On the stress depression under a sandpile

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Abstract

The observed minimum in the normal stress beneath the highest portion of a 'sandpile' (a pile of granular material) is a counter-intuitive result that has long remained unexplained. In this paper, we suggest that spatial size differentiation, where the larger particles are separated from smaller particles within the pile, creates an increase in the internal horizontal stress distribution, which in turn gives rise to a minimum in the normal stress, similar to what is observed experimentally. We develop a simple analytic model for the normal stress that encapsulates the existence of horizontal contact forces between adjacent colevel particles. We compare this analytic model to the results of an n-body simulation code, where we obtain close quantitative agreement for small sandpiles. Finally, experimental results from a real sandpile are replicated by an n-body code, but only for a distribution of material where the largest particles are at the top of the pile.

Introduction

What is the distribution of stress within and beneath a pile of granular material? This question, of great practical importance in the storage and handling of bulk particulate materials, is not without a certain charm as a problem in physics. The proverbial man or woman on the street (and most scientists polled by the authors) would predict that the pressure on the floor (more precisely, the normal stress) should be greatest beneath the highest point on the pile. This turns out not always to be so. In 1979 and 1981 respectively, Jotaki and Moriyama [1] and Šmíd and Novosad [2] measured the normal stress beneath conical piles of particles and found a dip in the normal stress on the floor under the highest part of the pile.

Various authors [3–9] have attempted to understand this phenomenon, but no physical reason for this counter-intuitive behaviour has been obtained. In this paper, we attempt to replicate the results of Smíd and Novosad [2] by using a pile of cylinders as a two-dimensional analogue of a conical pile of particulate material, which we call a sandpile. We study the force distribution via simple analytic models and computer simulation, thereby obtaining a physical description of the force structure within a sandpile.

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Analytic results

To begin our study, we analyze the force structure in a 2D equilateral pile of perfectly hard, identical cylinders (which we will refer to as 'particles' from here on). For this analytic treatment, we let the particles lie on a grid as shown in Fig. 1. Unlike the normal Cartesian system, the x and y axes are not mutually orthogonal. We take on this grid scheme, because the basic structural unit is a triangular system of three particles (see Fig. 1 and ref. [9]). The static forces will propagate through the contact points and so be at an angle to the gravitational force g.

If we analyze the forces acting on a particle at (x, y), we find by resolving the forces in the horizontal direction that

$$L(x, y-1)+R(x, y)-L(x, y)-R(x-1, y)=0 \tag{1}$$

where L stands for forces directed along the left diagonal and R stands for forces directed along the right diagonal. The vertical component of the force satisfies the equation

$$L(x, y)+R(x, y)-L(x, y-1)-R(x-1, y)=W \tag{2}$$

where $W=mg/\cos\,\theta$, m being the mass of each particle.

By adding and subtracting eqn. (1) from eqn. (2), and setting

$$L(x, 0)=R(0, y)=0 \tag{3}$$

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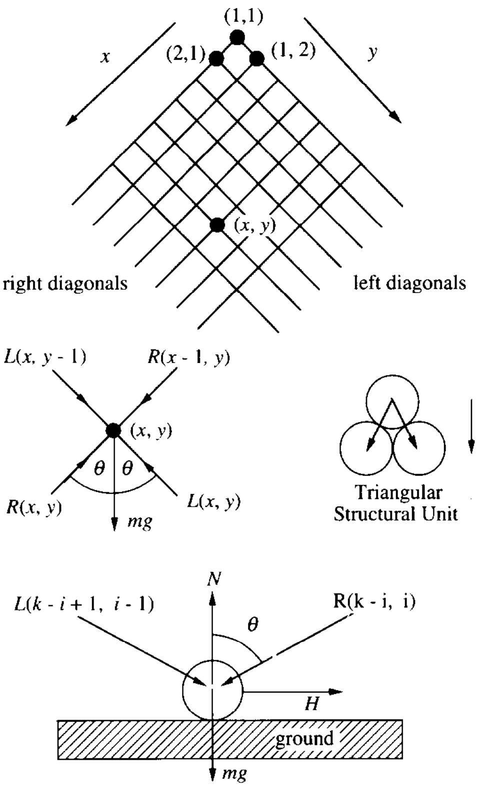


Fig. 1. To obtain an analytic expression for the distribution of forces in a sandpile, we have created a coordinate grid where every intersection point can contain a particle. The forces propagate through the particle contact points which lie along the coordinate lines. Unlike normal cartesian coordinates, our grid lines are not mutually orthogonal. We are motivated to assume this grid system by noting that the basic force unit is a system of three particles.

By labeling the forces propagating down the left and right diagonal grid lines by L and R, respectively, it is possible to obtain an analytic expression for the force structure within and at the base of a 2D equilateral pile of perfectly hard, identical particles. The labeling system for a particle in contact with the ground is also shown.

we obtain recurrence relations for L and R, which have the solutions

$$L(x, y) = Wy/2 \tag{4}$$

and

$$R(x, y) = Wx/2 \tag{5}$$

So, as expected, the weight force along the diagonal grid lines increases as one travels from the top of the pile to its base. A particle at the base of a static pile, will experience a normal force from the ground plus a friction force. If we progressively number the base particles from $i=1$ to k then the normal (N) and horizontal (H) stresses* at the base of the pile have the general form (Fig. 1)

$$N=mg+L(k-i+1, i-1) \cos \theta+R(k-i, i) \cos \theta \tag{6}$$

and

$$H=L(k-i+1, i-1) \sin \theta-R(k-i, i) \sin \theta \tag{7}$$

Using eqns. (4) and (5) from our idealized sandpile, we have the solutions

$$N=(k+1)mg/2 \tag{8}$$

and

$$H=mg \tan \theta[2i-k-1]/2 \tag{9}$$

For this ideal case, the normal force is a constant for every particle along the base of the pile, while the horizontal stress along the base has a minimum absolute value under the highest point of the pile, and has two maxima at the edges of the sandpile (a result first shown by Bagster [3]).

