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ON THE EXISTENCE OF HYDRODYNAMIC FLUCTUATION FORCES

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The form of fluctuating hydrodynamic forces has been derived recently [R.B. Jones, Physica **105A** (1981) 395] from the stochastic fluid equations of Landau and Lifshitz. We show by general physical arguments as well as by a direct calculation that such forces do not exist in the classical limit.

1. Introduction

The physical origins of the Van der Waals interactions between macroscopic bodies are to be found in the fluctuating electromagnetic fields generated by the translational, vibrational, rotational and electronic orbital motions of the constituent molecules of the bodies. The presence of one body in the vicinity of another alters the disposition of the fluctuating electromagnetic field and the subsequent changes in the energy associated with the field can be meaningfully defined as a distance-dependent interaction energy between the bodies. The quantitative evaluation of this dispersion interaction can be made by quantum field theoretic techniques[1]) or equivalently by a semiclassical method[2]) whereby the classical (continuum dielectric) field equations are solved in the geometry defined by the two bodies in the presence of a random electromagnetic field $\boldsymbol{K}(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega t}$. Equivalence of the methods is achieved by requiring $\boldsymbol{K}(\boldsymbol{r})$ to have the property

$$\langle \boldsymbol{K}(\boldsymbol{r})\boldsymbol{K}(\boldsymbol{r}')\rangle = 2\hbar\epsilon''(\omega)\coth\frac{\hbar\omega}{2kT}\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{I}, \tag{1.1}$$

where $\epsilon''(\omega)$ is the imaginary part of the dielectric response of the body and $\langle\ldots\rangle$ denotes the usual statistical average. For a fuller discussion of this equivalence see Landau and Lifshiftz[3]). By evaluating the resultant electromagnetic stress tensor on the surface of one of the bodies and integrating over all frequencies ω, the dispersion force $\boldsymbol{F}_\mathrm{D}$ exerted on that body by the other is obtained[1,2]).

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There are in real systems other fluctuating fields of a non-electromagnetic nature, e.g. flow fields, in the case of a fluid material or elastic displacement fields, in the case of an elastic solid. The question arises as to whether these fluctuations can give rise to a net force between bodies by analogy to the dispersion force. Dzyaloshinski et al.[1]) have suggested that acoustic fluctuations in the liquid film (thickness L) that separates two semi-infinite half spaces can give rise to a (hydrodynamic) force per unit area between the surfaces,

$$F_{\mathrm{H}} \sim \frac{kT}{L^3}\,. \tag{1.2}$$

The constant of proportionality in the above expression was expected to be a number of order unity – its precise value was not, however, determined. Since the Van der Waals interaction for a similar system has the same functional form as, and comparable magnitude to, the result in eq. (1.2), it is important to determine the exact magnitude of any fluctuation forces of non-electromagnetic origin.

Recently, Jones[4]) had analysed the nature of hydrodynamic fluctuations in a fluid using the random field formalism as applied to the Navier–Stokes equations. An expression for the pressure field near a rigid plane boundary generated by these acoustic fluctuations was derived and an approximate calculation of the net force exerted on a sphere by the plane boundary was performed. This study indicated that these hydrodynamic fluctuation forces might indeed be numerically very significant in interfacial phenomena. The important geometry of plane parallel half spaces separated by a liquid was not considered by Jones. It is the aim of the present paper to address ourselves to that problem.

Before we proceed, several philosophical reservations should be discussed. Firstly, the nature of these non-electromagnetic fluctuations is *not* completely analogous to their electromagnetic counterparts. The electromagnetic field equations are *linear* while the electromagnetic stress tensor is a *quadratic* function of the field variables. As we shall see below, the classical hydrodynamic field equations are *quadratic* while the hydrodynamic stress tensor is a *linear* function of the field variables. Indeed, it is precisely the non-linearity of the field equations which provides the net hydrodynamic fluctuation force as Jones[4]) has shown. This fundamental lack of correspondence does not allow for an easy method of calculation of hydrodynamic fluctuation forces as is available in the electromagnetic case, namely, the normal mode (via the dispersion relation) formalism of Van Kampen et al.[5]).

