Acoustic streaming around a sphere Supplementary Material

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This document contains supplementary material to the paper *Acoustic streaming around a sphere* – the main text. Starting from solution to the first order in ϵ primary flow in a compressible Newtonian fluid due to a sphere executing rectilinear oscillations in the z-direction given in the main text, the various special limits of: potential flow - small viscosity, thin boundary layer; Stokes flow - low frequency, thick boundary layer; large sphere - flow near a flat place and the case of a incompressible fluid with an infinite speed of sound are recovered.

1. Solution of the primary flow

Details of reduction of the general solution of the primary flow due to the rectilinear oscillation of a sphere in a compressible Newtonian are given in this document.

At small vibrating amplitudes of the sphere, all quantities in the continuity and Navier-Stokes equation

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho}\, \overline{\boldsymbol{v}}) = 0, \tag{1.1a}$$

$$\frac{\partial (\overline{\rho}\, \overline{\boldsymbol{v}})}{\partial t} + \nabla \cdot (\overline{\rho}\, \overline{\boldsymbol{v}}\, \overline{\boldsymbol{v}}) = -\nabla \overline{p} + \mu \nabla^2 \overline{\boldsymbol{v}} + \Big(\mu_B + \frac{1}{3}\mu\Big) \nabla(\nabla \cdot \overline{\boldsymbol{v}}). \tag{1.1b}$$

are linearised about their equilibrium values in terms of the small parameter, ϵ defined by: $\epsilon \equiv |\nabla \cdot (\overline{\rho}\, \overline{\boldsymbol{v}}\, \overline{\boldsymbol{v}})| / |\partial(\overline{\rho}\, \overline{\boldsymbol{v}})/\partial t| \sim U_0/(af) << 1$, that measures the ratio of the non-linear inertial term to the explicit time derivative. With the constant reference density ρ_0, and pressure, p_0 and noting that the reference velocity is zero, we have

$$\overline{\rho} = \rho_0 + \epsilon\, \overline{\rho}_1 + \epsilon^2\, \overline{\rho}_2 + ..., \quad \overline{p} = p_0 + \epsilon\, \overline{p}_1 + \epsilon^2\, \overline{p}_2 + ..., \quad \overline{\boldsymbol{v}} = \epsilon\, \overline{\boldsymbol{v}}_1 + \epsilon^2\, \overline{\boldsymbol{v}}_2 + ... \tag{1.2}$$

To order ϵ, we have the equations that govern the primary flow

$$\frac{\partial \overline{\rho}_1}{\partial t} + \rho_0 \nabla \cdot \overline{\boldsymbol{v}}_1 = 0, \tag{1.3a}$$

$$\rho_0 \frac{\partial \overline{\boldsymbol{v}}_1}{\partial t} = -\nabla \overline{p}_1 + \mu \nabla^2 \overline{\boldsymbol{v}}_1 + \Big(\mu_B + \frac{1}{3}\mu\Big) \nabla(\nabla \cdot \overline{\boldsymbol{v}}_1). \tag{1.3b}$$

With harmonic time dependence in all primary flow quantities with: $\overline{\rho}_1(\boldsymbol{x}, t) \sim \rho(\boldsymbol{x}) e^{-\mathrm{i}\omega t}$,

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$\overline{p}_1(\boldsymbol{x},t) \sim p(\boldsymbol{x})e^{-\mathrm{i}\omega t}$ and $\overline{\boldsymbol{v}}_1(\boldsymbol{x},t) \sim \boldsymbol{u}(\boldsymbol{x})e^{-\mathrm{i}\omega t}$ the order ϵ equations (1.3) become

$$-\mathrm{i}\omega\rho + \rho_0 \nabla\cdot\boldsymbol{u} = 0, \tag{1.4a}$$

$$-\mathrm{i}\omega\rho_0\boldsymbol{u} = -\nabla p + \mu\nabla^2\boldsymbol{u} + \Big(\mu_B + \frac{1}{3}\mu\Big)\nabla(\nabla\cdot\boldsymbol{u}). \tag{1.4b}$$