We now suppose (for reasons that will become clearer as we go further into this paper) that an additional horizontal force acts upon these particles. This force may arise due to a size difference between the particles and so there are now additional contact points in the horizontal direction as well as along the diagonal lines (see Fig. 2). In such a circumstance, our system becomes *over determined*, since we have at least three unknown forces – those along the left diagonal, right diagonal and horizontal directions, but we can only resolve the forces in two directions and thus have only two independent force equations.

One way to resolve this impasse is to allow one of the forces to be a free variable, we can then solve for the two remaining forces in terms of the unknown force. Our original force equation in the horizontal direction (eqn. (1)) can now be rewritten to incorporate the free variable force.

$$L(x, y-1)+R(x, y)-L(x, y)-R(x-1, y)=f(x, y) \tag{10}$$

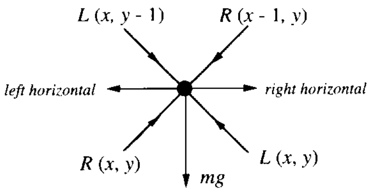


Fig. 2. The force structure on a particle that is subjected to an additional horizontal force.

*Note, as a naming convention in this paper, when we refer to a particular particle in a pile, we will consider the "forces" on that particle. However, when we refer to a large collection of particles (i.e. a pile), we will consider the "stresses" that act on the pile.

where the force $f(x, y)$ represents this hypothetical contact force from adjacent colevel particles, and it has the form $f(x, y) = Csgn(y-x)/\sin\theta$, where the factor C is an assumed free parameter representing the strength of this hypothetical contact force, and $sgn(y-x)$ is the assumed direction of the horizontal contact force. When C is positive, particles on the left hand side of the sandpile ($x>y$) see an applied horizontal force to the left, while particles on the right hand side ($y>x$) see an applied horizontal force to the right. The direction of the forces are reversed when C is negative. The vertical component of the force still satisfies eqn. (2), and the boundary conditions of eqn. (3) are still true, but the recurrence relations now become

$$R(x,y)=\begin{cases}(W-\alpha)x/2, & x<y\\ yW/2-(y-1)\alpha/2, & x=y\\ xW/2+(x-2y+1)\alpha/2 & x>y\end{cases} \tag{11}$$

and

$$L(x,y)=\begin{cases}(W-\alpha)y/2, & y<x\\ xW/2-(x-1)\alpha/2. & y=x\\ yW/2+(y-2x+1)\alpha/2 & y>x\end{cases} \tag{12}$$

with

$$\alpha = C/\sin\theta \tag{13}$$

From eqns. (6) and (7), the normal and horizontal forces at the base of the pile have a number of different formulae, dependent on whether we have an even or odd number of particles at the base of the sandpile.

For the case where k is an even number, then the normal and horizontal force have the forms

$$N=\begin{cases}\frac{1}{2}\{mg(k+1)+C(k-4i+2)\cot\theta\} & i\leqslant k/2\\ \frac{1}{2}\{mg(k+1)+C(4i-3k-2)\cot\theta\} & i\geqslant k/2+1\end{cases} \tag{14}$$

and

$$H=\begin{cases}\frac{1}{2}\{mg(2i-k-1)\tan\theta-C(k-2i)\} & i\leqslant k/2\\ \frac{1}{2}\{mg(2i-k-1)\tan\theta+C(2i-k-2)\} & i\geqslant k/2+1\end{cases} \tag{15}$$

Note that when $C>0$ the normal force N has a minimum at the points $i=k/2$ and $k/2+1$. When $C<0$ the normal force has a maximum at those points.

For the case where k is an odd number, we have

$$N=\begin{cases}\frac{1}{2}\{mg(k+1)+C(k-4i+2)\cot\theta\} & i<(k+1)/2\\ \frac{1}{2}\{mg(k+1)-C(k-1)\cot\theta\} & i=(k+1)/2\\ \frac{1}{2}\{mg(k+1)+C(4i-3k-2)\cot\theta\} & i>(k+1)/2\end{cases} \tag{16}$$

and

$$H=\begin{cases}\frac{1}{2}\{mg(2i-k-1)\tan\theta-C(k-2i)\} & i<(k+1)/2\\ 0 & i=(k+1)/2\\ \frac{1}{2}\{mg(2i-k-1)\tan\theta+C(2i-k-2)\} & i>(k+1)/2\end{cases} \tag{17}$$

Again, when $C>0$ or $C<0$, the normal force has a minimum or a maximum, respectively, at the base of the pile. To obtain some intuitive understanding of eqns. (14) and (15), we plot, in Fig. 3, the normal and horizontal forces for a trial case where k takes on an

