SURFACES IN THE FIGURE-8 KNOT COMPLEMENT

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ABSTRACT

We examine various closed normal surfaces immersed in the triangulated figure-8 knot complement. We give some conditions for surfaces to be regular (without branch points) and a sufficient condition for compressibility. We end with a criterion for incompressibility of normal surfaces in cubed 3-manifolds.

Keywords: Normal Surfaces, Immersions

1. Introduction

One of the most remarkable and interesting ways of constructing the complements of various alternating knots and links is to glue two polyhedra together along their faces and eliminate the neighborhoods of all vertices. A prime example of this construction is the figure-8 knot complement described by Thurston [20], which is a compact hyperbolic 3-manifold with one boundary torus. In this paper we use the polyhedral structure of the figure-8 knot complement – as a union of two tetrahedra minus the vertices – to examine various normal surfaces in the manifold.

The role of incompressible surfaces in the theory of 3-manifolds is extremely important in light of well-known and powerful results of Haken, Waldhausen, Thurston, and others. The existence of such surfaces in a prime 3-manifold often determines its geometric structure and its topological type. As we will see later, since the figure-8 knot complement is finitely covered by a cubed 3-manifold in a natural way, the lifts of normal surfaces in the figure-8 knot complement also shed some insight into incompressible surfaces in cubed 3-manifolds.

In [20], Thurston showed that there are no embedded closed incompressible surfaces in the figure-8 knot complement, except for the boundary torus. This led to the question of the existence of immersed closed incompressible surfaces. As we shall see later, following a suggestion of Aitchison and Rubinstein, Thurston discovered one such surface. Also, Skinner [19] later used the cubed finite cover of the figure-8 knot complement to obtain another incompressible surface. Both of these are in fact totally geodesic. Reid [18] has proved that there are infinitely
We now specialize to the manifold $M_8$. Any normal disk in a tetrahedron is one of four triangles and three quadrilaterals.

We label each of the 14 normal disks in $M_8$ (shown in Figure 2) as well as their edges. Note that each $T_i$ has four normal triangles and three normal quadrilaterals. In our notation, the faces are labeled $A, B, C, \ldots, H$, and the vertices (although removed in $M_8$) are labeled $1, 2, \ldots, 8$. These labelings provide a natural way to specify each arc in a face of $T_i$ joining distinct edges; "$a_1$" is the edge of the face $A$ cutting off the vertex 1, and so on. For example, as shown in Figure 3 (a), the normal quadrilateral $Q_1$ in $T_1$ has four edges labeled $a_4, c_1, d_3,$ and $b_3$. The face identification of the tetrahedra induces the identification of these arcs in pairs (Table 1).
1. So far the only known examples of incompressible surfaces are totally geodesic ones. Are there any incompressible but not totally geodesic surfaces? In such cases, what are some combinatorial conditions for incompressibility?

2. Closed incompressible surfaces without accidental parabolics immersed in knot and link complements remain incompressible under most surgeries (cf. [7]). Also, we know that totally geodesic surfaces (either in hyperbolic or polyhedral sense) have no accidental parabolics. Find a combinatorial way of deciding which normal surfaces have no such parabolics.

3. Is there some analogue for immersed closed surfaces of the result of Jaco-Oertel [14] that one can find embedded incompressible surfaces by looking at the vertices of the projective normal solution space? This fails for the Figure-8 knot complement, so what can be said about the location of immersed incompressible surfaces in this space?

4. The concept of normal and nearly normal surfaces in cubed manifolds seems to be a very worthwhile area for further studying the dichotomy between totally geodesic and non-totally geodesic quasi-Fuchsian surfaces.

5. Characterize those normal surfaces which are preserved geometrically under the symmetry $f^*$ of $M_8$.

6. For a closed normal surface $S$, a normal loop corresponding to a non-trivial element of $\pi_1(S)$ lifts to an infinite normal path in $\hat{S}$. This path has a unique development in $\hat{M}_8$. Characterize those paths which close in $\hat{M}_8$, i.e., which finite sequences of quadrilaterals and triangles, corresponding to a closed normal path, can be realized on some closed surface and give closed loops in $\hat{M}_8$?

7. Characterize the combinatorial types of surfaces in $M_8$: given a good decomposition of a surface $S$ into quadrilaterals and triangles, is it realizable as a normal surface in $M_8$?

8. Given a good decomposition of a surface with boundary, can we extend it to a good decomposition of some closed surface realizable in $M_8$?

9. These is an analogous notion of "face homogeneous" surfaces, corresponding to vertex homogeneity of the graph dual to the decomposition into quadrilaterals and triangles. Skelton’s surface is such an example. We are currently carrying out an analogous study of these surfaces.

10. Two closed surfaces which are commensurable will have universal cover with identical combinatorial structure. Is the converse true?

11. Given a closed normal surface, there is a well-defined ratio of triangles to quadrilaterals, which is an invariant of the commensurability class. What additional invariants, if any, characterize the commensurability class? Also, if two surfaces have the same ratio and one is incompressible, is the other also incompressible?

