Statistical mechanics of stochastic and deterministic cellular automata.

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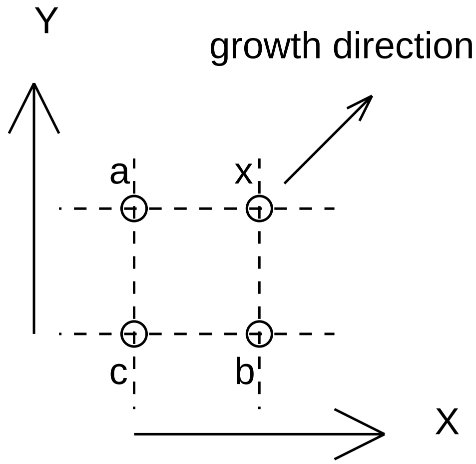
Overview

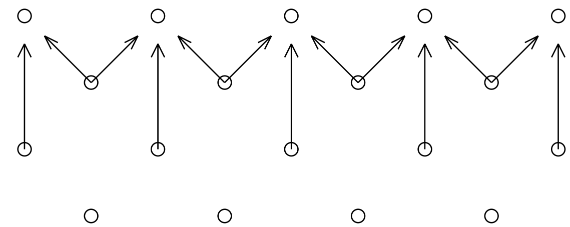
- Significance of cellular automata
- Random fields: Gibbs/Markov random fields
- Analysis of stochastic cellular automata (SCA)
  - Gibberd transformation
  - Significance of symmetry
- Symmetries of triangular lattice cases
- An example
- Future directions

SCA applications

- Statistics of disordered crystals (Welberry)
- Traffic flow
- Pest outbreaks
- Image processing
- Directed percolation and applications
- Design deterministic cellular automata (Rujan)

Some definitions





- $\Pr(\sigma_x = 1) = \frac{1}{2}[1 + \sigma_x(\alpha + \beta_x \sigma_a + \beta_y \sigma_b + \gamma \sigma_a \sigma_b)]$
  for $\sigma = \pm 1$ OR
- $\Pr(n_x = 1) = A' + B'_x n_a + B'_y n_b + C' n_a n_b$
  for $n = 0, 1$ OR
- $\Pr(1|00) = A$, $\Pr(1|01) = B_x$,
  $\Pr(1|01) = B_y$, $\Pr(1|11) = C$

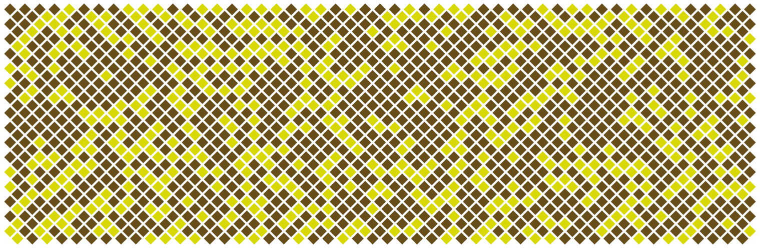
General case

For general square lattice case, need $\Pr(1|a,c,b)$

$$\Pr(1|000) = A \qquad \Pr(1|100) = B_X$$
$$\Pr(1|001) = B_Y \qquad \Pr(1|101) = C$$
$$\Pr(1|010) = D \qquad \Pr(1|110) = F_X$$
$$\Pr(1|011) = F_Y \qquad \Pr(1|111) = G$$

Plots list probabilities (ordered by inputs representing binary number) as: $[A, B_X, F_Y, D, B_X, C, F_X, G]$.

SCA examples



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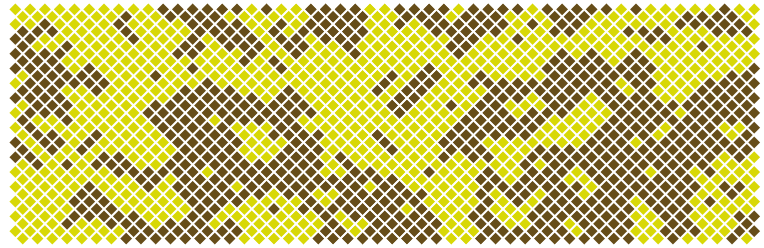

