An Introduction to Polyhedral Metrics of Non-Positive Curvature on 3-Manifolds

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§0 INTRODUCTION
Polyhedral differential geometry has been an active area of research for a long time. In general relativity it is often called Regge calculus. In the 1960's, work was done by T. Banchoff and D. Stone. More recently, beautiful results have been obtained by J. Cheeger [9], M. Gromov [13] and M. Gromov and W. Thurston [14].

The Geometrization Programme of Thurston (see [33], [34] and the survey article of P. Scott [29]) seeks to classify all closed 3-manifolds by dividing them canonically into pieces, which admit locally symmetric Riemannian metrics called geometries. There are eight geometries $S^3$, $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, Nil, Solv, $\text{PSL}(2,\mathbb{R})$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^3$.

Our aim is to introduce polyhedral metrics which are applicable to the geometries $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^3$. In particular, these metrics have non-positive curvature, in the sense of polyhedral differential geometry. It is easy to show that only the three geometries indicated admit such metrics. However this is sufficient to describe generic 3-manifolds. We quickly review the basic strategy for classifying 3-manifolds.

By Kneser [25] and Milnor [28], there is essentially a canonical way of decomposing any closed 3-manifold into a finite connected sum, i.e. $M = M_1 \# M_2 \# \ldots \# M_k$. Each $M_i$ is prime, i.e. if $M$ is expressed as a connected sum, $M = P \# Q$, then either $P$ or $Q$ is a 3-sphere. It is elementary to show that if $N$ is a prime 3-manifold, then either $N$ is a 2-sphere bundle over $S^1$ or $N$ is irreducible, meaning that every embedded 2-sphere in $N$ bounds a 3-cell. From now on we will always assume that 3-manifolds under consideration are irreducible and orientable, to simplify the discussion.
Suppose $V$ is a closed, embedded surface in $M$ and $V$ is not a 2-sphere or projective plane. Then $V$ is called *incompressible* if the map $\pi_1(V) \to \pi_1(M)$ of fundamental groups induced by the embedding is one-to-one. By Dehn’s lemma and the loop theorem [31], an orientable $V$ is incompressible if and only if whenever $D$ is a 2-disk embedded in $M$ with $D \cap V = \partial D$ (the boundary of $D$), then $\partial D$ is contractible in $V$. We will call a 3-manifold *Haken* if it contains such an incompressible surface.

The *characteristic variety* of a Haken 3-manifold $M$ comes from the work of W. Jaco and P. Shalen [23] and also K. Johannson [24]. A *Seifert fibered* 3-manifold $N$ has a foliation by circles. Each leaf has a foliated neighborhood which is a solid torus $D \times S^1$. The foliation can be described by gluing the ends together of $D \times [0,1]$, where $(z,0)$ is identified with $(\exp(2\pi i q/p)z,1)$. Note that here $p, q$ are relatively prime positive integers and $D$ is viewed as the unit disk in the complex plane. The leaves are the unions of finitely many arcs of the form $\{z\} \times [0,1]$. Suppose that $N$ is a compact orientable Seifert fibered 3-manifold and $\partial N$ is a non-empty collection of tori. If $N$ is embedded in $M$, we say that $N$ is incompressible if every torus of $\partial N$ is incompressible in $M$.

The *characteristic variety Theorem* ([23], [24]) can now be summarized. If $M$ is Haken, then either $M$ is a Seifert fiber space or $M$ has a maximal incompressible Seifert fibered submanifold $N$. $N$ is unique up to isotopy and every map $f: N' \to M$ where $N'$ is Seifert fibered and $f_*: \pi_1(N') \to \pi_1(M)$ is one-to-one, can be homotoped to have image in $N$. $N$ is called the characteristic variety of $M$.

If $M$ is a Seifert fiber space, then we can call $M$ itself the characteristic variety. Thurston’s uniformization theorem [33], [34], [35] then shows that if $M$ is a Haken 3-manifold with empty characteristic variety then $M$ has a metric of constant negative curvature, i.e. is hyperbolic. Also if $M$ is Haken, not Seifert fibered and has a non-empty characteristic variety $N$, then $M - N$ has a complete metric which is hyperbolic. Also $N$ admits a metric of type $H^2 \times R$, or $R^3$, i.e. of non-positive curvature. R. Schoen and P. Shalen (unpublished) have shown that if $N$ is of type $H^2 \times R$ and $M - N$ has finite volume then $M$ admits a Riemannian metric of non-positive curvature.

Finally the Geometrization Programme conjectures that if $M$ is irreducible, has infinite fundamental group and is neither Haken nor Seifert fibered, then $M$ always should admit a hyperbolic metric. So metrics of non-positive curvature should occur on most 3-manifolds which are irreducible and have infinite fundamental group.

Our approach is to describe polyhedral metrics with non-positive curvatures on
several large classes of 3-manifolds. Our constructions include various branched coverings over knots and links, Heegaard splittings, groups generated by reflections (Coxeter groups), surgery on knots and links and singular incompressible surfaces. The last method seems to have special significance. We are able to derive a strong topological rigidity result. Assume a 3-manifold $M$ admits a polyhedral metric of non-positive curvature coming from decomposing $M$ into regular Euclidean cubes. If $M'$ is irreducible and there is a homotopy equivalence between $M$ and $M'$ then $M$ is homeomorphic to $M'$.

Note that Mostow rigidity shows that if $M$ and $M'$ are both hyperbolic and homotopy equivalent then $M$ and $M'$ are isometric. Here we only need to assume one of the manifolds has a special polyhedral metric of non-positive curvature. Notice also that from a purely topological point of view, F. Waldhausen [39] proved that if $M$ is Haken, $M'$ is irreducible and $M$ is homotopy equivalent to $M'$ then $M$ and $M'$ are homeomorphic. In our case we do not suppose $M$ or $M'$ is Haken. Since A. Hatcher [19] has shown most surgeries on links yield non-Haken 3-manifolds, it is most useful to drop the assumption of existence of embedded incompressible surfaces.

To deal with surgery on knots and links, we discuss also polyhedral metrics of non-positive curvature on knot and link complements. One example is given in detail – a two component link in $\mathbb{RP}^3$ (real projective 3-space) which is obtained by identifying the faces of a single regular ideal cube in $\mathbb{H}^3$. We show how this link lifts to a simple four component link in $S^3$. Also by branched coverings we get an infinite collection of examples formed from gluing cubes with some ideal vertices. We sketch the interesting result that nearly all surgeries on all components of such links give 3-manifolds which satisfy the conclusions of the topological rigidity theorem.

In this paper, our aim is to give an overview of this subject. Arguments are only summarized. For more details, the reader is referred to [1], [2], [3]. In the final section, we give some extremely optimistic conjectures. It would certainly be possible to study polyhedral metrics adapted to the other five geometries. Especially in view of the work already done by R. Hamilton [15], [16] and P. Scott [30], this does not appear to be quite as significant as the case of non-positive curvature.

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§1 POLYHEDRAL METRICS OF NON-POSITIVE CURVATURE

We begin by discussing examples of such metrics in dimension two, i.e. on surfaces.
The boundary of the regular Euclidean cube defines a polyhedral metric on the 2-sphere. At each vertex, the three squares give total dihedral angle of $3\pi/2$. If we attribute a positive curvature of $\pi/2$ at each vertex, then the total curvature is $4\pi$, as given by Gauss-Bonnet. Similarly if a flat torus is formed as usual by identifying opposite edges of a regular square, then the dihedral angle at the vertex is $2\pi$ and so the vertex has zero curvature. Finally if an octagon has edges identified by the word $aba^{-1}b^{-1}cde^{-1}d^{-1}$, in cyclic order, the result is a closed orientable surface of genus two. We find it very useful to ascribe dihedral angles of $\pi/2$ at each vertex of the octagon. There is a natural way to do this, so that the metric is Euclidean except at the center of the octagon. Join the midpoint of each edge to the center of the octagon. This divides the polygon into eight quadrilaterals. We view each of the latter as a regular Euclidean square. The result is the dihedral angle at the center is $4\pi$, so there is curvature of $-2\pi$ there. Similarly after identification, the single vertex has dihedral angle $4\pi$ and curvature $-2\pi$ as well. Again the total curvature agrees with Gauss-Bonnet.

Note that a key feature of such metrics is that geodesics diverge at least linearly. If a line segment meets a point $p$ where the dihedral angle is $2\pi + k$, for $k > 0$, then it is easy to see that all possible geodesic extensions form a cone subtending an angle $k$ at $p$. So if two points move out along geodesic rays emanating from a single point, then the points travel away from each other at least at linear speed. In fact, working in the universal cover of the octagon surface described above, we see that such geodesic rays actually diverge exponentially. (See also M. Gromov [13].)

**Definition.** A polyhedral metric of non-positive curvature on a closed orientable surface is a metric which is locally Euclidean except at a finite number of points, where the dihedral angle is greater than $2\pi$.

