The continuous-time lace expansion

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May 23, 2019

Abstract

We derive a continuous-time lace expansion for a broad class of self-interacting continuous-time random walks. Our expansion applies when the self-interaction is a sufficiently nice function of the local time of a continuous-time random walk. As a special case we obtain a continuous-time lace expansion for a class of spin systems that admit continuous-time random walk representations.

We apply our lace expansion to the n-component $g|\varphi|^4$ model on $\mathbb{Z}^d$ when $n = 1, 2$, and prove that the critical Green's function $G_{\nu_c}(x)$ is asymptotically a multiple of $|x|^{2-d}$ when $d \geq 5$. As another application of our method we establish the analogous result for the lattice Edwards model at weak coupling.

1 Introduction

Many lattice spin systems are expected to exhibit mean-field behaviour on $\mathbb{Z}^d$ when $d > d_c = 4$, and several results along these lines have been proven by making use of *random walk representations* [1, 2, 16, 14]. In this paper we will establish a very precise statement about the mean-field behaviour of certain $O(n)$-invariant spin models by making use of the random walk representation that originated in the work of Symanzik [42] and was developed in [10, 13]. For recent developments regarding this representation see [3, 43], and for alternative random walk representations see [2, 22].

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To be more precise, this article is concerned with the asymptotics of the infinite volume critical two-point function $\langle \varphi_a \cdot \varphi_b \rangle$. Here $\langle \cdot \rangle$ denotes the expectation of an $O(n)$-invariant $g|\varphi|^4$ spin model; the spins φ take values in $\mathbb{R}^n$. The definitions of these models are given in Section 3, and our results mostly concern the cases $n = 1, 2$. Let $|x|$ and $|\varphi|$ denote the Euclidean norms of $x \in \mathbb{Z}^d$ and $\varphi \in \mathbb{R}^n$.

Theorem 1.1. *Let $d > d_c = 4$ and $n \in \{1, 2\}$. Let $\langle \cdot \rangle$ denote expectation with respect to the critical n-component $g|\varphi|^4$ model. For $g > 0$ sufficiently small there is a constant $C > 0$ such that*

$$\langle \varphi_a \cdot \varphi_b \rangle \sim \frac{C}{|b-a|^{d-2}}, \qquad \text{as } |b-a| \to \infty. \tag{1.1}$$

The relation $\sim$ in (1.1) means the ratio of the left-hand and right-hand sides tends to one in the designated limit. Our theorem exhibits mean-field behaviour in the sense that the exponent $d - 2$ in (1.1) is the exponent predicted by Landau's extension of mean-field theory [27, Chapter 2]. The right hand side of (1.1) is Euclidean invariant, so for weak coupling the conclusion strengthens existing triviality results [1, 16] by showing that the scaling limit of the two-point function of this model is Euclidean invariant and equals the massless free field two-point function. When $n = 1$ Theorem 1.1 was first proven by Sakai [35]. The case $n = 2$ is new. For $d = d_c$ the asymptotics in (1.1) have been established by a rigorous renormalization group technique for the n-component $g|\varphi|^4$ model for all $n \in \mathbb{N}$ [40].

Sakai's proof of the $n = 1$ case of Theorem 1.1 made use of the *lace expansion*, a technique originally introduced to prove mean-field behaviour for discrete-time weakly self-avoiding walk [8]. The lace expansion has since been reformulated in many different settings: unoriented and oriented percolation [19, 32, 50], the contact process [49], lattice trees and animals [20], Ising and $g|\varphi|^4$ models [34, 35], and various self-interacting random walk models [47, 17, 23, 45]. Within these settings the lace expansion has been applied to a variety of problems, ranging from proofs of weak convergence on path space for branching particle systems [11, 48, 25] to proofs of monotonicity properties of self-interacting random walks [46, 24, 26]. In each case the expansion is based on a discrete parameter that plays the role of time.

To prove Theorem 1.1 we introduce a lace expansion in continuous time. Our methods naturally apply to a broader class of problems than $g|\varphi|^4$ models, and to illustrate this we also analyze the lattice Edwards model. A precise formulation of our main results is given in Section 3, after the introduction of the basic objects of our paper.

2 Random walk and local times

To fix notation and assumptions, we define continuous-time random walk started at a point a in $\mathbb{Z}^d$ and killed outside of a finite subset Λ of $\mathbb{Z}^d$. These stochastic processes are central to the rest of the paper.

2.1 Infinite volume

We begin by defining the class of jump distributions that we will allow. Recall that a one-to-one map T from the vertex set of $\mathbb{Z}^d$ onto itself such that edges $\{x, y\}$ of $\mathbb{Z}^d$ are mapped to edges $\{Tx, Ty\}$ of $\mathbb{Z}^d$ is called an *automorphism*. Let $\mathrm{Aut}_0(\mathbb{Z}^d)$ denote the group of automorphisms that fix the origin 0. A function f on $\mathbb{Z}^d$ is $\mathbb{Z}^d$*-symmetric* if $f(Tx) = f(x)$ for all $x \in \mathbb{Z}^d$ and $T \in \mathrm{Aut}_0(\mathbb{Z}^d)$. Similarly, a function $f(x, y)$ of two variables is $\mathbb{Z}^d$*-symmetric* if $f(Tx, Ty) = f(x, y)$ for all $x, y \in \mathbb{Z}^d$ and $T \in \mathrm{Aut}_0(\mathbb{Z}^d)$.

Assumptions 2.1. *Assume $J\colon \mathbb{Z}^d \to \mathbb{R}$ satisfies*

(J1) $J(x) \geq 0$ for $x \neq 0$, and $J(0) := -\sum_{x\neq 0} J(x)$ is finite,

(J2) the set $\{x \in \mathbb{Z}^d \mid J(x) > 0\}$ is a generating set for $\mathbb{Z}^d$,

(J3) J is $\mathbb{Z}^d$-symmetric,

(J4) J has finite range $R > 0$, i.e., $J(x) = 0$ if $|x| \geq R$.

Henceforth we work with a fixed choice of J satisfying (J1)–(J4). Let $\Delta^{(\infty)}\colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ be the infinite matrix with entries

$$\Delta^{(\infty)}_{x,y} := J(y - x). \tag{2.1}$$

By (J3), $\Delta^{(\infty)}$ is symmetric, and (J1) implies that $\Delta^{(\infty)}$ has non-negative off-diagonal elements and that its row sums are all equal to zero. This implies $\Delta^{(\infty)}$ is the generator of a continuous-time random walk $X^{(\infty)}$ on $\mathbb{Z}^d$. The assumption (J2) ensures this walk is irreducible. Let

$$\hat{J} := -\Delta^{(\infty)}_{x,x} = -J(0), \qquad J_+(y) := J(y)\mathbb{1}_{\{y\neq 0\}}. \tag{2.2}$$

By (J1), $\hat{J}$ is finite. The walk $X^{(\infty)}$ has a mean $\hat{J}^{-1}$ exponential holding time at each x, and jumps from x to $y \neq x$ with probability $J_+(y-x)/\hat{J}$. We write P_a for a probability measure under which $X^{(\infty)}$ is a continuous-time random walk on $\mathbb{Z}^d$ started at $a \in \mathbb{Z}^d$, and E_a for the corresponding expectation.

Example 2.2. *The most important example is when*

$$J(x) = \mathbb{1}_{\{|x|=1\}} - 2d\mathbb{1}_{\{x=0\}}.$$

In this case $\Delta^{(\infty)}$ *is called the* lattice Laplacian, *and* $X^{(\infty)}$ *is a continuous-time nearest-neighbour random walk on* $\mathbb{Z}^d$*.*

2.2 Finite volume

Let Λ be a finite subset of $\mathbb{Z}^d$ and let $T^{(\Lambda)}$ be the first time $X^{(\infty)}$ exits Λ:

$$T^{(\Lambda)} \coloneqq \inf\{t \geq 0 : X_t^{(\infty)} \notin \Lambda\}. \tag{2.3}$$

Let $* \notin \mathbb{Z}^d$ be an additional "cemetery" state, and define $X_t^{(\Lambda)}$ by

$$X_t^{(\Lambda)} = \begin{cases} X_t^{(\infty)}, & t < T^{(\Lambda)}, \\ *, & t \geq T^{(\Lambda)}. \end{cases} \tag{2.4}$$

For each finite Λ the process $X^{(\Lambda)} = (X_t^{(\Lambda)})_{t\geq 0}$ is a continuous-time Markov chain on $\Lambda \cup \{*\}$ with absorbing state $*$. Note that this construction defines the processes $X^{(\Lambda)}$ on the same probability space for all finite $\Lambda \subset \mathbb{Z}^d$.

The generator $\Delta_*^{(\Lambda)}$ of $X^{(\Lambda)}$ is a $(|\Lambda|+1) \times (|\Lambda|+1)$ matrix with a row of zeros since $*$ is an absorbing state. We will let $\Delta^{(\Lambda)} \colon \Lambda \times \Lambda \to \mathbb{R}$ denote the matrix obtained by removing the row and column corresponding to transition rates to and from $*$, i.e., $(\Delta^{(\Lambda)})_{x,y} \coloneqq (\Delta^{(\infty)})_{x,y}$ for $x, y \in \Lambda$.

2.3 Local time and free Green's functions

For $x \in \Lambda$ and a Borel set $I \subset [0,\infty)$ the *local time of* $X^{(\Lambda)}$ *at* x *during* I is

$$\tau_{I,x}^{(\Lambda)} \coloneqq \int_I d\ell \, \mathbb{1}_{\{X_\ell^{(\Lambda)}=x\}}. \tag{2.5}$$

Let $\boldsymbol{\tau}_I^{(\Lambda)} := (\tau_{I,x}^{(\Lambda)})_{x\in\Lambda}$ denote the vector of all local times, and let $\tau_x^{(\Lambda)}$ denote $\tau_{[0,\infty),x}^{(\Lambda)}$. We will often omit the superscript Λ when there is no risk of confusion. For $a, b \in \Lambda$ we define the *free Green's function in* Λ by

$$S^{(\Lambda)}(a,b) := E_a[\tau_b^{(\Lambda)}] = \int_0^\infty d\ell\, P_a(X_\ell^{(\Lambda)} = b). \tag{2.6}$$

Note that $S^{(\Lambda)}(a,b) < \infty$ since the expected time for $X^{(\Lambda)}$ to exit Λ is finite. The next lemma, proved in Appendix A.1, explains why $S^{(\Lambda)}(a,b)$ is called the free Green's function.

Lemma 2.3. $\Delta^{(\Lambda)}$ *is invertible, and for* $a, b \in \Lambda$

$$S^{(\Lambda)}(a,b) = (-\Delta^{(\Lambda)})^{-1}_{a,b}. \tag{2.7}$$

There is also an infinite volume version of Lemma 2.3, and it is nicer because the infinite volume limit restores translation invariance. For $d \geq 3$, define $S(a,b)$ to be the expected time spent at b by the random walk $X^{(\infty)}$ started from a, i.e.,

$$S(a,b) := E_a[\tau_b^{(\infty)}], \tag{2.8}$$

where $\tau_b^{(\infty)}$ is defined as in (2.5) but with $X_\ell^{(\Lambda)}$ replaced by $X_\ell^{(\infty)}$. Since $T^{(\Lambda)} \uparrow \infty$ as $\Lambda \uparrow \mathbb{Z}^d$ a.s., monotone convergence implies that

$$S^{(\Lambda)}(a,b) = E_a[\tau^{(\infty)}_{[0,T^{(\Lambda)}],b}] \uparrow E_a[\tau^{(\infty)}_{[0,\infty),b}] = S(a,b) < \infty. \tag{2.9}$$

where the finiteness holds by transience. The translation invariance of the infinite volume random walk implies

$$S(x) := S(a, a+x), \qquad x, a \in \mathbb{Z}^d \tag{2.10}$$

is well-defined, i.e., independent of $a \in \mathbb{Z}^d$. Recall, see [28, Theorem 4.3.5], that there is a C_J depending on J such that

$$S(x) \sim \frac{C_J}{|x|^{d-2}}. \tag{2.11}$$

Since $S(x)$ is positive for all x this implies that there is a constant c_J depending on J such that $S(x) \geq \frac{c_J}{|x|^{d-2}}$ for $x \neq 0$.

Recall that the discrete convolution $f * h$ of functions f and h on $\mathbb{Z}^d$ is defined by $f * h(x) := \sum_{y\in\mathbb{Z}^d} f(y)h(x-y)$. For $n \in \mathbb{N}$ let f^{*n} denote the n-fold convolution of f with itself, and let $f^{*0}(x) = \mathbb{1}_{\{x=0\}}$. The next lemma is proved in Appendix A.1.

Lemma 2.4. *Suppose $d \geq 3$. For S defined by (2.10) and $x \in \mathbb{Z}^d$,*

$$J * S(x) = S * J(x) = -\mathbb{1}_{\{x=0\}}. \tag{2.12}$$

Moreover, recalling $\hat{J}$ and J_+ from (2.2),

$$S(x) = \sum_{n \geq 0} \hat{J}^{-(n+1)} \, J_+^{*n}(x), \tag{2.13}$$

where the right-hand side converges absolutely.

2.4 A convenient technical choice

In this section we make a specific choice for the measurable space $(\Omega_1, \mathcal{F})$ on which $X^{(\infty)}$ is defined so that the paths of $X^{(\infty)}$ have desirable regularity properties. This reduces the number of statements that have to be qualified as holding almost surely (a.s.). Let

$$\Omega_1 = \left\{ X^{(\infty)} \colon [0, \infty) \to \mathbb{Z}^d \,\middle|\, X^{(\infty)} \text{ is càdlàg} \right\}, \tag{2.14}$$

where we recall a function is càdlàg if it is right continuous with left limits. Let $(\mathcal{F}_t)_{t \geq 0}$ denote the natural filtration of $X^{(\infty)}$, i.e., $\mathcal{F}_t$ is the smallest σ-algebra on Ω_1 such that $\{X^{(\infty)} \mid X^{(\infty)}_s = y\} \in \mathcal{F}_t$ for each $s \in [0, t]$ and $y \in \mathbb{Z}^d$, and let $\mathcal{F}$ denote the smallest σ-algebra on Ω_1 containing $\cup_{t \geq 0} \mathcal{F}_t$. Henceforth we let P_a denote the probability measure on $(\Omega_1, \mathcal{F})$ under which $X^{(\infty)}$ is a continuous-time random walk on $\mathbb{Z}^d$ started at $a \in \mathbb{Z}^d$.

3 The Green's function

In this section we define the object $G^{(\Lambda)}_{\boldsymbol{t}}(a, b)$ at the center of our results. It is called the Green's function and it is an integral over all walks, of varying continuous-time duration ℓ, that start at a and end at b. We have two motivations for studying this Green's function. The first is that two-point correlations of lattice spin models such as the n-component $g|\varphi|^4$ model have random walk representations in terms of this Green's function. We state this representation below, see Definition 3.2 and Theorem 3.3. The second motivation is that it is a departure point for the study of random walks with self-interactions that are functions of local time. These are of interest in chemistry, physics, and probability; they include a canonical model of self-avoiding walk, the lattice Edwards model, as a special case. We define the lattice Edwards model in Definition 3.1.

Fix a finite set $\Lambda \subset \mathbb{Z}^d$. Let A^Λ denote the set of sequences $(x_v)_{v\in\Lambda}$ with each x_v in A, and let $Z\colon\ [0,\infty)^\Lambda \to (0,\infty)$, $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ be a bounded continuous positive function. For a random variable $\boldsymbol{\sigma}$ taking values in $[0,\infty)^\Lambda$, $Z_{\boldsymbol{\sigma}}$ denotes Z evaluated at the random point $\boldsymbol{\sigma}$. For $\boldsymbol{t}, \boldsymbol{s} \in [0,\infty)^\Lambda$ let

$$Y_{\boldsymbol{t},\boldsymbol{s}} := \frac{Z_{\boldsymbol{t}+\boldsymbol{s}}}{Z_{\boldsymbol{t}}}. \tag{3.1}$$

For $a, b \in \Lambda$ and $\boldsymbol{t} \in [0,\infty)^\Lambda$ define the *Green's function*

$$G_{\boldsymbol{t}}^{(\Lambda)}(a,b) := \int_{[0,\infty)} d\ell\ \ E_a\left[Y_{\boldsymbol{t},\boldsymbol{\tau}^{(\Lambda)}_{[0,\ell]}}\mathbb{1}_{\{X_\ell^{(\Lambda)}=b\}}\right]. \tag{3.2}$$

Note that $G_{\boldsymbol{t}}^{(\Lambda)}(a,b) > 0$ since $Z_{\boldsymbol{t}}$ is continuous and positive. We extend the definition (3.2) by setting $G_{\boldsymbol{t}}^{(\Lambda)}(a,b) = 0$ if a or b is the cemetery state $*$.

The free Green's function $S^{(\Lambda)}(a,b)$ is the special case of $G_{\boldsymbol{t}}^{(\Lambda)}$ when $Z := 1$, see (2.6). For each $\boldsymbol{t}$ the function $Y_{\boldsymbol{t},\boldsymbol{\tau}_{[0,\ell]}}$ is bounded as a function of $\omega \in \Omega_1$ and $\ell \in [0,\infty)$ because $Z_{\boldsymbol{t}}$ is bounded and positive. By (2.9) this implies

$$G_{\boldsymbol{t}}^{(\Lambda)}(a,b) < \infty, \qquad \boldsymbol{t} \in [0,\infty)^\Lambda. \tag{3.3}$$

Our primary interest in this paper is $G_{\boldsymbol{0}}^{(\Lambda)}(a,b)$ given by (3.2) when $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ is one of the choices described in the next two sections. Both choices involve parameters $g > 0$ and $\nu \in \mathbb{R}$ called *coupling constants*.

3.1 The Edwards model

Definition 3.1. *Fix $g > 0$ and $\nu \in \mathbb{R}$. The* Green's function $G_{\boldsymbol{t}}^{(\Lambda)}(a,b)$ of the (lattice) Edwards model *is given by* (3.2) *and* (3.1) *with the choice*

$$Z_{\boldsymbol{t}} := \exp\left\{-g\sum_{x\in\Lambda} t_x^2 - \nu \sum_{x\in\Lambda} t_x\right\}. \tag{3.4}$$

To explain Definition 3.1 note that

$$\sum_x \tau^2_{[0,\ell],x} = \iint_{[0,\ell]^2} ds\, dr\, \mathbb{1}_{\{X_s^{(\Lambda)}=X_r^{(\Lambda)}\}}$$

is the time $X^{(\Lambda)}$ spends intersecting itself up to time ℓ. Since $g > 0$, the choice of $Z_{\boldsymbol{t}}$ in Definition 3.1 weights a walk in (3.2) by the exponential of minus its self-intersection time: self-intersection is discouraged. The parameter $\nu \in \mathbb{R}$ is called the *chemical potential*, and it controls the expected length of a walk. Thus the Edwards model is a continuous time self-avoiding walk. See [5] for further details and background on this model.