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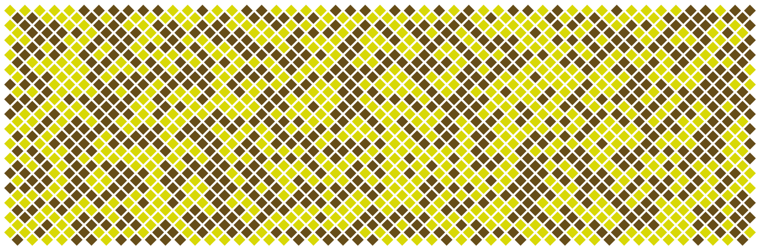

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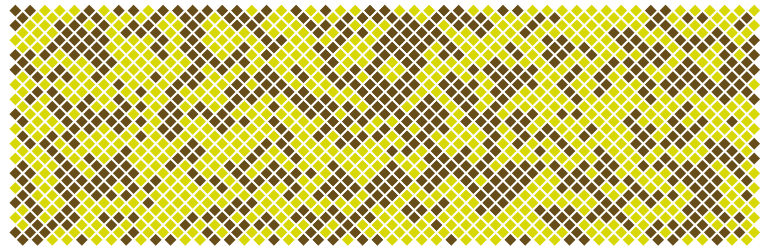

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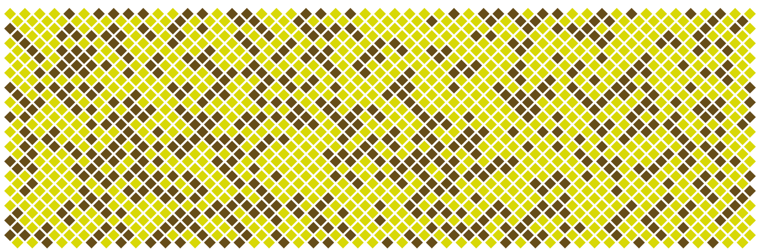
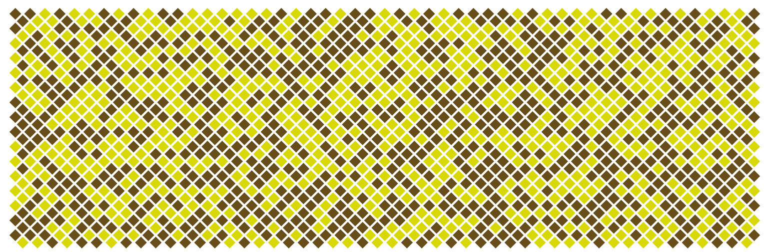
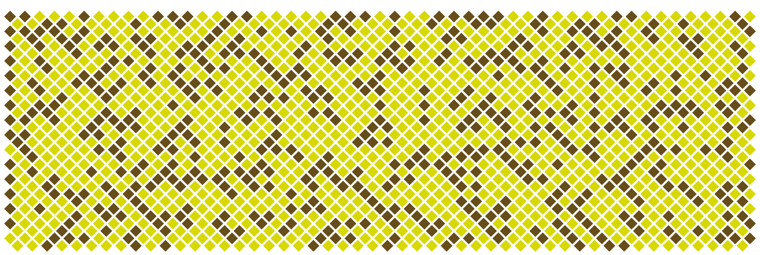


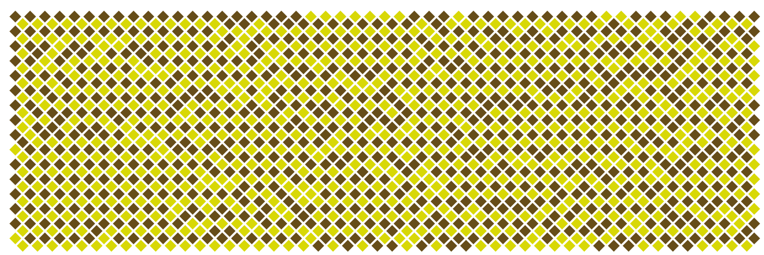
0.75 0.5 0.75 0.5 0.5 0.666667 0.5 0.666667



