agrees with the dimension of the corresponding irreducible highest weight module, *modulo k*.

It follows that for $\mathsf{SU}(2)$, the algebraic brane charge $Q_{\text{alg}}(\lambda)$, Equation (4.1.2), the quantum shifted brane charge $Q_F(\lambda)$, Equation (7.1.1), and the modified brane charge $Q_{F'}(\lambda)$, Equation (7.1.3), all coincide. Furthermore, the first two charge definitions predict the same charge group $\mathbb{Z}_{k+2}$, whereas the modified charge needs an *a posteriori* level

shift, $k \mapsto k + \mathsf{h}^\vee$, to bring its predicted charge group into line. It is reasonable, therefore, to suppose that these three approaches to defining a sensible brane charge are somehow equivalent. In the next section, this coincidence of charge definitions will be tested explicitly for the non-trivial case $\mathsf{SU}(3)$, and then proven rigorously for all compact, connected, simply-connected, simple Lie groups.

7.2. Charge Computations

7.2.1. $\mathsf{SU}(3)$: An Extended Example. Consider now the regular, untwisted symmetry-preserving branes in $\mathsf{SU}(3)$. The brane worldvolumes are six-dimensional regular conjugacy classes diffeomorphic to $\mathsf{SU}(3)/\mathsf{U}(1)^2$, hence diffeomorphic to $\mathcal{F}_3$, the complete flag manifold of $\mathbb{C}^3$ [**29**]. Therefore, the technology of Schubert calculus [**72**] may be applied to them. Complete flag manifolds may be explicitly decomposed as the disjoint union of even-dimensional cells X_w°, known as the Schubert cells, and indexed by the elements of the Weyl group W, in this case, S_3. The closures of the Schubert cells are called Schubert varieties, X_w, and these generate the integral homology of the flag manifold in degree $2\ell(w)$ (compare with Theorem C.1). As an example, the Schubert variety corresponding to the simple Weyl reflection X_{w_i} is, under the diffeomorphism with a regular conjugacy class, just the 2-sphere S_i given by Equation (6.3.12).

The computational utility of Schubert calculus becomes apparent upon taking Poincaré duals and considering the cohomology ring. The cohomology of a flag manifold may be determined directly [**29**, **72**] and is torsion-free and generated by classes of degree 2 (in agreement with Theorem C.1). The Poincaré duals of the Schubert varieties therefore generate the integral cohomology. But, the dual of the class of X_w has degree $6 - 2\ell(w) = 2\ell(w_\mathrm{L} w)$ (for $\mathcal{F}_3$), where $w_\mathrm{L} = w_1 w_2 w_1 = w_2 w_1 w_2$ is the longest element (Appendix A.2) of S_3. It is therefore convenient to denote the cohomology class Poincaré dual to X_w by $p_{w_\mathrm{L} w}$, so that $p_w \in \mathsf{H}^{2\ell(w)}(\mathcal{F}_3; \mathbb{Z})$.

The cohomology class p_w may therefore be expressed as a polynomial in the degree 2 cohomology classes p_{w_1} and p_{w_2}. It is, however, traditional to express them in terms of the classes $x_1 = p_{w_1}$, $x_2 = p_{w_2} - p_{w_1}$, and $x_3 = -p_{w_2}$, which naturally appear in the computation of the cohomology ring of $\mathcal{F}_3$. The polynomial is then referred to as the Schubert polynomial. The Schubert polynomials for $\mathcal{F}_3$ take the form

$$p_e = 1 \qquad p_{w_1} = x_1 \qquad p_{w_2} = x_1 + x_2$$
$$p_{w_1 w_2} = x_1 x_2 \qquad p_{w_2 w_1} = x_1^2 \qquad p_{w_1 w_2 w_1} = x_1^2 x_2$$

(any explicit x_3-dependence may be supressed using $x_1 + x_2 + x_3 = 0$). The classes x_i are permuted by the natural action of S_3 on $\mathsf{H}^2(\mathcal{F}_3; \mathbb{Z})$, and the cohomology ring takes the

form (compare Theorem C.4):

$$\begin{aligned}\mathsf{H}^*(\mathcal{F}_3;\mathbb{Z}) &= \frac{\mathbb{Z}[x_1,x_2,x_3]}{\langle x_1+x_2+x_3, x_1x_2+x_1x_3+x_2x_3, x_1x_2x_3\rangle}\\ &= \frac{\mathbb{Z}[x_1,x_2]}{\langle x_1^2+x_1x_2+x_2^2, x_1^2x_2+x_1x_2^2\rangle}.\end{aligned} \tag{7.2.1}$$

Note that the p_w given above do actually generate this cohomology ring.

Consider now the computation of the geometric charges, $Q_F(\lambda)$ and $Q_{F'}(\lambda)$ given by Equations (7.1.1) and (7.1.3) respectively. It is first necessary to determine the image of the $\mathsf{U}(1)$-flux in $\mathsf{H}^2(\mathcal{F}_3;\mathbb{Z})$ as a linear combination of the x_i. This is achieved by recalling the periods of the $\mathsf{U}(1)$-flux over the 2-spheres $S_i = X_{w_i}$. For the quantum-shifted flux,

$$\lambda_1+1 = \int_{X_{w_1}} F_\lambda = \int_{\mathcal{F}_3} F_\lambda \wedge p_{w_{\mathrm{L}}w_1} = \int_{\mathcal{F}_3} F_\lambda \wedge p_{w_1w_2} = \int_{\mathcal{F}_3} F_\lambda \wedge x_1x_2,$$

and similarly, $\lambda_2+1 = \int_{\mathcal{F}_3} F_\lambda \wedge x_1^2$. Putting $F_\lambda = ax_1+bx_2$ gives $a-b = \lambda_1+1$ and $b = \lambda_2+1$. That is,

$$F_\lambda = (\lambda_1+\lambda_2+2)\,x_1 + (\lambda_2+1)\,x_2.$$

It follows that $F'_\lambda = (\lambda_1+\lambda_2)\,x_1+\lambda_2 x_2$.

The quantum-shifted charge may now be computed:

$$\begin{aligned}Q_F(\lambda) &= \frac{1}{3!}\int_{\mathcal{F}_3}(ax_1+bx_2)^3 = \frac{1}{2}\int_{\mathcal{F}_3}\left(a^2b-ab^2\right)x_1^2x_2\\ &= \frac{1}{2}ab(a-b) = \frac{1}{2}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2).\end{aligned} \tag{7.2.2}$$

This is the dimension of the irreducible $\mathfrak{su}(3)$-module with highest weight λ, in agreement with the algebraic charge of Equation (4.1.2).

To compute the modified charge (of a regular brane), the Todd class of the tangent bundle of the flag manifold must be evaluated. Using the explicit expressions for the first few Todd polynomials given in Equation (C.2.4), this can be deduced from the Chern classes of this tangent bundle. Whilst these can probably be determined directly for flag manifolds, they follow[4] from Theorem C.5. It is easy to check that the Chern classes are given by

$$\begin{aligned}&\mathrm{c}_1(T(\mathcal{F}_3)) = 4x_1+2x_2,\\ &\mathrm{c}_2(T(\mathcal{F}_3)) = 6x_1^2+6x_1x_2,\\ \text{and}\qquad &\mathrm{c}_3(T(\mathcal{F}_3)) = 6x_1^2x_2.\end{aligned}$$

[4]It is necessary to compare the expression for F'_λ in terms of the x_i with that of Section 7.2.2. It should be evident that in the formalism of Appendix C.2, the x_i are represented by the weights of the $\mathfrak{su}(3)$-module with highest weight Λ_1.

The contributing part of the integrand of $Q_{F'}(\lambda)$ is

$$\begin{aligned}&\mathrm{Td}_3(\mathcal{F}_3)+\mathrm{Td}_2(\mathcal{F}_3)F'_\lambda+\frac{1}{2}\mathrm{Td}_1(\mathcal{F}_3)\left(F'_\lambda\right)^2+\frac{1}{3!}\mathrm{Td}_0(\mathcal{F}_3)\left(F'_\lambda\right)^3\\&\quad=\frac{1}{24}\mathrm{c}_1(\mathcal{F}_3)\mathrm{c}_2(\mathcal{F}_3)+\frac{1}{12}\left[\mathrm{c}_1(\mathcal{F}_3)^2+\mathrm{c}_2(\mathcal{F}_3)\right]F'_\lambda+\frac{1}{4}\mathrm{c}_1(\mathcal{F}_3)\left(F'_\lambda\right)^2+\frac{1}{6}\left(F'_\lambda\right)^3.\end{aligned}$$

Substituting the Chern classes gives

$$\begin{aligned}\frac{1}{24}\mathrm{c}_1(\mathcal{F}_3)\mathrm{c}_2(\mathcal{F}_3)&=x_1^2x_2,\\\frac{1}{12}\left[\mathrm{c}_1(\mathcal{F}_3)^2+\mathrm{c}_2(\mathcal{F}_3)\right]F'_\lambda&=\frac{3}{2}(\lambda_1+\lambda_2)x_1^2x_2,\\\frac{1}{4}\mathrm{c}_1(\mathcal{F}_3)\left(F'_\lambda\right)^2&=\frac{1}{2}\left(\lambda_1^2+4\lambda_1\lambda_2+\lambda_2^2\right)x_1^2x_2,\end{aligned}$$

and from Equation (7.2.2) that

$$\frac{1}{6}\left(F'_\lambda\right)^3=\frac{1}{2}\lambda_1\lambda_2(\lambda_1+\lambda_2)x_1^2x_2.$$

The modified charge is therefore

$$\begin{aligned}Q_{F'}(\lambda)&=\int_{\mathcal{F}_3}\left[1+\frac{3}{2}(\lambda_1+\lambda_2)+\frac{1}{2}\left(\lambda_1^2+4\lambda_1\lambda_2+\lambda_2^2\right)+\frac{1}{2}\lambda_1\lambda_2(\lambda_1+\lambda_2)\right]\\&=\frac{1}{2}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2),\end{aligned}\tag{7.2.3}$$

in agreement with the quantum-shifted charge and the algebraic charge.

7.2.2. The General Case. In this section, the geometric charges $Q_F(\lambda)$ and $Q_{F'}(\lambda)$ of Section 7.1 will be shown to coincide with the algebraic charge $Q_{\mathrm{alg}}(\lambda)$ of Section 4.1.2, for all regular, untwisted, symmetry-preserving branes in compact, connected, simply-connected, simple Lie groups. The computation which demonstrates the coincidence of these charge definitions can also, essentially, be found in [**24**], though there the motivation is of course purely mathematical.

It will be convenient to use the formalism introduced in Appendix C.2 in which the real cohomology of the space G/T is naturally represented by polynomials in the fundamental weights of G. Specifically, the fundamental weights $\{\Lambda_i\}_{i=1}^r$ will correspond to the basis of $\mathsf{H}^2(\mathsf{G}/\mathsf{T};\mathbb{R})$ dual (not Poincaré dual) to the basis $\{S_i\}_{i=1}^r$ of $\mathsf{H}_2(\mathsf{G}/\mathsf{T};\mathbb{Z})$ introduced in Section 6.3.6. With this choice, the cohomology class of the $\mathsf{U}(1)$-flux F_λ is represented in the most pleasing manner. For the periods of F_λ over the S_i are just the integers $\langle\lambda+\rho,\alpha_i^\vee\rangle$, the Dynkin labels of $\lambda+\rho$, so it follows that in this formalism, F_λ is identified with $\lambda+\rho$. Similarly, F'_λ is identified with λ.

THEOREM 7.2. *Let* G *be a compact, connected, simply-connected, simple Lie group with maximal torus* T*, and let* λ *denote an integral weight in the* interior *of the fundamental alcove of* G *at level k. Then, the quantum-shifted charge of the brane with worldvolume* $\mathcal{C}(\exp(2\pi\mathrm{i}(\lambda+\rho)/(k+\mathsf{h}^\vee)))$ *and the modified charge of the (regular) brane with*

worldvolume $\mathcal{C}(\exp(2\pi i\lambda/k))$ *coincide:*

$$\int_{\mathsf{G}/\mathsf{T}} e^{\lambda+\rho} = \int_{\mathsf{G}/\mathsf{T}} e^{\lambda}\mathrm{Td}\left(T(\mathsf{G}/\mathsf{T})\right).$$

PROOF. From the definition of the Todd class, Equation (C.2.3), it follows that the modified charge is

$$\begin{aligned} Q_{F'}(\lambda) &= \int_{\mathsf{G}/\mathsf{T}} \frac{e^{\lambda}}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})} \prod_{\alpha\in\Delta_+}\alpha \\ &= \int_{\mathsf{G}/\mathsf{T}} \frac{e^{\lambda}\prod_{\alpha\in\Delta_+}e^{\alpha/2}}{\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)} \prod_{\alpha\in\Delta_+}\alpha \\ &= \int_{\mathsf{G}/\mathsf{T}} e^{\lambda+\rho} \prod_{\alpha\in\Delta_+}\frac{\alpha}{e^{\alpha/2}-e^{-\alpha/2}}, \end{aligned} \tag{7.2.4}$$

where Δ_+ denotes the set of positive roots of G.

Consider the product in this last expression. As $x/(e^x-e^{-x})$ is an even function, any $\alpha\in\Delta_+$ could be replaced by its negative without changing the product. But, if $\alpha\in\Delta_+$, either $\alpha\in w(\Delta_+)$ or $-\alpha\in w(\Delta_+)$, but not both (for any $w\in\mathsf{W}$. It follows that the product is invariant under the action of W. By Theorem C.4, W-invariants of positive degree vanish in cohomology. This product defines an analytic power series (in the positive roots), so it follows that it is cohomologically equivalent to its constant term. By L'Hôpital's rule, this constant term is 1, hence

$$Q_{F'}(\lambda) = \int_{\mathsf{G}/\mathsf{T}} e^{\lambda+\rho} = Q_F(\lambda). \qquad \blacksquare$$

Therefore, the modification of the brane charge suggested in [**117**] is equivalent to taking into account the quantum shift of Section 7.1.1, at least when the (unshifted) brane is regular (λ was restricted to be in the interior of the fundamental alcove at level k so that the unshifted brane worldvolume was diffeomorphic to G/T). One might expect that an analogous calculation would extend this result to the case when the unshifted brane is singular. This extension is expected to be relatively straight-forward when the brane worldvolume is diffeomorphic to G/Z, where Z is the centraliser of a torus, as the cohomology of these spaces is well understood [**22**–**24**, **26**]. However, as noted in Proposition C.2, there exist singular (untwisted symmetry-preserving) branes where Z is not the centraliser of any (non-trivial) torus, and so these cases are expected to cause difficulty. In any case, the quantised branes are supposed to be all regular when the quantum shift is accounted for. In what follows, this viewpoint will be followed, so the singular calculation can be safely ignored. The coincident charges will both be referred to as the *geometric* brane charge:

$$Q_{\mathrm{geo}}(\lambda) = \int_{\mathsf{G}/\mathsf{T}} e^{\lambda+\rho} = \int_{\mathsf{G}/\mathsf{T}} e^{\lambda}\mathrm{Td}\left(T(\mathsf{G}/\mathsf{T})\right). \tag{7.2.5}$$

It remains to demonstrate that this geometric charge coincides with the algebraic charge.

LEMMA 7.3. *Let* G *be a compact, connected, simply-connected, simple Lie group with maximal torus* T*, let* W *be the Weyl group of* G*, and let* Δ_+ *denote the set of positive roots of* G*. Then, in the formalism of Appendix C.3,* $\prod_{\alpha\in\Delta_+}\alpha$ *is anti-invariant under the action of* W*, and is represented by* $|\mathsf{W}|$ *in the top cohomology group,* $\mathsf{H}^{2|\Delta_+|}(\mathsf{G}/\mathsf{T};\mathbb{R})=\mathbb{R}$*.*

PROOF. The anti-invariance follows from the fact that the length of $w\in\mathsf{W}$ is the number of negative roots in $w(\Delta_+)$. Thus,

$$w\left(\prod_{\alpha\in\Delta_+}\alpha\right)=(-1)^{\ell(w)}\prod_{\alpha\in\Delta_+}\alpha=\det w\prod_{\alpha\in\Delta_+}\alpha.$$

From Theorem C.5, $\prod_{\alpha\in\Delta_+}\alpha$ is the top Chern class of $T(\mathsf{G}/\mathsf{T})$ (as a complex vector bundle), which is the Euler class of $T(\mathsf{G}/\mathsf{T})$ (as a real vector bundle — see Appendix C.2). Thus,

$$\int_{\mathsf{G}/\mathsf{T}}\prod_{\alpha\in\Delta_+}\alpha=\int_{\mathsf{G}/\mathsf{T}}\mathrm{e}\left(T(\mathsf{G}/\mathsf{T})\right)=\chi\left(\mathsf{G}/\mathsf{T}\right),$$

the Euler characteristic of G/T. But G/T has no odd cohomology, and its homology classes are in bijection with the elements of W (Theorem C.1). Hence, $\chi(\mathsf{G}/\mathsf{T})=|\mathsf{W}|$. ∎

THEOREM 7.4. *Let* G *be a compact, connected, simply-connected, simple Lie group with maximal torus* T *and Lie algebra* $\mathfrak{g}$*. If* λ *denotes an integral weight in the fundamental alcove of* G *at level* k*, then the geometric charge of the brane with worldvolume* $\mathcal{C}\left(\exp\left(2\pi\mathrm{i}\left(\lambda+\rho\right)/\left(k+\mathsf{h}^\vee\right)\right)\right)$ *is just the dimension of the irreducible* $\mathfrak{g}$*-module with highest weight* λ*. Therefore, the geometric and algebraic charges coincide for untwisted symmetry-preserving D-branes.*

PROOF. By Theorem 7.2 and Equation (7.2.4), the geometric charge is

$$Q_{\mathrm{geo}}(\lambda)=\int_{\mathsf{G}/\mathsf{T}}\frac{e^{\lambda+\rho}}{\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)}\prod_{\alpha\in\Delta_+}\alpha.$$

By Lemma 7.3, the product $\prod_{\alpha\in\Delta_+}\alpha$ generates the top (real) cohomology group. If the prefactor $e^{\lambda+\rho}/\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)$ defined an analytic power series (in the fundamental weights), then the geometric charge could be computed by determining the constant term in this series. Alas, this prefactor clearly has a pole at 0.

However, this prefactor may be recognised as the character of the *Verma module* of $\mathfrak{g}$ with highest weight λ (Appendix A.3). Verma modules are infinite-dimensional, so it follows (again) that this power series is non-analytic. However, the character of the (finite-dimensional) irreducible module of highest weight λ is analytic, even polynomial, and is closely related to the Verma module character, which suggests how to proceed.

The only contributing part of the integrand is the component in the top cohomology group, and by Lemma 7.3, this component may be taken to be anti-invariant under the

action of W. It follows that[5]

$$\begin{aligned} Q_{\text{geo}}(\lambda) &= \frac{1}{|\mathsf{W}|}\sum_{w\in\mathsf{W}}\det w\int_{\mathsf{G}/\mathsf{T}} w\left(\frac{e^{\lambda+\rho}}{\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)}\prod_{\alpha\in\Delta_+}\alpha\right)\\ &= \frac{1}{|\mathsf{W}|}\int_{\mathsf{G}/\mathsf{T}}\sum_{w\in\mathsf{W}}\det w\, w\left(e^{\lambda+\rho}\right)\prod_{\alpha\in\Delta_+}\frac{\alpha}{e^{\alpha/2}-e^{-\alpha/2}}\\ &= \frac{1}{|\mathsf{W}|}\int_{\mathsf{G}/\mathsf{T}}\frac{\sum_{w\in\mathsf{W}}\det w\, e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)}\prod_{\alpha\in\Delta_+}\alpha, \end{aligned}$$

since $\prod_{\alpha\in\Delta_+}\alpha/\left(e^{\alpha/2}-e^{-\alpha/2}\right)$ was noted to be W-invariant in the proof of Theorem 7.2. The prefactor of $\prod_{\alpha\in\Delta_+}\alpha$ may now be recognised as the character of the irreducible module of highest weight λ. Since this prefactor is analytic and $\prod_{\alpha\in\Delta_+}\alpha$ generates the top cohomology group, the only contribution to the integral comes from the constant term of the prefactor. Thus,

$$Q_{\text{geo}}(\lambda) = \frac{1}{|\mathsf{W}|}\int_{\mathsf{G}/\mathsf{T}}\chi_\lambda(0)\prod_{\alpha\in\Delta_+}\alpha = \dim(\lambda)\,\frac{1}{|\mathsf{W}|}\int_{\mathsf{G}/\mathsf{T}}\prod_{\alpha\in\Delta_+}\alpha = \dim(\lambda),$$

by Lemma 7.3 again. This is exactly the algebraic definition of the charge. ∎

7.3. Charge Group Constraints

7.3.1. Ambiguities in λ. Having established that the geometric brane charges defined in Section 7.1 may be identified with the algebraic brane charge discussed in Section 4.1.2, it remains to consider any constraints that the geometric definitions may impose upon this charge. Recall that the algebraic charge was dynamically constrained by the fusion rules, Equation (4.1.3), leading to the interpretation that these charges are only defined modulo some integer x. The charge group for untwisted symmetry-preserving branes was then algebraically determined as $\mathbb{Z}_x$, and for $\mathsf{SU}(2)$, x was determined to be $k+2$ in Section 4.2.1. In Section 7.1.3, an ambiguity in the definition of the $\mathsf{SU}(2)$ geometric brane charge was also shown to imply that these charges were only consistently defined modulo $k+2$. That is, topological consistency also predicts that the charge group for untwisted symmetry-preserving branes in $\mathsf{SU}(2)$ is $\mathbb{Z}_{k+2}$. This suggests that it would be very interesting to compare the charge groups predicted by the dynamical fusion constraints with those predicted by the purely topological constraints arising through ambiguities in the geometric charge definition.

Consider therefore the geometric definition of brane charge:

$$Q_{\text{geo}}(\lambda) = \int_{\mathsf{G}/\mathsf{T}} e^{\lambda+\rho} = \dim(\lambda) \qquad (\text{mod } x).$$

The branes are labelled by an integral weight λ in the fundamental alcove at level k, so the shifted weight $\lambda+\rho$ lies in the interior of the fundamental alcove at level $k+$

[5]Alternatively, one may observe that G/T represents a regular conjugacy, and that conjugacy classes are preserved by W. The anti-invariance is manifested by the orientation-reversing nature of the Weyl reflections.

$\mathsf{h}^\vee$. Geometrically, this labelling arises from the identification of the brane worldvolume with the conjugacy class $\mathcal{C}(h_\lambda)$, where $h = \exp(2\pi i(\lambda+\rho)/(k+\mathsf{h}^\vee))$. Insisting that λ belongs to the fundamental alcove at level k fixes it uniquely, but it is clear that this is but a convenient choice.

Indeed, it should first be noted that describing the brane worldvolume as $\mathcal{C}(h_\lambda)$ only determines $h_\lambda \in \mathsf{T}$ up to the action of the Weyl group W. For determining the $g \in \mathsf{G}$ such that $gh_\lambda g^{-1} \in \mathsf{T}$ is equivalent to solving $h_\lambda \in \mathrm{AD}\left(g^{-1}\right)\mathsf{T}$ for g. The image of a maximal torus under an inner automorphism is another maximal torus, but h_λ is regular, and therefore belongs to a single maximal torus (Appendix A.1). It follows that $\mathrm{AD}\left(g^{-1}\right)\mathsf{T} = \mathsf{T}$, so $g \in \mathsf{N}(\mathsf{T})$. As $\mathsf{W} = \mathsf{N}(\mathsf{T})/\mathsf{T}$,

$$\mathcal{C}(h_\lambda) \cap \mathsf{T} = \mathsf{W}(h_\lambda),$$

proving the claim.

