Weak convergence of measure-valued processes and r-point functions.

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Abstract: We prove a sufficient set of conditions for a sequence of finite measures on the space of cadlag measure-valued paths to converge to the canonical measure of super-Brownian motion in the sense of convergence of *finite-dimensional distributions*. The conditions are convergence of the Fourier transform of the r-point functions and perhaps convergence of the "survival probabilities". These conditions have recently been shown to hold for a variety of statistical mechanical models including critical oriented percolation, the critical contact process, and lattice trees at criticality, all above their respective critical dimensions.

1. Motivation

In the last few years a number of rescaled models from interacting particle systems and statistical physics have been shown to converge to the canonical measure of super-Brownian motion. The models include critical oriented percolation above 4 dimensions [7], critical contact processes above 4 dimensions [6] and critical lattice trees above 8 dimensions [8], all for sufficiently spread-out kernels. In each of these cases what is actually proved is convergence of the Fourier transforms of the moment measures (or r-point functions). Our modest objective here is to translate this result into the more conventional probabilistic language of weak convergence of stochastic processes. To those well-versed in weak convergence arguments we fear this may be one of the proverbial much-needed gaps in the literature, but to others who have complained to us, it is an irritant that should be spelled out once and for all.

The limiting measure is a sigma-finite measure (not a probability) on the space of continuous measure-valued paths which presents some additional minor worries. The full convergence on path space remains open in all of the above settings due to the absence of any tightness result on path space. Even the natural statement of convergence of finite-dimensional distributions requires convergence of the survival probabilities (see Proposition 2.4 below), a result which was only recently discovered for critical oriented percolation [3, 4] and is currently being pursued in the other contexts mentioned above. So in the end we

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thought someone should advertise this state of affairs and we have acquiesced in the writing of this note. If you are reading this in a journal at least one editor and/or referee has agreed with us.

2. Introduction

Consider a discrete time, critical nearest neighbour branching random walk on $\mathbb{Z}^d$ starting with a single particle at the origin. That is at time $n \in \mathbb{Z}_+$ each individual gives birth to a random number of offspring which immediately take a step to a randomly chosen nearest neighbour of its parent. Assume each parent dies immediately after giving birth, that the offspring distribution has mean one and finite variance $\gamma > 0$, and each of the offspring laws and random walk steps are independently chosen.

Extend the branching random walk to all times $t \geq 0$ by making it a right-continuous step function. Let $\mathcal{B}_t = \{x_t^{(\alpha)} : \alpha \in I_t\}$ denote the set of locations of particles in $\mathbb{Z}^d$ alive at time t. We have suppressed the details of the labelling system (see e.g. section II.3 in [9]) but as multiple occupancies are allowed some labelling scheme is needed here.

In order to describe the scaling limit, we represent the model as a *cadlag* measure-valued process by setting

$$X_t^n = \frac{C_1}{n} \sum_{\alpha \in I_{nt}} \delta_{x_{nt}^{(\alpha)}/(C_2\sqrt{n})},$$

where $C_1 = \gamma^{-1/2}$ and $C_2 = d^{-1/2}$. If E and M are separable metric spaces, $M_F(E)$ denotes the space of finite Borel measures on E with the topology of weak convergence, and $D(M)$ denotes the space of *cadlag* M-valued paths with the Skorokhod topology. With probability 1, X_t^n is a finite measure for all $n \in \mathbb{Z}_+$ and $t \geq 0$, so that $\{X_t^n\}_{t \geq 0} \in D \equiv D(M_F(\mathbb{R}^d))$.

The *extinction time* $S : D \to [0, \infty]$ is defined by

$$S(X) \equiv \inf\{s > 0 : X_s = 0_M\},$$

where 0_M is the zero measure on $\mathbb{R}^d$ and $\inf \emptyset = \infty$. Next we define a sequence of measures $\mu_n \in M_F(D)$ by

$$\mu_n(\bullet) \equiv C_3 n \mathbb{P}(\{X_t^n\}_{t \geq 0} \in \bullet), \tag{1}$$

where $C_3 = 1$ for this branching random walk model.

Let $M_\sigma(D)$ denote the σ-finite measures on D which assign finite mass to $\{S > \epsilon\}$ for all $\epsilon > 0$, with the topology of weak convergence defined as follows:

Definition 2.1 (Weak convergence). *Let $\{\nu_n : n \in \mathbb{N} \cup \{\infty\}\} \subset M_\sigma(D)$. We write $\nu_n \xRightarrow{w} \nu_\infty$ if for every $\epsilon > 0$,*

$$\nu_n^\epsilon(\bullet) \equiv \nu_n(\bullet, S > \epsilon) \xRightarrow{w} \nu_\infty(\bullet, S > \epsilon) \equiv \nu_\infty^\epsilon(\bullet), \qquad as \ n \to \infty,$$

where the convergence is in $M_F(D)$.

It is a standard result in the superprocess literature (see for example [9] Theorem II.7.3.) that there exists $\mathbb{N}_0 \in M_\sigma(D)$, supported by the continuous paths in D which remain at 0_M after time S, and called the *canonical measure of super-Brownian motion* (CSBM), such that $\mu_n \stackrel{w}{\Longrightarrow} \mathbb{N}_0$. In [9] one is working with branching Brownian motion instead of branching random walk but it is trivial to modify the arguments. We have chosen our constants C_i so that the branching and diffusion parameters of our limiting super-Brownian motion are both equal to one. Much is known about $\mathbb{N}_0$, for example as in Theorem II.7.2(iii) of [9] we have for every $b > 0$,

$$\mathbb{N}_0 \left(X_b(1) \in A \setminus \{0\} \right) = \left(\frac{2}{b} \right)^2 \int_A e^{-\frac{2}{b}x} dx. \tag{2}$$