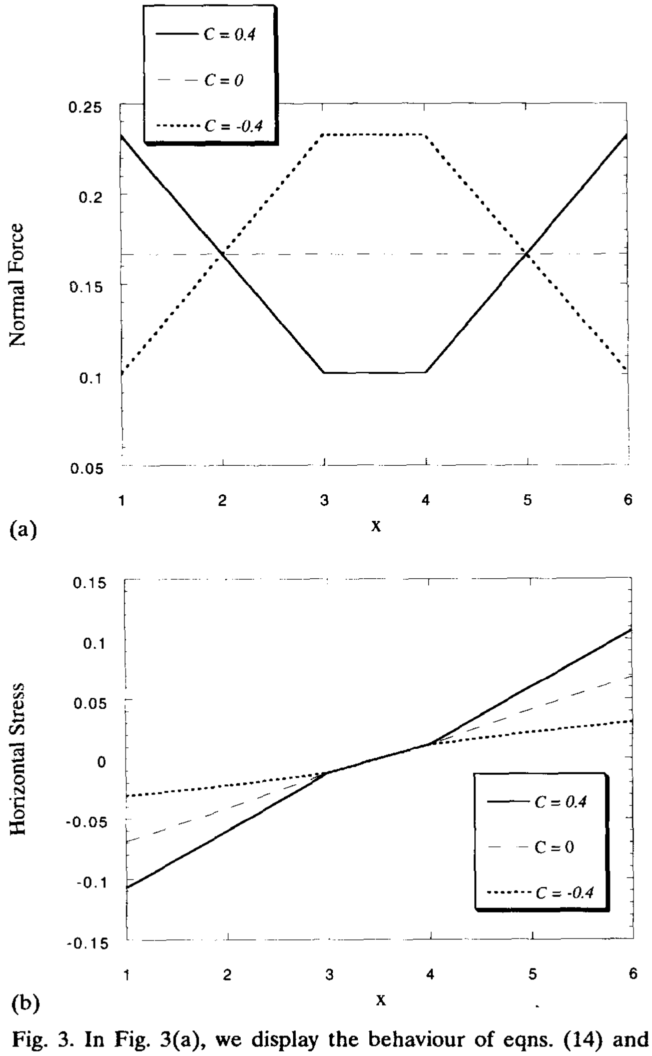
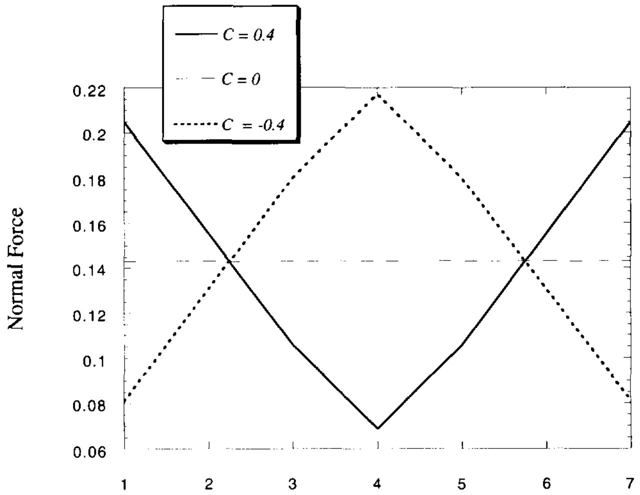


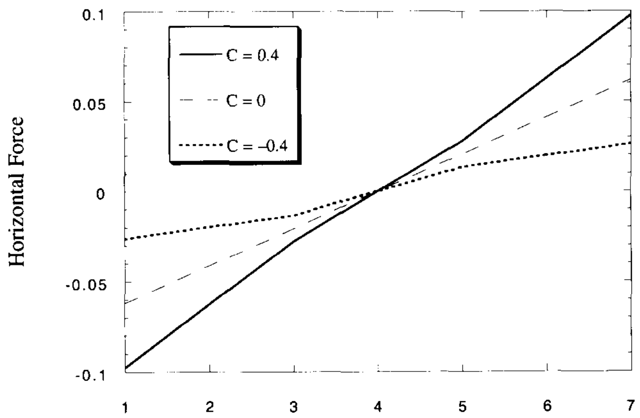
Fig. 3. In Fig. 3(a), we display the behaviour of eqns. (14) and (15) for $C=\pm 0.4$ and $C=0$ where C is expressed as a proportion of the weight of one particle (mg). The value $C=0.4$ corresponds to the case where the horizontal forces, between adjacent colevel particles, are pointing away from the centreline of the pile. In this case, we obtain a minimum in the normal stress at the base of the pile. We also obtain an increase in the horizontal stress at the base of the pile (Fig. 3(b)). When $C=-0.4$, the horizontal forces are now directed towards the centreline of the sandpile, and the normal force has a maximum under the highest part of the pile, while the horizontal shear stress decreases. Finally, for $C=0$ we retrieve the behaviour of our 'perfect sandpile' as expressed by eqns. (8) and (9).

even value. In Fig. 3(a), we show the normal force for three different values of C, where C is expressed in units of the weight (mg) of one particle, and k has a value of 6. In all the examples that we will study we shall set θ equal to $\pi/6$.

