Analysis of inhomogeneous optical systems by the use of ray tracing. II. Three-dimensional systems with symmetry

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We describe a new approach to the index reconstruction of three-dimensional optical systems with rotational symmetry, which is based on sampling ray paths that lie in the sagittal plane. Since the observed rays are distorted by the optical system itself, they cannot be used directly for index reconstruction. We present an iterative procedure to compute the true ray paths and then to find the index distribution. The utility of the method is verified on the model problem. © 1998 Optical Society of America

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1. Introduction

A rotationally symmetric inhomogeneous optical system may be obtained by the rotation of a generic planar system with the refractive index $n(x, y)$ around its axis of symmetry OX. A typical example of such a system is the crystalline lens of the eye, shown in Fig. 1. The determination of the refractive-index distribution is of great importance for vision science,[1–6] and several methods for index reconstruction have been reported.[1–3,6] The direct methods, such as that reported by Jagger,[2] allow one to find the refractive index on slices of an isolated crystalline lens and then to reconstruct the three-dimensional (3-D) index distribution. Currently available nondestructive methods of index reconstruction[3,6] compute the index $n(r)$ in the equatorial plane of the lens (by inversion of the optical path or deflection angle of the rays that propagate in the equatorial plane) and then use an elliptical model of the lens, in which the isoindicial curves are half-ellipses sharing the major axis.

In this paper we describe an alternative approach to nondestructive 3-D index reconstruction based on the observation of rays that propagate in the sagittal plane of the lens. In previous papers[7,8] we described a method of index reconstruction of an asymmetric planar optical system that used a ray-tracing analy-

sis. We apply this technique to reconstruct the index in the sagittal plane, and then we obtain the 3-D index distribution by using the rotational symmetry of the lens.

In the ray-tracing analysis, the paths of a set of sampling rays are observed at right angles to the sagittal plane and recorded. The coordinates of points that form sampling rays are the initial data for index reconstruction. If the paths of observed rays were the true ray paths, then one might apply the method from our previous papers[7,8] directly. However, the registered image of the sampling rays is distorted by the optical system itself, and the true positions of the rays must be found in order to apply the tracing analysis. In this paper we develop an iterative algorithm to compute the true ray paths and the refractive index.

To analyze the correctness of our method we have used the elliptical model of a crystalline lens. First we numerically generate a set of sampling rays as if it were observed in a real experiment. Then we use these rays to reconstruct the refractive index. Finally we compare the results of our method to the true refractive index used in the model.

2. Simulation of the Ray-Tracing Analysis

To compute the paths of sampling rays distorted by the lens, we need to solve two problems. First, given the model index distribution in the sagittal plane, we compute the true ray paths by the use of ray tracing.[7,9] Then we must perform 3-D ray tracing to find the distorted image of the sampling rays.

The ray tracing in three dimensions is similar to the ray tracing in a plane, but it involves a larger system of equations. Provided that the index has