Suppose a closed orientable $n$-dimensional manifold $M$ is formed by gluing together the codimension one faces of a finite collection of compact Euclidean polyhedra $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$. Assume that the face identifications are achieved by Euclidean isometries. If $Q$ is an $r$-dimensional face of $\Sigma_i$, let $B^{n-r}$ be a small ball of dimension $n-r$ which is orthogonal to $Q$ and is centered at a point $x$ in $\text{int} \ Q$. We call $\partial B^{n-r}$ the *link* of $Q$ and denote it by $\text{lk}(Q)$. Clearly the metric on the $(n-r-1)$-sphere $\text{lk}(Q)$ changes only by a homothety if we vary $x$ and the size of $B^{n-r}$. At any point of $\text{lk}(Q) \cap \text{int} \ \Sigma_i$ or $\text{lk}(Q) \cap \text{int} \ F$, where $F$ is a codimension one face of $\Sigma_i$, it is easy to see that $\text{lk}(Q)$ is locally spherical. So $\text{lk}(Q)$ is locally spherical away from a codimension two complex. We again require geodesic rays which meet $Q$. 
orthogonally to diverge as in the surface case.

**Definition.** $M$ has a polyhedral metric with non-positive curvature if every closed embedded geodesic loop in $\text{lk}(Q)$ has length at least $2\pi$, for every face $Q$, assuming that the metric on $\text{lk}(Q)$ is scaled to have curvature one at the locally spherical points.

**Theorem (Cartan-Hadamard).** If $M$ has a polyhedral metric with non-positive curvature, then the universal cover of $M$ is diffeomorphic to Euclidean space.

In [7], manifolds with such metrics are called Cartan-Hadamard spaces. Note that if a 3-manifold has a non-positively curved polyhedral metric then it is irreducible, as can be seen by lifting an embedded 2-sphere to the universal cover.

To finish this section we give two important methods for presenting 3-manifolds with these metrics. This viewpoint works well in all dimensions but is particularly useful in dimension three.

**I. Cubings with non-positive curvature.** Suppose $M$ is obtained by gluing together faces of a finite collection of regular Euclidean cubes, all with the same edge length. There are two conditions to achieve a polyhedral metric of non-positive curvature on $M$ from such a cubing:

(a) Each edge must belong to at least four cubes. (This is equivalent to the link of the edge has length at least $2\pi$, as each cube contributes $\pi/2$.)

(b) Let $F, F', F''$ be three faces of cubes at a vertex $v$. Assume the faces have edges $e_i, e_i', e_i''$ respectively at $v$, for $i = 1, 2$. Finally suppose all edges are oriented with $v$ as initial point and for (i)–(iii) below, identifications are orientation-preserving. Then we exclude the following three types of gluings of edges.

(i) $e_1$ is identified to $e_2$.

(ii) $e_1$ (resp. $e_2$) is identified to $e_1'$ (resp. $e_2'$).

(iii) $e_1$ (resp. $e_1', e_1''$) is identified to $e_2'$ (resp. $e_2'', e_2$).

**Remarks.** 1) Condition (b)(iii) above does not apply if $F, F', F''$ are all faces of a single cube and $e_1 = e_2', e_1' = e_2''$, $e_1'' = e_2$. 


2) It turns out that an embedded closed geodesic of length less than $2\pi$ can only arise on $\operatorname{lk}(v)$ if one of the gluings (i), (ii), (iii) occurs.

II. Generalized cubings with non-positive curvature. There is an obvious rotation of order three about the diagonal of a cube. The quotient space (orbifold) is homeomorphic to a ball and is locally Euclidean, except along the diagonal, where the dihedral angle is clearly $2\pi/3$. Suppose we now take a cyclic branched cover of this orbifold, with branch set the diagonal and having degree $d \geq 3$. The resulting space is called a flying saucer. It can be viewed as having top and bottom hemispheres, dividing up the 2-sphere boundary into two disks. Each disk consists of $d$ Euclidean squares with a common vertex (the two ends of the diagonal). For example, if $d = 4$ then each hemisphere has the same induced metric as for the octagon at the beginning of this section. If $d = 3$ we get a cube back. Note that all the dihedral angles at the edges of a flying saucer are still $\pi/2$ and all faces are squares.

We can now define a generalized cubing with non-positive curvature exactly as for cubings. The conditions which are needed for the gluing of faces are identical. Here $M$ is constructed by attaching together faces of a finite number of flying saucers, where all edges have the same length. Note again that some of these flying saucers could be cubes.

§2 CONSTRUCTING 3-MANIFOLDS WITH NICE METRICS
Our first result is a general criterion for constructing polyhedral metrics of non-positive curvature by gluing together a finite set of compact 3-dimensional Euclidean polyhedra $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$. Assume that for each $\Sigma_i$, no loop $C$ on $\partial \Sigma_i$ crosses at most three edges at one point each, unless $C$ bounds a disk on $\partial \Sigma_i$ containing a vertex of degree three and $C$ meets the three edges ending at $v$.

**Theorem 1.** Suppose a closed orientable 3-manifold $M$ is obtained by identifying faces of $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$. Assume:

(i) Each face of $\Sigma_i$ has at least four edges.

(ii) Each edge in $M$ belongs to at least four of the $\Sigma_i$.

(iii) Let $v$ be a vertex of $M$ and $F, F', F''$ any three faces at $v$. Suppose the same gluings of edges of these faces, as in I(b) of the previous section, are excluded. Note that of course $F, F', F''$ could be all the faces of some $\Sigma_i$ at $v$. Then $M$ has a polyhedral metric of non-positive curvature.
Remarks. This result is proved by dividing each $\Sigma_i$ into cubes (respectively flying saucers) if every vertex of $\Sigma_i$ has degree three (resp. degree three or greater). We illustrate this with the example of the Weber-Seifert hyperbolic dodecahedral space [40].

Example. Construct a regular hyperbolic dodecahedron with all dihedral angles $2\pi/5$. Identify opposite pairs of faces by rotation of $3\pi/5$. Then all edges have degree five in the resulting manifold $M$ which thus has a hyperbolic metric.

Now the dodecahedron can be decomposed into twenty cubes as follows. Join the center of each face to the midpoints of all edges of the face. This divides each pentagonal face into five quadrilaterals (squares – c.f. the octagon example in §1). Now join the center of each face to the center of the dodecahedron. The neighborhood of every original vertex on the boundary of the dodecahedron is three squares. when these three squares are “coned” to the center of the dodecahedron, the result is a cube. It is easy to check that every edge of these cubes has degree four or five in $M$, checking I(a) in §1. On the other hand, there is a unique vertex in $M$ coming from the vertices of the dodecahedron. This vertex has a link in $M$ which has the structure of an icosahedron (the dual tessellation to the dodecahedral tessellation of $H^3$). It is easy to check that the other vertices have links which are either octahedral or are formed from ten triangles arranged like two pentagons joined along their boundaries. Therefore I(b) follows immediately and $M$ has a polyhedral metric of non-positive curvature.

The next construction of nice metrics can be viewed as dual to Theorem 1.

As is well-known, every closed orientable 3-manifold $M$ can be constructed by identifying the boundaries of two 3-dimensional handlebodies, by an orientation-reversing homeomorphism. The genus is the number of handles and such a decomposition is called a Heegaard splitting of $M$. There is no procedure known for constructing hyperbolic metrics from such splittings.

Before stating the result, we need to define Heegaard diagrams. Assume $M = Y \cup Y'$, where $Y$, $Y'$ are handlebodies of genus $g$ with $\partial Y = \partial Y' = L$, a closed orientable surface. let $D_1, D_2, \ldots, D_m$ (resp. $D_1', \ldots, D_n'$) be a collection of disjoint meridian disks for $Y$ (resp. $Y'$). This means that $D_i$ (resp. $D_i'$) is properly embedded in $Y$ (resp. $Y'$) with $\partial D_i$ (resp. $\partial D_i'$) a non-contractible loop in $\partial Y$ (resp. $\partial Y' = \partial Y = L$). We say that the collection $D_1, D_2, \ldots, D_m$ is full if int $Y - D_1 \ldots D_m$ is a set of open 3-cells. The meridian disks will never be chosen so that $D_i$ and $D_k$ are parallel.
in $Y$, for $i \neq k$. Finally denote $\partial D_i$ by $C_i$, $\partial D'_j$ by $C'_j$ and let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$, $\mathcal{C}' = \{C'_1, C'_2, \ldots, C'_n\}$, $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$, $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_n\}$.

**Definition.** The triple $(L, \mathcal{C}, \mathcal{C}')$ is called a Heegaard diagram for $M$ if both collections of meridian disks $\mathcal{D}$ and $\mathcal{D}'$ are full.

Obviously $M$ can be assembled from the Heegaard diagram by attaching 2-handles to $L$ along $\mathcal{C}$ and $\mathcal{C}'$, then filling in by 3-handles. Without loss of generality, assume $\mathcal{C}$ and $\mathcal{C}'$ are isotoped to be transverse and to have minimal intersection, i.e. no pair of arcs in $\mathcal{C}$ and $\mathcal{C}'$ have common endpoints and bound a 2-gon in $L$.

Suppose $R$ is the closure of a component of $L - \mathcal{C} - \mathcal{C}'$. We call $R$ a region of the Heegaard diagram. A boundary component of $R$ can be viewed as a polygon, with an equal number of arcs coming from $\mathcal{C}$ and $\mathcal{C}'$.

**Theorem 2.** Suppose a Heegaard diagram $(L, \mathcal{C}, \mathcal{C}')$ for $M$ satisfies the following conditions:

(a) Every curve $C_i$ (resp. $C'_j$) meets $\mathcal{C}'$ (resp. $\mathcal{C}$) at least four times. Moreover, every non-contractible loop on $L$ must intersect $\mathcal{C} \cup \mathcal{C}'$ in at least four points.