Let $l \geq 1$, and $\vec{t} = \{t_1, \ldots, t_l\} \in [0, \infty)^l$. We use $\pi_{\vec{t}} \colon D \to M_F(\mathbb{R}^d)^l$ to denote the projection map satisfying $\pi_{\vec{t}}(X) = (X_{t_1}, \ldots X_{t_l})$. The *finite-dimensional distributions* of $\nu \in M_\sigma(D)$ are the measures $\nu^\epsilon \pi_{\vec{t}}^{-1} \in M_F(M_F(\mathbb{R}^d)^l)$ given by

$$\nu^\epsilon \pi_{\vec{t}}^{-1}(H) \equiv \nu^\epsilon \left(\{X : \pi_{\vec{t}}(X) \in H\} \right), \qquad H \in \mathcal{B}(M_F(\mathbb{R}^d)^l).$$

Definition 2.2 (Convergence of f.d.d.). *Let $\{\nu_n : n \in \mathbb{N} \cup \{\infty\}\} \subset M_\sigma(D)$. We write $\nu_n \stackrel{f.d.d}{\Longrightarrow} \nu_\infty$ if for every $\epsilon > 0$, $m \in \mathbb{N}$, and $\vec{t} \in [0, \infty)^m$,*

$$\nu_n^\epsilon \pi_{\vec{t}}^{-1}(\bullet) \stackrel{w}{\Longrightarrow} \nu_\infty^\epsilon \pi_{\vec{t}}^{-1}(\bullet), \qquad \textit{as } n \to \infty,$$

where the convergence is in $M_F(M_F(\mathbb{R}^d)^m)$.

If ν_∞ is supported by continuous paths in D it is easy to see that weak convergence to ν_∞ (Definition 2.1) implies convergence of the finite-dimensional distributions to ν_∞ (Definition 2.2). An additional tightness condition is needed for the converse.

We now work in a more abstract setting than the branching random walk described above, in which $\{\mu_n\}$ is any sequence of finite measures on D. For $k \in \mathbb{R}^d$, let $\phi_k(x) = e^{ik \cdot x}$ and write $E_{\mu_n}[Y]$ for $\int Y d\mu_n$, and $X_t(\phi)$ for $\int \phi(x) X_t(dx)$ respectively. Consider the following convergence condition on the moment measures of μ_n:

$$E_{\mu_n} \left[\prod_{i=1}^{r-1} X_{t_i}(\phi_{k_i}) \right] \to E_{\mathbb{N}_0} \left[\prod_{i=1}^{r-1} X_{t_i}(\phi_{k_i}) \right], \text{ for } r \geq 2, \vec{t} \in [0, \infty)^{r-1}, \vec{k} \in \mathbb{R}^{d(r-1)}. \tag{3}$$

An explicit formula for the right side of (3) can be found in Section 1.2.3 of [7].

Of course (3) does hold for the μ_n defined in (1) for branching random walk but our interest in this condition arises from a number of models in which $\mathcal{B}_t$ is the (finite) set of occupied sites in $\mathbb{Z}^d$ at time t. Examples include the critical contact process, critical oriented percolation or critically weighted lattice trees, all with the natural definitions of "occupied site". For $r \geq 2$ and $\vec{t} \in [0, \infty)^{r-1}$ the r-point functions for this model are $B_{\vec{t}}(\vec{x}) = \mathbb{P}(x_i \in \mathcal{B}_{t_i}, i = 1, \ldots, r-1)$,

while the $\widehat{r}$-*point functions* are the Fourier transforms of these quantities,

$$\widehat{B}_{\vec{t}}(\vec{k}) = \sum_{\vec{x}} e^{i\vec{k}\cdot\vec{x}} B_{\vec{t}}(\vec{x}), \ \vec{k} \in \mathbb{R}^{d(r-1)},$$

which are defined whenever $B_{\vec{t}}(\vec{x})$ is summable in $\vec{x}$. Define $X_t^n \in M_F(\mathbb{R}^d)$ by

$$X_t^n \equiv \frac{C_1}{n} \sum_{x: C_2\sqrt{n}x\in\mathcal{B}_{nt}} \delta_x,$$

and assume that μ_n given by (1) defines a finite measure on D. An easy calculation then shows that

$$\frac{C_1^{r-1}C_3}{n^{r-2}} \widehat{B}_{n\vec{t}}\left(\frac{\vec{k}}{C_2\sqrt{n}}\right) = E_{\mu_n}\left[\prod_{i=1}^{r-1} X_{t_i}(\phi_{k_i})\right],$$

whenever $B_{n\vec{t}}(\vec{x})$ is summable. Therefore, the asymptotic formulae for the $\widehat{r}$-point functions for sufficiently spread out critical rescaled oriented percolation ($d > 4$), critical rescaled lattice trees ($d > 8$), and critical rescaled contact processes ($d > 4$) derived in [7], [8] and work in progress in [6], respectively, immediately imply (3) in each of these cases. Moreover in each of these models it is known that μ_n is a finite measure supported by D as is required above.

In what follows we use $\mathcal{D}_F$ to denote the set of discontinuities of a function F. A function $Q : M_F(\mathbb{R}^d)^m \to \mathbb{R}$ is called a *multinomial* if $Q(\vec{X})$ is a real multinomial in $\{X_1(1), \ldots, X_m(1)\}$. A function $F : M_F(\mathbb{R}^d)^m \to \mathbb{C}$ is said to be *bounded by a multinomial* ($|F| \leq Q$) if there exists a multinomial Q such that $|F(\vec{X})| \leq Q(\vec{X})$ for every $\vec{X} \in M_F(\mathbb{R}^d)^m$. The main results of this paper are the following two propositions. By the above, the first result is applicable in each of the three settings [7], [8], [6].