As expected, when C is positive the horizontal force is 'repulsive' and points away from the centre of the sandpile, with the subsequent formation of a depression in the normal force. However, when C is negative and the horizontal force is 'attractive', i.e. pointing toward the centre of the sandpile, then the normal force assumes a maximum under the centre of the sandpile, while

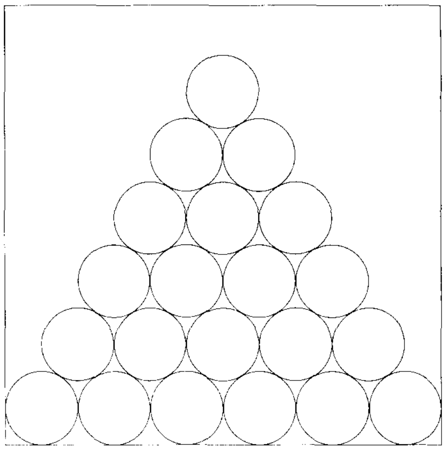


(a)

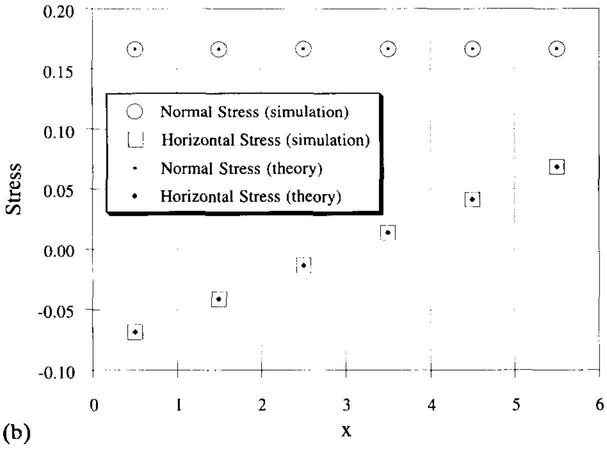


(b)

Fig. 4. In Fig. 4(a), we display the behaviour of eqns. (16) and (17) for $C = \pm 0.4$ and $C = 0$, where C is expressed as a proportion of the weight of one particle (mg). The value $C = 0.4$ corresponds to the case where the horizontal forces, between adjacent colevel particles, are pointing away from the centre of the pile. In this case, we obtain a minimum in the normal stress at the base of the pile. We also obtain an increase in the horizontal stress at the base of the pile (Fig. 4(b)). When $C = -0.4$, the normal force has a maximum under the highest part of the pile, while the horizontal shear stress decreases. Finally, for $C = 0$ we retrieve the behaviour of our 'perfect sandpile' as expressed by eqns. (8) and (9).



(a)

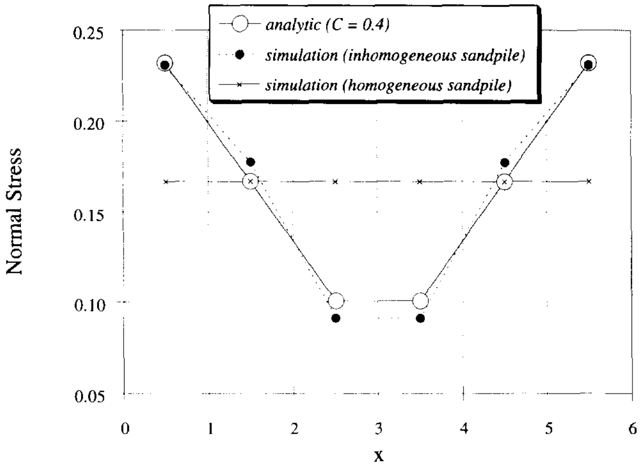


(b)

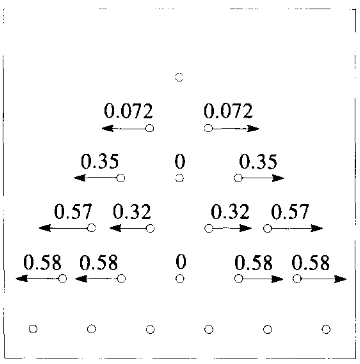
Fig. 5. A system of 21 particles arranged in the form of an equilateral triangle is the initial particle configuration for our computer simulation code (Fig. 5(a)). The normal and shear stresses obtained from our simulation of this equilateral sandpile (Fig. 5(b)) agree extremely well with the analytic results derived from eqns. (8) and (9). The stresses have been normalized such that the weight of the simulation pile is equal to one.

for $C = 0$ we obtain the flat normal force profile that one would expect from eqn. (8). In Fig. 3(b), we show the horizontal stress (H) for the three different C values. For positive C values, the H increases at the base of the pile, while H decreases for negative C values, and $C = 0$ gives the linear horizontal stress profile of eqn. (9).

In Fig. 4, we display the behaviour of eqns. (16) and (17), in which k takes on an odd value, in this case $k = 7$. The normal forces obtained for the different values of C are displayed in Fig. 4(a). As with the results shown in Fig. 3, the normal force is sharply peaked for $C < 0$, suffers a depression when $C > 0$ and is flat for $C = 0$. The horizontal forces are displayed in Fig. 4(b), where $C > 0$ gives an increase in the horizontal forces, $C < 0$ causes the horizontal forces to decrease, and $C = 0$ gives the linear solution of eqn. (9).