3.2 The $g|\varphi|^4$ models

Our second choice of $Z_{\boldsymbol{t}}$ requires some preparation. Let $\mathbb{R}^{n\Lambda} := (\mathbb{R}^n)^\Lambda$, $\boldsymbol{\varphi} := (\varphi_x)_{x\in\Lambda}$ be a point in $\mathbb{R}^{n\Lambda}$, and let $\varphi_x^{[i]}$ denote the ith component of $\varphi_x \in \mathbb{R}^n$. Define a Gaussian measure P on the Borel sets of $\mathbb{R}^{n\Lambda}$ in terms of a density with respect to Lebesgue measure $d\boldsymbol{\varphi}$ on $\mathbb{R}^{n\Lambda}$ by

$$dP(\boldsymbol{\varphi}) := Ce^{\frac{1}{2}(\boldsymbol{\varphi},\Delta^{(\Lambda)}\boldsymbol{\varphi})}\, d\boldsymbol{\varphi}, \tag{3.5}$$

where C normalises the measure to have total mass one and the quadratic form $(\boldsymbol{\varphi}, \Delta^{(\Lambda)}\boldsymbol{\varphi})$ is defined by:

$$(\Delta^{(\Lambda)}\boldsymbol{\varphi})_x^{[i]} := \sum_{y\in\Lambda} \Delta^{(\Lambda)}_{x,y}\boldsymbol{\varphi}_y^{[i]}, \qquad (f,h) := \sum_{x\in\Lambda}\sum_{i=1}^{n} f_x^{[i]} h_x^{[i]}. \tag{3.6}$$

The covariance of $\boldsymbol{\varphi}$ under P is the $n|\Lambda| \times n|\Lambda|$ positive definite matrix $(-\Delta^{(\Lambda)})^{-1}_{x,y}\delta_{i,j}$; positive definiteness follows from (A.3) in Appendix A.1. By Lemma 2.3,

$$\int_{\mathbb{R}^{n\Lambda}} dP(\boldsymbol{\varphi})\, \varphi_x^{[i]}\varphi_y^{[j]} = S^{(\Lambda)}(x,y)\delta_{i,j}. \tag{3.7}$$

Definition 3.2. *Fix $g > 0$, $\nu \in \mathbb{R}$, and $n \in \mathbb{N}_{\geq 1}$. The* Green's function $G^{(\Lambda)}_{\boldsymbol{t}}(a,b)$ of the n-component $g|\varphi|^4$ model *is given by* (3.2) *and* (3.1) *with the choice*

$$Z_{\boldsymbol{t}} := \int_{\mathbb{R}^{n\Lambda}} dP(\boldsymbol{\varphi})\ \exp\left\{-\sum_{x\in\Lambda}\Big(g(|\varphi_x|^2+2t_x)^2 + \nu(|\varphi_x|^2+2t_x)\Big)\right\}. \tag{3.8}$$

The justification for Definition 3.2 is given by the next theorem.

Theorem 3.3. *Let $G^{(\Lambda)}_{\boldsymbol{0}}(a,b)$ be given by Definition 3.2. Then*

$$G^{(\Lambda)}_{\boldsymbol{0}}(a,b) = \frac{1}{n}\langle\boldsymbol{\varphi}_a\cdot\boldsymbol{\varphi}_b\rangle_{g,\nu}, \tag{3.9}$$

where $\langle \cdot \rangle_{g,\nu}$ denotes expectation with respect to the probability measure

$$dQ(\boldsymbol{\varphi}) := \frac{1}{Z_{\mathbf{0}}} e^{-\frac{1}{2}(\boldsymbol{\varphi}, -\Delta^{(\Lambda)}\boldsymbol{\varphi})} \prod_{x\in\Lambda} \Big(e^{-V(|\varphi_x|^2)} d\varphi_x \Big), \qquad \boldsymbol{\varphi} \in \mathbb{R}^{n\Lambda}, \tag{3.10}$$

where $V(\psi) = g\psi^2 + \nu\psi$ and $Z_{\mathbf{0}}$ is defined by (3.8).

The quantity $\frac{1}{n}\langle \boldsymbol{\varphi}_a \cdot \boldsymbol{\varphi}_b \rangle_{g,\nu}$ in (3.9) is the standard definition of the n-component $g|\varphi|^4$ two-point function, see, e.g., [4, Section 1.6]. Note this references writes $\langle \boldsymbol{\varphi}_a^{[1]} \boldsymbol{\varphi}_b^{[1]} \rangle_{g,\nu}$ in place of $\frac{1}{n}\langle \boldsymbol{\varphi}_a \cdot \boldsymbol{\varphi}_b \rangle_{g,\nu}$ which is the same by $O(n)$ invariance.

Proof of Theorem 3.3. The theorem is a consequence of the BFS-Dynkin isomorphism as formulated in [4, Theorem 11.2.3][1] with $\beta_{xy} = J(x-y)$ and $F\colon [0,\infty)^\Lambda \to \mathbb{R}$ defined by

$$F(\boldsymbol{t}) = \exp\big[-\sum_{x\in\Lambda} \big(\gamma_x t_x - V(2t_x)\big)\big]. \tag{3.11}$$

The coefficient γ_x is such that the quadratic form $(\boldsymbol{\varphi}, \Delta^{(\Lambda)}\boldsymbol{\varphi})$ defined in (3.6) equals the quadratic form $(\boldsymbol{\varphi}, -\Delta_\beta \boldsymbol{\varphi})$ in [4, Theorem 11.2.3] plus $\sum_x \gamma_x \frac{1}{2}|\varphi_x|^2$. From (A.3) and [4, (1.3.3)], $\gamma_x = \sum_{y\notin\Lambda} J(x-y)$. ■

3.3 Main result

Our main result concerns the infinite volume limit of the Green's function for the examples in the previous sections.

Lemma 3.4 (Proof in Section 11)**.** *Let $a, b \in \Lambda$. $G_{\mathbf{0}}^{(\Lambda)}(a,b)$ is non-decreasing in Λ for the $n = 1, 2$-component $g|\varphi|^4$ and Edwards models.*

Lemma 3.4 implies that for each of our examples

$$G_{g,\nu}(x) = G_{g,\nu}^{(\infty)}(x) := \lim_{\Lambda\uparrow\mathbb{Z}^d} G_{\mathbf{0}}^{(\Lambda)}(a, a+x) \tag{3.12}$$

is either infinite or exists, is positive and translation invariant, and is $\mathbb{Z}^d$-symmetric. It is in fact known that the limit is finite, and our results prove this for $d \geq 5$ and $\nu \geq \nu_c$, where ν_c is defined below in (3.13). The subscripts g and ν denote the coupling constants in the examples. A closely related property of our models is expressed by

[1] Note an unfortunate clash of notation: in [4, Theorem 11.2.3] the notation $\tau_x = \frac{1}{2}|\varphi_x|^2$ is used, while the vector of local times of the walk up to time T is denoted therein by L_T.

Lemma 3.5 (Proof in Section 11)**.** *For each $x \in \mathbb{Z}^d$, $G_{g,\nu}(x)$ is non-increasing in ν for the $n = 1, 2$-component $g|\varphi|^4$ and Edwards models.*

This lemma motivates defining the *critical value* of ν by

$$\nu_c := \inf \{\nu \in \mathbb{R} \mid \sum_{x \in \mathbb{Z}^d} G_{g,\nu}(x) < \infty\}. \tag{3.13}$$

It will be a conclusion of our theorems that ν_c is finite. Since our models depend on g, ν_c is a function of g and, when necessary, we write $\nu_c = \nu_c(g)$. When $\nu = \nu_c$ we say that the Green's function is *critical*.

Our main result is the following more precise version of Theorem 1.1.

Theorem 3.6. *Suppose $d \geq 5$, J satisfies (J1)–(J4), and consider the Edwards and the $g|\varphi|^4$ models given by Definition 3.1 and Definition 3.2 with $n = 1, 2$. For each model there exists $g_0 = g_0(J) > 0$ such that for $g \in (0, g_0)$ $\nu_c(g)$ is finite and there exists $C = C(g, J) > 0$ such that*

$$G_{g,\nu_c}(x) \sim \frac{C}{|x|^{d-2}}, \qquad \text{as } |x| \to \infty. \tag{3.14}$$

Theorem 3.6 describes mean field asymptotics of the infinite volume Green's function at the critical point, c.f. (2.11). The restriction to $n = 1, 2$ for the $g|\varphi|^4$ models is necessary because our proof uses the Griffiths II inequality, which is not known to hold for $n > 2$.

The proof of Theorem 3.6 occupies Sections 4 through 11. Section 4 serves as an overview of lace expansion methods and reduces a key step of our argument to some auxiliary lemmas. The remainder of the argument is comprised of three parts: the derivation of a lace expansion in finite volume (Sections 5 and 6), establishing an infinite volume expansion (Section 7 through 9), the analysis of this expansion (Section 10), and the application of this analysis to our examples (Section 11). The contents of individual sections will be discussed locally.

3.4 Related lace expansion results

Sakai has proved similar results for the Green's function of the Ising model and the scalar (i.e., $n = 1$) $g|\varphi|^4$ model by lace expansion [34, 35]. Sakai first studied the Ising model by lace expanding the random current representation. By using the Griffiths-Simon trick [37], which represents the scalar $g|\varphi|^4$

model in terms of Ising models, Sakai then studied scalar $g|\varphi|^4$. Since it is unclear how to use the Griffiths-Simon approximation when $n = 2$, we have developed alternative methods.

4 Infrared bound and overview

A key step in the proof of Theorem 3.6 is to obtain the upper bound on $G_{g,\nu_c}(x)$ provided by Theorem 4.1. This section begins the proof of Theorem 4.1 by reducing it to lemmas which will be proved in later sections. Our reduction reviews the guiding ideas of proofs by lace expansion, which are explained in more detail and attribution in [38]. See also [21, 6].

Recall the definitions of the Edwards model and the $g|\varphi|^4$ model from Sections 3.1 and 3.2, and that the infinite volume Green's functions $G_{g,\nu}(x)$ for these models are given by (3.12). We are mainly interested in the case where $\nu = \nu_c$, the critical value given by (3.13). In this section we do not make our standing Assumptions 2.1 explicit in theorem statements.

Theorem 4.1. *Suppose $d \geq 5$. For the $n = 1, 2$-component $g|\varphi|^4$ and Edwards models there are $g_0 = g_0(d, J) > 0$ such that if $0 < g < g_0$ then $\nu_c(g)$ is finite and*

$$G_{g,\nu_c}(x) \leq 2S(x), \qquad x \in \mathbb{Z}^d. \tag{4.1}$$

This is called an *infrared bound*. Infrared bounds in Fourier space for nearest neighbour models (i.e., J as in Example 2.2) were first proved for $n \geq 1$ and $d > 2$ in [15] with 2 replaced by 1. The relation between Fourier infrared bounds and (4.1) is non-trivial, see [41, Appendix A] and [31, Example 1.6.2].

The proof of Theorem 4.1 also yields bounds on the critical value ν_c:

Proposition 4.2 (Proof in Section 11)**.** *For the $n = 1, 2$-component $g|\varphi|^4$ and Edwards models, $\nu_c = O(g)$ as $g \downarrow 0$.*

In Proposition 4.2 and in what follows, for functions f, r, the notation $f(x) = O(r(x))$ as $x \to a$ has its standard meaning, i.e., that there exists a $C > 0$ such that $|f(x)| \leq Cr(x)$ if x is sufficiently close to a.

4.1 The infrared bound

Recall that S is the free Green's function from (2.8).

Proposition 4.3. *Suppose $d \geq 5$. For the $n = 1, 2$-component $g|\varphi|^4$ and Edwards models, there are $g_0 = g_0(d, J) > 0$ such that if $0 < g < g_0$ then*

$$G_{g,\nu} \leq 2S, \qquad \text{for } \nu > \nu_c. \tag{4.2}$$

Here we include case $\nu_c(g) = -\infty$.

Before giving the proof we discuss the proof strategy and state some preparatory results. Lace expansion arguments have been reduced to three schematic steps, all for $\nu > \nu_c$. As we discuss these steps it will be helpful to recall (2.12), i.e., $J * S = -\mathbb{1}_{\{x=0\}}$. This is equivalent to

$$\hat{J}S(x) = \mathbb{1}_{\{x=0\}} + J_+ * S(x), \tag{4.3}$$

where $\hat{J}$ and J_+ were defined in (2.2).

Step one We will call a bound of the form

$$G_{g,\nu} \leq KS \tag{4.4}$$

a *K-infrared bound* or *K-IRB*. Step one assumes a 3-IRB and uses the assumption that g is sufficiently small to prove that there exist $L_{g,\nu} = O(g) \in \mathbb{R}$ and an $O(g)$ integrable function $\Psi_{g,\nu} \colon \mathbb{Z}^d \to \mathbb{R}$ such that for all x

$$(\hat{J} - L_{g,\nu})G_{g,\nu}(x) = \mathbb{1}_{\{x=0\}} + J_+ * G_{g,\nu}(x) + \Psi_{g,\nu} * G_{g,\nu}(x). \tag{4.5}$$

This is a generalization of (4.3). The term involving $L_{g,\nu}$ could be absorbed into $\Psi_{g,\nu}$, but (4.5) is more convenient for future developments. Step one also provides a formula for $\Psi_{g,\nu}$ that shows $\Psi_{g,\nu}$ is small relative to J. The formula for $\Psi_{g,\nu}$ is called the *lace expansion*.

Step two Step two assumes $\Psi_{g,\nu}$ is small relative to J and shows that (4.5) implies that $G_{g,\nu}(x)$ satisfies a 2-IRB. Steps one and two combined show that a 3-IRB implies a 2-IRB.

Step three Step three removes the 3-IRB assumption of step one: the desired (4.2) holds unconditionally. The removal of the 3-IRB assumption uses continuity of $G_{g,\nu}(x)$ in ν together with an auxiliary result that $G_{g,\nu}(x)$ satisfies a 2-IRB if ν is large enough. The 2-IRB cannot be lost as ν is decreased towards ν_c because step two implies that $G_{g,\nu}(x)$ cannot continuously become greater than $3S(x)$.

4.2 Proofs of Proposition 4.3 and Theorem 4.1

We now develop these steps into lemmas. This requires the following notation. For $|z| \le \hat{J}^{-1}$, define real-valued functions $\tilde{S}_z$ and $D_z^{\tilde{S}}$ on $\mathbb{Z}^d$ by

$$\tilde{S}_z(x) := \sum_{n \ge 0} (zJ_+)^{*n}(x), \quad D_z^{\tilde{S}}(x) := -\mathbb{1}_{\{x=0\}} + zJ_+(x). \tag{4.6}$$

$D_z^{\tilde{S}}$ is a variant of J such that when $z = \hat{J}^{-1}$ the jump rates are normalised by z to probabilities. The series defining $\tilde{S}_z(x)$ in (4.6) is absolutely convergent, see [28, Section 4.2], and it is easy to show that $D_z^{\tilde{S}} * \tilde{S}_z(x) = -\mathbb{1}_{\{x=0\}}$. Let

$$\tilde{S}(x) := \tilde{S}_{\hat{J}^{-1}}(x). \tag{4.7}$$

By comparing the definition of $\tilde{S}_z$ with Equation (2.13) and using (2.11)

$$\tilde{S}_z(x) \le \tilde{S}(x) = \hat{J} S(x) \le \frac{\tilde{C}_J}{|\!|\!|x|\!|\!|^{d-2}}, \tag{4.8}$$

for some $\tilde{C}_J > 0$, where $|\!|\!|x|\!|\!| := \max\{|x|, 1\}$.

Step one is formulated by the next lemma.

Lemma 4.4 (Proof in Section 11)**.** *Let $d \ge 5$, and consider the lattice Edwards model or the $g|\varphi|^4$ model with $n = 1, 2$. If $G_{g,\nu} \le 3S$ then there exist $\alpha > 0$, $g_0 > 0$, $L_{g,\nu}$ and $\Psi_{g,\nu}$ such that for all $g \in (0, g_0)$,*

(i) $|\Psi_{g,\nu}(x)| \le g\alpha |\!|\!|x|\!|\!|^{-3(d-2)}$.

(ii) (4.5) *holds.*

(iii) $\hat{J} - L_{g,\nu} \ge \frac{1+O(g)}{3S(0)}$.

The conclusions of the lemma enable us to rewrite equation (4.5) as

$$D_{g,\nu} * \tilde{G}_{g,\nu}(x) = -\mathbb{1}_{\{x=0\}}, \tag{4.9}$$

with the definitions

$$\begin{aligned} w(g,\nu) &:= (\hat{J} - L_{g,\nu})^{-1}, & \tilde{G}_{g,\nu}(x) &:= w(g,\nu)^{-1} G_{g,\nu}(x), \\ D_{g,\nu} &:= D_{w(g,\nu)}^{\tilde{S}} + \tilde{\Psi}_{g,\nu}, & \tilde{\Psi}_{g,\nu}(x) &:= w(g,\nu) \Psi_{g,\nu}(x). \end{aligned} \tag{4.10}$$

Lemma 4.5 (Proof in Section 11)**.** *With the same hypotheses as Lemma 4.4, if $\nu \in (\nu_c, g]$, there exists $C_0 > 1$ such that*

(I) $D = D_{g,\nu}$ satisfies (i) – (iii) in the statement of Lemma 4.6 below, with $C = C_0$;

(II) $|L_{g,\nu}| \le C_0 g$.

Step two is accomplished by Lemma 4.6 below, which is a generalization of [6, Lemma 2] to more general step distributions. We will use Lemma 4.6 with $D = D_{g,\nu}$ given by (4.10), but in the lemma D is arbitrary. The hypotheses say that D is a perturbation of $D_z^{\tilde{S}}$ for some z, and the conclusion is that D has a convolution inverse $-H$. The lemma also relates the decay of H to the decay of $\tilde{S}_\mu$ for a specified parameter μ; note that μ is not necessarily equal to z.

Lemma 4.6 (Proof in Section 10.2)**.** *Let $d \ge 5$, and $C > 0$. There exist $g_0 = g_0(d, J, C) > 0$ and $C' > 0$ such that for each $g \in (0, g_0)$ the following holds. Suppose $D\colon \mathbb{Z}^d \to \mathbb{R}$ satisfies*

(i) D is $\mathbb{Z}^d$-symmetric,

(ii) $\sum_{x\in\mathbb{Z}^d} D(x) \le 0$,

(iii) there exists $z = z(g, D) \in \left[0, \hat{J}^{-1}\right]$ such that $\left|D(x) - D_z^{\tilde{S}}(x)\right| \le Cg|||x|||^{-(d+4)}$.

Then there is an $H\colon \mathbb{Z}^d \to \mathbb{R}$ such that $D * H = -\mathbb{1}_{\{x=0\}}$ and

$$|H(x) - \tilde{S}_\mu(x)| \le C' g|||x|||^{-(d-2)}, \tag{4.11}$$

where $\mu := \hat{J}^{-1}\big(1 + \sum_{x\in\mathbb{Z}^d} D(x)\big) \in \left[-(2\hat{J})^{-1}, \hat{J}^{-1}\right]$.

The μ created by Lemma 4.6 with D as in (4.10) will from now on be written as $\mu(g, \nu)$.

Step three is based on the application of the next lemma to the function F defined in (4.12) below. The lemma is implied by the fact that the continuous image of a connected interval is connected. The use of this lemma to extend estimates up to the critical point in lace expansion analyses originated in [39]; a related application is in [9].

Lemma 4.7 ([21, Lemma 2.1])**.** *Let $F\colon\ (\nu_c, \nu_1] \to \mathbb{R}$. If* *(i)* $F(\nu_1) \le 2$,

(ii) F *is continuous on* $(\nu_c, \nu_1]$,

(iii) for $\nu \in (\nu_c, \nu_1]$ *the inequality* $F(\nu) \le 3$ *implies the inequality* $F(\nu) \le 2$,

then $F(\nu) \le 2$ *for all* $\nu \in (\nu_c, \nu_1]$.

The next two lemmas provide hypotheses (i) and (ii) of Lemma 4.7 with $\nu_1 = g$ for the function $F\colon (\nu_c, \infty) \to \mathbb{R}$ defined by

$$F(\nu) := \sup_{x \in \mathbb{Z}^d} \frac{G_{g,\nu}(x)}{S(x)}. \tag{4.12}$$

Lemma 4.8 (Proof in Section 11)**.** *For the lattice Edwards model and the* $n = 1, 2$*-component* $g|\varphi|^4$ *model, with* F *as in* (4.12)

$$F(\nu) \le 2 \text{ when } \nu = g. \tag{4.13}$$

Lemma 4.9 (Proof in Section 11)**.** *For the lattice Edwards model and the* $n = 1, 2$*-component* $g|\varphi|^4$ *model the function* F *in* (4.12) *is continuous on* $(\nu_c, g]$.

The next lemma ensures the interval $(\nu_c, g]$ is not empty for $g > 0$.