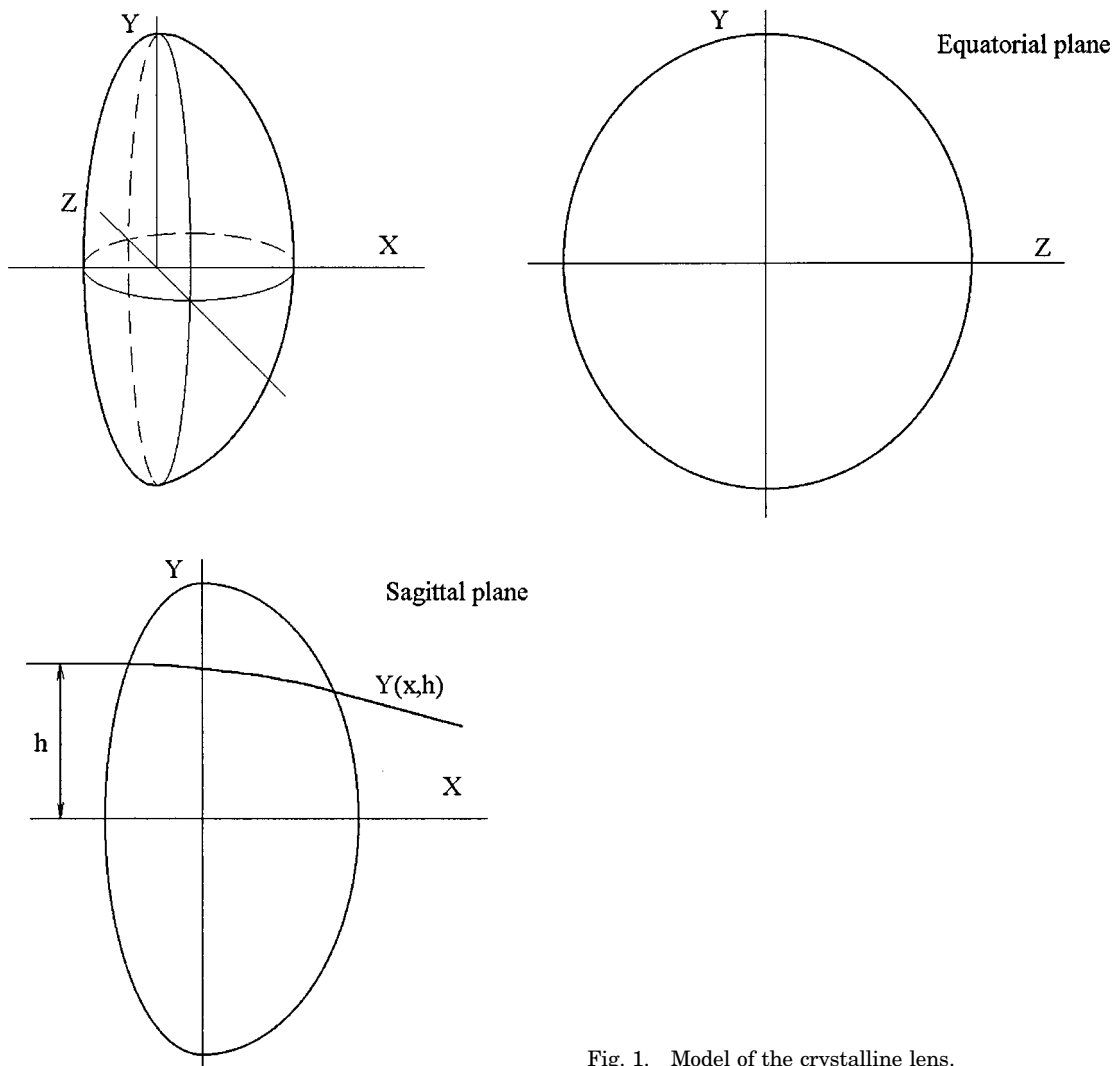The authors are with the Department of Mathematics, The University of Melbourne, Parkville VIC 3052, Australia.

Fig. 1. Model of the crystalline lens.

modest variation, the ray equation of geometric optics[10] relates a vector position of the ray **r** and refractive index function $n(\mathbf{r})$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right) = \nabla n,$$

in which $\mathrm{d}s$ denotes an element of the ray path. For a refractive-index function $n(x, y, z)$, we can write the above vector equation in component form,[7,10] i.e.,