(a)



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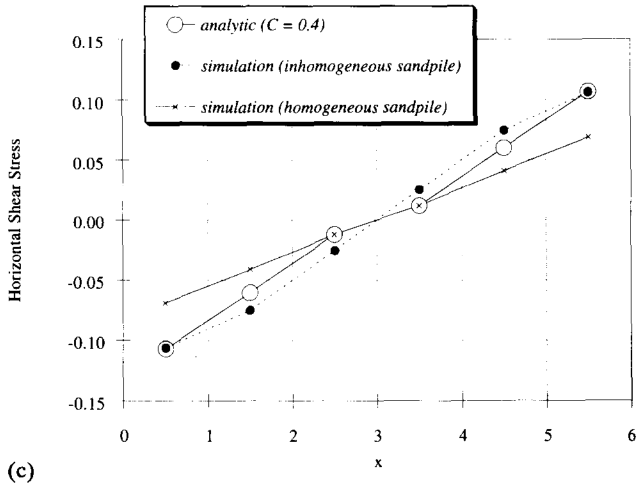


(c)

Fig. 6. In Fig. 6(a), we display the normal stress for an inhomogeneous version of the simulation sandpile shown in Fig. 5(a). All the particles, except for the base particles, have a diameter of $1.0001d$, where d is the diameter of the base particles. The normal stress has a minimum under the highest point of the pile. This agrees approximately with the results of eqn. (14) for an assumed nonzero horizontal contact force ($C=0.4$). These 'dips' in the normal stresses are markedly different from the flat normal stress obtained for a sandpile where all the particles are the same size (homogeneous sandpile). Figure 6(b) displays the horizontal contact forces from the simulation inhomogeneous sandpile, expressed in units of mg. The shear stress for the different sandpiles is shown in Fig. 6(c). The shear stress for the inhomogeneous pile is greater than for the homogeneous pile. The normal and shear stresses shown in Figs. 6(a) and 6(c) have been normalized such that the weight of each sandpile is equal to one.

In summary, eqns. (14) to (17) show that if the horizontal forces between the particles are repulsive then the shear stress at the base of the pile increases and the normal stress develops a minimum above the centre of the sandpile. However if the forces are attractive, the shear stress decreases and the normal stress has a maximum at the centre of the pile. To see if these results are in anyway realistic, we use a particle simulation code, previously described in ref. [9], to compare theory with simulation.

Simulation versus analytic results

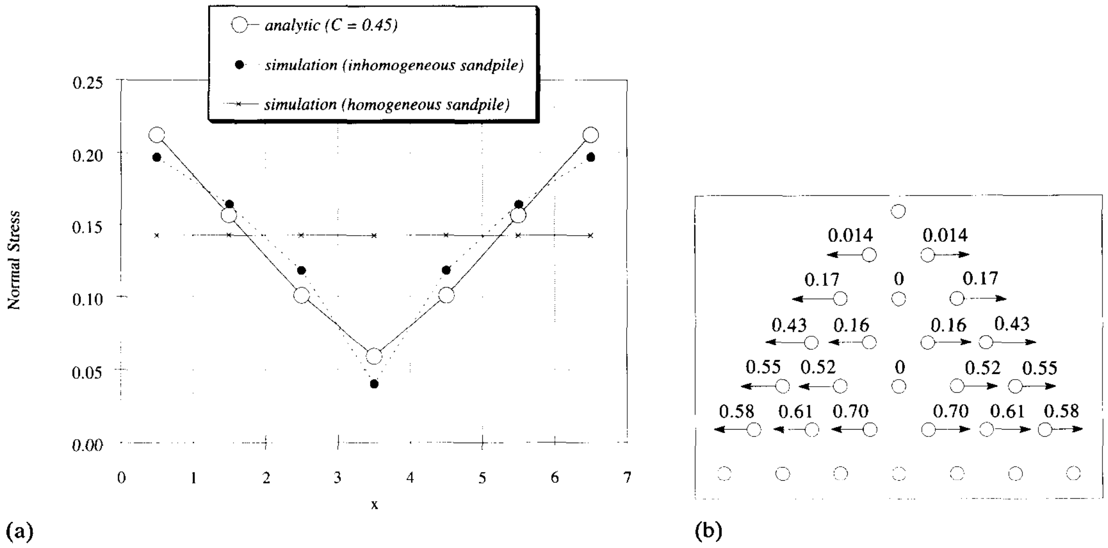
For our initial study, we simulate a small equilateral pile of 21 particles, where each particle has the same diameter d (see Fig. 5(a)). An equilateral pile has a slope, or angle of repose, of 60° with respect to the ground. Equivalently, we simply set $\theta=\pi/6$.

To overcome the possibility of mathematical indeterminacy in the contact forces between hard particles, we assume that interparticle interactions are modelled by using a very stiff, damped spring system, where the spring constant (K_s) of the particle satisfies the condition

$$mg/K_s d = 10^{-5} \tag{18}$$

i.e. a single computer particle will sink into the ground, under its own weight, a distance of $10^{-5}d$ (see ref. [9] for details).

The forces, from our computer simulation, are found to propagate along diagonal lines similar to the grid lines of Fig. 1. No horizontal contact forces are found to exist between adjacent colevel particles in the pile. In Fig. 5(b), we plot the resulting normal and horizontal stresses as a function of position along the base of this simulated pile and compare them with the results from eqns. (8) and (9). As was deduced, the normal stress is a constant for all the particles at the base of the pile, while the horizontal stress displays the predicted linear behaviour. We find the agreement between simulation and theory to be good down to a relative difference of 5×10^{-5}, i.e.