A more fundamental question arises as to exactly what part of the energy of the system we are calculating when we pursue the acoustic fluctuation formalism of Jones[4]). Hydrodynamic fluctuations are collective motions in which kinetic energy is dissipated, therefore in a classical system, such effects must be contained in the momentum integration of the phase space integral for the partition functions. However, for a system with velocity-*independent* potentials the momentum part of the phase integral has no separation or distance dependence which can give rise to a net *average* force on the bodies in the system. We expect, therefore that the hydrodynamic fluctuation formalism of Jones[4]) should yield a zero value for the coefficient in (1.2). To what extent the formalism fulfils this expectation is outlined below.

2. Formalism

Hydrodynamic fluctuations in a Newtonian fluid are governed by the stochastic equations of Landau and Lifshitz[6])

$$\eta \nabla^2 \boldsymbol{v} + (\eta/3 + \zeta)\nabla(\nabla \cdot \boldsymbol{v}) - \nabla p - \rho\left[\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\right] = -\nabla \cdot \boldsymbol{S}\,, \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0\,, \tag{2.2}$$

where heat transfer effects have been neglected for simplicity, and $\boldsymbol{v}(\boldsymbol{r}, t)$, $p(\boldsymbol{r}, t)$, $\rho(\boldsymbol{r}, t)$ are the velocity, pressure and density, respectively, η and ζ are shear and bulk viscosities. Stochastic fluctuations in the fluid are described by the random stress tensor $\boldsymbol{S}$ which has the Gaussian stochastic properties

$$\langle \hat{S}_{ij}(\boldsymbol{r}, \omega)\rangle = 0\,, \tag{2.3}$$

$$\langle \hat{S}_{ij}(\boldsymbol{r}, \omega)\hat{S}_{kl}(\boldsymbol{r}', \omega')\rangle = \frac{kT}{\pi}\,[\mathrm{Re}\,\eta(\omega)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \mathrm{Re}\,(\zeta(\omega) - \tfrac{2}{3}\eta(\omega)\delta_{ij}\delta_{kl}]\delta(\boldsymbol{r} - \boldsymbol{r}')\delta(\omega - \omega')\,, \tag{2.4}$$

where $\langle\ldots\rangle$ denotes an equilibrium ensemble average and $\hat{}$ the frequency Fourier transform

$$\hat{f}(\omega) = \int \mathrm{d}t\ \mathrm{e}^{\mathrm{i}\omega t} f(t)\,. \tag{2.5}$$

The random stress tensor $\boldsymbol{S}$ may be regarded as the driving force for the fluid motion. The force exerted on any solid body by the fluid can be computed in the usual way from the stress tensor $\boldsymbol{\sigma}$[6]),

$$\boldsymbol{\sigma} = \eta[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\mathrm{T}}] + [(\zeta - \tfrac{2}{3}\eta)(\nabla \cdot \boldsymbol{v}) - p]\boldsymbol{I}\,, \tag{2.6}$$

where $(\quad)^{\mathrm{T}}$ denotes the transpose and $\boldsymbol{I}$ the unit tensor. In view of eq. (2.3) it is easy to see that the non-linear term in eq. (2.1) must be considered in order to obtain a non-zero contribution to the average stress tensor $\langle \boldsymbol{\sigma}\rangle$ from hydrodynamic fluctuations.