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the surface cover \( \tilde{M} \) of \( M \) for which the component of \( T \) met by \( D \) lifts to an embedding. We can then assume that the lifted copy \( \tilde{T} \) of the component of \( T \) contains a boundary arc (denoted by \( \tilde{b} \)) of a lift \( \tilde{D} \) of \( D \) and the other arc \( \tilde{m} \) is on some lift \( \tilde{S} \) of \( S \). Now by Assumption 4, the lift \( \tilde{S} \) meets the region \( R \) consisting of all polyhedra adjacent to \( \tilde{T} \) in an annulus \( A \). Hence \( A \) intersects \( \tilde{T} \) in a single essential loop by Assumptions 4 and 1.

Now the arc \( \tilde{m} \) lies on \( A \) so there is a homotopy along the annulus between \( \tilde{m} \) and some arc \( \tilde{n} \) in the intersection curve of \( A \) and \( \tilde{T} \). We can view this homotopy as a mapped-in 2-gon \( D'' \). Then the union \( \tilde{D} \cup \tilde{D''} \) is a disk with boundary \( \tilde{I} \cup \tilde{n} \) lying on \( \tilde{T} \). Hence, since \( \tilde{T} \) is incompressible, this loop is contractible on \( T \) so bounds a mapped-in disk \( D' \) which is the required 2-gon.

![Figure 19. Replacing a compressing disk](image)

To complete the proof we replace \( D \) in the original disk by the projection of \( D' \) into \( M \) and push slightly off of \( T \). See Figure 19. It remains to show that this process decreases the number of arcs of intersection of the disk with \( T \). Specifically, we will show that the arc \( l \) is removed from \( D \) without introducing any new arcs on \( D \). Here we must use Assumptions 1 and 2. Since the intersection curves between \( S \) and \( T \) lift to lines meeting in single points in the universal cover \( T^* \) of \( T \), we can examine the lift \( D^* \) of \( D' \) to \( T^* \) readily. In fact \( D^* \) is a 2-gon between an arc \( n^* \) on a line covering a curve of intersection of \( S \) and \( T \) and an arc \( l^* \) which is a lift of \( l \). In the preimage of this immersed 2-gon, the points mapped to lines of intersection of planes over \( T \) will give embedded arcs and loops. Now any line of intersection of the planes of \( T \) which meets \( n^* \) must cross \( l^* \) since there are no 2-gons between these lines and those coming from lifts of \( S \) by Assumptions 1 and 2. Therefore, in the preimage of \( D^* \), the points mapping to lines between planes over \( T \) will give loops and arcs, but all arcs with one end on \( n^* \) must have the other end on \( l^* \).

Finally these lines between the planes over \( T \) divide \( T^* \) into polygons, each of which has at least 4 sides. So it is easy to homotope the mapping of \( D^* \) to decrease the number of intersections between arcs and loops mapping to these lines, if there is a polygon with fewer than 4 sides in the preimage of \( D' \) between them. After these homotopies are performed, all such polygons have 4 or more sides, implying opposite \( c \) of the 6 tetrahedra around \( e \). Then, the segment from \( v \) to each vertex of this hexagon can be labeled either up or down, depending on whether \( S \) intersects the 2-skeleton of \( M_6 \) in the edge above that vertex or in the edge below the vertex. So we can think of “up” and “down” as signs at the vertices of this hexagon. Now, if we have a normal triangle \( T \) whose one vertex is \( v \in e \), the other two vertices of \( T \) must be both up or both down. In a normal quadrilateral \( Q \), however, if one vertex of \( Q \) is \( v \in e \), the two adjacent vertices will have the property that one is up and the other is down. Hence, a quadrilateral reverses the sign while a triangle preserves it. If \( v \) is not a branch point and only if the sign-switching occurs an even number of times. Since \( S \) has no branch points, each vertex must have an even number of quadrilaterals around it.

**Definition 5.** An (abstract) decomposition of a surface \( S \) into triangles and quadrilaterals is said to be a good decomposition if it satisfies the two conditions of Lemma 4.

**Lemma 6.** Let \( M \) be any 3-manifold with a triangulation such that each edge is of degree at least 6. If \( S \) is a normal surface immersed in \( M \), then \( S \) admits an induced polyhedral metric of non-positive curvature from the inherited polygonal structure.

**Proof.** Declare \( S \) as a union of equilateral Euclidean triangles and Euclidean squares. The degree condition of \( M \) implies that each vertex of \( S \) has non-positive curvature. In fact, the curvature at a vertex \( v \) is

\[
\kappa(v) = -\frac{q}{2} - \frac{\pi}{3} = -\frac{\pi}{6} q,
\]

where \( q \) is the number of quadrilaterals at \( v \).

This metric on \( S \) enables us to easily recognize non-trivial elements of \( \pi_1(S) \) when we discuss compressibility. This also helps us determine \( \chi(S) \) as in the following lemma.