Proposition 2.3. *Let $\{\mu_n\}_{n\geq 1}$ be a sequence of finite measures on $D(M_F(\mathbb{R}^d))$ such that (3) holds. Then for every $s > 0$, $\lambda > 0$, $m \geq 1$, $\vec{t} \in [0,\infty)^m$ and every Borel measurable $F : M_F(\mathbb{R}^d)^m \to \mathbb{C}$ bounded by a multinomial and such that $\mathbb{N}_0\pi_{\vec{t}}^{-1}(\mathcal{D}_F) = 0$,*

1.

$$E_{\mu_n}\left[X_s(1)F(\vec{X}_{\vec{t}})\right] \to E_{\mathbb{N}_0}\left[X_s(1)F(\vec{X}_{\vec{t}})\right], \qquad (4)$$

*   and*

2.

$$E_{\mu_n}\left[F(\vec{X}_{\vec{t}})I_{\{X_s(1)>\lambda\}}\right] \to E_{\mathbb{N}_0}\left[F(\vec{X}_{\vec{t}})I_{\{X_s(1)>\lambda\}}\right]. \qquad (5)$$

For critical oriented percolation above the critical spatial dimension of 4 (and for sufficiently spread out kernels) [3, 4] show that

$$\mu_n(S > \epsilon) \to \mathbb{N}_0(S > \epsilon) \text{ for every } \epsilon > 0. \qquad (6)$$

The corresponding results for critical lattice trees and critical contact processes are conjectured to be true above the critical dimension, and are currently work in progress [5, 6]. The next result allows us to strengthen the conclusion of Proposition 2.3 under (6).

Proposition 2.4. *Let $\{\mu_n\}_{n\geq 0}$ be a sequence of finite measures on $D(M_F(\mathbb{R}^d))$ such that (3) and (6) hold. Then $\mu_n \stackrel{f.d.d}{\Longrightarrow} \mathbb{N}_0$.*

In particular the results of [7, 3, 4] together with Proposition 2.4 imply that above the critical dimension and at the critical occupation probability, the scaling limit (in the sense of finite-dimensional distributions) of sufficiently spread-out oriented percolation is CSBM. Tightness, and hence a full statement of weak convergence, remains an open problem.

The additional condition (6) is necessary (consider the test function 1) because μ_n and $\mathbb{N}_0$ are unnormalized. In [1] a conditional limit theorem for rescaled lattice trees above 8 dimensions is proved in which the limit distribution (ISE) is $\mathbb{N}_0\left(\int_0^\infty X_s ds \in \cdot \mid \int_0^\infty X_s(1)\, ds = 1\right)$. The conditioning means all the involved measures are probabilities and so (6) is not needed.

The following assumption will be in force throughout the rest of the paper.

Assumption 2.5. *$\mathcal{F}$ denotes a class of $\mathbb{C}$-valued bounded continuous functions that is closed under conjugation, is convergence determining for $M_F(\mathbb{R}^d)$, and contains the constant function 1.*

We show in Section 4 that both Propositions are consequences of standard exponential moment bounds for $\mathbb{N}_0$ and the following theorem. By convention, an empty product is defined to be 1.

Theorem 2.6. *Let $\mu_n, \mu \in M_F\left(D(M_F(\mathbb{R}^d))\right)$. Suppose that for every $l \in \mathbb{Z}_+$ and $\vec{t} \in [0,\infty)^l$, we have*

  1. there exists a $\delta = \delta(\vec{t}) > 0$ such that for all $\theta_i < \delta$, $E_\mu[e^{\sum_{i=1}^l \theta_i X_{t_i}(1)}] < \infty$,
  2. for every $\phi_i \in \mathcal{F}$

$$E_{\mu_n}\left[\prod_{i=1}^l X_{t_i}(\phi_i)\right] \to E_\mu\left[\prod_{i=1}^l X_{t_i}(\phi_i)\right] < \infty. \qquad (7)$$

Then for every $l \in \mathbb{N}$ and every $\vec{t} \in [0,\infty)^l$, $\mu_n \pi_{\vec{t}}^{-1} \stackrel{w}{\Longrightarrow} \mu \pi_{\vec{t}}^{-1}$.

Note that some of the t_i's may be the same in (7).

The remainder of this paper is organised as follows. In Section 3 we prove Theorem 2.6. In Section 4 we prove Propositions 2.3 and 2.4.

3. Proof of Theorem 2.6

In this section we prove Theorem 2.6 as a consequence of Lemmas (3.2–3.7). Lemma 3.2 is standard and states that if a sequence of finite measures is tight then every subsequence has a further subsequence that converges. Lemma 3.3 establishes tightness of the $\{\mu_n \pi_{\vec{t}}^{-1} : n \in \mathbb{N}\}$ for each fixed $\vec{t}$. Thus every subsequence of the $\mu_n \pi_{\vec{t}}^{-1}$ has a further subsequence that converges. Lemma 3.4 states that any limit point of the $\{\mu_n \pi_{\vec{t}}^{-1} : n \in \mathbb{N}\}$ must have the same moments (7) as $\mu \pi_{\vec{t}}^{-1}$. Lemma 3.5 extends equality of the moments on the right-hand side of

(7) for two measures μ, μ' to all $\phi_i \geq 0$ bounded and continuous. Lemmas 3.6 and 3.7 together imply that under condition 1. of Theorem 2.6, equality of the moments in Lemma 3.5 implies equality of the underlying finite measures on $M_F(\mathbb{R}^d)^m$. Taken together they show that since all subsequential limit points have the same moments (7), the limit points all coincide, and thus the whole sequence converges to that limit point. Thus, Theorem 2.6 follows immediately from the Lemmas proved in this section.

Recall the notion of *tightness* for finite measures.

