A unified formulation of wave phenomena in electromagnetics and elasticity

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A direct formulation of electromagnetic scattering based only on scalar Helmholtz equations is given. The solution, expressed as coupled non-singular boundary integral equations for the field components, provides the benefits of the reduction of dimensionality. For perfect conductors, consideration of induced surface current densities, central to standard methods, is not required. This approach has utility in high aspect ratio electromagnetic problems, surface plasmon spectra and dispersion force calculations of complex nano structures as near and far field values are given with equal precision. Extension to dielectric scatterers and elastic wave propagation in solids is immediate.

The foundations in the theory of wave propagation based on Maxwell's electromagnetic equations [1] and Navier's equation for elastic waves [2] were laid in the 19th and early 20th century. The classic analytical solution of the benchmark problem of the scattering of an electromagnetic plane wave by a single sphere uses two Debye potentials [3–6]. However, extending this approach that represents the Debye potentials as series expansions in orthogonal special functions to more complex geometries is not practical. On the other hand, the boundary integral method of solving elliptic partial differential equations developed by Green [7] while attractive because it reduced problems in 3D space to solving 2D problems on surfaces, was before its time and had to await 135 years for the development of suitable computational hardware and software [8, 9].

Today's technological needs that require general solutions of Maxwell's equations, ranging from accurate radar telemetry to controlling electromagnetic shielding in modern microelectronic packages to surface plasmon and dispersion force calculations are supported by the well-developed and mature engineering field of computational electromagnetics [10–12]. The theoretical basis of most computational methods rests on first finding the induced surface current densities on the boundaries. The electromagnetic fields, that are the quantities of physical interest, are then calculated subsequently from the induced surface current densities [13–16].

In this Letter, motivated by the general applicability of the combination of the boundary integral method with the elegant compactness of the Debye potential representation of the solution of electromagnetic wave equations by two scalar functions that satisfy the Helmholtz equation, we have developed a unified formulation of wave propagation that is applicable to both the Maxwell equations of electromagnetic waves and the Navier equation for elastic waves. This method works with the field variables directly so that in contrast to current electromagnetic computational approaches, it does not require the calculation of surface current densities as an intermediate quantity. In both electromagnetic and elastic problems, the solution is given in terms of the solution of scalar Helmholtz equations for components of the actual physical fields - electromagnetic or strain. Furthermore, using a recently developed analytical desingularization method [18] that has also been applied successfully to fluid mechanics problems [19] and to the Laplace equation in potential flow [20], the singular behavior of the Green's function [21] can be removed entirely. Consequently, higher order surface elements can be easily used to represent boundaries more accurately and the boundary integrals can be evaluated using standard quadrature to confer high numerical accuracy with fewer degrees of freedom.

Electromagnetic scattering by perfect conductors – We illustrate the concept of our unified formulation of wave phenomena with the example of electromagnetic scattering by a perfect electrical conductor (PEC). The case of dielectric scatters or elastic waves only require an extension of the same basic concepts.

In the frequency domain with harmonic time dependence $\exp(-i\omega t)$, the propagating electric field $\boldsymbol{E}$ in a source free region is given by the wave equation:

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = \boldsymbol{0}, \qquad \nabla \cdot \boldsymbol{E} = 0 \tag{1}$$

where $k^2 = \omega^2 \epsilon_r \epsilon_0 \mu_r \mu_0$. The condition $\nabla \cdot \boldsymbol{E} = 0$ means that the two independent components of $\boldsymbol{E}$ in eq. (1) [22] are found by specifying the incident field, $\boldsymbol{E}^{inc}$ and imposing the usual boundary conditions on the surface, S of the perfect electrical conductor, namely that the tangential components of $\boldsymbol{E}$ must vanish. Currently, this is achieved by finding the induced surface current density, $\boldsymbol{J}$ on the PEC in terms of the incident field, $\boldsymbol{E}^{inc}$ by solving the surface integral equation [13–16]

$$\frac{4\pi i}{\omega \mu_r \mu_0} \boldsymbol{n}(\boldsymbol{x}_0) \times \boldsymbol{E}^{inc}(\boldsymbol{x}_0) = \boldsymbol{n}(\boldsymbol{x}_0) \times \int_S G(\boldsymbol{x}_0, \boldsymbol{x}) \{ \boldsymbol{J}(\boldsymbol{x}) + \frac{1}{k^2} \nabla (\nabla \cdot \boldsymbol{J}(\boldsymbol{x})) d\boldsymbol{x} \} \tag{2}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the surface outward unit normal vector at