Lemma 4.10 (Proof in Section 11)**.** *For the lattice Edwards model and the* $n = 1, 2$*-component* $g|\varphi|^4$ *models,* $\nu_c \le 0$.

Let $\ell^p = \ell^p(\mathbb{Z}^d)$ denote the set of $f\colon \mathbb{Z}^d \to \mathbb{R}$ with $\sum_{x \in \mathbb{Z}^d} |f(x)|^p$ finite.

Proof of Proposition 4.3. By hypothesis, $d \ge 5$ and g is small enough that the results we have stated above in this section are applicable. Since $(\nu_c, g]$ is not empty by Lemma 4.10, by referring to the definition (4.12) and Lemma 3.5, it suffices to prove

$$F(\nu) \le 2 \qquad \text{for } \nu \in (\nu_c, g]. \tag{4.14}$$

To prove (4.14) it is enough to prove that the statement $F(\nu) \le 3$ implies $F(\nu) \le 2$ when $\nu \in (\nu_c, g]$, as this is hypothesis (iii) in Lemma 4.7, and the hypotheses (i) and (ii) have been verified in Lemmas 4.8 and 4.9.

Assume $F(\nu) \le 3$. Then Lemma 4.5 applies, $|L_{g,\nu}| \le C_0 g$, and the hypotheses of Lemma 4.6 are satisfied providing us with H as in (4.11). Since $|\mu| \le \hat{J}^{-1}$ and $d \ge 5$, (4.8) and (4.11) imply that H is an ℓ^2 convolution inverse of $-D$; by the Fourier transform H is the unique ℓ^2 convolution inverse of $-D$. By (4.9) $\tilde{G}_{g,\nu}$ is also a convolution inverse of $-D$. Moreover, $\tilde{G}_{g,\nu}$ is in ℓ^2 because it is proportional to $G_{g,\nu}$ by (4.10) and $G_{g,\nu} \in \ell^1 \subset \ell^2$ by the hypothesis $\nu > \nu_c$. Therefore $\tilde{G}_{g,\nu} = H$. By (4.11), $|L_{g,\nu}| \le C_0 g$, and the lower bound below (2.11), $|\tilde{G}_{g,\nu}(x)| \le (1+O(g))\tilde{S}(x)$. By (4.10) and the equality in (4.8) this is the same as

$$G_{g,\nu}(x) \le \frac{\hat{J}}{\hat{J} - O(g)}\big(1 + O(g)\big)S(x) = \big(1 + O(g)\big)S(x). \tag{4.15}$$

By taking g smaller if necessary we have $G_{g,\nu}(x) \le 2S(x)$. ■

Lemma 4.11 (Proof in Section 11)**.** *For the lattice Edwards model and the $n = 1, 2$-component $g|\varphi|^4$ models, $L_{g,\nu} \to \infty$ as $\nu \to -\infty$.*

We use this result to prove that ν_c is finite as claimed in Theorem 4.1. By Lemma 4.10 it is enough to rule out $\nu_c = -\infty$. Towards a contradiction, suppose $\nu_c = -\infty$. Then Proposition 4.3 implies a 2-IRB holds for all $\nu \le g$, and hence Lemma 4.5 (II) implies $|L_{g,\nu}| \le C_0 g$ for all $\nu \le g$. This contradicts Lemma 4.11.

Proof of Theorem 4.1. For future reference, we note that the remainder of this proof deduces the desired (4.1) from (4.2), (i) $G^{(\Lambda)}$ is monotone in Λ and (ii) the *finite volume* $G^{(\Lambda)}_{\mathbf{0}}(0, x)$ is continuous in ν at ν_c. Claim (i) is Lemma 3.4. Claim (ii) is deferred to the end of the proof.

By (i) and (4.2), for $\nu > \nu_c$,

$$G^{(\Lambda)}_{\mathbf{0}}(0, x) \le G_{g,\nu}(x) \le 2S(x) \tag{4.16}$$

By (ii) $G^{(\Lambda)}_{\mathbf{0}}(0, x)$ is bounded above by $2S(x)$ when $\nu = \nu_c$. By taking the infinite volume limit as in (3.12) with $\nu = \nu_c$ we obtain $G_{g,\nu_c}(x) \le 2S(x)$.

To prove (ii) for the Edwards model, observe from (3.4) that $Z_{\boldsymbol{t}}$ is continuous in ν pointwise in $\boldsymbol{t}$ and uniformly bounded in $\boldsymbol{t}$ for each ν. By dominated convergence it follows from (3.2) that the *finite* volume $G^{(\Lambda)}_{\mathbf{0}}(0, x)$ is continuous in ν at ν_c. A similar argument applies to the $g|\varphi|^4$ model. ■

Remark 4.12. *Lemma 4.6 and 4.7 are* model-independent*. We also classify Lemmas 4.4 and 4.5 as model-independent even though the hypotheses refer to our two examples, because they will be subsumed within a general framework. We classify Lemmas 3.4, 3.5, and 4.8 – 4.11 as* model dependent*. The proofs of Proposition 4.3 and Theorem 4.1 only used the lemmas just listed. Furthermore, for the model-dependent lemmas, it is enough to have only the conclusions. We will explain how this plays a role in the next section.*

4.3 Outline of the remainder of the paper

Our analysis is done in a general context that includes the Edwards and the $g|\varphi|^4$ models with $n = 1, 2$ as special cases. The general context is determined by a set of hypotheses on the function $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ that enters into the definition (3.2) of the Green's function; see Section 10.1 for a full list of hypotheses. In the course of the paper we introduce these hypotheses on $Z_{\boldsymbol{t}}$ as they are needed. In some initial sections we use hypotheses that will eventually be superseded; these are indicated by ending in a 0, e.g., (G0) below. We verify that the Edwards and the $g|\varphi|^4$ models with $n = 1, 2$ satisfy the hypotheses in Section 11. We have based our proof on hypotheses on Z to facilitate extending the continuous-time lace expansion to other models: isolating properties that currently play a role should help the search for more appealing hypotheses.

In Section 5 and Section 6 we develop a lace expansion for Green's functions as in (3.2) in *finite volumes* $\Lambda \subset \mathbb{Z}^d$. Working in a finite volume is essential, as we have only defined the $g|\varphi|^4$ model as the infinite volume limit of finite volume models.

The next part of the paper, Sections 7 through 9, develops estimates on our finite volume lace expansion, under the hypothesis that the Green's function satisfies a 3-IRB. These estimates establish the infinite-volume lace expansion equation (4.5) under the general hypotheses on $Z_{\boldsymbol{t}}$ and provide the key inputs for the proofs of Lemma 4.4 and Lemma 4.5.

In Section 10.2 and 10.3 we complete the proofs of the lemmas we have used in the last two sections, and thus establish the conclusions of Theorem 4.1 for any $Z_{\boldsymbol{t}}$ satisfying our hypotheses. We then make use of this result, in conjunction with a theorem of Hara [18], to obtain the Gaussian asymptotics of Theorem 3.6.

5 Functions on a set of intervals

In this section we begin to derive the lace expansion needed for step one of Section 4.1. The main result of this section is Theorem 5.2, which is an expansion for a function $\mathcal{Y}\colon \mathcal{D} \to \mathbb{R}$, where

$$\mathcal{D} := \{(s,t) : 0 \le s \le t < \infty\} \subset [0,\infty)^2. \tag{5.1}$$

Theorem 5.2 will be used in the next section to derive our lace expansion for Green's functions of the form (3.2). We begin with notation and minimal assumptions on $\mathcal{Y}$ needed for the main result of the section.

For a function $\mathcal{Y}$ on $\mathcal{D}$ and $t \in (0,\infty)$, we denote by $\mathcal{Y}_{\cdot,t}\colon [0,t] \to \mathbb{R}$ the function $s \mapsto \mathcal{Y}_{s,t}$, and for each $s \in [0,\infty)$, we denote by $\mathcal{Y}_{s,\cdot}\colon [s,\infty) \to \mathbb{R}$ the function $t \mapsto \mathcal{Y}_{s,t}$. We will write ∂_1 and ∂_2 to denote partial differentiation with respect to the first and second coordinates, respectively.

We will assume $\mathcal{Y}$ satisfies the following assumptions. The *almost every* (a.e.) statements in the assumptions are with respect to Lebesgue measure.

Assumptions 5.1.

1. *$\mathcal{Y}$ is continuous and strictly positive on $\mathcal{D}$, and $\mathcal{Y}_{s,s} = 1$ for all $s \geq 0$.*

2. *For each $t \in (0,\infty)$, $\mathcal{Y}_{\cdot,t}$ is absolutely continuous. For each $s \in [0,\infty)$, $\mathcal{Y}_{s,\cdot}$ is absolutely continuous on bounded subintervals of $[s,\infty)$.*

3. *For a.e. $t \in (0,\infty)$, the function $(\partial_2\mathcal{Y})_{\cdot,t}$ is absolutely continuous on $(0,t)$. For a.e. $s \in [0,\infty)$, the function $(\partial_1\mathcal{Y})_{s,\cdot}$ is absolutely continuous on bounded subintervals of (s,∞).*

4. *$\partial_1\partial_2\mathcal{Y} = \partial_2\partial_1\mathcal{Y}$ a.e. on the interior of $\mathcal{D}$.*

Absolutely continuous functions are uniformly continuous on bounded intervals and therefore the derivative $\partial_2\mathcal{Y}_{\cdot,t}$ in 3. extends to an absolutely continuous function on $[0,t]$ and similarly $\partial_1\mathcal{Y}_{s,\cdot}$ extends to $[s,\infty)$. When we write derivatives on the boundaries of their domains we are referring to these extensions. Note that the first and second derivatives of $\mathcal{Y}$ are measurable because they are pointwise limits of measurable functions.

5.1 The expansion

In this section we introduce the objects that enter into our expansion, and state the expansion in Theorem 5.2 below.

Define *vertex functions* by

$$\begin{aligned} r_s &\coloneqq -\lim_{t\downarrow s} \partial_1 \log \mathcal{Y}_{s,t}, \qquad 0 \leq s < \infty, \\ r_{s,t} &\coloneqq -\partial_1\partial_2 \log \mathcal{Y}_{s,t}, \qquad 0 \leq s < t < \infty. \end{aligned} \tag{5.2}$$

These are a.e. equalities. By Assumptions 5.1 and the paragraph that follows them, the chain rule, which applies to a smooth function composed with an absolutely continuous function, proves these derivatives exist a.e. and

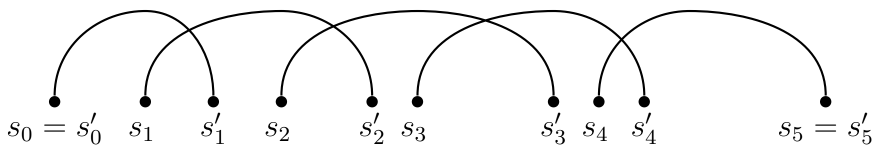


Figure 1: A lace with $m = 5$ intervals

provides formulas for them. The limit defining r_s is well-defined by the paragraph before this subsection.

For $m \in \mathbb{N}$ a *lace* L is a sequence

$$L = \big((s_i, s'_{i+1})\big)_{i=0,\dots,m-1} \tag{5.3}$$

of m intervals (s_i, s'_{i+1}) with $s_0 \coloneqq s'_0$, $s_m \coloneqq s'_m$, and

$$0 \le s'_0 < s_1 < s'_1 < s_2 < s'_2 < \cdots < s'_{m-1} < s_m < \infty. \tag{5.4}$$

The meaning of (5.4) is illustrated by Figure 1. The union of all of the intervals of a lace is (s_0, s'_m), and if any single interval is excluded from the union the resulting set does not cover (s_0, s'_m).

Let $\mathcal{L}_m$ be the region in $\mathbb{R}^{2m}$ defined by the inequalities in (5.4). We identify $\mathcal{L}_m$ with the collection of all laces containing m intervals. Let

$$\mathcal{L}_0 \coloneqq \{s_0 \mid 0 \le s_0 < \infty\}. \tag{5.5}$$

We associate to a lace L a product

$$r(L) \coloneqq \begin{cases} r_{s_0}, & \text{if } L = \{s_0\} \in \mathcal{L}_0, \\ \prod_{i=0}^{m-1} r_{s_i, s'_{i+1}}, & \text{if } L = \big((s_i, s'_{i+1})\big)_{i=0,\dots,m-1} \in \mathcal{L}_m,\ m \ge 1, \end{cases} \tag{5.6}$$

of vertex functions. The *weight* $L \mapsto w(L)$ of a lace is defined to be

$$w(L) \coloneqq r(L) \times \begin{cases} 1, & L \in \mathcal{L}_0, \\ P(L), & L \in \mathcal{L}_m,\ m \ge 1, \end{cases} \tag{5.7}$$

where for $L = \big((s_i, s'_{i+1})\big)_{i=0,\dots,m-1}$

$$P(L) \coloneqq \mathcal{Y}_{s'_0, s'_1} \prod_{i=0}^{m-2} \frac{\mathcal{Y}_{s'_i, s'_{i+2}}}{\mathcal{Y}_{s'_i, s'_{i+1}}}. \tag{5.8}$$

For $m = 1$ the empty product in (5.8) is defined to be one by convention.

For $m \geq 0$ we define integration $\int_{\mathcal{L}_m} dL$ over $\mathcal{L}_m$ to be integration with respect to Lebesgue measure on $\mathcal{L}_m$. For example, if $m = 0$ then $\int_{\mathcal{L}_m} dL = \int_{[0,\infty)} ds_0$. Let $\mathcal{L}_{m,\ell}$ be the subset of $\mathcal{L}_m$ such that $s_m \leq \ell$, and let $\mathcal{D}_\ell$ be the subset of $\mathcal{D}$ with $t \leq \ell$.

Theorem 5.2. *Let $\mathcal{Y}$ be such that (i) $\mathcal{Y}$ satisfies Assumption 5.1, (ii) the function $r_{s,t}$ defined in* (5.2) *is Lebesgue a.e. bounded on $\mathcal{D}_\ell$. Then for $\ell > 0$*

$$\mathcal{Y}_{0,\ell} = 1 + \sum_{m \geq 0} \int_{\mathcal{L}_{m,\ell}} dL \; w(L) \; \mathcal{Y}_{s'_m,\ell} \; , \tag{5.9}$$

and the sum is absolutely convergent.

The proof of this theorem is given in Section 5.4 and is based on two identities, which are given in the next two subsections. We will also need the following fact from real analysis, whose proof we omit.

Lemma 5.3. *Let $I \subset \mathbb{R}$ be bounded. If $f \colon I \to \mathbb{R}$ is Lipschitz continuous and $g \colon I \to I$ is absolutely continuous, then $f \circ g$ is absolutely continuous.*

5.2 The identity that starts the expansion

Lemma 5.4. *Under Assumptions 5.1,*

$$\mathcal{Y}_{0,\ell} = 1 + \int_{(0,\ell)} ds_0 \; r_{s_0} \; \mathcal{Y}_{s_0,\ell} + \int_{(0,\ell)} ds_0 \int_{(s_0,\ell)} ds'_1 \; r_{s_0,s'_1} \; \mathcal{Y}_{s_0,\ell}. \tag{5.10}$$

Proof. By Assumption 2,

$$\mathcal{Y}_{0,\ell} = \mathcal{Y}_{\ell,\ell} - \int_{(0,\ell)} ds_0 \; \partial_1 \mathcal{Y}_{s_0,\ell} \; . \tag{5.11}$$

$\mathcal{Y}$ is bounded below by a positive constant on $\mathcal{D}_\ell$ by Assumption 1 and the compactness of $\mathcal{D}_\ell$. Hence, by $\mathcal{Y}_{\ell,\ell} = 1$, (5.11) can be rewritten as

$$\mathcal{Y}_{0,\ell} = 1 - \int_{(0,\ell)} ds_0 \; \frac{\partial_1 \mathcal{Y}_{s_0,\ell}}{\mathcal{Y}_{s_0,\ell}} \; \mathcal{Y}_{s_0,\ell}. \tag{5.12}$$

Since the range of $\mathcal{Y}$ is the continuous image of a compact set and $z \mapsto 1/z$ is Lipschitz on compact subsets of $(0, \infty)$, it follows from Lemma 5.3 that $t \mapsto$

$\mathcal{Y}_{s_0,t}^{-1}$ is absolutely continuous in t for t in a bounded interval whose closure does not contain 0. Combined with Assumption 3 this shows $\mathcal{Y}_{s_0,t}^{-1}\,\partial_1\mathcal{Y}_{s_0,t}$ is absolutely continuous in $t \in (s_0,\ell)$ for a.e. s_0, hence for a.e. $s_0 > 0$

$$\partial_1 \log \mathcal{Y}_{s_0,\ell} = (\partial_1 \log \mathcal{Y})_{s_0,s_0} + \int_{(s_0,\ell)} ds_1'\ \partial_2\partial_1 \log \mathcal{Y}_{s_0,s_1'}. \tag{5.13}$$

In this equation $(\partial_1 \log \mathcal{Y})_{s_0,s_0}$ is by definition the continuous extension of $(\partial_1 \log \mathcal{Y})_{s_0,t} = \mathcal{Y}_{s_0,t}^{-1}\,\partial_1\mathcal{Y}_{s_0,t}$ as a function of $t > 0$ to the boundary $t = s_0$ of its domain. This is an instance of the convention we declared below Assumption 5.1. By inserting (5.13) into (5.12) we obtain

$$\begin{aligned}\mathcal{Y}_{0,\ell} = 1 - &\int_{(0,\ell)} ds_0\ \mathcal{Y}_{s_0,\ell}\ (\partial_1 \log \mathcal{Y})_{s_0,s_0} \\ &- \int_{(0,\ell)} ds_0 \int_{(s_0,\ell)} ds_1'\ \mathcal{Y}_{s_0,\ell}\ \partial_2\partial_1 \log \mathcal{Y}_{s_0,s_1'}.\end{aligned} \tag{5.14}$$

The proof is completed by substituting in the definitions of r_{s_0} and $r_{s_0,s_1'}$. For the last term the interchange of derivatives is justified by Assumption 4. ■

5.3 The identity that generates the expansion

Lemma 5.5. *Suppose $\mathcal{Y}$ satisfies Assumptions 5.1. If $(u,v) \in \mathcal{D}_\ell$ then*

$$\mathcal{Y}_{u,\ell} = \mathcal{Y}_{u,v}\ \mathcal{Y}_{v,\ell}\ + \int_{(u,v)} ds_+ \int_{(v,\ell)} ds_+'\ r_{s_+,s_+'}\ \mathcal{Y}_{u,s_+'}\ \frac{\mathcal{Y}_{v,\ell}}{\mathcal{Y}_{v,s_+'}}. \tag{5.15}$$

Proof. As $\mathcal{Y}$ is bounded below by a positive constant on $\mathcal{D}_\ell$, we can rewrite the left-hand side:

$$\mathcal{Y}_{u,\ell} = \mathcal{Y}_{u,v}\mathcal{Y}_{v,\ell} + \left(\frac{\mathcal{Y}_{u,\ell}}{\mathcal{Y}_{v,\ell}} - \mathcal{Y}_{u,v}\right)\mathcal{Y}_{v,\ell}. \tag{5.16}$$

By Assumption 2, $\mathcal{Y}_{s,s} = 1$, and the absolute continuity of $\mathcal{Y}_{v,\ell}^{-1}$ in ℓ noted after (5.12), this can be rewritten as

$$\mathcal{Y}_{u,\ell} = \mathcal{Y}_{u,v}\mathcal{Y}_{v,\ell} + \left(\int_{(v,\ell)} ds_+'\ \partial_2 \frac{\mathcal{Y}_{u,s_+'}}{\mathcal{Y}_{v,s_+'}}\right)\mathcal{Y}_{v,\ell}. \tag{5.17}$$

Since exp and log are Lipschitz on compact subsets of their open domains we can compute the derivative in (5.17) using $f(x) = \exp\log f(x)$:

$$\mathcal{Y}_{u,\ell} = \mathcal{Y}_{u,v}\mathcal{Y}_{v,\ell} + \left(\int_{(v,\ell)} ds'_+ \left(\partial_2 \log\left(\frac{\mathcal{Y}_{u,s'_+}}{\mathcal{Y}_{v,s'_+}}\right)\right)\frac{\mathcal{Y}_{u,s'_+}}{\mathcal{Y}_{v,s'_+}}\right)\mathcal{Y}_{v,\ell} \tag{5.18}$$

$$= \mathcal{Y}_{u,v}\mathcal{Y}_{v,\ell} - \left(\int_{(v,\ell)} ds'_+ \left(\int_{(u,v)} ds_+\, \partial_1\partial_2 \log \mathcal{Y}_{s_+,s'_+}\right)\frac{\mathcal{Y}_{u,s'_+}}{\mathcal{Y}_{v,s'_+}}\right)\mathcal{Y}_{v,\ell}, \tag{5.19}$$

where we have used Assumption 3 in the second step. By Fubini's theorem this can be rewritten as the desired result. ∎

5.4 Proof of Theorem 5.2

Recall the definitions of $w(L)$, $r(L)$ and $P(L)$ in (5.6)–(5.8) and define

$$R_n := \int_{\mathcal{L}_{n,\ell}} dL\; w(L)\, \frac{\mathcal{Y}_{s'_{n-1},\ell}}{\mathcal{Y}_{s'_{n-1},s'_n}}, \qquad n \geq 1. \tag{5.20}$$

Lemma 5.6. *Suppose $\mathcal{Y}$ satisfies Assumptions 5.1. Then*

$$\mathcal{Y}_{0,\ell} = 1 + \sum_{m=0}^{n-1} \int_{\mathcal{L}_{m,\ell}} dL\; w(L)\, \mathcal{Y}_{s'_m,\ell} \; + R_n, \qquad n \geq 1. \tag{5.21}$$

Proof. We first prove (5.21) with $n = 1$. By Lemma 5.4

$$\mathcal{Y}_{0,\ell} = 1 + \int_{(0,\ell)} ds_0\, r_{s_0}\, \mathcal{Y}_{s_0,\ell} + \int_{(0,\ell)} ds_0 \int_{(s_0,\ell)} ds'_1\, r_{s_0,s'_1}\, \mathcal{Y}_{s_0,\ell}. \tag{5.22}$$

By the definition (5.7) of $w(L)$ and the definition of integration over $\mathcal{L}_{m,\ell}$ given below (5.8) (recall also that $s'_0 := s_0$), this can be rewritten as

$$\mathcal{Y}_{0,\ell} = 1 + \int_{\mathcal{L}_{0,\ell}} dL\; w(L)\, \mathcal{Y}_{s'_0,\ell} \; + \int_{\mathcal{L}_{1,\ell}} dL\; r_{s'_0,s'_1}\, \mathcal{Y}_{s'_0,\ell}. \tag{5.23}$$

This establishes (5.21) when $n = 1$ as the final term is R_1.