0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5

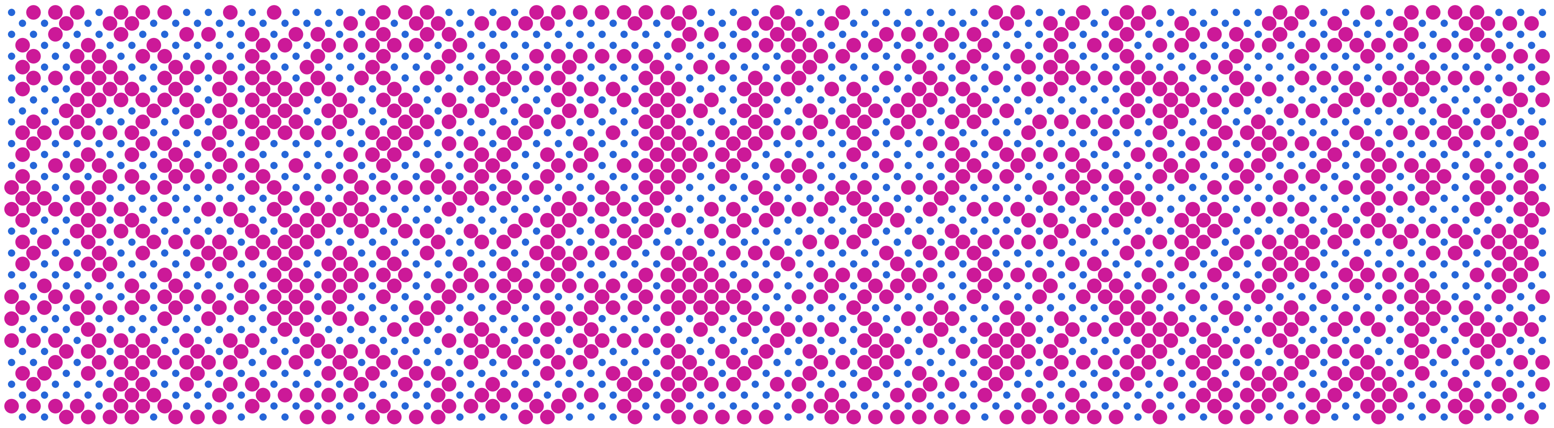


1.0 0.5 1.0 0.5 0.5 0.75 0.5 0.75

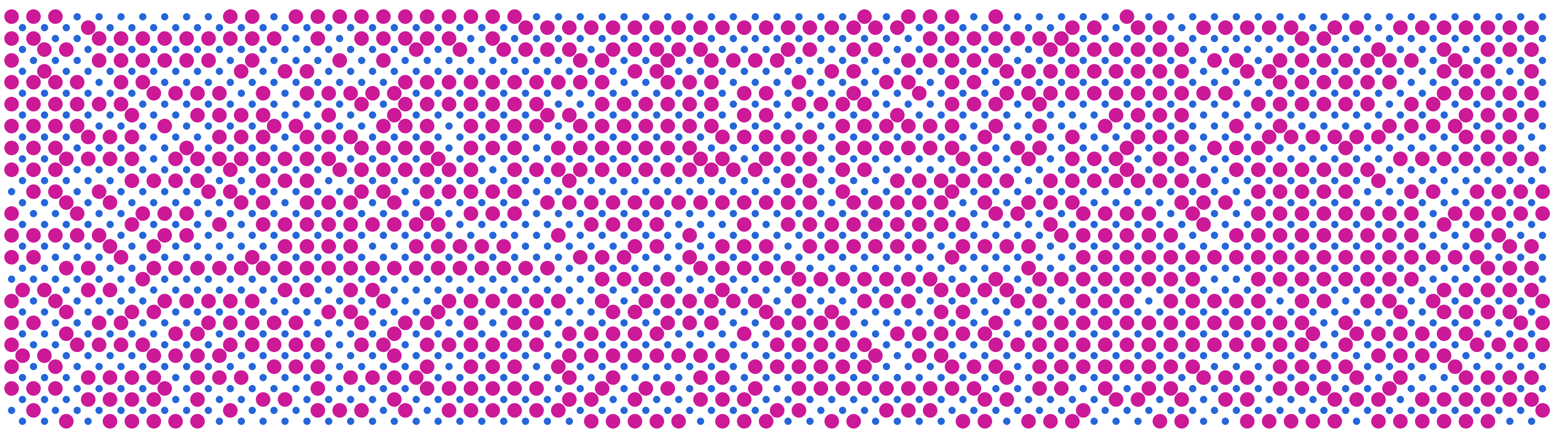


0.95 0.5 0.95 0.5 0.5 0.05 0.5 0.05

Implicit symmetry



0.2 0.8 0.2 0.8 0.8 0.2 0.8 0.2



0.985 0.8 0.985 0.8 0.8 0.2 0.8 0.2

Lower example has implicit symmetry under reversal of growth direction.

