Series Analysis of Kosterlitz–Thouless Transitions

I. G. Enting and A. J. Guttmann

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Summary

- Potts models: standard and planar
- Planar Potts
- Kosterlitz-Thouless Transitions
- 6-state model: Monte Carlo and high-field
- 6-state model: Low-T series by finite lattice method
- Two-parameter 5-state model: series analysis and phase-space

Potts Models

Potts model variables, x_j, at each lattice site take one of q values: $1, 2 \ldots q$, generalising $q = 2$ (Ising) case.
For $q > 3$ two natural generalisations (with scalar product interactions):

Standard: The x_j are vectors in $(q-1)$-D space. Only two distinct pair-energies: x_j, x_k are the same or different.
Widely-studied, generalised to non-integer q.
First-order transition for $q > 4$ on 2-D lattices (Baxter).

Planar: The x_j are treated as vectors in 2-D space. Pair energy $\propto -\cos(2\pi(x_j - x_k))$.

Planar Potts models

The q-state Planar Potts model has low-T ordered state.

Ordered state disappears as $q \to \infty$ (planar rotator).

Planar rotator has low-T spin-wave states (as does larger-q planar Potts, above ordered phase).

Disordered phase at high-temperatures.

Expect 3 phases for q *sufficiently large*, in practice for $q \geq 5$.

Spin-wave states

Primary excitation of planar rotator (and planar Potts analogues) is spin-wave.

Spin-waves preclude long-range order in planar rotator (Mermin-Wagner theorem).

Spin-wave spectrum exhibits algebraic decay of correlations.

Bound vortex-pairs are additional excitations, ultimately leading to breakdown of spin-wave phase and a transition to a disordered state.

K-T transitions

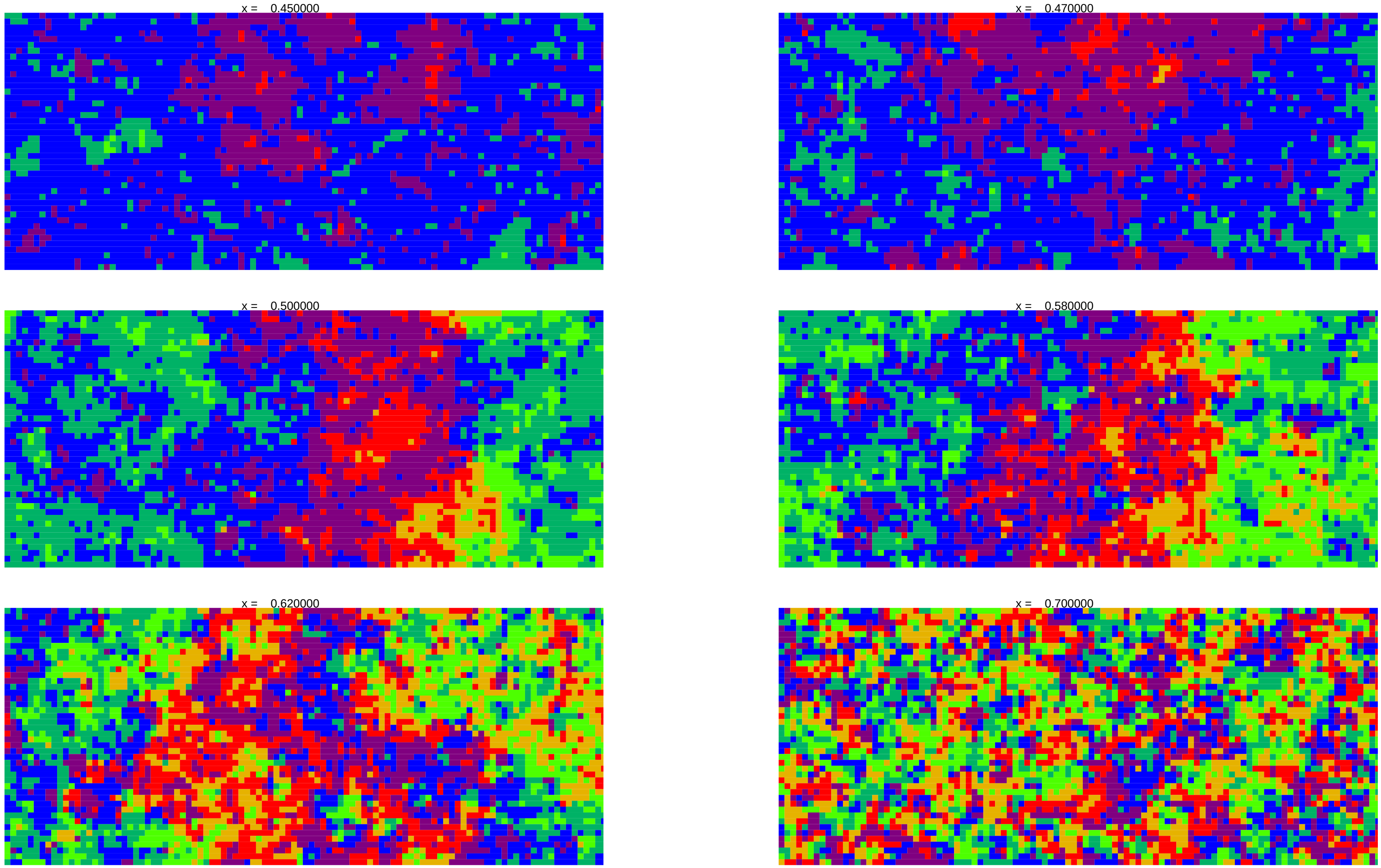
Phase between ordered and disordered phases has long-range algebraic decay of correlations (with varying exponent η). Termed ‘massless’ phase, based on particle physics identification of correlation length with inverse mass.
Onset of massless phase has expected critical behaviour with