We now prove (5.21) holds for $n \geq 1$ by induction, using (5.21) as the inductive hypothesis. By Lemma 5.5 with (u, v) replaced by (s'_{n-1}, s'_n),

$$\mathcal{Y}_{s'_{n-1},\ell} = \mathcal{Y}_{s'_{n-1},s'_n}\, \mathcal{Y}_{s'_n,\ell} + \int_{(s'_{n-1},s'_n)} ds_+ \int_{(s'_n,\ell)} ds'_+\, r_{s_+,s'_+}\, \mathcal{Y}_{s'_{n-1},s'_+}\, \frac{\mathcal{Y}_{s'_n,\ell}}{\mathcal{Y}_{s'_n,s'_+}}. \tag{5.24}$$

We insert (5.24) into the definition (5.20) of R_n and use the definition (5.7) of $w(L)$ for the contribution from the first term $\mathcal{Y}_{s'_{n-1},s'_n}\,\mathcal{Y}_{s'_n,\ell}$:

$$R_n = \int_{\mathcal{L}_{n,\ell}} dL\; w(L)\mathcal{Y}_{s'_n,\ell} \;+ \Bigg(\int_{\mathcal{L}_{n,\ell}} dL\; r(L)\; P(L) \\ \times \int_{(s'_{n-1},s'_n)} ds_+ \int_{(s'_n,\ell)} ds'_+\; r_{s_+,s'_+}\, \frac{\mathcal{Y}_{s'_{n-1},s'_+}}{\mathcal{Y}_{s'_{n-1},s'_n}}\, \frac{\mathcal{Y}_{s'_n,\ell}}{\mathcal{Y}_{s'_n,s'_+}} \Bigg). \tag{5.25}$$

Renaming s_+, s'_+ as s_n, s'_{n+1} and combining the integrals in the second term into an integral over $\mathcal{L}_{n+1,\ell}$ yields

$$R_n = \int_{\mathcal{L}_{n,\ell}} dL\; w(L)\; \mathcal{Y}_{s'_n,\ell} \;+ \int_{\mathcal{L}_{n+1,\ell}} dL\; r(L)\; P(L)\, \frac{\mathcal{Y}_{s'_n,\ell}}{\mathcal{Y}_{s'_n,s'_{n+1}}} \tag{5.26}$$

For the second term on the right of (5.25), $r_{s_+,s'_+} := r_{s_n,s'_{n+1}}$ became part of $r(L)$ and the ratio of $\mathcal{Y}$'s became part of $P(L)$ when the range of integration became $\mathcal{L}_{n+1,\ell}$. By the definition (5.20) of R_{n+1}, this is

$$R_n = \int_{\mathcal{L}_{n,\ell}} dL\; w(L)\; \mathcal{Y}_{s'_n,\ell} \;+ R_{n+1}. \tag{5.27}$$

Inserting (5.27) into the inductive hypothesis completes the proof. ■

Proof of Theorem 5.2. We justify taking the $n \to \infty$ limit of Lemma 5.6.

The factors $\mathcal{Y}_{s,t}$ under the integrals in (5.21) are bounded above and below because they are strictly positive and continuous functions on the compact domain $\mathcal{D}_\ell$. Together with the assumption that $r_{s,t}$ is uniformly bounded this proves that there is a constant $C = C(\ell)$ such that $\left|w(L)\mathcal{Y}_{s'_n,\ell}\right| \leq C^{n+1}$ for $L \in \mathcal{L}_{n,\ell}$ and $n \geq 1$, where w was defined in (5.7). Similarly the integrand in R_n is bounded by C^{n+1} for $L \in \mathcal{L}_{n,\ell}$. Because $\mathcal{Y}_{s,t}$ is bounded above and below we also have that $\left|w(L)\mathcal{Y}_{s'_0,\ell}\right|$ is bounded by a constant $C' = C'(\ell)$ when $L \in \mathcal{L}_{0,\ell}$.

The Lebesgue measure of $\mathcal{L}_{n,\ell}$ is the Lebesgue measure of all $2n$-tuples of ordered points in $(0,\ell)$, which is $\frac{1}{(2n)!}\ell^{2n}$. Therefore $|R_n| \leq C^{n+1}\frac{1}{(2n)!}\ell^{2n}$ and the mth term in the sum over m in (5.21) is bounded by $C^{m+1}\frac{1}{(2m)!}\ell^{2m}$. Therefore the series in the right hand side of (5.9) is absolutely convergent and equals $\mathcal{Y}_{0,\ell}$ as claimed because $\lim_{n\to\infty} R_n = 0$ in (5.21). ■

6 The lace expansion in finite volume

In this section we continue with step one of Section 4.1. The main result is Proposition 6.2, which provides a finite-volume version of (4.5) as desired for step one. This proposition involves a function $\Pi^{(\Lambda)}(x,y)$, and the formula (6.3) for $\Pi^{(\Lambda)}(x,y)$ is called a lace expansion for reasons to be explained following Proposition 7.4. We begin by introducing some further definitions and assumptions.

Given $Z^{(\Lambda)}\colon [0,\infty)^\Lambda \to (0,\infty)$, $\boldsymbol{u} \mapsto Z^{(\Lambda)}_{\boldsymbol{u}}$, define a function $\mathcal{Y}\colon \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Y}_{s,t} = \mathcal{Y}^{(\Lambda)}_{s,t} := \left(\frac{Z^{(\Lambda)}_{\cdot}}{Z^{(\Lambda)}_{\boldsymbol{0}}}\right) \circ \boldsymbol{\tau}^{(\Lambda)}_{[s,t]} = \frac{1}{Z^{(\Lambda)}_{\boldsymbol{0}}} Z^{(\Lambda)}_{\boldsymbol{\tau}^{(\Lambda)}_{[s,t]}}. \tag{6.1}$$

Recall (3.1) and note that $\mathcal{Y}_{s,t} = Y_{\boldsymbol{0},\boldsymbol{\tau}^{(\Lambda)}_{[s,t]}}$. $\mathcal{Y}$ is random as it depends on the local time $\boldsymbol{\tau}$, and henceforth we assume $\mathcal{Y}$ is of the form in (6.1). Let $G^{(\Lambda)}$ be the Green's function determined by (3.2) with this choice of $Z^{(\Lambda)}$. Recall that the weight $w(L)$ of a lace is defined in terms of $\mathcal{Y}_{s,t}$ in (5.7). We write $w^{(\Lambda)}(L) := w(L)$ for the weight that arises with the choice (6.1).

Let $\mathcal{L}_m(s) \subset \mathcal{L}_m$ be the hypersurface defined by $s'_0 = s$. For $m \geq 1$ we write $\int_{\mathcal{L}_m(s)} dL$ for integration with respect to Lebesgue measure on $\mathcal{L}_m(s)$. Then, by (5.4), we have

$$\int_{\mathcal{L}_m} dL = \int_{[0,\infty)} ds \int_{\mathcal{L}_m(s)} dL. \tag{6.2}$$

For $m = 0$, since $\mathcal{L}_0(s)$ consists of the single point $s'_0 = s$ we let dL in the inner integral denote a unit Dirac mass at s.

In the following assumptions, and hereafter, we write Z in place of $Z^{(\Lambda)}$ when Λ is contextually clear.

Assumptions 6.1. *For all $a \in \Lambda$,*

(Z1) $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ *is strictly positive and continuous on* $[0,\infty)^\Lambda$.

(Z2) $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ *is* $\mathcal{C}^2$ *on* $[0,\infty)^\Lambda$.

(G0) $\int_{[0,\infty)} d\ell\, E_a\left[\mathcal{Y}^{(\Lambda)}_{0,\ell}\right] < \infty$.

(F0) $\sum_{m\geq 0} \int_{\mathcal{L}_m(0)} dL\, E_a\big[|w^{(\Lambda)}(L)|\,\big] < \infty$.

In (Z2), $\mathcal{C}^2$ at the boundary means that the derivatives have continuous extensions to the boundary. Lemma 6.3 below proves that Assumptions 6.1 are consistent, and this enables us to define a function $\Pi^{(\Lambda)} \colon \Lambda \times \Lambda \to \mathbb{R}$ by

$$\Pi^{(\Lambda)}(x,y) = \sum_{m \geq 0} \int_{\mathcal{L}_m(0)} dL \; E_x \left[w^{(\Lambda)}(L) \mathbb{1}_{\{X^{(\Lambda)}_{s'_m} = y\}} \right]. \tag{6.3}$$

Proposition 6.2. *Under Assumptions 6.1,*

$$G_{\mathbf{0}}^{(\Lambda)}(a,b) = S^{(\Lambda)}(a,b) + \sum_{x,y \in \Lambda} S^{(\Lambda)}(a,x) \Pi^{(\Lambda)}(x,y) G_{\mathbf{0}}^{(\Lambda)}(y,b). \tag{6.4}$$

The proof of this proposition occupies the rest of this section.

6.1 Derivatives of local time

For each $x \in \Lambda$, (2.5) defines a local time $\boldsymbol{\tau}_{[s,s'],x}$ that is absolutely continuous in s for $s \leq s'$ with s' fixed, and similarly is absolutely continuous in s' for fixed s when s' is restricted to a bounded interval. Note that

$$\partial_2 \tau_{[s,s'],x} = \partial_2 \int_{[s,s']} \mathbb{1}_{\{X^{(\Lambda)}_r = x\}} \, dr \; = \mathbb{1}_{\{X^{(\Lambda)}_{s'} = x\}}, \tag{6.5}$$

$$\partial_1 \tau_{[s,s'],x} = -\mathbb{1}_{\{X^{(\Lambda)}_s = x\}}, \tag{6.6}$$

$$\partial_2 \partial_1 \tau_{[s,s'],x} = \partial_1 \partial_2 \tau_{[s,s'],x} = 0. \tag{6.7}$$

The first equation holds a.e. in s' for $s \leq s'$. The derivative does not depend on s, so it is absolutely continuous in s. By similar reasoning (6.6) holds a.e. in s for $s \leq s'$, and as a consequence (6.7) holds a.e. in $\{s \leq s'\}$.

6.2 Proof of Proposition 6.2

Lemma 6.3. *If $Z_{\boldsymbol{t}}$ satisfies (Z1) and (Z2) then $w^{(\Lambda)}(L)$ is well-defined and $\mathcal{Y}^{(\Lambda)}_{s,t}$ defined by* (6.1) *satisfies the hypotheses of Theorem 5.2.*

Proof. The hypothesis that $\mathcal{Y}^{(\Lambda)}_{s,s} = 1$ holds as $\boldsymbol{\tau}_{[s,s]} = \mathbf{0}$. By (Z2) and the compactness of $[0,\ell]^\Lambda$ the function $Z_{\boldsymbol{t}}$ is Lipschitz in $\boldsymbol{t}$. For each s, $\boldsymbol{\tau}_{[s,t]}$ is absolutely continuous as a function of t when t is restricted to a bounded interval, and vice-versa by Section 6.1. By Lemma 5.3 this implies that for each s, $Z_{\boldsymbol{\tau}_{[s,t]}}$ is absolutely continuous as a function of t when t is restricted to a bounded interval and vice-versa. Combined with (Z1) this proves the first of Assumptions 5.1. Furthermore, by the chain rule, the composition $Z_{\boldsymbol{\tau}_{[s,t]}}$ is differentiable in t at points (s,t) where $\boldsymbol{\tau}_{[s,t]}$ has this property. Therefore for each s, $Z_{\boldsymbol{\tau}_{[s,t]}}$ is differentiable in t at all but a countable number of points (recall that simple random walk takes only finitely many jumps in any finite time interval), and hence is absolutely continuous in t, and vice-versa. This verifies the second item of Assumptions 5.1, and an analogous argument verifies the third item. The fourth follows by (Z2) and (6.7).

Lastly, we must prove that $r_{s,t}$ is a.e. bounded on $\mathcal{D}_\ell$. This follows by a direct calculation, using that $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ is $\mathcal{C}^2$ and strictly positive, (6.5)–(6.7), and the compactness of $[0,\ell]^\Lambda$. ■

Proof of Proposition 6.2. We omit the superscript Λ on Green's functions, etc., since Λ is fixed. At two points in the proof we will use the Markov property; the justifications for these applications are given in Appendix A.2.

By definition (3.2) and (6.1),

$$G_{\mathbf{0}}(a,b) = \int_{[0,\infty)} d\ell \; E_a \left[\mathcal{Y}_{0,\ell} \mathbb{1}_{\{X_\ell = b\}} \right]. \tag{6.8}$$

Expanding $\mathcal{Y}_{0,\ell}$ by Theorem 5.2 and using the definition (2.6) of $S(a,b)$,

$$G_{\mathbf{0}}(a,b) = S(a,b) + \int_{[0,\infty)} d\ell \; E_a \Big[\sum_{m\geq 0} \int_{\mathcal{L}_{m,\ell}} dL \; w(L) \; \mathcal{Y}_{s'_m,\ell} \; \mathbb{1}_{\{X_\ell = b\}} \Big], \tag{6.9}$$

where s'_m is defined by the lace L as explained in (5.4). For convenience, define $U_{a,b} := G_{\mathbf{0}}(a,b) - S(a,b)$. Using (6.2) and (F0) we obtain

$$U_{a,b} = \int_{(0,\infty)} ds \sum_{m\geq 0} \int_{\mathcal{L}_m(s)} dL \int_{[s'_m,\infty)} d\ell \; E_a \left[w(L) \; \mathcal{Y}_{s'_m,\ell} \; \mathbb{1}_{\{X_\ell = b\}} \right]. \tag{6.10}$$

By a change of variable in the integral with respect to ℓ, this is equal to

$$\int_{(0,\infty)} ds \sum_{m\geq 0} \int_{\mathcal{L}_m(s)} dL \int_{[0,\infty)} d\ell \; E_a \left[w(L) \; \mathcal{Y}_{s'_m,s'_m+\ell} \; \mathbb{1}_{\{X_{s'_m+\ell} = b\}} \right]. \tag{6.11}$$

For $\ell > 0$, let $h(\ell,y,b) := E_y \left[\mathcal{Y}_{0,\ell} \; \mathbb{1}_{\{X_\ell = b\}} \right]$. By conditioning on $\mathcal{F}_{s'_m}$ in the last expectation, using $w(L) \in \mathcal{F}_{s'_m}$, and applying the Markov property to

$E_a\big[\mathcal{Y}_{s'_m,s'_m+\ell}\ \mathbb{1}_{\{X_{s'_m+\ell}=b\}}\big|\mathcal{F}_{s'_m}\big]$,

$$U_{a,b} = \int_{(0,\infty)} ds \sum_{m\geq 0} \int_{\mathcal{L}_m(s)} dL \int_{[0,\infty)} d\ell\ E_a\Big[w(L)\ h(\ell, X_{s'_m}, b)\Big]. \tag{6.12}$$

By (F0) the right-hand side converges absolutely and likewise for the following equations. We bring the integral with respect to ℓ inside the expectation and rewrite $\int_{[0,\infty)} d\ell\ h(\ell, X_{s'_m}, b)$ using the definition of $G_{\mathbf{0}}(a,b)$:

$$U_{a,b} = \int_{(0,\infty)} ds \sum_{m\geq 0} \int_{\mathcal{L}_m(s)} dL\ E_a\Big[w(L)\ G_{\mathbf{0}}(X_{s'_m}, b)\Big]. \tag{6.13}$$

By changing variables in the integral over $\mathcal{L}_m(s)$ so that it becomes an integral over $\mathcal{L}_m(0)$ we rewrite this as

$$U_{a,b} = \int_{(0,\infty)} ds \sum_{m\geq 0} \int_{\mathcal{L}_m(0)} dL\ E_a\Big[w(L+s)\ G_{\mathbf{0}}(X_{s'_m+s}, b)\Big], \tag{6.14}$$

where for $L \in \mathcal{L}_m(0)$, $L+s$ is defined to be the lace in $\mathcal{L}_m(s)$ obtained from L by adding s to each s_i, s'_i. For $L \in \mathcal{L}_m(0)$ define $f(x,b,L) := E_x\Big[w(L)\ \ G_{\mathbf{0}}(X_{s'_m}, b)\Big]$. By conditioning on $\mathcal{F}_s$ inside the expectation in (6.14) and applying the Markov property to $E_a\Big[w(L+s)\ G_{\mathbf{0}}(X_{s'_m+s}, b)\big|\mathcal{F}_s\Big]$ we obtain

$$U_{a,b} = \int_{(0,\infty)} ds \sum_{m\geq 0} \int_{\mathcal{L}_m(0)} dL\ E_a\big[f(X_s, b, L)\big]. \tag{6.15}$$

The expectation is equal to

$$\sum_{x,y\in\Lambda} E_a\Big[E_{X_s}\big[w(L) G_{\mathbf{0}}(X_{s'_m}, b)\mathbb{1}_{\{X_{s'_m}=y\}}\big]\mathbb{1}_{\{X_s=x\}}\Big] \tag{6.16}$$

$$= \sum_{x,y\in\Lambda} G_{\mathbf{0}}(y,b)\, E_a\big[\mathbb{1}_{\{X_s=x\}}\big] E_x[w(L)\mathbb{1}_{\{X_{s'_m}=y\}}], \tag{6.17}$$

where we have used the fact that the sums over $x, y \in \Lambda$ are finite to take them outside the expectation in the first line. Recalling the definition (6.3) we see that (6.15) can be written as

$$U_{a,b} = \int_{(0,\infty)} ds \sum_{x,y\in\Lambda} G_{\mathbf{0}}(y,b)\, E_a\big[\mathbb{1}_{\{X_s=x\}}\big]\ \Pi(x,y). \tag{6.18}$$

By the definitions of $S(x,y)$ and $U_{a,b}$ this is the same as

$$G_{\mathbf{0}}(a,b) = S(a,b) \ + \sum_{x,y\in\Lambda} S(a,x)\Pi(x,y)G_{\mathbf{0}}(y,b). \qquad \blacksquare$$

7 The terms $\Pi^{(\Lambda)}_m$ of the lace expansion

Recall the definitions below (6.1) and define

$$\Pi^{(\Lambda)}_m(x,y) := \int_{\mathcal{L}_m(0)} dL\, E_x\left[w^{(\Lambda)}(L)\mathbb{1}_{\{X^{(\Lambda)}_{s'_m}=y\}}\right], \qquad m\geq 0. \tag{7.1}$$

Thus $\Pi^{(\Lambda)}_m$ is the m^{th} term in the series (6.3) that defines $\Pi^{(\Lambda)}(x,y)$. This section has two parts. The first computes formulas for the weights $w^{(\Lambda)}$, and the second derives bounds on $\Pi^{(\Lambda)}_m$ for $m\geq 1$. The main result is Proposition 7.4, which bounds $\Pi^{(\Lambda)}_m$ in terms of $G^{(\Lambda)}_{\mathbf{0}}$; these bounds are used in implementing step one of Section 4.1.