$\boldsymbol{x}$ and $G(\boldsymbol{x}_0, \boldsymbol{x}) = \exp(ik|\boldsymbol{x}_0 - \boldsymbol{x}|)/|\boldsymbol{x}_0 - \boldsymbol{x}|$ is the Green's function that gives rise to an integrable singularity in the surface integral at $\boldsymbol{x} = \boldsymbol{x}_0$. Eq. (2) is the essence of the current state of the art in computational electromagnetic applications for PEC in which the Rao-Wilton-Glisson (RWG) basis functions [23] are used to represent the surface current density, $\boldsymbol{J}$ and enable eq. (2) to be solved. The scattered field is then calculated from $\boldsymbol{J}$.

In spite of the above well established standard of solving Maxwell's equations, this approach is not ideal or optimal. It is highly desirable from physical and pedagogical points of view to be able to calculate the scattered field directly without the intermediate step of having to first calculate the induced surface current density by solving eq. (2) and then use that to calculate the scattered field. More seriously, an inherent limitation of this approach is that the numerical accuracy of field values near the surface is adversely affected by the singular nature of the integral relation between the surface current and the field. Also the use of the RWG basis functions for the surface current density requires the surface of the scatterer to be represented by *planar* triangular elements so that any extension to high order surface elements becomes more complex. Finally for small particles of dimension a, such as in nanoscale problems, the $1/k^2$ term in eq. (2) poses numerical challenges for small ka.

We now develop a formulation of the scattering problem that works directly with the field quantities without the use of surface currents and thereby avoids all the issues canvassed above. First we observe that since $\boldsymbol{E}$ obeys the Helmholtz equation, the condition $\nabla \cdot \boldsymbol{E} = 0$ can be replaced using a vector identity for $(\boldsymbol{x} \cdot \boldsymbol{E})$, where $\boldsymbol{x}$ is the position vector, to recast eq. (1) as (see Electronic Supplement [13])

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = \boldsymbol{0} \tag{3}$$

$$2(\nabla \cdot \boldsymbol{E}) \equiv \nabla^2 (\boldsymbol{x} \cdot \boldsymbol{E}) + k^2 (\boldsymbol{x} \cdot \boldsymbol{E}) = 0 \tag{4}$$

In other words, the constraint $\nabla \cdot \boldsymbol{E} = 0$ can be replaced by a Helmholtz equation for the scalar function $(\boldsymbol{x} \cdot \boldsymbol{E})$ [24]. Thus the standard Maxwell equation for the scattered electric field $\boldsymbol{E}$, eq. (1), is now replaced by 4 scalar Helmholtz equations:

$$\nabla^2 p_i(\boldsymbol{x}) + k^2 p_i(\boldsymbol{x}) = 0, \qquad i = 1..4 \tag{5}$$

where the scalar functions p_i denote one of the 3 components of $\boldsymbol{E}$ or $(\boldsymbol{x} \cdot \boldsymbol{E})$.

The 3 scalar equations that originate from eq. (3) furnish 3 equations between the 6 unknowns, namely: E_α and $\partial E_\alpha / \partial n$, $(\alpha = x, y, z)$. Eq. (4) between $(\boldsymbol{x} \cdot \boldsymbol{E})$ and $\partial(\boldsymbol{x} \cdot \boldsymbol{E})/\partial n$ provides one more relation between E_α and $\partial E_\alpha / \partial n$ since: $\partial(\boldsymbol{x} \cdot \boldsymbol{E})/\partial n = \boldsymbol{n} \cdot \boldsymbol{E} + \boldsymbol{x} \cdot \partial \boldsymbol{E}/\partial n$. The electromagnetic boundary conditions on the continuity of the tangential components of $\boldsymbol{E}$ and the normal component of the displacement field: $\epsilon_r \epsilon_0 \boldsymbol{E}$, provide the remaining equations to determine $\boldsymbol{E}$ and $\partial \boldsymbol{E}/\partial n$ completely.