(b) Every region of the Heegaard diagram is a disk and has at least six boundary edges.

Then $M$ has a polyhedral metric of non-positive curvature. Moreover $M$ has a cubing (resp. generalized cubing) if all regions are hexagons and every curve $C'_j$ meets $\mathcal{C}$ exactly in four points (resp. regions have six or more edges and $C'_j \cap \mathcal{C}$ has at least four points).

**Remarks.** Note that we are not assuming that any region $R$ is an embedded disk. So an edge may occur twice in the boundary of $R$. Also splittings as in the theorem are often reducible, i.e. have trivial handles. So there can be curves $C_i$ and $C'_j$ in the diagram which cross exactly once.

**Examples.** 1) Heegaard diagrams corresponding to cubings come from torsion-free subgroups of finite index in the Coxeter group constructed from the tesselation of the hyperbolic plane by the regular right-angled hyperbolic hexagon. Note that such subgroups yield similar tilings of Riemann surfaces by right-angled hexagons, which is condition (b) in the theorem, in the case of a cubing. For more information, see also [5].
2) Generally, suppose as in Theorem 1 that $M$ is formed by identifying faces of $\Sigma_1, \Sigma_2, \ldots, \Sigma_i$. Let us truncate each $\Sigma_i$ by chopping off a piece at each vertex $v$ by a plane chosen to pass through each edge $e$ at $v$ at $3/4$ of the distance along $e$ away from $v$. Then we obtain a truncated polyhedron $\hat{\Sigma}_i$ as illustrated in Figure 1 for the cube.

![Figure 1](image)

This shows how to obtain the surface $L$ (the hexagonal faces in Figure 1). $Y'$ is the union of the truncated polyhedra $\hat{\Sigma}_i$, and $Y$ is the closure of $M - Y'$. The disks $P'$ are the quadrilateral faces in Figure 1 and the disks $D$ are dual to the edges of the polyhedra $\Sigma_i$. Moreover, to go from the Heegaard diagram to the cubing or generalized cubing, we merely reverse the process of truncation, i.e. expand $\hat{\Sigma}_i$ back to $\Sigma_i$.

Another standard method of constructing 3-manifolds is via branched coverings. W. Thurston [37] has introduced the concept of a universal knot or link $\mathcal{L}$ in $S^3$, which is disjoint union of embedded circles so that every closed orientable 3-manifold is some branched cover of $S^3$ over $\mathcal{L}$. H. Hilden, M. Lozano and J. Montesinos [20], [21] have shown that various knots and links are universal. Among these are the Whitehead link, Borromean rings, Figure 8 knot and the $5_2$ knot.
Theorem 3. Suppose \( \mathcal{L} \) is a link in \( S^3 \) and a 3-manifold \( M \) is constructed which is a branched coverings of \( S^3 \) over \( \mathcal{L} \), with all components of the branch set having degree at least \( d \). Then \( M \) has a polyhedral metric of non-positive curvature if \( \mathcal{L} \) is the Whitehead link or \( 5_2 \) knot and \( d = 4 \), \( \mathcal{L} \) is the Figure 8 knot and \( d = 3 \) or \( \mathcal{L} \) is the Borromean rings and \( d = 2 \). Moreover if \( \mathcal{L} \) is the Whitehead link, \( 5_2 \) knot or Borromean rings (resp. Figure 8 knot) then all such \( M \) have cubings (resp. generalized cubings) as in §1.

Remarks. We sketch the argument for the Whitehead link \( \mathcal{L} \). The hyperbolic dodecahedral space \( M \) has a cyclic group action of order five given by rotation of the dodecahedron about an axis through the centers of opposite faces. This is shown in [40] to exhibit \( M \) as a 5-fold cyclic branched cover of \( S^3 \) over \( \mathcal{L} \). Clearly this action permutes the twenty cubes of \( M \) described previously. So we see that \( S^3 \) can be built from four cubes. Also \( \mathcal{L} \) is contained in the edges of these cubes and each edge in \( \mathcal{L} \) has dihedral angle \( \pi/2 \). All other edges in \( S^3 \) still have degree four or five. We conclude that any branched cover of \( S^3 \) over \( \mathcal{L} \), where all components have degree at least four, yields a cubing with non-positive curvature.

It is also sometimes useful to construct metrics by gluing together hyperbolic polyhedra and also to look at branched coverings for polyhedral hyperbolic metrics.

Suppose \( \mathcal{L} \) is a simple link which is not a 2-bridge link, torus link or Montesinos link. (Here we include knots as examples of links.) Also assume that if an embedded 2-sphere \( S \) meets \( \mathcal{L} \) in four points, then \( S \) bounds a 3-cell \( B \) with \( \mathcal{L} \cap B \) consisting of two unknotted arcs. Then by the orbifold theorem of W. Thurston [36] (see also [22]), the 2-fold branched cover \( M \) of \( S^3 \) over \( \mathcal{L} \) has a hyperbolic metric and the covering transformation is an isometry. Projecting to \( S^3 \), we conclude that \( S^3 \) has a polyhedral hyperbolic metric with singular set \( \mathcal{L} \) and dihedral angle \( \pi \) along \( \mathcal{L} \). Using the smoothing technique of [14], we easily deduce the following result.

Theorem 4. Any branched cover of \( S^3 \) over \( \mathcal{L} \), where all components of the branch set have degree at least two, has a Riemannian metric with strictly negative curvature.

Corollary. Any such a branched cover has universal cover \( \mathbb{R}^3 \), by the Cartan-Hadamard theorem.

An important class of hyperbolic 3-manifolds are surface bundles over the circle with pseudo-Anosov monodromy (see [38]). We describe two closely related constructions of such bundles with polyhedral metrics of non-positive curvature.
The first is given in the article of D. Sullivan [32] on Thurston's work. The 2-fold cyclic branched cover of $S^3$ over the Borromean rings $\mathcal{L}$ is a flat 3-manifold $\hat{M}$. In Figure 2, $S^3$ is formed by folding a cube along six edges bisecting the faces. This gives a polyhedral metric which is flat except along $\mathcal{L}$, where the dihedral angle is $\pi$. The flat 3-manifold $\hat{M}$ has a fibering as a 2-torus bundle over a circle. The fibers are easily seen to be transverse to the pre-image $\hat{\mathcal{L}}$ of $\mathcal{L}$.

![Figure 2](image)

Then this fibering lifts to any branched covering $\hat{M}$ of $M$ over $\hat{\mathcal{L}}$. Such an $\hat{M}$ can be shown to be a hyperbolic bundle [38] and $\hat{M}$ obviously has a cubing satisfying the conditions of §1. This completes the first construction.

For the second method, start with a product $L \times S^1$, where $L$ is a closed orientable surface of genus at least two. As in §1, we put a polyhedral metric of non-positive curvature on $L$ by decomposing it into Euclidean squares, with all vertices having degree at least four. Then $L \times S^1$ has a cubing in an obvious way, by dividing $S^1$ into intervals. Let us form a link $\mathcal{L}$ in $L \times S^1$ as a union of embedded geodesic loops in this metric. Start with a diagonal of a cube and note that at an end which lies over a vertex with degree greater than four, there is a cone of choices of continuing the geodesic. In this way, we can build complicated loops which are braids, in the sense that the loops are transverse to the fibering by copies of $L$. So any branched cover $M$ of $L \times S^1$ over $\mathcal{L}$ is a fiber bundle over $S^1$ with a polyhedral metric of non-positive curvature. Negative curvature occurs along the pre-image $\hat{\mathcal{L}}$ of $\mathcal{L}$ and along the vertical loops projecting to vertices with degree greater than four in $L$.

To finish the discussion, we need to decide when the bundle $M$ is hyperbolic, i.e. has pseudo-Anosov monodromy. If the monodromy is not pseudo-Anosov, then a
finite power takes some non-contractible loop $C$ in the fiber to itself, homotopically. Hence there is a map $f: T^2 \rightarrow M$ so that $f_*: \pi_1(T^2) \rightarrow \pi_1(M)$ is one-to-one. By standard methods, we can homotop $f$ to a map $g$ realizing the smallest area in the homotopy class. Then $g$ is a minimal immersion, so by Gauss-Bonnet $g(T^2)$ is flat, due to the non-positivity of the curvature. In particular $g(T^2)$ cannot cross the graph, described above, along which negative curvature is concentrated. However it is straightforward to choose $L$ so that the complement of the graph is a handlebody which contains no (singular) incompressible tori. So we can arrange that $M$ is of hyperbolic type.

There is a nice connection between hyperbolic Coxeter groups and polyhedral metrics of non-positive curvature. Suppose that $G$ is a group generated by a finite number of hyperbolic reflections and that $G$ is cocompact, i.e. $\mathbb{H}^3/G$ is compact. (We will briefly discuss the finite volume case in §5.) Assume that a fundamental domain for $G$ is a hyperbolic polyhedron $\Sigma$ with all dihedral angles of the form $\pi/m$, for some integer $m \geq 2$. Andreev's theorem ([6], [33]) gives a characterization of such $\Sigma$. There are three conditions to be satisfied:

1. At each vertex of $\Sigma$, the sum of the dihedral angles is greater than $\pi$.

2. If a loop $C$ on $\partial \Sigma$ meets three edges exactly once each and $C$ does not bound a disk with a single vertex belonging to these edges, then the sum of the dihedral angles at the edges is less than $\pi$.