$$\frac{nx''}{1 + x'^2 + y'^2} + x'n_z - n_x = 0,$$

$$\frac{ny''}{1 + x'^2 + y'^2} + y'n_z - n_y = 0, \tag{1}$$

where $\{x(z), y(z)\}$ denotes the ray path parameterized by z, the prime denotes a differentiation with respect to z, and $n_x, n_y,$ and n_z are partial derivatives of n; see Fig. 2. The refraction of rays on the interface between two media is described by the Snell law.
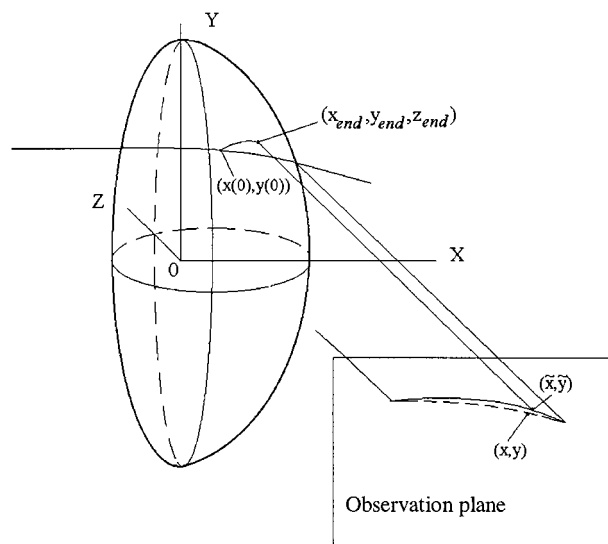

Fig. 2. Formation of a distorted image by the crystalline lens. On the observation plane the dashed curve denotes the true ray path, and the solid curve denotes the observed distorted ray path.

Let the set of true sampling rays[7] be given by

$$\{(x_{ij}, y_{ij}): \quad y_{ij} = Y(x_{ij}, h_j), \quad i = 1, 2, \ldots, N_j,$$

$$j = 1, 2, \ldots, M\}, \tag{2}$$

where x_{ij} is the abscissa of the ith point on the ray j, y_{ij} is the ordinate of this point, N_j is the number of registered points on the ray j, and M is the number of sampling rays. The sampling rays lie in the plane OXY (Fig. 2) and are observed at a right angle from a point at infinity. The observed coordinates of points that form sampling rays, $(\tilde{x}_{ij}, \tilde{y}_{ij})$, differ from the true positions (x_{ij}, y_{ij}) and can be calculated as follows. Consider rays coming from the point (x_{ij}, y_{ij}). One of these rays is parallel to the Z axis when it leaves the lens at the point $(x_{\text{end}}, y_{\text{end}}, z_{\text{end}})$, and its point of intersection with a plane parallel to OXY (call it the observation plane) is the required image $(\tilde{x}_{ij}, \tilde{y}_{ij}) = (x_{\text{end}}, y_{\text{end}})$.

The path of the ray inside the lens is computed by solving Eqs. (1) numerically, with n given by the model and with the initial condition $[x(0), y(0)] = (x_{ij}, y_{ij})$. In contrast to the usual ray tracing, however, the direction of the ray at the point of incidence (x_{ij}, y_{ij}) is unknown. Instead, we have a condition on the other end,

$$[x'(z_{\text{end}}), y'(z_{\text{end}})] = (x_{\text{end}}', y_{\text{end}}'),$$

where $[x(z_{\text{end}}), y(z_{\text{end}}), z_{\text{end}}] = (x_{\text{end}}, y_{\text{end}}, z_{\text{end}})$ is a point on the lens boundary in which the ray leaves the lens. In the index-matched case, i.e., when the index on the lens boundary is constant and coincides with the index of the surrounding media, $(x_{\text{end}}', y_{\text{end}}') = (0, 0)$, the ray leaving the lens is parallel to O_z, and no refraction takes place. In the case of index mismatch, the direction of the ray inside the lens $(x_{\text{end}}', y_{\text{end}}')$ is computed by applying the Snell law on the lens boundary, provided that after refraction the ray is parallel to Oz.

Hence, instead of an initial value problem (a so-called Cauchy problem) for Eqs. (1), in which all the initial conditions are given on one end of the interval $[0, z_{\text{end}}]$, we have a boundary value problem, in which one of the conditions is given at $z = 0$, and the other at an unknown point $[x(z_{\text{end}}), y(z_{\text{end}}), z_{\text{end}}]$. We can solve the problem numerically by using the shooting method,[11] which consists of specifying both conditions on one end and solving the corresponding initial value problem, and then checking whether the condition on the other end is satisfied. For every point (x_{ij}, y_{ij}) from the set given Eq. (2), we vary the x, y coordinates of the point $[x(z_{\text{end}}), y(z_{\text{end}}), z_{\text{end}}]$ (z_{end} is expressed through x, y in the equation of the lens boundary), and we solve an initial value problem for Eqs. (1) in the interval $[z_{\text{end}}, 0]$ with the initial conditions