In this paper, we follow Jones' treatment[4]) of the non-linear term in eq. (2.1). We seek a solution of eqs. (2.1) and (2.2) in the form of a power series in the random stress $\boldsymbol{S}$:

$$\boldsymbol{v} = \sum_{n=0}^{\infty} \boldsymbol{v}^{(n)}, \tag{2.7a}$$

$$p = \sum_{n=0}^{\infty} p^{(n)}, \tag{2.7b}$$

$$\rho = \sum_{n=0}^{\infty} \rho^{(n)}, \tag{2.7c}$$

$$\boldsymbol{\sigma} = \sum_{n=0}^{\infty} \boldsymbol{\sigma}^{(n)}, \tag{2.7d}$$

where (n) denotes a term of order n in $\boldsymbol{S}$. The equilibrium solutions corresponding to $\boldsymbol{S} = 0$ are $\boldsymbol{v}^{(0)} = 0$, $p^{(0)} = p_0$ (constant), $\rho^{(0)} = \rho_0$ (constant) and $\boldsymbol{\sigma}^{(0)} = -p_0\boldsymbol{I}$. Assuming local equilibrium, p and ρ can be related by

$$p^{(1)} = C_0^2\rho^{(1)}, \tag{2.8}$$

$$p^{(2)} = C_0^2\rho^{(2)} + b_0\rho^{(1)}\rho^{(1)}, \tag{2.9}$$

where $C_0 = (\partial p/\partial\rho)_0^{1/2}$ is the adiabatic velocity of sound and $b_0 = \frac{1}{2}(\partial^2 p/\partial\rho^2)_0$ the second virial coefficient. Combining eqs. (2.1), (2.2) and (2.7)–(2.9) we have to order 1 in powers of $\boldsymbol{S}$, the linearized equations

$$\eta\nabla^2\boldsymbol{v}^{(1)} + (\eta/3 + \zeta)\nabla(\nabla\cdot\boldsymbol{v}^{(1)}) - \nabla p^{(1)} - \rho_0\frac{\partial\boldsymbol{v}^{(1)}}{\partial t} = -\nabla\cdot\boldsymbol{S}, \tag{2.10}$$

$$\nabla\cdot\boldsymbol{v}^{(1)} + \frac{1}{C_0^2}\frac{\partial p^{(1)}}{\partial t} = 0. \tag{2.11}$$

To second order, we have

$$\begin{aligned}\eta\nabla^2\boldsymbol{v}^{(2)} &+ (\eta/3 + \zeta)\nabla(\nabla\cdot\boldsymbol{v}^{(2)}) - \nabla p^{(2)} - \rho_0\frac{\partial\boldsymbol{v}^{(2)}}{\partial t}\\ &= \rho_0\boldsymbol{v}^{(1)}\cdot\nabla\boldsymbol{v}^{(1)} + \frac{1}{C_0^2}\frac{\partial(p^{(1)}\boldsymbol{v}^{(1)})}{\partial t},\end{aligned} \tag{2.12}$$

$$\nabla\cdot\boldsymbol{v}^{(2)} + \frac{1}{\rho_0C_0^2}\frac{\partial p^{(2)}}{\partial t} = -\frac{1}{\rho_0C^2}\nabla\cdot(p^{(1)}\boldsymbol{v}^{(1)}) + \frac{b_0}{\rho_0C_0^6}\frac{\partial(p^{(1)}p^{(1)})}{\partial t}. \tag{2.13}$$

From eqs. (2.10) and (2.11), the first order solutions $\boldsymbol{v}^{(1)}$ and $p^{(1)}$ will be linear in the random field $\boldsymbol{S}$, and because of the stochastic nature of the random field, eq. (2.3), the averages $\langle\boldsymbol{v}^{(1)}\rangle$ and $\langle p^{(1)}\rangle$ vanish. Since the stress tensor $\boldsymbol{\sigma}$ is linear in $\boldsymbol{v}$ and p, $\langle\boldsymbol{\sigma}^{(1)}\rangle$ also vanishes. Thus the first non-vanishing term of $\boldsymbol{\sigma}$ which depends on $\boldsymbol{S}$ and hence gives rise to fluctuation forces is $\boldsymbol{\sigma}^{(2)}$. Consequently, we shall require averages of the second order terms $\langle \boldsymbol{v}^{(2)} \rangle$ and $\langle p^{(2)} \rangle$. By taking the average of equations (2.12) and (2.13) we see that averages of the products of first order terms, namely $\langle \boldsymbol{v}^{(1)}\boldsymbol{v}^{(1)} \rangle$, $\langle p^{(1)}\boldsymbol{v}^{(1)} \rangle$ and $\langle p^{(1)}p^{(1)} \rangle$ are required. These averages can be related to the Green's function of the linear Navier–Stokes equations with the usual stick boundary conditions at any solid boundaries[4]). The ensemble average of the second order equations, (2.12) and (2.13), can be expressed as a pair of 'static' equations