$$M \sim \exp(-c/\sqrt{T_{\mathsf{C}} - T})$$

known as a Kosterlitz-Thouless transition.

In geology, K-T transition means Cretaceous-Tertiary, time of extinction of dinosaurs, with iridium-rich layer from asteroid impact.

Monte Carlo: 6-state planar



Simulations for 0.45, 0.47, 0.50, 0.58, 0.62, 0.70.
c.f. $x_1 \approx 0.49$, $x_2 \approx 0.60$ series: Barber and Enting (1981).

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Series: 6-state planar model

Barber and Enting (1981) analysed massless phase $T_1 < T < T_2$ using high-field series for varying δ where $M \sim H^{1/\delta}$ as $H \to 0$, expecting $\delta = 1$ for $T > T_2$ and $\delta = \infty$ for $T < T_1$ (actually diagnosed by poor convergence of estimates).
Hyperscaling $(\delta - 1)/(\delta + 1) = (2 - \eta)/d$.

Series to μ^9 ($\mu = \exp(-H/k_{\mathrm{B}}T)$) from code method (partial generating functions: from sublattice summations).

Low temperature series for M to x^{16} too short to analyse (and wrong). With extension (2005) to x^{31}, analysis is still problematic.

Finite Lattice Method

Expansions start (formally or actually) from partition function Z, or limit of $Z^{1/N}$, often as multi-variate series so derivatives can be taken.

Approximations from inclusion-exclusion relations extract $Z^{1/N}$ behaviour from finite rectangular lattices with $Z_{m,n}$ calculated by transfer matrix techniques: e.g.

$$Z^{1/N} \approx [Z_{m+1,m+1} Z_{m,m}]/[Z_{m+1,m} Z_{m,m+1}]$$

More efficient cutoff uses $Z_{m,n}$ with $m + n \leq k$.

6-state results (1/12/05)

Order parameter series extended to x^{41}.

Series inconsistent with power-law.