Descending to the Lie algebra, it follows that $\lambda+\rho \in \mathfrak{t}^*_{\mathbb{R}}$ is only determined up to the usual W-action. Therefore, λ is determined up to the shifted W-action. It follows that for the geometric charge to be well-defined, the following constraint must be imposed[6]:

$$\dim(w\cdot\lambda) = \det w\ \dim(\lambda) \qquad (\mathrm{mod}\ x) \tag{7.3.1}$$

for each integral weight λ in the fundamental alcove at level k and each $w \in \mathsf{W}$. The $\det w$ factor arises as w may reverse the orientation of the brane manifold, contributing a relative sign to the integration. However, it follows easily from Equation (A.3.6) that these constraints on the geometric brane charge are automatically accounted for. That is, these constraints are satisfied identically (over $\mathbb{Z}$).

To derive some more interesting constraints, recall that $\ker\exp = 2\pi i\mathsf{Q}^\vee$ (since G is assumed simply-connected). It follows that h_λ only determines $(\lambda+\rho)/(k+\mathsf{h}^\vee)$ up to $\mathsf{Q}^\vee$, hence λ up to translation by an element of $(k+\mathsf{h}^\vee)\mathsf{Q}^\vee$. Together with the W-ambiguity, this shows that the brane only determines λ up to the shifted action of the *affine* Weyl group, $\widehat{\mathsf{W}}_k$ (at level k). Hence for the geometric charge to be well-defined,

$$\dim(\widehat{w}\cdot\lambda) = \det\widehat{w}\ \dim(\lambda) \qquad (\mathrm{mod}\ x), \tag{7.3.2}$$

for each $\widehat{w} \in \widehat{\mathsf{W}}_k$ and each integral weight λ in the fundamental alcove at level k (hence each integral weight in the interior of a shifted affine alcove at level $k+\mathsf{h}^\vee$). Note however, that the interesting constraints are generated by the cases where $\widehat{w}$ is restricted to be a pure translation by $(k+\mathsf{h}^\vee)\alpha_i^\vee$.

[6]Note that if the weight λ labelling the brane is chosen outside the fundamental chamber, then the interpretation of the geometric charge as the dimension of the corresponding highest weight $\mathfrak{g}$-module is lost. Instead, the proof of Theorem 7.4 gives the geometric charge as the constant term of the expression $\chi_\lambda(0)$ obtained by formally substituting λ into the Weyl character formula, Equation (A.3.3). It is convenient to disregard the fact that this is not a dimension, and instead redefine dim as a function

$$\dim\colon \mathsf{P} \to \mathbb{Z}, \qquad \dim(\lambda) = \chi_\lambda(0),$$

where χ_λ is the formal expression given in Weyl's character formula.

As always, it is instructive to analyse these constraints when $\mathsf{G} = \mathsf{SU}(2)$. The generating constraints take the form

$$\begin{aligned} a+1 &= \dim(a\Lambda_1) = \dim\left(a\Lambda_1 + (k+2)\,\alpha_1^\vee\right) \\ &= \dim\left((a+2(k+2))\,\Lambda_1\right) = a+1+2(k+2) \qquad (\text{mod } x), \end{aligned}$$

where $a+1 \in \mathbb{Z}$ does not divide $k+2$. These equations therefore all reduce to the statement $2(k+2) = 0 \pmod{x}$, hence they predict a charge group $\mathbb{Z}_{2(k+2)}$. This does not contradict the computation of Section 7.1.3 (which predicted $\mathbb{Z}_{k+2}$) as the result there was derived using a different set of constraints, and it has not been claimed that the constraints derived in this section are exhaustive. Indeed, this mismatch indicates that the constraints given by Equation (7.3.2) are not exhaustive.

7.3.2. Boundary Constraints. Of course, one can ask whether the affine constraints, Equation (7.3.2), should be augmented. The integral weight in these constraints is restricted to the interior of the shifted affine alcoves because only these weights correspond to an allowed D-brane in the fully quantised theory (Sections 6.3.6 and 7.1.1). There are therefore no (explicit) constraints on the weights lying on the shifted affine alcove boundaries. It is extremely tempting to declare constraints on these boundary weights λ to the effect that $\dim(\lambda) = 0 \pmod{x}$. For such weights correspond to no brane, and the absence of a brane must surely give zero brane charge. *This reasoning is facetious.* These boundary weights do not correspond to the absence of a brane — they correspond to the absence of the entire theory! There is no action discussed in Chapter 6 which corresponds to such a boundary weight, so these weights cannot be honestly assigned a brane charge. Boundary weights should instead be viewed as present only by convenience in the above analysis. One may assign them a number $\dim(\lambda)$, but this cannot be interpreted as a charge in any sense.

However, this does not mean that one cannot *derive* constraints on the boundary weights from Equation (7.3.2). The obvious identity (over $\mathbb{Z}$),

$$\dim(\lambda)\dim(\mu) = \sum_\nu N_{\lambda\mu}{}^\nu \dim(\nu)$$

(where $N_{\lambda\mu}{}^\nu$ represents the tensor-product coefficients), is useful for this purpose. For example, in $\mathsf{SU}(2)$, the identity

$$\dim(a\Lambda_1) = \dim((a-1)\Lambda_1)\dim(\Lambda_1) - \dim((a-2)\Lambda_1),$$

which follows from the familiar tensor-product rules, implies that

$$\begin{aligned} \dim((a+k+2)\Lambda_1) - \dim(a\Lambda_1) = {} & [\dim((a+k+1)\Lambda_1) - \dim((a-1)\Lambda_1)]\dim(\Lambda_1) \\ & - [\dim((a+k)\Lambda_1) - \dim((a-2)\Lambda_1)]. \end{aligned}$$

If $a\Lambda_1$ is a boundary weight and $k > 0$, it follows that $(a-1)\Lambda_1$ and $(a-2)\Lambda_1$ are not boundary weights, so $\dim((a+k+2)\Lambda_1) = \dim(a\Lambda_1) \pmod{x}$. Therefore, the

constraints, Equation (7.3.2), *algebraically generate* the corresponding constraint on the boundary weights, at least for $\mathsf{SU}(2)$. This result seems to persist for general G.

CONJECTURE 7.1. *When the level is sufficiently large, the constraints given by Equation (7.3.2) for integral weights in the interior of the shifted affine alcoves, algebraically generate identical "constraints" for boundary weights.*

"Sufficiently large", in this conjecture, appears to be about 2. Indeed, numerical experimentation at low rank and level suggests that the interior constraints generate the corresponding boundary constraints except when the group is $\mathsf{Spin}(r)$ or E_r, with rank $r = 2^n$, $n \geqslant 2$, and the level is 1.

The relevance of these boundary constraints is seen by recalling the (unjustified) temptation to set $\dim(\lambda) = 0 \pmod{x}$ when λ is a boundary weight. Assuming Conjecture 7.1, it follows now from the existence of an affine Weyl reflection $\widehat{w}$ fixing λ that

$$\dim(\lambda) = \det\widehat{w}\,\dim(\widehat{w}\cdot\lambda) = -\dim(\lambda) \qquad (\mathrm{mod}\ x).$$

Hence, $2\dim(\lambda) = 0 \pmod{x}$. Note however, that a weight λ on a shifted *chamber* boundary must have $\dim(\lambda) = 0$ identically. This follows from the identities over $\mathbb{Z}$, Equation (7.3.1), and the fact that such weights are fixed by the shifted action of a (finite) Weyl reflection. It is now clear that any boundary weight λ, which is the $(k+\mathsf{h}^\vee)\mathsf{Q}^\vee$-translate of a weight on a shifted chamber boundary, satisfies $\dim(\lambda) = 0 \pmod{x}$. Exactly which weights are such translates is the content of the following result.

LEMMA 7.5. *If* G *is* not *symplectic, then every boundary weight is the* $(k+\mathsf{h}^\vee)\mathsf{Q}^\vee$*-translate of a weight on a shifted chamber boundary. If* G *is symplectic, then a boundary weight* λ *must satisfy either*

$$(\alpha, \lambda+\rho) \in \mathbb{Z}, \qquad \textit{for some short root } \alpha,$$
$$\textit{or } (\alpha, \lambda+\rho) \in 2\mathbb{Z}, \qquad \textit{for some long root } \alpha,$$

to be such a translate.

PROOF. Let μ be on a shifted chamber boundary, so $(\alpha, \mu+\rho) = 0$ for some root α. λ is a $(k+\mathsf{h}^\vee)\mathsf{Q}^\vee$-translate of μ if and only if

$$(\alpha, \lambda+\rho) = (\alpha, \lambda-\mu) \in \left(k+\mathsf{h}^\vee\right)\left\langle\alpha, \mathsf{Q}^\vee\right\rangle.$$

To determine the subgroup $\langle\alpha, \mathsf{Q}^\vee\rangle \subseteq \mathbb{Z}$, note that α is transformed into a simple root α_i (of the same length) by some element of W. As $\mathsf{Q}^\vee$ is W-invariant, $\langle\alpha, \mathsf{Q}^\vee\rangle = \langle\alpha_i, \mathsf{Q}^\vee\rangle = m_i\mathbb{Z}$, where

$$m_i = \gcd\left\{\left\langle\alpha_i, \alpha_j^\vee\right\rangle : j = 1, \ldots, \mathrm{rank}\,\mathsf{G}\right\}.$$

Perusing the entries of the Cartan matrices leads to the conclusion that $m_i = 1$ unless G is symplectic and α_i (and hence α) is long, in which case $m_i = 2$. Note that in this latter case, $\alpha/2$ is an element of the weight lattice. ∎

Note that $\mathsf{SU}(2) = \mathsf{Sp}(2)$ should be considered to be symplectic for the purposes of these results. This lemma guarantees that if Conjecture 7.1 holds, and G is not symplectic, then any boundary weight λ necessarily satisfies $\dim(\lambda) = 0 \pmod{x}$. This conclusion is extremely interesting as it implies the (dynamical) fusion constraints on the algebraic charge, Equation (4.1.3), of Fredenhagen and Schomerus.

PROPOSITION 7.6. *Suppose that* G *is not symplectic. Then, the geometric constraints, Equation (7.3.2), and Conjecture 7.1 together imply the algebraic constraints*

$$\dim(\lambda)\dim(\mu) = \sum_{\nu\in\widehat{\mathsf{P}}_k} \mathcal{N}_{\lambda\mu}{}^{\nu} \dim(\nu) \qquad (\mathrm{mod}\ x),$$

where $\mathcal{N}_{\lambda\mu}{}^{\nu}$ *denotes the fusion coefficients, and* $\widehat{\mathsf{P}}_k$ *denotes the set of integral weights in the fundamental alcove at level* k.

PROOF. The geometric constraints and the Kac-Walton formula, Proposition 5.2, give

$$\sum_{\nu\in\widehat{\mathsf{P}}_k} \mathcal{N}_{\lambda\mu}{}^{\nu} \dim(\nu) = \sum_{\nu\in\widehat{\mathsf{P}}_k} \sum_{\widehat{w}\in\widehat{\mathsf{W}}_k} \det\widehat{w}\, N_{\lambda\mu}{}^{\widehat{w}\cdot\nu} \dim(\nu)$$

(where $N_{\lambda\mu}{}^{\nu}$ denote the tensor product coefficients)

$$\begin{aligned} &= \sum_{\nu\in\widehat{\mathsf{P}}_k} \sum_{\widehat{w}\in\widehat{\mathsf{W}}_k} N_{\lambda\mu}{}^{\widehat{w}\cdot\nu} \dim(\widehat{w}\cdot\nu) \qquad (\mathrm{mod}\ x) \\ &= \sum_{\nu} N_{\lambda\mu}{}^{\nu} \dim(\nu) \qquad (\mathrm{mod}\ x), \end{aligned}$$

where the sum is over all integral weights not on a shifted boundary at level $k+\mathsf{h}^\vee$. As G is not symplectic, $\dim(\nu) = 0 \pmod{x}$ when ν is on such a boundary, hence these boundary weights may be included in the sum without affecting its value. It follows that

$$\sum_{\nu\in\widehat{\mathsf{P}}_k} \mathcal{N}_{\lambda\mu}{}^{\nu} \dim(\nu) = \sum_{\nu\in\mathsf{P}} N_{\lambda\mu}{}^{\nu} \dim(\nu) = \dim(\lambda)\dim(\mu) \qquad (\mathrm{mod}\ x). \qquad \blacksquare$$

Therefore, the dynamical fusion constraints are consequences of the purely topological constraints given by Equation (7.3.2), at least when G is not symplectic. It follows that the charge group $\mathbb{Z}_x$ predicted by these affine constraints has the property that x divides that given by the fusion constraints (Equation (4.2.11)). In fact, the fusion ideal contains all anti-invariant polynomials under $\widehat{\mathsf{W}}_k$ (Proposition 5.5), so the fusion constraints, in turn, imply the affine constraints. That is, when G is not symplectic, the affine constraints are equivalent to the fusion constraints (assuming Conjecture 7.1). Indeed, in this case, numerical computation at low rank and level indicates that the parameter x derived from the affine constraints coincides with that derived from the fusion constraints.

When G is symplectic, the fusion constraints may be strictly stronger than the affine constraints. This is exemplified by $\mathsf{G} = \mathsf{SU}(2)$ where it has already been noted in Section 7.3.1 that the affine constraints predict $x = 2(k+2)$ (the fusion constraints predict

$x = k+2$). In this case, $\lambda = (k+1)\Lambda_1$ provides an example of a boundary weight satisfying $2\dim(\lambda) = 0 \pmod{2(k+2)}$ but $\dim(\lambda) \neq 0 \pmod{2(k+2)}$. Generally, numerical computation for the symplectic groups $\mathsf{Sp}(2r)$ at low rank r and low level suggests that the affine constraints predict the same charge group as the fusion constraints, except when r is a power of 2 and $k+\mathsf{h}^\vee$ a multiple of r. Then, the affine prediction for x is twice the fusion prediction.

It was, however, noted in Section 7.3.1 that the affine constraints are not expected to be exhaustive, even among the geometric constraints. There are other such constraints, and these were shown in Section 7.1.3 to predict $x = k+2$ for $\mathsf{SU}(2)$, in agreement with the fusion constraints. It remains to elucidate and analyse these constraints in the general case.

7.3.3. Ambiguities in F_λ. Recall from Section 7.1.2 that the $\mathsf{U}(1)$-flux F_λ entering the geometric brane charge, Equation (7.1.1), was determined by constructing a complex $\mathfrak{C}_\lambda$. This construction involved attaching 3-cells to the regular conjugacy class describing the brane worldvolume along a basis of homology 2-spheres. It was noted that choosing homotopically different 3-cells results in a cohomologically different $\mathsf{U}(1)$-flux. These choices led to the realisation that the periods of F_λ are only well-defined modulo $k+\mathsf{h}^\vee$.

In Section 7.2.2, it was shown that in the formalism of Appendix C.2, F_λ may be identified with $\lambda+\rho$. The ambiguity in the periods of F_λ now implies that this identification is only valid modulo translations by $(k+\mathsf{h}^\vee)\mathsf{P}$. This should be compared with the ambiguities of Section 7.3.1 where it was shown that the weight λ labelling the brane worldvolume is only well-defined modulo translations by $(k+\mathsf{h}^\vee)\mathsf{Q}^\vee$. Since $\mathsf{Q}^\vee \subseteq \mathsf{P}$, it follows that these weight lattice ambiguities lead to stronger constraints than the affine constraints of Equation (7.3.2). Combining with the identities of Equation (7.3.1), these weight lattice constraints may be expressed as

$$\dim(\tilde{w}\cdot\lambda) = \det\tilde{w}\ \dim(\lambda) \qquad (\mathrm{mod}\ x), \tag{7.3.3}$$

where $\tilde{w}$ is an element of the group $\widetilde{\mathsf{W}}_k = \mathsf{W} \ltimes \mathsf{P}$ acting by $\tilde{w}\cdot\lambda = w\cdot\lambda + (k+\mathsf{h}^\vee)\mu$ (where $w \in \mathsf{W}$ and $\mu \in \mathsf{P}$).

Of course, it was these weight lattice constraints that were used in Section 7.1.3 (and indeed, [**6**, **59**, **117**]) to predict the charge group, $\mathbb{Z}_{k+2}$, for $\mathsf{SU}(2)$, in full agreement with the prediction of the fusion constraints, Equation (4.1.3). It should also be noted that these constraints correspond exactly to the *weight lattice* charge symmetries observed empirically in Section 4.3.1. There it was noticed that the algebraic brane charges determined by the fusion constraints seemed to be invariant under weight lattice translations, except when the group was symplectic. In particular, when $\mathsf{G} = \mathsf{Sp}(2r)$ with r *not* a power of 2 (and various equispaced k). Indeed, the weight lattice constraints were remarked to predict a charge group $\mathbb{Z}_\xi$ with ξ *half* that predicted by the fusion constraints (in these cases), so it follows that these constraints are strictly stronger than the fusion constraints. Moreover, the charge group predicted by the weight lattice constraints (for a general group G)

admits an aesthetically pleasing universal formula, namely $\mathbb{Z}_\xi$ with Equation (4.3.7):

$$\xi = \frac{k+\mathsf{h}^\vee}{\gcd\{k+\mathsf{h}^\vee, y\}}, \qquad \text{where } y = \operatorname{lcm}\{1,2,\ldots,\mathsf{h}-1\},$$

and h is the Coxeter number of $\mathfrak{g}$.

It appears then that brane charge groups *are* determined by constraints which are stronger (for the symplectic groups) than the fusion constraints. This therefore explains the "mysterious" weight lattice charge symmetries that were noted in Section 4.3.1, which could not be related to the centre of the group. It also has a purely aesthetic advantage in that it brings the symplectic groups in line with a universal formula for the charge groups, Equation (4.3.7). However, this conclusion was noted already in Section 4.3.2 to be in conflict with the popular wisdom, which holds that brane charges are classified by twisted K-theory (which is isomorphic to the fusion ring). The conflict arises because the torsion order of the K-group ${}^{k+\mathsf{h}^\vee}\mathsf{K}^*(\mathsf{G})$ is known to agree with the charge group parameter x derived from the fusion constraints (and given by Equation (4.2.11) and Table 4.1).

One conclusion that might be drawn from these results is therefore that the popular wisdom is incorrect, and that brane charges are not classified by twisted K-theory after all. Indeed, there are already signs in the literature that more esoteric abelian groups may be relevant in classifying the branes of general string theories [**109**]. However, there is an alternative possibility. Recall that the charge group $\mathbb{Z}_\xi$ for untwisted symmetry-preserving branes is but a subgroup of the full brane charge group, which is supposed to be given by ${}^{k+\mathsf{h}^\vee}\mathsf{K}^*(\mathsf{G}) \cong \mathbb{Z}_x^{\oplus 2^{r-1}}$ (see Section 4.3.2). The conflict between ξ and x may be resolved by the following supposition:

> When G is symplectic and r is not a power of two, suppose that the inclusion $\mathbb{Z}_\xi \hookrightarrow {}^{k+\mathsf{h}^\vee}\mathsf{K}^*(\mathsf{G})$ maps the generator of $\mathbb{Z}_\xi$ to *twice* a generator of ${}^{k+\mathsf{h}^\vee}\mathsf{K}^*(\mathsf{G})$.

This supposition is of course, pure speculation. One might further speculate that such a supposition may be a consequence of the fact noted in the proof of Lemma 7.5 that symplectic groups are uniquely distinguished in possessing roots which are twice an integral weight. In any case, one should be able to test this supposition by investigating sets of branes which *generate* the full group ${}^{k+\mathsf{h}^\vee}\mathsf{K}^*(\mathsf{G})$. At this time, however, very little is known regarding such generating branes.

Conclusions

In this thesis, charge groups were computed for the untwisted symmetry-preserving D-branes of the Wess-Zumino-Witten models over all compact, connected, simply-connected, simple Lie groups. This computation was carried out twice using quite different methods, with *a priori* quite different charge definitions. However, both methods led to the formulation of constraints on the charges, which were then evaluated to determine the form of the charge group. The results of these evaluations for the different methods turned out to be in almost complete agreement, with only a discrepancy by a factor of two in a small number of cases. The thesis finished with a possible resolution of this discrepancy.

The first method employed in the computation of the brane charge groups is due to Fredenhagen and Schomerus [**63**]. This is based on a proposed dynamical process for branes called condensation. In the formalism of boundary conformal field theory, a brane was identified with a boundary condition. A consistency condition on these boundary conditions, Equation (3.2.9), leads to constraints on the (untwisted symmetry-preserving) brane charges, given by Equation (4.1.3). Using an explicit presentation of the fusion ring, I evaluated these constraints for the Lie groups $\mathsf{SU}(r+1)$ and $\mathsf{Sp}(2r)$, and rigorously determined the (largest) charge group they imply. For the other compact, connected, simply-connected, simple Lie groups, I had to resort to extensive numerical computations which suggested the form of the corresponding charge groups. The result was that the charge groups are of the form $\mathbb{Z}_x$, where x is given by Equation (4.2.11) and Table 4.1.

These rigorous computations for $\mathsf{SU}(r+1)$ and $\mathsf{Sp}(2r)$ relied on the well known fact that their fusion rings may be described by a fusion potential. Unfortunately, I have found no genuine proofs of this fact in the literature, despite many claims to the contrary. I therefore gave two rigorous proofs of this fact. The first relies on ideas from commutative algebra and appears to be quite different from the usual "proofs" in the literature. It shows that that the fusion potentials correctly describe the corresponding fusion rings when the scalar ring is $\mathbb{Z}$ (which is required for the application to brane charge groups). The second proof may be viewed as a completion of the proofs in the literature, in that it is based on the same idea, but it only holds when the scalar ring is $\mathbb{C}$.

The second method for determining brane charge groups relies only on analysing the (geometric) definition of the charges. That is, no mention of any dynamical processes is made. Just as ambiguities in the Wess-Zumino-Witten actions leads to quantisation phenomena (Chapter 6), ambiguities in the brane charge definitions lead to constraints on the charges which must be satisfied if they are to be well-defined. Such ambiguities

had been analysed in the literature for $\mathsf{SU}(2)$, but the extension of this analysis to more general Lie groups requires a way to evaluate these brane charges.

After discussing the rôle of certain quantum shifts, I showed that by taking the (appropriate) shifts into account, the standard geometric D-brane charge of Polchinski coincides with the modified charge of Minasian and Moore. That is, the modification suggested by Minasian and Moore reduces, in the special case of these Wess-Zumino-Witten models, to a simple quantum shift. As far as I am aware, the quantum shifts used above are empirically observed facts, so I think that this suggests that more work should be expended in trying to verify and generalise them (in the formalism of geometric quantisation for example). The modification of Minasian and Moore was motivated by the study of a certain quantum anomaly, so it would be of interest to investigate if more general anomalies can be circumvented in a full quantisation prescription. In particular, it would be interesting to see what further quantisation effects show up in more general models.

I then gave an explicit, and quite general, computation of the modified charge of Minasian and Moore, showing that it reduced to the dynamical charge postulated by Fredenhagen and Schomerus. The rather satisfying conclusion is therefore that all three brane charges considered in this thesis in fact coincide.

I then investigate the ambiguities inherent in Polchinski's (equivalently, Minasian and Moore's) brane charge definition. The untwisted symmetry-preserving branes may be labelled by weights in the fundamental alcove, and I note that the choice of alcove leads to an ambiguity in the brane charges. This ambiguity implies the constraint that the brane charge must be invariant (up to a sign) under the action of the affine Weyl group on the alcoves. Using the Kac-Walton formula (Proposition 5.2), which relates the affine Weyl group and the fusion ring, I then demonstrated that these constraints are *equivalent* to the dynamical constraints of Fredenhagen and Schomerus, provided that the Wess-Zumino-Witten model was over a non-symplectic group. It follows that in these (non-symplectic) cases, this ambiguity in the charge definition leads to constraints which predict the same charge group as before!

