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Phonon Interaction Energy of an Elastic Layer on an Infinite Half Space Solid

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This brief note seeks to establish whether there is a significant contribution to the interaction energy of thin layers on a large substrate from surface mechanical or elastic vibrations. Recent conjectures in the literature (Ruckenstein and Churaev, *J. Colloid Interface Sci.* 147, 535 (1991)) have claimed that a significant error occurs in the standard Van der Waals theory (see, e.g., Dzyaloshinskii *et al., Sov. Phys. JETP* 37, 10 (1960); Lifshitz, *Sov. Phys. JETP* 2, 73 (1956)) due to the neglect of this contribution, since only the energies associated with electromagnetic waves are incorporated. The results of this investigation have entirely confirmed the traditional perspective, and specific calculations are presented as evidence that the phonon contribution is negligible. © 1994 Academic Press, Inc.

1. INTRODUCTION

Various methods of calculating the interaction energy in multilayer problems have been well established in the case of electromagnetic excitations, but few if any calculations have been performed using mechanical vibrations. The methods of Van Kampen *et al.* (4) and Ninham *et al.* (5) for electromagnetic waves carry over to the acoustic case quite simply, and are invoked here for convenience.

The method involves solving the boundary value problem for the mechanical excitations and performing a sum of the quantized phonon energies corresponding to the allowed frequencies. The boundary value problem solution is in the form of a dispersion relation, i.e., a relation connecting the frequency of surface modes to their wave number, and the sum is weighted by a suitable thermodynamic distribution function (which will of course be the Bose–Einstein distribution for bosons). The final result is naturally a function of the geometrical parameters of the system, since these determine the allowed frequencies of the excitations. The remaining free parameters in the result are those describing the elastic response of the media involved, such as densities and longitudinal or transverse sound velocities.

We consider a three-layer problem, consisting of a "sandwich" oriented perpendicular to the z-axis. The half-space $z < 0$ is vacuum and the region $0 < z < L$ is an elastic solid (or ideal fluid) with density ρ_1 and Lame coefficients λ_1 and μ_1 (where naturally $\mu_1 = 0$ if the layer is an ideal fluid). The half-space $z > L$ is a different elastic solid with density ρ_2 and Lame coefficients λ_2 and μ_2. The longitudinal and transverse velocities of sound in each medium ($i = 1, 2$) are, respectively, $\alpha_i = \sqrt{(\lambda_i + 2\mu_i)/\rho_i}$ and $\beta_i = \sqrt{\mu_i/\rho_i}$, and the inequality $\alpha_i > \sqrt{2}\beta_i$ discussed in (6) is of interest. If the region $0 < z < L$ is an ideal fluid, we need only set $\mu_1 = 0$, so $\beta_1 = 0$ and $\alpha_1 = \sqrt{\lambda_1/\rho_1}$.

1.1. *Wave Equations*

The problem of finding the elastic modes of vibration is reviewed fully in Ewing *et al.* (7), where it is solved in terms of the displacement potentials ϕ and $\mathbf{V}$, the displacement $\mathbf{u} = (u_x, u_y, u_z)$ being given by

$$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{V}.$$

As is usual in this formalism, Navier's equation for the displacement field,

$$\mu\nabla^2\mathbf{u} + (\mu + \lambda)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2},$$

in the absence of an external body force (such as might arise, for example, from Van der Waals forces at short distances), can be factored into two wave equations for the displacement potentials representing longitudinal and transverse waves, viz

$$\nabla^2\phi = \frac{1}{\alpha_i^2}\frac{\partial^2\phi}{\partial t^2}$$

and

$$\nabla^2\mathbf{V} = \frac{1}{\beta_i^2}\frac{\partial^2\mathbf{V}}{\partial t^2}.$$

A complete representation of the surface wave solutions would be of the form $h(z)e^{i(\omega t - px - qy)}$ but the cylindrical

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symmetry of the problem means that an ansatz of form $h(z)e^{i(\omega t-kx)}$ contains all the information in the problem; we need only regard k as the magnitude of a two dimensional vector $\mathbf{k} = (p, q)$, and perform a $\int d^2\mathbf{k}$ integral when summing over all possible modes. Accordingly, the trial solution

$$\phi = h(z)e^{i(\omega t-kx)}$$
$$\psi = g(z)e^{i(\omega t-kx)}$$

is introduced, where ψ is the y-component of the vector potential $\mathbf{V}$. Only one component of the vector potential need be considered, since the axial symmetry of the problem clearly indicates that only two components of displacement are independent (one in the z direction and one perpendicular to it) and these can be fully accounted for by the scalar potential and the y component of the vector potential. The differential equations obtained for h and g have exponential solutions, and to allow for the possibility of positive exponential growth in the region $0 < z < L$, a solution of the form