To formulate the boundary integral equations for the common problem of scattering by a perfect electrical conductor (PEC), it is more convenient to work in terms the tangential, $\boldsymbol{E}_t = (E_{t1}, E_{t2}) = \boldsymbol{n} \times \boldsymbol{E}$, and normal, $E_n = \boldsymbol{n} \cdot \boldsymbol{E}$, components of the electric field at the surface. There are 4 unknowns to be determined, namely: $E_n, \partial E_x/\partial n, \partial E_y/\partial n, \partial E_z/\partial n$ because the tangential components of the electric field must vanish on the surface of a PEC (see Electronic Supplement [13] for details). The number of unknowns to be found is the same as for the classic solution of the scattering problem by a PEC using a pair of Debye potentials in which the 2 unknown functions and their derivatives have to be found [3–6]. However, in the Debye potential approach, the electromagnetic boundary conditions are expressed as combinations of the two potentials and components of their gradients on the surface of the PEC and give rise to equations that are not straightforward to solve.

The boundary integral formulation of the solution of the Helmholtz equation, eq. (5), based on Green's Second Identity [7, 21], provides the following relation between $p_i(\boldsymbol{x})$ and its normal derivative $\partial p_i/\partial n \equiv \boldsymbol{n}(\boldsymbol{x}) \cdot \nabla p_i(\boldsymbol{x})$ at points $\boldsymbol{x}$ and $\boldsymbol{x}_0$ on the boundary, S with outward unit normal $\boldsymbol{n}(\boldsymbol{x})$ and $G \equiv G(\boldsymbol{x}_0, \boldsymbol{x})$ [18]

$$\int_S [p_i(\boldsymbol{x}) - p_i(\boldsymbol{x}_0) g(\boldsymbol{x}) - \frac{\partial p(\boldsymbol{x}_0)}{\partial n} f(\boldsymbol{x})] \frac{\partial G}{\partial n} d\boldsymbol{x} = \int_S G[\frac{\partial p_i(\boldsymbol{x})}{\partial n} - p_i(\boldsymbol{x}_0) \frac{\partial g(\boldsymbol{x})}{\partial n} - \frac{\partial p_i(\boldsymbol{x}_0)}{\partial n} \frac{\partial f(\boldsymbol{x})}{\partial n}] d\boldsymbol{x} \tag{6}$$

where the requirement on $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ is that they satisfy the Helmholtz equation with boundary conditions: $f(\boldsymbol{x}) = 0, \boldsymbol{n} \cdot \nabla f(\boldsymbol{x}) = 1, g(\boldsymbol{x}) = 1, \boldsymbol{n} \cdot \nabla g(\boldsymbol{x}) = 0$ at $\boldsymbol{x} = \boldsymbol{x}_0$ on the surface, S [18]. Thus if p_i (or $\partial p_i/\partial n$) is given then eq. (6) can be solved for $\partial p_i/\partial n$ (or p_i) in a straightforward manner because both integrands are regular and consequently the integrals can be evaluated accurately by quadrature [13, 18].

The advantage of the boundary integral formulation, eq. (6), is the reduction in the dimension of the problem since it is only necessary to solve for unknowns on the boundary where the physical boundary conditions are prescribed. With the removal of all singular behavior and without the need to represent surface current densities, it becomes very easy to use higher order surface elements to evaluate the surface integral. This can provide orders of magnitude of improvement in the numerical results for the same number of degrees of freedom [18]. And once the field quantities are known on the boundary, values in the 3D solution domain, even at locations close to the boundaries can be obtained easily and accurately since the boundary integral equation has been desingularized [13, 18].