$$\eta_0 \nabla^2 \boldsymbol{V}(\boldsymbol{r}) - \nabla P(\boldsymbol{r}) + \boldsymbol{H}(\boldsymbol{r}) = 0\,, \tag{2.14}$$

$$\nabla \cdot \boldsymbol{V}(\boldsymbol{r}) = 0\,, \tag{2.15}$$

where

$$\langle \hat{\boldsymbol{v}}^{(2)}(\boldsymbol{r}, \omega) \rangle = \delta(\omega) \boldsymbol{V}(\boldsymbol{r})\,, \tag{2.16}$$

$$\langle \hat{p}^{(2)}(\boldsymbol{r}, \omega) \rangle = \delta(\omega) P(\boldsymbol{r})\,, \tag{2.17}$$

and $\eta_0 \equiv \eta(\omega = 0)$ is the low frequency shear viscosity. The cartesian components of the 'external' force density $\boldsymbol{H}(\boldsymbol{r})$ are defined by

$$H_j(\boldsymbol{r}) = \frac{\rho_0 kT}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\eta(\omega)} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} G_{ij}(\boldsymbol{r}, \boldsymbol{r}; \omega)\,, \tag{2.18}$$

where the subscripts i, $j = 1$, 2, 3 denote the three cartesian coordinates or components. The dyadic Green's function is the solution of ($\gamma = 1, 2, 3, 4$)

$$\begin{bmatrix} \nabla^2 + \dfrac{\mathrm{i}\omega\rho_0}{\eta(\omega)} & 0 & 0 & \xi \dfrac{\partial}{\partial x} \\ 0 & \nabla^2 + \dfrac{\mathrm{i}\omega\rho_0}{\eta(\omega)} & 0 & \xi \dfrac{\partial}{\partial y} \\ 0 & 0 & \nabla^2 + \dfrac{\mathrm{i}\omega\rho_0}{\eta(\omega)} & \xi \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} & -\lambda \end{bmatrix} \begin{bmatrix} G_{1\gamma} \\ G_{2\gamma} \\ G_{3\gamma} \\ G_{4\gamma} \end{bmatrix} = \begin{bmatrix} \delta_{1\gamma} \\ \delta_{2\gamma} \\ \delta_{3\gamma} \\ 0 \end{bmatrix} \delta(\boldsymbol{r} - \boldsymbol{r}')\,, \tag{2.19}$$

where

$$\lambda = \frac{\mathrm{i}\omega\eta(\omega)}{\rho_0 C_0^2}\,, \tag{2.20}$$

$$\xi = \frac{\mathrm{i}\omega}{\rho_0 C_0^2}\left(\frac{\eta(\omega)}{3} + \zeta(\omega)\right). \tag{2.21}$$

A detailed derivation of the above results is given in ref. 4.