3. If a loop $C$ on $\partial \Sigma$ meets four edges in one point each, then either the sum of the dihedral angles of the edges is less than $2\pi$ or $C$ bounds a disk containing two vertices of degree three joined by an edge and the dihedral angles add to $2\pi$.

The following result is easy to check.

**Lemma.** If three planes of faces of $\Sigma$ meet in a triple point in $\mathbb{H}^3$ or $S^3_\infty$, then the triple point is a vertex of $\Sigma$. If the planes do not intersect then there is a common perpendicular plane which crosses $\Sigma$ in a triangular region.

**Corollary.** If $\Sigma$ has a face of degree three, then either $\Sigma$ is one of nine possible hyperbolic simplices (see [8]) or $\Sigma$ contains a triangular hyperbolic disk which is properly embedded and orthogonal to $\partial \Sigma$.

Assume $\Sigma$ has a disk $D$ as in the corollary. It is clear that any 3-manifold $M$ whose fundamental group is a torsion free subgroup of finite index in $G$ will have
an embedded totally geodesic incompressible surface built from copies of \( D \). So we leave this case aside, since it is difficult to handle and it is not so interesting to construct nice polyhedral metrics, as Waldhausen's theorem applies.

On the other hand, we have:

**Theorem 5.** Suppose \( G \) is a hyperbolic Coxeter group with fundamental domain \( \Sigma \) having dihedral angles of the form \( \pi/m \), for \( m \geq 2 \). Assume that \( \Sigma \) is not a simplex and \( \Sigma \) contains no properly embedded hyperbolic triangular disk perpendicular to \( \partial \Sigma \). Then if \( M \) is a 3-manifold with \( \pi_1(M) \) isomorphic to a torsion free subgroup of finite index in \( G \) then \( M \) has a polyhedral metric of non-positive curvature and a cubing.

**Remarks.** 1) The theorem follows immediately from Theorem 1. Compare also with the example of the Weber-Seifert hyperbolic dodecahedral space.

2) Six of the nine simplices can be dealt with very simply. A fundamental domain for the action of the dihedral group of order six on the cube is a Euclidean simplex \( \Delta \) with dihedral angles \( (2, 2, 4, 2, 3, 4) \), using the notation of [8]. The rotation of order three is about a diagonal and the reflection is in a plane containing this diagonal of the cube. The integers \( m \) in the bracket are dihedral angles of \( \pi/m \) for the six edges of \( \Delta \).

Now it is easy to verify that for the simplices \( T3, T4, T6, T8, T9 \) on the list of [8], take dihedral angles can be increased to give \( \Delta \). In other words, the simplices can be given the metric of \( \Delta \) and negative curvature will be concentrated along edges which are on the cube. This latter observation turns out to be important in the next section. Finally \( T1 \) is constructed by gluing two copies of \( \Delta \) by reflection in the face with dihedral angles \( (2, 2, 4) \).

The simplices \( T2, T5 \) and \( T7 \) are more complex. Note that 120 copies of \( T2 \) give the regular hyperbolic icosahedron with dihedral angle \( 2\pi/3 \) (see [8]). This is the fundamental domain for the fivefold cyclic branched cover \( M \) of \( S^3 \) (see [41]), over the Figure 8 knot. Hence there is an induced polyhedral metric of non-positive curvature on \( M \), which is more difficult to relate back to the simplex \( T2 \). Note also that the Weber Seifert hyperbolic dodecahedral manifold \( M \) has \( \pi_1(M) \) a torsion free subgroup of index 120 in the Coxeter group for \( T4 \) (see [8]). The polyhedral metric we have just described using \( \Delta \) is the same as previously.
§3 SINGULAR INCOMPRESSIBLE SURFACES

Definition. Suppose $V$ is a closed surface. If $f: V \to M$ is continuous and $f_*: \pi_1(V) \to \pi_1(M)$ is one-to-one then we call $f$ or $f(V)$ a (singular) incompressible surface.

Our aim is to analyze the relationship between polyhedral metrics of non-positive curvature arising from cubings or generalized cubings and singular incompressible surfaces. Also some remarks will be made about the characteristic variety in 3-manifolds with (generalized) cubings.

Let $p: \tilde{M} \to M$ be the universal covering. By M. Freedman, J. Hass and P. Scott [12], if an incompressible surface $f$ has least area in its homotopy class, then the preimage $p^{-1}(f(V))$ consists of embedded planes. The following important properties were introduced by P. Scott [30].

Definition. Suppose $f: V \to M$ is a (singular) incompressible surface. $f$ has the 4-plane property if for any least area map $g$ homotopic to $f$ and four planes $P_1$, $P_2$, $P_3$, $P_4$ in $p^{-1}(g(V))$, at least one pair of planes are disjoint. $f$ has the 1-line property if $P_1 \cap P_2$ is either empty or a single line, for all pairs of planes in $p^{-1}(g(V))$.

Remark. These properties depend only on the homotopy class of $f$, i.e. the subgroup $f_*\pi_1(V)$ in $\pi_1(M)$.

Definition. Suppose $f: V \to M$ is a (singular) incompressible surface. $f$ has the triple point property if for any map $g$ homotopic to $f$ so that $p^{-1}(g(V))$ consists of planes, any three such planes meeting pairwise must intersect in at least one triple point.

Remark. This property can also be stated in terms of $f_*\pi_1(V)$. Suppose $H_1$, $H_2$, $H_3$ are subgroups of $\pi_1(M)$ which are conjugate to $f_*\pi_1(V)$ in $\pi_1(M)$ and have non-trivial intersections in pairs. Then $f$ has the triple point property is equivalent to $H_1 \cap H_2 \cap H_3 = \{1\}$.

Theorem 6. Suppose $M$ has a cubing (respectively generalized cubing) giving a polyhedral metric of non-positive curvature. Then $M$ has a (singular) incompressible surface $f: V \to M$ satisfying the 4-plane, 1-line and triple point (resp. 4-plane and triple point) properties. In addition, $f$ can be realized as a totally geodesic surface in the polyhedral metric.
Remarks. 1) In the case of a cubing, to build $f(V)$ start with a square parallel to and midway between opposite faces of a cube. Follow this square to similar squares in adjacent cubes, via the exponential map. The result is eventually a totally geodesic closed orientable surface immersed in $M$, since there are only finitely many squares of this type.

For a generalized cubing, the method is similar except we begin with a square $D$ which is a face of a flying saucer. Choose a preferred normal direction to this square which we call “up” for convenience. Let $e$ be an edge of $D$. Then $e$ belongs to $d$ squares, where $d \geq 4$. If $d = 4$ then there is a unique choice of adjacent square to $D$ at $e$ so as to obtain a totally geodesic surface. If $d > 4$ we proceed as follows. Let $F_1$ be the flying saucer on the “up” side of $D$ and let $D_1$ be the second face of $F_1$ containing $e$. Let $F_2$ be the adjacent flying saucer containing $D_1$ and let $D_2$ be the other face of $F_2$ including $e$. We continue on from $D$ to $D_2$.

Roughly speaking, make a “turn” of $\pi$ on the “up” side at each edge to go from a square to the next. Again this constructs a totally geodesic surface which is immersed but is in general not self transverse as in the case of a cubing. The surface can use a square twice, since the “up” direction may reverse as a loop is traversed in the manifold.

2) We discuss the 4-plane and triple point properties first, in the context of a cubing. Suppose four planes $P_1$, $P_2$, $P_3$, $P_4$ all intersect in pairs. It is easy to see that one of the planes, say $P_4$, can be chosen so that the other three planes meet $P_4$ in lines which cross in a triangle. However the planes meet at right angles (the squares are parallel to cubical faces). So a totally geodesic right angled triangle is obtained which contradicts Gauss-Bonnet, since the curvature is non-positive.

Similarly if $P_1$, $P_2$, $P_3$ are planes so that each pair meets in lines but there is no triple point, then there are infinite triangular prisms. Choose a plane orthogonal to these three planes. This yields a similar contradiction.

For a generalized cubing, it can be verified that if two planes meet, then they intersect in a 2-complex which has a spine which is a tree. We could say that $f$ has the 1-tree property rather than the 1-line property. However by essentially the same arguments as above, the 4-plane and triple point properties follow. Finally, for a cubing, if two planes meet in at least two lines, then choosing a plane orthogonal to both planes gives a right angled
2-gon, a contradiction. Note that, since the planes are totally geodesic, they meet along geodesics which are lines, not loops in the universal cover, as the curvature is non-positive.

**Theorem 7.** Suppose \( M \) has a cubing with non-positive curvature and \( M' \) is irreducible with \( M \) homotopy equivalent to \( M' \). Then \( M \) and \( M' \) are homeomorphic.

**Remarks.** 1) This follows immediately from the result of J. Hass and P. Scott [18]. In fact they only need to assume \( M \) has an incompressible surface satisfying the 4-plane, 1-line properties and must work hard to deal with triangular prisms.

2) The same result should be true for generalized cubings, but dropping the 1-line property causes considerable problems in the technique in [18].

We now turn to the converse of Theorem 7.

**Definition.** An immersed surface \( f: V \to M \) is called filling if \( M - f(V) \) consists of open cells. Note that we do not assume that \( V \) is connected.