$$\left.\begin{aligned}\phi_1 &= Ae^{i(\omega t-kx)-\nu_1 z} + Be^{i(\omega t-kx)+\nu_1 z}\\ \psi_1 &= Ce^{i(\omega t-kx)-\nu_1' z} + De^{i(\omega t-kx)+\nu_1' z}\end{aligned}\right\} 0 < z < L$$

$$\left.\begin{aligned}\phi_2 &= Ee^{i(\omega t-kx)-\nu_2 z}\\ \psi_2 &= Fe^{i(\omega t-kx)-\nu_2' z}\end{aligned}\right\} z > L$$

is required, where

$$\nu_i^2 = k^2 - (\omega/\alpha_i)^2 \quad \nu_i'^2 = k^2 - (\omega/\beta_i)^2.$$

If the region $0 < z < L$ is an ideal fluid, the expressions involving the vector potential ψ_1 drop out, since no shear can be sustained.

1.2. *Boundary Conditions*

For a solid–solid interface, the assumption of continuity in the normal and tangential displacements leads to the pair of equations

$$u_x(L) = \left(\frac{\partial\phi_1}{\partial x} - \frac{\partial\psi_1}{\partial z}\right)\bigg|_{z=L} = \left(\frac{\partial\phi_2}{\partial x} - \frac{\partial\psi_2}{\partial z}\right)\bigg|_{z=L} \qquad [1]$$

and

$$u_z(L) = \left(\frac{\partial\phi_1}{\partial z} + \frac{\partial\psi_1}{\partial x}\right)\bigg|_{z=L} = \left(\frac{\partial\phi_2}{\partial z} + \frac{\partial\psi_2}{\partial x}\right)\bigg|_{z=L}, \qquad [2]$$

whereas for a solid–fluid interface only the normal component of displacement is required to be continuous:

$$u_z(L) = \left(\frac{\partial\phi_1}{\partial z}\right)\bigg|_{z=L} = \left(\frac{\partial\phi_2}{\partial z} + \frac{\partial\psi_2}{\partial x}\right)\bigg|_{z=L}. \qquad [3]$$

The normal stress in material i may be written as

$$p_{zz}^{(i)} = (\lambda_i + 2\mu_i)\nabla^2\phi_i - 2\mu_i\left(\frac{\partial^2\phi_i}{\partial x^2} - \frac{\partial^2\psi_i}{\partial x\partial z}\right), \qquad [4]$$

and the tangential stress as

$$p_{zx}^{(i)} = \mu_i\left(2\frac{\partial^2\phi_i}{\partial x\partial z} - \frac{\partial^2\psi_i}{\partial z^2} + \frac{\partial^2\psi_i}{\partial x^2}\right). \qquad [5]$$

If the middle layer is a solid, both stresses vanish at the solid–vacuum interface, so we obtain two conditions by setting Eqs. [4] and [5] to zero (with the subscript $i = 1$):

$$p_{zz}^{(1)}|_{z=0} = 0$$
$$p_{zx}^{(1)}|_{z=0} = 0. \qquad [6]$$

Furthermore, both the normal and tangential stresses must match at $z = L$, a condition which provides the two equations

$$p_{zz}^{(1)}|_{z=L} = p_{zz}^{(2)}|_{z=L}$$
$$p_{zx}^{(1)}|_{z=L} = p_{zx}^{(2)}|_{z=L}. \qquad [7]$$

If the middle layer is an ideal fluid, the shear stress is zero and the normal pressure is continuous, so we require

$$p_{zz}^{(1)}|_{z=L} = p_{zz}^{(2)}|_{z=L}$$
$$p_{zx}^{(2)}|_{z=L} = 0, \qquad [8]$$

and the necessity of zero pressure at the free surface gives

$$p_{zz}^{(1)}|_{z=0} = 0. \qquad [9]$$

1.3. *Zero Temperature Interaction*

Solution of the wave equations subject to the relevant boundary conditions leads to a dispersion relation $\Delta(\omega, k, L)$ which has real zeros at the allowed (eigen) frequencies of the surface modes, and has, in addition, no poles in the complex ω plane. At zero temperature, each eigenmode contributes an energy $\hbar\omega_i/2$, so following Van Kampen *et al.* (4), the total energy may be written (using the cylindrical symmetry) as

$$U(L) = \frac{\hbar}{4\pi}\int_0^\infty kdk\left(\sum_{Z_L}\omega_i(k) - \sum_{Z_\infty}\omega_i(k)\right),$$

where Z_L denotes the set of zeros of $\Delta(\omega, k, L)$. If C denotes a contour enclosing all of the zeros of Δ, then this result may conveniently be rewritten as