3. The two-wall problem

We consider hydrodynamic fluctuations in a fluid confined between two solid half-spaces located at $z = 0$ and at $z = L$. Due to the translational invariance in the transverse (x–y) direction, we seek a solution of eq. (2.19) in the form

$$G_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}'; \omega) = \frac{1}{(2\pi)^2} \int \mathrm{d}^2\boldsymbol{k}\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}-\boldsymbol{s}')} \tilde{G}_{\alpha\beta}(z, z'; k, \omega)\,, \tag{3.1}$$

with $\boldsymbol{s} = (x, y)$ and $\boldsymbol{k} = (k_x, k_y)$. By defining the dimensionless quantities

$$\tilde{U}_\gamma = k_x\tilde{G}_{1\gamma} + k_y\tilde{G}_{2\gamma}\,, \tag{3.2}$$

$$\tilde{V}_\gamma = k_y\tilde{G}_{1\gamma} - k_x\tilde{G}_{2\gamma}\,, \tag{3.3}$$

$$\tilde{Z}_\gamma = \mathrm{i}k\tilde{G}_{3\gamma}\,, \tag{3.4}$$

$$\tilde{W}_\gamma = -\xi\tilde{G}_{4\gamma}\,, \tag{3.5}$$

eq. (2.19) becomes

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - q^2\right)\tilde{U}_\gamma - \mathrm{i}k^2\tilde{W}_\gamma = (k_x\delta_{1\gamma} + k_y\delta_{2\gamma})\delta(z - z')\,, \tag{3.6}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - q^2\right)\tilde{V}_\gamma = (k_y\delta_{1\gamma} - k_x\delta_{2\gamma})\delta(z - z')\,, \tag{3.7}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - q^2\right)\tilde{Z}_\gamma - \mathrm{i}k\frac{\mathrm{d}\tilde{W}_\gamma}{\mathrm{d}z} = \mathrm{i}k\delta_{3\gamma}\delta(z - z')\,, \tag{3.8}$$

$$\frac{\mathrm{d}\tilde{Z}_\gamma}{\mathrm{d}z} - k\tilde{U}_\gamma - \mathrm{i}k\alpha\tilde{W}_\gamma = 0\,, \tag{3.9}$$

where

$$q^2 = k^2 - \frac{\mathrm{i}\omega\rho_0}{\eta(\omega)}\,, \tag{3.10}$$

$$\alpha = -\lambda/\xi\,. \tag{3.11}$$

The boundary conditions for these differential equations are

$$\tilde{Z}_\gamma, \tilde{U}_\gamma, \tilde{V}_\gamma = 0, \quad \text{at} \quad z = 0, L\,, \tag{3.12}$$

which correspond to the usual 'stick' boundary conditions.

The solution of eqs. (3.6)–(3.12) proceeds as follows. Eq. (3.7) for $\tilde{V}_\gamma$ is not coupled to the other functions and can be solved. However, the function $\tilde{V}_\gamma$ makes no contribution to $\boldsymbol{H}(\boldsymbol{r})$ in this problem and therefore need not be considered further. To solve the remaining coupled equations we begin by differentiating equation (3.8) with respect to z and then eliminate the functions $\tilde{U}_\gamma$ and $(\partial\tilde{Z}_\gamma/\partial z)$

using eqs. (3.6) and (3.9) to obtain an equation for $\tilde{W}_\gamma$,

$$\frac{\mathrm{d}^2\tilde{W}_\gamma}{\mathrm{d}z^2} - m^2\tilde{W}_\gamma = \frac{-1}{(1-\alpha)}[\delta_{3\gamma} + \mathrm{i}(k_x\delta_{1\gamma} + k_y\delta_{2\gamma})]\delta(z-z'), \tag{3.13}$$

$$m^2 = k^2 - \omega^2\left[C_0^2 - \frac{\mathrm{i}\omega}{\rho_0}(\tfrac{4}{3}\eta(\omega) + \zeta(\omega))\right]^{-1}, \tag{3.14}$$

with solution

$$\tilde{W}_\gamma = C\,\mathrm{e}^{-mz} + D\,\mathrm{e}^{m(z-L)} + \frac{e^{-m|z-z'|}}{2m(1-\alpha)}[\mathrm{i}(k_x\delta_{1\gamma} + k_y\delta_{2\gamma}) - m\delta_{3\gamma}\,\mathrm{sgn}(z-z')], \tag{3.15}$$

where C and D are constants of integration. This result is then substituted into eq. (3.8) to give a differential equation for $\tilde{Z}_\gamma$ which can be readily solved. The expression so obtained for $\tilde{Z}_\gamma$ will contain two further constants of integration in addition to C and D. These four constants can be determined by applying the boundary conditions $\tilde{U}_\gamma$, $\tilde{V}_\gamma = 0$ at $z = 0$, L where $\tilde{U}_\gamma$ is obtained from eq. (3.9).