Elastic waves – The propagation of elastic waves in an isotropic medium that avoids hypersingular dyadic Green's functions can be formulated for the displacement

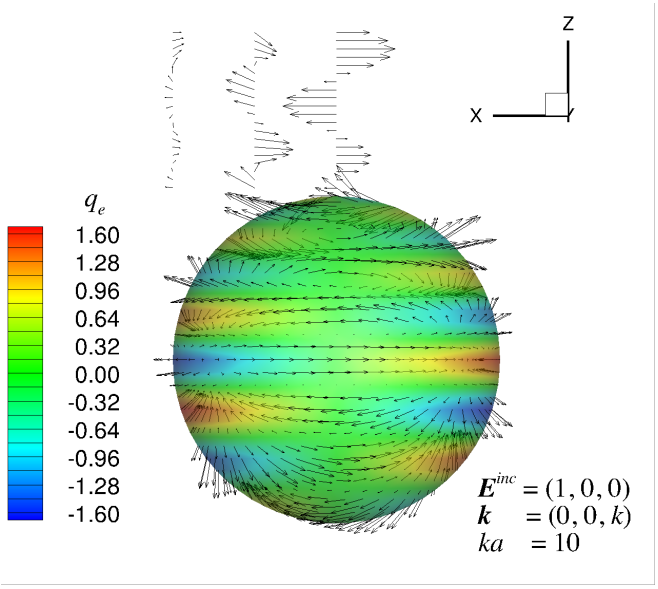

FIG. 1: The scattered electric field (arrows) on and near the surface of a perfect conducting sphere of radius, a, and the induced surface charge density, q_e (color scale), for a unit incident electric field in the x-direction, propagating in the z-direction with $ka = 10$. See supplementary material [13] for the phase variation of these quantities. (Color on-line)

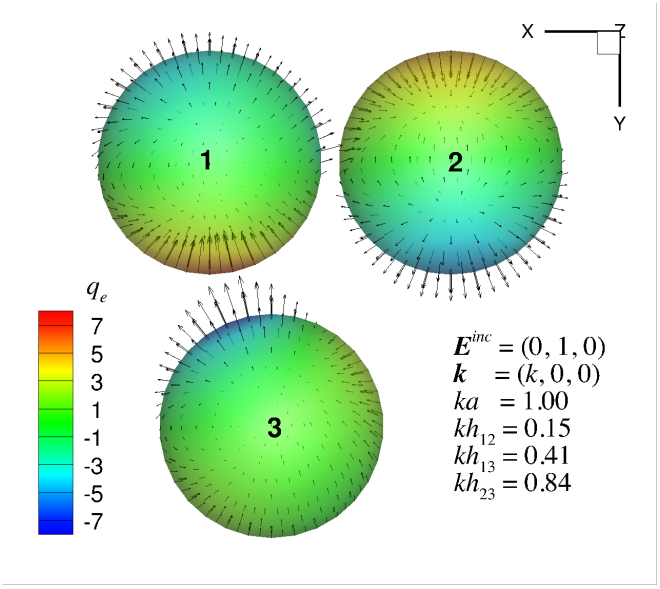

FIG. 2: The scattered electric field (arrows) on the surfaces of 3 identical perfect conducting spheres of radius, a, and the induced surface charge density, q_e (color scale), for a unit incident electric field in the y-direction, propagating in the x-direction with $ka = 1.0$ and h_{ij} is the distance of closest approach between each pair of spheres. See supplementary material [13] for the phase variation of these quantities. (Color on-line)

field, $\boldsymbol{u}$ in the frequency domain Navier equation [2]

$$c_t^2\nabla^2\boldsymbol{u} + \omega^2\boldsymbol{u} + (c_l^2 - c_t^2)\nabla(\nabla\cdot\boldsymbol{u}) = \boldsymbol{0} \qquad (7)$$

with "divergence free", $c_t = [E/(2\rho(1+\sigma))]^{1/2}$ and "irrotational", $c_l = c_t[2(1-\sigma)/(1-2\sigma)]^{1/2}$ wave velocities defined in terms of the Young's modulus, E, Poisson's ratio, σ and material density, ρ. Using the Helmholtz decomposition, the displacement can be written in general as the sum two vector fields: $\boldsymbol{u} = \boldsymbol{u}_l + \boldsymbol{u}_t$ that satisfy

$$\nabla^2\boldsymbol{u}_t + k_t^2\boldsymbol{u}_t = \boldsymbol{0}, \qquad \nabla\cdot\boldsymbol{u}_t = 0 \qquad (8)$$

$$\nabla^2\boldsymbol{u}_l + k_l^2\boldsymbol{u}_l = \boldsymbol{0}, \qquad \nabla\times\boldsymbol{u}_l = \boldsymbol{0} \qquad (9)$$

with $k_l = \omega/c_l$ and $k_t = \omega/c_t$.