Definition 3.1. *A set of finite Borel measures $F \subset M_F(E)$ on a metric space E is* tight *if $\sup_{\mu \in F} \mu(E) < \infty$ and for every $\eta > 0$ there exists a compact $K \subset E$ such that $\sup_{\mu \in F} \mu(K^c) < \eta$.*

Lemma 3.2. *If $F \subset M_F(E)$ is tight, then every sequence in F has a further subsequence which converges in $M_F(E)$ (weak convergence).*

Lemma 3.3. *Let $\mu_n, \mu \in M_F(D)$. Suppose that $E_{\mu_n}[1] \to E_\mu[1] < \infty$ and that for every $t \in [0, \infty)$, and every $\phi \in \mathcal{F}$,*

$$E_{\mu_n}[X_t(\phi)] \to E_\mu[X_t(\phi)] < \infty. \tag{8}$$

Then for each $m \in \mathbb{Z}_+$ and every $\vec{t} \in [0, \infty)^m$, the set of measures $\{\mu_n \pi_{\vec{t}}^{-1} : n \in \mathbb{N}\}$ is tight on $M_F(\mathbb{R}^d)^m$.

Proof. That $\sup_n \mu_n \pi_{\vec{t}}^{-1}\left(M_F(\mathbb{R}^d)^m\right) < \infty$ for every $m, \vec{t}$ is trivial (as is the $m = 0$ case) since $E_{\mu_n}[1] \to E_\mu[1] < \infty$. It remains to prove the existence of the appropriate compact set for $m \geq 1$.

For $m = 1$, let $\epsilon > 0$ and $t \geq 0$. Define the mean measures $\nu_n, \nu \in M_F(\mathbb{R}^d)$ by $\nu_n(A) = E_{\mu_n}[X_t(A)]$, and $\nu(A) = E_\mu[X_t(A)]$. Then (8) implies $\nu_n \to \nu$ in $M_F(\mathbb{R}^d)$ and $\sup_n \nu_n(\mathbb{R}^d) \equiv L < \infty$. Choose M such that $L/M < \epsilon/2$. Then by Markov's inequality,

$$\sup_n \mu_n(X_t(\mathbb{R}^d) > M) \leq \frac{L}{M} < \frac{\epsilon}{2}. \tag{9}$$

Fix $\eta > 0$. There exists $K_{-1} \subset \mathbb{R}^d$ compact such that $\nu(K_{-1}^c) < \eta/2$. Furthermore there exists $K_0 \subset \mathbb{R}^d$ compact such that $\nu(\overline{K_0^c}) \leq \nu(K_{-1}^c)$ (e.g. the set $K_0 = \{x : d(x, K_{-1}) \leq 1\}$). Since $\nu_n \to \nu$ in $M_F(\mathbb{R}^d)$ and $\overline{K_0^c}$ is closed,

$$\limsup_n \nu_n(\overline{K_0^c}) \leq \nu(\overline{K_0^c}) < \frac{\eta}{2}.$$

It follows easily that there is a compact subset K of $\mathbb{R}^d$ such that $\sup_n \nu_n(K^c) < \eta$. Another application of Markov's inequality implies that

$$\sup_n \mu_n\left(X_t(K^c) > \eta^{1/4}\right) \leq \eta^{-1/4} \sup_n E_{\mu_n}[X_t(K^c)] < \eta^{3/4}.$$

Choose $\eta^{1/4} = 2^{-j}$. Then there exists $K_j \subset \mathbb{R}^d$ compact such that

$$\sup_n \mu_n\left(X_t(K_j^c) > \frac{1}{2^j}\right) \leq \frac{1}{2^{3j}}. \tag{10}$$

Choose N so that $8^{1-N} < \epsilon/2$ and let

$$\mathbf{K} \equiv \bigcap_{j \geq N} \left\{ Y : Y(K_j^c) \leq \frac{1}{2^j} \right\} \bigcap \{ Y : Y(\mathbb{R}^d) \leq M \}.$$

Now $\mathbf{K}$ is compact in $M_F(\mathbb{R}^d)$ (see for example in the proof of Theorem II.4.1 of [9]), and

$$\mathbf{K}^c = \bigcup_{j \geq N} \left\{ Y : Y(K_j^c) > \frac{1}{2^j} \right\} \bigcup \left\{ Y : Y(\mathbb{R}^d) > M \right\}.$$

Thus, (10) and (9) imply

$$\sup_n \mu_n(X_t \in \mathbf{K}^c) \leq \sum_{j=N}^{\infty} \frac{1}{2^{3j}} + \frac{\epsilon}{2} \leq \frac{1}{8^{N-1}} + \frac{\epsilon}{2} < \epsilon, \tag{11}$$

which verifies that the $\mu_n \pi_t^{-1}$, $n \geq 1$ are tight.

For $m > 1$ and $\vec{t} \in [0, \infty)^m$, we have from (11) that for each $i \in \{1, \ldots, m\}$ there exists $\mathbf{K}_i \subset M_F(\mathbb{R}^d)$ compact such that $\sup_n \mu_n \pi_{t_i}^{-1}(\mathbf{K}_i^c) < \epsilon/m$. Let $\mathbf{K} = \mathbf{K}_1 \times \mathbf{K}_2 \times \cdots \times \mathbf{K}_m$. Then $\mathbf{K} \subset M_F(\mathbb{R}^d)^m$ is compact and

$$\sup_n \mu_n \pi_{\vec{t}}^{-1}(\mathbf{K}^c) \leq \sup_n \sum_{i=1}^{m} \mu_n \pi_{t_i}^{-1}(\mathbf{K}_i^c) < \epsilon,$$

which gives the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Lemma 3.4. *Suppose that $\mu_n, \mu \in M_F(D)$ satisfy the second hypothesis of Theorem 2.6. Fix $l \geq 0$ and $\vec{t} \in [0, \infty)^l$. If, for a given subsequence μ_{n_k}, we have $\mu_{n_k} \pi_{\vec{t}}^{-1} \overset{w}{\Longrightarrow} \nu$ in $M_F\left(M_F(\mathbb{R}^d)^l\right)$, then for each $\vec{m} \in \mathbb{Z}_+^l$ and $\phi_{ij} \in \mathcal{F}$,*

$$E_\nu \left[\prod_{i=1}^{l} \prod_{j=1}^{m_i} Y_i(\phi_{ij}) \right] = E_\mu \left[\prod_{i=1}^{l} \prod_{j=1}^{m_i} X_{t_i}(\phi_{ij}) \right]. \tag{12}$$