7.1 Formulas for weights

We compute formulas for $\Pi^{(\Lambda)}_0$ and the factors $r^{(\Lambda)}$ that enter into $w^{(\Lambda)}$. Both computations are applications of the chain rule to our choice (6.1) of $\mathcal{Y}$ together with the formulas (6.5)–(6.7) for derivatives of the local time.

7.1.1 The term $\Pi^{(\Lambda)}_0$

For $x\in\Lambda$ define $L^{(\Lambda)}_x\in\mathbb{R}$ by

$$L^{(\Lambda)}_x := \lim_{\boldsymbol{t}\downarrow\mathbf{0}} \partial_{t_x}\log Z^{(\Lambda)}_{\boldsymbol{t}}, \tag{7.2}$$

where $Z^{(\Lambda)}$ is the function entering in the definition (6.1) of $\mathcal{Y}$. The limit $L^{(\Lambda)}_x$ exists and is finite by (Z1)-(Z2) and Lemma 6.3.

Lemma 7.1. *For all finite Λ, $\Pi^{(\Lambda)}_0(x,y) = L^{(\Lambda)}_x\mathbb{1}_{\{x=y\}}$.*

Proof. From (7.1),

$$\Pi^{(\Lambda)}_0(x,y) = \int_{\mathcal{L}_0(0)} dL\, E_x\left[r^{(\Lambda)}_{s_0}\mathbb{1}_{\{X^{(\Lambda)}_{s'_0}=y\}}\right]. \tag{7.3}$$

By definition, dL for $L \in \mathcal{L}_0(0)$ is a unit mass at $s_0 = 0$, and by definition $s_0' = s_0$. Moreover, $X_0^{(\Lambda)} = x$ under the measure E_x. Hence (7.3) becomes

$$\Pi_0^{(\Lambda)}(x, y) = \mathbb{1}_{\{x=y\}} E_x \left[r_0^{(\Lambda)} \right] \tag{7.4}$$

By the definition (5.2) of $r_0^{(\Lambda)}$, (6.1), and $\partial_{t_x} Z_{\mathbf{0}}^{(\Lambda)} = 0$,

$$r_0^{(\Lambda)} = - \lim_{s' \downarrow 0} \sum_{v \in \Lambda} \partial_{t_v} \log Z_{\boldsymbol{t}}^{(\Lambda)} \Big|_{\boldsymbol{t} = \boldsymbol{\tau}_{[0,s']}^{(\Lambda)}} \partial_1 \tau_{[0,s'],v}^{(\Lambda)} \tag{7.5}$$

$$= \sum_{v \in \Lambda} \partial_{t_v} \log Z_{\boldsymbol{t}}^{(\Lambda)} \Big|_{\boldsymbol{t} = \mathbf{0}} \mathbb{1}_{\{X_0^{(\Lambda)} = v\}}, \qquad \text{a.s.} \tag{7.6}$$

In obtaining (7.6) we used (6.6) and the right-continuity of the random walk. Since $X_0^{(\Lambda)} = x$ a.s. under E_x, the lemma follows. ■

7.1.2 The vertex weight $r^{(\Lambda)}$

Recall from (5.2) that $r_{u,v} \coloneqq -\partial_1 \partial_2 \log \mathcal{Y}_{u,v}$. Define, for $x, y \in \Lambda$ and $u < v$,

$$r_{u,v}^{(\Lambda)}(x, y) \coloneqq -\partial_{t_x} \partial_{t_y} \log Z_{\boldsymbol{t}}^{(\Lambda)} \Big|_{\boldsymbol{t} = \boldsymbol{\tau}_{[u,v]}^{(\Lambda)}}. \tag{7.7}$$

Lemma 7.2. *For all finite* Λ *and all* $u < v$,

$$r_{u,v}^{(\Lambda)} = \sum_{x,y \in \Lambda} r_{u,v}^{(\Lambda)}(x, y) \mathbb{1}_{\{X_u^{(\Lambda)} = x\}} \mathbb{1}_{\{X_v^{(\Lambda)} = y\}}. \tag{7.8}$$

Proof. This follows from a calculation similar to the proof of Lemma 7.1; we omit the details. ■

7.2 Bounds on $\Pi_m^{(\Lambda)}$, $m \geq 1$

Our bounds on $\Pi_m^{(\Lambda)}$ for $m \geq 1$ will rely on two assumptions. Recall the definition (7.7) of $r_{u,v}^{(\Lambda)}(x, y)$.

Assumptions 7.3.

(G1) For all $\boldsymbol{t} \in [0, \infty)^\Lambda$, $G_{\boldsymbol{t}}^{(\Lambda)} \leq G_{\mathbf{0}}^{(\Lambda)}$.

(R0) There exists $\bar{r}^{(\Lambda)} \colon \Lambda \times \Lambda \to \mathbb{R}$ *such that* $\left| r_{u,v}^{(\Lambda)}(x, y) \right| \leq \bar{r}^{(\Lambda)}(x, y)$ *for all* $x, y \in \Lambda$ *and* $0 \leq u < v < \infty$.

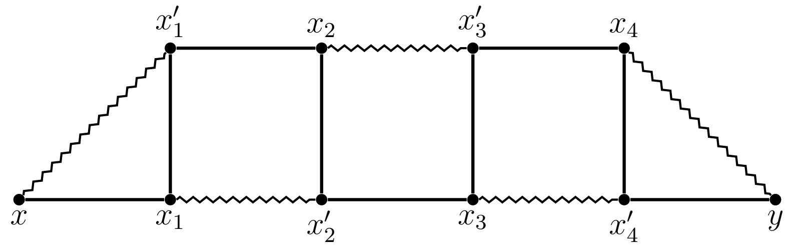


Figure 2: The upper bound on $\Pi_5^{(\Lambda)}(x, y)$ from Proposition 7.4. All vertices except x and y are summed over Λ. Lines connecting vertices represent functions: wavy lines represent $\bar{r}^{(\Lambda)}$ and straight lines represent $G_{\mathbf{0}}$.

Given vertices x and y in Λ, and $m \geq 1$, define

$$\Lambda_{x,y}^{2m-2} := \{((x_i, x_i'))_{i=0,\ldots,m} \mid x_0 = x_0' = x \text{ and } x_m = x_m' = y\}. \tag{7.9}$$

Generic elements of $\Lambda_{x,y}^{2m-2}$ will be denoted by $\boldsymbol{x} = ((x_i, x_i'))_{i=0,\ldots,m}$.

Proposition 7.4. *Suppose (Z1)–(Z2), (G0), and Assumptions 7.3 hold. For $m \geq 1$ and $x, y \in \Lambda$*

$$|\Pi_m^{(\Lambda)}(x, y)| \leq \sum_{\boldsymbol{x} \in \Lambda_{x,y}^{2m-2}} G_{\mathbf{0}}^{(\Lambda)}(x, x_1) \bar{r}^{(\Lambda)}(x, x_1') \times \prod_{j=1}^{m-1} G_{\mathbf{0}}^{(\Lambda)}(x_j, x_j') G_{\mathbf{0}}^{(\Lambda)}(x_j', x_{j+1}) \bar{r}^{(\Lambda)}(x_j, x_{j+1}'). \tag{7.10}$$

See Figure 2 for a diagrammatic representation of the upper bound, which explains our use of the term *lace expansion*: the upper bound is of exactly the form that occurs in discrete-time lace expansion analyses of self-avoiding walk [8, 38]. In more detail, for the Edwards model, a computation (see (11.1)) shows that we can choose $\bar{r}^{(\Lambda)}$ to be a constant times $\mathbb{1}_{\{x=y\}}$. This amounts to shrinking the wavy edges in Figure 2 to points, and these are the diagrams occurring in [8, 38].

The next two subsections prove Proposition 7.4. As Λ is fixed it will be omitted from the notation.

7.2.1 A preparatory lemma

Recall the definition (6.1) of $\mathcal{Y}_{s,t}$ and define

$$\bar{\mathcal{Y}}_{u,v}(w) := \frac{\mathcal{Y}_{u,w}}{\mathcal{Y}_{u,v}}, \qquad u \leq v \leq w. \tag{7.11}$$

By the definitions (5.7) and (5.8) of $w(L)$ and $P(L)$ for $L \in \mathcal{L}_m$ with $m \geq 1$,

$$\begin{aligned} w(L) &= r(L)P(L), \\ P(L) &= \prod_{i=-1}^{m-2} \bar{\mathcal{Y}}_{s'_i, s'_{i+1}}(s'_{i+2}), \qquad s'_{-1} := s'_0. \end{aligned} \tag{7.12}$$

The term $\bar{\mathcal{Y}}_{s'_{-1},s'_0}(s'_1)$ in the product is the factor $\mathcal{Y}_{s'_0,s'_1}$ in (5.8) (recall that $\mathcal{Y}_{s'_0,s'_0} = 1$). For $k = 1, 2$ define $P_{-k}(L)$ by replacing the upper limit $m - 2$ in (7.12) by $m - 2 - k$. By convention empty products are defined to be one.

When we use the next lemma it will help to remember that on the left-hand side u_3 is the time that appears in the σ-algebra and lower limit of integration, while u_1 and u_2 are the subscript parameters in $\bar{\mathcal{Y}}$.

Lemma 7.5. *Let $0 \leq u_1 \leq u_2 \leq u_3$, and let H be $\mathcal{F}_{u_3}$-measurable. Then almost surely*

$$\int_{[u_3,\infty)} d\ell \, E_x \left[H \, \bar{\mathcal{Y}}_{u_1,u_2}(\ell) \, \mathbb{1}_{\{X_\ell = y\}} \middle| \mathcal{F}_{u_3} \right] = H \, \bar{\mathcal{Y}}_{u_1,u_2}(u_3) \, G_{\boldsymbol{\tau}_{[u_1,u_3]}}(X_{u_3}, y) \tag{7.13}$$

Proof. By the definition (7.11) of $\bar{\mathcal{Y}}_{u_1,u_2}(u_3)$

$$\bar{\mathcal{Y}}_{u_1,u_2}(\ell) = \frac{\mathcal{Y}_{u_1,\ell}}{\mathcal{Y}_{u_1,u_2}} = \frac{\mathcal{Y}_{u_1,u_3}}{\mathcal{Y}_{u_1,u_2}} \frac{\mathcal{Y}_{u_1,\ell}}{\mathcal{Y}_{u_1,u_3}} = \bar{\mathcal{Y}}_{u_1,u_2}(u_3) \bar{\mathcal{Y}}_{u_1,u_3}(\ell). \tag{7.14}$$

Insert (7.14) into the left-hand side of (7.13). After taking the $\mathcal{F}_{u_3}$-measurable factor $H \, \bar{\mathcal{Y}}_{u_1,u_2}(u_3)$ outside of the conditional expectation and integration what remains is

$$\int_{[u_3,\infty)} d\ell \, E_a \left[\bar{\mathcal{Y}}_{u_1,u_3}(\ell) \mathbb{1}_{\{X_\ell = y\}} \middle| \mathcal{F}_{u_3} \right] = G_{\boldsymbol{\tau}_{[u_1,u_3]}}(X_{u_3}, y) \qquad \text{a.s.,} \tag{7.15}$$

by the Markov property as stated in Lemma A.2. ∎

7.2.2 Proof of Proposition 7.4

Before giving the proof of Proposition 7.4, we recall the following consequence of the Fubini–Tonelli theorem that will be used in the proof. If X_u is a real-valued stochastic process satisfying $E[\int_{[a,b]} du \, |X_u|] = \int_{[a,b]} du \, E[|X_u|] < \infty$, then integration and conditional expectation can be interchanged:

$$\int_{[a,b]} du \, E\left[X_u \mid \mathcal{G}\right] = E\left[\int_{[a,b]} du \, X_u \mid \mathcal{G} \right], \qquad \text{a.s.} \tag{7.16}$$

Proof of Proposition 7.4. Let $L \in \mathcal{L}_m(0)$. Given a sequence $\boldsymbol{x} \in \Lambda_{x,y}^{2m-2}$ as in (7.9) and a time $u \in [0,\infty)$ define the indicator function

$$\mathcal{I}_{L,\boldsymbol{x},u} := \prod_{s_j \le u} \mathbb{1}_{\{X_{s_j}=x_j\}} \prod_{s'_{j+1} \le u} \mathbb{1}_{\{X_{s'_{j+1}}=x'_{j+1}\}} \tag{7.17}$$

of the event that the path X is at the points (x_i, x'_{i+1}) at the times (s_i, s'_{i+1}) in $L = ((s_i, s'_{i+1}))_{i=0,\ldots,m-1}$. See Figures 1 and 2 and think of the solid lines in the latter figure as a representation of paths X with $X_0 = x$, $X_{s_1} = x_1$, $X_{s'_1} = x'_1$, etc. For $y \in \Lambda$ we have

$$\mathbb{1}_{\{X_0=x\}} \mathbb{1}_{\{X_{s'_m}=y\}} = \sum_{\boldsymbol{x} \in \Lambda_{x,y}^{2m-2}} \mathcal{I}_{L,\boldsymbol{x},s'_m} \tag{7.18}$$

since X cannot be at the absorbing state $*$ at times earlier than s'_m on the event $\{X_{s'_m} = y\}$. We have also used that $\{X_0 = x\} = \{X_{s_0} = x\}$ since $L \in \mathcal{L}_m(0)$. Define

$$\Gamma_{m,\boldsymbol{x}} := \int_{\mathcal{L}_m(0)} dL\, E_x\Big[\mathcal{I}_{L,\boldsymbol{x},s'_m}\, P(L)\Big]. \tag{7.19}$$

Since $\mathbb{1}_{\{X_0=x\}} = 1$ a.s. under E_x, we can insert (7.18) into the definition (7.1) of Π_m to obtain

$$|\Pi_m(x,y)| = \Big| \sum_{\boldsymbol{x} \in \Lambda_{x,y}^{2m-2}} \int_{\mathcal{L}_m(0)} dL\, E_x\Big[\mathcal{I}_{L,\boldsymbol{x},s'_m}\, r(L) P(L)\Big]\Big| \tag{7.20}$$

$$\le \sum_{\boldsymbol{x} \in \Lambda_{x,y}^{2m-2}} \prod_{j=0}^{m-1} \bar{r}(x_j, x'_{j+1}) \Gamma_{m,\boldsymbol{x}}, \tag{7.21}$$

where we have used the triangle inequality and (R0) to bound the vertex functions in $r(L)$, and $P(L) > 0$ to remove absolute values. This reduces Proposition 7.4 to proving

$$\Gamma_{m,\boldsymbol{x}} \le G_{\mathbf{0}}(x_0, x_1) \prod_{j=1}^{m-1} G_{\mathbf{0}}(x_j, x'_j) G_{\mathbf{0}}(x'_j, x_{j+1}), \tag{7.22}$$

for $m \ge 1$ and $\boldsymbol{x} \in \Lambda_{x,y}^{2m-2}$, which we will do by induction on m. The base case $m = 1$ follows by noting that $\mathcal{I}_{L,\boldsymbol{x},s'_1} = \mathbb{1}_{\{X_{s'_1}=y\}}$ under E_x and recalling that $\int_{\mathcal{L}_1(0)} dL = \int_0^\infty ds'_1$, so $\Gamma_{1,\boldsymbol{x}} = G_{\mathbf{0}}(x,y)$.

Suppose (7.22) holds when $m = n$ for some $n \geq 1$. For $L \in \mathcal{L}_{n+1}(0)$ the measure dL factorizes as $dL'\, ds'_n\, ds'_{n+1}$, where dL' is Lebesgue measure on

$$\mathcal{L}'_n(0) = \{(s_1, s'_1, s_2, \ldots, s_n) \mid s_1 < s'_1 < \cdots < s_n\} \tag{7.23}$$

and $ds'_n\, ds'_{n+1}$ is Lebesgue measure on the set of (s'_n, s'_{n+1}) such that $s_n < s'_n < s'_{n+1}$. Rewriting $\Gamma_{n+1,\boldsymbol{x}}$ using this factorization yields

$$\Gamma_{n+1,\boldsymbol{x}} = \int_{\mathcal{L}'_n(0)} dL' \int_{[s_n,\infty)} ds'_n \int_{[s'_n,\infty)} ds'_{n+1}\, E_x\Big[\mathcal{I}_{L,\boldsymbol{x},s'_{n+1}} P(L)\Big]. \tag{7.24}$$

The induction step involves estimating the integrals over s'_{n+1} and s'_n by Lemma 7.5 and (G1). To bound the s'_{n+1} integral note the range of integration starts at s'_n and accordingly insert a conditional expectation with respect to $\mathcal{F}_{s'_n}$ under the expectation E_x. Bringing the s'_{n+1} integral inside the expectation yields

$$\Gamma_{n+1,\boldsymbol{x}} = \int_{\mathcal{L}'_n(0)} dL' \int_{[s_n,\infty)} ds'_n\, E_x\Big[J_{L,\boldsymbol{x},s'_n}\Big], \quad \text{where} \tag{7.25}$$

$$J_{L,\boldsymbol{x},s'_n} := \int_{[s'_n,\infty)} ds'_{n+1}\, E_x\Big[\mathcal{I}_{L,\boldsymbol{x},s'_{n+1}} P(L)\Big|\mathcal{F}_{s'_n}\Big]. \tag{7.26}$$

Recall that $P_{-1}(L)$ was defined below (7.12), and note that

$$\mathcal{I}_{L,\boldsymbol{x},s'_{n+1}} = \mathcal{I}_{L,\boldsymbol{x},s'_n} \mathbb{1}_{\{X_{s'_{n+1}} = x'_{n+1}\}}, \qquad P(L) = P_{-1}(L)\bar{\mathcal{Y}}_{s'_{n-1},s'_n}(s_{n+1}). \tag{7.27}$$