$M \sim \exp(-c/\sqrt{x_{\mathrm{c}} - x})$ implies

$X = \ln M \sim (x_{\mathrm{c}} - x)^{-0.5}$

Fit differential approximants to series $x^{-4}X$

No indication of confluent singularity.

	x_c	exponent
1st order DA	0.4881 ± 0.0006	0.55 ± 0.05
2nd order DA	0.4888 ± 0.0008	0.50 ± 0.05

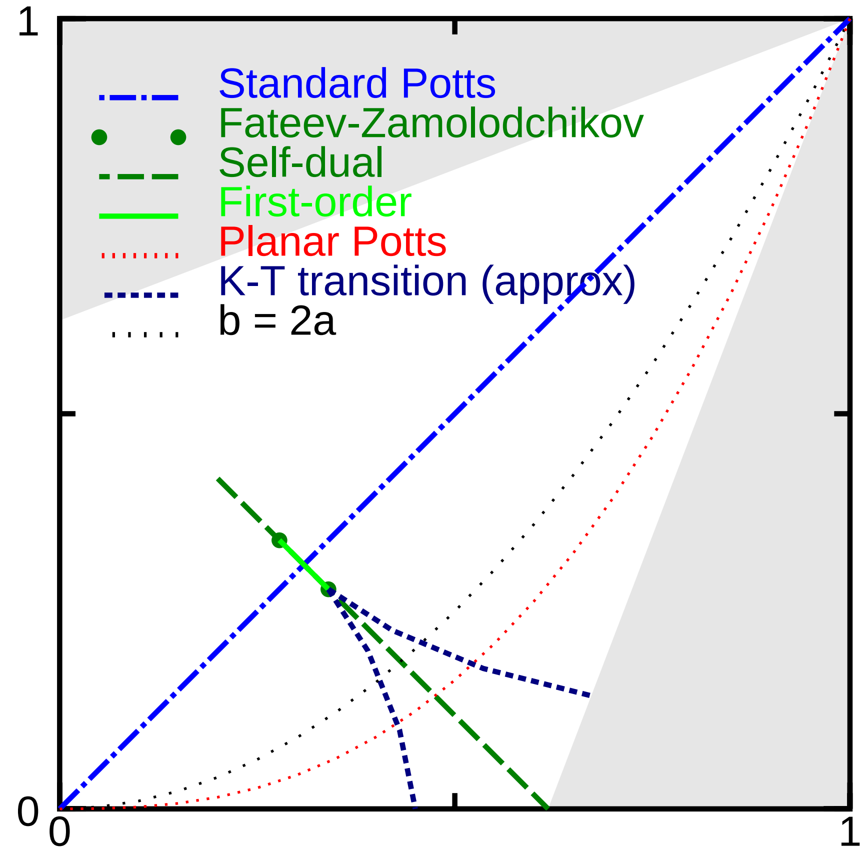
Padé approximants estimate $c = 0.02750$

6-state susceptibility: 4/12/05

- Low-T series for χ to order 35 — more field terms, so fewer temperature terms.
- Use known x_c from M.
- Not a conventional singularity
- $\chi \sim \exp(A/(x_{\mathrm{c}} - x)^{\alpha})$
- estimate from DAs to $\ln \chi$ suggest $\alpha \approx 0.9$
- probably $\alpha = 1$?????

Phase space of 5-state models

Rotation/reflection symmetries imply 2 possible pair energies (relative to $x_j = x_k$). For $|x_j - x_k| = 1$ or 2, weights are z_1 and z_2. If energies sufficiently different, expect 2 phase transitions (Wu, Cardy). High-T vs low-T (and associated transition lines) related by duality (Wu).



Solution by Fateev and Zamolodchikov may mark bifurcation point.

Analysis: 5-state models

Low-T series for order parameter using FLM expanded in x with $z_1 = x^a$, $z_2 = x^b$.

M to order x^{41} for $a/b = 1/2, 1/3, 1/4$.