$$U(L) = \frac{\hbar}{8\pi^2 i}\int_0^\infty kdk \int_C \omega \frac{d}{d\omega} \ln D(\omega, k, L),$$

where

$$D(\omega, k, L) \equiv \frac{\Delta(\omega, k, L)}{\Delta(\omega, k, \infty)}$$

is the dispersion relation normalized by its dominant contribution as $L \to \infty$. Now D is an even function of ω, and also approaches unity exponentially as $|\omega| \to \infty$ in the right-half plane, so we may choose a contour up the imaginary axis and closed to the right for C and double the result. A simple integration by parts then yields

$$U(L) = \frac{\hbar}{4\pi^2}\int_0^\infty kdk \int_0^\infty \ln D(i\xi, k, L)d\xi. \qquad [10]$$

1.4. *Finite Temperature Interaction*

Following the method of Ninham *et al.* (5), the film energy at finite temperature may be derived from the relation

$$U(L, T) = \frac{1}{2\pi}\int_0^\infty [G_L(k) - G_\infty(k)]kdk,$$

where

$$G_L(k) = \mathrm{k}T \sum_{Z_L} \ln[\sinh(\hbar\beta\omega_i/2)]$$

and k is Boltzmann's constant, with $\beta \equiv 1/\mathrm{k}T$ as usual. Now for any holomorphic function $f(\omega)$, the identity

$$\sum_{Z_L} f(w_i) = \frac{1}{2\pi i}\int_C f(\omega)\frac{1}{D(\omega)}\frac{dD(\omega)}{d\omega} d\omega$$

holds, so (following closely the derivation given in (5)) the free energy may be expressed as

$$U(L, T) = \frac{\mathrm{k}T}{4\pi}\int_0^\infty kdk \sum_{n=-\infty}^{\infty} \ln D(i\xi_n, k, L), \qquad [11]$$

and $\xi_n = 2\pi n\mathrm{k}T/\hbar$ defines a sequence of "sample" frequencies spaced at about 10^{14} Hz at room temperature. At frequencies of this order solids are simply unable to propagate the excitations (see, e.g., typical dispersion graphs in Chap. 4 of Kittel (8)), which amounts to zero speed of sound. Similarly, the modes in liquids are completely damped within atomic dimensions (see (9) and Section 2.1). Hence the exponential terms which will occur in $D(\omega)$ are vanishingly small (see Sections 2 and 3), and thus the free energy comes entirely from the $n = 0$ term in Eq. [11]. The evaluation of the k integral is then greatly simplified:

$$U(L, T) = \frac{\mathrm{k}T}{4\pi}\int_0^\infty kdk \ln D(0, k, L). \qquad [12]$$

2. CALCULATIONS FOR THE SOLID–SOLID PROBLEM

Using the convenient notations $a_i = \omega/\alpha_i$, $b_i = \omega/\beta_i$ and

$$F(k) = (2k^2 - b_1^2)^2 - 4k^2\nu_1\nu_1'$$
$$f(k) = (2k^2 - b_1^2)^2 + 4k^2\nu_1\nu_1',$$

the dispersion relation (i.e., the determinant of the matrix representing the six linear equations [1], [2], [6], and [7]) may be written as

$$\Delta(\omega, k, L) \equiv e^{(\nu_1+\nu_1')L}[A_1 + A_2e^{-2\nu_1 L} + A_3e^{-2\nu_1' L} + A_4e^{-(\nu_1+\nu_1')L} + A_5e^{-2(\nu_1+\nu_1')L}].$$

The coefficients A_i are given by

$$A_1 = F\Delta_{12}$$
$$A_2 = -f\Delta_{23}$$
$$A_3 = f\Delta_{14}$$
$$A_4 = 4k(2k^2 - b_1^2)(\nu_1\Delta_{13} - \nu_1'\Delta_{24})$$
$$A_5 = F\Delta_{34},$$

where the factors Δ_{ij} are the following determinants:

$$\Delta_{12} = \begin{vmatrix} -k & \nu_1' & k & \nu_2' \\ \nu_1 & -k & \nu_2 & k \\ -2k\nu_1 & 2k^2 - b_1^2 & -2k\nu_2\mu_2/\mu_1 & -(2k^2 - b_2^2)\mu_2/\mu_1 \\ 2k^2 - b_1^2 & -2k\nu_1' & -(2k^2 - b_2^2)\mu_2/\mu_1 & -2k\nu_2'\mu_2/\mu_1 \end{vmatrix}$$

$$\Delta_{14} = \begin{vmatrix} -k & -\nu_1' & k & \nu_2' \\ \nu_1 & -k & \nu_2 & k \\ -2k\nu_1 & 2k^2 - b_1^2 & -2k\nu_2\mu_2/\mu_1 & -(2k^2 - b_2^2)\mu_2/\mu_1 \\ 2k^2 - b_1^2 & -2k\nu_1' & -(2k^2 - b_2^2)\mu_2/\mu_1 & -2k\nu_2'\mu_2/\mu_1 \end{vmatrix}$$