We insert these identities into $J_{L,\boldsymbol{x},s'_n}$ and apply Lemma 7.5 with $(u_1, u_2, u_3) = (s'_{n-1}, s'_n, s'_n)$ and $H = \mathcal{I}_{L,\boldsymbol{x},s'_n} P_{-1}(L) \geq 0$. The result is, after using (G1),

$$J_{L,\boldsymbol{x},s'_n} \leq \mathcal{I}_{L,\boldsymbol{x},s'_n} P_{-1}(L)\, G_{\mathbf{0}}(x'_n, x'_{n+1}) \tag{7.28}$$

because $\bar{\mathcal{Y}}_{s'_{n-1},s'_n}(s'_n) = 1$. Hence by (7.25) and that $x'_{n+1} = x_{n+1}$ by (7.9),

$$\Gamma_{n+1,\boldsymbol{x}} \leq \int_{\mathcal{L}'_n(0)} dL' \int_{[s_n,\infty)} ds'_n\, E_x\Big[\mathcal{I}_{L,\boldsymbol{x},s'_n} P_{-1}(L)\Big]\, G_{\mathbf{0}}(x'_n, x_{n+1}). \tag{7.29}$$

For the s'_n integral in (7.29) the procedure is similar so we will be brief. Insert a conditional expectation with respect to $\mathcal{F}_{s_n}$ under the expectation in (7.29), bring the integral over s'_n inside E_x, and then insert

$$\mathcal{I}_{L,\boldsymbol{x},s'_n} = \mathcal{I}_{L,\boldsymbol{x},s_n} \mathbb{1}_{\{X_{s'_n} = x'_n\}}, \qquad P_{-1}(L) = P_{-2}(L)\bar{\mathcal{Y}}_{s'_{n-2},s'_{n-1}}(s'_n). \tag{7.30}$$

We apply Lemma 7.5 with $(u_1, u_2, u_3) = (s'_{n-2}, s'_{n-1}, s_n)$ and $H = \mathcal{I}_{L,\boldsymbol{x},s_n}$, with the result, again after using (G1),

$$\Gamma_{n+1,\boldsymbol{x}} \leq \int_{\mathcal{L}'_n(0)} dL'\, E_x\Big[\mathcal{I}_{L,\boldsymbol{x},s_n}\, P_{-2}(L)\, \bar{\mathcal{Y}}_{s'_{n-2},s'_{n-1}}(s_n)\Big] \times G_{\mathbf{0}}(x_n, x'_n) G_{\mathbf{0}}(x'_n, x_{n+1}). \tag{7.31}$$

By (7.23) the measure spaces $(\mathcal{L}'_n(0), \mathcal{B}(\mathcal{L}'_n(0)), dL')$ and $(\mathcal{L}_n(0), \mathcal{B}(\mathcal{L}_n(0)), dL)$ are the same. Using this identification and changing the integration variable s_n to s'_n in (7.31) allows $P_{-2}(L)\, \bar{\mathcal{Y}}_{s'_{n-2},s'_{n-1}}(s_n)$ to be rewritten as $P(L)$ for $L \in \mathcal{L}_n(0)$ since s'_n is the final time in such a lace. The result is

$$\Gamma_{n+1,\boldsymbol{x}} \leq \left(\int_{\mathcal{L}_n(0)} dL\, E_x\Big[\mathcal{I}_{L,\boldsymbol{x},s'_n}\, P(L)\,\Big]\right) G_{\mathbf{0}}(x_n, x'_n) G_{\mathbf{0}}(x'_n, x_{n+1}). \tag{7.32}$$

Replacing $\boldsymbol{x}$ in (7.32) with $\boldsymbol{x}' := (x_i, x'_i)_{i=0,\dots,n}$ with $x'_n := x_n$ shows the quantity in brackets is $\Gamma_{m,\boldsymbol{x}}$ with $m = n$. Applying the inductive hypothesis (7.22) to this term completes the proof. ■

8 Preparation for the infinite volume limit

This section continues with step one of Section 4.1. The main result is Corollary 8.6, which is a bound on $\Psi^{(\Lambda)}$, which is the finite volume version of the term Ψ in (4.5). A crucial aspect of the bound given by Corollary 8.6 is that it is uniform in Λ. The bound relies on Assumptions 8.4, which play a continuing role in the remainder of the paper.

8.1 Convolution estimates

Recall that $|||x||| = \max\{|x|, 1\}$. The next lemma says that when the sum over $w \in \mathbb{Z}^d$ is sufficiently convergent there is a bound as if $w = 0$.

Lemma 8.1 (Equation (4.17) of [21])**.** *Let $d \geq 5$, $u, v \in \mathbb{Z}^d$. There exists a $C > 0$ such that*

$$\sum_{w \in \mathbb{Z}^d} |||w|||^{4-2d} |||w - v|||^{2-d} |||w - u|||^{2-d} \leq C |||v|||^{2-d} |||u|||^{2-d}.$$

The next estimate says the convolution of two functions decays according to whichever has the weaker decay, provided at least one of them is integrable.

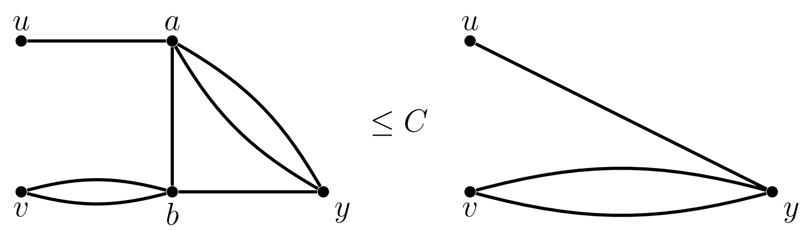


Figure 3: A diagrammatic depiction of Lemma 8.3. Solid lines represents factors $|||x_2 - x_1|||^{2-d}$. The vertices u, v, y are fixed, but a and b are summed over $\mathbb{Z}^d$.

Lemma 8.2 (Proposition 1.7(i) of [21])**.** *Let $f, g\colon \mathbb{Z}^d \to \mathbb{R}$ be such that $|f(x)| \le |||x|||^{-a}$, $|g(x)| \le |||x|||^{-b}$, $a \ge b > 0$. There exists a $C > 0$ such that*

$$|(f * g)(x)| \le \begin{cases} C|||x|||^{-b}, & a > d, \\ C|||x|||^{d-(a+b)}, & a < d \text{ and } a + b > d. \end{cases}$$

Figure 3 gives a diagrammatic representation of the next lemma.

Lemma 8.3. *Fix $u, v, y \in \mathbb{Z}^d$, $d \ge 5$. There exists a $C > 0$ such that*

$$\sum_{a,b \in \mathbb{Z}^d} |||u - a|||^{2-d} |||y - a|||^{4-2d} |||b - a|||^{2-d} |||v - b|||^{4-2d} |||y - b|||^{2-d} \qquad (8.1)$$
$$\le C|||y - u|||^{2-d} |||y - v|||^{4-2d}.$$

Proof. Lemma 8.1 can be applied to the sum over a to upper bound the left-hand side of (8.1) as if $a = y$, that is, by a constant $C > 0$ times

$$|||y - u|||^{2-d} \sum_{b \in \mathbb{Z}^d} |||y - b|||^{2-d} |||v - b|||^{4-2d} |||y - b|||^{2-d}.$$

The sum over b is a convolution of two functions that decay at rate $\gamma = 2d - 4$. As γ exceeds d when $d \ge 5$, Lemma 8.2 implies the claimed upper bound. ■

8.2 $\Psi^{(\Lambda)}$ and uniform bounds on $\Psi^{(\Lambda)}$

Assumptions 8.4. *For all $\Lambda \subset \mathbb{Z}^d$ finite,*

(G2) For $\Lambda' \subset \Lambda$ and $x, y \in \Lambda'$, $G_{\mathbf{0}}^{(\Lambda')}(x, y) \le G_{\mathbf{0}}^{(\Lambda)}(x, y)$, and

(R1) There exists $\eta > 0$ independent of Λ, such that for $0 \le u < v < \infty$

$$\left|r_{u,v}^{(\Lambda)}(x,y)\right| \le \eta(\mathbb{1}_{\{x=y\}} + G_{\mathbf{0}}^{(\Lambda)}(x,y)^2), \qquad x,y \in \Lambda. \tag{8.2}$$

The assumption (R1) supersedes (R0) by stipulating a specific form for the bound $\bar{r}^{(\Lambda)}$ of (R0). This form is motivated by our applications, as will become clear in Section 11. Note that (G2) implies $G_{\mathbf{0}}^{(\infty)} := \lim_{\Lambda \uparrow \mathbb{Z}^d} G_{\mathbf{0}}^{(\Lambda)}$ exists as a function taking values in $[0,\infty]$.

The propositions of this section will be made under the assumption that $G_{\mathbf{0}}^{(\infty)}$ satisfies a K-IRB, i.e., that

$$G_{\mathbf{0}}^{(\infty)}(x,y) \le KS(y-x), \qquad x,y \in \mathbb{Z}^d. \tag{8.3}$$

The key aspect of the next proposition is that the bound is independent of Λ and proportional to $(c\eta)^m$.

Proposition 8.5. *Suppose $d \ge 5$, and that (Z1)–(Z2), (G1)–(G2), and (R1) hold, and that a K-IRB (8.3) holds. Then there are constants $c_1, c_2 > 0$ depending only on d, J and K such that for each $m \ge 1$*

$$|\Pi_m^{(\Lambda)}(x,y)| \le c_1(c_2\eta)^m |||y-x|||^{-3(d-2)}. \tag{8.4}$$

Proof. The basic input in our estimates is that by (G1) and (G2), $G_{\boldsymbol{t}}^{(\Lambda)}(x,y) \le G_{\mathbf{0}}^{(\infty)}(x,y)$, so the K-IRB and (4.8) imply

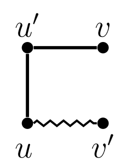
$$G_{\boldsymbol{t}}^{(\Lambda)}(x,y) \le K\tilde{C}_J |||y-x|||^{2-d}, \tag{8.5}$$

and hence, letting $K_1 = \max\{K\tilde{C}_J, 1\}$, by (R1) and (8.5),

$$\left|r_{u,v}^{(\Lambda)}(x,y)\right| \le \eta(\mathbb{1}_{\{x=y\}} + K_1^2)|||y-x|||^{4-2d}. \tag{8.6}$$

For $u, u', v, v' \in \mathbb{Z}^d$, define

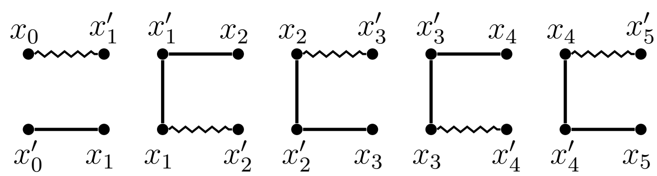
$$A^{(\Lambda)}(u,u';v,v') := G_{\mathbf{0}}^{(\Lambda)}(u,u')\bar{r}^{(\Lambda)}(u,v')G_{\mathbf{0}}^{(\Lambda)}(u',v) =: \tag{8.7}$$



where the right-hand side follows the diagrammatic notation of Figure 2. Recall the notation $\boldsymbol{x}$ defined in (7.9), and note that (G2) combined with a K-IRB implies (G0) holds. Hence we can apply Proposition 7.4, and this proposition can be rewritten as

$$|\Pi_m^{(\Lambda)}(x,y)| \le \sum_{\boldsymbol{x}\in\Lambda_{x,y}^{2m-2}} G_{\mathbf{0}}^{(\Lambda)}(x_0',x_1)\bar{r}^{(\Lambda)}(x_0,x_1') \prod_{j=1}^{m-1} A^{(\Lambda)}(x_j,x_j';x_{j+1},x_{j+1}'). \tag{8.8}$$

To check this claim compare Figure 2 with



(8.9)

and recall that $x_0 = x_0' = x$ and $x_m = x_m' = y$ and there is a sum over the remaining x_i, x_i'.

Inserting the bounds (8.5) (with $\boldsymbol{t} = \mathbf{0}$) and (8.6) into $A^{(\Lambda)}(u,u';v,v')$ we obtain

$$A^{(\Lambda)}(u,u';v,v') \le \eta \bar{A}(u,u';v,v'), \tag{8.10}$$

$$G_{\mathbf{0}}^{(\Lambda)}(x_0',x_1)\bar{r}(x_0,x_1') \le \eta K_1^{-1}\bar{A}(x_0,x_0';x_1,x_1'), \tag{8.11}$$

where

$$\bar{A}(u,u';v,v') := \frac{K_1^2(1+K_1^2)}{|||u-u'|||^{d-2}|||u-v'|||^{2d-4}|||u'-v|||^{d-2}}. \tag{8.12}$$

Inserting (8.10)–(8.11) into (8.8) yields

$$|\Pi_m^{(\Lambda)}(x,y)| \le K_1^{-1}\eta^m U_m(x,y) \tag{8.13}$$

where

$$U_m(x,y) := \sum_{\boldsymbol{x}\in\mathbb{Z}_{x,y}^{d(2m-2)}} \prod_{j=0}^{m-1} \bar{A}(x_j,x_j';x_{j+1},x_{j+1}'). \tag{8.14}$$

In obtaining (8.13) sums over vertices in Λ have been extended to sums over $\mathbb{Z}^d$, which gives an upper bound as all terms are non-negative. To prove (8.4) holds for $m \ge 1$ it therefore suffices to establish the upper bound

$$U_m(x,y) \le c_1 c_2^m |||y-x|||^{-3(d-2)}. \tag{8.15}$$

We prove (8.15) with $c_1 = K_1^2(1+K_1^2)$ and $c_2 = \max\{1, C(K_1)\}$ (where C is a constant defined in (8.16) below) by induction. For $m = 1$ there is no sum in (8.14), so the bound follows from $x_0 = x'_0 = x$, $x_1 = x'_1 = y$, and (8.12).

Suppose the upper bound has been established for some $n-1 \geq 1$. By Lemma 8.3 there is a $C = C(K_1) > 0$ such that for $x_1, x'_1, x_2, x'_2 \in \mathbb{Z}^d$,

$$\sum_{u,v\in\mathbb{Z}^d} \bar{A}(x_1, x'_1; u, v)\bar{A}(u, v; x'_2, x'_2) \leq C\bar{A}(x_1, x'_1; x'_2, x'_2). \tag{8.16}$$

For $m = n$ using (8.16) to estimate the sum over x_{n-1}, x'_{n-1} in the definition of U_n, and then using the induction hypotheses, implies

$$U_n(x, y) = \sum_{\boldsymbol{x}\in\mathbb{Z}^{d(2n-2)}_{x,y}} \prod_{j=0}^{n-1} \bar{A}(x_j, x'_j; x_{j+1}, x'_{j+1}) \tag{8.17}$$

$$\leq C \sum_{\boldsymbol{x}\in\mathbb{Z}^{d(2n-4)}_{x,y}} \prod_{j=0}^{n-2} \bar{A}(x_j, x'_j; x_{j+1}, x'_{j+1}) \tag{8.18}$$

$$\leq c_1 c_2^n \| y - x \|^{-3(d-2)},$$

where in the second line we have redefined $x_{n-1} := x_n$ and $x'_{n-1} := x'_n$. The final line follows by recalling that $x_n = x'_n = y$. ■

Define $\Psi^{(\Lambda)}$, the finite-volume precursor to Ψ from Section 4.1, by

$$\Psi^{(\Lambda)}(x, y) := \sum_{m\geq 1} \Pi^{(\Lambda)}_m(x, y) = \Pi^{(\Lambda)}(x, y) - \Pi^{(\Lambda)}_0(x, y). \tag{8.19}$$

Summing (8.4) over $m \geq 1$ immediately gives the following.

Corollary 8.6. *Under the hypotheses of Proposition 8.5, if $c_2\eta < 1$ then*

$$|\Psi^{(\Lambda)}(x, y)| \leq \frac{c_1 c_2 \eta}{1 - c_2\eta} \| y - x \|^{-3(d-2)}, \qquad x, y \in \mathbb{Z}^d. \tag{8.20}$$

9 The lace expansion in infinite volume

The main result of this section is Proposition 9.9, which constructs $L_{g,\nu}$ and $\Psi_{g,\nu}$ such that (4.5) holds. This completes a key part of step one of Section 4.1. The proof uses Corollary 8.6 and the algebraic structure of Proposition 6.2 to take the infinite volume limit of Proposition 6.2. In particular we prove the existence of the infinite volume limit $\Pi^{(\infty)}$ of $\Pi^{(\Lambda)}$.

9.1 The infinite volume limit of $G^{(\Lambda)}$

We begin by establishing some properties of $G^{(\Lambda)}$.

Lemma 9.1. *The infinite volume limit* $G^{(\infty)}_{\mathbf{0}}(x,y) := \lim_{\Lambda\uparrow\mathbb{Z}^d} G^{(\Lambda)}_{\mathbf{0}}(x,y)$ *exists in* $[0,\infty]$ *pointwise in* $x,y\in\mathbb{Z}^d$*, is translation invariant, and* $G^{(\infty)}_{\mathbf{0}}(x) := G^{(\infty)}_{\mathbf{0}}(0,x)$ *is* $\mathbb{Z}^d$*-symmetric. If a K-IRB* (8.3) *holds, then* $G^{(\infty)}_{\mathbf{0}}(x)\in[0,\infty)$.

Proof. By monotone convergence provided by (G2) the limit $G^{(\infty)}_{\mathbf{0}}(a,b) := \lim_{\Lambda\uparrow\mathbb{Z}^d} G^{(\Lambda)}_{\mathbf{0}}(a,b)$ exists for any choice of exhaustion $\Lambda_n\uparrow\mathbb{Z}^d$. The limit is independent of the limiting sequence: given two sequences Λ_n and Λ'_n the sequences can be interlaced (under inclusion) with one another.

This also implies $G^{(\infty)}_{\mathbf{0}}(a,b)$ is translation invariant: the limit of $G^{(\Lambda)}_{\mathbf{0}}(a,b)$ through Λ_n equals the limit of $G^{(\Lambda)}_{\mathbf{0}}(a',b')$ through (Λ_n+e), where $a'=a+e$, $b'=b+e$, and e a unit vector in $\mathbb{Z}^d$. Simultaneously, this latter limit is the same as the limit of $G^{(\Lambda)}_{\mathbf{0}}(a',b')$ through (Λ_n). This proves translation invariance. A similar argument shows $G^{(\infty)}_{\mathbf{0}}(x)$ is $\mathbb{Z}^d$-symmetric.

If a K-IRB holds, then $G^{(\Lambda)}_{\mathbf{0}}(a,b)$ is bounded uniformly in Λ and a,b, and hence $G^{(\infty)}_{\mathbf{0}}$ cannot take the value ∞. ■

Since $G^{(\Lambda)}_{\mathbf{0}}(x,y)$ is translation invariant, a K-IRB of the form (8.3) implies $G^{(\Lambda)}_{\mathbf{0}}(x)$ satisfies a K-IRB of the form (4.4). Thus in the sequel there is no ambiguity when we say $G^{(\Lambda)}_{\mathbf{0}}$ satisfies a K-IRB without further specification.

9.2 The infinite volume limit of $\Pi^{(\Lambda)}$

In this section we prove the existence of the infinite volume limit of $\Pi^{(\Lambda)}$ under the following assumption; recall the definition of $L^{(\Lambda)}_x$ from (7.2).

Assumptions 9.2.

(Z3) If a K-IRB holds, then $L^{(\Lambda)}_x$ *is bounded uniformly in* x *and* Λ*, and the limit* $L^{(\infty)}_x := \lim_{\Lambda\uparrow\mathbb{Z}^d} L^{(\Lambda)}_x$ *exists and is independent of* x.

For $A,B\colon\Lambda\times\Lambda\to\mathbb{R}$ we write $AB(x,y)=\sum_{u\in\Lambda}A(x,u)B(u,y)$. Our first lemma is the well-known algebraic fact that left and right inverses coincide for algebraic structures with an associative product. We state it here in a language convenient for our purposes.

Lemma 9.3. *Let* $A,B,C\colon\mathbb{Z}^d\times\mathbb{Z}^d\to\mathbb{R}$, $AB(x,y)=CA(x,y)=\mathbb{1}_{\{x=y\}}$*, and suppose* $C(AB)=(CA)B$*. Then* $B=C$.