The above process is straightforward to execute but the algebraic manipulations involved are extremely long and tedious. It turns out that for the particular geometry considered here, the only non-zero component of $\boldsymbol{H}(\boldsymbol{r})$ is the z-component, $H_3(z)$ which is only a function of z. The only contribution to $H_3(z)$ is from G_{33}. The final expression for $H_3(z)$ is

$$H_3(z) = \frac{\rho_0 kT}{\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\eta(\omega)}\int_0^{\infty}\mathrm{d}k\,k\,\frac{m^2}{1-\alpha}\frac{(q+m\beta)}{(q^2-m^2)(q-m\beta)}\frac{Y}{\mathscr{D}}, \tag{3.16}$$

where

$$\beta = 1 + \frac{\alpha(q^2-m^2)}{m^2}, \tag{3.17}$$

$$\mathscr{D} = \left[1 - e^{-(q+m)L} - \left(\frac{q+m\beta}{q-m\beta}\right)(\mathrm{e}^{-qL} - \mathrm{e}^{-mL})\right] \times \left[1 - e^{-(q+m)L} + \left(\frac{q+m\beta}{q-m\beta}\right)(\mathrm{e}^{-qL} - \mathrm{e}^{-mL})\right], \tag{3.18}$$

$$\begin{aligned} Y = {} & (1 - e^{-2qL})\,\mathrm{e}^{-qL}\sinh q(2z-L) + (1 - e^{-2qL})\,\mathrm{e}^{-mL}\sinh m(2z-L) \\ & - (1+\beta)\left(\frac{q+m}{q+m\beta}\right)(1 - \mathrm{e}^{-(m+q)L})\,\mathrm{e}^{-(q+m)L/2}\sinh\tfrac{1}{2}(q+m)(2z-L) \\ & - (1+\beta)\left(\frac{q-m}{q-m\beta}\right)(\mathrm{e}^{-qL} - \mathrm{e}^{-mL})\,\mathrm{e}^{-(q+m)L/2}\sinh\tfrac{1}{2}(q-m)(2z-L), \end{aligned} \tag{3.19}$$

and both q and m are taken to have positive real parts. In the limit $L \to \infty$ we recover Jones' result for a single wall[4]).

Before we analyse the results for some simple cases, we should recall the limitations of the present method of studying hydrodynamic fluctuations. The Navier–Stokes equations represent a continuum description of fluid motion. As such they are only appropriate for describing low frequency and long wavelength phenomena, that is, fluid motions that vary slowly over molecular dimensions and are slow on a molecular timescale. Consequently, we need to exercise some care in interpreting our results which have been derived on the tacit assumption that the Navier–Stokes description remains valid in the limits $|\omega| \to \infty$, $k \to \infty$.

To calculate the average hydrodynamic fluctuation force exerted on either half-space that confines the fluid, we only consider the contribution from the second order term of the stress tensor $\langle \boldsymbol{\sigma}^{(2)} \rangle$. As mentioned earlier, the first order term $\langle \boldsymbol{\sigma}^{(1)} \rangle$ vanishes. The averaged stress tensor $\langle \boldsymbol{\sigma}^{(2)} \rangle$ is given in terms of the averaged second order fluctuations in the velocity, $\langle \boldsymbol{v}^{(2)} \rangle$ and pressure, $\langle p^{(2)} \rangle$ which are defined by eqs. (2.6), (2.14)–(2.17). In the present two wall problem, the sole non-zero component of the force density $\boldsymbol{H}(\boldsymbol{r})$ is the z-component $H_3(z)$ which is only a function of z. Therefore, the solution of eqs. (2.14) and (2.15) is