When the underlying Lie group is symplectic, I found through numerical experimentation that these ambiguities predict a charge group of the form $\mathbb{Z}_\xi$ where ξ either agrees with, or is twice as large as, that given by Equation (4.2.11). However, there is a further ambiguity in the brane charge definitions, and it yields somewhat stronger constraints on the charge group. After taking these constraints into account, I found that the predicted charge group $\mathbb{Z}_\xi$ now has ξ agreeing with Equation (4.2.11) in the non-symplectic cases, and either agreeing with, or is *half*, that given by Equation (4.2.11) in the symplectic case. This is the discrepancy referred to above. I also noted that these stronger constraints precisely correspond to certain symmetries which were empirically observed in the charges of Fredenhagen and Schomerus in Section 4.3.1 (again excluding certain symplectic cases).

The research reported on in this thesis suggests various directions for generalisation. In particular, the *twisted* symmetry-preserving branes (Section 6.3.2) are obvious candidates for consideration. In [**76**], the charges of these twisted branes were studied using the dynamical constraints, Equation (4.1.1), of Fredenhagen and Schomerus. This is complicated by the fact that the NIMrep coefficients (see Proposition 3.3) appearing in these constraints no longer coincide with the fusion coefficients. A direct consequence of this is that these constraints no longer suggest the form of the brane charge (as in Equation (4.1.2)). Nevertheless, it was there guessed that the charge is given by the dimension of a (certain) twisted representation, and then shown that this guess satisfies the constraints (perhaps uniquely). It would be very interesting to try to *derive* this in the geometric approach to brane charges. Presumably, such a derivation would involve some fascinating mathematics and should shed some light on what is really going on.

Another obvious generalisation is to extend these results to Wess-Zumino-Witten models over more general groups. In particular, to relax the simply-connected assumption. Again, some investigations have been made [**37**, **62**, **75**], but the conclusions are not entirely convincing (and do not seem to agree). The situation appears to be even more delicate than the twisted case mentioned before, and it seems to me that a geometric understanding of the branes of the simplest case $\mathsf{SO}(3)$ would help to clarify our understanding.

There have also been attempts [**62**, **77**, **78**] to extend the Fredenhagen and Schomerus constraints to the non-symmetry-preserving branes of [**134**,**135**] (which do not correspond to an automorphism in Section 3.2.1). The idea here is to try to find enough branes to justify the multiplicity of $\mathbb{Z}_x$ in the appropriate twisted K-theory (see Section 4.3.2). Unfortunately, no attention seems to have paid so far to the question of whether these branes are actually independent (in a K-theoretic sense). For example, it has not even been shown that the twisted branes discussed above give charges independent of the untwisted branes. Similarly, no attention seems to have been paid to the multiplicative structure (generalising that of the fusion ring) for these more general branes, nor has the question of how to identify branes as (K-theoretic) generators of the charge group been addressed.

In view of these criticisms, my general feeling is that we are still very far from having anything more than a rudimentary understanding of the general state of the brane charge $\leftrightarrow$ twisted K-theory correspondence. Of course, that may change once the full result of Freed, Hopkins, and Teleman, rigorously linking twisted equivariant K-theory to the fusion ring, is finally proven (generalising that stated in Theorem 4.3) and understood.

APPENDIX A

Finite-Dimensional Simple Lie Algebras

This thesis will make almost constant use of (characteristic zero) Lie theory, both in concepts and calculations. In this appendix, some standard concepts and results are introduced for the finite-dimensional simple Lie algebras. This serves, partly, as an attempt to achieve some measure of completeness, but mostly to fix notation and convention. Useful references for this theory include [**15**, **30**, **69**, **73**, **96**].

A Lie algebra $\mathfrak{g}$ with Lie bracket $[\cdot,\cdot]$ is said to be *simple* if it is non-abelian[1] and has no non-trivial ideals. This section concerns itself with the theory of simple *complex* Lie algebras. The Lie algebras of interest in this thesis are actually the real Lie algebras which correspond to the compact simple (real) Lie groups considered in Appendix C. These real algebras are the *compact real forms* of the simple complex Lie algebras, and so all the theory developed below for complex Lie algebras is valid for them, after complexification. One therefore often talks about root vectors for the real Lie algebra $\mathfrak{su}(2)$ (for instance), despite the fact that said root vector is an element of $\mathfrak{sl}(2;\mathbb{C})$. The relationship between a simple complex Lie algebra and its compact real form will be discussed further at the end of Appendix A.1.

A.1. Basics

A homomorphism of $\mathfrak{g}$ into $\mathrm{End}\,V$, where V is some (complex) vector space, is called a *representation*. V is called a representation space or *module*. The archetypal example is the *adjoint* representation $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$, defined by $\mathrm{ad}(x)\,y = [x,y]$. A simple Lie algebra admits a symmetric bilinear form called the *Killing form*, given by

$$\kappa(x,y) = \frac{1}{I_{\mathrm{ad}}}\,\mathrm{tr}\left[\mathrm{ad}(x)\,\mathrm{ad}(y)\right], \tag{A.1.1}$$

where I_{ad} is a normalisation to be fixed later (Equation (A.1.3)). This form is also *associative* (or *invariant*), $\kappa([x,y],z) = \kappa(x,[y,z])$, and this property uniquely characterises it (up to normalisation) among symmetric bilinear forms.

If $\{t_a\}$ is a (vector space) basis of the simple Lie algebra $\mathfrak{g}$, then the entries of the matrix representing $\mathrm{ad}(t_a)$ (in this basis) are the *structure constants*: $[t_a,t_b] = \sum_c f_{abc}t_c$. It is sometimes useful to take this basis to be orthonormal with respect to the Killing form[2]. Then, $f_{abc} = \kappa([t_a,t_b],t_c)$, and associativity immediately implies that the structure constants are completely antisymmetric in a, b, and c (in this orthonormal basis). Note

[1] This serves to exclude the one-dimensional abelian algebra $\mathfrak{u}(1)$.

[2] It will be clear that such bases exist when compact real forms are discussed at the end of this section.

that it also follows that

$$\sum_{b,c} f_{abc} f_{cbd} = \sum_{b,c} f_{abc} f_{dcb} = \operatorname{tr}[\operatorname{ad}(t_a)\operatorname{ad}(t_d)] = I_{\mathrm{ad}}\kappa(t_a,t_d) = I_{\mathrm{ad}}\delta_{ad}. \tag{A.1.2}$$

A simple complex Lie algebra $\mathfrak{g}$ always contains elements $x \neq 0$ such that $\operatorname{ad}(x)$ is a diagonalisable (semisimple) endomorphism. A subalgebra consisting of such elements is called a *toral subalgebra*, and must be abelian. Choose a maximal toral subalgebra (also known as a *Cartan subalgebra*) $\mathfrak{t}$. Any chosen $\mathfrak{t}$ is also maximal abelian, and its dimension is independent of the choice, and is known as the *rank* of $\mathfrak{g}$, denoted by r. It follows that there is a (vector space) basis of $\mathfrak{g}$ consisting of simultaneous eigenvectors of the elements of $\operatorname{ad}(\mathfrak{t})$. The eigenvalues are therefore functions on $\mathfrak{t}$, hence elements of $\mathfrak{t}^*$:

$$\operatorname{ad}(x)\, e_\alpha = \langle \alpha, x\rangle\, e_\alpha \qquad \text{for all } x \in \mathfrak{t}.$$

(The pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$ will be denoted by $\langle\cdot,\cdot\rangle$.) When the eigenvectors e_α do not belong to the maximal toral subalgebra, $\alpha \in \mathfrak{t}^*$ is non-zero (by maximality), and is referred to as a *root* of $\mathfrak{g}$. The set of roots, also called the *root system*, of $\mathfrak{g}$ is denoted by Δ, and the e_α with $\alpha \neq 0$ are called *root vectors*. The root system gives a grading on $\mathfrak{g}$, meaning that $[e_\alpha, e_\beta]$ is proportional to $e_{\alpha+\beta}$ (assuming $\alpha+\beta \in \Delta$).

The root system Δ of a simple Lie algebra is an object that has been intensively studied. It is closed under negation. Furthermore, it turns out that one can choose a basis $\{\alpha_i : i = 1,2,\ldots,r\}$ of $\mathfrak{t}^*$ consisting of *simple roots* such that every root of $\mathfrak{g}$ has the form

$$\alpha = \sum_{i=1}^{r} m_i \alpha_i,$$

where the coefficients m_i are not only integers, but are either all positive or all negative (when non-zero). Such a choice of simple roots therefore partitions Δ into a set of *positive roots* Δ_+ and its negation Δ_-. This also induces a *triangular decomposition* of $\mathfrak{g}$ as a (vector space) direct sum,

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{t} \oplus \mathfrak{g}_+,$$

where $\mathfrak{g}_\pm$ are (nilpotent) subalgebras spanned by the root vectors e_α with $\alpha \in \Delta_\pm$. The *height* of a root α is defined to be the sum of the corresponding m_i. There is then a unique *highest root* θ, whose corresponding m_i are known as the *marks* of $\mathfrak{g}$, and will be denoted by a_i. The height of θ defines the *Coxeter number* h of $\mathfrak{g}$ by $\sum_i a_i = \mathsf{h} - 1$. The set of all linear combinations of the simple roots with integer coefficients defines the *root lattice*, denoted by Q.

Since the root system spans $\mathfrak{t}^*$, the action of an arbitrary $\lambda \in \mathfrak{t}^*$ on $\mathfrak{t}$ is determined. The Killing form is also non-degenerate when restricted to $\mathfrak{t}$, so it induces a canonical isomorphism $\iota : \mathfrak{t} \to \mathfrak{t}^*$ by $\langle \iota(x), y\rangle = \kappa(x,y)$ (for all $x,y \in \mathfrak{t}$). Lifting the Killing form to $\mathfrak{t}^*$ by this isomorphism gives an inner product $(\cdot,\cdot)$ on the real subspace $\mathfrak{t}^*_{\mathbb{R}}$ spanned by the roots. Using this inner product, the length of the roots of $\mathfrak{g}$ are found to have (at most) two possible values. The highest root θ is always a "long" root. It is convenient to normalise this inner product by setting $(\theta,\theta) = 2$. The image of $\mathfrak{t}^*_{\mathbb{R}}$ under ι^{-1} defines

the real subalgebra $\mathfrak{t}_\mathbb{R}$, which contains (among other things) the lattice Q^* which is dual to the root lattice.

One may repeat the decomposition of $\mathfrak{g}$ under the adjoint representation of $\mathfrak{t}$ for other representations. If $\pi\colon \mathfrak{g} \to \mathsf{End}\, V$ is a representation, V decomposes into (simultaneous) eigenspaces of $\pi(\mathfrak{t})$. The eigenvalues $\lambda \in \mathfrak{t}^*$ are called the *weights* of the representation. The roots of $\mathfrak{g}$ together with $0 \in \mathfrak{t}^*_\mathbb{R}$ (of multiplicity r) therefore constitute the weights of the adjoint representation. If the representation is irreducible (and all finite-dimensional representations are completely reducible), then there is a unique *highest weight*, meaning that all weights can be obtained from this one by (repeatedly) subtracting simple roots. The adjoint representation turns out to be irreducible with highest weight θ.

It turns out that the weights of every (finite-dimensional) representation may be expressed as linear combinations of r weights, called *fundamental weights* and denoted by Λ_i, with integer coefficients. These integer coefficients are called the *Dynkin labels* of the weight λ and are denoted by λ_i. The sum of the fundamental weights is called the *Weyl vector* ρ, which also happens to be related to the positive roots through

$$\rho = \sum_{i=1}^{r} \Lambda_i = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

Since roots are weights (and they form a spanning set), the fundamental weights span $\mathfrak{t}^*_\mathbb{R}$, which is accordingly called the *weight space*. The set of all linear combinations of the fundamental weights with integer coefficients defines the *weight lattice*, denoted by P. The lattice dual to the weight lattice is called the *coroot lattice* $\mathsf{Q}^\vee \subset \mathfrak{t}_\mathbb{R}$, and is spanned (over $\mathbb{Z}$) by the *simple coroots* $\alpha_i^\vee$, $i = 1, 2, \ldots, r$. The simple coroots and simple roots are related by

$$\alpha_i^\vee = \frac{2}{\|\alpha_i\|^2} \iota^{-1}(\alpha_i).$$

Extending this relation to all roots α defines the corresponding coroots $\alpha^\vee \in \mathsf{Q}^\vee$. In particular, one has the coroot $\theta^\vee = \iota^{-1}(\theta)$ whose decomposition into simple coroots defines the *comarks* $a_i^\vee$ by $\theta^\vee = \sum_i a_i^\vee \alpha_i^\vee$. The comarks in turn define the *dual Coxeter number* $\mathsf{h}^\vee$ by $\sum_i a_i^\vee = \langle \rho, \theta^\vee \rangle = \mathsf{h}^\vee - 1$.

The simple roots and simple coroots together define the entries of the *Cartan matrix* A by $A_{ij} = \left\langle \alpha_i, \alpha_j^\vee \right\rangle$. This $r \times r$ matrix in fact completely characterises a simple complex Lie algebra. The diagonal entries are all 2, and the off-diagonal entries are non-positive integers which are constrained in various ways. Analysing these constraints leads to the celebrated Cartan-Killing classification of the finite-dimensional complex simple Lie algebras. The allowed Cartan matrices may be conveniently displayed by exhibiting the corresponding graphs whose adjacency matrices are $2\,\mathrm{id} - A$. These graphs are called *Dynkin diagrams* and the complete list is given in Figure A.1. The restrictions on r given there are assumed because of the isomorphisms $\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{sp}(2)$, $\mathfrak{so}(5) \cong \mathfrak{sp}(4)$, and $\mathfrak{su}(4) \cong \mathfrak{so}(6)$, and the fact that $\mathfrak{so}(2) \cong \mathfrak{u}(1)$ and $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ are not simple. A simple Lie algebra is said to be *simply laced* if its Dynkin diagram is undirected.

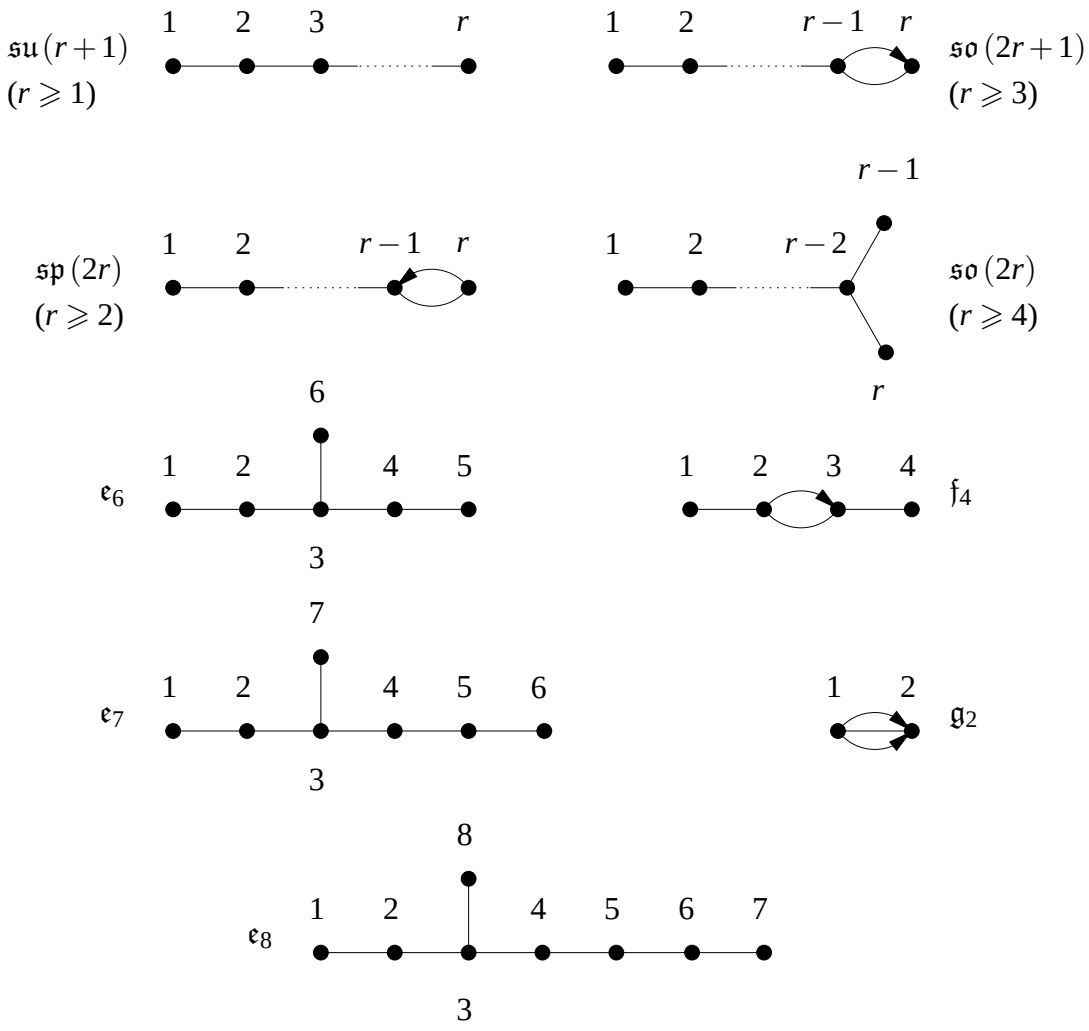

FIGURE A.1. The Dynkin diagrams of the simple complex Lie algebras, labelled by their compact real forms. The numbers indicate the ordering of the nodes (hence simple roots) used in this thesis, and the arrows always point from a long root to a short root.

Note that the rows of the Cartan matrix therefore consist of the Dynkin labels of the simple roots. It follows that $\det A = |\mathsf{P}/\mathsf{Q}|$. Similarly, one can consider the *symmetrised Cartan matrix* $A^\vee$, whose rows are the Dynkin labels of the simple coroots. Thus, $\det A^\vee = |\mathsf{P}/\mathsf{Q}^\vee|$. Since the dual of the coroot lattice is the weight lattice, the inverse of the symmetrised Cartan matrix is the matrix whose entries are $F_{ij} = \left(\Lambda_i, \Lambda_j\right)$. This is sometimes called the *quadratic form matrix*. Some other useful data is collected in Table A.1.

One can now tidy up a few details that will be used in the thesis. The normalisation of the Killing form (the following argument appears in [**110**]) given in Equation (A.1.1) is determined by the normalisation of the root lengths:

$$2 = (\theta, \theta) = \kappa\left(\theta^\vee, \theta^\vee\right) = \frac{1}{I_{\text{ad}}} \operatorname{tr}\left[\operatorname{ad}\left(\theta^\vee\right) \operatorname{ad}\left(\theta^\vee\right)\right].$$

$\mathfrak{g}$	θ	h $\mathsf{h}^\vee$	$\{a_i\}$ $\{a_i^\vee\}$	Exponents
$\mathfrak{su}(r+1)$	$\Lambda_1+\Lambda_r$	$r+1$	$\{1,\ldots,1\}$	$\{1,2,\ldots,r\}$
$\mathfrak{so}(2r+1)$ $(r\geqslant 2)$	Λ_2	$2r$ $2r-1$	$\{1,2,\ldots,2\}$ $\{1,2,\ldots,2,1\}$	$\{1,3,\ldots,2r-1\}$
$\mathfrak{sp}(2r)$	$2\Lambda_1$	$2r$ $r+1$	$\{2,\ldots,2,1\}$ $\{1,\ldots,1\}$	$\{1,3,\ldots,2r-1\}$
$\mathfrak{so}(2r)$ $(r\geqslant 3)$	Λ_2	$2r-2$	$\{1,2,\ldots,2,1,1\}$	$\{1,3,\ldots,2r-3,r-1\}$
$\mathfrak{e}_6$	Λ_6	12	$\{1,2,3,2,1,2\}$	$\{1,4,5,7,8,11\}$
$\mathfrak{e}_7$	Λ_1	18	$\{2,3,4,3,2,1,2\}$	$\{1,5,7,9,11,13,17\}$
$\mathfrak{e}_8$	Λ_1	30	$\{2,3,4,5,6,4,2,3\}$	$\{1,7,11,13,17,19,23,29\}$
$\mathfrak{f}_4$	Λ_1	12 9	$\{2,3,4,2\}$ $\{2,3,2,1\}$	$\{1,5,7,11\}$
$\mathfrak{g}_2$	Λ_1	6 4	$\{2,3\}$ $\{2,1\}$	$\{1,5\}$

TABLE A.1. The highest root, Coxeter and dual Coxeter number, marks and comarks, and exponents for the simple complex Lie algebras (labelled by their compact real forms). For the simply laced algebras, the Coxeter number coincides with the dual Coxeter number, as do the marks and comarks. They are therefore not repeated.

The eigenvalues of $\mathrm{ad}(\theta^\vee)$ are 0 on $\mathfrak{t}$ and $\langle\alpha,\theta^\vee\rangle=(\alpha,\theta)$ on e_α. It is not hard to show that $(\alpha,\theta)\in\{0,1\}$ for all $\alpha\in\Delta_+\setminus\{\theta\}$. Thus,

$$\mathrm{tr}\left[\mathrm{ad}\left(\theta^\vee\right)\mathrm{ad}\left(\theta^\vee\right)\right]=2\sum_{\alpha\in\Delta_+}(\alpha,\theta)^2=2\left[4+\sum_{\alpha\in\Delta_+\setminus\{\theta\}}(\alpha,\theta)\right]=2\left[2+(2\rho,\theta)\right]=4\mathsf{h}^\vee.$$

It follows that the normalisation of the Killing form is given by

$$I_{\mathrm{ad}}=2\mathsf{h}^\vee. \tag{A.1.3}$$

Equation (A.1.1) is a special case of a *representation independent* definition of the Killing form. In this definition, the adjoint representation ad is replaced by an arbitrary representation π. This new form is again bilinear, symmetric, and associative, hence by uniqueness must be a multiple of the Killing form:

$$\kappa(x,y)=\frac{1}{I_\pi}\mathrm{tr}\left[\pi(x)\,\pi(y)\right]. \tag{A.1.4}$$

The normalisation I_π is called the *Dynkin index* of the representation π, and may be evaluated with the help of the *quadratic Casimir* Q. This is an element of the *universal*

enveloping algebra of $\mathfrak{g}$, $\mathfrak{U}(\mathfrak{g})$, defined most simply by $Q = \sum_a t_a t_a$ where $\{t_a\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form. The quadratic Casimir commutes (in $\mathfrak{U}(\mathfrak{g})$) with every element of $\mathfrak{g}$, and so in an irreducible representation it acts as a multiple of the identity. Indeed, if π is the irreducible representation of highest weight λ, then

$$\pi(Q) = (\lambda, \lambda + 2\rho)\,\mathrm{id}. \tag{A.1.5}$$

Let $\dim(\lambda)$ denote the dimension of the irreducible representation of highest weight λ, and suppose that $\{t_a\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form. Then one has

$$\begin{aligned}\dim(\theta) = \sum_a \kappa(t_a, t_a) = \frac{1}{I_\pi}\sum_a \mathrm{tr}\left[\pi(t_a)\,\pi(t_a)\right] = \frac{1}{I_\pi}\mathrm{tr}\,\pi(Q) = \frac{(\lambda, \lambda+2\rho)\dim(\lambda)}{I_\pi} \\ \Rightarrow \qquad I_\pi = (\lambda, \lambda+2\rho)\frac{\dim(\lambda)}{\dim(\theta)}.\end{aligned} \tag{A.1.6}$$

Putting $\pi = \mathrm{ad}$ (so $\lambda = \theta$) recovers Equation (A.1.3).