Proof. $C = C(AB) = (CA)B = B$. ■

Lemma 9.4. *Assume the hypotheses of Proposition 8.5 and (Z3). If η is sufficiently small, then for $x, y \in \mathbb{Z}^d$*

$$|\Pi^{(\Lambda)}(x,y)| = O(|||y-x|||^{-3(d-2)}), \tag{9.1}$$

uniformly in x, y and Λ.

Proof. This is immediate from Corollary 8.6, Lemma 7.1, and (Z3). ■

Lemma 9.5. *Assume the hypotheses of Proposition 8.5 and (Z3) and that η is sufficiently small. Then for any sequence of volumes $\Lambda_n \uparrow \mathbb{Z}^d$ there exists a subsequence Λ_{n_k} such that $\Pi(x,y) := \lim_{k\to\infty} \Pi^{(\Lambda_{n_k})}(x,y)$ exists pointwise in $x, y \in \mathbb{Z}^d$.*

Proof. Extend the definition of $\Pi^{(\Lambda)}\colon \Lambda \times \Lambda \to \mathbb{R}$ to $\mathbb{Z}^d \times \mathbb{Z}^d$ by letting $\Pi^{(\Lambda)}(x,y) = 0$ if $x \notin \Lambda$ or $y \notin \Lambda$. By Lemma 9.4, $|\Pi^{(\Lambda)}(x,y)|$ is $O(|||y-x|||^{-3(d-2)})$ uniformly in Λ. Thus for any $x, y \in \mathbb{Z}^d$ and any increasing sequence of volumes $\Lambda_n \uparrow \mathbb{Z}^d$, there exists a subsequence $\Lambda_{n_k(x,y)}$ such that $\Pi^{(\Lambda_{n_k(x,y)})}(x,y)$ converges as $k \to \infty$. By a diagonal argument we can refine this sequence such that the limit exists for all $x, y \in \mathbb{Z}^d$. ■

Lemma 9.6. *Assume the hypotheses of Proposition 8.5 and (Z3) and that η is sufficiently small. For a sequence $\Lambda_n \uparrow \mathbb{Z}^d$ for which $\Pi^{(\Lambda_n)}$ converges pointwise to Π,*

$$S^{(\Lambda_n)}\Pi^{(\Lambda_n)}G_{\mathbf{0}}^{(\Lambda_n)}(x,y) \to S\Pi G_{\mathbf{0}}^{(\infty)}(x,y) \qquad x, y \in \mathbb{Z}^d, \tag{9.2}$$

and the product on the right-hand side is absolutely convergent, so there is no ambiguity in the order of the products.

Proof. By Lemma 9.4, $\Pi^{(\Lambda)}(x,y)$ is uniformly bounded above by a multiple of $U(x,y) := |||y-x|||^{-3(d-2)}$. $S^{(\Lambda)}$ is bounded above by S, and by (G2) $G_{\mathbf{0}}^{(\Lambda)}$ is bounded above by $G_{\mathbf{0}}^{(\infty)}$.

Both $S(x,y)$ and $G_{\mathbf{0}}^{(\infty)}(x,y)$ are non-negative and bounded above by a multiple of $|||y-x|||^{-d+2}$. Hence the products $SU(x,y)$ and $UG_{\mathbf{0}}^{(\infty)}(x,y)$ are both absolutely convergent by Lemma 8.2, and decay at least as fast as a multiple of $|||y-x|||^{-d+2}$. Applying Lemma 8.2 once more with $d \geq 5$ shows $SUG_{\mathbf{0}}^{(\infty)}(x,y)$ is given by an absolutely convergent double sum. These applications of Lemma 8.2 to $G_{\mathbf{0}}^{(\infty)}$ are valid by Lemma 9.1.

As $\Pi^{(\Lambda_n)} \to \Pi$ pointwise by hypothesis, $S^{(\Lambda_n)} \to S$ pointwise (see (2.9)), and $G_{\mathbf{0}}^{(\Lambda_n)} \to G_{\mathbf{0}}^{(\infty)}$ pointwise by Lemma 9.1, (9.2) follows by the dominated convergence theorem. ■

Recall the definition of $\Delta^{(\infty)}$ from (2.1).

Lemma 9.7. *Assume the hypotheses of Proposition 8.5 and (Z3), that η is sufficiently small, and that $\Lambda_n \uparrow \mathbb{Z}^d$ such that $\Pi^{(\Lambda_n)} \to \Pi$ pointwise. Then $-(\Delta^{(\infty)} + \Pi)$ is a two-sided inverse of $G_{\mathbf{0}}^{(\infty)}$.*

Proof. By (6.4), Lemma 9.1 and Lemma 9.6,

$$G_{\mathbf{0}}^{(\infty)} = S + S\Pi G_{\mathbf{0}}^{(\infty)}. \tag{9.3}$$

Multiplying (9.3) on the left by $-\Delta^{(\infty)}$ and using (2.12) yields

$$-(\Delta^{(\infty)} + \Pi) G_{\mathbf{0}}^{(\infty)}(x,y) = \mathbb{1}_{\{x=y\}}. \tag{9.4}$$

In applying (2.12) we have used that $\Delta^{(\infty)}(S\Pi G_{\mathbf{0}}^{(\infty)}) = (\Delta^{(\infty)}S)(\Pi G_{\mathbf{0}}^{(\infty)})$, which holds as $\Delta^{(\infty)}(x,\cdot)$ is finite range by (J4).

Letting A^t denote the transpose of a matrix A, note that

$$-\mathbb{1}_{\{x=y\}} = (G_{\mathbf{0}}^{(\infty)})^t (\Delta^{(\infty)} + \Pi)^t(x,y) = G_{\mathbf{0}}^{(\infty)}(\Delta^{(\infty)} + \Pi)^t(x,y), \tag{9.5}$$

as $(G_{\mathbf{0}}^{(\infty)})^t = G_{\mathbf{0}}^{(\infty)}$ by Lemma 9.1. Note $\Pi(x,y) = O(|\!|\!|y-x|\!|\!|^{-3(d-2)})$, as Π is a pointwise limit of functions satisfying this uniform bound by Lemma 9.4. Since $\Delta^{(\infty)}(x,\cdot)$ is finite-range by (J4), this implies $(\Delta^{(\infty)}+\Pi)(x,y)$ is $O(|\!|\!|y-x|\!|\!|^{-3(d-2)})$, and hence $(\Delta^{(\infty)}+\Pi)^t(x,y)$ is also $O(|\!|\!|y-x|\!|\!|^{-3(d-2)})$. Thus $(\Delta^{(\infty)}+\Pi)G_{\mathbf{0}}^{(\infty)}(\Delta^{(\infty)}+\Pi)^t$ is absolutely convergent and unambiguously defined by Lemma 8.2, so Lemma 9.3 implies

$$\Delta^{(\infty)} + \Pi = (\Delta^{(\infty)} + \Pi)^t. \qquad \blacksquare$$

Proposition 9.8. *Assume the hypotheses of Proposition 8.5 and (Z3) and that η is sufficiently small. The limit $\Pi^{(\infty)}(x,y) := \lim_{\Lambda\uparrow\mathbb{Z}^d} \Pi^{(\Lambda)}(x,y)$ exists. Moreover, it is translation invariant and $\Pi^{(\infty)}(0,x)$ is $\mathbb{Z}^d$-symmetric.*

Proof. Lemma 9.3 implies that if there are two-sided inverses A_1, A_2 of $G_{\mathbf{0}}^{(\infty)}$ such that $(A_1 G_{\mathbf{0}}^{(\infty)})A_2 = A_1(G_{\mathbf{0}}^{(\infty)}A_2)$, then $A_1 = A_2$. By Lemma 9.7 and its proof, if $\Lambda \uparrow \mathbb{Z}^d$ then any pair of subsequential limits $\Pi^{(\infty)}$ and $\tilde{\Pi}^{(\infty)}$ give rise to left and right inverses $A_1 = -(\Delta^{(\infty)} + \Pi^{(\infty)})$ and $A_2 = -(\Delta^{(\infty)} + \tilde{\Pi}^{(\infty)})$ of $G_{\mathbf{0}}^{(\infty)}$ satisfying the above conditions, so $A_1 = A_2$ and hence $\Pi^{(\infty)} = \tilde{\Pi}^{(\infty)}$. Since any sequence of volumes $\Lambda_n \uparrow \mathbb{Z}^d$ has a pointwise limit (by Lemma 9.5), and all the limit points coincide, this implies that $\lim_{\Lambda\uparrow\mathbb{Z}^d} \Pi^{(\Lambda)}$ exists.

Let $T \in \mathrm{Aut}_0(\mathbb{Z}^d)$; the matrix representation[2] of T acts by conjugation, so $TG_{\mathbf{0}}^{(\infty)}T^{-1} = G_{\mathbf{0}}^{(\infty)}$ by Lemma 9.1. Thus

$$-\mathbb{1}_{\{x=y\}} = T(\Delta^{(\infty)} + \Pi^{(\infty)})G_{\mathbf{0}}^{(\infty)}T^{-1}(x,y) = T(\Delta^{(\infty)} + \Pi^{(\infty)})T^{-1}G_{\mathbf{0}}^{(\infty)}(x,y),$$

but since $T(\Delta^{(\infty)} + \Pi^{(\infty)})T^{-1}(x,y) = O(|\!|\!| y-x |\!|\!|^{-3(d-2)})$ as automorphisms preserve $|\!|\!|\cdot|\!|\!|$, Lemma 9.3 and Lemma 9.7 imply $T(\Delta^{(\infty)} + \Pi^{(\infty)})T^{-1} = \Delta^{(\infty)} + \Pi^{(\infty)}$. ■

9.3 The infinite-volume lace expansion equation

In the next proposition $\Psi^{(\infty)}(x) \coloneqq \Psi^{(\infty)}(0,x)$.

Proposition 9.9. *Assume the hypotheses of Proposition 8.5 and (Z3) hold. Then there exist $\alpha > 0$, $\eta_0 > 0$, $L^{(\infty)}$ and $\Psi^{(\infty)}$ such that for all $\eta \in (0, \eta_0)$,*

(i) $\Psi^{(\infty)}(x)$ *is a $\mathbb{Z}^d$-symmetric function of x.*

(ii) $|\Psi^{(\infty)}(x)| \le \alpha\eta |\!|\!| x |\!|\!|^{-3(d-2)}$.

(iii) $(\hat{J} - L^{(\infty)})G_{\mathbf{0}}^{(\infty)}(x) = \mathbb{1}_{\{x=0\}} + J_+ * G_{\mathbf{0}}^{(\infty)}(x) + \Psi^{(\infty)} * G_{\mathbf{0}}^{(\infty)}(x)$.

(iv) $\hat{J} - L^{(\infty)} \ge \frac{1+O(\eta)}{KS(0)}$.

Proof. Item (i): Note $\Psi^{(\Lambda)}(x,y) = \Pi^{(\Lambda)}(x,y) - \Pi_0^{(\Lambda)}(x,y)$. Both terms on the right-hand side have translation invariant and $\mathbb{Z}^d$-symmetric infinite volume limits by Proposition 9.8, Lemma 7.1 and (Z3) as desired.

Item (ii) follows from the finite-volume estimate given by Corollary 8.6.

Item (iii): Corollary 8.6 (and its proof) and (Z3) imply (F0) holds, and hence by taking the limit as $\Lambda \uparrow \mathbb{Z}^d$ of Proposition 6.2 we obtain

$$G_{\mathbf{0}}^{(\infty)}(x) = S(x) + L^{(\infty)}S * G_{\mathbf{0}}^{(\infty)}(x) + S * G_{\mathbf{0}}^{(\infty)} * \Psi^{(\infty)}(x). \tag{9.6}$$

Recall from (2.2) that $J(x-y) = -\hat{J}\mathbb{1}_{\{x=y\}} + J_+(x-y)$. We apply $-J$ to (9.6) followed by $-J * S(x) = \mathbb{1}_{\{x=0\}}$ from Lemma 2.4. By item (ii) we have absolute convergence of all sums. By re-arranging we obtain (iii).

[2] If T is an automorphism, then the matrix representation of T (abusing notation we also write this as T), is given by $T(x,y) \coloneqq \delta_{T(x),y}$

Item (iv): We evaluate (iii) at $x = 0$, insert item (ii) using $d \geq 5$ to obtain $|G_{\mathbf{0}}^{(\infty)} * \Psi^{(\infty)}(0)| = O(\eta)$ and insert $J_+ * G_{\mathbf{0}}^{(\infty)}(0) \geq 0$. The result is the desired bound

$$\hat{J} - L^{(\infty)} \geq \frac{1 + O(\eta)}{G_{\mathbf{0}}^{(\infty)}(0)} \geq \frac{1 + O(\eta)}{KS(0)}, \tag{9.7}$$

where the second inequality is implied by the K-IRB hypothesis. ■

10 Final hypotheses and proof of asymptotic behaviour

This section reduces the proof of Theorem 4.1, and in turn the proof of our main Theorem 3.6, to verifying that the Edwards model and the $n = 1, 2$-component $g|\varphi|^4$ models satisfy Assumptions 10.1 and Assumptions 10.2 below. Assumptions 10.1 is the accumulation of all the assumptions from Section 5 to this point on the interaction Z and the jump law J. Assumptions 10.2 constitute further hypotheses.

10.1 Final hypotheses

This subsection summarizes the hypotheses under which we will draw conclusions about the asymptotic behaviour of the Green's function. For each finite $\Lambda \subset \mathbb{Z}^d$ and two parameters $g > 0$ and $\nu \in \mathbb{R}$ let $Z_{g,\nu}^{(\Lambda)} \colon [0, \infty)^\Lambda \to (0, \infty)$, i.e. $Z_{g,\nu,\boldsymbol{t}}^{(\Lambda)} \in (0, \infty)$.

Assumptions 10.1. *Assume J is such that (J1)–(J4) hold, and*

(i) for each $g > 0, \nu \in \mathbb{R}$, (Z1), (Z2), (Z3), (G1)–(G2) hold for $Z_{g,\nu}^{(\Lambda)}$.

(ii) there exists $c_* > 0$ such that for each $g > 0, \nu \in \mathbb{R}$, (R1) holds for $Z_{g,\nu}^{(\Lambda)}$ with $\eta = c_* g$.

Assumptions 10.1 and Lemma 9.1 imply that for each $g > 0, \nu \in \mathbb{R}$ the infinite-volume Green's function $G_{g,\nu} \coloneqq G_{g,\nu,\mathbf{0}}^{(\infty)}$ exists. Define

$$\chi_g(\nu) \coloneqq \sum_{x \in \mathbb{Z}^d} G_{g,\nu}(x), \tag{10.1}$$

which is called the *susceptibility*. The *critical value* $\nu_c(g)$ of ν is defined as

$$\nu_c(g) \coloneqq \inf\{\nu \in \mathbb{R} \mid \chi_g(\nu) < \infty\}. \tag{10.2}$$

Assumptions 10.2. *Assume that $g > 0$, that $\nu_c(g) \leq 0$, and that*

(G3) $G_{g,\nu}(x)$ *is non-increasing in* $\nu \in \mathbb{R}$. *Moreover,*

- *(a) For* $\nu \in (\nu_c(g), \infty)$ *and* $x \in \mathbb{Z}^d$, $G_{g,\nu}(x)$ *is continuous in* ν.
- *(b) If* $G_{g,\nu} \leq 3S$ *for some* $\nu \in \mathbb{R}$, *then* $\{G_{g,\nu'}(x)\}_{x \in \mathbb{Z}^d}$ *is a uniformly equicontinuous family of functions for* $\nu' \in [\nu, \infty)$.

(G4) For $\nu \in (\nu_c(g), \infty)$, $\sup_{|x| \geq r} G_{g,\nu}(x)/S(x) \to 0$ *as* $r \to \infty$.

(G5) $G_{g,g} \leq 2S$.

(Z4) For each finite Λ *and* $g > 0$, $Z^{(\Lambda)}_{g,\nu}$ *is continuous in* $\nu \in \mathbb{R}$.

(Z5)
- *(a) For* $\nu \in (\nu_c(g), \infty)$, $L_{g,\nu}$ *is continuous.*
- *(b) If* $G_{g,\nu} \leq 3S$ *for some* ν, *then* $L_{g,\nu'}$ *is continuous for* $\nu' \in [\nu, \infty)$. *Moreover, if* $L_{g,\nu} \leq 0$ *and* $\nu \in (\nu_c(g), g]$, *then* $L_{g,\nu} = O(g)$.

(Z6) $L_{g,\nu} \to \infty$ *as* $\nu \to -\infty$.

Our assumptions are certainly not optimal. In particular, at the price of more involved arguments, (J4) and (R1) could be relaxed.

10.2 Model independent lemmas

This section begins with the promised proof of Lemma 4.6. This lemma is an extension of a lemma in [6], but the proof is nearly identical. Hence we only describe where care must be taken in obtaining the extension.

Proof of Lemma 4.6. This is, *mutatis mutandis*, the proof of [6, Lemma 2]. Note this reference uses $-\Delta$ for what we denote by D, and that the formula for μ and its range is stated in the body of the proof of the lemma.

The most significant step to check is [6, Lemma 4]. The necessary Edgeworth expansion follows from [44, Theorem 2, (1.5b)], which applies by Assumptions 2.1, and (J3) implies the norm $\| \cdot \|$ of [44] is $| \cdot |$ and that $U_1 = U_3 = 0$. The factor $U_2(\tilde{\omega}^x)$ in [44, (1.5b)] can be computed and is compatible with the bounds used in [6, Proof of Lemma 4]. ∎

Lemma 10.3. *The conclusions of Lemma 4.4 and Lemma 4.5 hold for all models satisfying Assumptions 10.1 and (Z5) in Assumptions 10.2.*

Proof. In this proof we drop the infinite volume superscripts on $\Psi_{g,\nu}^{(\infty)}$ etc. By Assumptions 10.1 and Proposition 9.9 $\Psi_{g,\nu}(x)$ is $\mathbb{Z}^d$-symmetric,

$$|\Psi_{g,\nu}(x)| \leq g\alpha c_* \|\!|x|\!\|^{-3(d-2)}, \tag{10.3}$$

(4.5) holds and $w(g,\nu) := (\hat{J} - L_{g,\nu})^{-1}$ is positive and bounded uniformly in g for g small. In particular, all conclusions of Lemma 4.4 hold, as desired. The objects $D_{g,\nu}, G_{g,\nu}(x), D^{\tilde{S}}_{w(g,\nu)}, \tilde{\Psi}_{g,\nu}(x)$ in the remainder of this proof are defined in (4.10).

Proof of conclusions of Lemma 4.5, part (I). We sequentially prove the three conclusions; the arguments are similar to part of the proof of [6, Lemma 1]. Item (i) follows by the $\mathbb{Z}^d$ -symmetry of $\Psi_{g,\nu}$ and the $\mathbb{Z}^d$-symmetry of J from (J3), which implies $D^{\tilde{S}}_{w(g,\nu)}$ is $\mathbb{Z}^d$-symmetric.

Item (ii). Sum (4.9) over x and interchange the sum over x with the sum in the convolution in (4.9). Since $\nu > \nu_c$ both sums are absolutely convergent, so the interchange is legal. We obtain the desired item (ii):

$$\sum_{x\in\mathbb{Z}^d} D_{g,\nu}(x) = -\left(\sum_{x\in\mathbb{Z}^d} \tilde{G}_{g,\nu}(x)\right)^{-1} < 0, \tag{10.4}$$

where the inequality follows from $G_{g,\nu}(x) > 0$ and $w(g,\nu) > 0$.