$$V(\boldsymbol{r}) = 0\,, \tag{3.20}$$

$$\frac{\mathrm{d}P(z)}{\mathrm{d}z} = H_3(z)\,. \tag{3.21}$$

3.1. *Incompressible fluids*

Consider the case of an imcompressible fluid for which $C_0 \to \infty$. We shall further neglect dispersion effects and take $\eta(\omega) = \eta_0$ (const) the low frequency viscosity. The force density becomes

$$H_3(z) = \frac{\mathrm{i}kT}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\omega} \int_0^{\infty} \mathrm{d}k\, k^3 \left(\frac{q+k}{q-k}\right) \frac{Y_\infty}{\mathscr{D}_\infty}\,, \tag{3.22}$$

where

$$\begin{aligned} Y_\infty &= Y(C_0 \to \infty) \\ &= (1 - e^{-2kL})\, \mathrm{e}^{-qL} \sinh q(2z - L) + (1 - e^{-2qL})\, \mathrm{e}^{-kL} \sinh k(2z - L) \\ &\quad - 2(1 - e^{-(q+k)L})\, \mathrm{e}^{-(q+k)L/2} \sinh \tfrac{1}{2}(k + q)(2z - L) \\ &\quad - 2(e^{-kL} - \mathrm{e}^{-qL})\, \mathrm{e}^{-(q+k)L/2} \sinh \tfrac{1}{2}(k - q)(2z - L)\,, \end{aligned} \tag{3.23}$$

$$
\begin{aligned}
\mathscr{D}_\infty &= \mathscr{D}(C_0 \to \infty) \\
&= \left[1 - e^{-(q+k)L} - \left(\frac{q+k}{q-k}\right)(\mathrm{e}^{-qL} - \mathrm{e}^{-kL})\right] \\
&\quad \times \left[1 - e^{-(q+k)L} + \left(\frac{q+k}{q-k}\right)(\mathrm{e}^{-qL} - \mathrm{e}^{-kL})\right].
\end{aligned} \tag{3.24}
$$

The ω-integral in eq. (3.22) can be evaluated by changing the contour to a large semi-circle in the upper half of the complex frequency plane. This procedure is possible because the integrand of the ω-integral is a Green's function (see eq. (2.18)) and by the usual causality argument, the integrand is an analytic function of ω in the upper half plane. It is easy to show that for the integral in eq. (3.22), the integrand behaves like $1/\omega$ as $\omega \to \mathrm{i}\infty$, thus the ω-integral can be evaluated to give

$$
H_3(z) = \frac{kT}{2\pi} \int_0^\infty \mathrm{d}k\, k^3 \frac{\sinh k(2z - L)}{\sinh kL}. \tag{3.25}
$$

Using this result in eq. (3.21) we find

$$
P(z) = P_0 + \frac{kT}{4\pi} \int_0^\infty \mathrm{d}k\, k^2 \frac{\cosh k(2z - L)}{\sinh kL}, \tag{3.26}
$$

where P_0 is a constant of integration. The force per unit area or disjoining pressure exerted on the surface at $z = L$ say, is

$$
\Pi(L) = P(z = L) - \lim_{L \to \infty} P(z = L) \tag{3.27}
$$

$$
= \frac{kT}{8\pi L^3} \zeta(3). \tag{3.28}
$$

The second term on the rhs of eq. (3.27) represents the pressure exerted on the 'back' of the plate. The function $\zeta(3) = \Sigma_{n=1}^{\infty} n^{-3} = 1.202$ is a Riemann ζ-function. We note that in an incompressible fluid, the hydrodynamic fluctuation force is a *repulsion* and has exactly the same functional form as that for Van der Waals interaction but is *independent* of the material properties of various media. From eq. (3.28) we see that the hydrodynamic disjoining pressure has an 'effective Hamaker constant' of $\approx 0.9kT$ whereas the Hamaker constant for, say, a silica–water–silica system is $\approx 2kT$. It appears, therefore, that hydrodynamic fluctuations are rather important in determining colloidal stability in incompressible fluids.