Proof. Assume first $\nu(1) > 0$. Since $\mu_{n_k} \pi_{\vec{t}}^{-1}(1) \to \nu(1)$, by normalization we may assume $\mu_{n_k} \pi_{\vec{t}}^{-1}$ are probabilities on $M_F(\mathbb{R}^d)^l$. Let $\vec{m}$ and ϕ_{ij} be as in the Lemma and set $W = \prod_{i=1}^{l} \prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})$ and $W_1 = \prod_{i=1}^{l} \prod_{j=1}^{m_i} X_{t_i}(1)$. Condition 2 from Theorem 2.6 implies $\sup_k \mu_{n_k}(|W|^2) \leq C_{\vec{\phi}} \sup_k \mu_{n_k}(W_1^2) < \infty$ (recall we can repeat t_i's). The assumed weak convergence implies that $\mu_{n_k} W^{-1} \overset{w}{\Longrightarrow} \nu W^{-1}$ as measures in $M_F(\mathbb{C})$. It follows from a standard result in weak convergence (see, e.g., Proposition 2.3 in the Appendix of [2]) that the left-hand side of (12) is equal to $\lim_{k \to \infty} E_{\mu_{n_k}}[W]$. The same is true of the right-hand side by the second hypothesis in Theorem 2.6, where we use a base vector $\vec{t'}$ with appropriately repeated t_i's.

If $\nu(1) = 0$, then $\mu_{n_k}(D) \to 0$ and so if W is as above,

$$\left| \int W \, d\mu_{n_k} \right|^2 \leq \mu_{n_k}(D) \int |W|^2 \, d\mu_{n_k} \to 0,$$

where L^2 boundedness of $|W|$ follows as above. Therefore the right-hand side of (12) is 0 by hypothesis (as above), and thus equals the left-hand side. □

Lemma 3.5. *Suppose $l \geq 0$, $\vec{m} \in \mathbb{Z}_+^l$ and $\mu, \mu' \in M_F\left(M_F(\mathbb{R}^d)^l\right)$. If*

$$E_\mu \left[\prod_{i=1}^l \prod_{j=1}^{m_i} Y_i(\phi_{ij}) \right] = E_{\mu'} \left[\prod_{i=1}^l \prod_{j=1}^{m_i} Y_i(\phi_{ij}) \right] \tag{13}$$

holds (and both quantities are finite) for every $\phi_{ij} \in \mathcal{F}$, then (13) holds for all bounded, continuous $\phi_{ij} \geq 0$.

Proof. If $l = 0$ or $\sum m_i = 0$ then the conclusion is trivial so we may assume that $l > 0$ and $\sum m_i > 0$. Since $1 \in \mathcal{F}$, we have $E_\mu[\prod_{i=1}^l \prod_{j=1}^{m_i} Y_i(1)] < \infty$. Let $\phi_{ij} \in \mathcal{F}$ and $\varphi((x_{ij})) = \prod_{i=1}^l \prod_{j=1}^{m_i} \phi_{ij}(x_{ij})$. Applying Fubini to (13), using the fact that the $\phi_{ij} \in \mathcal{F}$ are bounded, we have

$$\int \cdots \int \varphi((x_{ij})) E_\mu \left[\prod_{i=1}^l \prod_{j=1}^{m_i} Y_i(dx_{ij}) \right] = \int \cdots \int \varphi((x_{ij})) E_{\mu'} \left[\prod_{i=1}^l \prod_{j=1}^{m_i} Y_i(dx_{ij}) \right].$$
$$\tag{14}$$

We claim that for any $r \geq 1$, the set of functions $\mathcal{F}_r \equiv \{\prod_{i=1}^r \phi_i(x_i) : \phi_i \in \mathcal{F}\}$ is a determining class for $M_F(\mathbb{R}^{dr})$. For real-valued functions this is Proposition 3.4.6 of [2]. The fact that $\mathcal{F}$ is closed under conjugation easily implies that it is a determining class for complex-valued measures. This allows us to apply the proof in [2] to verify the claim.

Therefore the products of ϕ_{ij} in (14) uniquely determine the measure ν on $\mathbb{R}^{d \sum m_i}$ defined by $\nu(d\vec{x}) = E_\mu[\prod_{i=1}^l \prod_{j=1}^{m_i} Y_i(dx_{ij})]$. Thus (14) holds for all ϕ_{ij} bounded, continuous. Now apply Fubini again to get (13) for all ϕ_{ij} bounded, continuous as required. □

In the following lemma, $\mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+)$ denotes the bounded, non-negative real-valued functions on $\mathbb{R}^d$, and $\overline{D_0}^{bp}$ denotes the bounded pointwise closure of $D_0 \subset \mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+)$, i.e. the smallest set containing D_0 that is closed under bounded pointwise convergence.

Lemma 3.6. *Suppose $\mu, \mu' \in M_F\left(M_F(\mathbb{R}^d)^m\right)$ and assume $D_0 \subset \mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+)$ satisfies $\overline{D_0}^{bp} = \mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+)$. If for all $h_j \in D_0$*

$$E_\mu \left[e^{-\sum_{j=1}^m Y_j(h_j)} \right] = E_{\mu'} \left[e^{-\sum_{j=1}^m Y_j(h_j)} \right], \tag{15}$$

then $\mu = \mu'$.