$$\Delta_{13} = \begin{vmatrix} -\nu'_1 & \nu'_1 & k & \nu'_2 \\ -k & -k & \nu_2 & k \\ 2k^2 - b_1^2 & 2k^2 - b_1^2 & -2k\nu_2\mu_2/\mu_1 & -(2k^2 - b_2^2)\mu_2/\mu_1 \\ 2k\nu'_1 & -2k\nu'_1 & -(2k^2 - b_2^2)\mu_2/\mu_1 & -2k\nu'_2\mu_2/\mu_1 \end{vmatrix}$$

$$\Delta_{24} = \begin{vmatrix} -k & -k & k & \nu'_2 \\ -\nu_1 & \nu_1 & \nu_2 & k \\ 2k\nu_1 & -2k\nu_1 & -2k\nu_2\mu_2/\mu_1 & -(2k^2 - b_2^2)\mu_2/\mu_1 \\ 2k^2 - b_1^2 & 2k^2 - b_1^2 & -(2k^2 - b_2^2)\mu_2/\mu_1 & -2k\nu'_2\mu_2/\mu_1 \end{vmatrix}$$

$$\Delta_{23} = \begin{vmatrix} -k & -\nu'_1 & k & \nu'_2 \\ -\nu_1 & -k & \nu_2 & k \\ 2k\nu_1 & 2k^2 - b_1^2 & -2k\nu_2\mu_2/\mu_1 & -(2k^2 - b_2^2)\mu_2/\mu_1 \\ 2k^2 - b_1^2 & -2k\nu'_1 & -(2k^2 - b_2^2)\mu_2\mu_1 & -2k\nu'_2\mu_2/\mu_1 \end{vmatrix}$$

$$\Delta_{34} = \begin{vmatrix} -k & -\nu'_1 & k & \nu'_2 \\ -\nu_1 & -k & \nu_2 & k \\ 2k\nu_1 & 2k^2 - b_1^2 & -2k\nu_2\mu_2/\mu_1 & -(2k^2 - b_2^2)\mu_2/\mu_1 \\ 2k^2 - b_1^2 & 2k\nu'_1 & -(2k^2 - b_2^2)\mu_2/\mu_1 & -2k\nu'_2\mu_2/\mu_1 \end{vmatrix}$$

Note that the last two columns are identical for all these determinants. The normalized dispersion relation is then

$$D(\omega, k) = 1 + \frac{A_2}{A_1} e^{-2\nu_1 L} + \frac{A_3}{A_1} e^{-2\nu'_1 L} + \frac{A_4}{A_1} e^{-(\nu_1+\nu'_1)L} + \frac{A_5}{A_1} e^{-2(\nu_1+\nu'_1)L}.$$

2.1. *Effects of Material Properties*

The elastic material properties are characterized by an absorption frequency which leads to a natural length scale when divided into the speed of sound. This in turn leads to two domains of theoretical tractability, one corresponding to a "slab" thickness much greater than this natural length, the other corresponding to a layer thickness much less than the length scale. Now if ω_0 is the principal absorption frequency of the middle layer with corresponding wavelength $\lambda = \alpha_1/\omega_0$, the exponentials in the dispersion relation are typically of the form

$$\exp[-2(k^2L^2 + \xi^2L^2/\alpha_1^2)^{1/2}] = \exp[-2(k^2L^2 + (\xi/\omega_0)^2(L/\lambda)^2)^{1/2}]$$

Hence if $L \gg \lambda$, the normalized dispersion relation approaches unity well before any material properties come into play, so for the purposes of evaluating the integral (Eq. [10]), the material constants α_i, β_i may be supposed independent of frequency. For solids, ω_0 is around 100 kHz–1 MHz (giving $\lambda \sim 1$ cm), whereas in liquids, ω_0 is usually about 10^{12} Hz with absorbtion wavelength $\lambda \sim 10 - 100$ Å (see (9)). For very many liquids, the ratio of the absorption coefficient κ to the frequency squared is a constant of order 10^{-14} neper $\cdot$ m^{-1} s^2, so the modes subject to significant damping ($\kappa L > 0.1$ say) are precisely those modes for which $\omega^2L^2/\alpha^2 > 1$ if $L > \lambda$, i.e., these damped modes contribute very little to the interaction energy (see (9)).