For small amplitude acoustic waves, we assume the equation of state: $\nabla p(\boldsymbol{x}) = c_0^2\nabla\rho(\boldsymbol{x})$ where $c_0 > 0$ is the constant speed of sound in the fluid. The pressure, p can thus be eliminated from (1.4) to give (Hahn *et al.* 2013)

$$[(k_T^2/k_L^2) - 1]\nabla(\nabla\cdot\boldsymbol{u}) + \nabla^2\boldsymbol{u} + k_T^2\boldsymbol{u} = \boldsymbol{0}, \tag{1.5}$$

with transverse, k_T and longitudinal, k_L wave numbers defined by

$$k_T^2 \equiv \mathrm{i}\frac{\rho_0\omega}{\mu} \quad\text{and}\quad k_L^2 \equiv \omega^2\Big/\left[c_0^2 - \frac{\mathrm{i}\omega}{\rho_0}\Big(\mu_B + \frac{4}{3}\mu\Big)\right]. \tag{1.6}$$

The solution of the order ϵ equation (1.5) due to the oscillatory motion of the sphere along the z-direction with velocity amplitude, U_0 can only depend on the vector $\boldsymbol{U}_0$ and the position vector $\boldsymbol{x}$ with the origin at the centre of the sphere (Landau & Lifshitz 1970). Symmetry consideration implies that the solution of (1.5) has the general form

$$\boldsymbol{u}(\boldsymbol{x}) = u_r(r)\cos\theta\,\boldsymbol{e}_r + u_\theta(r)\sin\theta\,\boldsymbol{e}_\theta \tag{1.7a}$$

$$= \left\{-\frac{2}{r}h(r) + \frac{\mathrm{d}\phi(r)}{\mathrm{d}r}\right\}U_0\cos\theta\,\boldsymbol{e}_r + \left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(rh(r)\Big) - \frac{\phi(r)}{r}\right\}U_0\sin\theta\,\boldsymbol{e}_\theta \tag{1.7b}$$

where the functions $u_r(r)$ and $u_\theta(r)$ are only functions of the radial distance, r from the centre of the sphere and $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are unit vectors in the direction of increasing radial and polar coordinated relative to the z-direction. The zero divergence transverse and irrotational longitudinal parts of $\boldsymbol{u}(\boldsymbol{x})$ are represented by the components in $h(r)$ and $\phi(r)$, respectively.

Equation (1.5) has the same mathematical form as that of the equation that governs the propagation of linear elastic waves in a solid where $\boldsymbol{u}$ is the material displacement. The solution of (1.5) can be represented as the Green's function (Stokelet) and a dipole field, so explicit solutions for the velocity components are (Klaseboer *et al.* 2019):

$$\frac{u_\theta(r)}{U_0} = C_1\frac{a}{r}\,[1 + G(k_T r)]\,e^{\mathrm{i}k_T r} + C_2\,\frac{a}{r}\,G(k_L r)\,e^{\mathrm{i}k_L r} \tag{1.8a}$$

$$\frac{u_r(r)}{U_0} = 2\,C_1\,\frac{a}{r}\,G(k_T r)\,e^{\mathrm{i}k_T r} + C_2\frac{a}{r}\,[1 + 2G(k_L r)]\,e^{\mathrm{i}k_L r} \tag{1.8b}$$

with $G(x) \equiv \mathrm{i}/x - 1/x^2$. The functions $h(r)$ and $\phi(r)$ defined in (1.7) are given by

$$h(r) = -C_1\,a\,G(k_T r)\,e^{\mathrm{i}k_T r}, \qquad \phi(r) = -C_2\,a\,G(k_L r)\,e^{\mathrm{i}k_L r}. \tag{1.9}$$

The constants C_1 and C_2 that satisfy the no-slip boundary condition, $u_r(a) = U_0$ and $u_\theta(a) = -U_0$ are:

$$C_1 = -\frac{[1 + 3G(k_L a)]}{[1 + G(k_T a) + 2G(k_L a)]}e^{-\mathrm{i}k_T a}, \quad C_2 = \frac{[1 + 3G(k_T a)]}{[1 + G(k_T a) + 2G(k_L a)]}e^{-\mathrm{i}k_L a}. \tag{1.10}$$

2. Small viscosity limit, thin boundary layer: potential flow

If viscous effects are small, we expect a very thin boundary layer. From (1.6), a vanishing viscosity corresponds to the limit $|k_T a| >> |k_L a|$ and hence $e^{\mathrm{i}k_T r} << 1$ for