**Theorem 8.** Suppose \( M \) is closed, orientable, irreducible and contains a singular incompressible surface \( f: V \to M \) so that \( f \) is filling and satisfies the 4-plane, 1-line and triple point properties. Then \( M \) admits a cubing which induces a polyhedral metric with non-positive curvature.

**Remarks.** The proof proceeds by checking that the closures of the components of \( M - f(V) \) are polyhedra satisfying the conditions of Theorem 1. Note that the union of all the surfaces constructed in Theorem 6 is filling.

The analogous result for generalized cubings should be that a filling incompressible surface satisfying the 4-plane, "1-tree" and triple point properties, suffices. However there are difficulties in moving such a surface to a canonical position as in [18]. To precisely define the 1-tree property, note we need to give a characterization depending only on the homotopy class of \( f \).

To complete this section, we comment on characteristic varieties in a 3-manifold \( M \) with a (generalized) cubing. Let \( T \) be an embedded incompressible torus in \( M \). Then \( T \) can be isotoped to be minimal relative to the polyhedral metric. By Gauss-Bonnet, as the curvature is non-positive it follows that this minimal surface, which we again denote by \( T \), is totally geodesic and flat. This implies the following:
- $T$ is disjoint from the interior of any edge at which negative curvature is concentrated, unless $T$ contains such an edge and locally has dihedral angle $\pi$ at the edge.

- If $T$ passes through a vertex $v$, then $T \cap \text{lk}(v)$ is a geodesic loop of length $2\pi$.

Using these two facts, we can work out where the characteristic variety is located. To be precise, remove the interiors of all edges with negative curvature from $M$. Also delete all vertices $v$ for which there is no geodesic loop in $\text{lk}(v)$ with length $2\pi$. Finally suppose a number of negatively curved edges end at some vertex $v$ and $\text{lk}(v)$ has a finite number of geodesic loops of length $2\pi$, denoted by $C_1, C_2, \ldots, C_k$. (It is easy to show infinitely many such curves is impossible.) There are two possibilities for a pair of negatively curved edges $e_1$ and $e_2$ which meet $\text{lk}(v)$ at $x_1$ and $x_2$ respectively.

If $x_1$ and $x_2$ are separated by some $C_i$, then we "split apart" $e_1$ and $e_2$ at $v$. Conversely, if $x_1$ and $x_2$ are on the same side of every $C_j$, then we join $e_1$ and $e_2$ at $v$. In this way a graph $\Gamma$ is constructed in $M$ by joining or pulling slightly apart the edges of negative curvature at such vertices. The closure of the complement $M - \Gamma$ is the region in which the characteristic variety is located. For example, in many cases it is very easy to observe that the characteristic variety must be empty, since $M - \Gamma$ is an open handlebody or is a product of a surface and an open interval. Note also that the characteristic variety can "touch" $\Gamma$, but not cross it. So we must take the closure of $M - \Gamma$.

§4 STRUCTURE OF THE SPHERE AT INFINITY

We begin by describing the ideal boundary or sphere at infinity, then will discuss some important properties of it. Let $M$ be a closed orientable 3-manifold with a cubing or generalized cubing of non-positive curvature. Let $\widetilde{M}$ denote the universal cover of $M$. Following M. Gromov [13], let $C(\widetilde{M})$ be the continuous functions on $\widetilde{M}$ with the compact open topology. The map $x \to d_x$ defines an embedding of $\widetilde{M}$ in $C(\widetilde{M})$, where $d_x(y) = d(x, y)$ is the distance between points in $\widetilde{M}$. Finally let $C_*(\widetilde{M})$ be the quotient of $C(\widetilde{M})$ by dividing out by the subspace of constant functions. Then the ideal boundary of $\widetilde{M}$ is $\text{bd}(\widetilde{M}) = \text{cl}(\widetilde{M}) - \widetilde{M}$ in $C_*(\widetilde{M})$, where $\text{cl}$ denotes closure. A function $h$ in $C(\widetilde{M})$ which projects to $\bar{h}$ in $\text{bd}(\widetilde{M})$ is called a horofunction centered at $\bar{h}$.

In P. Eberlein, B. O'Neill [11], $\text{bd} \widetilde{M}$ is defined more geometrically as follows. Suppose geodesics $c_1, c_2: \mathbb{R} \to \widetilde{M}$ are parametrized by arc length. Then $c_1, c_2$ are asymptotic if $d(c_1(t), c_2(t))$ is a bounded function of $t$. The equivalence classes of
geodesics using this relation are the points at infinity and are denoted \( \widetilde{M}(\infty) \). It is easy to verify that these points are in one-to-one correspondence with all the geodesic rays \( c(t) \) starting at any fixed point \( x \) in \( \widetilde{M} \). To put a topology on \( \widetilde{M}(\infty) \), we need to build the tangent space \( T(\widetilde{M})_x \) as a topological space (which is homeomorphic to \( \mathbb{R}^3 \)).

Suppose for convenience that \( x \) is in the interior of some cube. Locally, the geodesics starting at \( x \) are straight lines. As these lines are extended, they hit vertices and edges with negative curvature. As described in §1, for surfaces, the geodesic segments which continue a line of this type form a cone. As viewed from \( x \), an edge (resp. a vertex) pulls back to an arc of a great circle (resp. a point) in the unit sphere \( S^2 \) of the usual tangent space at \( x \). We cut \( S^2 \) open along this arc (resp. point) and insert a suitable disk, representing the cone of extensions of lines to the edge (resp. line to the vertex). The size of the cone is bounded by the maximal degree of edges (resp. vertices) in \( \widetilde{M} \).

Since \( \widetilde{M} \) covers \( M \), which is compact, such degrees are bounded. Also the inserted disk must be scaled down by dividing by the distance of the edge (resp. vertex) from \( x \). In this way, \( S^2 \) is modified by adding infinitely many disks with diameters converging to zero. The result of all the insertions is a new 2-sphere \( \widetilde{S}^2 \), which is called the unit sphere of the tangent space of \( \widetilde{M} \) at \( x \).

The topology on \( \widetilde{M}(\infty) \) is induced by the one-to-one correspondence with \( \widetilde{S}^2 \). Note that \( T(\widetilde{M})_x \) can be viewed as the infinite cone on \( \widetilde{S}^2 \). It follows that \( \widetilde{M}(\infty) \) and \( \partial \widetilde{M} \) are homeomorphic, as in [7]. This completes the description of the sphere at infinity for \( M \). Finally, note that a horofunction at \( c(\infty) \), where \( c \) is a geodesic ray from \( x \), can also be described as a Busemann function, \( h_c(y) = \lim_{t \to -\infty} (d(y, c(t)) - t) \).

Now the Geometrization Programme conjectures that if \( M \) is irreducible, has infinite fundamental group and is atoroidal (i.e. has empty characteristic variety) then \( M \) should admit a hyperbolic metric. In [11] it is shown there is a close connection between metrics of strictly negative curvature and visibility manifolds.

**Definition.** If \( M \) has a (generalized) cubing of non-positive curvature, then \( M \) is a visibility manifold if given any two points \( z_1 \) and \( z_2 \) on \( \tilde{M}(\infty) \), there is a geodesic \( c \) in \( \tilde{M} \) with \( z_1 = c(\infty) \) and \( z_2 = c(-\infty) \).

We note the following simple result.

**Proposition.** Suppose \( M \) has a (generalized) cubing of non-positive curvature. Then \( M \) is a visibility manifold if and only if \( M \) is atoroidal.
Proof: Suppose $M$ has a non-trivial characteristic variety. Then $M$ has an immersed incompressible torus. (If $M$ is Seifert fibered, there may not be any embedded such tori.) As previously, using least area maps, we may assume $f: T^2 \to M$ is totally geodesic and flat. If $\tilde{f}: \mathbb{R}^2 \to \tilde{M}$ is a lift of $f$ to $\tilde{M}$, then $\tilde{f}$ is a totally geodesic, flat embedding. It is simple to check that if $c_1, c_2$ are non-parallel geodesics in $\tilde{f}(\mathbb{R}^2)$, then $z_1 = c_1(\infty)$ and $z_2 = c_2(\infty)$ cannot be joined by a geodesic. Consequently $M$ is not a visibility manifold.

Conversely, if $M$ does not have the visibility property, as in [7] or [11] two geodesic rays $c_1$ and $c_2$ can be found so that $c_1(0) = c_2(0) = x$ and the horofunction $h_{c_1}$ is bounded on $c_2$. We can choose $x$ to be an interior point of a cube. Let $v_1$ and $v_2$ be unit vectors in the directions of $c_1$ and $c_2$ respectively at $x$. Also let $C$ denote the union of all geodesic rays from $x$ in the direction of $\lambda v_1 + (1 - \lambda) v_2$, for $0 \leq \lambda \leq 1$. If $C$ meets edges (and vertices) of negative curvature transversely at a sequence of points with distance to $x$ converging to infinity, then it is easy to see that $c_1(t)$ and $c_2(t)$ diverge faster than any linear function in $t$. Consequently $\lim_{t \to \infty} h_{c_1}(c_2(t)) = \infty$, a contradiction.

We conclude that $C_t = \{ y: y \text{ is in } C \text{ and } d(y, x) \geq t \}$ is flat and totally geodesic, for $t$ sufficiently large. Suppose $C_t$ is projected to $M$. Then it can be shown that either the image is compact or by taking limit points, a compact totally geodesic flat surface is obtained. In either case, this shows that the characteristic variety of $M$ is non-empty and the proposition is established.