$M \sim \exp(-c/\sqrt{x_c - x})$ implies

$X = \frac{d}{dx} \ln M \sim (x_c - x)^{-1.5}$

Fit differential approximants (DAs) to series X

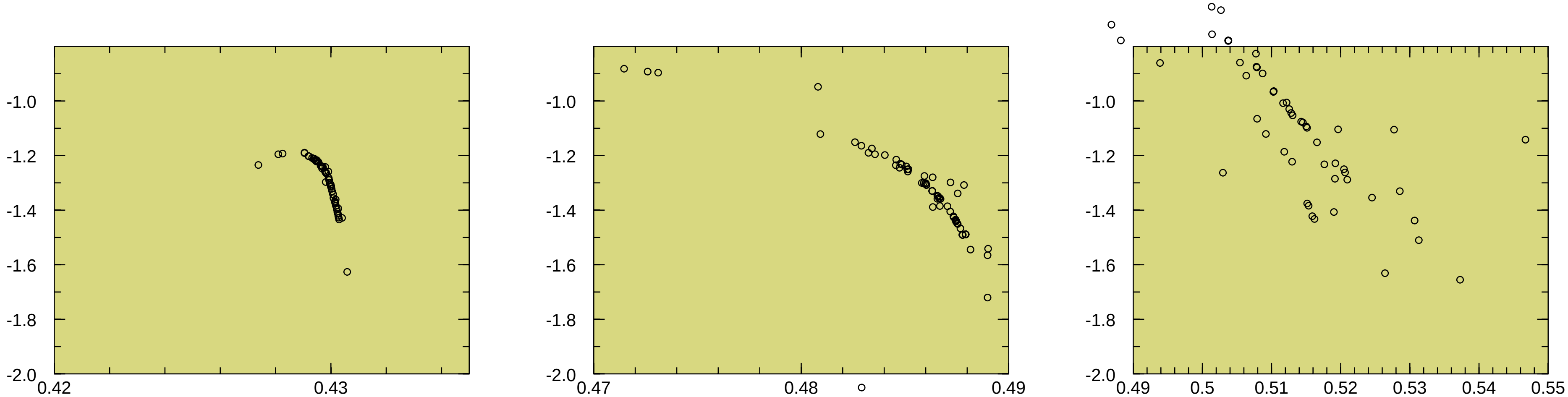
$$P_2(x)\frac{d^2}{dx^2}X + P_2(x)\frac{d}{dx}X + P_0(x)X = 0$$

Roots of P_2 give x_c and indicial equation gives exponent (-1.5 expected) (Guttmann 1989).

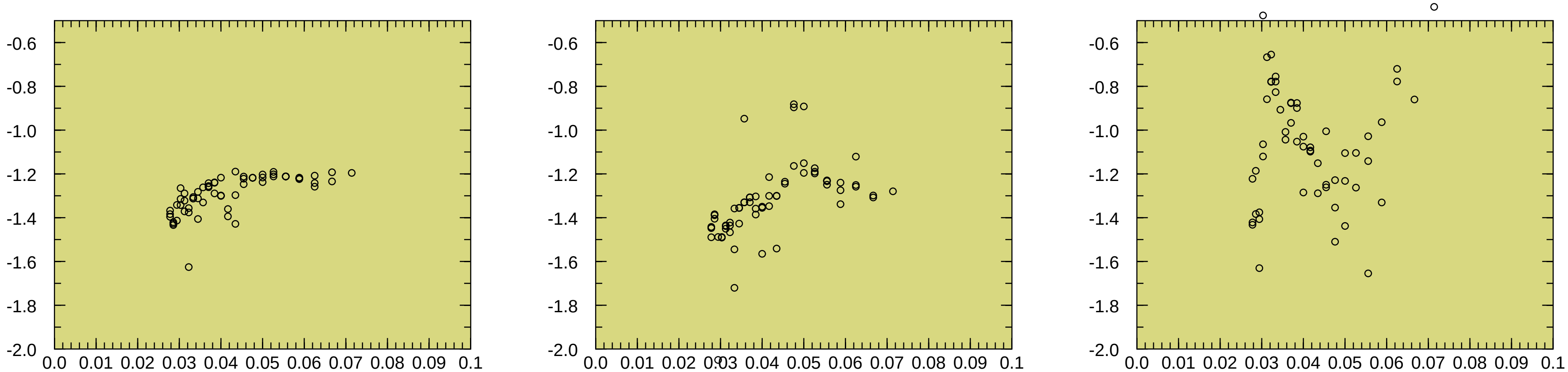
Poor convergence, especially if a/b small.