The equation for $\boldsymbol{u}_t$ can be treated as eq. (1) for the electric field $\boldsymbol{E}$ and be cast as a problem of 4 scalar Helmholtz equations for the 3 components of $\boldsymbol{u}_t$ and the scalar function $(\boldsymbol{x}\cdot\boldsymbol{u}_t)$.

For the equation with $\boldsymbol{u}_l$, we can use a vector identity to replace the vector condition $\nabla\times\boldsymbol{u}_l = \boldsymbol{0}$ by a Helmholtz equation of the vector function $(\boldsymbol{x}\times\boldsymbol{u}_l)$ constructed from the displacement field because of the identity

$$2(\nabla\times\boldsymbol{u}_l) \equiv \nabla^2(\boldsymbol{x}\times\boldsymbol{u}_l) + k_l^2(\boldsymbol{x}\times\boldsymbol{u}_l) = \boldsymbol{0}. \qquad (10)$$

This formulation of propagating waves in elastic media replaces the scalar condition: $\nabla\cdot\boldsymbol{u}_t = 0$ and the vector condition $\nabla\times\boldsymbol{u}_l = \boldsymbol{0}$ by a scalar or vector Helmholtz equation respectively and works directly with the displacement field without the use of dyadic Green's functions [17].

Numerical examples – Owing to the ubiquity of scattering problem involving perfect electrical conductors (PEC) we will illustrate the utility of our method by considering 3 examples: *i*) the Mie scattering by a PEC sphere of radius a that has an analytical solution [3–5] and we also show that our methods is robust in the limit $ka \to 0$; *ii*) the scattering from 3 PEC spheres in which 2 are very close together to illustrate the stability of our method for problems with very different length scales as a consequence of the absence of singular kernels in our formulation; and *iii*) the scattering by axisymmetric and general 3D PEC objects for which the characteristic dimensions vary by about a factor of 10. Quadratic elements are used in all cases to represent the surfaces. In the supplementary material [13], we show animations of the variation of the electric field and the induced surface charge density over one cycle of the incident field. See [13] for details of numerical implementations.

In Fig. 1, we show the induced surface charge density (that is proportional to the normal total electric field) and the scattered electric field at the surface of a PEC sphere of radius, a at $ka = 10$. These surface quantities are difficult to calculate using the conventional formulation of electromagnetic scattering because of singularities of the Green's function. In contrast, the present formulation removed all such singularities and works directly with the physical fields. In the electronic supplement [13] we also see that the radar cross-section that depends on