The final topic for this section is the limit circles of the singular incompressible surfaces described in §3. The first result shows these surfaces are quasi-Fuchsian with regard to a hyperbolic metric. Next, the 4-plane, 1-line, triple point and filling properties are translated to the intersection pattern for the limit circles. We would like to thank Peter Scott for a very helpful conversation which led to this viewpoint. The final theorem gives an interesting converse; the picture of the circles on the 2-sphere at infinity completely determines a 3-manifold with a cubing of non-positive curvature. So this class of 3-manifolds can be derived from appropriate group actions on the 2-sphere, with invariant graphs which are the union of all the limit circles.

Suppose $f: V \to M$ is a surface in a cubing with non-positive curvature, obtained from squares which are parallel to faces and bisect cubes. Then we know that a component $P$ of the pre-image of $f(V)$ in the universal cover $\tilde{M}$ is an embedded plane. The geodesics $c$ lying in $P$ define a limit circle.
**Definition.** \( \text{bd}(P) = P(\infty) = \{ c(\infty) : c \text{ is in } P \} \) is a limit circle of \( f : V \to M \).

Similarly, if \( M \) has a generalized cubing, then \( f : V \to M \) can be chosen as a union of squares which are faces of flying saucers, as discussed in §3. Then the same definition of limit circle applies.

**Lemma.** If \( M \) has a hyperbolic metric and a (generalized) cubing of non-positive curvature and \( f : V \to M \) is a totally geodesic surface in the polyhedral metric as above, then \( f \) is quasi-Fuchsian in the hyperbolic metric.

**Proof:** By a result of W. Thurston [33], either \( f \) is quasi-Fuchsian or \( f \) lifts to \( \bar{f} : \bar{V} \to \bar{M} \), where \( \bar{M} \) is a fiber bundle finitely covering \( M \) and \( \bar{f} \) is an embedding giving a fiber. In the latter case, \( \bar{M} \) has a (generalized) cubing given by lifting the structure in \( M \). Let \( C \) be the union of all the line segments in the squares of \( \bar{f}(\bar{V}) \), which bisect the squares and are parallel to the edges. Then \( C \) is a collection of immersed geodesics.

Cut \( \bar{M} \) along \( \bar{f}(\bar{V}) \) to obtain \( \bar{V} \times [0, 1] \). We can form a surface in \( \bar{V} \times [0, 1] \) by starting with squares intersecting \( \bar{f}(\bar{V}) \) orthogonally along \( C \), then following via the exponential map to construct a properly immersed totally geodesic surface in \( \bar{V} \times [0, 1] \). Then it can be shown that this surface is homeomorphic to \( C \times [0, 1] \) and meets both \( \bar{V} \times \{0\} \) and \( \bar{V} \times \{1\} \) in \( C \). (Suppose some arc in the surface has endpoints in, say, \( \bar{V} \times \{1\} \), but is not homotopic along the surface into \( \bar{V} \times \{1\} \), keeping its ends fixed. Then a right angled 2-gon can be formed by projecting the arc to \( \bar{V} \times \{1\} \) in \( \bar{V} \times [0, 1] \). This is a contradiction, by Gauss Bonnet.) Hence \( \bar{V} \times [0, 1] \) has the product polyhedral metric and the monodromy for \( \bar{M} \) is periodic. This implies that neither \( \bar{M} \) nor \( M \) has a hyperbolic metric.

**Remark.** There is a simple proof that each circle \( \text{bd}(P) \) in \( \bar{M}(\infty) \) satisfies \( \bar{M}(\infty) - \text{bd}(P) \) is a pair of open disks. In fact, the Gauss or normal map, when suitably defined over \( P \), gives a homeomorphism between \( P \) and each of these two regions.

Suppose that \( f : V \to M \) is a totally geodesic surface constructed as above from a cubing with non-positive curvature. Let \( \{ P_i : i \in I \} \) denote the plane components of the pre-image \( p^{-1}(f(V)) \) in \( \bar{M} \).

**Definition.** \( f \) satisfies the 4 circle property, if given limit circles \( P_1(\infty), P_2(\infty), P_3(\infty), P_4(\infty) \), at least one pair has no transverse intersections. \( f \) satisfies the 2 point property if given limit circles \( P_1(\infty) \) and \( P_2(\infty) \), then \( P_1(\infty) \) and \( P_2(\infty) \) have either zero or two transverse intersection points. \( f \) satisfies the triple region property if whenever limit circles \( P_1(\infty), P_2(\infty) \) and \( P_3(\infty) \) meet in pairs with two transverse
crossing points, then their intersection pattern is as in Figure 3(a) and never as in Figure 3(b). The limit circles of \( f \) are called filling if for any \( z \) in \( \overline{M}(\infty) \), \( z = \cap \{D_j; D_j \text{ is a disk bounded by } P_j(\infty) \text{ in } \overline{M}(\infty) \text{ and } z \in \text{int } D_j \} \). Finally the limit circles of \( f \) are said to be discrete if they form a discrete subset of the space of embedded circles in \( \overline{M}(\infty) \).

**Remark.** It is easy to show the limit circles have no transverse triple points, by the filling and 4-plane properties.

![Figure 3](image)

**Theorem 9.** Assume \( M \) has a cubing of non-positive curvature and \( f: V \to M \) is a standard filling totally geodesic surface. Then \( f \) has the 4 circle, 2 point and triple region properties. Moreover, the limit circles of \( f \) are filling and discrete.

**Proof:** The first three properties of \( f \) follow immediately from the corresponding properties of the plane components of \( p^{-1}(f(V)) \) (see Theorem 6). To see that the limit circles are filling, assume first that \( M \) is atoroidal. Then by the proposition above, \( M \) is a visibility manifold. Assume there are points \( z, z' \) in the intersection of all disks \( D_j \) satisfying \( \partial D_j = P_j(\infty) \) and \( z \) is in \( \text{int } D_j \). Let \( c \) be the geodesic in \( \overline{M} \) with \( c(\infty) = z \) and \( c(-\infty) = z' \). It is easy to see that \( c \) cannot meet any component of \( p^{-1}(f(V)) \) transversely, since no circle \( P_j(\infty) \) can separate \( z \) and \( z' \). Consequently the closure of the component of \( \overline{M} - p^{-1}(f(V)) \) containing \( c \) is non-compact. This contradicts the assumption that \( f \) is filling, since then the complementary regions of \( f(V) \) and \( p^{-1}(f(V)) \) are all (compact) cells.

Assume \( M \) has a non-empty characteristic variety \( N \). Then \( p^{-1}(N) \) consists of disjoint copies of the universal cover \( \tilde{N} \) of \( N \) embedded in \( \overline{M} \). Again we suppose that there are distinct points \( z, z' \) in the intersection of all disks bounded by limit circles and containing a fixed point in their interiors. The limit circles corresponding to the tori boundary components of \( N \) divide \( \overline{M}(\infty) \) into two regions, one of which
is a union of copies of $\tilde{N}(\infty)$. (Note we can have $M = N$, so $N$ has no boundary.) If $z$, $z'$ lie outside these copies of $\tilde{N}(\infty)$, the same argument as before applies, as $M - N$ has the visibility property. On the other hand, if $z$, $z'$ are in a copy of $\tilde{N}(\infty)$, we can use the fact that $N$ has an $R^3$ or $H^2 \times R$ geometric structure. (By H. Lawson, S. T. Yau [26], the latter polyhedral metric on $N$ splits as a product metric.) So we can explicitly separate $z$ and $z'$ by a limit circle.

Finally, suppose there is a sequence of limit circles $P_n(\infty)$ converging to a limit circle $P_0(\infty)$. Let $\gamma_n$ be covering transformations in $\pi_1(M)$ such that $P_n = \gamma_n P_0$. $\gamma_n$ is defined modulo the subgroup $f_+ \pi_1(V)$ in $\pi_1(M)$, i.e. lies in a coset of this subgroup. Coset representatives can be chosen so that for some $x$ in $P_0$, $\gamma_n x \to x$ as $n \to \infty$. This contradicts the fact that $\gamma_n$ is a covering transformation, for all $n$.

**Theorem 10.** Suppose a 2-sphere $S^2$ has a family of embedded circles $\{C_i: i \in I\}$ satisfying the 4 circle, 2 point, triple region, filling and discrete properties. Then there is a canonical way of constructing a 3-cell $B^3$ with $\partial B^3 = S^2$ so that each $C_i$ bounds a properly embedded disk $\bar{P}_i$ in $B^3$, the disks intersect minimally and all the complementary regions of $\bigcup \bar{P}_i$ are 3-cells. In particular, if the $C_i$ are circles for a standard, filling totally geodesic surface $f: V \to M$ in a cubed manifold with non-positive curvature, then $\pi_1(M)$ acts as a covering transformation group on $int B^3$. Moreover, $\bigcup \bar{P}_i$ is invariant and there is a homeomorphism from $int B^3/\pi_1(M)$ to $M$ mapping $\bigcup P_i/\pi_1(M)$ to $f(V)$, where $P_i = int \bar{P}_i$.