3.2. *Compressible fluids*

We can examine the effects of compressibility on hydrodynamic fluctuation forces by keeping C_0 finite. We can evaluate the ω-integral in eq. (3.16) in the same manner as we did for eq. (3.22). That is, we can replace the contour along the real axis by a large semi-circle in the upper half of the complex frequency plane. This procedure is possible because the ω-integral is a Green's function (see eq. (2.18)) and by the usual causality argument, the integrand is an analytic function of ω in the upper half plane. It is straightforward to show that for the integral in eq. (3.16), the integrand vanishes exponentially as $\omega \rightarrow i\infty$. As a consequence the ω-integral is identically zero! In other words, when compressibility effects are included, $H_z(z)$ is identically zero and the net hydrodynamic fluctuation force vanishes.

4. Discussion

The rather dramatic consequence of compressibility effects may be attributed to the assumption of incompressibility. The 'incompressible fluid' is really a low frequency idealization of the behaviour of fluids; at sufficiently high frequencies, all fluids become compressible. Also, as mentioned earlier, the Navier–Stokes equations, which are used here to describe the hydrodynamic fluctuations, are themselves only valid for describing low frequency phenomena. Thus within the framework of a continuum description it is not possible to be more positive about the nature of hydrodynamic fluctuations, never the less we expect that a better treatment of the high frequency behaviour of fluid motion will not alter our conclusion about nature of hydrodynamic fluctuation forces.

It is interesting to note, however, that within the present continuum description, a non-zero hydrodynamic fluctuation force may be obtained in a compressible fluid if quantum effects are included. To see this, we recall that if quantum effects are allowed, the r.h.s. of eq. (2.4) has to be multiplied by the factor $(\hbar\omega/2kT)\coth(\hbar\omega/2kT)$[3]). With the inclusion of this term, the integrand of the ω-integral for $H_3(z)$ will no longer be analytic in the upper half plane but will contain an infinite number of poles along the imaginary frequency axis at

$$\omega_n = i\frac{2kT}{\hbar}n, \quad n = 1, 2, \ldots . \tag{4.1}$$

These poles correspond to the singularities of $\coth(\hbar\omega/2kT)$. As a consequence, the ω-integral will not be zero but will be given by an infinite sum of terms evaluated at imaginary frequencies from the residues at these poles. This is analogous to the imaginary frequency sum in the general formula for Van der

Waals forces[1,2]) and the contributions from these terms are of the order $\exp(-2kTL/C_0)$. At separations larger than molecular dimensions, these terms are negligibly small. In contrast with similar terms in electromagnetic fluctuations, such quantum contributions are negligibly small even at separations comparable to molecular dimensions because of the very small ratio of the speed of sound to that of light. (Strictly speaking, a proper treatment of quantum effects should also include quantum corrections in the field equations (2.1) and (2.2).)

There is one further fundamental difference between hydrodynamic and electromagnetic fluctuations. In the latter case, the expression for the correlation formula for the random electromagnetic field (see eq. (1.1)) is proportional to $(\hbar/2kT)\coth(\hbar\omega/2kT)$ as distinct from the $(\hbar\omega/2kT)\coth(\hbar\omega/2kT)$ dependence for the random hydrodynamic stress tensor when quantum effects are included. As a consequence, in the final expression for Van der Waals forces, there is a contribution from the pole at $\omega = 0$ from the coth function – the so-called $n = 0$ term[1,2]). In the hydrodynamic problem, such a term does not arise because of an extra factor of ω multiplying the coth function. This is the basic mathematical reason why the average hydrodynamic force vanishes for a *compressible* fluid in the acoustic fluctuation formalism – a result which can be foreseen on more physical grounds from the arguments of section 1 above.

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