Some origins

- Models of spatial processes (Bartlett)
- Disorder in mixed crystals – interpretation of X-ray data (Welberry)
- Solutions of special Ising model cases (Gibberd, Verhagen, Enting and extensive later work)
- Special factorisation of eigenvectors of Ising transfer matrix (Baxter)
- Pickard random fields – conditions to allow statistically stationary random fields

Pickard approach

Objective

- Explicit construction for stationary finite random fields
- i.e. avoid issues of rate of convergence to stationarity
- Specify boundaries to have the stationary distribution

Approach

- $\Pr(c, a, b, x) = \Pr(c) \Pr(a|c) \Pr(b|ac) \Pr(x|abc)$
- with consistency $\Pr(b|ac) = \Pr(b|c)$
- $\Pr(b|ax) = \Pr(b|x)$ gives stationarity with MC boundary
- as does $\Pr(x|bc) = \Pr(x|b)$ and $\Pr(x|ac) = \Pr(x|a)$

Gibbs random field

Energies specified as $\mathcal{H} = \sum_{i,j} \mathcal{H}_{i,j}$ with

$$\mathcal{H}_{i,j} = H\sigma_{x:i,j} + J_x\sigma_{x:i,j}\sigma_{b:i,j} + J_y\sigma_{x:i,j}\sigma_{a:i,j}$$

$$+J_2\sigma_{a:i,j}\sigma_{b:i,j} + L\sigma_{x:i,j}\sigma_{a:i,j}\sigma_{b:i,j}$$

Joint probability of spin states, $\sigma = \pm 1$, are:

$$\Pr(\sigma_1 \ldots \sigma_N) = \frac{1}{Z}\exp[-\mathcal{H}/k_\mathrm{B}T]$$

The normalising factor, Z, is called the partition function, and acts as a generating function for moments of the distribution.