As an example of how the Dynkin index may arise in calculations, consider the sum of the terms $(\lambda, \mu)(\mu, \nu)$, over all the weights μ of the irreducible representation π of highest weight Λ. Using Equation (A.1.4), this evaluates to

$$\begin{aligned}\sum_\mu (\lambda, \mu)(\mu, \nu) &= \sum_\mu \left\langle \mu, \imath^{-1}(\lambda)\right\rangle \left\langle \mu, \imath^{-1}(\nu)\right\rangle = \mathrm{tr}\left[\pi\left(\imath^{-1}(\lambda)\right)\pi\left(\imath^{-1}(\nu)\right)\right] \\ &= I_\pi \kappa\left(\imath^{-1}(\lambda), \imath^{-1}(\nu)\right) = I_\pi(\lambda, \nu).\end{aligned} \tag{A.1.7}$$

It remains to discuss the compact real form $\mathfrak{g}_{\mathrm{comp}}$ of the complex Lie algebra $\mathfrak{g}$. This is defined to be the real Lie algebra spanned by

$$\left\{\mathrm{i}\alpha_i^\vee, \mathrm{i}(e_\alpha + e_{-\alpha}), e_\alpha - e_{-\alpha} \colon i = 1, 2, \ldots, r;\ \alpha \in \Delta_+\right\}.$$

Noting that the coroots $\alpha^\vee$ may be used to normalise the root vectors by $[e_\alpha, e_{-\alpha}] = \alpha^\vee$, the structure coefficients of $\mathfrak{g}_{\mathrm{comp}}$ may be checked to be real. It also follows from this normalisation that

$$\kappa\left(e_\alpha, e_\beta\right) = \frac{2}{\|\alpha\|^2}\delta_{\alpha+\beta,0},$$

and so $-\kappa(\cdot,\cdot)$ may be checked to be a (positive definite) inner product on $\mathfrak{g}_{\mathrm{comp}}$. With respect to this inner product, $\mathrm{ad}(x)$ is skew-symmetric for all $x \in \mathfrak{g}_{\mathrm{comp}}$ by associativity, hence has purely imaginary eigenvalues. The subalgebra $\mathfrak{it}_{\mathbb{R}}$ therefore acts as a Cartan subalgebra for $\mathfrak{g}_{\mathrm{comp}}$, and its action decomposes $\mathfrak{g}_{\mathrm{comp}}$ into the zero eigenspace $\mathfrak{it}_{\mathbb{R}}$, and a two-dimensional eigenspace for each $\alpha \in \Delta_+$ corresponding to the pair of complex conjugate eigenvalues $\mathrm{i}\alpha$ and $-\mathrm{i}\alpha$.

A.2. Automorphisms of $\mathfrak{g}$

The automorphism group $\mathrm{Aut}\,\mathfrak{g}$ of a simple complex Lie algebra $\mathfrak{g}$ plays a vital rôle in the theory. An important class of automorphisms is induced by the group of symmetries of the Dynkin diagram (equivalently, of the Cartan matrix). A transformation of the "node"

$\mathfrak{g}$	$\mathsf{Out}\,\mathfrak{g}$	W	$\omega_+(1,2,\ldots,r)$
$\mathfrak{su}(r+1)$ $(r \geqslant 2)$	$\mathbb{Z}_2$	S_{r+1}	$(r,r-1,\ldots 1)$
$\mathfrak{so}(2r+1)$ $(r \geqslant 2)$	$\{\mathrm{id}\}$	$\mathsf{S}_r \ltimes \mathbb{Z}_2^r$	$(1,2,\ldots,r)$
$\mathfrak{sp}(2r)$ $(r \geqslant 1)$	$\{\mathrm{id}\}$	$\mathsf{S}_r \ltimes \mathbb{Z}_2^r$	$(1,2,\ldots,r)$
$\mathfrak{so}(8)$	S_3	$\mathsf{S}_4 \ltimes \mathbb{Z}_2^3$	$(1,2,\ldots,r)$
$\mathfrak{so}(2r)$ $(r \geqslant 5$, odd)	$\mathbb{Z}_2$	$\mathsf{S}_r \ltimes \mathbb{Z}_2^{r-1}$	$(1,2,\ldots,r-2,r,r-1)$
$\mathfrak{so}(2r)$ $(r \geqslant 6$, even)	$\mathbb{Z}_2$	$\mathsf{S}_r \ltimes \mathbb{Z}_2^{r-1}$	$(1,2,\ldots,r)$
$\mathfrak{e}_6$	$\mathbb{Z}_2$	—	$(5,4,3,2,1,6)$
$\mathfrak{e}_7$	$\{\mathrm{id}\}$	—	$(1,2,3,4,5,6,7)$
$\mathfrak{e}_8$	$\{\mathrm{id}\}$	—	$(1,2,3,4,5,6,7,8)$
$\mathfrak{f}_4$	$\{\mathrm{id}\}$	$\mathsf{S}_3 \ltimes \mathsf{S}_4 \ltimes \mathbb{Z}_2^3$	$(1,2,3,4)$
$\mathfrak{g}_2$	$\{\mathrm{id}\}$	D_6	$(1,2)$

TABLE A.2. The group of outer automorphisms $\mathsf{Out}\,\mathfrak{g}$, the Weyl group W (that of $\mathfrak{e}_r$ is too complicated to list), and the action of the conjugation automorphism $\omega_+ \in \mathsf{Out}\,\mathfrak{g}$ as a permutation of the nodes of the Dynkin diagram. S_n denotes the symmetric group on n elements, and $\mathsf{D}_n = \mathbb{Z}_2 \ltimes \mathbb{Z}_n$ the dihedral group (symmetry group of an n-sided regular polygon). Note that $\mathsf{Out}\,\mathfrak{su}(2) = \mathsf{Out}\,\mathfrak{sp}(2)$ is trivial.

labels which leaves the diagram invariant gives a corresponding permutation of the simple roots, and therefore of the simple coroots, root system, and root vectors. Because the Cartan matrix is invariant, these *Dynkin symmetries* act orthogonally on the weight space. Another (complementary) class of automorphisms consists of those of the form

$$\mathrm{Ad}(x) = e^{\mathrm{ad}(x)} \colon \mathfrak{g} \longrightarrow \mathfrak{g}$$

(recall that the exponential of an endomorphism of a finite-dimensional vector space is always defined [**105**]). The subgroup of $\mathsf{Aut}\,\mathfrak{g}$ that is generated by such automorphisms is called the group of *inner automorphisms*, $\mathsf{Int}\,\mathfrak{g}$. This is a normal subgroup, and the corresponding quotient group turns out to be finite. It is called the group of *outer automorphisms*, $\mathsf{Out}\,\mathfrak{g} = \mathsf{Aut}\,\mathfrak{g}/\mathsf{Int}\,\mathfrak{g}$, and turns out to be naturally isomorphic to the group of Dynkin symmetries introduced above. This group is listed for each $\mathfrak{g}$ in Table A.2.

The identification of the Dynkin symmetries with the outer automorphisms leads to a decomposition of $\mathsf{Aut}\,\mathfrak{g}$ as a *semidirect product*. A group H is the semidirect product of its subgroups K and N, written $\mathsf{H} = \mathsf{K} \ltimes \mathsf{N}$, if H is generated by K and N, N is normal, and $\mathsf{K} \cap \mathsf{N} = \{\mathrm{id}\}$. For then, normality means that any $h \in \mathsf{H}$ may be decomposed as $h = kn = n'k$, where $k \in \mathsf{K}$ and $n, n' \in \mathsf{N}$, and the trivial intersection means that these decompositions are unique. H therefore has the structure $\mathsf{K} \times \mathsf{N}$ as a set, with multiplication defined by

$$(k_1, n_1)(k_2, n_2) = \left(k_1 k_2, \left[k_2^{-1} n_1 k_2\right] n_2\right).$$

It follows from this abstract definition that $\mathsf{Aut}\,\mathfrak{g} = \mathsf{Out}\,\mathfrak{g} \ltimes \mathsf{Int}\,\mathfrak{g}$.

Recall that before decomposing $\mathfrak{g}$ to define roots, one had to *choose* a maximal toral subalgebra $\mathfrak{t}$. This arbitrariness does not cause any concern because of the important fact that any two maximal toral subalgebras are related by the *conjugate* action of some automorphism. In fact, as Dynkin symmetries permute the coroots, they preserve any chosen maximal toral subalgebra, so this conjugation may be performed by an inner automorphism. Choosing a particular maximal toral subalgebra therefore breaks much of the $\mathsf{Aut}\,\mathfrak{g}$ symmetry. What remains is the finite subgroup of automorphisms which preserve $\mathfrak{t}$. This subgroup, which will be denoted by $\mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g}$, obviously consists of $\mathsf{Out}\,\mathfrak{g}$ and the subgroup of inner automorphisms which preserve $\mathfrak{t}$. The latter group is the famous *Weyl group* W. W is clearly a normal subgroup of $\mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g}$, so one has $\mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g} = \mathsf{Out}\,\mathfrak{g} \ltimes \mathsf{W}$. The form of W is listed in Table A.2.

The action of the Weyl group may be restricted to $\mathfrak{t}$. This (restricted) group is generated by the r elements $w_i = \mathrm{Ad}\,(e_{\alpha_i})\,\mathrm{Ad}\,(-e_{-\alpha_i})\,\mathrm{Ad}\,(e_{\alpha_i})$, whose (dual) action on $\mathfrak{t}^*$ is given by

$$w_i\,(\lambda) = \lambda - \left\langle \lambda, \alpha_i^\vee \right\rangle \alpha_i.$$

The action of W thus restricts further to the weight space $\mathfrak{t}^*_{\mathbb{R}}$. The w_i are called the *simple Weyl reflections*, and they reflect about the hyperplane orthogonal to the simple root α_i. The properties of W are legion:

- It preserves the root system Δ. More generally, it preserves the set of weights of any finite-dimensional representation of $\mathfrak{g}$.
- It contains a reflection w_α for each $\alpha \in \Delta_+$ which reflects about the hyperplane orthogonal to α. Indeed, all reflections in W have this form.
- Removing these reflection hyperplanes from $\mathfrak{t}^*_{\mathbb{R}}$ divides it into $|\mathsf{W}|$ congruent *open Weyl chambers*, which are permuted by the action of W.
- It acts freely and transitively on the set of choices of simple roots.
- Its elements act orthogonally on $\mathfrak{t}^*_{\mathbb{R}}$.
- It admits a length $\ell\,(w)$, defined as the minimal number of simple Weyl reflections that w can be decomposed into. Equivalently,

$$\ell\,(w) = \left|\{\alpha \in \Delta_+ \colon w\,(\alpha) \in \Delta_-\}\right|. \tag{A.2.1}$$

- The determinant of $w \in \mathsf{W}$, as a linear transformation on $\mathfrak{t}^*_{\mathbb{R}}$, is given by $\det w = (-1)^{\ell(w)}$.
- There is a unique (hence involutive) element w_{L} of maximal length $|\Delta_+|$, called the *longest element* of W.

The closure (in $\mathfrak{t}^*_{\mathbb{R}}$) of the open Weyl chambers is just called a *Weyl chamber*. The Weyl chamber which contains the fundamental weights is referred to as the *fundamental Weyl chamber*. It consists of the elements with non-negative Dynkin labels, and is mapped to its geometric opposite (all Dynkin labels non-positive) by the longest element of W,

w_{L}. Elements of the fundamental Weyl chamber are said to be *dominant*, and the set of weights in the fundamental Weyl chamber will be denoted by P_+.

One important automorphism of $\mathfrak{g}$ that has not yet been mentioned is the *Chevalley automorphism* ω_{C}, given by multiplication by -1 on $\mathfrak{t}$ and $\omega_{\mathrm{C}}(e_\alpha) = -e_{-\alpha}$ on the root vectors. Obviously, $\omega_{\mathrm{C}} \in \mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g}$. Given that w_{L} sends the fundamental chamber to the chamber geometrically opposite, it is tempting to conclude that the Chevalley automorphism coincides with the longest element of W. However this is not true in general, as is shown by the fact that there are representations whose weights are not invariant under negation[3]. Thus ω_{C} need not be an element of W. In fact, it decomposes under $\mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g} = \mathsf{Out}\,\mathfrak{g} \ltimes \mathsf{W}$ into the product $\omega_+ w_{\mathrm{L}} = w_{\mathrm{L}}\omega_+$, where $\omega_+ \in \mathsf{Out}\,\mathfrak{g}$ is called the *conjugation automorphism*.

Upon restricting to the weight space $\mathfrak{t}^*_{\mathbb{R}}$, it follows that conjugation acts as $\omega_+ = -w_{\mathrm{L}}$. The image of a weight λ under ω_+ is called the weight *conjugate* to λ, denoted by λ^+. Similarly, if the set of weights of a representation is $\{\lambda\}$, then the representation with weights $\{\lambda^+\}$ is called the *conjugate representation*. The form of ω_+ (restricted to $\mathfrak{t}^*_{\mathbb{R}}$) is given in Table A.2.

A.3. Representations and Characters

Recall from Appendix A.1 that any finite-dimensional irreducible representation of $\mathfrak{g}$ possesses weights in the integral lattice P, and that there is a unique highest weight λ. In terms of the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{t} \oplus \mathfrak{g}_+$, this means that the corresponding (non-degenerate) eigenvector $|\lambda\rangle$ of $\mathfrak{t}$ is annihilated by $\mathfrak{g}_+$. $|\lambda\rangle$ is called the *highest weight vector* of the representation. This highest weight must be dominant, and conversely, any dominant (integral) weight is the highest weight of some irreducible representation. The finite-dimensional irreducible representations are therefore in bijection with the set of dominant (integral) weights.

When $\lambda \in \mathfrak{t}^*_{\mathbb{R}}$ is not dominant integral, one can still construct an abstract representation of $\mathfrak{g}$ with highest weight λ by invoking the existence of a corresponding highest weight vector $|\lambda\rangle$, and letting $\mathfrak{U}(\mathfrak{g}_-)$ act upon it freely. The abstract vectors thus obtained span a vector space upon which $\mathfrak{g}$ acts (the action of $\mathfrak{t}$ and $\mathfrak{g}_+$ is obtained inductively by commuting the action through $\mathfrak{U}(\mathfrak{g}_-)$ until they act on the highest weight vector). As $\mathfrak{U}(\mathfrak{g}_-)$ is an infinite-dimensional algebra, this vector space is an infinite-dimensional $\mathfrak{g}$-module of highest weight λ, called a *Verma module*. In fact, one can even show that this module is irreducible. The corresponding (infinite-dimensional) Verma module may be constructed for λ dominant integral in exactly the same manner. However, this Verma module is no longer irreducible. It contains a unique maximal proper submodule which may be quotiented out from the Verma module to get the finite-dimensional irreducible highest weight module discussed above.

[3]When $\mathfrak{g} = \mathfrak{su}(3)$, the irreducible representation of highest weight Λ_1 provides a simple example.

These finite-dimensional irreducible highest weight modules turn out to be *unitary*, meaning that one can equip them with an inner product (remembering that these are complex vector spaces) with respect to which the elements of the compact real form of $\mathfrak{g}$ are represented by skew-symmetric matrices[4], and normalised by setting the norm of the highest weight vector to 1. Referring back to Appendix A.1, it follows that the elements of $\mathfrak{t}_{\mathbb{R}}$ will be represented by symmetric matrices, as will the combinations $e_\alpha + e_{-\alpha}$ and $\mathfrak{i}(e_\alpha - e_{-\alpha})$. This gives the following definition of the *adjoint* in a finite-dimensional irreducible representation π:

$$\pi\left(\alpha_i^\vee\right)^\dagger = \pi\left(\alpha_i^\vee\right) \qquad \text{and} \qquad \pi(e_\alpha)^\dagger = \pi(e_{-\alpha}). \tag{A.3.1}$$

Note that on the real span of the root vectors and $\mathfrak{t}_{\mathbb{R}}$, this adjoint coincides with the action of $-\omega_{\mathrm{C}}$, the opposite of the Chevalley automorphism (Appendix A.2). As adjointing is an antiautomorphism, it follows that this coincidence persists if $-\omega_{\mathrm{C}}$ is extended antilinearly to all of $\mathfrak{g}$. This defines the *Chevalley antiautomorphism* $\widetilde{\omega}_{\mathrm{C}}$, and demonstrates the relation

$$\pi(x)^\dagger = \pi(\widetilde{\omega}_{\mathrm{C}}(x)) \qquad \text{for all } x \in \mathfrak{g}.$$

It turns out that when one tries to impose these adjoints on the Verma modules corresponding to a dominant integral highest weight, every element of the unique maximal proper submodule ends up having zero norm. The situation is even worse when the highest weight is not dominant integral, as then the (irreducible) Verma module necessarily contains elements of negative norm. It follows that the unitary highest weight modules of $\mathfrak{g}$ are precisely the finite-dimensional irreducible ones.

A finite-dimensional representation π of $\mathfrak{g}$ is uniquely determined, up to isomorphism, by its set of weights Δ^π (including multiplicities). These weights (with multiplicities) are conveniently encoded in the *character* of π, an element of the group ring $\mathbb{Z}\left[e^{\mathsf{P}}\right]$ which is given by

$$\chi_\pi = \sum_{\mu \in \Delta^\pi} e^\mu,$$

where the exponential is formal. The set of weights (with multiplicity) of the irreducible representation of highest weight λ will be denoted by Δ^λ, and the corresponding character by χ_λ. Since weights are functionals on $\mathfrak{t}_{\mathbb{R}}$, which is canonically isomorphic to $\mathfrak{t}_{\mathbb{R}}^*$, one can naturally define an action of χ_π on $\mathfrak{t}_{\mathbb{R}}^*$ by $\chi_\pi(\nu) = \sum_{\mu \in \Delta^\pi} e^{(\mu,\nu)}$. Note that the dimension of the representation is given by $\dim \pi = \chi_\pi(0)$.

One particularly useful fact about characters is that they nicely describe the tensor product operation of representations. Given two representations $\pi_1 \colon \mathfrak{g} \to \mathsf{End}\, V_1$ and $\pi_2 \colon \mathfrak{g} \to \mathsf{End}\, V_2$, one defines the tensor product $\pi_1 \otimes \pi_2 \colon \mathfrak{g} \to \mathsf{End}\,(V_1 \otimes V_2)$ by

$$(\pi_1 \otimes \pi_2)(x)(v_1 \otimes v_2) = [\pi_1(x) v_1] \otimes v_2 + v_1 \otimes [\pi_2(x) v_2].$$

[4]After exponentiating, the elements of the corresponding compact group would be represented by unitary matrices. This is the case of interest in this thesis. Other (non-compact) simple Lie groups correspond to other real forms of $\mathfrak{g}$, and therefore other definitions of unitarity.

Along with the direct sum operation, the tensor product defines a ring structure on the set of (finite-dimensional) representations of $\mathfrak{g}$, called the *representation ring* of $\mathfrak{g}$. The set of weights of the tensor product is then the sum over all pairs of weights, one from π_1 and the other from π_2. It follows from familiar properties of the exponential that the corresponding characters behave multiplicatively: $\chi_{\pi_1\otimes\pi_2} = \chi_{\pi_1}\chi_{\pi_2}$. The map $\pi \mapsto \chi_\pi$ therefore gives an injective homomorphism from the representation ring into the group ring $\mathbb{Z}\left[e^{\mathsf{P}}\right]$. The image of this ring, sometimes called the *character ring*, is then isomorphic to the representation ring. It turns out to be a free polynomial ring $\mathbb{Z}[\chi_1,\chi_2,\ldots,\chi_r]$ on the characters $\chi_i \equiv \chi_{\Lambda_i}$ of the irreducible representations of highest weight Λ_i.

Since the set of weights of a finite-dimensional representation is invariant under the Weyl group W, it follows that the characters are W-invariant functions on $\mathfrak{t}^*_{\mathbb{R}}$:

$$\chi_\pi(w(\nu)) = \sum_{\mu\in\Delta^\pi} e^{(\mu,w(\nu))} = \sum_{\mu\in\Delta^\pi} e^{\left(w^{-1}(\mu),\nu\right)} = \sum_{\mu\in\Delta^\pi} e^{(\mu,\nu)} = \chi_\pi(\nu)\,.$$

It is very important that there is a converse to this rather trivial computation. Every W-invariant element of $\mathbb{Z}\left[e^{\mathsf{P}}\right]$ may be expressed as a linear combination of characters, hence as a polynomial in the characters χ_i. That is, the W-invariants in $\mathbb{Z}\left[e^{\mathsf{P}}\right]$ are precisely the elements of the character ring. Furthermore, the elements *anti-invariant* under W (meaning they are invariant under $w\in\mathsf{W}$ up to the sign $\det w$) are precisely these invariant elements multiplied by the *primitive anti-invariant element*

$$\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right) = e^\rho \prod_{\alpha\in\Delta_+}\left(1-e^{-\alpha}\right).$$

One can easily give an explicit formula for the character of a Verma module. The generators of $\mathfrak{g}_-$ are the negative root vectors $e_{-\alpha}$, and these act freely on the highest weight vector $|\lambda\rangle$ to give a basis of the module. One only needs to keep a track of the number of times each negative root vector acts to specify these basis vectors, and these numbers are completely unrestricted. The character of the Verma module of highest weight λ is therefore just

$$\chi^{\text{Verma}}_\lambda = e^\lambda \prod_{\alpha\in\Delta_+}\left(1+e^{-\alpha}+e^{-2\alpha}+\ldots\right) = \frac{e^{\lambda+\rho}}{e^\rho\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})}, \tag{A.3.2}$$

which is to be understood as a formal power series in the e^α. One notes that the denominator coincides with the primitive anti-invariant element.

When λ is dominant integral, one would like a similar explicit expression for the character of the irreducible representation of highest weight λ. Recalling that such characters are invariant under W, an inspired guess might be to antisymmetrise the numerator of the corresponding Verma module character. This guess is entirely accurate and leads to the *Weyl character formula*:

$$\chi_\lambda = \frac{\sum_{w\in\mathsf{W}}\det w\; e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Delta_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)} = \frac{\sum_{w\in\mathsf{W}}\det w\; e^{w(\lambda+\rho)}}{\sum_{w\in\mathsf{W}}\det w\; e^{w(\rho)}}. \tag{A.3.3}$$

The second equality follows by applying the first form of the character formula to the trivial representation of highest weight 0. By evaluating this character formula at $t\rho$ and taking the limit as $t \to 0$, one deduces *Weyl's dimension formula* for the dimension $\dim(\lambda)$ of the irreducible representation of highest weight λ:

$$\dim(\lambda) = \prod_{\alpha \in \Delta_+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}. \tag{A.3.4}$$

Repeating this exercise for the formal derivative of $\chi_\lambda(t\rho)$ at 0 (as a function of t) yields the so-called "strange formula"

$$\|\rho\|^2 = \frac{h^\vee \dim \mathfrak{g}}{12}. \tag{A.3.5}$$

The expression, Equation (A.3.3), does not itself require that the weight λ be dominant integral. Of course, for other weights the corresponding highest weight representations are infinite-dimensional, and Weyl's character formula does not apply. Nevertheless, it is sometimes useful to extend the notion of a character to all integral weights $\lambda \in P$ by using this formula. When this is done, one finds that no new "characters" are discovered because of the identity

$$\chi_{w(\lambda+\rho)-\rho} = \det w \, \chi_\lambda.$$

If $\lambda + \rho$ is on the boundary of a Weyl chamber, then it is fixed by some reflection $w \in W$, so the corresponding character vanishes. Otherwise, there is a w taking $\lambda + \rho$ into the (open) fundamental Weyl chamber, so the corresponding character is (up to a sign) equal to the character of an irreducible representation. The W-action suggested by this identity, however, is often found whenever the character formula (or the dimension formula) is being used. It is known as the *shifted action* of W, denoted thus:

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

One often therefore talks about *shifted* reflections, *shifted* Weyl chambers, and so on. For example, the above "character" identity may be interpreted as anti-invariance of $\lambda \mapsto \chi_\lambda$ under the shifted Weyl action:

$$\chi_{w\cdot\lambda} = \det w \, \chi_\lambda. \tag{A.3.6}$$

APPENDIX B

Untwisted Affine Lie Algebras

The *affine* Lie algebras form a special class of the *Kac-Moody* algebras (discovered independently by Kac and Moody), which includes the finite-dimensional simple Lie algebras discussed in Appendix A. In this thesis, only the *untwisted* affine Lie algebras will be needed. Despite being infinite-dimensional, their theory is surprisingly analogous to the finite-dimensional case. Indeed, each untwisted affine Lie algebra corresponds to a unique finite-dimensional simple Lie algebra from which it may be constructed. The notation introduced in Appendix A will therefore be used without comment in this appendix. The classic reference for affine Lie algebras is [**99**], and other treatments may be found in [**15**, **16**, **61**, **67**, **69**, **120**].