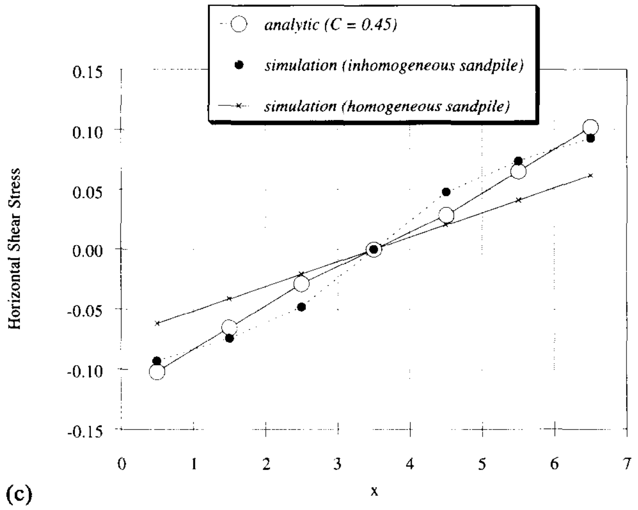


Fig. 7. The normal stress for an inhomogeneous equilateral sandpile containing 28 particles, with seven particles at the base, is shown in Fig. 7(a). All the particles, except for the base particles, have a diameter of 1.0001d, where d is the diameter of the base particles. The normal stress has a minimum under the highest point of the pile. This agrees approximately with the results of eqn. (16) for an assumed non-zero horizontal contact force ($C=0.45$). These 'dips' in the normal stresses are markedly different from the flat normal stress obtained for a sandpile where all the particles are the same size (homogeneous sandpile). Figure 7(b) displays the horizontal contact forces from the simulation inhomogeneous sandpile, expressed in units of mg. The shear stress for the different sandpiles is shown in Fig. 7(c). The shear stress for the inhomogeneous pile is greater than for the homogeneous pile. The normal and shear stresses shown in Figs. 7(a) and 7(c) have been normalized such that the weight of each sandpile is equal to one.

$$\max\left\{\left|\frac{V_{\text{simulation}}-V_{\text{theory}}}{V_{\text{simulation}}}\right|\right\}\approx 5\times 10^{-5} \tag{19}$$

where $V_{simulation}$ is the 'value' of the simulation variable and similarly with V_{theory}.

Similar results, with the same code, have been found for equilateral sandpiles containing thousands of particles [9]. This provides us with some confidence that our simulation code is replicating, to a good approximation, the behaviour of hard particles.

In our second example, we again use the simulation pile of Fig. 5(a) where particles at the base of the pile still have a diameter d, but all other particles in the pile are given a slightly enlarged diameter of 1.0001d. This slight size variation gives rise to horizontal contact forces within this inhomogeneous sandpile, since the colevel particles now push against each other. At first glance, an increase in the diameter of 10^{-4} should produce negligible horizontal contact forces, but if a simulation particle were pressed that distance into the ground, it would produce a counter-force equal to 10 times its own weight. Thus, an increase in particle diameter of 10^{-4} should produce non-negligible contact forces. We note that particles at the base of the pile are fixed in position by frictional forces with the ground, but all the other particles are free to move. As we increase the size of the particles, the pile will expand slightly.

Figure 6(a) displays the normal stress simulation results. The small size inhomogeneity gives rise to a large depression in the structure of the normal stress. We attempt to replicate this result analytically via eqns. (14) and (15), where we set $C=0.4mg$, and obtain general, but not exact, agreement between simulation and theory (Fig. 6(a)). The difference between the analytic and simulation results is due to our assumption, in the analytic model, of a contact force, $f(x, y)$, which is constant in magnitude throughout the sandpile. However, the simulation results show that $f(x, y)$ decreases