$$[x(z_{\text{end}}), y(z_{\text{end}})] = (x, y),$$

$$[x'(z_{\text{end}}), y'(z_{\text{end}})] = (x_{\text{end}}', y_{\text{end}}')$$

by using the Runge–Kutta method,[11] until the solution on the other end, $[x(0), y(0)]$, is within a given neighborhood of the point (x_{ij}, y_{ij}). Then the initial condition (x, y) used in the latest iteration is the required image $(\tilde{x}_{ij}, \tilde{y}_{ij})$.

To compute the images of all points constituting the sampling rays, we have to repeat the process described above many times. Let us denote this process by operator D, so that for any point in the sagittal plane (x, y), its image $(\tilde{x}, \tilde{y}) = D[(x, y)]$. Operator D^{-1} is the inverse of D, i.e., $D^{-1}[(\tilde{x}, \tilde{y})]$ denotes a point in the sagittal plane whose image is $(\tilde{x}, \tilde{y})$. Repetitive application of operator D to all the points on the sampling rays gives the images of the rays that would be observed at a right angle to the sagittal plane that model the experimental data.

Because sampling rays contain many points, this process of generation of the model of experimental data is inefficient. To improve the method, we observe that the application of operator D^{-1} is far less time-consuming than the application of D to any given point. This is because to find the image of a point in the sagittal plane we have to solve a boundary value problem, whereas to find the preimage of a point in the observation plane we solve an initial value problem, whose numerical solution is straightforward. Then it is possible to create a look-up table, the preimage of the rectangular mesh $\{(\tilde{a}_i, \tilde{b}_j), i = 1, \ldots, I, j = 1, \ldots, J\}$ in the observation plane, by using $(a_i, b_j) = D^{-1}[(\tilde{a}_i, \tilde{b}_j)]$ and taking a sufficiently large number of points I and J in each direction. After the points (a_i, b_j) have been computed, we can find the image $(\tilde{x}_{ij}, \tilde{y}_{ij})$ of a given point on the true ray path (x_{ij}, y_{ij}) by looking up for the knot (a_k, b_l) nearest to (x_{ij}, y_{ij}) and by putting $(\tilde{x}_{ij}, \tilde{y}_{ij}) \approx (\tilde{a}_k, \tilde{b}_l)$. The precision depends on the number of knots in the mesh, I and J, and on how accurately the preimage of the mesh has been computed.

The main advantage of computing the look-up table is that the large numerical computation, repetitive solution of Eqs. (1) with different initial conditions, has to be carried out only once, whereas the direct application of the shooting method would have required doing this for every point in the set given in Eq. (2). Figure 3 shows the true ray paths in the plane OXY, and their distorted image in the observation plane, computed as described above. The same pattern is clearly seen on the actual pictures of the ray paths, like those in Ref. 2. The generated distorted image of the ray paths in Fig. 3(b) represents a model of experimental data that one obtains in observing ray paths lying in the sagittal plane. For Fig. 3 the refractive index n is given by $n(x, y) = n_{\max}[1 - \delta(p^2 x^2 + y^2)]^{1/2}$, with $\delta = 0.2$, $n_{\max} = (1 - \delta)^{-1/2}$, and

$$p = \begin{cases} 1.8, & \text{if } x \leq 0, \\ 1.4, & \text{if } x > 0. \end{cases}$$