**Proof:** We sketch the construction of the 3-cell $B^3$ and the disks $\bar{P}_i$. The homeomorphism between $int B^3/\pi_1(M)$ and $M$ follows from [18], as in Theorem 7. The model to have in mind is a collection of totally geodesic planes in hyperbolic 3-space. The strategy is to show $\bigcup \bar{P}_i$ can be realized as a 2-complex so that each face lies in a pair of 2-spheres, one on each side of the face. Capping off these 2-spheres with polyhedral 3-cells (the complementary regions of $\bigcup \bar{P}_i$) gives $B^3$.

(i) A vertex $v$ of a polyhedral 3-cell corresponds to a triple region of three limit circles, as in Figure 3a. We view a disk $D_j$ bounded by a circle $C_j$ in $S^2$ as representing a half-space of $B^3$ bounded by $\bar{P}_j$. Similarly two circles with transverse intersections, $C_i$ and $C_j$, define a line $c$ where $\bar{P}_i$ and $\bar{P}_j$ cross. $c$ has endpoints at the two transverse points of $C_i \cap C_j$. So a triple region corresponds to three such lines meeting at a triple point $v$. We can schematically project onto the 2-sphere $S^2$ "at infinity" as in Figure 4.
(ii) An edge $e$ of a polyhedral 3-cell is an arc of a line $c$ defined by intersecting circles $C_i$ and $C_j$. $e$ ends at vertices $v$ and $v'$ defined by three circles $C_i, C_j, C_k$ and $C_i, C_j, C_m$ respectively. The picture is drawn in Figure 5. Note by the 4-circle property that $C_k$ and $C_m$ have no transverse crossing points.

(iii) A face $f$ of a polyhedral 3-cell is a finite-sided polygon bounded by edges $e_1, e_2, \ldots, e_p$ running around a circle $C_i$. (See Figure 6.) $f$ is uniquely specified by a choice of a single vertex $v$ and a triple region as in Figure 4 containing $v$. Consider the point $z$ in Figure 4. By the filling property, there is a circle $C_m$ with $z$ in $\text{int} \ D_m$, where $D_m$ is a disk bounded by $C_m$ in $S^2$. By the 4 circle property, $C_m$ cannot transversely cross $C_k$ and if $C'_m$ is another such circle, then $C_m$ and $C'_m$ have no transverse intersection points. So these circles are nested and by discreteness, we can choose $C_m$ so
that $D_m$ is maximal. This gives a unique prescription for forming the edge $e$ from $v$ in the direction of the specified line $c$. (See Figure 5.) $e$ runs from $v$ (corresponding to $C_i, C_j, C_k$) to $v'$ (corresponding to $C_i, C_j, C_m$). By this process, we can generate the picture of $f$ as in Figure 6. We need only check that there is a finite chain of circles $C_1, C_2, \ldots, C_p$ produced, so that $f$ is a finite polygon.

Assume, on the contrary, as in Figure 7, that an infinite pattern of circles is constructed. By the 4-circle property, the circles are linearly ordered along $C_i$ as in Figure 7.

Let $C_1, C_2, C_3, \ldots$ denote the circles and let $D_1, D_2, D_3, \ldots$, be the disk bounded by the respective circles, as in Figure 7. Assume that a point $y$ is chosen in $D_j$ so that $y_j \to y$ as $j \to \infty$. By the filling property, there is a circle $C'$ bounding a disk $D'$ with $y$ in int $D'$. Now $C'$ must transversely meet some circle $C_k$. Since $C'$ intersects $C_i$, by the 4 circle property, $C'$ does.
Figure 7

not transversely cross $C_{k-1}$ and $C_{k+1}$. It can be checked that $C_{k+1}$ must be inside $D'$. This contradicts our choice of $C_{k+1}$ as bounding a maximal disk. So $f$ is finite-sided.

(iv) To complete the construction, all the adjoining faces to $f$ are combined to give a finite-sided polygonal 2-sphere. See Figure 8.

We need only check finitely many adjacent faces are obtained which cover $S^2$. As before, suppose faces $f_1, f_2, f_3, \ldots$ are produced containing a sequence $y_1, y_2, y_3, \ldots$, of points with $y_n \to y$ as $n \to \infty$. By the filling property, $y$ is in the interior of a disk $D'$ bounded by $C'$. This gives a contradiction to the maximality of the chosen circles, by the 4-circle property.

To show the result of attaching all the complementary 3-cells to $\cup \tilde{P}_i$ is $B^3$ we employ the argument in [17].

Remarks. We would like to thank B. Maskit for a helpful comment about non-transverse intersections of the limit circles. Also there is an analogous construction of $n$-manifolds which are cubed with non-positive curvature, for $n \geq 4$, using
codimension one spheres in $S^{n-1}$ which are equivariant under a torsion free group action.

§5 COMPLETE POLYHEDRAL METRICS OF FINITE VOLUME AND DEHN SURGERY ON CUSPS

In this section, we describe the polyhedral analogue of complete non-compact hyperbolic 3-manifolds with finite volume. A brief discussion is given of the translation of results in the previous sections to this setting. Finally a detailed description is given of a specific example, a two component link in $\mathbb{RP}^3$ formed by identifying faces of a single cube. This example is double covered by a simple four component link in the 3-sphere and also has a hyperbolic metric given by the regular ideal cube in hyperbolic 3-space. Negatively curved Dehn surgery of M. Gromov is described and applied to this example. In particular, we show the remarkable result that for all but a very small number of surgeries on each component of the link, the result is a closed 3-manifold with a Riemannian metric of strictly negative curvature and the surgered manifold satisfies the topological rigidity result of Theorem 7. In [3], a large class of alternating links in $S^3$ are discussed which have similar properties to this example.
Assume $M$ is a compact orientable 3-manifold with finitely many singularities with
the link type of a torus. This means that there is a finite number of points
$v_1, v_2, v_3, \ldots, v_k$ in $M$, where a neighborhood of $v_i$ is a cone on a torus rather
than an open 3-cell. We say that $M$ has a polyhedral metric with non-positive
curvature if $M$ is formed by gluing together finitely many Euclidean polyhedra
$\Sigma_1, \Sigma_2, \ldots, \Sigma_t$ as in §1, so that each $v_i$ is a vertex and the link of each edge and
vertex, different from all $v_i$, has no closed geodesic loops of length strictly less than
$2\pi$. If $M$ is neither a union of a solid torus and a cone on a torus nor the suspension
of a torus, nor the union of a twisted line bundle over a Klein bottle and a cone on
a torus, then we say $M$ has finite volume (by analogy with the hyperbolic case).
Note that this includes also the possibility that $M$ has a characteristic variety. (For
such manifolds, there are incompressible tori which are embedded and not parallel
to links of the $v_i$.)

We will be interested in the case that $M$ has a (generalized) cubing of non-positive
curvature and finite volume. Denote $M - \{v_1, v_2, \ldots, v_k\}$ by $M_0$, the manifold part
of $M$. Then $M_0$ has universal cover $\mathbb{R}^3$ and is a Cartan-Hadamard space. (We can
rescale near $v_i$ so that the $v_i$ are at "infinity", i.e. the metric on $M_0$ is complete.)
The conditions that the (generalized) cubing must satisfy are as in Theorem 1, where
vertices refer to points in $M_0$. Notice that at a singular vertex $v_i$, in the case of a
cubing the average degree of edges at $v_i$ must be six, so that the Euler characteristic
of $\text{lk}(v_i)$ is zero. For a generalized cubing, $\text{lk}(v_i)$ may contain many-sided polygons,
so the average degree can be much less.

There is a nice counterpart to Theorem 2. Suppose $L$ is a closed orientable surface
of positive genus. A compression body is the result of attaching disjoint 2-handles
to $L \times \{0\}$ in $L \times [0, 1]$ and capping off any boundary 2-spheres by adding 3-cells.
Suppose $M$ has a finite number of toral singularities. It is easy to show that $M$ can
be formed by a Heegaard splitting, i.e. a compression body with boundary consisting
of $k$ tori and a copy of $L$, can be glued to a handlebody along $L$. Then cones on
the tori can be attached to the remaining boundary components, resulting in $M$.
A Heegaard diagram $(L, C, C')$ for $M$ then has the property that $C$ is full (i.e. every
component of $L - C$ is planar) but $L - C'$ has $k$ regions which are punctured tori
and the rest are planar. With this modification, the remaining conditions on the
diagram are as in Theorem 2 to ensure a polyhedral metric of non-positive curvature
on $M$, except that the only non-contractible loops on $L$ which can cross $C \cup C'$ in
three or fewer points, are non-separating curves in $L - C'$. 
An illustration of the technique in Theorem 3 will be given in the following example of the two component link in $\mathbb{R}P^3$. Theorem 5 also has an analogue; consider hyperbolic Coxeter groups where the fundamental domain has some ideal vertices. Again a cubing can be constructed with the toral cone singularities at the ideal vertices and non-positive curvature, so long as the domain is not an ideal simplex and there are no orthogonal totally geodesic triangular disks in the domain. The case of ideal simplices requires a special argument, as for the compact case.

A very important result is that if $M$ has a (generalized) cubing with finite volume and non-positive curvature, then $M$ has a singular incompressible totally geodesic surface $f: V \to M$ missing the singular vertices and satisfying the same properties as in Theorem 6. $f$ is called filling if the closures of the components of $M - f(V)$ are cells and cones on tori. Then, exactly as in Theorem 8, a finite volume $M$ with a cubing of non-positive curvature arises from a filling singular incompressible surface $f: V \to M$ satisfying the 4-plane, 1-line and triple point properties. The characteristic variety of $M$ can be located as in §3.