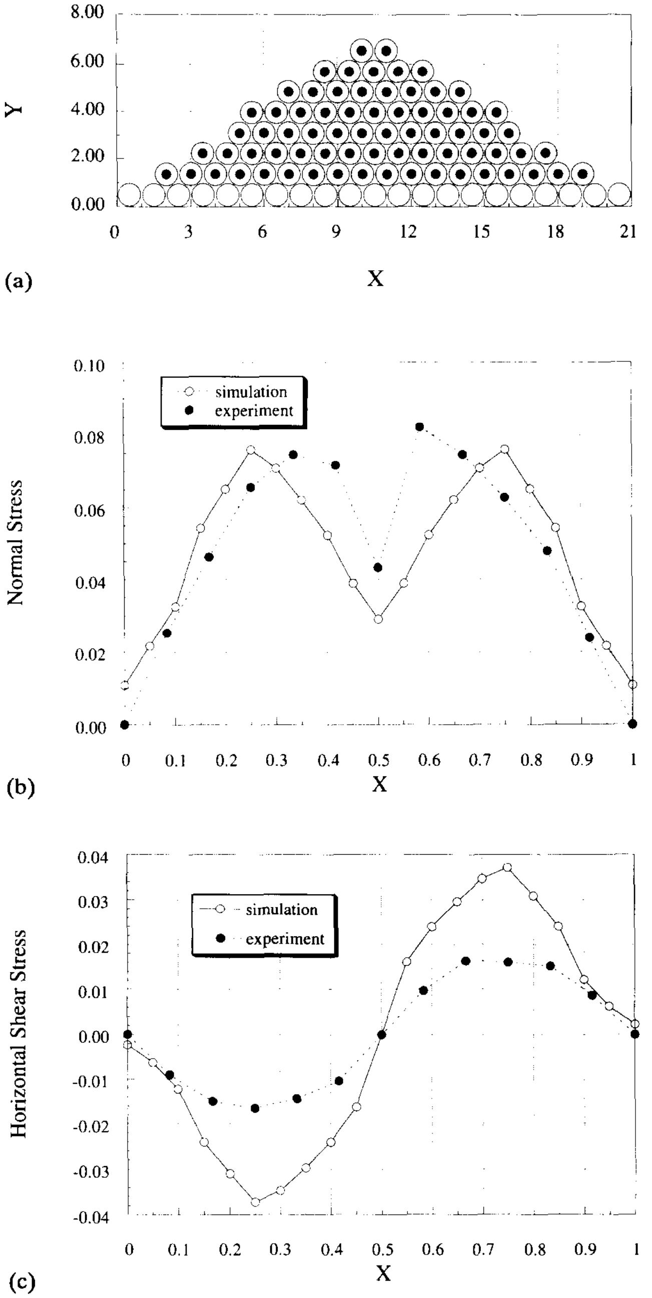


Fig. 8. We construct a simulation sandpile, which has the same slope as the experimental sandpile of ref. [2]. In this sandpile, all the particles, except the particles at the base of the pile, have been given a diameter of 1.0001*d*, where *d* is the diameter of a base particle. These larger particles are denoted by spotted circles in Fig. 8(a). The normal and shear stresses for this inhomogeneous simulation pile are compared to those observed experimentally, and are found to be qualitatively similar, although the simulation sandpile has a much broader and deeper normal stress minimum than does the experimental sandpile (Fig. 8(b)). In Fig. 8(c), the horizontal stress profiles are computed with the same scaling as the normal stress profiles. The horizontal stress for the simulation sandpile is seen to have a maximum value which is approximately three times the horizontal shear stress of the experimental sandpile. The normal stresses have been normalized such that the weights of the simulation and experimental sandpiles are equal to one.

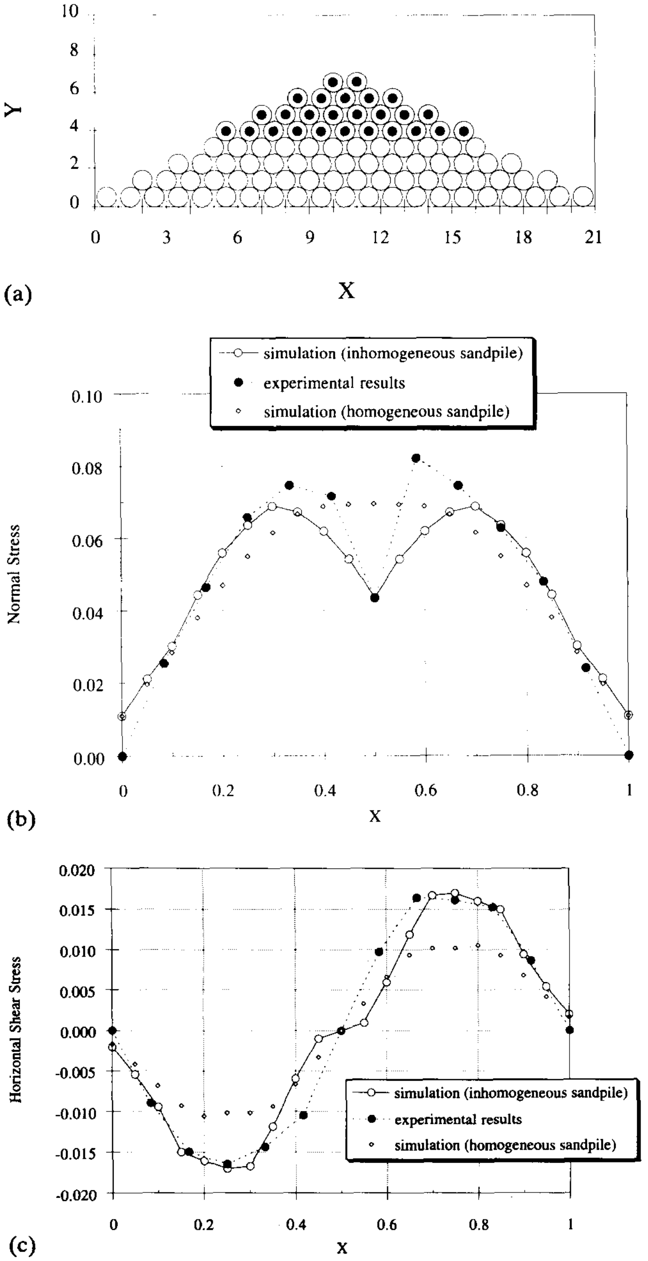


Fig. 9. We construct a simulation sandpile, where the larger particles (denoted by spotted circles in Fig. 9(a)) are near or at the top of a sandpile. The normal and horizontal shear stresses for this inhomogeneous simulation pile are compared to those observed experimentally (Figs. 9(b) and 9(c)), and are found to be similar. The normal stress for the homogeneous sandpile (where all the particles now have the same size) does not display the depression in the normal stress, while the shear stresses for the homogeneous pile are considerably less than the shear stresses obtained from the inhomogeneous or experimental sandpiles. The stresses have been normalized such that the weights of the simulation and experimental sandpiles are equal to one.

in magnitude as one travels from the base to the top of the pile (Fig. 6(b)). Note that our assumption, in the analytic model, of horizontal forces pointing away from the centre of the pile has been verified by the simulation results. The dip in the normal stress is accompanied by an increase in horizontal stress within the sandpile. This can be observed by comparing the horizontal stress on the base particles in the homogeneous and inhomogeneous sandpiles (Fig. 6(c)).