B.1. Basics

Given a finite-dimensional simple Lie algebra $\mathfrak{g}$, a concrete realisation of the corresponding untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ may be constructed as follows. One motivation for this construction comes from quantum field theory. First, generalise the generators of $\mathfrak{g}$ to formal Laurent polynomials with coefficients in $\mathfrak{g}$. This gives the *loop algebra* $\mathfrak{g}\otimes\mathbb{C}\left[t,t^{-1}\right]$, whose Lie bracket is defined by

$$[x\otimes t^n, y\otimes t^m] = [x,y]\otimes t^{m+n}.$$

When t is interpreted as the coordinate on a circle, this becomes the set of maps from the circle to $\mathfrak{g}$ ($\mathfrak{g}$-valued *fields* on the circle), hence the name. Note that the power of t induces a $\mathbb{Z}$-grading on the loop algebra. As noted in Section 2.1.1, when quantising a classical system with such a symmetry, one may extend the symmetry algebra by a central extension. There is a unique (up to isomorphism) non-trivial central extension, given by

$$[x\otimes t^n, y\otimes t^m] = [x,y]\otimes t^{m+n} + n\delta_{n+m,0}\kappa(x,y)K,$$

where K is the adjoined central element, and $\kappa(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}$.

This centrally extended Lie algebra suffices for many applications. It has an abelian subalgebra of ad-diagonalisable elements $\mathfrak{t}\otimes t^0\oplus\operatorname{span}\{K\}$ with respect to which one may decompose into root spaces. However, it suffers from the problem that the corresponding roots do not depend on the power of t, and are therefore infinitely degenerate. This may be circumvented by introducing a further element L_0 which commutes with the abelian subalgebra, but whose eigenvalues in the adjoint representation take into account the power of t (also called the *grade*). This is most conveniently achieved by defining $\operatorname{ad}(L_0)$ to be

the derivation[1] $-t\mathrm{d}/\mathrm{d}t$ of $\mathbb{C}\left[t,t^{-1}\right]$. One therefore (finally) gets the *untwisted affine Lie algebra*

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t,t^{-1}\right] \oplus \operatorname{span}\{K, L_0\},$$

whose Lie bracket is given by

$$[x\otimes t^n, y\otimes t^m] = [x,y]\otimes t^{m+n} + n\delta_{n+m,0}\kappa(x,y)K \quad \text{and} \quad [L_0, x\otimes t^n] = -nx\otimes t^n. \quad \text{(B.1.1)}$$

The original Lie algebra $\mathfrak{g}$ is embedded in $\widehat{\mathfrak{g}}$ as the *horizontal subalgebra* $\mathfrak{g}\otimes t^0$.

With this embedding, the *Cartan subalgebra* is defined to be that of the horizontal subalgebra, augmented by the central element and the derivation: $\widehat{\mathfrak{t}} = \mathfrak{t} \oplus \operatorname{span}\{K, L_0\}$. The adjoint action of this abelian subalgebra may now be used to decompose $\widehat{\mathfrak{g}}$. Since K is central, $\operatorname{ad}(K) = 0$, so the *roots* may be expressed as the triple $\widehat{\alpha} = (\lambda, 0, n)$ of eigenvalues of $\operatorname{ad}(\mathfrak{t})$, $\operatorname{ad}(K)$, and $-\operatorname{ad}(L_0)$. In fact, the *root system* has the form

$$\widehat{\Delta} = \{(\alpha, 0, n) : \alpha \in \Delta, n \in \mathbb{Z}\} \cup \{(0,0,n) : n \in \mathbb{Z}\setminus\{0\}\}.$$

Those of the form $\widehat{\alpha} = (\alpha, 0, n)$ correspond to the (non-degenerate) *root vector* $e_\alpha \otimes t^n$, and those of the form $\widehat{\alpha} = (0,0,n)$ correspond to $\mathfrak{t}\otimes t^n$ (and so are r-fold degenerate). The former are said to be *real* roots, and the latter *imaginary*. The *primitive imaginary root* $(0,0,1)$ is denoted by $\widehat{\delta}$. Note that affine Lie algebras have no highest root.

As affine Lie algebras are infinite-dimensional, the Killing form cannot be defined as a trace (Equation (A.1.1)). However, one can extend the Killing form on the horizontal subalgebra to a bilinear symmetric form $\widehat{\kappa}(\cdot,\cdot)$ on $\widehat{\mathfrak{g}}$. Demanding associativity gives

$$\widehat{\kappa}(x\otimes t^n, y\otimes t^m) = \kappa(x,y)\,\delta_{n+m,0}, \qquad \widehat{\kappa}(x\otimes t^n, K) = 0, \qquad \widehat{\kappa}(x\otimes t^n, L_0) = 0,$$
$$\widehat{\kappa}(K,K) = 0, \qquad \widehat{\kappa}(K, L_0) = -1, \qquad \text{and} \quad \widehat{\kappa}(L_0, L_0) = 0.$$

This *extended Killing form* may be checked to be non-degenerate, and it restricts to a non-degenerate form on the Cartan subalgebra too. It therefore lifts to an (non-degenerate) symmetric bilinear form on the *weight space* $\widehat{\mathfrak{t}}^*_{\mathbb{R}}$. Writing elements of this space as a triple $\widehat{\lambda} = (\lambda, k_\lambda, n_\lambda)$ of eigenvalues of $\mathfrak{t}$, K, and $-L_0$ (in some representation), this bilinear form is given by

$$\left(\widehat{\lambda}, \widehat{\mu}\right) = (\lambda,\mu) + k_\lambda n_\mu + k_\mu n_\lambda.$$

Note that the imaginary affine roots have zero length with respect to this form.

The roots $\widehat{\alpha}_i = (\alpha_i, 0, 0)$ ($i = 1, 2, \ldots, r$) do not generate the root system $\widehat{\Delta}$. Any basis of the root system must include a root at non-zero grade. It is convenient to include the *lowest* root of grade 1, $\widehat{\alpha}_0 = (-\theta, 0, 1)$, in the basis of *simple roots* as this definition preserves the property that no difference of simple roots is a root. Note that $\|\widehat{\alpha}_0\|^2 = 2$. The *marks* of $\widehat{\mathfrak{g}}$ are now defined by the decomposition $\widehat{\delta} = \sum_{i=0}^r a_i \widehat{\alpha}_i$. Obviously, $a_0 = 1$ and for $i \neq 0$, the a_i coincide with the marks of $\mathfrak{g}$. The simple roots partition $\widehat{\Delta}$ into *positive*

[1] The sign in this definition is only necessary so that L_0 may be identified with the zero-mode of the Virasoro algebra (Section 3.1.2). Note that one could redefine L_0 by adding an arbitrary multiple of K, without affecting its desired properties, but then it would not coincide with its Virasoro counterpart.

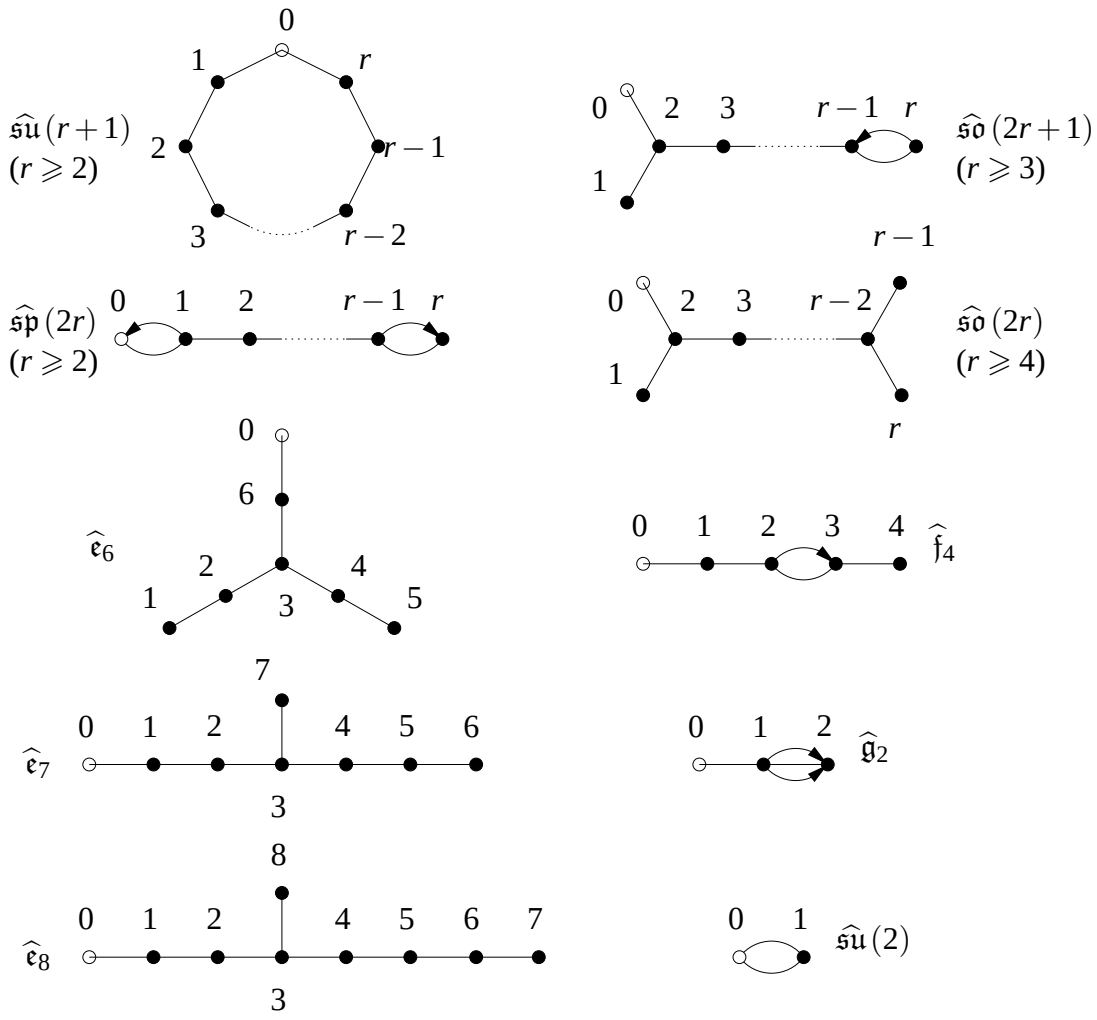

FIGURE B.1. The Dynkin diagrams of the untwisted affine Lie algebras, labelled through the compact real forms of their horizontal subalgebras. The numbers indicate the ordering of the nodes (hence simple roots) used in this thesis, and the arrows always point from a long root to a short root. The white node is the (new) affine node, labelled 0.

and *negative roots*:

$$\widehat{\Delta}_+ = \{(\alpha,0,0) : \alpha \in \Delta_+\} \cup \{(\alpha,0,n) : n > 0\} \cup \{(0,0,n) : n > 0\}\,.$$

Affine Lie algebras therefore admit a *triangular decomposition*, $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \widehat{\mathfrak{t}} \oplus \widehat{\mathfrak{g}}_+$, extending that of $\mathfrak{g}$.

The real roots define corresponding *coroots* as in the case of finite-dimensional simple Lie algebras. The same is not true for the imaginary roots as they have zero length. The *Cartan matrix* is given again by $\widehat{A}_{ij} = \left\langle \widehat{\alpha}_i, \widehat{\alpha}_j^\vee \right\rangle$, and completely characterises $\widehat{\mathfrak{g}}$ as an $(r+1)\times(r+1)$ matrix. The *Dynkin diagram* of $\widehat{\mathfrak{g}}$ is defined as before, and the complete list (for untwisted affine Lie algebras) is given in Figure B.1.

The *fundamental weights* are defined to be the elements dual to the simple coroots. Because the K-eigenvalue of the roots must always vanish, the $-L_0$-eigenvalue (grade) of

the fundamental weights cannot be determined[2]. It may be (arbitrarily) chosen to be 0. The fundamental weights are thus

$$\widehat{\Lambda}_0 = (0,1,0) \qquad \text{and} \qquad \widehat{\Lambda}_i = \left(\Lambda_i, a_i^\vee, 0\right) \quad \text{for } i = 1,2,\ldots,r.$$

The sum of these fundamental weights is the *affine Weyl vector* $\widehat{\rho} = (\rho, \mathsf{h}^\vee, 0)$. Since $\widehat{\alpha}_0$ is a long root, the zeroth comark is $a_0^\vee = a_0 = 1$. Hence $\left\langle \widehat{\Lambda}_i, K \right\rangle = a_i^\vee$, and the *comarks* of $\widehat{\mathfrak{g}}$ are the coefficients in the decomposition $K = \sum_i a_i^\vee \widehat{\alpha}_i^\vee$.

The *Dynkin labels* are defined as before, $\widehat{\lambda} = \sum_i \lambda_i \widehat{\Lambda}_i$. The zeroth Dynkin label of $\widehat{\lambda} = (\lambda, k, 0)$ may therefore be conveniently expressed as

$$\lambda_0 = k - (\lambda, \theta). \tag{B.1.2}$$

The K-eigenvalue of a weight is called the *level*. Note that the fundamental weights do not form a basis of the weight space $\widehat{\mathfrak{t}}_{\mathbb{R}}^*$. One has to add the element $\widehat{\delta}$ to complete the basis. However, as the grade of the fundamental weights is indeterminate, it is often harmless to ignore this additional element.

B.2. Automorphisms of $\widehat{\mathfrak{g}}$

Every simple root $\widehat{\alpha}_i$ of $\widehat{\mathfrak{g}}$ gives rise to a reflection $\widehat{w}_i$ on the weight space $\widehat{\mathfrak{t}}_{\mathbb{R}}^*$ by

$$\widehat{w}_i\left(\widehat{\lambda}\right) = \widehat{\lambda} - \left\langle \widehat{\lambda}, \widehat{\alpha}_i^\vee \right\rangle \widehat{\alpha}_i.$$

These *simple Weyl reflections* leave fixed the hyperplane orthogonal to $\widehat{\alpha}_i$, and together generate the *affine Weyl group* $\widehat{\mathsf{W}}$ of $\widehat{\mathfrak{g}}$. Analogously to Appendix A.2, there is a similar Weyl reflection $\widehat{w}_{\widehat{\alpha}}$ for each *real* root $\widehat{\alpha}$. In constrast, every element of $\widehat{\mathsf{W}}$ preserves each *imaginary* root $n\widehat{\delta}$, and has no effect on the level (K-eigenvalue).

Writing $\widehat{\lambda} = (\lambda, k, n)$ and $\widehat{\alpha} = (\alpha, 0, m)$, the action of $\widehat{w}_{\widehat{\alpha}}$ becomes

$$\widehat{w}_{(\alpha,0,m)}(\lambda,k,n) = \left(w_\alpha\left(\lambda + km\,\iota\left(\alpha^\vee\right)\right), k, n - m\left[\left\langle \lambda, \alpha^\vee \right\rangle + \frac{2km}{\|\alpha\|^2}\right]\right).$$

In particular,

$$\widehat{w}_0(\lambda,k,n) = (w_\theta(\lambda) + k\theta, k, n - \lambda_0) \quad \text{and} \quad \widehat{w}_i(\lambda,k,n) = (w_i(\lambda), k, n) \quad \text{when } i \neq 0.$$

One notices that the induced action on the weight space of the horizontal algebra is *affine*, being a translation by an element of the coroot lattice $\mathsf{Q}^\vee$ followed by a (finite) Weyl reflection. This translation may be isolated by forming the combination $\widehat{t}_{m\alpha^\vee} = \widehat{w}_{(\alpha,0,0)}\widehat{w}_{(\alpha,0,m)}$ which acts by

$$\widehat{t}_{m\alpha^\vee}(\lambda,k,n) = \left(\lambda + k\,\iota\left(m\alpha^\vee\right), k, n - \left\langle \lambda, m\alpha^\vee \right\rangle - \frac{1}{2}k\kappa\left(m\alpha^\vee, m\alpha^\vee\right)\right).$$

The translations $\left\{\widehat{t}_{q^\vee} : q^\vee \in \mathsf{Q}^\vee\right\}$ in fact form a normal subgroup of $\widehat{\mathsf{W}}$, isomorphic to $\mathsf{Q}^\vee$. The subgroup of $\widehat{\mathsf{W}}$ generated by the $\widehat{w}_i$ for $i = 1,2,\ldots,r$ is clearly isomorphic to

[2]This reflects the fact that L_0 may be redefined as $L_0 + aK$ for any a without changing the Lie bracket.

W, and has trivial intersection with the subgroup of translations. The decomposition $\widehat{w}_{(\alpha,0,m)} = \widehat{w}_{(\alpha,0,0)}\widehat{t}_{m\alpha^\vee}$ then shows that the affine Weyl group has the semidirect product structure

$$\widehat{\mathsf{W}} = \mathsf{W} \ltimes \mathsf{Q}^\vee.$$

Note that $\widehat{\mathsf{W}}$ is an infinite group.

As in Appendix A.2, the affine Weyl group acts orthogonally on the (affine) weight space $\widehat{\mathfrak{t}}^*_{\mathbb{R}}$. Removing the hyperplanes orthogonal to the real roots partitions $\widehat{\mathfrak{t}}^*_{\mathbb{R}}$ into an infinite number of congruent *open affine Weyl chambers* which are permuted by the action of $\widehat{\mathsf{W}}$. The closures are the *affine Weyl chambers*, and the *fundamental affine Weyl chamber* is distinguished by consisting of the *dominant* elements (those with non-negative Dynkin labels).

It is frequently the case that when working with untwisted affine Lie algebras, the relevant representations all share the same level k. Under these circumstances, a weight $\widehat{\lambda} \in \widehat{\mathfrak{t}}^*_{\mathbb{R}}$ is completely determined (up to harmless factors of $\widehat{\delta}$) by its projection onto the weight space $\mathfrak{t}^*_{\mathbb{R}}$ of the horizontal subalgebra. The level k projection of the $\widehat{\mathsf{W}}$-action is given by the semidirect product structure:

$$\widehat{w}(\lambda) = w(\lambda) + kq^\vee,$$

for some (unique) $w \in \mathsf{W}$ and $q^\vee \in \mathsf{Q}^\vee$. The level k projection of the affine Weyl group is sometimes denoted by $\widehat{\mathsf{W}}_k$ when the k-dependence is to be emphasised. The projection of the fundamental affine Weyl chamber is given by the inequalities

$$\lambda_0 = k - (\lambda, \theta) \geqslant 0 \qquad \text{and} \qquad \lambda_i \geqslant 0 \qquad \text{for } i \neq 0.$$

That is, λ must be dominant *and* $(\lambda, \theta) \leqslant k$.

The projection of the fundamental affine Weyl chamber is therefore a compact subset of $\mathfrak{t}^*_{\mathbb{R}}$, and is accordingly referred to as the *fundamental affine* (or *Weyl*) *alcove*. The set of integral weights (of $\mathfrak{g}$) contained in the level k fundamental affine alcove will be denoted by $\widehat{\mathsf{P}}_k$. The fundamental affine alcove is a simplex with vertices 0 and $k\Lambda_i/a_i^\vee$ for $i = 1, 2, \ldots, r$.

More generally, the reflection hyperplane orthogonal to the real root $(\alpha, 0, m)$ projects (at level k) onto the hyperplane $(\lambda, \alpha) = -mk$. The set of these projected reflection hyperplanes therefore form a grid, of a size specified by k, which divides the Weyl chambers of $\mathfrak{t}^*_{\mathbb{R}}$ into congruent *affine alcoves* (which are permuted by the $\widehat{\mathsf{W}}_k$-action). An element $\lambda \in \mathfrak{t}^*_{\mathbb{R}}$ is thus a vertex of an affine alcove if and only if $(\lambda, \alpha) \in k\mathbb{Z}$ for all α in a subset of Δ_+ which *spans* the weight space $\mathfrak{t}^*_{\mathbb{R}}$. Equivalently, before projecting, the corresponding elements of $\widehat{\mathfrak{t}}^*_{\mathbb{R}}$ must be orthogonal to a subset of $\widehat{\Delta}_+$ which, when augmented by $\widehat{\delta}$, spans the root space (the level 0 subspace of the weight space).

One also must mention the *shifted* action of $\widehat{\mathsf{W}}$. This is defined analogously to the W case: $\widehat{w} \cdot \widehat{\lambda} = \widehat{w}\left(\widehat{\lambda} + \widehat{\rho}\right) - \widehat{\rho}$. The level k projection of this action on $\mathfrak{t}^*_{\mathbb{R}}$ may then be

determined from

$$\begin{aligned}\widehat{w}\cdot(\lambda,k,n) &= \widehat{w}\left(\lambda+\rho,k+\mathsf{h}^\vee,n\right)-\left(\rho,\mathsf{h}^\vee,0\right)\\ &= \left(w(\lambda+\rho),k+\mathsf{h}^\vee,n\right)-\left(\rho,\mathsf{h}^\vee,0\right)\\ &\qquad +\left(\left(k+\mathsf{h}^\vee\right)q^\vee,0,-\left(\lambda+\rho,q^\vee\right)-\frac{1}{2}\left(k+\mathsf{h}^\vee\right)\kappa\left(q^\vee,q^\vee\right)\right),\end{aligned}$$

giving

$$\widehat{w}\cdot\lambda = w\cdot\lambda+\left(k+\mathsf{h}^\vee\right)q^\vee, \tag{B.2.1}$$

for some $w\in\mathsf{W}$ and $q^\vee\in\mathsf{Q}^\vee$. Note the level shift $k\mapsto k+\mathsf{h}^\vee$.

As in Appendix A.2, the symmetries of the Dynkin diagram of $\widehat{\mathfrak{g}}$ induce automorphisms of the root system. The marks and comarks are invariant under these *Dynkin symmetries*, hence they leave the primitive imaginary root $\widehat{\delta}$ and the central element K invariant. Such automorphisms uniquely extend to all of $\widehat{\mathfrak{g}}$, and preserve $\widehat{\mathfrak{t}}$ (leaving L_0 invariant). They also act orthogonally on the weight space. As they permute the fundamental weights $\widehat{\Lambda}_i$, they also preserve the fundamental affine Weyl chamber (and hence the fundamental affine alcove). The (non-trivial) Dynkin symmetries are therefore outer automorphisms (and are never elements of $\widehat{\mathsf{W}}$, in contrast to the claims of [**61**, **126**]). The group of Dynkin symmetries of $\widehat{\mathfrak{g}}$ will be denoted by $\mathsf{Out}\,\widehat{\mathfrak{g}}$, and the form of these groups is listed in Table B.1. Note that there is a (generally non-normal) subgroup isomorphic to $\mathsf{Out}\,\mathfrak{g}$, consisting of those Dynkin symmetries which preserve the zero node.