**Lemma 7.** Suppose \( S \) is a closed normal surface in \( M_6 \) and has \( Q \) normal quadrilaterals. Then, \( \chi(S) = -Q/3 \). In particular, \( Q \) must be a multiple of \( 3 \), and \( S \) is not \( S^2 \) or \( \mathbb{R}P^2 \). Moreover, \( S \) cannot be a Klein bottle, and the only tori are multiples of the peripheral torus.
basically two ways to explicitly construct more surfaces in \( M_8 \). Firstly, given a particular class, one can use a computer program to examine every possible gluing which gives a regular surface, as in the work of Rannard [17]. Since his lists give a complete collection of all possible surfaces in a given class, the program is quite useful in providing data and many small examples for further study. The difficulty is that the number of gluings increases exponentially fast, and it is virtually impossible to run the program for larger classes.

![Figure 18. Skinner's incompressible surface in \( M_8 \)](image)

The other way is to look at the 5-fold covering space \( \tilde{M}_8 \) of \( M_8 \), which admits a cubing of non-positive curvature. Cubed manifolds have a canonical surface (through the middle of the cubes, parallel to two opposite faces) which is totally geodesic in that cubing and hence is automatically incompressible [5]. This is how Skinner [19] described an incompressible surface (Figure 18). Simply take the canonical surface in \( \tilde{M}_8 \); the gluing of the two cubes shows that there is only one connected component, which passes through each cube 3 times. Project this to \( M_8 \) and call it \( S_g \). Note that \( S_g \) belongs to the class \( 3A + B \) and is non-orientable with \( \chi(S_g) = -2 \). To this point, \( S_0 \) and \( S_3 \) are the only explicit examples of non-trivial incompressible surfaces immersed in \( M_8 \).

6. Criterion for Incompressibility

In this section we give a criterion for a normal surface in a 3-manifold admitting a cubing of non-positive curvature (we will simply call them cubed 3-manifolds) to be incompressible. The idea is to model the case of hyperbolically totally geodesic surfaces. Our method works for various types of cubings: ideal cubings where the vertices are at infinity [2], cubings of 3-manifolds with boundary ([1] and [7]) and cubings of closed manifolds [5].

If a normal surface \( S \) is incompressible and least weight, then it will lift to a collection \( S \) of embedded planes meeting in lines in the universal cover of \( M \) (see [16] and [10] for the definition of "least weight"). We also see a collection of planes \( T \) in the universal cover coming from the canonical surface \( T \) of the cubing [6]. The two families of planes will also meet in lines, i.e. no simple closed curves will occur. If all the planes are hyperbolically totally geodesic or totally geodesic in surfaces. In other words, these classes give a necessary but not sufficient condition for the existence of regular surfaces. For example, neither the class \( D \) nor any of its multiples can give a regular surface because the class forces every vertex to have precisely 3 quadrilaterals. The same is true for \( D' \). Inspection of these classes provides an insight into Thurston's result that the only embedded incompressible surface in \( M_8 \) is the peripheral torus: we cannot use any basic vectors which contain a pair of intersecting quadrilaterals, which leaves only triangles.

Consider now the 6 vectors in light of the symmetry group (generated by \( f \) and \( h \)) of \( M_8 \) mentioned earlier. It is easy to verify that the symmetries \( f \) and \( f^3 \) interchange \( C \) and \( C' \) while \( h \) and \( f^3h \) interchange \( D \) and \( D' \) and \( fh \) and \( f^3h \) interchange both of these pairs. This gives rise to the following, somewhat surprising observation.

**Lemma 8.** There exists a non-trivial symmetry of \( M_8 \) which preserves the algebraic class of every regular surface.

**Proof.** The symmetry \( f^2 = (12)(34)(56)(78) \) clearly preserves all of these 6 basic vectors and thus any linear combination of them.

**Remark.** Geometrically, \( f^2 \) is the rotation by \( \pi \) along the axis \( I \) shown in Figure 3 (b). Note that this symmetry preserves each quadrilateral type in the tetrahedron. Since we have both tetrahedra with this type of symmetry, every quadrilateral coordinate is preserved by \( f^2 \). On the other hand, by Tolxcell's Q-normal surface theory [21], any normal class is determined by the quadrilateral coordinates. So this makes clear why \( f^2 \) acts trivially. Rannard has generalized this example to all punctured torus bundles over \( S^1 \) by finding similar symmetries and triangulations.

This also leads to the fact such bundles and their sibling manifolds (where the monodromy is multiplied by \(-I\)) have the same projective solution space.

**Definition 9.** Let \( v \) be a vertex of an immersed normal surface \( S \). The vertex type of \( v \) is the cyclic ordering of the six normal disk types around \( v \) (labeled as \( Q \) or \( T \)). A type is well-defined up to the action of \( D_6 \) (the dihedral group of order 12).