Thus for $L \gg \lambda$, the dimensionless variables $\Omega = \xi/(k\alpha_1)$ and $x = kL$ may be used to render the coefficients A_i and the dispersion relation D functions of Ω and x only, so

$$D(\Omega, x) = 1 + \frac{A_2(\Omega)}{A_1(\Omega)} e^{-2\nu_1 x/k} + \frac{A_3(\Omega)}{A_1(\Omega)} e^{-2\nu'_1 x/k} + \frac{A_4(\Omega)}{A_1(\Omega)} e^{-(\nu_1+\nu'_1)x/k} + \frac{A_5(\Omega)}{A_1(\Omega)} e^{-2(\nu_1+\nu'_1)x/k}.$$

The exponentials are then typically of the form

$$e^{-2\nu_1 x/k} = e^{-2x\sqrt{1+\Omega^2}},$$

which shows that $D \to 1$ for large x or Ω. Note that both $A_1(\Omega)$ and $\Delta(\Omega)$ go as Ω^6 near the origin, so there is no problem with the normalization.

2.2. *Zero Temperature Results*

The interaction energy at zero temperature for $L \gg \lambda$ is then

$$U(L) = \frac{\hbar\alpha_1}{4\pi^2L^3} \int_0^\infty d\Omega \int_0^\infty x^2 \ln D(\Omega, x)dx \qquad [13]$$

and the logarithm can be expanded as a Taylor series to give

$$U(L) \approx \frac{\hbar\alpha_1}{4\pi^2L^3} \int_0^\infty d\Omega \int_0^\infty x^2(D(\Omega, x) - 1)dx.$$

The x integrals are all of the form $\int_0^\infty x^2 e^{-q(\Omega)x}dx = 2/q(\Omega)^3$, which means that the interaction energy reduces to an expression involving only one integral:

$$U(L) \approx \frac{\hbar\alpha_1}{16\pi^2 L^3}\int_0^\infty d\Omega\left[\frac{A_2(\Omega)}{A_1(\Omega)}\frac{1}{(1+\Omega^2)^{3/2}} + \frac{A_3(\Omega)}{A_1(\Omega)}\frac{1}{(1+(\Omega\alpha_1/\beta_1)^2)^{3/2}} + \frac{8A_4(\Omega)+A_5(\Omega)}{A_1(\Omega)} \times \frac{1}{((1+\Omega^2)^{1/2}+(1+(\Omega\alpha_1/\beta_1)^2)^{1/2})^3}\right].$$

This integral is an order one number, and if one chooses typical material constants, say equivalent to those used in Fig. 4-35 of (7) ($\alpha_1/\beta_1 = \alpha_2/\beta_2 = \sqrt{3}$, $\beta_2/\beta_1 = 3.147$, $\rho_2/\rho_1 = 1.39$), the integral can be numerically evaluated to have value 0.47. In this case the potential is roughly

$$U(L) \approx \frac{0.003\hbar\alpha_1}{L^3},$$

which is the "retarded" result well predicted by Dzyaloshinskiĭ *et al.* (2). We recall that α_1 is the longitudinal velocity of sound in medium one ($0 < z < L$), and observe that the potential $U(L)$ is truly a function of all the material constants, since their dimensionless ratios enter directly into the numerical coefficient.

If $L \ll \bar{\lambda}$, the material absorption properties now guarantee the convergence of the integral in Eq. [10]. The energy is then

$$U(L) = \frac{\hbar}{4\pi^2 L^2}\int_0^\infty x dx \int_0^\infty \ln D(i\xi, x)d\xi$$

and it is clear that the double integral is of order ω_0, so

$$U(L) \sim \frac{\hbar\omega_0}{4\pi^2 L^2} \sim \frac{\hbar\alpha_1}{L^2\bar{\lambda}},$$

which again is precisely the prediction of Dzyaloshinskiĭ *et al.* (2). From the perspective of energetics, clearly the interaction energy of fluctuating elastic modes ($\sim\hbar\alpha$) is negligible compared to the electromagnetic contribution ($\sim\hbar c$).

It is easy to see that the energy for a free plate can be obtained by "switching" off the substrate (i.e., sending μ_2 and λ_2 to zero) and that this interaction potential is always positive. In this limit the dispersion relation may be written as a product, corresponding to symmetric and antisymmetric modes,

$$D_{\text{plate}} = D_1 D_2,$$

where, for $L \gg \bar{\lambda}$

$$D_1 = 1 - e^{-x(\sqrt{1+\Omega^2}+\sqrt{1+(\alpha_1\Omega/\beta_1)^2})} + \frac{P+Q^2}{P-Q^2}\left(e^{-x\sqrt{1+(\alpha_1\Omega/\beta_1)^2}} - e^{-x\sqrt{1+\Omega^2}}\right)$$