Consider the (projected level k) action of an automorphism $\widehat{\omega}\in\mathsf{Out}\,\widehat{\mathfrak{g}}$ on the vertices of the fundamental affine alcove (which is preserved by this action). Before projection, these vertices correspond to the elements $k\widehat{\Lambda}_i/a_i^\vee\in\widehat{\mathfrak{t}}^*_{\mathbb{R}}$, $i=0,1,\ldots,r$, and the vertex labelled by i is clearly orthogonal to the simple roots $\widehat{\alpha}_j$, $j\neq i$ (which with $\widehat{\delta}$ span the root space). Define a transformation of the level k subspace of $\widehat{\mathfrak{t}}^*_{\mathbb{R}}$ by

$$\Omega\left(\widehat{\lambda}\right)=\widehat{\omega}\left(\widehat{\lambda}\right)+k\left(\widehat{\Lambda}_0-\widehat{\Lambda}_{\widehat{\omega}(0)}\right)=\widehat{\omega}\left(\widehat{\lambda}+k\left(\widehat{\Lambda}_{\widehat{\omega}^{-1}(0)}-\widehat{\Lambda}_0\right)\right).$$

This preserves $k\widehat{\Lambda}_0$, so after projection will preserve the origin.

The aim is now to show that Ω maps the other vertices $k\widehat{\Lambda}_i/a_i^\vee$ ($i\neq 0$) to vertices of *some other* affine alcove. Consider therefore the simple roots $\widehat{\alpha}_j$ for $j\neq i$. By orthogonality,

$$\left(\Omega\left(\frac{\widehat{\Lambda}_i}{a_i^\vee}\right),\widehat{\omega}\left(\widehat{\alpha}_j\right)\right)=\left(\frac{\widehat{\Lambda}_i}{a_i^\vee}+\widehat{\Lambda}_{\widehat{\omega}^{-1}(0)}-\widehat{\Lambda}_0,\widehat{\alpha}_j\right)=\delta_{\widehat{\omega}^{-1}(0),j}-\delta_{0j}.$$

When $\widehat{\omega}$ fixes the affine node 0, this vanishes for all $j\neq i$. In general, one defines the roots $\widehat{\beta}_j=\widehat{\omega}\left(\widehat{\alpha}_j\right)-\left(\delta_{\widehat{\omega}^{-1}(0),j}-\delta_{0j}\right)\widehat{\delta}$, and notes that

$$\left(\Omega\left(\frac{\widehat{\Lambda}_i}{a_i^\vee}\right),\widehat{\beta}_j\right)=0\qquad\text{for all } j\neq i.$$

$\widehat{\mathfrak{g}}$	$\mathsf{Out}\,\widehat{\mathfrak{g}}$	$\mathsf{Q}^*/\mathsf{Q}^\vee$ $\langle$Presentation$\rangle$	P/Q^* $\langle$Presentation$\rangle$	$\mathsf{P}/\mathsf{Q}^\vee$ $\langle$Presentation$\rangle$
$\widehat{\mathfrak{su}}(r+1)$ $(r \geqslant 2)$	D_{r+1}	$\mathbb{Z}_{r+1}$ $\langle \Lambda_1 \colon i\Lambda_1 = \Lambda_i \rangle$	$\{\mathrm{id}\}$	$\mathbb{Z}_{r+1}$
$\widehat{\mathfrak{so}}(2r+1)$ $(r \geqslant 2$, even)	$\mathbb{Z}_2$	$\mathbb{Z}_2$ $\langle \Lambda_1 = \Lambda_3 = \cdots = \Lambda_{r-1} \rangle$	$\mathbb{Z}_2$ $\langle \Lambda_r \rangle$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\langle \Lambda_1, \Lambda_r \rangle$
$\widehat{\mathfrak{so}}(2r+1)$ $(r \geqslant 2$, odd)	$\mathbb{Z}_2$	$\mathbb{Z}_2$ $\langle \Lambda_1 = \Lambda_3 = \cdots = \Lambda_{r-2} \rangle$	$\mathbb{Z}_2$ $\langle \Lambda_r \rangle$	$\mathbb{Z}_4$ $\langle \Lambda_r \colon 2\Lambda_r = \Lambda_1 \rangle$
$\widehat{\mathfrak{sp}}(2r)$ $(r \geqslant 1)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$ $\langle \Lambda_r \rangle$	$\mathbb{Z}_2^{r-1}$ $\langle \Lambda_1, \ldots, \Lambda_{r-1} \rangle$	$\mathbb{Z}_2^r$ $\langle \Lambda_1, \ldots, \Lambda_r \rangle$
$\widehat{\mathfrak{so}}(8)$	S_4	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\langle \Lambda_3, \Lambda_4 \colon \Lambda_3 + \Lambda_4 = \Lambda_1 \rangle$	$\{\mathrm{id}\}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\widehat{\mathfrak{so}}(2r)$ $(r \geqslant 5$, odd)	$\mathbb{Z}_2 \ltimes \mathbb{Z}_4$	$\mathbb{Z}_4$ $\langle \Lambda_r \colon 2\Lambda_r = \Lambda_1, 3\Lambda_r = \Lambda_{r-1} \rangle$	$\{\mathrm{id}\}$	$\mathbb{Z}_4$
$\widehat{\mathfrak{so}}(2r)$ $(r \geqslant 6$, even)	$\mathbb{Z}_2 \ltimes \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\langle \Lambda_{r-1}, \Lambda_r \colon \Lambda_{r-1} + \Lambda_r = \Lambda_1 \rangle$	$\{\mathrm{id}\}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\widehat{\mathfrak{e}}_6$	S_3	$\mathbb{Z}_3$ $\langle \Lambda_1 \colon 2\Lambda_1 = \Lambda_5 \rangle$	$\{\mathrm{id}\}$	$\mathbb{Z}_3$
$\widehat{\mathfrak{e}}_7$	$\mathbb{Z}_2$	$\mathbb{Z}_2$ $\langle \Lambda_6 \rangle$	$\{\mathrm{id}\}$	$\mathbb{Z}_2$
$\widehat{\mathfrak{e}}_8$	$\{\mathrm{id}\}$	$\{\mathrm{id}\}$	$\{\mathrm{id}\}$	$\{\mathrm{id}\}$
$\widehat{\mathfrak{f}}_4$	$\{\mathrm{id}\}$	$\{\mathrm{id}\}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\langle \Lambda_3, \Lambda_4 \rangle$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\widehat{\mathfrak{g}}_2$	$\{\mathrm{id}\}$	$\{\mathrm{id}\}$	$\mathbb{Z}_3$ $\langle \Lambda_2 \rangle$	$\mathbb{Z}_3$

TABLE B.1. The groups $\mathsf{Out}\,\widehat{\mathfrak{g}}$, $\mathsf{Q}^*/\mathsf{Q}^\vee$, P/Q^*, and $\mathsf{P}/\mathsf{Q}^\vee$ for each untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ and its horizontal subalgebra $\mathfrak{g}$. Presentations of the quotient groups in terms of fundamental weights are also given (that of $\mathsf{P}/\mathsf{Q}^\vee$ is only given if it differs from both other quotient groups). S_n denotes the symmetric group on n elements, and $\mathsf{D}_n = \mathbb{Z}_2 \ltimes \mathbb{Z}_n$ the dihedral group (symmetry group of an n-sided regular polygon).

The roots $\left\{\widehat{\beta}_j \colon j \neq i\right\}$ are therefore orthogonal to the images of the fundamental affine alcove vertices under Ω. Furthermore, when augmented with $\widehat{\delta}$, this set of roots spans the same subspace as the set $\left\{\widehat{\omega}\left(\widehat{\alpha}_j\right) \colon j \neq i\right\} \cup \left\{\widehat{\delta}\right\}$, which is the whole root space. Thus Ω maps the fundamental affine alcove vertices to other affine alcove vertices.

Now project onto the weight space of the horizontal subalgebra. As noted above, Ω now preserves the origin. In fact, Ω acts on $\mathfrak{t}^*_{\mathbb{R}}$ by

$$\Omega(\lambda) = \sum_{i=1}^{r} \lambda_{\widehat{\omega}^{-1}(i)}\Lambda_i - k\left(1 - \delta_{\widehat{\omega}(0),0}\right)\Lambda_{\widehat{\omega}(0)}.$$

If $\widehat{\omega}(0) = 0$, then this reduces to $\Omega(\lambda) = \sum_{i=1}^{r} \lambda_{\widehat{\omega}^{-1}(i)}\Lambda_i = \widehat{\omega}(\lambda)$, and $\widehat{\omega}$ acts (linearly) as a Dynkin symmetry of $\mathfrak{g}$. When $\widehat{\omega}(0) \neq 0$,

$$\Omega(\lambda) = \lambda_0\Lambda_{\widehat{\omega}(0)} + \sum_{\substack{i=1 \\ i\neq\widehat{\omega}(0)}}^{r} \lambda_{\widehat{\omega}^{-1}(i)}\Lambda_i - k\Lambda_{\widehat{\omega}(0)} = \sum_{\substack{i=1 \\ i\neq\widehat{\omega}(0)}}^{r} \lambda_{\widehat{\omega}^{-1}(i)}\Lambda_i - (\lambda,\theta)\Lambda_{\widehat{\omega}(0)},$$

by Equation (B.1.2), so it can be seen that in this case, Ω also acts *linearly* on $\mathfrak{t}^*_{\mathbb{R}}$.

It follows from these two facts that Ω maps the fundamental Weyl *chamber* to another Weyl chamber, linearly, and therefore is an element of the Weyl group of $\mathfrak{g}$, possibly composed with a Dynkin symmetry of $\mathfrak{g}$. That is, $\Omega \in \mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g}$. The level k projected action of $\widehat{\omega} \in \mathsf{Out}\,\widehat{\mathfrak{g}}$ on $\mathfrak{t}^*_{\mathbb{R}}$ is thus given by

$$\widehat{\omega}(\lambda) = \Omega(\lambda) + k\Lambda_{\widehat{\omega}(0)} \qquad \text{for some } \Omega \in \mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g}, \tag{B.2.2}$$

where it is understood that $\Lambda_0 = 0$. Since $\mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g} = \mathsf{Out}\,\mathfrak{g} \ltimes \mathsf{W}$, one can decompose Ω uniquely as ωw or $w'\omega$, where $\omega \in \mathsf{Out}\,\mathfrak{g}$ and $w, w' \in \mathsf{W}$. The Weyl group element might depend on the ordering, but its determinant does not. This determinant may be computed [**57**], and turns out to be

$$\det w = (-1)^{2\left(\Lambda_{\widehat{\omega}(0)},\rho\right)}. \tag{B.2.3}$$

One therefore has the unique decomposition $\widehat{\omega} = t_{\Lambda_i}\omega w$, where t_{Λ_i} is translation by $k\Lambda_i$ ($i = \widehat{\omega}(0)$). Using the normality of W in $\mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g}$, one can check that the subgroup V of $\mathsf{Out}\,\widehat{\mathfrak{g}}$ corresponding to $\omega = \mathrm{id}$ is normal. That is,

$$\mathsf{Out}\,\widehat{\mathfrak{g}} = \mathsf{Out}\,\mathfrak{g} \ltimes \mathsf{V}.$$

The structure of the subgroup V is not difficult to guess. The node $\widehat{\omega}(0)$ determines the translation uniquely, and this determines the Weyl chamber which the fundamental one is mapped to, hence the Weyl element. The fundamental weights $\Lambda_{\widehat{\omega}(0)}$ therefore may be used to label the elements of V (the weight 0 then labels the identity of V). Note that $\alpha_{\widehat{\omega}(0)}$ is always a long root, so the corresponding fundamental weights belong to the dual root lattice Q^*. Now, $\mathsf{Out}\,\widehat{\mathfrak{g}} \cap \widehat{\mathsf{W}} = \{\mathrm{id}\}$, so translations by (non-trivial) elements of $\mathsf{Q}^\vee$ cannot correspond to Dynkin symmetries of $\widehat{\mathfrak{g}}$. This suggests the structure

$$\mathsf{V} \cong \mathsf{Q}^*/\mathsf{Q}^\vee.$$

That this is indeed correct may be seen by defining a map from $\mathsf{V} \subseteq \mathsf{Out}\,\widehat{\mathfrak{g}}$ which associates to each $\widehat{\omega} = t_{\Lambda_i}w \in \mathsf{V}$, the class $[\Lambda_i] \in \mathsf{Q}^*/\mathsf{Q}^\vee$. The class associated with the product $\widehat{\omega}'\widehat{\omega}$ is then $[w(\Lambda_{i'}) + \Lambda_i]$. One can check (inductively) that as $\Lambda_{i'} \in \mathsf{Q}^*$, $w(\Lambda_{i'}) = \Lambda_{i'} \pmod{\mathsf{Q}^\vee}$, and so this map is a group homomorphism. It is clearly injective, and

surjectivity is most easily checked by comparing the orders $|\mathsf{V}| = |\mathsf{Out}\,\widehat{\mathfrak{g}}|\,/\,|\mathsf{Out}\,\mathfrak{g}|$ and $|\mathsf{Q}^*/\mathsf{Q}^\vee|$. These orders can be determined from Table A.2 and Table B.1.

Finally, consider the group $\mathsf{Out}\,\widehat{\mathfrak{g}} \ltimes \widehat{\mathsf{W}}$. Both components are themselves semidirect products, so an arbitrary element of this group may be uniquely decomposed as

$$\left(t_{\Lambda_i}\omega w\right)\left(t_{q^\vee}w'\right) = t_{q^\circ}\,\omega\left(ww'\right),$$

where t_{Λ_i} is translation by $\Lambda_i \in \mathsf{Q}^* \pmod{\mathsf{Q}^\vee}$, $\omega \in \mathsf{Out}\,\mathfrak{g}$, $w, w' \in \mathsf{W}$, $t_{q^\vee}$ is translation by $q^\vee \in \mathsf{Q}^\vee$, and t_{q° is translation by some $q^\circ \in \mathsf{Q}^*$. It is easily checked that this gives

$$\mathsf{Out}\,\widehat{\mathfrak{g}} \ltimes \widehat{\mathsf{W}} = \mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g} \ltimes \mathsf{Q}^*.$$

In other words, the Dynkin symmetries which are induced by the symmetries of the horizontal subalgebra augment W to $\mathsf{Aut}_{\mathfrak{t}}\,\mathfrak{g}$, whereas the purely affine symmetries augment the coroot lattice translations to translations by elements of the dual root lattice.

B.3. Representations, Characters, and Modularity

As in Appendix A.3, the triangular decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \widehat{\mathfrak{t}} \oplus \widehat{\mathfrak{g}}_+$ defines *highest weight* representations to be those of the form $\mathfrak{U}(\widehat{\mathfrak{g}}_-)\left|\widehat{\lambda}\right\rangle$, where $\left|\widehat{\lambda}\right\rangle$ is an eigenvector of $\widehat{\mathfrak{t}}$ (of eigenvalue $\widehat{\lambda} \in \widehat{\mathfrak{t}}^*$) which is annihilated by the action of $\widehat{\mathfrak{g}}_+$. If $\mathfrak{U}(\widehat{\mathfrak{g}}_-)$ acts freely on $\left|\widehat{\lambda}\right\rangle$, then one has the *Verma module* of highest weight λ. As before, a Verma module has a unique proper maximal submodule (which may be $\{0\}$), and quotienting out this submodule gives the (unique) irreducible highest weight module of highest weight $\widehat{\lambda}$.

Recall that a sesquilinear form could be defined on a representation of $\mathfrak{g}$ so that the adjoint operation was given by the action of the Chevalley antiautomorphism $\widetilde{\omega}_{\mathrm{C}}$ (Appendix A.3). This antiautomorphism swapped the nilpotent subalgebras $\mathfrak{g}_-$ and $\mathfrak{g}_+$ of the triangular decomposition of $\mathfrak{g}$. The Chevalley antiautomorphism of the horizontal subalgebra may be extended to an antiautomorphism $\widehat{\omega}_{\mathrm{C}}$ on $\widehat{\mathfrak{g}}$ by

$$\widehat{\omega}_{\mathrm{C}}\left(x \otimes t^n\right) = \widetilde{\omega}_{\mathrm{C}}\left(x\right) \otimes t^{-n}, \qquad \widehat{\omega}_{\mathrm{C}}\left(K\right) = K, \qquad \text{and} \qquad \widehat{\omega}_{\mathrm{C}}\left(L_0\right) = L_0.$$

This now swaps $\widehat{\mathfrak{g}}_-$ and $\widehat{\mathfrak{g}}_+$, and defines the adjoint of an element of $\widehat{\mathfrak{g}}$ (in an arbitrary highest weight representation) to be its image under $\widehat{\omega}_{\mathrm{C}}$. As before, the sesquilinear form defined by this adjoint (and normalised by setting the norm of the highest weight vector to 1) is not usually positive-definite.

The weights of a highest weight representation of $\widehat{\mathfrak{g}}$ have the form $\widehat{\lambda} - \sum_{i=0}^{r} m_i \widehat{\alpha}_i$ for some non-negative integers m_i, where $\widehat{\lambda}$ is the highest weight. The corresponding weight spaces are all finite-dimensional. The most important class of highest weight representations are those which satisfy an additional finiteness constraint, that the weight system contains no infinite subset of the form $\{\widehat{\mu}, \widehat{\mu} - \widehat{\alpha}_i, \widehat{\mu} - 2\widehat{\alpha}_i, \ldots\}$. Such representations are said to be *integrable*.

This is quite a restrictive requirement. In fact, the integrable highest weight representations are precisely the irreducible highest weight representations with dominant integral highest weight. Thus, if the highest weight is $\widehat{\lambda} = (\lambda, k, 0)$, then λ is in the fundamental

Weyl chamber of $\mathfrak{g}$, and $\lambda_0 = k - (\lambda, \theta)$ is a non-negative integer. It follows that the level k must be a non-negative integer, and λ must belong to the fundamental affine alcove (at level k), $\widehat{\mathsf{P}}_k$. At each level k, there are therefore only a finite number of integrable highest weight representations.

The importance of these representations lies in the fact that they are only irreducible highest weight representations which are *unitary*, meaning the sesquilinear form constructed above is positive-definite. In other words, the sesquilinear form on a Verma module is positive-semidefinite if and only if the highest weight is dominant integral, and the maximal proper submodule then coincides with the elements of zero norm. If this highest weight state of the Verma module is $\left|\widehat{\lambda}\right\rangle$, then this maximal proper submodule is in fact generated by the action of $\mathfrak{U}(\widehat{\mathfrak{g}}_-)$ on the states

$$e_{-\widehat{\alpha}_i}^{\lambda_i+1}\left|\widehat{\lambda}\right\rangle \qquad (i=1,2,\dots,r), \qquad \text{and} \qquad e_{-\widehat{\alpha}_0}^{\lambda_0+1}\left|\widehat{\lambda}\right\rangle = e_{\theta-\widehat{\delta}}^{k+1-(\lambda,\theta)}\left|\widehat{\lambda}\right\rangle.$$

Denoting the set of weights (with multiplicity) of the integrable highest weight representation of highest weight $\widehat{\lambda} = (\lambda, k, 0)$ by $\widehat{\Delta}^{\lambda}$, one can define the character of this representation to be

$$\mathrm{ch}_{\widehat{\lambda}} = \sum_{\widehat{\mu}\in\widehat{\Delta}^{\lambda}} e^{\widehat{\mu}}.$$

This is a formal power series in the (formal exponentials) of the fundamental weights. This character is invariant under the affine Weyl group $\widehat{\mathsf{W}}$, and may alternatively be expressed (analogously to Equation (A.3.3)) in the forms

$$\mathrm{ch}_{\widehat{\lambda}} = \frac{\sum_{\widehat{w}\in\widehat{\mathsf{W}}} \det \widehat{w}\; e^{\widehat{w}\left(\widehat{\lambda}+\widehat{\rho}\right)}}{e^{\widehat{\rho}}\prod_{\widehat{\alpha}\in\widehat{\Delta}_+}\left(1-e^{-\widehat{\alpha}}\right)^{\mathrm{mult}\,\widehat{\alpha}}} = \frac{\sum_{\widehat{w}\in\widehat{\mathsf{W}}} \det \widehat{w}\; e^{\widehat{w}\left(\widehat{\lambda}+\widehat{\rho}\right)}}{\sum_{\widehat{w}\in\widehat{\mathsf{W}}} \det \widehat{w}\; e^{\widehat{w}(\widehat{\rho})}},$$

where $\mathrm{mult}\,\widehat{\alpha}$ is the multiplicity of the root $\widehat{\alpha}$ (1 for real roots, r for imaginary roots). These formulae are known as the *Weyl-Kac character formula*.

Again, characters may be evaluated at elements of the weight space $\widehat{\mathfrak{t}}_{\mathbb{R}}^*$. When the character is restricted to multiples of $\widehat{\Lambda}_0$, then it is said to be *specialised*. This will be denoted by

$$\mathrm{ch}_{\lambda}^{\mathfrak{Vir}}(q) = \mathrm{ch}_{\widehat{\lambda}}\left(-2\pi\mathrm{i}\tau\widehat{\Lambda}_0\right) = \sum_{\widehat{\mu}\in\widehat{\Delta}^{\lambda}} e^{2\pi\mathrm{i}\langle\widehat{\mu},L_0\rangle} = \mathrm{tr}_{\widehat{\lambda}}\, q^{L_0},$$

where $q = e^{2\pi\mathrm{i}\tau}$, and the trace is over the integrable highest weight representation[3] of $\widehat{\mathfrak{g}}$ of highest weight $\widehat{\lambda}$.

The semidirect product structure $\widehat{\mathsf{W}} = \mathsf{W} \ltimes \mathsf{Q}^{\vee}$ may be used to decompose the $\widehat{\mathsf{W}}$ sums in the character formula into (finite) sums over W and (infinite) sums over $\mathsf{Q}^{\vee}$. The infinite sums over translations by $\mathsf{Q}^{\vee}$ turn out to give *generalised theta functions*, up to a multiplicative factor. These in turn have nice transformation properties under the *modular*

[3]The superscript "$\mathfrak{Vir}$" (for Virasoro) signifies that this specialisation only accounts for the eigenvalues of the Virasoro mode L_0 and therefore corresponds to the character of this $\widehat{\mathfrak{g}}$-module as a representation of the Virasoro algebra.

group $\mathsf{SL}(2;\mathbb{Z})$. Taking out the multiplicative factors defines the *normalised* characters,

$$\widehat{\chi}_{\widehat{\lambda}} = e^{-m_\lambda \widehat{\delta}} \mathrm{ch}_{\widehat{\lambda}}, \quad \text{where} \quad m_\lambda = \frac{\|\lambda+\rho\|^2}{2(k+\mathsf{h}^\vee)} - \frac{\|\rho\|^2}{2\mathsf{h}^\vee} = \frac{(\lambda,\lambda+2\rho)}{2(k+\mathsf{h}^\vee)} - \frac{k\dim\mathfrak{g}}{24(k+\mathsf{h}^\vee)},$$

where m_λ is the *modular anomaly*, and the last equality uses the strange formula, Equation (A.3.5). The specialisation of these normalised characters is then just

$$\widehat{\chi}_\lambda^{\mathfrak{Vir}}(q) = \mathrm{tr}_{\widehat{\lambda}}\, q^{L_0+m_\lambda}.$$

Of all these affine characters, it is only these specialised, normalised characters which will be (explicitly) used in this thesis[4].