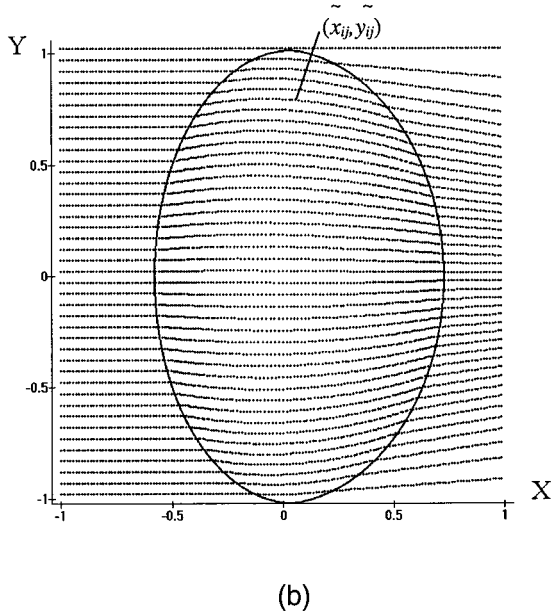In what follows this information is used as the initial data for index reconstruction.
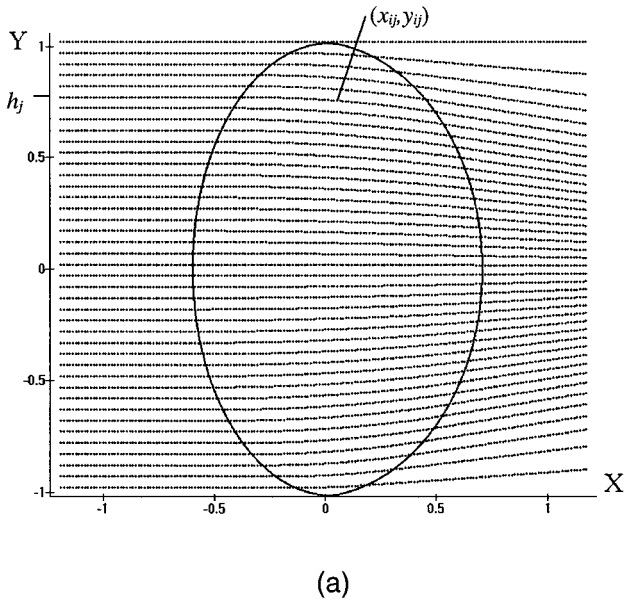
(a)

(b)

Fig. 3. (a) True ray paths. (b) Observed ray paths distorted by the optical system. Note the nonmonotonicity of the rays inside the lens.

3. Refractive-Index Reconstruction

At this point we assume that the refractive index is unknown, and the experimental data are given in the form of distorted images of the points on sampling rays:

$$\tilde{X} = \{(\tilde{x}_{ij}, \quad \tilde{y}_{ij}), \quad i = 1, 2, \ldots, N_j, \quad j = 1, 2, \ldots, M\}. \quad (3)$$

Let operator $N[X]$ denote the solution of the index-reconstruction problem given by the method in Ref. 7, where X denotes the set of analyzed rays, and let operator $D_n[X]$ denote the distortion of the rays X by the optical system whose index is n.

Since the paths $\tilde{X}$ are not the true ray paths, we cannot apply the algorithm from Ref. 7 to find the refractive index. We may write, however, the following fixed point equation:

$$n = N[D_n^{-1}[\tilde{X}]]. \quad (4)$$

It says that if the true ray paths X were distorted by the system $n(x, y, z)$, $\tilde{X} = D_n[X]$, then the inverse operator D_n^{-1} applied to the distorted rays would give the true ray paths, which in turn would determine the refractive index. Then our aim is to create on the basis of Eq. (4) an iterative algorithm that would converge to the fixed point n—the true refractive index.

One difficulty is that the transformation $f(n) \equiv N[D_n^{-1}[\tilde{X}]]$ is not a contraction, i.e., the condition $\|f(n_1) - f(n_2)\| < \|n_1 - n_2\|$ for any n_1, n_2 is not satisfied. In other words, if two refractive-index functions n_1 and n_2 are similar, then this inverse method does not guarantee that the reconstructed index functions will necessarily be close together. This is because the method of ray-tracing analysis[7,8] uses the first and second derivatives of the ray paths. Given small variations of the ray paths, the variations of the second derivative and the consequential variations of the computed index may be large, and this fact does not depend on the method of numerical derivation being used. As with most inverse problems, the problem of index reconstruction is ill conditioned, i.e., small errors in the initial data may lead to large errors in the solution. As a consequence, the direct iterative algorithm in the form

$$n_{k+1} = N[D_{n_k}^{-1}[\tilde{X}]],$$

where k denotes the iteration number, does not converge. Moreover, it diverges even if the initial iteration n_0 is picked up in a small neighborhood of the solution.