**Corollary 10.** There are exactly 8 possible vertex types in regular normal surfaces \( S \) in \( M_8 \).

**Proof.** By Lemma 4, the number \( Q \) of quadrilaterals around a given vertex is 0, 2, 4, or 6.

If \( Q = 0 \), the type is \( TTTTTT \).

If \( Q = 2 \), the three types \( QQTTTT, QTQTTT \), and \( QQTQTT \) are possible.

Similarly, if \( Q = 4 \), we have three types: \( QQQQTT, QQQTQT \), and \( QQTQQT \).

Finally, \( Q = 6 \) gives \( QQQQQQ \).

4. Compressibility and Homogeneous Surfaces

We begin by giving a sufficient condition for compressibility. By a row of quadrilaterals, we mean a collection of quadrilaterals \( P_i \) such that there is a path entering each \( P_i \) from one edge and exiting from the opposite edge, passing through all \( P_i \) (see Figure 9).
regular ideal simplex. Note that not all polygonal shapes can be obtained in this way.

For surfaces with a high degree of symmetry, existence can in many cases be argued by beginning with a corresponding highly symmetric combinatorial decomposition of \$$\mathbb{E}^2\$$, such as the tessellation by right-angle hexagons (underlying Escher’s picture *Heaven and Hell* [4]). This tessellation can then be modified to produce other candidates for local structure of normal surfaces. For example, the surfaces \$$S_5, S_6, S_7, S_8\$$ below are all obtained in this way.

We define a surface \$$S\$$ to be (vertex) homogeneous if each vertex of \$$S\$$ has the same vertex type (one of the \$$8\$$ listed earlier). We consider only the minimal surface in each commensurability class, i.e., those surfaces that are not proper covers of other surfaces. In this section, we examine the existence of homogeneous surfaces of given vertex type. Uniqueness does not hold in general. Where we show existence, we proceed to determine compressibility.

1. Type TTTTT

The eight normal triangles in \$$M_8\$$ can be joined together uniquely to form \$$S_1\$$, the incompressible torus which is parallel to \$$\partial M_8\$$. That all \$$8\$$ triangles are required is clear from the generating vectors. \$$A\$$ is the only one without any quadrilaterals.

2. Type QQTQT

There is such a surface, \$$S_2\$$, consisting of \$$32\$$ triangles and \$$12\$$ quadrilaterals: the ratio of these must be \$$8:3\$$ because of the vertex type. This lies in the class \$$2(2A + B)\$$, as shown in Figure 11.

**Lemma 12.** Any homogeneous surface of the type QQTQT in particular, \$$S_2\$$ is compressible.

**Proof.** Let \$$v\$$ be one of the vertices of such a surface \$$S\$$. As \$$v\$$ is of the type QQTQT (Figure 12), \$$v_1\$$ must have another quadrilateral \$$Q_3\$$ adjacent to \$$Q_2\$$, and similarly \$$v_2\$$ must have another quadrilateral \$$Q_4\$$ adjacent to \$$Q_2\$$. This pattern must continue, giving infinite rows of quadrilaterals in the tessellation associated with \$$S\$$. So by Theorem 11, \$$S\$$ is compressible. \$$\square$$

3. Type QTQQT

There are many surfaces of this type, but one particular surface, which we will refer to as \$$S_3\$$, has been known for some time. Thurston discovered the tessellation of \$$\mathbb{H}^2\$$ which gives rise to \$$S_3\$$ and verified that it is totally geodesic (and thus incompressible) in the hyperbolic geometry by using a computer program and symmetry. \$$S_3\$$ consists of \$$16\$$ triangles and \$$6\$$ quadrilaterals, in the class \$$A + 2D\$$. As \$$\chi(S_3) = -2\$$ and it is non-orientable, \$$S_3\$$ is homeomorphic to the connected sum of two Klein bottles. Figure 13 gives the fundamental domain of \$$S_3\$$. It has some interesting rotational symmetries by construction: other totally geodesic normal surfaces constructed in similar ways will be investigated in a subsequent paper.

4. Type QTQQT
5. Type **QQQQT**

Just as in the previous case, no homogeneous surfaces exist of this type. The proof is similar.

**Lemma 14.** There is no homogeneous surface of vertex type **QQQQT** in $M_6$.

6. Type **QQQTQ**

As in the third case, there are many surfaces of this type. One of these surfaces, which we will call $S_6$, is shown in Figure 15, and it belongs to the class $A + 2B$. $S_6$ is the non-orientable surface with $\chi(S_6) = -4$. It has very nice properties in that every quadrilateral has exactly 2 adjacent quadrilaterals, and in that there are at most 2 consecutive quadrilaterals.