The results in §4 also apply in the finite volume case. Here $M$ is atoroidal means that any embedded incompressible torus in $M_0$ is parallel to the link of some singular vertex $v_i$. As in Theorems 9 and 10, we can characterize the limit circles of the totally geodesic, filling surface $f: V \to M$ and can reconstruct $M$ from these circles.

Example. Suppose a cube has faces identified in pairs as in Figure 9. The resulting 3-manifold $M$ has two vertices $v_1$ and $v_2$ with toral links and two edges $a$ and $b$, both having degree six. It is immediate that $M_0$ is a complete hyperbolic 3-manifold with finite volume and two cusps, by using the regular hyperbolic cube metric with all dihedral angles $\pi/3$.

$M$ can be identified by drawing a neighborhood of the dual 1-skeleton as a genus three handlebody and attaching a pair of 2-handles dual to $a$ and $b$. This is a Heegaard splitting as discussed above. By a sequence of handle slides, it can be shown that $M$ is the complement of a two component link in $\mathbb{R}P^3$. We draw its double cover in Figure 10, as a four component link in $S^3$. (See [2] for more details.)

We content ourselves here with algebraically describing $\pi_1(M)$ in terms of generators and relations. Also we identify the cusps as subgroups of $\pi_1(M)$ and give the (meridional) surgery yielding $Z_2 = \pi_1(\mathbb{R}P^3)$. The generators $X$, $Y$, $Z$ of $\pi_1(M)$ are the face identifications shown in Figure 9. They can also be viewed as loops built by joining the centers of a pair of matched faces to the center of the cube. So $X$,
$1', Z$ generate the fundamental group of the dual 1-skeleton, which is a bouquet of three circles. Next, the attaching circles of the 2-handles dual to $a$ and $b$ can be
pushed into the dual 1-skeleton to give relations:

\[
XZ^{-1}Z^{-1}XY^{-1}Y^{-1} = 1, \quad \text{for the dual to } a
\]
\[
XZY^{-1}X^{-1}Z^{-1}Y = 1, \quad \text{for the dual to } b
\]

It is immediate that \( H_1(M) = Z \oplus Z \oplus Z_2 \). The peripheral or cusp subgroups are the free abelian subgroups of rank two in \( \pi_1(M) \) given by loops on the links of \( v_1 \) and \( v_2 \). For \( \text{lk}(v_1) \) we compute generators \( \{Z^{-1}X^{-1}, Z^{-1}Y\} \), by choosing a suitable base point. Similarly for \( \text{lk}(v_2) \), a generating set is \( \{Y^2X^{-1}Y, YZX^{-1}Y\} \). Finally if solid tori are attached to \( M_0 \) at \( v_1 \) and \( v_2 \) so that meridian disks have boundaries \( Z^{-1}Y \) and \( Y^2X^{-1}Y \), the result is a closed orientable 3-manifold \( M^* \) with \( \pi_1(M^*) = Z_2 \). By handle slides, \( M^* \) is actually homeomorphic to \( \mathbb{RP}^3 \).

Consider the hyperbolic, totally geodesic surface \( f: V \to M \) with image the three squares bisecting the cube and equidistant from opposite faces. \( V \) has three faces, six edges and two vertices, hence Euler characteristic \( -1 \). We conclude that \( V \) is non-orientable with three cross caps.

To complete the discussion of this example, we briefly indicate the idea of negatively curved Dehn surgery attributed to M. Gromov. Suppose a horospherical neighborhood of one of the cusps of \( M \) is removed. Choose a geodesic loop on the flat horotorus boundary with length strictly larger than \( 2\pi \). Then a suitable negatively curved solid torus can be attached with appropriate smoothing along a collar of the torus boundary, so that the geodesic curve bounds a disk in the solid torus and the resulting metric has strictly negative curvature everywhere. The key point is that by Gauss-Bonnet, the boundary of a negatively curved disk on a horotorus which has constant mean curvature one must be longer than \( 2\pi \). (The boundary loop will have geodesic curvature one in \( M_0 \).) Now since the cusp neighborhood can be chosen disjoint from \( f(V) \) and all sufficiently long geodesics are permissible, we deduce the main result of this section.

**Theorem 11.** Let \( M_* \) be the closed orientable 3-manifold obtained by Dehn surgery on both cusps of \( M \). Then for all but a finite number of choices for each cusp, \( M_* \) has a Riemannian metric of strictly negative curvature. Moreover, if \( M' \) is an irreducible 3-manifold which is homotopy equivalent to such an \( M_* \), then \( M' \) is homeomorphic to \( M_* \).

**Proof:** The idea is that since the metric is negatively curved and has not changed near \( f(V) \), we see that \( f \) is still totally geodesic and hence incompressible. In
addition, the angles of intersection between the plane components of the pre-image of $f(V)$ in the universal cover of $M_s$ are all still $\pi/2$. We conclude that $f$ still satisfies the 4-plane, 1-line and triple point properties. Therefore, the topological rigidity Theorem 7 applies (i.e. the main result of [18]). Note that $f$ is not filling in $M_s$, but this is unnecessary here.

Remarks. 1) We can construct infinitely many other examples from this one as follows. Let $C$ be the vertical line through the center of the cube in Figure 9, i.e. $C$ joins the centers of the pair of faces identified by $X$. Then $C$ is a geodesic loop. Take any branched cover of $M$ over $C$. The result is a 3-manifold with a cubing of non-positive curvature and finite volume again. Note that here we cannot directly use a regular hyperbolic cube metric. However, we can divide such a cube into eight congruent cubes, using the usual three bisecting squares. Put this metric on each cube in the branched covering manifold. This is exactly like a polyhedral metric in that sums of dihedral angles around edges are at least $2\pi$, etc. To find $f(V)$, we must use the flying saucer technique, since the cubes are no longer regular. However, the 1-line property is valid here, since we can also use Euclidean cubes and push $f(V)$ into the middle of the Euclidean cubes! So Theorem 7 applies to all these examples.

2) Using a computer, we have generated all single cube manifolds with toral links and metrics of non-positive curvature (see [2]).

§6 PROBLEMS

1) Two basic pieces have been used to construct polyhedral metrics of non-positive curvature; the cube and the flying saucer. Are there any other suitable Euclidean polyhedra? It is clearly an advantage for a polyhedron to be regular, i.e. have congruent faces, and to have dihedral angles associated with a tessellation of Euclidean space.

2) Describe appropriate theories of polyhedral metrics for the other five geometries of Thurston.

3) Suppose $M$ has a polyhedral metric of non-positive curvature and $M$ is atoroidal. There are three ways one might tackle the problem of constructing a hyperbolic metric on $M$.

a) The Ricci flow of R. Hamilton [15] may deform the polyhedral metric to the hyperbolic one. Note that the polyhedral metric is instanta-
neously smoothed by the Ricci flow to a Riemannian metric. If $M$ is atoroidal, is this metric strictly negatively curved? There is also an interesting “cross-curvature” flow discussed in notes of B. Chow and R. Hamilton. This parabolic flow has a quadratic expression for the time derivative of the metric and only makes sense in dimension three! This appears to be a desirable feature, given the results of M. Gromov and W. Thurston [14] showing that no flows can deform even nearby metrics to hyperbolic ones in higher dimensions.

b) The action of $\pi_1(M)$ on $\tilde{M}$ extends to an action on the 2-sphere at infinity $\tilde{M}(\infty)$. If an invariant conformal structure on $\tilde{M}(\infty)$ could be found, then we would obtain an embedding of $\pi_1(M)$ in $\text{PSL}(2, \mathbb{C})$. So there would be a hyperbolic 3-manifold $M'$ with the same fundamental group as $M$. By the rigidity Theorem 7, at least if $M$ is cubed then we obtain that $M$ and $M'$ are homeomorphic. Consequently $M$ is hyperbolic.

c) D. Long [27] has recently shown that if a closed orientable 3-manifold $M$ is hyperbolic and has an immersed totally geodesic surface $V$, then $M$ has a finite sheeted cover $\tilde{M}$ in which $V$ lifts to an embedding. Suppose an analogous result could be established for cubed $M$ with a polyhedral metric of non-positive curvature. By Thurston’s uniformization theorem [34], $\tilde{M}$ has a hyperbolic metric since it is atoroidal. By a result of M. Culler, P. Shalen [10], $\pi_1(M)$ embeds in $\text{PSL}(2, \mathbb{C})$ as it is a finite extension of $\pi_1(\tilde{M})$. Again as in b), we conclude that $M$ is hyperbolic.

4) Does every hyperbolic 3-manifold admit a (generalized) cubing of non-positive curvature? We do not know any obvious obstruction.

5) Extend the result of J. Hass, P. Scott [18] to cover generalized cubings. So the 1-line property is replaced by the 1-tree property.

6) Does every Haken 3-manifold have a (generalized) cubing? Does every surface bundle over a circle with pseudo Anosov monodromy have such a structure?

7) We have several constructions of extensive classes of 4-manifolds with cubings of non-positive curvature [4]. Such a 4-manifold $M^4$ has an immersed cubed 3-manifold $V^3$ and the map $\pi_1(V) \to \pi_1(M)$ is one-to-one. By Theorem 7
we know that $V$ is essentially determined by $\pi_1(V)$. Also $V$ has a variety of special properties, such as the 5-plane property. Is there a topological rigidity result for such 4-manifolds?

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