Proof. By Dominated Convergence the equality (15) extends to all bounded, non-negative Borel measurable h_j. The result follows by using a standard Monotone Class argument (e.g. see Theorem 4.3 in the Appendix of [2]) on

$$\mathcal{H} \equiv \left\{ \Phi \in \mathcal{B}_b(M_F(\mathbb{R}^d)^m, \mathbb{R}) : E_\mu[\Phi(\vec{Y})] = E_{\mu'}[\Phi(\vec{Y})] \right\}. \tag{16}$$

$\square$

Lemma 3.7. *Let $\mu \in M_F\left(M_F(\mathbb{R}^d)^m\right)$. Suppose there exists a $\delta > 0$ such that for all $\theta_i < \delta$,*

$$E_\mu\left[e^{\sum_{i=1}^m \theta_i Y_i(1)}\right] < \infty. \tag{17}$$

Then for every bounded continuous $0 \leq \psi_i$, the quantity $E_\mu\left[e^{-\sum_{i=1}^m Y_i(\psi_i)}\right]$ is uniquely determined by the collection of mixed moments $\{E_\mu[\prod_{i=1}^m Y_i(h_i)^{n_i}] : 0 \leq h_i \leq 1$ is continuous, $i = 1, \ldots, m.\}$

Proof. Without loss of generality we may assume that $m > 0$. Given bounded continuous $\psi_i \geq 0$, define $h_i = \psi_i / \|\psi_i\|_\infty \in [0, 1]$ (set $h_i = 0$ if $\psi_i \equiv 0$).

For $\text{Re}z_i < \delta$, $i = 1, \ldots, m$, let

$$f(z_1, \ldots, z_m) = E_\mu[e^{\vec{z} \cdot \vec{Y}(\vec{h})}].$$

Use (17), the Taylor expansion for the exponential function and Fubini's theorem to see that for $\|\vec{z}\|_\infty < \delta$,

$$f(z_1, \ldots, z_m) = \sum_{l=0}^\infty \frac{1}{l!} E_\mu\left[\sum_{\substack{\vec{n} \in \mathbb{Z}_+^m : \\ \sum n_i = l}} \frac{l!}{\prod_{i=1}^m n_i!} \prod_{i=1}^m (z_i Y_i(h_i))^{n_i}\right].$$

Hence the mixed moments of the form

$$E_\mu\left[\prod_{i=1}^m Y_i(h_i)^{n_i}\right], \; n_i \in \mathbb{Z}_+ \tag{18}$$

uniquely determine $f(z)$ for $\|\vec{z}\|_\infty < \delta$. A simple application of Dominated Convergence and (17) allows us to differentiate through the expectation and show that for fixed $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m$ satisfying $\text{Re}z_i < \delta$ for $i \neq j$, $f(z)$ is analytic in $\text{Re}z_j < \delta$ (and not just $|z_j| < \delta$). Now use induction on $j \leq m$ to see that moments of the form (18) uniquely determine $f(z_1, \ldots, z_m)$ for $\text{Re}z_1, \ldots, \text{Re}z_{j-1} < \delta, |z_j| \vee \cdots \vee |z_m| < \delta$. Here one used the aforementioned analyticity in $\text{Re}z_j < \delta$ in the induction step. Apply this result at $z_i = -\|\psi_i\|_\infty$ to complete the proof. $\square$

4. Applications of Theorem 2.6

In this section we prove Propositions 2.3 and 2.4, which relate the asymptotic formulae for the $\hat{r}$-point functions for various spread-out models above their critical dimensions to the convergence to CSBM. Recall that $\phi_k(x) = e^{ik \cdot x}$. In this section we fix our convergence determining class of functions for $M_F(\mathbb{R}^d)$ to be

$$\mathcal{F} = \{\phi_k : k \in \mathbb{R}^d\}, \tag{19}$$

which clearly satisfies Assumption 2.5.

The following Lemma will be used to verify the exponential moment hypothesis of Theorem 2.6 for $\mathbb{N}_0$. The branching and diffusion parameters for $\mathbb{N}_0$ are taken to be 1. The Lemma is well-known but we include a proof for completeness.

Lemma 4.1. *For every $b \geq 0$ the following hold.*

1. *For every $\lambda > 0$, $\mathbb{N}_0(X_b(1) = \lambda) = 0$.*
2. *For every l and $\vec{t} \in [0,\infty)^l$ there exists a $\delta = \delta(\vec{t}, b) > 0$ such that for $\theta_i < \delta$*
$$E_{\mathbb{N}_0}\left[X_b(1)e^{\sum_{i=1}^l \theta_i X_{t_i}(1)}\right] < \infty. \tag{20}$$

3. *For every m and $\vec{t} \in [0,\infty)^m$ and every $\epsilon > 0$ there exists a $\delta = \delta(\vec{t}, \epsilon) > 0$ such that for $\theta_i < \delta$*
$$E_{\mathbb{N}_0}\left[e^{\sum_{i=1}^m \theta_i X_{t_i}(1)}I_{\{S>\epsilon\}}\right] < \infty. \tag{21}$$

Proof. The first assertion is trivial by (2) and the fact that $\mathbb{N}_0(X_0(1) > 0) = 0$.

As above we may assume $b > 0$ in part 2. The fact that $X_t = 0_M$ for $t \geq S$ $\mathbb{N}_0$-a.e. implies that for each $\eta > 0$,

$$X_b(1) \leq I_{\{S>b\}}C_\eta e^{\eta X_b(1)}, \ \mathbb{N}_0 - \text{a.e.}$$

Therefore part 2 will follow from part 3 with $\epsilon = b = t_{l+1}$ and $m = l + 1$.