Although our analytic solutions (eqns. (16) and (17)) suggest that the normal stress distribution will have a maximum when $C<0$, we have found no way of simulating this by using only size differences between the particles. Instead, we consider an equilateral 2D sandpile containing 28 particles and therefore 7 particles at its base, where the particles at the base of the pile have a diameter d, and all the other particles are given a diameter of $1.0001d$.

The data for the normal stress at the base of this $k=7$ sandpile is displayed in Fig. 7(a). The normal stress as predicted by eqn. (15) is found to approximately describe the results obtained from the simulation code. The reason for the divergence between the analytic and simulation results can be seen in Fig. 7(b), which shows the direction and magnitude of the forces between the adjacent colevel particles within the pile. In our analytic model, we assumed that such forces would be constant and diverge away from the centreline of the sandpile, and while we obtained the correct force direction, the magnitudes of these forces are not constant as a function of position within the sandpile. The horizontal shear stresses for the different sandpiles are shown in Fig. 7(c), where the simulation inhomogeneous sandpile is seen to have a greater horizontal shear stress than the homogeneous sandpile.

Simulation versus experimental results

To compare simulation with experiment, we take the experimental results for a particular 3D pile of sand [2] which had a height of 60 cm and a 32.6° angle of repose. We then compare these data with those obtained from our simulation code, where our 2D computer sandpile now contains 92 particles in the pile with an angle of repose of approximately 33°. All the particles in our simulation sandpile have a diameter of $1.0001d$, except for the particles at the base of the sandpile (Fig. 8(a)).

To enable a quantitatively correct comparison, we normalize the normal stress such that each pile has unit weight and a unit base length. Once this is done, we find that the dip in the normal stress for the simulation pile is deeper and broader than the experimental observations (Fig. 8(b)). In Fig. 8(c), the horizontal stress profiles are computed with the same scaling as the normal stress profiles. The horizontal stress for the simulation sandpile is seen to have a maximum value which is approximately three times the horizontal shear stress of the experimental sandpile.

A trial and error search was undertaken, to obtain comparable experimental and simulation results. The optimum configuration of particles within the simulation sandpile is shown in Fig. 9(a), where the larger particles are located near or at the top of the sandpile. The resulting normal stress distributions for the simulation pile and experimental piles are shown in Fig. 9(b), where we also show the normal stress obtained from a simulation pile where the particles all have the same size (homogeneous pile). In a homogeneous sandpile there is no depression in the normal stress. In Fig. 9(c), the horizontal stress profiles, obtained without further scaling, for both the experimental and simulation results are shown. Again we obtain reasonable agreement between the experimental data and the simulation results for an inhomogeneous sandpile. The particle size inhomogeneity has produced not only a dip in the normal stress, but also an increase in the horizontal shear stress under the pile.

Conclusions

Our analytic work and computer simulations show that a depression in the normal stress at the base of a sandpile occurs when additional horizontal shear stress exists within the pile. In our computer simulations of two dimensional sandpiles, we can generate this extra shear stress by making the pile of material out of differently sized particles, where the smaller particles are at the base of the pile.

Care should be taken in generalizing these theoretical results to real, three dimensional sandpiles. Nonetheless, the 'dipped' normal stress distribution observed in experimental sandpiles may be, in part, due to an inhomogeneity in grain sizes. We are not suggesting that these sandpiles all have smaller particles at their base, but the sandpiles in refs. [1] and [2] were created by pouring the granular material onto a measuring platform from an overhead bin, and we hypothesize that the agitation of the granular material as it settles into a conical pile causes size differentiation and a subsequent increase in internal shear stress to take place.

It has been shown both experimentally and via computer simulation [10], that a mixture of different sized particles will suffer spatial size separation when agitated. The dip in the normal stress is perhaps symptomatic of this size differentiation and increased internal shear stress.

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References

1 T. Jokati and R. Moriyama, *J. Soc. Powder Technol. Jpn., 60* (1979) 184.
2 J. Šmíd and J. Novosad, *Ind. Chem. Eng. Symp., 63,* D3/V/1-D3/V/12 (1981).
3 D.F. Bagster, *J. Powder Bulk Solids Technol., 2* (1978) 42.
4 D.F. Bagster, *J. Powder Bulk Solids Technol., 6* (1982) 1.
5 G.R. Brooks and D.F. Bagster, *J. Powder Bulk Solids Technol., 8* (1984) 18.
6 D.F. Bagster and R. Kirk, *J. Powder Bulk Solids Technol., 9* (1985) 19.
7 S.F. Edwards and R.B.S. Oakeshott, *Physica D, 38* (1989) 88.
8 E. Li and D.F. Bagster, *Powder Technol., 63* (1990) 277.
9 K. Liffman, D.Y.C. Chan and B.D. Hughes, *Powder Technol., 72* (1992) 255.
10 R. Jullien, P. Meakin and A. Pavlovitch, *Phys. Rev. Lett., 69* (1992) 640.