**Lemma 15.** $S_5$ is compressible.

**Proof.** We will explicitly describe a compressing disk for $S_6$ in the universal cover $	ilde{M}_6$, which arises from the tessellation of $\mathbb{H}^3$ by hyperbolic ideal tetrahedra. From the point of $\infty$ in the upper-half space model, we may view $\tilde{M}_6$ as shown in Figure 17. Lift one of the triangles, say $T_i$, of $S_6$ as shown in Figure 18 (a) so that it cuts off the vertex at $\infty$. It is surrounded by 3 quadrilaterals situated like Figure 18 (c), so we can draw arcs indicating where $S_6$ intersects the "bottom" faces of these...
Theorem 11. Suppose $\mathcal{S}$ is a normal surface immersed in $M_8$. $\mathcal{S}$ is compressible if there is an infinite row of quadrilaterals in the universal cover $\tilde{\mathcal{S}}$ of $\mathcal{S}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9}
\caption{$t$ through infinite quadrilaterals}
\end{figure}

Proof. Suppose $\mathcal{S}$ is a normal surface in $M_8$ such that $\tilde{\mathcal{S}}$ has an infinite row of quadrilaterals, labeled $P_i$, $i \in \mathbb{Z}$. Let $t$ be an infinite line ($\equiv \mathbb{R}$) passing through the middle of the $P_i$, parallel to two edges of each quadrilateral (Figure 9) and projecting down to $t \subset \mathcal{S}$. Say $P_0$ projects down to $Q \subset \mathcal{T}_1$. Then, $Q$ is disjoint from two edges, say $e_1$ and $e_2$ of $\mathcal{T}_1$, and one of them (say $e_1$) is parallel to $t \cap \mathcal{T}_1$. Take the other edge $e_2$ and examine its link. As shown in Figure 10, $t$ stays "perpendicular" to $e_2$ as it passes through the projections of the $P_i$, and the fact that there is an infinite row of quadrilaterals in $\tilde{\mathcal{S}}$ implies that $t$ returns to its base point in $\mathcal{S}$ after $6n$ quadrilaterals (for some $n \in \mathbb{Z}$). Hence, $t$ is a loop in $M_8$, clearly bounding a disk $D$ whose center meets $e_2$ perpendicularly.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10}
\caption{$t$ bounding a compressing disk}
\end{figure}

In the induced polyhedral metric on $\mathcal{S}$ described in Lemma 6, $t$ is a totally geodesic loop. Therefore, $t$ represents a non-trivial element of $\pi_1(\mathcal{S})$ which is an element of the kernel of the inclusion map $\pi_1(\mathcal{S}) \rightarrow \pi_1(M_8)$ as $t$ bounds $D$ in $\mathcal{S}$. This implies that $\mathcal{S}$ is compressible. \hfill $\Box$

This suggests that, heuristically, compressibility arises if the ratio of quadrilaterals to triangles is large. The converse of Theorem 11 is not true: there exist compressible normal surfaces without having 6 quadrilaterals in a row. The example $S_8$ below is of this type (Lemma 16).

It is natural to ask whether all triangles and quadrilaterals of a given normal surface $\mathcal{S}$ can be given the structure of hyperbolic polygons in such a way as to make $\tilde{\mathcal{S}}$ isometric to the hyperbolic plane $\mathbb{H}^2$. Generally there will be an associated deformation space (Teichmüller space) of such geometric structures. $\mathcal{S}$ will be totally geodesic in $M_8$ if and only if all shapes of triangles and quadrilaterals can be simultaneously realized as the intersection of a hyperbolic plane in $\mathbb{H}^2$ with a homogeneous surface of this type, $S_7$, is in the class $C$, and the tessellation consists of infinitely many lines of consecutive quadrilaterals and disjoint "trees" of triangles where each triangle is connected to three other triangles at the vertices. One can easily produce a fundamental domain of this surface, which is completely determined by the vertex type. The edge identification shows that this is an orientable surface. As $\chi(S_7) = -\frac{1}{2}(6) = -2$, $S_7$ is homeomorphic to the genus-2 surface. It is easy to check, exactly as in the proof of Lemma 12, that this pattern always gives infinite rows of quadrilaterals in the universal cover of the surface. Hence, we obtain the following.

Lemma 16. Any homogeneous surface of this type is compressible.

8. Type QQQQQ

This last homogeneous surface, $S_8$, partially motivated Theorem 11. All six of the normal quadrilaterals in $M_8$ can be uniquely joined together to create the non-orientable surface with $\chi(S_8) = -2$. It is clearly in the class $B$, and by Theorem 11, it is compressible.

5. Other Immersed Surfaces

While it is natural to examine these 8 homogeneous cases, it is also evident that most normal surfaces immersed in $M_8$ are not homogeneous. There have been
Proof. Using the metric on $S$ described in Lemma 6, we see that each "corner" of a quadrilateral contributes curvature $-\frac{\pi}{6}$ at a vertex. There are $4Q$ such corners in $S$, so the total curvature of $S$ is

$$\kappa(S) = 4Q(-\frac{\pi}{6}) = -\frac{1}{3}(2\pi)Q.$$ 

But then, the Gauss-Bonnet Theorem gives us

$$\chi(S) = \frac{1}{2\pi}\kappa(S) = -\frac{1}{3}Q.$$ 

Note that $\chi(S) = 0$ if and only if $Q = 0$. In this case, $S$ is a union of triangles only; in other words, it is a multiple of the peripheral torus. This is because a union of triangles can easily be seen to be transverse to the $I$-factor, there is a projection of the surface onto the embedded peripheral torus, which is thus a submersion and hence a covering map. Thus no Klein bottles can occur.