Results: 5-state models

Exponent vs critical point



$a/b = 1/2$ $a/b = 1/3$ $a/b = 1/4$



Exponent vs 1/number of terms fitted

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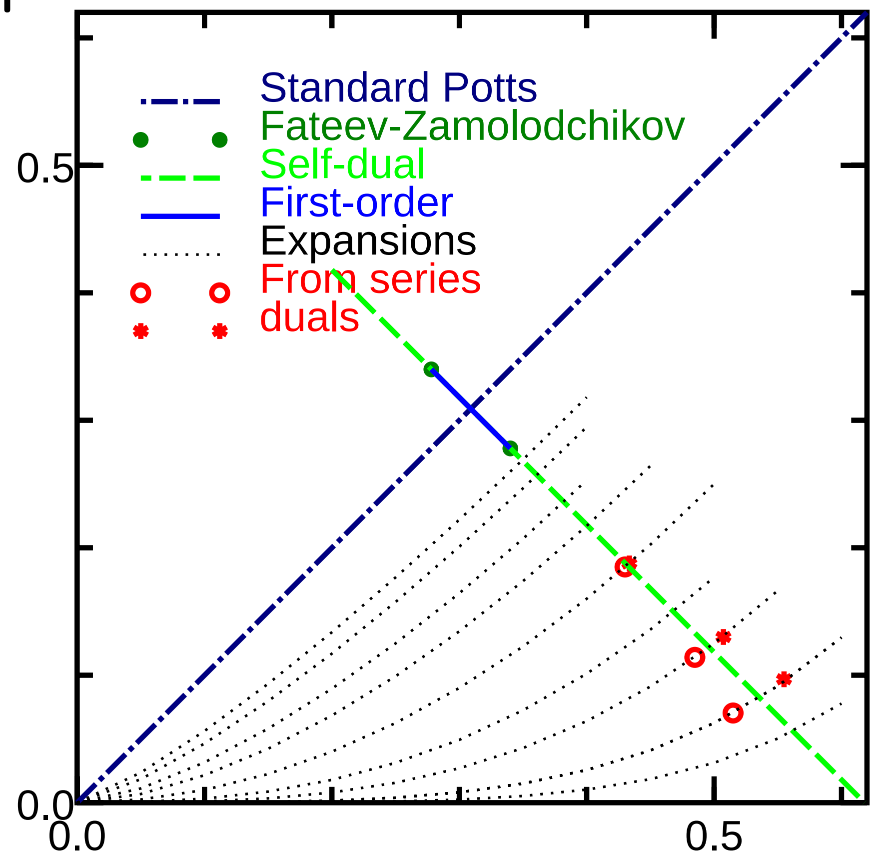
Series: 5-state models

Low-T expansions for order parameter using FLM. Expanded in x with $z_1 = x^a$, $z_2 = x^b$ (a, b small integers).

Estimated critical points for $a/b = 1/2$, $1/3$, $1/4$ suggest massless phase is narrow for $q = 5$.



Paths of fixed a/b dotted.

Implications

Kosterlitz-Thouless Transitions can be successfully studied by series analysis of favourable cases.

Prediction of $-1/2$ exponent in exponential confirmed.

It is likely to be hard to confirm the role of the Fateev-Zamolodchikov point.

Possible future directions

- Longer low-T series;
- Revisit high-field series analysis (as per Barber and Enting)
- More insight into structure of singularity: $\exp(-c/\sqrt{T_{\mathsf{c}} - T})$ vs $(T_{\mathsf{c}} - T)^{\beta} \exp(-c/\sqrt{T_{\mathsf{c}} - T})$
- Cases that are easier (for series): Z6 to Z5 (to Ashkin-Teller?).