For the last claim of the Lemma we abuse our notation and let E_{X_0} also denote expectation for our standard super-Brownian motion starting at X_0. Let $\mathcal{G}_t$ denote the canonical filtration generated by the coordinates X_s of our super-Brownian motion for $s \leq t$. If $H : M_F(\mathbb{R}^d) \to [0,\infty)$ is continuous, then for $t \geq s > 0$,

$$E_{\mathbb{N}_0}[H(X_t)|\mathcal{G}_s] = E_{X_s}[H(X_{t-s})], \ \mathbb{N}_0 - \text{a.e.} \tag{22}$$

This is easily derived, for example, from the convergence of branching random walk to $\mathbb{N}_0$ mentioned in Section 2, the Markov property for branching random walk, and the analogous convergence result for super-Brownian motion (e.g. Theorem II.5.2 of [9]).

We may assume without loss of generality that $0 < \epsilon < t_i < t_{i+1}$ for each i. Observe from (22) that

$$
\begin{aligned}
E_{\mathbb{N}_0}\left[e^{\sum_{i=1}^m \theta_i X_{t_i}(1)}I_{\{S>\epsilon\}}\right] &= E_{\mathbb{N}_0}\left[E_{X_{t_{m-1}}}\left[e^{\theta_m X_{t_m-t_{m-1}}(1)}\right]e^{\sum_{i=1}^{m-1}\theta_i X_{t_i}(1)}I_{\{S>\epsilon\}}\right] \\
&\leq E_{\mathbb{N}_0}\left[e^{2\theta_m X_{t_{m-1}}(1)}e^{\sum_{i=1}^{m-1}\theta_i X_{t_i}(1)}I_{\{S>\epsilon\}}\right],
\end{aligned}
\tag{23}
$$

where the inequality holds for θ_m sufficiently small depending on $t_m - t_{m-1}$ by Lemma III.3.6. of [9]. The last line of (23) has no t_m dependence and proceeding by induction it is enough to show that for sufficiently small $\theta > 0$,

$$E_{\mathbb{N}_0}\left[e^{\theta X_{t_1}(1)}I_{\{S>\epsilon\}}\right] < \infty. \tag{24}$$

For $\theta > 0$ small enough (depending on (ϵ, t_1)), as in (23), the left-hand side is

$$E_{\mathbb{N}_0}\left[E_{\mathbb{N}_0}\left[e^{\theta X_{t_1}(1)}|\mathcal{G}_\epsilon\right]I_{\{S>\epsilon\}}\right] \leq E_{\mathbb{N}_0}\left[e^{2\theta X_\epsilon(1)}I_{\{S>\epsilon\}}\right]$$

$$\leq E_{\mathbb{N}_0}\left[e^{2\theta X_\epsilon(1)}I_{\{X_\epsilon(1)>0\}}\right] \qquad (25)$$

$$= \left(\frac{2}{\epsilon}\right)^2\int_0^\infty e^{2\theta x}e^{-\frac{2x}{\epsilon}}\,dx,$$

where the last equality holds by (2) . The last line of (25) is finite for sufficiently small $\theta > 0$ (depending on ϵ) and the result follows. $\qquad\square$

4.1. Proof of Proposition 2.3.

Define $\mu_{n,s}, \mathbb{N}_{0,s} \in M_F\left(D(M_F(\mathbb{R}^d))\right)$ by

$$\mu_{n,s}(A) = \int_A X_s(1)d\mu_n, \quad \mathbb{N}_{0,s}(A) = \int_A X_s(1)d\mathbb{N}_0. \qquad (26)$$

That these measures are finite follows from the fact that for $s > 0$,

$$\mu_{n,s}(D) = E_{\mu_n}[X_s(1)] \to E_{\mathbb{N}_0}[X_s(1)] < \infty. \qquad (27)$$

For all $l \geq 0$ and $\vec{k} \in \mathbb{R}^{dl}$,

$$E_{\mu_{n,s}}\left[\prod_{i=1}^l X_{t_i}(\phi_{k_i})\right] = E_{\mu_n}\left[X_s(1)\prod_{i=1}^l X_{t_i}(\phi_{k_i})\right]$$

$$\to E_{\mathbb{N}_0}\left[X_s(1)\prod_{i=1}^l X_{t_i}(\phi_{k_i})\right] \qquad (28)$$

$$= E_{\mathbb{N}_{0,s}}\left[\prod_{i=1}^l X_{t_i}(\phi_{k_i})\right],$$

where even in the $l = 0$ case, the presence of the factor $X_s(1)$ ensures that the convergence in (28) follows from (3).

By Lemma 4.1 we have that

$$E_{\mathbb{N}_{0,s}}\left[e^{\sum_{i=1}^m \theta_i X_{t_i}(1)}\right] < \infty, \qquad (29)$$

for $\theta_i > 0$ sufficiently small depending on $\vec{t}$ and s. In view of (27), (28) and (29) we may apply Theorem 2.6 to the measures $\mu_{n,s}, \mathbb{N}_{0,s}$ to get

$$\mu_{n,s}\pi_{\vec{t}}^{-1} \overset{w}{\Longrightarrow} \mathbb{N}_{0,s}\pi_{\vec{t}}^{-1}.$$

Thus (4) holds for every bounded continuous F. The extension to bounded, Borel measurable F satisfying $\mathbb{N}_{0,s}\pi_{\vec{t}}^{-1}(\mathcal{D}_F) = 0$ is standard. For F as in the

theorem we may assume that $F \geq 0$. The extension to F dominated by a multinomial Q is obtained by an easy uniform integrability argument since $\lim_{n\to\infty} E_{\mu_{n,s}}[Q(\vec{X}_{\vec{t}})] = E_{\mathbb{N}_{0,s}}[Q(\vec{X}_{\vec{t}})]$.