A way around this difficulty is to use a suitable method of regularization[12] to smooth out the sensitivity of the ray-tracing method to small variations in the ray paths. In a previous paper[7] we used polynomial smoothing splines to approximate the ray paths. Since smoothing splines are the functions belonging to the Sobolev space,[12]

$$W_2^m(\Omega) = \{f(x): \quad f \in C^{m-1}(\Omega), f^{(m)} \in L_2(\Omega)\},$$

which is the space or family of functions that are $m - 1$ times continuously differentiable (in the generalized sense) and whose mth derivative is square integrable on Ω, then for the splines of the fifth degree we have used the second derivative is continuous, and therefore the variations of n, given finite errors in the data, are bounded. Moreover, the growth of the derivatives is suppressed by the smoothing parameter. Therefore we may find a constant $\alpha \in [0, 1]$ such that the operator $\hat{N}[X] = \alpha N[X] + (1 - \alpha)n$ is a contraction, and the iterative scheme converges.

The resulting iterative scheme takes the form

$$n_{k+1} = \alpha N[D_{n_k}^{-1}[\tilde{X}]] + (1 - \alpha)n_k, \quad 0 < \alpha < 1. \quad (5)$$

The value of α should be sufficiently small to guarantee the convergence but still be large enough to make the algorithm converge at an acceptable rate. This value, of course, depends on the particular refractive index and particular set of sampling rays. For the model elliptical refractive index we have used, and the set of 40 sampling rays with 30–100 points on each and small, uniformly distributed noise with amplitude in the interval [−0.0005, 0.0005] to simulate experimental errors, the value $\alpha = 0.15$ requires only eight iterations for convergence. The maximum absolute error of the refractive index was 0.0108, and the average error was 0.00256. These results are shown in Fig. 4.

4. Implementation

Here we discuss how the iterative scheme of Eq. (5) was implemented. The detailed description of the index reconstruction by the ray tracing, denoted here by the operator $N[X]$, is given elsewhere.[7,8] It involves a numerical solution of a first-order partial differential equation by integration along the characteristic lines. The projections of the characteristic lines on the plane OXY coincide with the wave fronts, which are the curves orthogonal to the sampling rays. Consequently, the resulting refractive index is given by means of its values on the wave fronts. This representation is not convenient for our purpose, because on each iteration in Eq. (5) the rays, and consequently the wave fronts, change. Therefore, we translate the results of Ref. 7 into the refractive index given on a rectangular mesh $n_{ij} = n(x_0 + ih_x, y_0 + jh_y)$, $i = 0, \ldots, I, j = 0, \ldots, J$, which covers the area of interest. Then we construct a tensor–product polynomial spline $S(x, y)$ that approximates n between the knots of the mesh. This is an established approximation methodology,[13] and hereafter we assume that the index reconstruction algorithm[7] gives the refractive index represented by the spline $S(x,y)$.

The iterative algorithm of Eq. (5) is implemented in three steps. First, given an approximation $n_k(x, y)$, we compute the approximation to the true sampling rays by using the observed rays, $X_k = D_{n_k}^{-1}[\tilde{X}]$. We do this by solving numerically, for each point $\tilde{X}$, an initial value problem for Eqs. (1) with the initial conditions

$$\{x(z_{end}), y(z_{end})\} = \{\tilde{x}_{ij}, \tilde{y}_{ij}\},$$

$$\{x'(z_{end}), y'(z_{end})\} = \{x_{end}', y_{end}'\},$$

where z_{end} is computed from the lens boundary equation and $\{x_{end}', y_{end}'\}$ are computed with the Snell law.