The normalised characters (specialised or otherwise) of the integrable highest weight representations are then ratios of finite sums (over W) of generalised theta functions, and therefore might be expected to have nice modular properties. This is indeed the case. The modular group $\mathsf{SL}(2;\mathbb{Z})$ is generated by elements s and t which act on the level k normalised characters by

$$\widehat{\chi}_{\widehat{\lambda}}\left(\xi/\tau, -1/\tau, h+\|\xi\|^2/2\tau\right) = \sum_{\widehat{\mu}\in\widehat{\mathsf{P}}_k} S_{\lambda\mu}\widehat{\chi}_{\widehat{\mu}}(\xi,\tau,h),$$

$$\text{and} \qquad \widehat{\chi}_{\widehat{\lambda}}(\xi,\tau+1,h) = \sum_{\widehat{\mu}\in\widehat{\mathsf{P}}_k} T_{\lambda\mu}\widehat{\chi}_{\widehat{\mu}}(\xi,\tau,h),$$

respectively. The coefficients in each case therefore constitute matrices, unimaginatively called the modular S-matrix and T-matrix (respectively). The T-matrix turns out to be diagonal, with eigenvalues $e^{2\pi\mathrm{i}m_\lambda}$, and the S-matrix is given by the *Kac-Peterson* formula,

$$S_{\lambda\mu} = \frac{\mathrm{i}^{|\Delta_+|}}{\sqrt{|\mathsf{P}/\mathsf{Q}^\vee|\,(k+\mathsf{h}^\vee)^r}} \sum_{w\in\mathsf{W}} \det w\, e^{\left(w(\lambda+\rho),\xi_\mu\right)} \qquad \text{where } \xi_\mu = -2\pi\mathrm{i}\frac{\mu+\rho}{k+\mathsf{h}^\vee}.$$

The groups $\mathsf{P}/\mathsf{Q}^\vee$ are given in Table B.1.

Both the S and T-matrices turn out to be symmetric (under transposition *without* conjugation) and unitary. S^2 sends $\widehat{\chi}_{\widehat{\mu}}(\xi,\tau,h)$ to $\widehat{\chi}_{\widehat{\mu}}(-\xi,\tau,h)$. On the weight space of $\mathfrak{g}$, -1 coincides with the Chevalley automorphism $\omega_\mathrm{C} = \omega_+ w_\mathrm{L}$ (Appendix A.2), so W-invariance of the characters gives $\widehat{\chi}_{\widehat{\mu}}(-\xi,\tau,h) = \widehat{\chi}_{\widehat{\mu}^+}(\xi,\tau,h)$. That is, S^2 acts on the integrable highest weight characters by conjugation.

[4] In Chapter 3, the modular anomaly will be recognised as $m_\lambda = h_\lambda - c/24$, where h_λ is the L_0-eigenvalue of the highest weight vector $\left|\widehat{\lambda}\right\rangle$ (Equation (3.1.21)) and c is the central charge (Equation (3.1.15)). There, the Virasoro mode L_0 is identified with a multiple of the quadratic Casimir of the horizontal subalgebra, removing the ambiguity in its definition, and thus any way of setting its highest weight eigenvalue to 0. To compare with this appendix, the L_0-eigenvalue of every weight in $\widehat{\Delta}^\lambda$ must be reduced by h_λ. This cancels that factor in the modular anomaly, and so the specialised normalised characters which will actually be used in this thesis have the form

$$\widehat{\chi}_\lambda^{\mathfrak{Vir}}(q) = \mathrm{tr}_{\widehat{\lambda}}\, q^{L_0-c/24}. \tag{B.3.1}$$

Considering the Kac-Peterson formula for $\lambda = 0$, and the two forms of the Weyl character formula, Equation (A.3.3), one may write

$$\begin{aligned} S_{0\mu} &= \frac{\mathrm{i}^{|\Delta_+|}}{\sqrt{|\mathsf{P}/\mathsf{Q}^\vee|\,(k+\mathsf{h}^\vee)^r}} \sum_{w\in\mathsf{W}} \det w \; e^{\left(w(\rho),\xi_\mu\right)} \\ &= \frac{2^{|\Delta_+|}}{\sqrt{|\mathsf{P}/\mathsf{Q}^\vee|\,(k+\mathsf{h}^\vee)^r}} \prod_{\alpha\in\Delta_+} \sin\frac{\pi\,(\alpha,\mu+\rho)}{k+\mathsf{h}^\vee}. \end{aligned}$$

Since $0 < (\alpha,\rho) \leqslant (\alpha,\mu+\rho) \leqslant (\theta,\mu+\rho) < k+\mathsf{h}^\vee$ (as $\alpha \in \Delta_+$ and $\mu \in \widehat{\mathsf{P}}_k$), it follows that $S_{0\mu} > 0$. One may therefore consider the ratio

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = \frac{\sum_{w\in\mathsf{W}} \det w \; e^{\left(w(\lambda+\rho),\xi_\mu\right)}}{\sum_{w\in\mathsf{W}} \det w \; e^{\left(w(\rho),\xi_\mu\right)}} = \chi_\lambda\left(-2\pi\mathrm{i}\frac{\mu+\rho}{k+\mathsf{h}^\vee}\right). \tag{B.3.2}$$

This rather surprising relationship between the modular S-matrix of $\widehat{\mathfrak{g}}$ and the characters of $\mathfrak{g}$ is fundamental in the theory of fusion (Chapter 5).

APPENDIX C

Compact Lie Groups

In this appendix, some results from the theory of compact Lie groups will be presented. The groups of interest in this thesis are the (simply-connected) groups corresponding to the compact real forms of the simple complex Lie algebras discussed in Appendix A. Accordingly, they are said to be simple. This does not quite follow the usual definition in group theory, where a group is simple if it has no non-trivial normal subgroups. A Lie group is said to be *simple* if it possesses no non-trivial *connected, closed*, normal subgroups. Some standard references for the theory of compact Lie groups are [**1**, **39**, **52**, **73**].

In this thesis, use will be made of the topology of the underlying manifold of the compact Lie group. In particular, certain basic facts regarding the homology and cohomology of these Lie groups will be required, as will a rather more extensive understanding of their conjugacy classes. This is a classical subject which helped guide much of the early developments in algebraic topology, and has involved some of the most respected mathematical luminaries of the twentieth century. However, it seems that there are few dedicated texts treating this subject in any detail (one could mention [**116**] however), perhaps because it has been subsumed under such mantles as Schubert theory, the theory of spectral sequences, and algebraic geometry. Nevertheless, [**21**, **139**] provide useful (if rather old) introductions to this field. A rather nice recent reference is [**136**].

C.1. Basics

The *adjoint* action of a real Lie group G on itself is given by conjugation: $\mathrm{AD}(h)g = hgh^{-1}$. Suppose now that $g\colon \mathbb{R} \to \mathsf{G}$ is a (smooth) group homomorphism. Differentiating g at the identity then gives an element y of the tangent space $T_{\mathrm{id}}(\mathsf{G})$. Since $\mathrm{AD}(h)$ is an (inner) automorphism, $\mathrm{AD}(h)g$ is a smooth group homomorphism, and its derivative at id is denoted by $\mathrm{Ad}(h)y$. This defines the *adjoint* action of G on $T_{\mathrm{id}}(\mathsf{G})$, $\mathrm{Ad}\colon \mathsf{G} \to \mathsf{Aut}\, T_{\mathrm{id}}(\mathsf{G})$. Finally, differentiating Ad at the identity yields a map $\mathrm{ad}\colon T_{\mathrm{id}}(\mathsf{G}) \to \mathsf{End}\, T_{\mathrm{id}}(\mathsf{G})$, which defines the *adjoint* action of $T_{\mathrm{id}}(\mathsf{G})$ on itself. Putting $[x,y] = \mathrm{ad}(x)y$ gives a Lie bracket on $T_{\mathrm{id}}(\mathsf{G})$, which defines the Lie algebra of G, henceforth to be denoted by $\mathfrak{g}$.

This then relates the Lie algebra to the Lie group. Given an $x \in \mathfrak{g} = T_{\mathrm{id}}(\mathsf{G})$, there is a unique group homomorphism $\varphi_x\colon \mathbb{R} \to \mathsf{G}$ whose derivative at id is x. The *exponential* map $\exp\colon \mathfrak{g} \to \mathsf{G}$ may now be defined by $\exp(x) = \varphi_x(1)$. It coincides with the matrix exponential when G is the general linear group $\mathsf{GL}(n)$ (or one of its Lie subgroups). This is a smooth map, locally invertible at id and surjective when G is connected and compact,

with the *naturality* property, $f(\exp(x)) = \exp(f_*(x))$, where $f_*\colon \mathfrak{g} \to \mathfrak{h}$ is the derivative at the identity of the group homomorphism $f\colon \mathsf{G} \to \mathsf{H}$. Applying this naturality to the various adjoint actions gives

$$\begin{aligned} &\mathrm{AD}(h)\exp(x) = \exp(\mathrm{Ad}(h)x) \qquad \text{and} \qquad \mathrm{Ad}(\exp(x))y = e^{\mathrm{ad}(x)}y, \\ \Rightarrow \qquad &\exp(x)\exp(y)\exp(-x) = \exp\{\mathrm{Ad}(\exp(x))y\} = \exp\left(e^{\mathrm{ad}(x)}y\right), \end{aligned} \tag{C.1.1}$$

where e^x denotes the usual matrix exponential on $\mathsf{End}\,\mathfrak{g}$. This shows how conjugation on G descends to $\mathfrak{g}$. A related result is the *Baker-Campbell-Hausdorff* identity,

$$\exp(x)\exp(y) = \exp\left(x+y+\frac{1}{2}[x,y]+\frac{1}{12}[x,[x,y]]+\frac{1}{12}[[x,y],y]+\ldots\right), \tag{C.1.2}$$

where "…" refers to higher order (nested) Lie brackets of x and y. This identity expresses the group multiplication in terms of the Lie bracket.

From here on, suppose that G is compact and connected (so the exponential map is surjective). The counterpart of a maximal toral subalgebra (Cartan subalgebra) of $\mathfrak{g}$ is a *maximal torus* in G. That is, a maximal torus T has Lie algebra $\mathfrak{t}$ which is a maximal toral subalgebra of $\mathfrak{g}$. Every maximal torus turns out to be conjugate to any given maximal torus, and every element of G is contained in some maximal torus. A consequence of this is that the centre $\mathsf{Z}(\mathsf{G})$ must coincide with the intersection of all the maximal tori. A fixed maximal torus T is its own centraliser (by maximality), and its normaliser $\mathsf{N}(\mathsf{T})$ has finite index over T. The quotient group of elements preserving T (under AD), modulo those which act trivially, is called the *Weyl group* $\mathsf{W} = \mathsf{N}(\mathsf{T})/\mathsf{T}$.

Using Equation (C.1), one can check that the element $n_\alpha = \exp(e_\alpha)\exp(-e_{-\alpha})\exp(e_\alpha)$ satisfies

$$\mathrm{AD}(n_\alpha)\exp(x) = \exp\left(x - \langle\alpha, x\rangle\,\alpha^\vee\right), \tag{C.1.3}$$

for all $x \in \mathfrak{t}$. The AD-action of the (coset represented by) $n_\alpha \in \mathsf{W}$ therefore corresponds to the Weyl reflection w_α on $\mathfrak{t}$.

For each simple Lie algebra $\mathfrak{g}$, there is a unique compact, connected, *simply-connected* Lie group G whose Lie algebra is $\mathfrak{g}$. Indeed, a semisimple compact connected Lie group G is determined by its Lie algebra and its *fundamental group* $\pi_1(\mathsf{G})$. The compact, connected, simply-connected simple Lie groups are:

- $\mathsf{SU}(r+1)$, the *special unitary groups*,
- $\mathsf{Sp}(2r)$, the *symplectic groups*,
- $\mathsf{Spin}(n)$, the *spin groups*, and
- E_6, E_7, E_8, F_4, and G_2, the *exceptional (Lie) groups*.

The other compact connected Lie groups with a given simple Lie algebra are obtained by factoring out a subgroup of the centre, and this subgroup becomes the fundamental group of the quotient. For example, the *special orthogonal groups* have the form[1]

[1] These groups explain the names given to the simple Lie algebras and their untwisted affine extensions in Appendices A and B

$\mathsf{SO}(r) \equiv \mathsf{Spin}(r)/\mathbb{Z}_2$. In a sense, the simply-connected group is the "largest" (compact connected) group with this Lie algebra. It is further distinguished by the fact that *every* finite-dimensional representation π of $\mathfrak{g}$ (Appendix A.3) can be exponentiated to a corresponding representation of G. By naturality, $\exp(x)$ is then represented by $e^{\pi(x)}$.

The exponential map is locally invertible, hence its kernel is a discrete subset of $\mathfrak{g}$. Restricting to $\mathfrak{t}$, exp becomes a group homomorphism by Equation (C.1.2) (treating $\mathfrak{t}$ as an abelian group). The kernel of $\exp\colon \mathfrak{t} \to \mathsf{T}$ is therefore a discrete subgroup of $\mathfrak{t}$, called the *integral lattice*. Unless otherwise indicated, G will now refer to a compact, connected, simply-connected, simple Lie group. In any (finite-dimensional) representation π, an element x of the integral lattice must satisfy $e^{\pi(x)} = \mathrm{id}$. But the eigenvalues of $\pi(x)$ are the weights $\langle \lambda, x\rangle$, so it follows that $\langle \lambda, x\rangle \in 2\pi\mathrm{i}\mathbb{Z}$ for all[2] weights λ. Thus $x \in 2\pi\mathrm{i}\mathsf{Q}^\vee$, the coroot lattice. In fact, since every compact Lie group has a faithful representation, the integral lattice coincides with (a multiple of) the coroot lattice. That is,

$$\ker\exp = 2\pi\mathrm{i}\mathsf{Q}^\vee.$$

Note that from Appendix A.1, $2\pi\mathrm{i}\mathsf{Q}^\vee$ is contained in the compact real form. Note also that the W-action on the group, Equation (C.1.3), is now seen to correspond to a $\widehat{\mathsf{W}}_1$-action on the algebra, where $\widehat{\mathsf{W}}_1$ is the affine (or extended) Weyl group (at level 1) introduced in Appendix B.2. An element $x \in \mathfrak{t}$ (or its image under exp) is said to be *regular* if it is contained in the interior of an affine alcove (at level 1), and *singular* otherwise.

Consider now the centre of the (simply-connected) Lie group G. From Equation (C.1) it follows that $\exp(x) \in \mathsf{Z}(\mathsf{G})$ if and only if $\exp(y) = \exp\left(e^{\mathrm{ad}(x)}y\right)$ for all $y \in \mathfrak{g}$. Thus, $y = e^{\mathrm{ad}(x)}y \pmod{2\pi\mathrm{i}\mathsf{Q}^\vee}$ for all $y \in \mathfrak{g}$. The eigenvalues of $e^{\mathrm{ad}(x)}$ (on the complexification $\mathfrak{g}^{\mathbb{C}}$) are just 1 and $e^{\langle\alpha,x\rangle}$, for each root α. Taking y to be the corresponding eigenvectors (of varying length), it follows that $x \in 2\pi\mathrm{i}\mathsf{Q}^*$, the dual root lattice. Conversely, if $x \in 2\pi\mathrm{i}\mathsf{Q}^*$, then $e^{\mathrm{ad}(x)} = \mathrm{id}$. Therefore,

$$\exp(x) \in \mathsf{Z}(\mathsf{G}) \qquad \Longleftrightarrow \qquad x \in 2\pi\mathrm{i}\mathsf{Q}^*.$$

The map $\mathsf{Q}^* \to \mathsf{Z}(\mathsf{G})$ given by $x \mapsto \exp(2\pi\mathrm{i}x)$ is then a group homomorphism (by Equation (C.1.2)) with kernel $\mathsf{Q}^\vee$. Therefore, $\mathsf{Z}(\mathsf{G}) \cong \mathsf{Q}^*/\mathsf{Q}^\vee$. Referring back to Appendix B.2, it follows that the centre of G may be identified with the subgroup V appearing in the decomposition of $\mathsf{Out}\,\widehat{\mathfrak{g}}$:

$$\mathsf{Out}\,\widehat{\mathfrak{g}} = \mathsf{Out}\,\mathfrak{g} \ltimes \mathsf{V} \cong \mathsf{Out}\,\mathfrak{g} \ltimes \mathsf{Z}(\mathsf{G}).$$

Multiplication by an element of $\mathsf{Z}(\mathsf{G})$ therefore corresponds to the action of an affine outer automorphism on the weight space.

Similarly, one can consider the centraliser $\mathsf{Z}(\exp(x))$ of an arbitrary point $\exp(x) \in \mathsf{G}$. Since the maximal tori cover G, there is no loss in generality in assuming that $x \in \mathfrak{t}$. By Equation (C.1), $\exp(y) \in \mathsf{Z}(\exp(x))$ if and only if $\exp(y) = \exp\left(e^{\mathrm{ad}(x)}y\right)$. This

[2]The simply-connected hypothesis is necessary to ensure that every representation of $\mathfrak{g}$ exponentiates to a representation of G

centraliser is a closed subgroup of G, hence is a Lie group in its own right. Replacing y with ty and differentiating at $t = 0$, one finds that the Lie algebra of this centraliser $\mathfrak{z}(\exp(x))$ consists of those $y \in \mathfrak{g}$ satisfying $y = e^{\mathrm{ad}(x)}y$. It is easy to see that

$$\mathfrak{z}(\exp(x)) = \mathfrak{t} \oplus \mathrm{span}\{\mathrm{i}(e_\alpha + e_{-\alpha}), e_\alpha - e_{-\alpha} : \langle \alpha, x\rangle \in 2\pi\mathrm{i}\mathbb{Z}\}.$$

One simple consequence of this is that if $\exp(x)$ is singular, $\mathfrak{z}(\exp(x))$ is strictly larger than $\mathfrak{t}$, and there is a maximal torus containing both $\exp(x)$ and the exponential of any element in $\mathfrak{z}(\exp(x)) \ominus \mathfrak{t}$. It follows that singular elements correspond to those that belong to more than one maximal torus, and regular elements belong to exactly one.

It is a theorem of Steinberg [**149**] that the centraliser of any element of a compact, connected, simply-connected, simple Lie group is connected (and reductive). $\mathsf{Z}(\exp(x))$ is therefore connected and compact, hence determined by its Lie algebra $\mathfrak{z}(\exp(x))$ and its fundamental group. The latter can be determined as the quotient of the integral lattice of $\mathsf{Z}(\exp(x))$ by the coroot lattice of $\mathsf{Z}(\exp(x))$. Since $\mathsf{Z}(\exp(x))$ and G share the same maximal torus and (suitably restricted) exponential map, it follows that

$$\pi_1(\mathsf{Z}(\exp(x))) = \frac{\mathsf{Q}^\vee}{\mathsf{Q}^\vee_{\mathsf{Z}(\exp(x))}}.$$

If the centraliser is semisimple, then it is completely determined by its Lie algebra and how its Lie algebra sits inside $\mathfrak{g}$. For general reductive centralisers, some ambiguity remains. In any case, the abstract form of the centralisers $\mathsf{Z}(\exp(x))$ will be invariant under the action of an automorphism of $\mathfrak{g}$ on x. The exponential map extends this by invariance under translations by $\mathsf{Q}^\vee$, and the obvious invariance under multiplication by an element of the centre of G extends this further to translations by Q^*. Summarising then, the abstract form of the centralisers $\mathsf{Z}(\exp(x))$ is in fact invariant under the action of the affine automorphism group $\mathsf{Out}\,\widehat{\mathfrak{g}} \ltimes \widehat{W}$ (Appendix B.2) on x.

The centralisers $\mathsf{Z}(\exp(x))$ (with x in the fundamental affine alcoves) of the rank 2 Lie groups are shown in Figure C.1, up to some finite (reductive) ambiguities. When x is regular, the centraliser is just the maximal torus. Note that for $\mathsf{Sp}(4)$, this corrects a similar (incorrect) figure in [**119**]. Note also that there is a centraliser for G_2 whose fundamental group is torsion.

C.2. A Little Topology

Before introducing the topological properties of Lie groups which will be required in Chapters 6 and 7, a few results from algebraic topology will be mentioned. It is also convenient to introduce a few of the characteristic classes which will be encountered. This section therefore serves as a guide to the topology that is assumed in this thesis. More information may be found in [**29**, **94**, **144**].

The standard notions of homotopy, homology, and cohomology will be used throughout without comment. As is often the case in physics, the homology is *singular homology* (with coefficients in $\mathbb{Z}$), $\mathsf{H}_*(X;\mathbb{Z})$, and the cohomology is *deRham cohomology*,

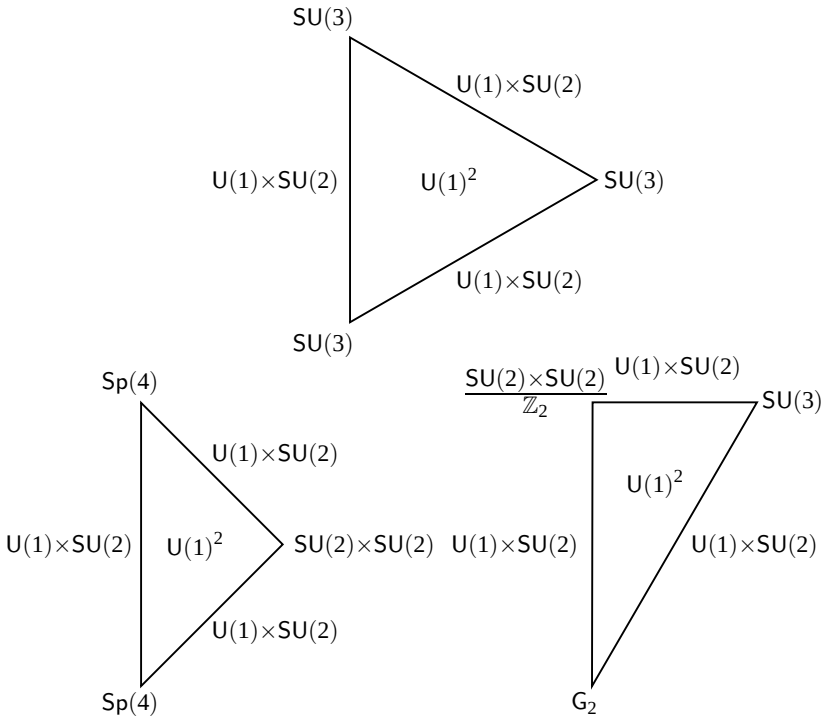

FIGURE C.1. The structures of the centralisers $\mathsf{Z}(\exp(x))$ for the Lie groups $\mathsf{SU}(3)$, $\mathsf{Sp}(4)$, and G_2. The point $x \in \mathfrak{t}$ may be restricted to the fundamental alcove without any loss in generality. Note the symmetry under the affine automorphisms (action of the centre). Note also that the direct product in the reductive centralisers is to be interpreted topologically, as there is still some unresolved ambiguity concerning the group structure. For example, $\mathsf{U}(2)$ is homeomorphic to $\mathsf{U}(1)\times\mathsf{SU}(2)$, but they are not isomorphic as groups (despite having isomorphic Lie algebras) as a computation of their centres shows.