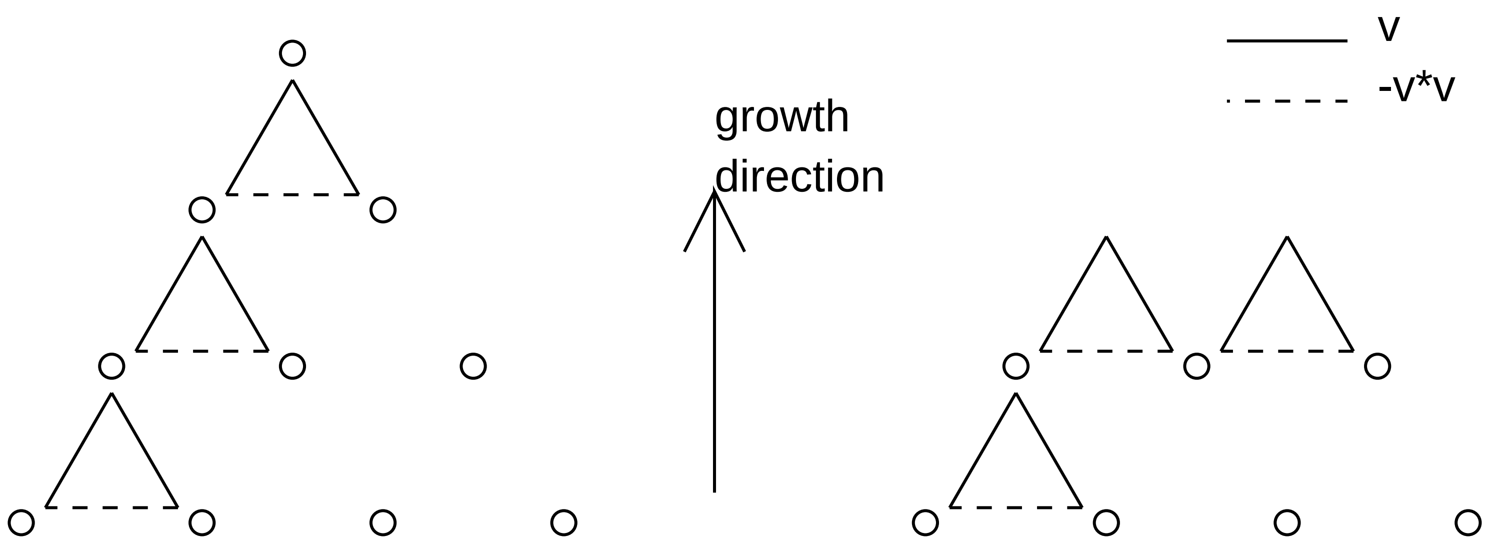
Relating SCA to Gibbs fields

- $\Pr(\sigma_1 \ldots \sigma_N) = \exp[-\mathcal{H}/k_{\rm B}T]/Z$
- Use $\exp(a\sigma) = \cosh(a)[1 + \sigma \tanh(a)]$ for $\sigma = \pm 1$
- $\Pr(\sigma_1 \ldots \sigma_N) = \prod_{i,j} [\prod_\alpha \cosh(J_\alpha/k_{\rm B}T)[1 + p(\sigma_{x:i,j}, \sigma_{a:i,j}, \sigma_{b:i,j}) \tanh(J_\alpha/k_{\rm B}T)]]$
- SCA has $\Pr(\sigma_1 \ldots \sigma_N) = \prod_{i,j} \Pr(\sigma_{x:i,j} | \sigma_{a:i,j}, \sigma_{b:i,j})$
- need 'tanh' expansion of $\exp[-\mathcal{H}_{i,j}/k_{\rm B}T]$ to have all terms either spin-independent or proportional to $\sigma_{x:i,j}$
- On triangular lattice, SCA are 4-parameter subset of 5-parameter Gibbs fields

Gibberd constructions

- Construct Z as sum over all spin states of un-normalised probability
- $Z = \sum_\sigma \exp(-\mathcal{H}/k_{\rm B}T) \sim \cosh(J_\alpha/k_{\rm B}T)^{n^2}[1 + f(v_\alpha)]^{n^2}$
- The $f(v_\alpha)$ is a combinatorial contribution from combinations of ‘tanh’ terms with all spins occuring with even powers.
- Apart from normalising factor, SCA have no closed loop contributions proportional to lattice size
- Potential closed loop factors come in cancelling pairs (or groups). e.g. $(1 + v\sigma_x\sigma_a)(1 + v\sigma_x\sigma_b)(1 - v^2\sigma_a\sigma_b)$

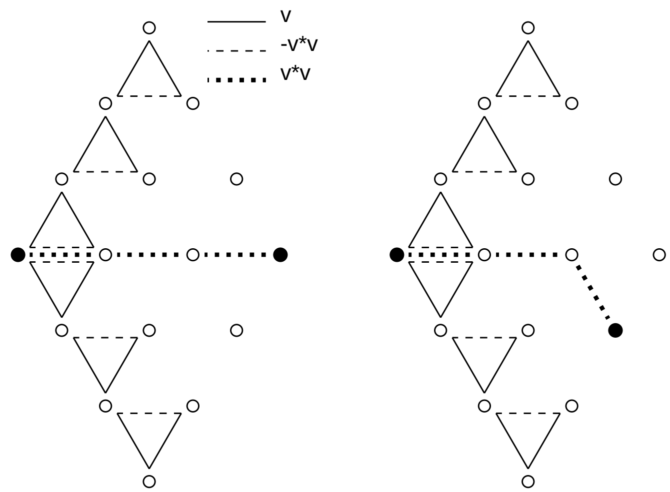
Gibberd construction



- Left: Loop segments in the growth direction never close
- Right: Loop segments in other directions have cancelling pairs

Solutions from symmetries

Gibbs representation revealed symmetries (Enting)and led to new solutions for correlations (Welberry)