$r > a$ due to the imaginary part of k_T. From (1.10) we find

$$\lim_{k_T a \to \infty} C_2 = \frac{1}{1 + 2G(k_L a)} e^{-\mathrm{i}k_L a}$$

and (1.8) becomes:

$$\lim_{k_T a \to \infty} \boldsymbol{u} = e^{\mathrm{i}k_L(r-a)} \frac{a}{r} U_0 \Big[\cos\theta \, \boldsymbol{e}_r + \frac{G(k_L r)}{1 + 2G(k_L a)} \sin\theta \, \boldsymbol{e}_\theta \Big]. \tag{2.1}$$

If in addition, $k_L a \to 0$, we recover the potential flow solution for the fluid velocity around a sphere moving at a constant velocity:

$$\lim_{k_L a \to 0 \; ; \; k_T a \to \infty} \boldsymbol{u} = \frac{a^3}{r^3} U_0 \Big[\cos\theta \, \boldsymbol{e}_r + \frac{1}{2} \sin\theta \, \boldsymbol{e}_\theta \Big]. \tag{2.2}$$

3. Large radius: flat plate limit

If the radius of the sphere, a, is very large, there are two locations of special interest: the front of the sphere at $\theta = 0$ and the side of the sphere at $\theta = \pi/2$.

Consider first the side at which $\cos\theta = 0$ and $\sin\theta = 1$. From (1.7) we then find $\boldsymbol{u} = -u_\theta(r)\boldsymbol{e}_z$ since $\boldsymbol{U}_0 = -U_0\boldsymbol{e}_z$. And setting $r/a \to$ in (1.8), we find

$$\frac{u_\theta(r)}{U_0} = C_1 e^{\mathrm{i}k_T r}[1 + G(k_T r)] + C_2 e^{\mathrm{i}k_L r} G(k_L r). \tag{3.1}$$

For a large sphere, $|k_L a|, |k_T a| >> 1$, so $G(k_L r), G(k_T r) \to 0$ and $C_1 \to -e^{\mathrm{i}k_T a}$ and the velocity becomes:

$$\lim_{|k_L a|, |k_T a| \to \infty \; ; \; \theta = \pi/2} \boldsymbol{u} = -e^{\mathrm{i}k_T(r-a)} \boldsymbol{U}_0. \tag{3.2}$$

In the time domain, this is equivalent to the well-known Stokes oscillatory boundary thickness equation (Stokes 1851; Schlichting 1955): $\boldsymbol{u} = -\boldsymbol{U}_0 e^{-ky} \cos(\omega t - ky)$, with $k = \sqrt{\omega\rho_0/(2\mu)}$, the real part of $k_T = (1+\mathrm{i})\sqrt{\omega\rho_0/(2\mu)}$, and $y = r - a$ being the distance from the a flat surface. Thus, the solution at the side of the sphere tends towards the Stokes vibrating boundary layer theory for a flat plate when the radius of the sphere is large enough.

The solution in front of the sphere at $\theta = 0$, in the large a or flat plate limit with $\boldsymbol{U}_0 = U_0\boldsymbol{e}_z$, the velocity in (1.7) becomes $\boldsymbol{u} = u_r(r)\boldsymbol{e}_z$ and with $a/r \to 1$, we have:

$$\frac{u_r(r)}{U_0} = 2e^{\mathrm{i}k_T r} C_1 G(k_T r) + C_2 e^{\mathrm{i}k_L r}[1 + 2G(k_L r)]. \tag{3.3}$$

Again with $G(k_L r), G(k_T r) \to 0$ for a large sphere, $|k_L a|, |k_T a| >> 1$, we have the limiting solution

$$\lim_{|k_L a|, |k_T a| \to \infty \; ; \; \theta = 0} \boldsymbol{u} = e^{\mathrm{i}k_L(r-a)} \boldsymbol{U}_0 \tag{3.4}$$

which is a plane sound wave propagating out of the oscillating plate as could be expected.