$$D_2 = 1 - e^{-x(\sqrt{1+\Omega^2}+\sqrt{1+(\alpha_1\Omega/\beta_1)^2})} - \frac{P+Q^2}{P-Q^2}\left(e^{-x\sqrt{1+(\alpha_1\Omega/\beta_1)^2}} - e^{-x\sqrt{1+\Omega^2}}\right)$$

and

$$P = 4\sqrt{1+(\alpha_1\Omega/\beta_1)^2}\sqrt{1+\Omega^2}$$

$$Q = 2 + (\alpha_1\Omega/\beta_1)^2.$$

Now clearly

$$\ln D_{\text{plate}} < D_1 + D_2 - 2 = -2e^{-x(\sqrt{1+\Omega^2}+\sqrt{1+(\alpha_1\Omega/\beta_1)^2})},$$

hence

$$U(L) < \frac{-\hbar\alpha_1}{2\pi^2 L^3}\int_0^\infty d\Omega \int_0^\infty dx\, x^2 e^{-x(\sqrt{1+\Omega^2}+\sqrt{1+(\alpha_1\Omega/\beta_1)^2})}$$

$$= \frac{-\hbar\alpha_1}{\pi^2 L^3}\frac{((\alpha_1/\beta_1)^2-5)(\alpha_1/\beta_1)^3 + 5(\alpha_1/\beta_1)^2 - 1}{5((\alpha_1/\beta_1)^2-1)^3},$$

and the right-hand side is always negative for $\alpha_1 > \sqrt{2}\beta_1$. The least attraction is obtained with an incompressible substance (Poisson's ratio $\sigma = \frac{1}{2}$ so $(\alpha_1/\beta_1)^2 = 2$), giving

$$U(L)_{\max} = \frac{-3\hbar\alpha_1}{\pi^2 L^3}.$$

2.3. *Finite Temperature Results*

At $\omega = 0$, the dispersion relation may be written as

$$D(\omega = 0, x) = 1 + (B_1 + B_2 x^2)e^{-2x} + Ce^{-4x},$$

where

$$B_1 + B_2 x^2 \equiv \lim_{\Omega\to 0}\frac{A_2(\Omega)e^{-2\nu_1 x/k} + A_3(\Omega)e^{-2\nu_1' x/k} + A_4(\Omega)e^{-(\nu_1+\nu_1')x/k}}{A_1(\Omega)},$$

so

$$B_1 = \frac{\begin{array}{r}-2[(\alpha_2^2+\beta_2^2)(\alpha_1^2-\beta_1^2)^2\mu_1^2 + 2\mu_1\mu_2\beta_1^2\beta_2^2(\alpha_1^2-\beta_1^2) \\ - \mu_2^2(\alpha_1^4+\beta_1^4)(\alpha_2^2-\beta_2^2)]\end{array}}{\begin{array}{r}(\alpha_1^2-\beta_1^2)[(\alpha_1^2-\beta_1^2)\mu_1 + (\alpha_1^2+\beta_1^2)\mu_2] \\ \times [(\alpha_2^2+\beta_2^2)\mu_1 + (\alpha_2^2-\beta_2^2)\mu_2]\end{array}}$$

$$B_2 = \frac{\begin{array}{r}-4(\alpha_1^2-\beta_1^2)[(\alpha_2^2+\beta_2^2)\mu_1^2 \\ - 2\mu_1\mu_2\beta_2^2 - \mu_2^2(\alpha_2^2-\beta_2^2)]\end{array}}{\begin{array}{r}[(\alpha_1^2-\beta_1^2)\mu_1 + (\alpha_1^2+\beta_1^2)\mu_2] \\ \times [(\alpha_2^2+\beta_2^2)\mu_1 + (\alpha_2^2-\beta_2^2)\mu_2]\end{array}}$$

and

$$C = \lim_{\Omega\to 0} \frac{A_5(\Omega)e^{-2(\nu_1+\nu_1')x/k}}{A_1(\Omega)}$$
$$= \frac{\begin{array}{r}(\mu_1-\mu_2)[(\alpha_1^2-\beta_1^2)(\alpha_2^2+\beta_2^2)\mu_1 \\ - (\alpha_2^2-\beta_2^2)(\alpha_1^2+\beta_1^2)\mu_2]\end{array}}{\begin{array}{r}[(\alpha_1^2-\beta_1^2)\mu_1 + (\alpha_1^2+\beta_1^2)\mu_2] \\ \times [(\alpha_2^2+\beta_2^2)\mu_1 + (\alpha_2^2-\beta_2^2)\mu_2]\end{array}}.$$

Hence we have

$$U(L, T) = \frac{\mathrm{k}T}{4\pi L^2}\int_0^\infty x\,dx\,\ln(1 + (B_1 + B_2x^2)e^{-2x} + Ce^{-4x}).$$

In the limit where the two materials are identical (i.e., the subscripts can be rubbed out) there should be no interaction energy, and a brief consideration of the expressions for B_i and C given above show that this is indeed true. It is also clear that the interaction can be either attractive or repulsive, depending on the choice of material constants and hence the signs of A and B_i. For the same typical data as used in the zero temperature calculation, the constants have the values $B_1 = 2.07$, $B_2 = 1.79$, and $C = 0.725$, and the integral is approximately $+1.026$, giving the repulsive potential

$$U(L, T) \approx 0.082\,\frac{\mathrm{k}T}{L^2},$$

a result whose form is exactly predicted in (2).