Remark. Suppose we let $T$ be the number of normal triangles in $S$, and let $V, E, F$ be the numbers of vertices, edges, and faces of $S$, respectively. Then, $F = Q + T$, and $E = \frac{3T + 4Q}{2}$. Since $\chi(S) = V - E + F$, we have

$$-\frac{1}{2}Q = V = \frac{3T + 4Q}{2} + (Q + T),$$

which gives us $V = (3T + 4Q)/6$.

To obtain a closed normal surface $S$ immersed in $M_8$, there is a set of equations that must be satisfied. These equations arise from matching the numbers of arc types coming from the normal disks so that a gluing is possible. In particular, we can define an (algebraic) class of unions of normal disks to be the 14-tuple $(a_1, \ldots, a_{14})$, indicating the numbers of the disks $T_1, T_2, T_3, T_4, Q_1, Q_2, Q_3, Q_4, \ldots, Q_6$ in that order (we omit commas between the digits). Richard Rannard [17] has worked out the solution space using linear algebra and reduced the possible classes to some "basic vectors":

$$A = (1111 000 1111 000)$$
$$B = (0000 111 0000 111)$$
$$C = (0011 102 0011 102)$$
$$C' = (1100 120 1100 120)$$
$$D = (1111 000 0000 111)$$
$$D' = (0000 111 1111 000)$$

Since we cannot use negative numbers, these vectors are not linearly independent in the usual sense, nor is the decomposition of a class into basic vectors always unique: In [17], Rannard examines various classes and considers possible gluings by computer: some linear combinations of these vectors give a class with no regular

The polyhedral metric coming from the cubing, then the lines are all geodesics and so have the 1-point property [13], i.e., intersecting lines meet in exactly one point. Moreover in the surface cover of a component of the canonical surface, all the lifts of $S$ meeting the compact lift of the canonical surface are annuli. We make a local version of this our criterion for incompressibility.

Theorem 17. Suppose that an immersed normal surface $S$ in a cubed 3-manifold $M$ satisfies the following conditions:

1. in the universal cover $\tilde{M}$ of $M$, the preimage of $S$ meets every plane lying over the canonical surface $T$ in a collection of embedded lines.
2. each of the above lines meets every line coming from intersections of two planes of $T$ in at most one point.
3. the intersection of $S$ with any complementary cell $\tilde{C}$ of $T$ is a collection of disks (each such a cell is just the link of a vertex in the cubing).
4. in the cover corresponding to a component of $T$, any lift of $S$ which intersects the compact lift $\tilde{T}$ of this component of $T$ meets the region $R$ consisting of all polyhedra adjacent to $\tilde{T}$ in annuli crossing $\tilde{T}$ in single curves.

Then the surface $S$ is incompressible. Moreover, the preimage of $S$ in the universal cover consists of embedded planes meeting in lines, i.e., $S$ behaves like a least weight normal surface in its homotopy class. Finally, $S$ and $T$ have the mutual boundary property: intersecting planes over $S$ and $T$ meet in a single line. Thus the common stabilizers of intersecting planes of $S$ and $T$ are cyclic.

Remark. The hypotheses of this theorem appear rather restrictive; however, Assumptions 1 and 3 are true for any incompressible least area surface in the geometry induced by the cubing, and Assumption 4 also holds for all such surfaces with the additional homotopic/algebraic condition (up to conjugacy) that $\text{Stab}(\pi_1(S)) \cap \text{Stab}(\pi_1(T)) = \mathbb{Z}$ or 1 for every component $T_i$ of $T$. Assumption 2 is the crucial geometric condition that gives a constraint on $S$, forcing it to be "sufficiently flat" like a totally geodesic surface. This concept will be pursued in another paper.

Proof. Suppose that $S$ is compressible. The main idea is to homotopically "squeeze" a compressing disk for $S$ (which is not embedded in general) until the entire disk is disjoint from the canonical surface $T$, implying that it resides entirely in a complementarly element. This contradicts the third assumption as the surface $S$ itself remains unchanged throughout the process.

Consider the intersection of a compressing disk for $S$ with the canonical surface $T$ for the cubing. In the universal cover we see that the preimage of a lift of this disk gives a collection of embedded arcs and loops where the disk crosses the planes lying over $T$. Exactly as in [12] we modify the disk to eliminate any loops without increasing the number of planes met by the disk. Next choose an outermost arc in the disk. This defines a 2-gon subdisk $D$ between $S$ and $T$. Our aim is to decrease the number of such arcs in the disk.