$\mathsf{H}^*(X;\mathbb{R})$. The boundary and coboundary operators will be denoted by ∂ and d respectively. Elements of the integral cohomology ring will generally be identified with their image in deRham cohomology, $\mathsf{H}^*(X;\mathbb{Z}) \to \mathsf{H}^*(X;\mathbb{R})$ (when they are non-torsion). The pairing of $M \in \mathsf{H}_p(X;\mathbb{Z})$ and $\omega \in \mathsf{H}^p(X;\mathbb{R})$ will be denoted by $\int_M \omega$.

The corresponding *relative* homology and cohomology will be useful in Section 6.3. The relative homology of X with respect to $Y \subseteq X$ may be defined through the modified boundary operator

$$\widetilde{\partial} M = [\partial M] \qquad (\text{in } X/Y).$$

It is denoted by $\mathsf{H}_*(X,Y;\mathbb{Z})$. If the complex of forms on X is denoted by $\Omega^*(X)$, the relative (deRham) cohomology $\mathsf{H}^*(X,Y;\mathbb{R})$ is the cohomology of the complex $\Omega^p(X) \oplus \Omega^{p-1}(Y)$, $p \geqslant 0$, with respect to the coboundary operator

$$\widetilde{\mathrm{d}}(\omega,\eta) = (\mathrm{d}\omega, \omega - \mathrm{d}\eta).$$

The pairing is given by

$$\int_{[M]} (\omega,\eta) = \int_M \omega - \int_{\partial M} \eta. \tag{C.2.1}$$

One long exact sequence which will be used is the *Mayer-Vietoris sequence for attaching n-cells*. An n-cell E_n is just a closed n-dimensional disc, and it may be attached to a space X via $f\colon \partial E_n \to X$ by forming

$$X \cup_f E_n = \frac{X \sqcup E_n}{x \sim f(x)},$$

where $\sqcup$ denotes disjoint union, and $\sim$ identification. The homology of X and $X \cup_f E_n$ can be related by a Mayer-Vietoris long exact sequence which gives $\mathsf{H}_p\left(X \cup_f E_n; \mathbb{Z}\right) = \mathsf{H}_p(X; \mathbb{Z})$ when $p \neq n-1, n$, and the exact sequence

$$0 \to \mathsf{H}_n(X; \mathbb{Z}) \to \mathsf{H}_n\left(X \cup_f E_n; \mathbb{Z}\right) \to \mathbb{Z} \xrightarrow{f_*} \mathsf{H}_{n-1}(X; \mathbb{Z}) \to \mathsf{H}_{n-1}\left(X \cup_f E_n; \mathbb{Z}\right) \to 0. \tag{C.2.2}$$

Attaching an n-cell therefore yields an injection of the degree n homology and a surjection of the degree $n-1$ homology.

Consider now the characteristic classes of a (finite-rank) complex vector bundle $\pi\colon E \to M$ over a smooth manifold M. These are classes in $\mathsf{H}^*(M; \mathsf{A})$ (with some coefficient group A) which help to characterise the bundle. The most fundamental of these are the *Chern classes*. These may be defined in several ways (see [**24**] for seven!), of which one is through a connection on E. Any connection ∇ defines a curvature F_∇ which is a closed 2-form on M taking values in $\mathsf{End}\, E$. The *total Chern class* $\mathrm{c}(E)$ may now be defined as

$$\mathrm{c}(E) = \det\left(\mathrm{id} - \frac{1}{2\pi \mathrm{i}} F_\Lambda\right),$$

or rather, its image in $\mathsf{H}^*(M; \mathbb{R})$. The total Chern class (in the cohomology ring) turns out to be independent of the connection used. The homogeneous component of $\mathrm{c}(E)$ of degree $2j$ is called the j^{th} Chern class of E, and is denoted by $\mathrm{c}_j(E)$. These classes are actually integral cohomology classes.

When E is a complex vector bundle of rank n, the top Chern class $\mathrm{c}_n(E) \in \mathsf{H}^{2n}(M; \mathbb{Z})$ coincides with the *Euler class* $\mathrm{e}(E)$ of the realification of E (obtained by treating E as a real vector bundle of rank $2n$). The Euler class is so named because the Euler class of the tangent bundle of M is given by the *Euler characteristic*:

$$\int_M \mathrm{e}(T(M)) = \chi(M) = \sum_p (-1)^p \dim \mathsf{H}^p(M; \mathbb{R}).$$

This demonstrates directly the integrality of the top Chern class.

Given a rank n complex vector bundle $\pi\colon E \to M$ over a smooth manifold, there exists a manifold $\mathfrak{S}(E)$ and a smooth map $\iota\colon \mathfrak{S}(E) \to M$ such that $\iota^* E$ is a direct sum of n line bundles over $\mathfrak{S}(E)$, and $\iota^*\colon \mathsf{H}^*(M; \mathbb{R}) \to \mathsf{H}^*(\mathfrak{S}(E); \mathbb{R})$ is injective. Thus, as far as the cohomology ring is concerned, E behaves like a sum of line bundles. This important fact is known as the *splitting principle*. One consequence of this principle is that if $L_1, L_2, \ldots, L_n$ denote the line bundles that E splits into, then

$$\mathrm{c}(E) = \prod_{i=1}^n \mathrm{c}(L_i) = \prod_{i=1}^n (1 + \mathrm{c}_1(L_i)).$$

The j^{th} Chern class of E is therefore the j^{th} elementary symmetric polynomial in the $\mathrm{c}_1(L_i)$. It follows that every symmetric polynomial (and power series) in the $\mathrm{c}_1(L_i)$ may be expressed in terms of the Chern classes of E, $\mathrm{c}_j(E)$.

This makes it easy to define more general characteristic classes. The two (closely related) symmetric power series

$$\prod_{i=1}^{n} \frac{\mathrm{c}_1(L_i)/2}{\sinh\left(\mathrm{c}_1(L_i)/2\right)} \qquad \text{and} \qquad \prod_{i=1}^{n} \frac{\mathrm{c}_1(L_i)}{1-e^{-\mathrm{c}_1(L_i)}} \tag{C.2.3}$$

define power series in the $\mathrm{c}_j(E)$ called the *A-roof genus* $\widehat{\mathrm{A}}(E)$ and the *Todd class* $\mathrm{Td}(E)$, respectively. Note that $\mathrm{Td}(E) = e^{\mathrm{c}_1(E)/2}\widehat{\mathrm{A}}(E)$. The *Todd polynomials* $\mathrm{Td}_i(E)$ are then defined to be homogeneous components of degree i of the Todd class. The first few are [**95**]:

$$\begin{aligned} &\mathrm{Td}_0(E) = 1 && \mathrm{Td}_1(E) = \frac{1}{2}\mathrm{c}_1(E) \\ &\mathrm{Td}_2(E) = \frac{1}{12}\left[\mathrm{c}_1(E)^2 + \mathrm{c}_2(E)\right] && \mathrm{Td}_3(E) = \frac{1}{24}\mathrm{c}_1(E)\,\mathrm{c}_2(E). \end{aligned} \tag{C.2.4}$$

C.3. The Topology of Lie Groups

Consider now the conjugacy classes (AD orbits) of a Lie group G. The conjugacy class containing g will be denoted by $\mathcal{C}(g)$. Since the element $\mathrm{AD}(h)g = hgh^{-1} \in \mathcal{C}(g)$ is invariant under the transformation $h \mapsto hz$ for any $z \in \mathsf{Z}(g)$, it follows that the map

$$\frac{\mathsf{G}}{\mathsf{Z}(g)} \longrightarrow \mathcal{C}(g), \qquad h\mathsf{Z}(g) \longmapsto \mathrm{AD}(h)g,$$

is well-defined. In fact, it is a homeomorphism between $\mathcal{C}(g)$ and $\mathsf{G}/\mathsf{Z}(g)$. The Lie algebra $\mathfrak{z}(g)$ of the centraliser of g is clearly the subspace of $\mathfrak{g}$ preserved by $\mathrm{Ad}(g)$. Its orthogonal complement (with respect to the Killing form) is the set of elements of the form $y - \mathrm{Ad}(g)y$, where $y \in \mathfrak{g}$. Right-translating this set to $T_g(\mathsf{G})$, it becomes the set $\{yg - gy : y \in \mathfrak{g}\}$ which may be recognised as the tangent space at g to the conjugacy class $\mathcal{C}(g)$. In this way, one gets the orthogonal splitting

$$T_g(\mathsf{G}) = T_g(\mathsf{Z}(g)) \oplus T_g(\mathcal{C}(g)).$$

It is a general fact [**158**] that any closed subgroup Z of a Lie group G defines a *fibre bundle* $\pi\colon \mathsf{G} \to \mathsf{G}/\mathsf{Z}$ with fibre Z. There is a long exact sequence in homotopy for fibre bundles [**29**] which includes the sequence (assuming G is connected)

$$\cdots \longrightarrow \pi_1(\mathsf{G}) \longrightarrow \pi_1(\mathsf{G}/\mathsf{Z}) \longrightarrow \pi_0(\mathsf{Z}) \longrightarrow 0.$$

When G is simply-connected, $\mathsf{Z}(g)$ is connected for all $g \in \mathsf{G}$ by a theorem of Steinberg [**149**]. The above sequence then demonstrates that the conjugacy classes of a simply-connected Lie group are themselves simply-connected: $\pi_1(\mathcal{C}(g)) = 0$.

Similarly, the second homotopy group $\pi_2(\mathsf{G})$ always vanishes when G is a Lie group [**39**], so one gets the exact sequence

$$0 \longrightarrow \pi_2(\mathsf{G}/\mathsf{Z}) \longrightarrow \pi_1(\mathsf{Z}) \longrightarrow \pi_1(\mathsf{G}) \longrightarrow \cdots.$$

It follows that if G is simply-connected, $\pi_2(\mathsf{G}/\mathsf{Z}) = \pi_1(\mathsf{Z})$. In particular, when G is simply-connected, each conjugacy class $\mathcal{C}(g)$ is too, so this result and the Hurewicz isomorphism [**144**] imply that $\mathsf{H}_2(\mathcal{C}(g);\mathbb{Z}) = \pi_2(\mathcal{C}(g)) = \pi_1(\mathsf{Z}(g))$. Figure C.1 indicates that there is a centraliser in G_2 which has the form $(\mathsf{SU}(2)\times\mathsf{SU}(2))/\mathbb{Z}_2$, hence its fundamental group is $\mathbb{Z}_2$. It follows that the corresponding conjugacy class has torsion in its homology, thus also in its cohomology (by the universal coefficient theorem [**94**]). That is, there is a $g \in \mathsf{G}_2$ with

$$\mathsf{H}_2(\mathcal{C}(g);\mathbb{Z}) = \mathbb{Z}_2 \qquad \text{and} \qquad \mathsf{H}^3(\mathcal{C}(g);\mathbb{Z}) = \mathbb{Z}^m \oplus \mathbb{Z}_2$$

(for some $m \geqslant 0$).

Finally, the third homotopy group $\pi_3(\mathsf{G})$ is $\mathbb{Z}$ when G is a (connected) simple Lie group [**38**]. A compact, connected, simply-connected, simple Lie group G therefore has $\mathsf{H}^3(\mathsf{G};\mathbb{Z}) = \mathsf{H}_3(\mathsf{G};\mathbb{Z}) = \pi_3(\mathsf{G}) = \mathbb{Z}$.

For fibre bundles, the counterpart in (co)homology of the homotopy long exact sequence is the spectral sequence of Leray (see [**29**, **93**]) which turns out to be extremely powerful. For example, $\mathsf{SU}(r+1)/\mathsf{SU}(r) \cong S^{2r+1}$ (the $2r+1$-dimensional sphere) defines a fibre bundle whose spectral sequence yields an easy inductive proof that the cohomology ring of $\mathsf{SU}(r+1)$ is an exterior algebra:

$$\mathsf{H}^*(\mathsf{SU}(r+1);\mathbb{Z}) = \wedge_{\mathbb{Z}}[s_3, s_5, \ldots, s_{2r+1}],$$

where the generators s_i have degree i. A similar result is true for the symplectic groups $\mathsf{Sp}(2r)$, but the other (compact, connected, simply-connected) simple Lie groups have torsion in their cohomology rings [**21**]. However, the real cohomology rings of these groups are always exterior algebras — this is Hopf's theorem — and the number of generators (excluding the unit) coincides with the rank of the group. These generators necessarily have odd degree $2m_i+1$, and the integers m_i, $i = 1, 2, \ldots, r$ coincide with the *exponents* of $\mathfrak{g}$ (listed in Table A.1).

To apply a spectral sequence, one needs an appropriate quotient G/Z as well as (co)homological information about this quotient and the group Z. One of the first general results giving detailed information about the cohomology of (a class of) these quotients was obtained by Bott [**26**]. His proof, surprisingly, was obtained through an application of Morse theory [**113**, **114**].

THEOREM C.1. *Let* G *be a compact, connected, simply-connected, simple Lie group, and let* Z *be the centraliser of a* non-trivial *torus in* G. *Then, the (co)homology of* G/Z *has no elements of odd degree and no torsion. When* $\mathsf{Z} = \mathsf{T}$ *is a maximal torus of* G, *then there is a bijection between the set of generators of the cohomology ring and the Weyl*

group W*, such that the degree of the generators is twice the length of the corresponding Weyl transformation.*

There is a simple proof when $\mathsf{Z} = \mathsf{T}$ [**136**]. Choose x regular and define a Morse function $\mathcal{M}\colon \mathsf{G}/\mathsf{T} \to \mathbb{R}$ by

$$\mathcal{M}(g\mathsf{T}) = -\kappa(\mathrm{Ad}(g)x, x).$$

By considering a generating set of vector fields on G/T, one shows that the critical points of this function are precisely the $n_w\mathsf{T} \in \mathsf{N}(\mathsf{T})/\mathsf{T} = \mathsf{W}$, the Weyl group. The hessian matrix of $\mathcal{M}$ is non-degenerate at these critical points, and a careful analysis (using the basis of the compact real form given in Appendix A.1) shows that the eigenvalues are precisely given by

$$\langle \alpha, x\rangle \left\langle w^{-1}(\alpha), x\right\rangle, \qquad \alpha \in \Delta,$$

where w is the Weyl transformation corresponding to the critical point $n_w\mathsf{T}$. The index of the critical point (the number of negative eigenvalues) is then given by

$$2\left|\left\{\alpha \in \Delta_+ \colon w^{-1}(\alpha) \in \Delta_-\right\}\right| = 2\ell\left(w^{-1}\right) = 2\ell(w).$$

Morse theory now states that G/T is homotopic to a complex built out of cells labelled by the $w \in \mathsf{W}$, and whose dimension is $2\ell(w)$. The homology of G/T now follows from the Mayer-Vietoris sequence for attaching, Equation (C.2.2), and it is easily seen that the even-dimensional nature of the cells implies that each $w \in \mathsf{W}$ contributes a free factor $\mathbb{Z}$ to the homology in degree $2\ell(w)$. Indeed, the closure of these cells forms a basis of homology cycles. The cohomology now follows from Poincaré duality (or the universal coefficient theorem).

The 2-cells in this construction correspond to the simple Weyl reflections w_i, $i = 1, 2, \ldots, r$. For each i, the eigenvectors corresponding to the negative eigenvalues are $\mathrm{i}(e_{\alpha_i} + e_{-\alpha_i})$ and $e_{\alpha_i} - e_{-\alpha_i}$ which suggests that the homology 2-cycles of G/T should be realised in the $\mathsf{SU}(2)$ subgroup of G corresponding to the simple root α_i. In fact, Bott and Samelson [**28**] show that these homology 2-cycles may be taken to be a two-dimensional conjugacy class of these $\mathsf{SU}(2)$ subgroups (translated into the regular conjugacy class homeomorphic to G/T).

The centraliser of any given $g \in \mathsf{G}$ obviously coincides with the centraliser of the cyclic subgroup it generates. Indeed, since multiplication is continuous, $\mathsf{Z}(g)$ coincides with the centraliser of the *closure* of the cyclic subgroup it generates. Since g belongs to some maximal torus T, it follows that the closure of the cyclic subgroup it generates will be a (non-trivial) torus *unless* some power of g is the identity (in which case the cyclic subgroup will be finite). At the Lie algebra level, the centraliser of $\exp(x)$ is then the centraliser of a torus unless some integer multiple of x is in the integral (coroot) lattice $\mathsf{Q}^\vee$. It seems logical to call such x *rational* and all others *irrational*, so irrational elements have centralisers which are centralisers of tori.

But, it was shown in Appendix A.1 that $\mathsf{Z}(\exp(x))$ is completely determined by the set of roots α which satisfy $\langle\alpha, x\rangle \in 2\pi\mathrm{i}\mathbb{Z}$. It follows that unless x is a vertex of some affine alcove, there will be an irrational element arbitrarily close to x which shares this same set of roots, hence shares the same centraliser. Therefore $\mathsf{Z}(\exp(x))$ is the centraliser of a torus unless x is a vertex of an affine alcove. Indeed, it is easy to see that when x is a vertex, it is necessarily rational, and the corresponding centraliser cannot be the centraliser of any torus in G. This proves the following:

PROPOSITION C.2. *If G is a compact, connected, simply-connected, simple Lie group of rank r, then with exactly $r+1$ exceptions (corresponding to the vertices of the fundamental affine alcove), every centraliser $\mathsf{Z}(g)$ is the centraliser of some torus in G.*

COROLLARY C.3 (to Theorem C.1). *If G is a compact, connected, simply-connected, simple Lie group of rank r, then with at most r exceptions, the cohomology rings of the conjugacy classes of G have no odd degree elements and no torsion.*

The vertex 0 (as with any vertex in Q^*) corresponds to a zero-dimensional conjugacy class which certainly has no odd cohomology or torsion. Thus there are at most r exceptions[3]. It was remarked earlier that there is a (unique) conjugacy class in G_2 whose homology and cohomology have torsion. This shows that there are genuine counterexamples to the common claim that conjugacy classes have non-torsion cohomology.

Whilst Bott's theorem gives a detailed understanding of the *integral* cohomology groups of G/T, it does not directly aid the understanding of the ring structure (although Morse theory can be used to determine this). The structure of the cohomology ring was investigated, quite generally, by Borel who employed detailed properties of the spectral sequences of various fibre bundles that can be associated with a given Lie group G. It is convenient at this stage to introduce an extremely elegant formalism with which one can discuss cohomology in terms of Lie-theoretic data. This formalism is nicest for the quotient G/T and may be expressed through the following diagram of (natural) isomorphisms:

$$\begin{array}{ccccccc}
\mathsf{H}_2(\mathsf{G}/\mathsf{T};\mathbb{Z}) & \xrightarrow{\tau} & \mathsf{H}_1(\mathsf{T};\mathbb{Z}) & & & & \\
\downarrow & & \downarrow & & & & \\
\pi_2(\mathsf{G}/\mathsf{T}) & \longrightarrow & \pi_1(\mathsf{T}) & \longrightarrow & \ker\{\exp\colon \mathfrak{t}\to\mathsf{T}\} & \longrightarrow & \mathsf{Q}^\vee.
\end{array}$$

Here, τ denotes the transgression (in the spectral sequence), which is followed by the isomorphism resulting from the facts that the first homology group is the abelianisation of the fundamental group, and the fundamental group of a Lie group is already abelian.

[3]It was remarked at the end of Appendix C.1 that Figure C.1 corrects a similar figure for $\mathsf{Sp}(4)$ in [**119**]. The centralisers of the form $(\mathsf{U}(1)\times\mathsf{SU}(2))/\mathbb{Z}_2$ given there do not, despite appearances, contradict Proposition C.2 and thus Bott's theorem, Theorem C.1. Indeed, $\pi_1((\mathsf{U}(1)\times\mathsf{SU}(2))/\mathbb{Z}_2) = \mathbb{Z}$, and in fact,

$$\frac{\mathsf{U}(1)\times\mathsf{SU}(2)}{\mathbb{Z}_2} \cong \mathsf{U}(2) \cong \mathsf{U}(1)\times\mathsf{SU}(2).$$

Note that this centraliser is *reductive*, not semisimple.

The other isomorphisms in the square are the Hurewicz isomorphism followed by the isomorphism from the long exact sequence in homotopy.

Upon taking duals, one has the natural isomorphism

$$\mathsf{H}^2\left(\mathsf{G}/\mathsf{T};\mathbb{Z}\right) = \mathsf{Hom}\left(\mathsf{H}_2\left(\mathsf{G}/\mathsf{T};\mathbb{Z}\right)\right) \cong \left(\mathsf{Q}^\vee\right)^* = \mathsf{P}.$$

In other words, the second integral cohomology of G/T (which is torsion free) can be naturally associated with the weight lattice. The generators of this cohomology then correspond to the fundamental weights Λ_i ($i = 1, 2, \ldots, r$), and general elements of the cohomology ring (which is even and torsion free) correspond to (formal) polynomials in the fundamental weights. In this formalism, Borel's famous result on the cohomology of G/T is [**19**]:

THEOREM C.4. *Let* G *be a compact, connected, simply-connected, simple Lie group of rank* r*, and* W *be its Weyl group. Then the* rational *cohomology ring of* G/T *is generated by the fundamental weights (and the unit) modulo the* W*-invariant polynomials of positive degree. Specifically,*

$$\mathsf{H}^*\left(\mathsf{G}/\mathsf{T};\mathbb{Q}\right) = \frac{\mathbb{Q}\left[\Lambda_1, \Lambda_2, \ldots, \Lambda_r\right]}{\mathcal{I}_+}, \tag{C.3.1}$$

where $\mathcal{I}_+$ *is the ideal generated by the* W*-invariants of positive degree.*

When the cohomology of G *has no torsion, that is* $\mathsf{G} = \mathsf{SU}(r+1)$ *or* $\mathsf{Sp}(2r)$*, then the result also holds over the integers.*

If Z is a subgroup of G with the same maximal torus, then Borel has generalised this result to G/Z [**19**, **22**]. The result is that the rational cohomology is isomorphic to $\mathcal{I}^{\mathsf{Z}}/\mathcal{I}_+$, where $\mathcal{I}^{\mathsf{Z}}$ is the ring of invariants of the Weyl group of Z (in $\mathbb{Q}\left[\Lambda_1, \Lambda_2, \ldots, \Lambda_r\right]$). Note that T has no roots, hence its Weyl group is trivial and $\mathcal{I}^{\mathsf{T}} = \mathbb{Q}\left[\Lambda_1, \Lambda_2, \ldots, \Lambda_r\right]$.

Using Theorem C.4, the rational cohomology of a compact, connected, simply-connected, simple Lie group can be computed in various ways [**20**, **136**], though it will not be needed in this thesis. The integral cohomology ring is much harder to compute. In particular, the form of this ring for the exceptional groups resisted attack for quite some time. Results for these groups and the corresponding quotients with the maximal tori (except E_8) can be found in [**20**, **21**, **27**, **123**, **150**, **151**].

There is one further result that will be required in this thesis. This is a determination of the characteristic classes for the tangent bundle of G/T. This tangent bundle is a real vector bundle of rank $\dim \mathsf{G}/\mathsf{T} = 2\left|\Delta_+\right|$, and may be considered a complex vector bundle of rank $\left|\Delta_+\right|$. Given this rank and the formalism of Equation (C.3.1), one is led to conjecture that the characteristic classes have something to do with the positive roots of G. The relationship is made precise by a theorem of Borel and Hirzebruch [**23**]:

THEOREM C.5. *With the above formalism, the first Chern classes of the line bundles associated to* $T(\mathsf{G}/\mathsf{T})$ *under the splitting principle (Appendix C.2), are precisely the*

positive roots of G. *The total Chern class of* $T(\mathsf{G}/\mathsf{T})$ *is therefore given by*

$$\mathsf{c}\left(T(\mathsf{G}/\mathsf{T})\right) = \prod_{\alpha\in\Delta_+} (1+\alpha)\,.$$

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