The energy for a free elastic plate can be obtained from this result by sending the elastic constants μ_2 and λ_2 to zero, a limiting procedure which returns the exact values $B_1 = -2$, $B_2 = -4$, and $C = 1$. This corresponds to the potential

$$U(L, T) = \frac{\mathrm{k}T}{4\pi L^2}\int_0^\infty x\,dx\,\ln(1 - (2 + 4x^2)e^{-2x} + e^{-4x})$$
$$\approx -0.273\,\frac{\mathrm{k}T}{L^2},$$

which is independent of the elastic constants of the plate and attractive as expected.

3. RESULTS FOR AN IDEAL FLUID LAYER ON A SOLID SUBSTRATE

If the top layer is now an ideal fluid characterized by a longitudinal velocity of sound α_1 and density ρ_1, the trial solution and the boundary condition equations [3], [8], and [9] lead to the dispersion relation

$$\Delta(k, \omega) = \nu_1[4k^2\nu_2\nu_2' - (k^2 + \nu_2'^2)^2]\cosh(\nu_1 L)$$
$$+ (\rho_1\alpha_1^2/\rho_2\beta_2^2)\nu_2 a_1^2 b_1^2 \sinh(\nu_1 L) = 0,$$

which is equivalent to the result displayed as Eq. [4–154] in (7), where the same problem is treated in the context of waves propagating through multilayer systems from a point source. Using the dimensionless variables Ω and x defined previously (for $L \gg \lambda\!\!\!^{-}$), the normalized dispersion relation may then be written as

$$D(\Omega, x) = 1 + A(\Omega)e^{-x\sqrt{1+\Omega^2}}, \qquad [14]$$

where

$$A(\Omega) \equiv \frac{t_1 - t_2}{t_1 + t_2}$$

and

$$t_1 \equiv (1 + \Omega^2)^{1/2}[(2 + (\Omega\alpha_1/\beta_2)^2)^2$$
$$- 4(1 + (\Omega\alpha_1/\beta_2)^2)^{1/2}(1 + (\Omega\alpha_2/\alpha_2)^2)^{1/2}]$$
$$t_2 \equiv f\Omega^4((1 + (\Omega\alpha_1/\alpha_2)^2)^{1/2}$$

with $f \equiv (\rho_1\alpha_1^4/\rho_2\beta_2^4)$.

3.1. *Zero Temperature Results*

At zero temperature, where $L \gg \lambda\!\!\!^{-}$, the interaction energy is clearly

$$U(L) = \frac{\hbar\alpha_1}{4\pi^2L^3} \times \int_0^\infty d\Omega \int_0^\infty x^2\ln(1 + A(\Omega)e^{-x\sqrt{1+\Omega^2}})dx \qquad [15]$$

from Eq. [10]. The inequality $\alpha_2 > \beta_2$ implies that $t_1 > 0$, and since $t_2 > 0$, we have $|A(\Omega)| < 1$. The interaction energy can then be bounded by taking the worst case $A = -1$,

$$\int_0^\infty d\Omega \int_0^\infty x^2\ln(1 - e^{-x\sqrt{1+\Omega^2}})dx = -2\zeta(4) = -\pi^4/45$$

so

$$|U| < \frac{\pi^2\hbar\alpha_1}{180L^3} \approx 0.055\hbar\alpha_1/L^3.$$

For interest, values typical of water and glass were inserted into Eq. [15] ($\rho_2/\rho_1 = 2.4$, $\alpha_2/\alpha_1 = 3.4$, $\beta_2/\alpha_1 = 1.9$) and the integral was evaluated numerically, giving the repulsion

$$U(L) = \frac{0.04497\hbar\alpha_1}{L^3}. \qquad [16]$$

If $L \ll \bar{\lambda}$, the change of variable $\omega \rightarrow \Omega$ is again no longer appropriate, so the energy is then

$$U(L) \sim \frac{\hbar\alpha_1}{L^2\bar{\lambda}}$$

by the same arguments advanced in Section 2.2.