The first step is to use Assumption 4 to show that there is a 2-gon subdisk $D'$ on $T$ with one boundary arc $\alpha$ from $D$ and the other arc $\beta$ lying on $S$. Consider
types of normal disks in one of the cubes – altogether, there are 8 normal surface types, giving 63 such surfaces in a cube. Additionally there are a few types of "nearly normal surfaces" one can get when lifting a normal surface from $M_8$ (see Figure 7): there are 3 types of such surfaces in a cube. We will later use this idea of normal surfaces in cubed manifolds (Section 6). Observe that totally geodesic surfaces always give normal disks in $\widehat{M}_8$.

![Figure 6. Normal disk types in the cube](image)

![Figure 7. Nearly normal disk types in the cube](image)

3. Regular Surfaces in $M_8$

We now begin with a few basic lemmas concerning immersed surfaces in $M_8$. By the term regular surface, we mean a (normal) surface without branch points. Each normal triangle in a tetrahedron $T$ cuts off a vertex of $T$ and hence is determined uniquely by any one of its edges. Similarly, each normal quadrilateral is also determined by any one of its edges.

Lemma 4. Suppose $S$ is a closed normal surface immersed in $M_8$. Then,
1. each vertex of $S$ is of degree 6, and
2. each vertex has an even number (possibly 0) of quadrilaterals around it.

Proof. The first is clear from the fact that the degree of each edge of $M_8$ is 6. The second statement is obvious once we look at the link of an edge of $M_8$, where 6 tetrahedral edges meet (Figure 8). Let $v$ be a vertex of $S$, so $v$ is an interior point of some edge $e$ of $M_8$. Fix an orientation of $e$ vertically so that we can refer to “up” and “down” with respect to the (horizontal) hexagon obtained by the edges that all the loops are eliminated and all arcs run from $i^*$ to $n^*$. In fact, these arcs do not even cross each other; they run parallel to one another from $i^*$ to $n^*$ (see [8] for details of this procedure). Hence, the replacement of $D$ by the projection of $D'$ slightly off $T$ decreases the number of arcs of intersection of the disk with $T$ as required.

Eventually the compression disk becomes disjoint from $T$, wholly contained in a complementary cell to the canonical surface $T$, contradicting Assumption 3. This completes the proof of Theorem 17.

Remark. We can check the properties of $S$ on a computer as follows. First of all, it is easy to see if the components of $S$ in the polyhedral cells (the complement of $T$) and in the region $R$ are disks and annuli, respectively, by computing Euler characteristics. Finding all the curves of intersection of $S$ and $T$ is also fine, and then one has the problem of checking whether there are any extra double points between the curves. This can be done easily by deforming the curves to geodesics in the squaring structure and seeing if the number of double points goes down.

Example 18. If we examine Thurston’s surface $S_8$, then we can easily show it satisfies the criterion in the above theorem and so is incompressible. So this gives an argument depending only on the combinatorial structure of $S_8$ as a normal surface and does not depend on hyperbolic geometry. All other surfaces whose existence has been shown by Reid can be similarly treated if they could be described as normal surfaces. This is because these surfaces, as well as all components of the canonical surface $T$, are totally geodesic in the hyperbolic geometry; hence, the picture in the universal covering space clearly satisfy all four assumptions.

Example 19. In [1] (cf. also [7]), a general construction of a cubing of a non-positive curvature is given on any simple alternating link complement. This cubing is for the compact link complement, and the boundary tori become faces in this cubing. Hence, the canonical surface is a properly immersed (but not closed) incompressible surface. The criterion of Theorem 17 applies well in this case and so gives a method of looking for immersed closed incompressible surfaces in all such link complements. This will be explored in another paper.

The following simple example of a compressible normal surface (in $\widehat{M}_8$) does not satisfy these conditions.

Example 20. Avoiding nearly normal surfaces in $\widehat{M}_8$ is not sufficient for incompressibility. In other words, there is a compressible normal surface in $M_8$ such that its lift into $\widehat{M}_8$ consists entirely of normal surfaces. Consider the “flat” hexagonal normal surfaces in a cube having an edge on each of the 6 faces of the cube. There are 4 of these in each cube, so we have 8 hexagons in $\widehat{M}_8$. There is a unique way of joining them together along their edges, resulting in a closed surface. It turns out to be the orientable genus-5 surface and, when lifted to $M_8$, has infinite rows of consecutive quadrilaterals. Hence, by Theorem 11, this surface is compressible.
This triangulation of $M_8$ is geometrically realizable via the tessellation of $E^3$ by ideal regular tetrahedra. One crucial combinatorial property of $M_8$ is that each edge is of degree 6.

It is known that $M_8$ has symmetry group isomorphic to the dihedral group $D_4 = \langle f, h \mid f^4 = h^2 = 1, fhf^{-1} = f^{-1} \rangle$.