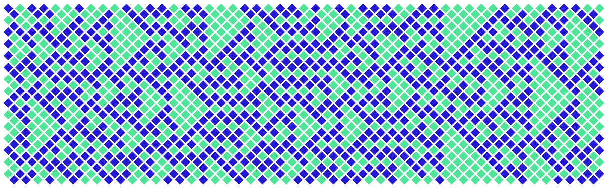
- $\mathrm{E}(\sigma_j\sigma_k)$ calculated with pseudo-interaction $J^*\sigma_j\sigma_k$ in 'tanh' expansion
- Symmetry-derived solutions understood as allowing a 2-way Gibberd construction
- Only contributions to $\mathrm{E}(\sigma_j\sigma_k)$ are corrections (if any) at boundary between growth directions (i.e. 1-D)

Some symmetry classes

TGn: denotes 'triangular growth model class, with n free parameters, classified by symmetry

- TG4: Generic. Z from Wu, no correlation solutions
- TG3:sc1 Special case 1 of Welberry and Galbraith: second triangular growth direction
- TG3:R Invariance under 180° rotation. Pickard case 1
- TG2:tri. X-Y symmetry combined with sc1. Three equivalent growth directions. Analysed by Enting. Pickard case 2.
- TG2:$\pm$ Invariance under spin reversal (anisotropic Ising) Gibberd solutions

Triangular symmetry case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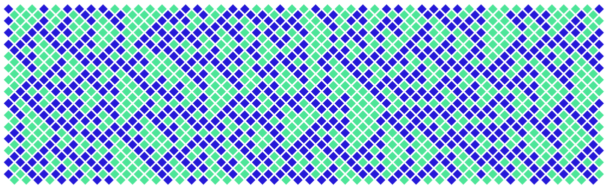
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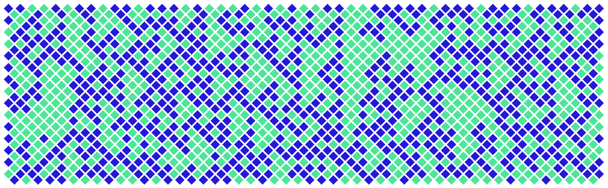
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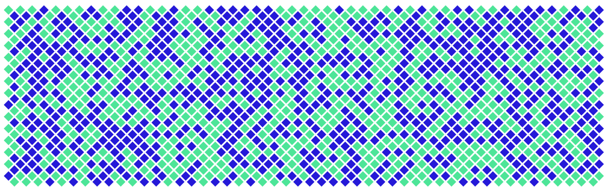
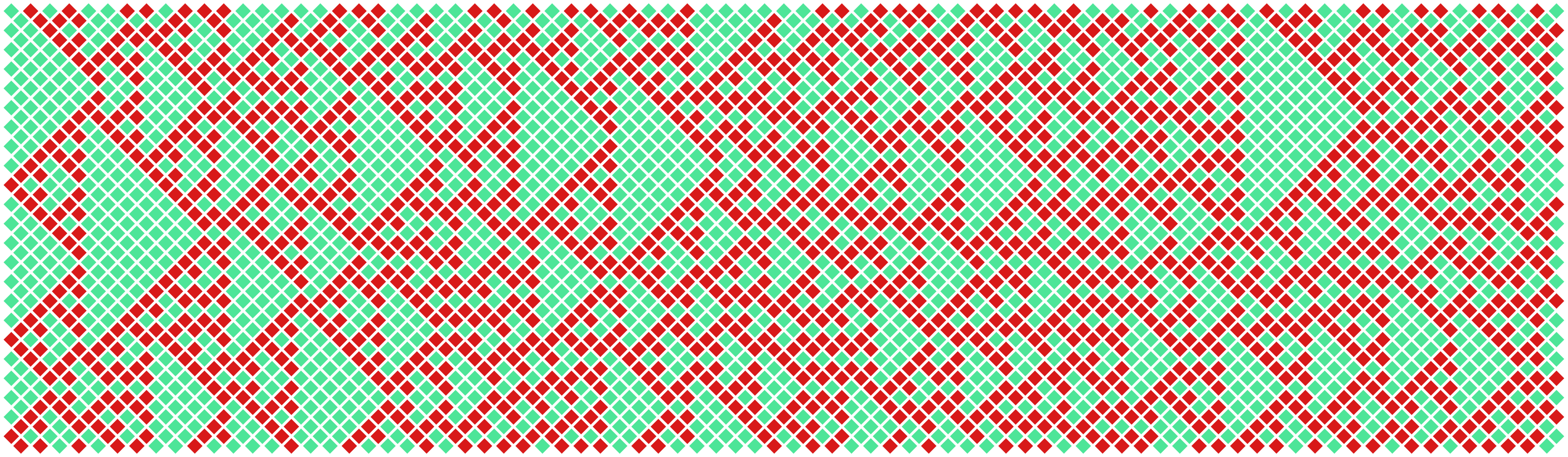