3.2. *Finite Temperature Results*

Using Eq. [11] and taking the limit $\omega \rightarrow 0$ of Eq. [14], we have

$$\begin{aligned} U(L, T) &= \frac{\mathrm{k}T}{4\pi L^2}\int_0^\infty x\ln(1 + e^{-x})dx \\ &= \frac{3\mathrm{k}T}{16\pi L^2}\zeta(3) \approx 0.0717\mathrm{k}T/L^2. \end{aligned}$$

Thus the interaction is independent of the material parameters, a phenomenon which is clearly a consequence of the choice of an ideal fluid.

4. CONCLUSIONS

The results of these calculations show clearly that for the case of an elastic solid film "glued" to an infinite elastic solid substrate, interactions arising from mechanical or elastic fluctuation phenomena are small compared to typical Van der Waals interactions of electromagnetic origin. Elastic fluctuations are most important in a free film and contribute to an effective Hamaker constant of around 0.3kT. This is of the order of the so-called zero frequency term in electromagnetic fluctuations. It was also found that the imposition of rigid (sticking) boundary conditions at the free surface prevents surface modes' arising, and in particular the three-layer "rigid–elastic–rigid" problem displays only bulk oscillation modes, and hence no interaction energy. Calculations seeking long-range interactions via other formalisms (e.g., (10)) may well be finding negative results because such boundary conditions are applied. There is potential to perform these calculations for a nonviscous fluid film as well, but the introduction of any viscosity implies damped oscillations, and therefore the concept of a "normal mode" is somewhat dubious in this case. The best formalism to handle this problem is probably that of Jones (11).

5. APPENDIX: USEFUL APPROXIMATIONS FOR THE FLUID–SOLID PROBLEM

Although the integrals in Eq. [15] are analytically intractable in general, it is possible to construct a reasonable approximation to $A(\Omega)$ and thus obtain a closed-form expression for U. Clearly the principal contribution to the integral comes from the region near the origin, so it is desirable to match the function $A(\Omega)$ closely here. From the readily derived properties

$$\lim_{\Omega\rightarrow 0} A(\Omega) = 1$$

$$\lim_{\Omega\rightarrow\infty} A(\Omega) = \frac{\rho_2\alpha_2 - \rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1}$$

$$A''(0) = \frac{-2\alpha_1^2\alpha_2^2\rho_1/\rho_2}{\beta_2^2(\alpha_2^2 - \beta_2^2)}$$

and the manifest evenness of A, it is clear that an ansatz of form

$$g(\Omega) = \frac{m}{p^2 + \omega^2} + c$$

is suitable, since $A(\omega)$ is monotonically decreasing for most realistic choices of material constants. The constants c, p^2, and m are easily obtained,

$$c = \frac{\rho_2\alpha_2 - \rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1}$$

$$p^2 = \frac{2\beta_2^2(1 - \beta_2^2/\alpha_2^2)}{\alpha_1\alpha_2(1 + \rho_1\alpha_1/\rho_2\alpha_2)}$$

$$m = p^2(1 - c),$$

and the integral can then be explicitly evaluated by expanding the logarithm as a Taylor series,

$$U(L) \approx U_{\text{app}}(L) = \frac{\hbar\alpha_1}{4\pi^2L^3}\sum_{k=1} E_k,$$

where

$$\begin{aligned} E_k &= \frac{(-1)^{k+1}}{k}\int_0^\infty d\Omega\int_0^\infty x^2\left(\frac{m}{p^2 + \omega^2} + c\right)^k e^{-kx\sqrt{1+\Omega^2}})dx \\ &= \frac{-(-c)^k\sqrt{\pi}}{k^4}\sum_{n=0}^{k}\binom{k}{n}\frac{(m/cp^2)^n n!}{\Gamma(n + 3/2)} \\ &\qquad \times F(n, 1/2; n + 3/2; 1 - 1/p^2) \end{aligned}$$

and $F(\alpha, \beta; \gamma; \delta)$ is a hypergeometric function. The first term has the useful closed form expression

$$E_1 = \frac{2m}{1-p^2}\left(\frac{\cos^{-1}p}{p(1-p^2)^{1/2}} - 1\right) + 2c.$$

In most realistic situations, both the density and the longitudinal sound velocity in the solid are greater than their fluid counterparts, so the constant c is close to unity and hence A is virtually unity over the significant part of the integrand. The first order approximation E_1 can then be weighted by

$$\frac{\int_0^\infty d\Omega \int_0^\infty x^2 \ln(1 + e^{-x\sqrt{1+\Omega^2}})dx}{\int_0^\infty d\Omega \int_0^\infty x^2 e^{-x\sqrt{1+\Omega^2}}dx} = 7\pi^4/720$$

in order to obtain a better approximation; in fact, for the water–glass configuration, comparison with Eq. [16] shows that such an approximation is nearly correct to three places:

$$\frac{7\pi^4}{720}\frac{E_1}{4\pi^2} = 0.04485.$$

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