4. Infinite sound speed: (oscillatory) incompressible Stokes flow limit

The incompressibility of the fluid implies that $c_0 \to \infty$. In this case, we can take the limit of $|k_L a| \to 0$ while keeping $|k_T a|$ finite. As such, we have

$$\begin{aligned}\lim_{|k_L a|\to 0} C_1 &= -\frac{3G(k_L a)}{2G(k_L a)} e^{-\mathrm{i}k_T a} \to -\frac{3}{2} e^{-\mathrm{i}k_T a},\\ \lim_{|k_L a|\to 0} C_2 &= \frac{1+3G(k_T a)}{e^{ik_L a}[2G(k_L a)]} \to -\frac{(k_L a)^2}{2}[1+3G(k_T a)].\end{aligned} \tag{4.1}$$

Introducing (4.1) into (1.8) we have

$$\frac{u_\theta(r)}{U_0} = -\frac{3}{2}\,\frac{a}{r}\,\big[1+G(k_T r)\big]\,e^{\mathrm{i}k_T(r-a)} + \frac{1+3G(k_T a)}{2}\frac{a^3}{r^3}, \tag{4.2a}$$

$$\frac{u_r(r)}{U_0} = -3\,\frac{a}{r}\,G(k_T r)\,e^{\mathrm{i}k_T(r-a)} + [1+3G(k_T a)]\frac{a^3}{r^3}. \tag{4.2b}$$

This solution is identical to the solution given by Landau & Lifshitz (1987), where the velocity was written as $\boldsymbol{u} \equiv \nabla\times\nabla\times[f_L(r)\boldsymbol{U}_0]$, then $u_r(r) = -(2/r)\,\mathrm{d}f_L(r)/\mathrm{d}r$ and $u_\theta(r) = (1/r)\,\mathrm{d}f_L(r)/\mathrm{d}r + \mathrm{d}^2 f_L(r)/\mathrm{d}r^2$. They showed that $\frac{\mathrm{d}f_L(r)}{\mathrm{d}r} = a_L\left(\frac{1}{r} - \frac{1}{\mathrm{i}k_T r^2}\right)e^{\mathrm{i}k_T r} + \frac{b_L}{r^2}$. This corresponds to our solution with $a_L = 3\mathrm{i}ae^{-\mathrm{i}k_T a}/(2k_T)$ and $b_L = -[1+G(k_T a)]a^3/2$. It is also consistent with the solution of Eq. (9) of Riley (1966).

In order to get back the Stokes limit, we have to take the limit $|k_T a| \to 0$ as well. Note that

$$\begin{aligned}\lim_{|k_T a|\to 0}[1+G(k_T r)]e^{\mathrm{i}k_T r} &= 1/2 - 1/(k_T^2 r^2),\\ \lim_{|k_T a|\to 0} G(k_T r)e^{\mathrm{i}k_T r} &= -1/2 - 1/(k_T^2 r^2),\\ \lim_{|k_T a|\to 0}[1+3G(k_T a)]e^{\mathrm{i}k_T a}/e^{\mathrm{i}k_T a} &= [-1/2 - 3/(k_T^2 a^2)]/e^{\mathrm{i}k_T a} = -1/2 - 3/(k_T^2 a^2)\end{aligned} \tag{4.3}$$

Then the velocity components become:

$$\frac{u_\theta(r)}{U_0} = -\frac{3}{4}\,\frac{a}{r} + \frac{3}{2k_T^2 r^2}\frac{a}{r} - \frac{1}{4}\frac{a^3}{r^3} - \frac{3}{k_T^2 a^2}\,\frac{a^3}{r^3} = -\frac{3}{4}\,\frac{a}{r} - \frac{1}{4}\frac{a^3}{r^3}, \tag{4.4a}$$

$$\frac{u_r(r)}{U_0} = \frac{3}{2}\,\frac{a}{r} + \frac{3}{k_T^2 r^2}\,\frac{a}{r} - \frac{1}{2}\frac{a^3}{r^3} - \frac{3}{k_T^2 a^2}\,\frac{a^3}{r^3} = \frac{3}{2}\,\frac{a}{r} - \frac{1}{2}\frac{a^3}{r^3} \tag{4.4b}$$

which is the velocity field for a sphere moving in Stokes flow. Note that in the above results, terms with $1/k_T^2$ cancel each other exactly out.

5. Steady acoustic streaming flow field and the primary flow field

Included in the Supplementary Material is a movie file of animations of the primary flow fields corresponding the numerical examples given in the main text and the corresponding steady acoustic streaming flow fields.

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