0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5

- Case TG1:tri4
- Symmetry under spin-reversal on any 2 sublattices
- $\mathrm{Pr}(\sigma_x) = \frac{1}{2}[1 + \gamma\sigma_x\sigma_a\sigma_b]$
- All 2-site correlations are zero.
- $\mathrm{E}[\sigma_{n,n}\sigma_{0,n}\sigma_{n,0}] = \gamma^{3^K}$ for $n = 2^K$
- Sequence: $\gamma = 0, -0.7, -0.9, -0.98$

CA limit (0/1 addition mod 2)

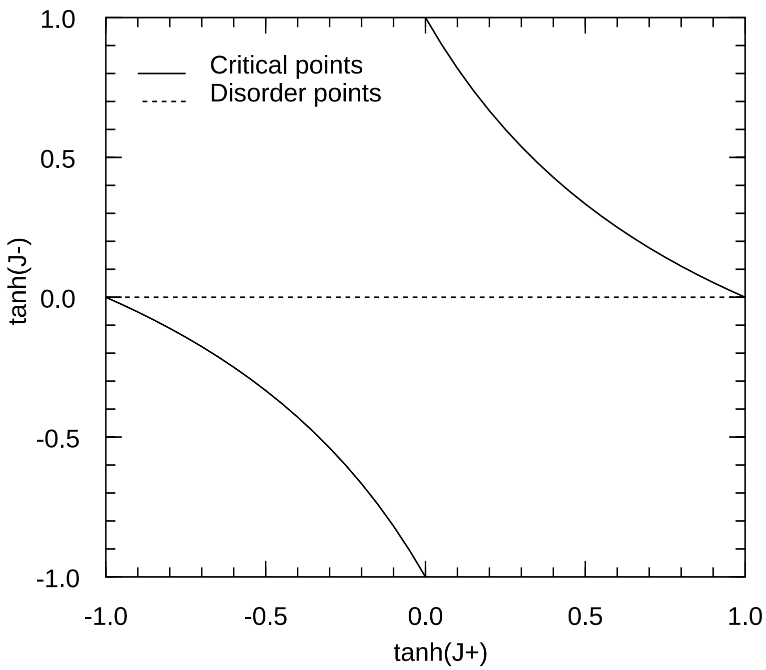


- Limit cycle sensitive to boundary conditions (width)
- Width 2^j has ‘all zero’ as only limit cycle
- Certain primes have long cycles: $2^{(W-1)/2} - 1$
- Sensitivity suggests relation to phase transition

Martin et al, 1984; Commun Math Phys 93: 219

Triple model phase diagram

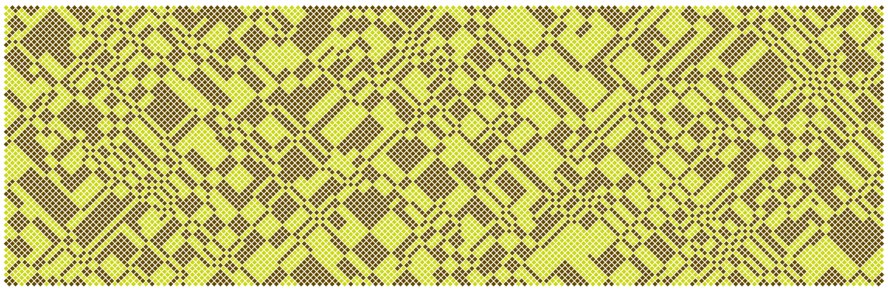
$$\mathcal{H} = \sum_{ij} [L^{[+]} \sigma_{x:i,j} \sigma_{a:i,j} \sigma_{b:i,j} + L^{[-]} \sigma_{c:i,j} \sigma_{a:i,j} \sigma_{b:i,j}]$$