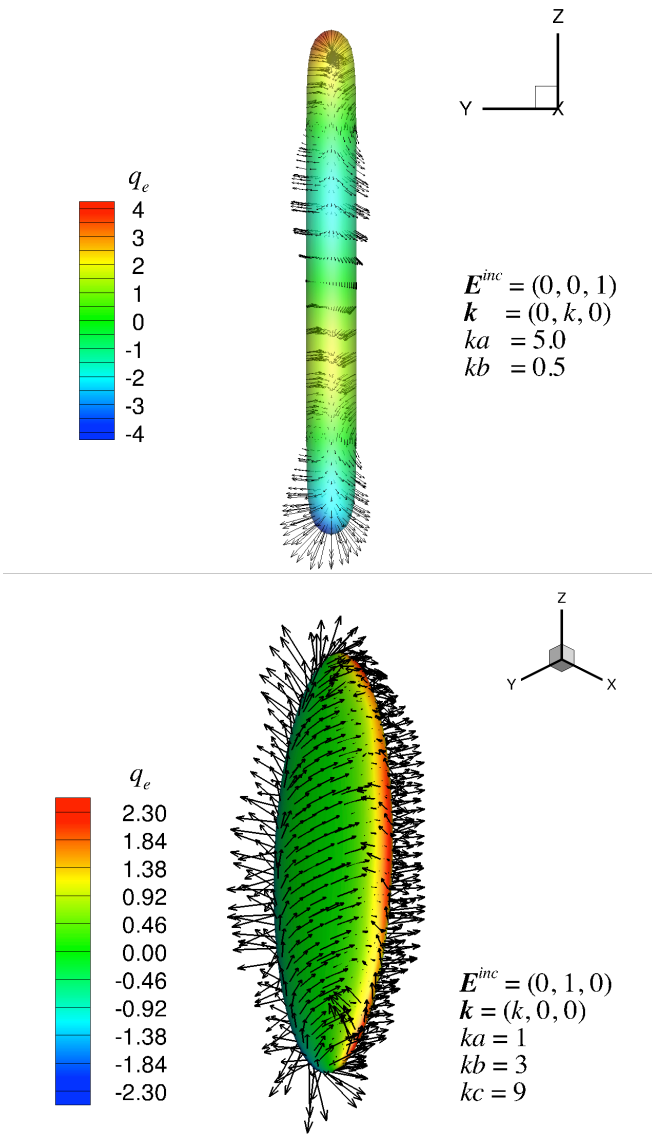

FIG. 3: The scattered electric field (arrows) and the induced surface charge density, q_e (color scale), on the surface of (upper) a long needle [13] with semi-major axes, a, b and unit incident electric field polarized parallel to the z-direction and travelling perpendicular to the long axis in the positive y-direction at $ka = 5$ and $kb = 0.5$, and (lower) an ellipsoid with semi-major axes a, b, c and unit incident electric field polarized in the y-direction and travelling in the x-direction at $ka = 1$, $kb = 3$ and $kc = 9$. See supplementary material [13] for the phase variation of these quantities for the needle and different perspectives of the ellipsoid. (Color on-line)

the far field quantities can be obtained to comparable precision results obtain by direct implementation of the infinite series solution of Mie [25]. Furthermore, in the limit $ka \to 0$ our approach reproduces known analytical results without numerical problems [13].

In Fig. 2. we show the induced surface charge density and the scattered electric field around 3 identical PEC spheres. The absence of singular terms in our formulation means that closely spaced surfaces will not cause degradation of numerical precision.

In Fig. 3. we show the induced surface charge density and the electric field at the surface of a long thin needle [13, 26] PEC conductor with an aspect ratio 10 and a 3D ellipsoid with aspect ratio 1:3:9. Our formulation can readily handle scattering by such high aspect ratio bodies.

Conclusions – We gave a formulation of the propagation of electromagnetic and elastic waves using the boundary integral method that is completely free of singularities that have been one of the inherent difficulties of boundary integral methods [18–20]. For electromagnetic scattering, the integral equations are expressed only in terms of the surface electric field rather than the induced surface current density. Both near and far field values are obtained with comparable precision. As the physical problem contains no singularities, the present exposition is more satisfactory from aesthetic and pedagogic view points. Being able to avoid calculating the surface current density, there are no numerical problems associated with the long wavelength, $k \to 0$, limit. For perfect conductors, the normal component of the electric field is one of quantities that emerges naturally from the boundary integral solution. Generalisation to dielectric scatterers is straightforward with the present approach. One needs to set up boundary integral equations for the tangential: $\boldsymbol{E}_t = (E_{t1}, E_{t2}) = \boldsymbol{n} \times \boldsymbol{E}$, and normal: $E_n = \boldsymbol{n} \cdot \boldsymbol{E}$ components of the electric field as well as their normal derivatives on both sides of the surface, then the solution follows after enforcing the continuity of the tangential components, $\boldsymbol{E}_t = (E_{t1}, E_{t2})$ of the electric field and of the normal displacement field, $\epsilon_r \epsilon_0 E_n$.

Due to the practical importance of electromagnetic scattering, considerable progress has already been made in developing fast numerical algorithms for the traditional formulation of this problem. In contrast, only rather elementary numerical schemes have been used to demonstrate the key physical features of the theoretical framework developed in this Letter. Therefore there is considerable scope for developing efficient algorithms to improve the numerical efficiency of the present new physical approach.

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