In the next step, we compute the refractive index by using ray X_k. Its values are translated into the values in the spline knots. In the last step, we compute the new approximation to the refractive index by using the reconstructed index, its previous value n_k, and a prescribed weight α. As the initial approximation n_0, it is convenient to take the elliptical index computed by means of its profile in the equatorial plane, as in Ref. 6.
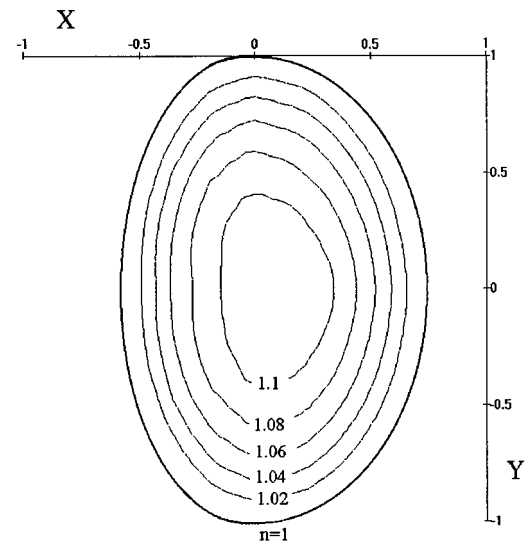
(a)
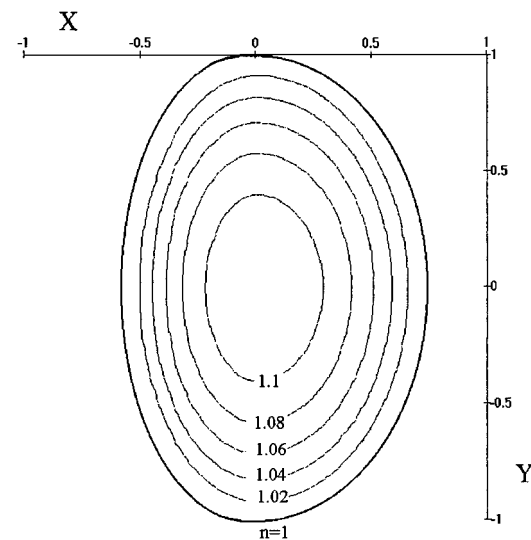
(b)

Fig. 4. Results of index reconstruction in the sagittal plane, compared with the model index: (a) reconstructed index, (b) model index, given by the formula for Fig. 3. The initial data consisted of the distorted image of the true rays in Fig. 3(b). As the initial approximation to the index, the constant value $n(x, y) = 1$ was taken. The algorithm converged in eight iterations. The maximum absolute error of reconstruction is 0.0108, and the average absolute error is 0.00256.

The most time-consuming step is the first one. It involves multiple applications of the Runge–Kutta method[11] to Eqs. (1), in which the index and its derivatives are given as a tensor–product spline. It takes us approximately 4 h of computing time on a

fast IBM R6000 workstation, compared with a few seconds for steps 2 and 3. Since the first step may be efficiently parallelized, the use of a multiprocessor supercomputer can drastically improve the computation time.

5. Conclusions

We have presented an approach to nondestructive diagnostics of 3-D optical systems with rotational symmetry by using a ray-tracing analysis. It is based on the observation of the rays that lie in a sagittal plane. Since the image of the ray paths is distorted by the system, the true ray paths are computed iteratively by using the index distribution obtained with the previous iteration.

The proposed method can solve two problems. First, given a model of 3-D index distribution, such as an elliptical model, one may check the consistency of the model with the experimental data. While an ellipical model for the index function has been used to illustrate the algorithm, the method is general and can be applied to any refractive-index function that has rotational symmetry. Second, the method allows one to compute the refractive index iteratively, starting with the initial approximation given by a suitable model, or, in the worst case, with a constant index.

It is important to note that the method is very sensitive to errors in the data, but the use of smoothing splines together with parameter α in the iteration step proved useful in regularizing or controlling the ill-conditioned nature of the problem. On one hand, the number of data points for the observed ray path and the amount of noise in the data are important factors in determining the ultimate precision of the index distribution. On the other hand, the attractiveness of this approach is that it does not require any underlying model of index distribution, nor of the lens boundaries (the elliptical model of Fig. 4 is used purely as an example). The ability of our method to detect nonregular, or lumpy, inhomogeneous structures[7] and its further regularization represent a topic for further research.

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