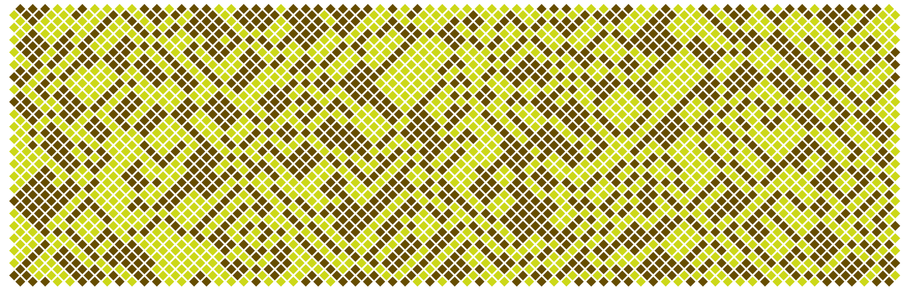
- Solved for $L^{[+]} = L^{[-]}$ by Baxter and Wu
- Transition is breaking of 4-fold symmetry
- Self-dual line (critical) along: $v^{[+]} = \tanh(L^{[+]}/k_\mathrm{B}T) = (1 - v^{[-]})/(1 + v^{[-]})$

Deterministic CA occurs where stochastic CA line meets phase transition line.

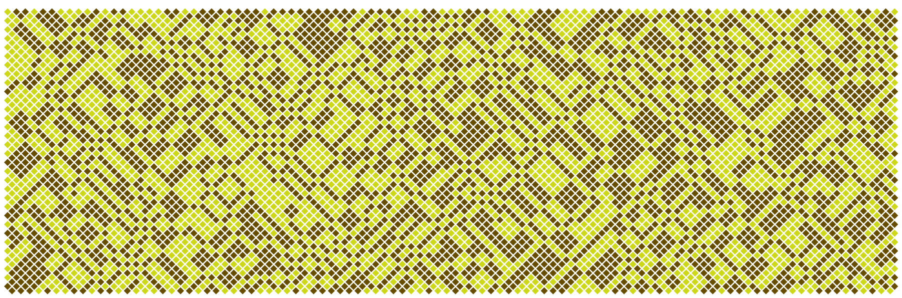
Scaling



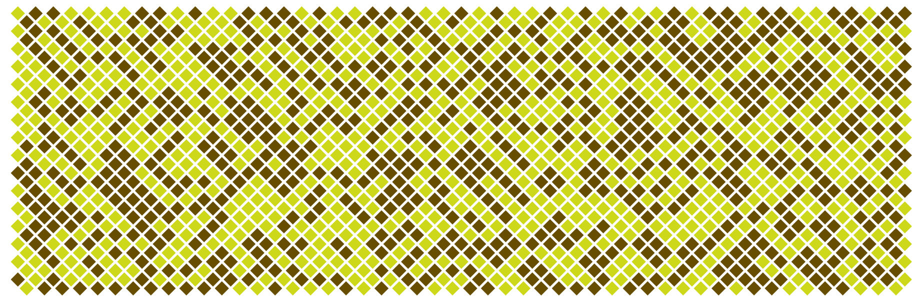
0.03 0.945 0.898 0.055 0.945 0.102 0.055 0.97



0.145 0.827 0.792 0.173 0.827 0.208 0.173 0.855



0.068 0.902 0.858 0.098 0.902 0.142 0.098 0.932



0.267 0.715 0.695 0.285 0.715 0.305 0.285 0.733

Rescaling of lengths by $\sqrt{2}$, adjusting probabilities so as to leave 2-site and 4-site correlations invariant in terms of actual length.

Future directions

- Classification of 4-site (rectangular lattice) cases
- Extension to higher order
  - Potts model cases
  - overlapping clusters
- Relation to variational approximations
  - insight into CTM?????
- Application of renormalisation group.