In the combinatorial structure of $M_8$, $h$ corresponds to the action interchanging $T_1$ and $T_2$, and $f$ corresponds to the reflection in $Q_1$ (and in $Q_2$) followed by the $\frac{\pi}{2}$-rotation along the axis $l$ shown in Figure 3 (b). Note that every element in this group can be represented by a permutation of the 8 vertices; in cycle notation, the 2 generators of this group are

\[ h = (15)(26)(37)(48), \quad \text{and} \quad f = (1324)(5867). \]

The 5 involutions of $M_8$ correspond to the 5 elements of order 2 ($h, f^2, fh, f^2h,$ and $f^3h$).

**Theorem 3.** There is a five-fold covering space of $M_8$ which consists of two cubes identified along their boundary faces.

**Proof.** Let $C_1$ and $C_2$ be two Euclidean cubes and partition each $C_i$ into five tetrahedra, as shown in Figure 4. Let the middle tetrahedron of $C_i$ be a copy of $T_i$ ($i = 1, 2$), and fix their positions therein. Now, each of the four surrounding tetrahedra in $C_i$ meets the middle tetrahedron in exactly one face and the middle tetrahedron has a fixed labeling of its vertices; hence, there is a unique way to glue these four tetrahedra to the middle one to obtain local structure compatible with $M_8$. Note that $C_1$ has one copy of $T_1$ and four copies of $T_2$, and $C_2$ similarly. Each face of the cubes is divided into two right triangles. $\partial C_1$ thus has $2 \times 6 = 12$ triangles, and the faces $E, F, G,$ and $H$ each occur three times. It is not hard to verify that these labels again give a unique way to identify each face of $\partial C_1$ preserving the local structure of $M_8$. Each pair of squares in $\partial C_1$ is identified by an orientation-reversing isometry, the reflection along a diagonal. Now, remove the small neighborhood of each vertex of the cubes, and refer to this compact space as $\widetilde{M}_8$.

Note that the diagonals drawn on $\partial C_1$ to embed tetrahedra are pre-images of the edges of $M_8$, and after gluing the faces of the cubes, edges have degree 6. Again, it is easy to check that the $2 \times 12 = 24$ edges are classified into four equivalence classes by the face identification (Figure 4), and each class contains six edges. Hence, there is no branch along the edges, and we see explicitly that $\widetilde{M}_8$ is a 5-fold covering space of $M_8$.

**Remark.** The $2 \times 8 = 16$ vertices are grouped into three equivalence classes after the identification (Figure 5). $v_1$ and $v_2$ each consists of four vertices, but each of these four vertices actually represents four vertices of $M_8$ (since four tetrahedra meet there). Hence, $\ker(v_i)$, for $i = 1, 2$, consists of 16 triangles, giving a double cover of the regular neighborhood $N(K)$. The other eight vertices make up $v_3$, which is a trivial cover of $N(K)$. As the latest version of Snappea shows, or by direct group-theoretic calculation, there is only one 5-fold cover of $\widetilde{M}_8$ with three cusps, necessarily the irregular 5-fold dihedral cover. Amazingly this is a very

**References**

many commensurability classes of totally geodesic surfaces immersed in this manifold, but there is no explicit description of them. These are the only known closed incompressible surfaces so far.

We begin with the definitions of some key terms and the labeling of the polyhedral structure of the figure-8 knot complement used throughout this paper.

2. Preliminaries and Definitions

Let $K$ be the figure-8 knot in $S^3$ and $N(K)$ be its regular (open) neighborhood. Following Thurston’s description [20], we can think of the figure-8 knot complement $S^3 - N(K)$ as two tetrahedra $T_1$ and $T_2$ glued together along their faces as shown in Figure 1 and then remove a small neighborhood of the vertex. Here, the two types of arrows shown in the figure and the face identification (A with E, B with F, C with G, and D with H) uniquely determine the gluing, resulting in two equivalence classes of edges. We will refer to this manifold, together with this polyhedral structure, as $M_8$.

![Figure 1. Figure-8 knot complement as a union of two tetrahedra](image-url)

Suppose more generally we have a polyhedral decomposition of a manifold $M^3$. Our aim is to examine immersed surfaces with reference to the underlying combinatorial structure. The surfaces $S$ we consider are closed but not necessarily orientable or connected, unless otherwise stated.

**Definition 1.** A normal disk in a polyhedron is a properly embedded disk, with boundary meeting each face of the polyhedron in at most one arc joining distinct edges of the face.

Note that a normal disk in a polyhedron for which adjacent faces have one edge in common is determined by the cyclic ordering of its edges.

**Definition 2.** Suppose we are given an immersion $f : S \to M^3$. We say that $S$ is a normal surface in $M^3$ if the the image $f(S)$ intersects each polyhedron in a finite collection of normal disks.

There is thus an induced polygonal structure on $S$, obtained by pulling back the normal disks by $f$. Clearly not all polygonal decompositions of $S$ arise from normal immersions into $M^3$. Similarly the universal cover $\tilde{S}$ inherits a polygonal decomposition.