To prove the second claim we define

$$G_s \equiv \begin{cases} 0 & , \text{ if } X_s(1) = 0 \\ \frac{I_{\{X_s(1)>\lambda\}}}{X_s(1)} & , \text{ otherwise.} \end{cases}$$

Then G_s is continuous except when $X_s(1) = \lambda$, and is bounded above by $\frac{1}{\lambda}$. Thus, Lemma 4.1 and (4) show that for F as in Proposition 2.3,

$$E_{\mu_n}\left[X_s(1)G_sF(\vec{X}_{\vec{t}})\right] \to E_{\mathbb{N}_0}\left[X_s(1)G_sF(\vec{X}_{\vec{t}})\right],$$

that is,

$$E_{\mu_n}\left[I_{\{X_s(1)>\lambda\}}F(\vec{X}_{\vec{t}})\right] \to E_{\mathbb{N}_0}\left[I_{\{X_s(1)>\lambda\}}F(\vec{X}_{\vec{t}})\right].$$

$\square$

4.2. Proof of Proposition 2.4.

We apply Theorem 2.6 to the finite measures μ_n^ϵ and $\mathbb{N}_0^\epsilon$ defined by

$$\mu_n^\epsilon(\bullet) = \mu_n(\bullet, S > \epsilon), \quad \mathbb{N}_0^\epsilon(\bullet) = \mathbb{N}_0(\bullet, S > \epsilon). \tag{30}$$

Fix $l \in \mathbb{Z}_+$ and $\vec{t} \in [0,\infty)^l$. By Lemma 4.1, for $\delta(\vec{t},\epsilon) > 0$ sufficiently small and for $\theta_i < \delta$,

$$E_{\mathbb{N}_0^\epsilon}\left[e^{\sum_{i=1}^l \theta_i X_{t_i}(\mathbb{R}^d)}\right] < \infty,$$

so that the first condition of Theorem 2.6 is satisfied. The second condition is trivially true if any $t_i = 0$ so we assume that $t_i > 0$ for each i.

Let $\eta > 0$. Fix $l \in \mathbb{Z}_+$, $\vec{k} \in \mathbb{R}^{dl}$, and write $F(\vec{X}_{\vec{t}}(\vec{\phi})) \equiv \prod_{i=1}^l X_{t_i}(\phi_{k_i})$. By hypothesis (repeat t_i's in (3)) we have

$$E_{\mu_n}\left[F^2(\vec{X}_{\vec{t}}(\vec{1}))\right] \to E_{\mathbb{N}_0}\left[F^2(\vec{X}_{\vec{t}}(\vec{1}))\right] < \infty,$$

so there exists $C_0(\vec{t})$ such that $\sup_n E_{\mu_n}\left[F^2(\vec{X}_{\vec{t}}(\vec{1}))\right]^{\frac{1}{2}} \leq C_0$. Choose $\lambda_0 = \lambda_0(\eta, C_0)$ sufficiently small so that

$$\mathbb{N}_0\left(X_\epsilon(1) \in (0,\lambda_0]\right) < \left(\frac{\eta}{6C_0}\right)^2. \tag{31}$$

By part 2 of Proposition 2.3 with $F \equiv 1$ we have

$$\mu_n\left(X_\epsilon(1) > \lambda_0\right) \to \mathbb{N}_0\left(X_\epsilon(1) > \lambda_0\right).$$

Combining this with (6) gives $\mu_n\left(X_\epsilon(1) \in (0, \lambda_0]\right) \to \mathbb{N}_0\left(X_\epsilon(1) \in (0, \lambda_0]\right)$. It follows from (31) that there exists n_0 such that for all $n \geq n_0$,

$$\mu_n\left(X_\epsilon(1) \in (0, \lambda_0]\right) < \left(\frac{\eta}{3C_0}\right)^2.$$

Using $I_{\{S>\epsilon\}} = I_{\{X_\epsilon(1)>\lambda_0\}} + I_{\{X_\epsilon(1)\in(0,\lambda_0]\}}$, $\mathbb{N}_0$-a.e. , we have

$$\left| E_{\mu_n}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{S>\epsilon\}}\right] - E_{\mathbb{N}_0}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{S>\epsilon\}}\right] \right|$$
$$\leq \left| E_{\mu_n}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{X_\epsilon(1)>\lambda_0\}}\right] - E_{\mathbb{N}_0}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{X_\epsilon(1)>\lambda_0\}}\right] \right| \qquad (32)$$
$$+ \left| E_{\mu_n}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{X_\epsilon(1)\in(0,\lambda_0]\}}\right] \right| + \left| E_{\mathbb{N}_0}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{X_\epsilon(1)\in(0,\lambda_0]\}}\right] \right|.$$

We bound the right-hand side of (32) as follows. By part 2 of Proposition 2.3, the first absolute value is less than $\eta/3$ for n sufficiently large. On the second term we use Cauchy-Schwarz to get

$$E_{\mu_n}\left[|F(\vec{X}_{\vec{t}})|I_{\{X_\epsilon(1)\in(0,\lambda_0]\}}\right] \leq E_{\mu_n}\left[F^2(\vec{X}_{\vec{t}}(\vec{1}))\right]^{\frac{1}{2}} \mu_n\left(X_\epsilon(1) \in (0, \lambda_0]\right)^{\frac{1}{2}} \leq \frac{C_0\eta}{3C_0}.$$

The third term is handled similarly. Thus for n sufficiently large,

$$\left| E_{\mu_n^\epsilon}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))\right] - E_{\mathbb{N}_0^\epsilon}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{S>\epsilon\}}\right] \right| < \eta,$$

which proves the second condition of Theorem 2.6 for $\{\mu_n^\epsilon\}_{n\geq 0}$ and $\mathbb{N}_0^\epsilon$. The result follows by Theorem 2.6. $\qquad\square$

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