Item (iii). By the triangle inequality, (10.3), the uniform bound on $w(g,\nu)$ and $D_{g,\nu}(x) := D^{\tilde{S}}_{w(g,\nu)} + \tilde{\Psi}_{g,\nu}$, it suffices to find a $0 \leq z \leq \hat{J}^{-1}$ such that

$$\left| D^{\tilde{S}}_{w(g,\nu)}(x) - D^{\tilde{S}}_{z}(x) \right| \leq Cg\|\!|x|\!\|^{-d-4}. \tag{10.5}$$

The left-hand side of (10.5) is $|w(g,\nu) - z|\, J_+(x)$, so by (J4) it is enough to show that it is possible to choose z such that

$$|w(g,\nu) - z| \leq cg \tag{10.6}$$

for some $c > 0$. If $w(g,\nu) \leq \hat{J}^{-1}$, then we can take $z = w(g,\nu)$. Otherwise, by the definition of $D^{\tilde{S}}_{w(g,\nu)}$,

$$1 - w(g,\nu)\hat{J} = -\sum_{x\in\mathbb{Z}^d} D^{\tilde{S}}_{w(g,\nu)}(x),$$

which implies, by the definition of $D_{g,\nu}$ and (4.10), that

$$0 \geq 1 - w(g,\nu)\hat{J} = \sum_{x\in\mathbb{Z}^d} (-D_{g,\nu}(x) + w(g,\nu)\Psi_{g,\nu}(x))$$

$$\geq - \left| \sum_{x\in\mathbb{Z}^d} w(g,\nu)\Psi_{g,\nu}(x) \right| \geq -Cw(g,\nu)g. \qquad (10.7)$$

The first inequality follows from item (ii), and the second follows from (10.3).

The lower bound in (10.7) is equivalent to $1 \geq w(g,\nu)(\hat{J} - Cg)$. Therefore for $g \leq \hat{J}/(2C)$ we have $w(g,\nu) \leq 2/\hat{J}$ and putting this back into the lower bound of (10.7) gives

$$0 \geq 1 - w(g,\nu)\hat{J} \geq -2(C/\hat{J})g \qquad (10.8)$$

which is equivalent to (10.6) with $z = \hat{J}^{-1}$ and $c = 2(C/\hat{J}^2)$. The proof of item (iii) and therefore of part (I) is complete.

Proof of conclusions of Lemma 4.5, part (II): We must show that $L_{g,\nu} = O(g)$. By (Z5) this holds if $L_{g,\nu} \leq 0$. If $L_{g,\nu} \geq 0$ this follows by inserting $w(g,\nu) = (\hat{J} - L_{g,\nu})^{-1}$ into (10.8) and solving the inequality for $L_{g,\nu}$. ■

10.3 Model-dependent lemmas and proof of Theorem 4.1

In this section we show that the conclusions of the model-dependent lemmas listed in Remark 4.12 hold under Assumptions 10.1 and 10.2.

Conclusion of Lemma 3.4. This is (G2) in Assumptions 10.1 ■

Conclusion of Lemma 3.5. This is a consequence of (G3). ■

Conclusion of Lemma 4.8. This is (G5), as $\nu = g$ is in the domain of F by the conclusions of Lemma 4.10. ■

Conclusion of Lemma 4.9. We follow [21, Proposition 2.2, (i)]. Let

$$F(\nu) := \sup_{x\in\mathbb{Z}^d} \frac{G_{g,\nu}(x)}{S(x)}. \qquad (10.9)$$

We must show $F(\nu)$ is continuous for $\nu \in (\nu_c, g]$. By (G4), the supremum of $G_{g,\nu}(x)/S(x)$ is obtained on a finite set of x. Since $G_{g,\nu}(x)$ is monotone in ν by Lemma 3.5, and each term $G_{g,\nu}(x)/S(x)$ is continuous in ν by (G3) part (a), continuity of (10.9) follows. ■

Conclusion of Lemma 4.10. This is part of Assumptions 10.2. ■

Conclusion of Lemma 4.11. This is (Z6). ■

By Remark 4.12 in Section 4.2 the conclusion (4.2) of Proposition 4.3 and the conclusions of Theorem 4.1 hold for models satisfying Assumptions 10.1 and 10.2. Thus we have proved

Theorem 10.4. *The conclusions of Theorem 4.1 hold for models satisfying Assumptions 10.1 and 10.2. Namely, there is $g_0 = g_0(d, J) > 0$ such that if $0 < g < g_0$ then $\nu_c(g)$ is finite and*

$$G_{g,\nu_c}(x) \leq 2S(x), \qquad x \in \mathbb{Z}^d. \tag{10.10}$$

10.4 Proof of asymptotic behaviour

We begin with two lemmas. The first, Lemma 10.5, is an extension of a lemma from [6], and hence we only describe where care must be taken in obtaining this extension.

Lemma 10.5. *Let D and H be as in Lemma 4.6. If $\sum_{x\in\mathbb{Z}^d} D(x) < 0$ then $H(x) \in \ell^1(\mathbb{Z}^d)$.*

Proof. In the proof of [6, Lemma 2], it is shown that

$$H = (-D * S_\mu)^{-1} * S_\mu,$$

where we have expressed the equation preceding [6, Equation (32)] in the notation of the present paper. In [6] it is shown that the first term $(-D{*}S_\mu)^{-1}$ is an element of the Banach algebra defined at the beginning of [6, Section 4], i.e., the set of functions f on $\mathbb{Z}^d$ that are ℓ^1 and have $\sup_x |f(x)||x|^d$ finite. Since $\sum_{x\in\mathbb{Z}^d} D(x) < 0$, the μ defined in Lemma 4.6 is subcritical for the free Green's function, and hence S_μ is an element of the Banach algebra as well since it decays exponentially in $|||x|||$. This decay is a standard fact and follows from, e.g., [28, Section 4.2]. Hence the convolution defining H is an element of the Banach algebra with finite norm; in particular it is ℓ^1. ■

Lemma 10.6. *Consider a model satisfying Assumptions 10.1 and 10.2. If $d \geq 5$ and g is sufficiently small, then $\{D_{g,\nu}(x)\}_{x\in\mathbb{Z}^d}$ is an equicontinuous family of functions for $\nu \in [\nu_c, \infty)$. Moreover, $\sum_{x\in\mathbb{Z}^d} D_{g,\nu}(x)$ is continuous in $\nu \in [\nu_c, \infty)$.*

Proof. To prove that $D_{g,\nu}(x)$ is defined and continuous in $\nu \in [\nu_c, \infty)$ we now discuss the definitions in (4.10) for $\nu = \nu_c$ as well as $\nu > \nu_c$. By Theorem 10.4 and (G3) the infrared bound $G_{g,\nu_c} \leq 2S$ holds for $\nu \geq \nu_c$. This implies the hypotheses of Proposition 9.9 hold with $\eta = c_* g$ for $\nu \geq \nu_c$. By item (iv) of Proposition 9.9 we conclude that $D_{g,\nu} := D^{\tilde{S}}_{w(g,\nu)} + \tilde{\Psi}_{g,\nu}$ exists for $\nu \geq \nu_c$ as desired. This definition together with item (iii) of Proposition 9.9 asserts that

$$D_{g,\nu} * \tilde{G}_{g,\nu}(x) = -\mathbb{1}_{\{x=0\}}. \tag{10.11}$$

Moreover, by (J4), items (ii) and (iv) of Proposition 9.9, and the lower bound on $w(g,\nu)$ following (10.3), there is a $c_1 > 0$ such that for $\nu \geq \nu_c$,

$$|D_{g,\nu}(x)| \leq c_1 |\!|\!| x |\!|\!|^{-d-4}. \tag{10.12}$$

For $\nu_1, \nu_2 \in [\nu_c, \infty)$, (10.11) implies that

$$D_{\nu_2} * (\tilde{G}_{\nu_1} - \tilde{G}_{\nu_2}) * D_{\nu_1} + (D_{\nu_1} - D_{\nu_2}) = 0, \tag{10.13}$$

where we have omitted the subscript g. Note that the omission of the order of the convolutions in this equation is valid as the iterated convolutions are absolutely convergent by (10.12), the infrared bound $G_{g,\nu_c} \leq 2S$, and Lemma 8.2. Therefore

$$\begin{aligned} |D_{\nu_2}(x) - D_{\nu_1}(x)| &\leq \sup_{y \in \mathbb{Z}^d} \left| \tilde{G}_{\nu_1}(y) - \tilde{G}_{\nu_2}(y) \right| \|D_{\nu_1}\|_1 \|D_{\nu_2}\|_1 \\ &\leq C \sup_{y \in \mathbb{Z}^d} \left| \tilde{G}_{\nu_1}(y) - \tilde{G}_{\nu_2}(y) \right| \end{aligned} \tag{10.14}$$

for some $C > 0$ by (10.12). By (Z5) part (b), $w(g,\nu)$ is continuous in ν for $\nu \geq \nu_c$. Therefore the functions $\tilde{G}_{g,\nu}(x)$ are equicontinuous on $[\nu_c, \infty)$ by (G3) part (b). This proves that the functions $D_{g,\nu}(x)$ are equicontinuous on $\nu \in [\nu_c, \infty)$ as desired.

The second claim, that $\nu \mapsto D_{g,\nu}$ is continuous in ℓ^1, follows from the first. This is so because $\sum_{x \in \mathbb{Z}^d} |D_{g,\nu}(x)|$ is uniformly convergent by item (ii) of Proposition 9.9 and (Z5) part (b). ■

Lemma 10.7. *Consider a model satisfying Assumptions 10.1 and 10.2. If $d \geq 5$ and g is sufficiently small then $\sum_{x \in \mathbb{Z}^d} D_{g,\nu_c}(x) = 0$.*

Proof. This follows from Lemmas 10.5 and 10.6 and the definition of ν_c. ■

Theorem 10.8. *For models satisfying Assumptions 10.1 and 10.2, if $d \geq 5$ there is a $g_0 = g_0(d, J)$ such that if $0 < g < g_0$, then there are constants $C > 0$, $\epsilon > 0$ such that*

$$G_{g,\nu_c}(x) \sim \frac{C}{|||x|||^{d-2}} + O\left(\frac{1}{|||x|||^{d-2+\epsilon}}\right) \tag{10.15}$$

Proof. By Theorem 10.4, (Z5) part (b), and Lemma 4.5 part (ii), $L_{g,\nu} = O(g)$ for $\nu \in [\nu_c, g]$, and hence $w(g,\nu) = \hat{J}^{-1}(1 + O(g))$. Hence it suffices to prove (10.15) for $\tilde{G}_{g,\nu_c}$.

Let $Q_{g,\nu}(x) \coloneqq w(g,\nu)\big(J_+(x) + \Psi_{g,\nu}(x)\big)$, and note that

$$D_{g,\nu}(x) = -\mathbb{1}_{\{x=0\}} + Q_{g,\nu}(x), \tag{10.16}$$

by the definition of $D_{g,\nu}$, see (4.10). By [21, Lemma 2.3] the Fourier transform satisfies, for $k \in [-\pi, \pi]^d$,

$$\hat{D}_{g,\nu}(k) = \hat{D}_{g,\nu}(0) + \frac{|k|^2}{2d}\nabla^2 \hat{D}_{g,\nu}(0) + O(|k|^4 \log |k|^{-1}), \tag{10.17}$$

and by the Lemma 4.5 part of Lemma 10.3, $\hat{D}_{g,\nu}(0) = \sum_{x\in\mathbb{Z}^d} D_{g,\nu}(x) \leq 0$. Note that $-\nabla^2 \hat{D}_{g,\nu}(0)$ is given by $\sum_{x\in\mathbb{Z}^d} |x|^2 D_{g,\nu}(x)$, which is finite by item (ii) of Proposition 9.9 and positive by (J4), if g is small enough. Here we have also used that $w(g,\nu) = \hat{J}^{-1}(1+O(g))$, as noted in the first paragraph. Thus there is a $b > 0$ such that for all $\nu \geq \nu_c$

$$\frac{1}{-\hat{D}_{g,\nu}(k)} \leq \frac{b}{|k|^2}. \tag{10.18}$$

This implies

$$\tilde{G}_{g,\nu_c}(x) = \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} \frac{e^{ik\cdot x}}{1 - \hat{Q}_{g,\nu_c}(k)}. \tag{10.19}$$

This equality is valid as (10.11) shows that the Fourier transform of $\tilde{G}_{g,\nu_c}$ is $-1/\hat{D}_{g,\nu_c}$, and this latter function is in $L^2 \cap L^1$. This implies that the inverse Fourier transform of the ℓ^2 function $\tilde{G}_{g,\nu_c}$ is given by the standard integral definition in (10.19). A formal justification of this last claim follows by approximating $\tilde{G}_{g,\nu_c}$ through functions in $\ell^1 \cap \ell^2$.

In what follows we focus on $\nu = \nu_c$, and write $Q = Q_{g,\nu_c}$ and $\hat{Q} = \hat{Q}_{g,\nu_c}$. By [18, Theorem 1.4], (10.15) holds for $\tilde{G}_{g,\nu_c}$ if there is a $\rho > 0$ such that

(H1) $\hat{Q}(0) = 1$,

(H2) $|Q(x)| \leq K_1 |||x|||^{-(d+2+\rho)}$, some $K_1 > 0$,

(H3) $\sum_{x \in \mathbb{Z}^d} \|x\|_2^{2+\rho} |Q(x)| \leq K_2$, some $K_2 > 0$,

(H4) there is a $K_0 > 0$ such that $\hat{Q}(0) - \hat{Q}(k) \geq K_0 \|k\|_2^2$, $k \in [-\pi, \pi]^d$.

By Theorem 10.4 G_{g,ν_c} satisfies an infrared bound. Hence (H2)–(H3) follow from Proposition 9.9, (J4), and the assumption $d \geq 5$. Furthermore (H1) follows from $\sum_{x \in \mathbb{Z}^d} D_{g,\nu_c} = 0$, i.e., Lemma 10.7.

Let $\hat{J}_+$ be the Fourier transform of J_+. Using $w(g, \nu_c) > 0$ and $\hat{J} = \hat{J}_+(0)$, (H4) can be re-expressed as

$$\hat{J} - \hat{J}_+(k) - (\hat{\Psi}_{g,\nu_c}(0) - \hat{\Psi}_{g,\nu_c}(k)) \geq K_0 \|k\|_2^2.$$

Thus to prove (H4) it suffices to show there is a $c > 0$, uniform in g, such that

$$\hat{\Psi}_{g,\nu_c}(0) - \hat{\Psi}_{g,\nu_c}(k) \geq -cg(\hat{J} - \hat{J}_+(k)),$$

because $\hat{J} - \hat{J}_+(k) \geq c' \|k\|_2^2$ for some $c' > 0$; the desired bound then follows by taking g small enough. The stated lower bound on $\hat{J} - \hat{J}_+(k)$ follows from [28, Lemma 2.3.2], whose hypotheses are provided by (J4) and the irreducibility assumption (J2). By the $\mathbb{Z}^d$-symmetry of Ψ_{g,ν_c}

$$\left| \hat{\Psi}_{g,\nu_c}(0) - \hat{\Psi}_{g,\nu_c}(k) \right| = \left| \sum_{x \in \mathbb{Z}^d} \Psi_{g,\nu_c}(x)(1 - \cos(k \cdot x)) \right|.$$

Since $1 - \cos(k \cdot x) \leq c_1 (k \cdot x)^2$, the claim follows from item (ii) of Proposition 9.9. ■

11 Verification of hypotheses

In conjunction with Theorem 10.8, the next lemma completes the proof of Theorem 3.6. By Section 10.3 it also completes the proofs of Lemmas that were promised in Sections 3 and 4.

Lemma 11.1. *The Edwards and $n = 1, 2$ $g|\varphi|^4$ models defined in Sections 3.1 and 3.2 satisfy Assumptions 10.1 and 10.2.*

The remainder of this section proves this lemma, first for the Edwards model and then for the $g|\varphi|^4$ model. In each section we also give the proof of Proposition 4.2 that was promised in Section 4.

11.1 Edwards model

Recall the Edwards model as defined in Definition 3.1. From this definition it is clear that (Z1), (Z2), and (Z4) hold. By the definitions of $L^{(\Lambda)}_{g,\nu}$ and $r^{(\Lambda)}_{s,s'}(x,y)$ in (7.2) and (7.7), respectively, short calculations yield

$$L^{(\Lambda)}_{g,\nu} = -\nu, \quad r^{(\Lambda)}_{s,s'}(x,y) = -g\mathbb{1}_{\{x=y\}}. \tag{11.1}$$

and from (11.1) we immediately obtain (Z3), (Z5), (Z6), and (R1).

By (3.4) and (3.1) it follows that

$$Y_{\boldsymbol{t},\boldsymbol{s}} := \exp\left\{-g\sum_{x\in\Lambda}(2t_x s_x + s_x^2) - \nu\sum_{x\in\Lambda} s_x\right\} \tag{11.2}$$

which is decreasing in t_x for each x, so (G1) holds. To verify (G2), note that $\{X^{(\Lambda)}_\ell = b\} = \{T^{(\Lambda)} > \ell, X^{(\infty)}_\ell = b\}$ and on this event $\tau^{(\Lambda)}_{[0,\ell],x} = \tau^{(\infty)}_{[0,\ell],x}$. Hence

$$Y_{\mathbf{0},\boldsymbol{\tau}^{(\Lambda)}_{[0,\ell]}}\mathbb{1}_{\{X^{(\Lambda)}_\ell = b\}} = Y_{\mathbf{0},\boldsymbol{\tau}^{(\infty)}_{[0,\ell]}}\mathbb{1}_{\{X^{(\infty)}_\ell = b\}}\mathbb{1}_{\{T^{(\Lambda)}>\ell\}}, \tag{11.3}$$

which is non-negative and increasing in Λ since $T^{(\Lambda)}$ is.

(G3). Monotonicity in ν is clear from (3.4). We defer the proofs of (G3) parts (a) and (b) until after Lemma 11.5 below, as they are very similar to the detailed proof of Lemma 11.5.

(G4) is well-known in the discrete setting, see, e.g., [38, Equation 2.23]. The proof adapts *mutatis mutandis* to the continuous-time setting by using a Simon inequality [36]. (G5) is clear since $G^{(\infty)}_{g,g} \leq S$.

We complete the proof of Assumptions 10.2 by showing $\nu_c \leq 0$. This is immediate from Definition 3.1: if $\nu > 0$ the Green's function is dominated by the Green's function of a simple random walk with non-zero killing.

Proof of Proposition 4.2 for Edwards model. This is immediate from $L_{g,\nu_c} = O(g)$ by (Z5) part (b) and Lemma 4.5 part (ii) since $L_{g,\nu} = -\nu$. ■

11.2 $g|\varphi|^4$ theory with $n = 1, 2$

From Definition 3.2 it is clear that (Z1), (Z2) and (Z4) hold. By the definitions of $L^{(\Lambda)}_{g,\nu}$ and $r^{(\Lambda)}_{s,s'}(x,y)$ in (7.2) and (7.7), respectively, calculations yield

$$L^{(\Lambda)}_{g,\nu} = -2\nu - 4g \left\langle |\varphi_x|^2 \right\rangle^{(\Lambda)}_{\mathbf{0}} \tag{11.4}$$

$$r^{(\Lambda)}_{s,s'}(x,y) = -4g \left(2\mathbb{1}_{\{x=y\}} - 4g \left\langle |\varphi_x|^2 ; |\varphi_y|^2 \right\rangle^{(\Lambda)}_{\boldsymbol{\tau}^{(\Lambda)}_{[s,s']}} \right), \tag{11.5}$$

where $\langle A; B\rangle \coloneqq \langle AB\rangle - \langle A\rangle \langle B\rangle$. (Z3) follows by (11.4) and Lemma 9.1. The continuity statements in (Z5) follow from (G3), which we will verify below. (Z6) follows from (11.4). The assumption that $L_{g,\nu} = O(g)$ holds if a K-IRB holds and $\nu \in (\nu_c(g), g]$ follows from (11.4).

To verify the remaining hypotheses, the following correlation inequalities will be useful. For $A \subset \Lambda$ and $i = 1, 2$ let $\varphi^A \coloneqq \prod_{x\in A} \varphi^{[i]}_x$.

Proposition 11.2 (GKS II inequality)**.** *Consider the n-component $g|\varphi|^4$-model with $n = 1, 2$. For $\boldsymbol{t} \in [0,\infty)$ and $A, B \subset \Lambda$, $\left\langle \varphi^A; \varphi^B \right\rangle^{(\Lambda)}_{\boldsymbol{t}} \geq 0$.*

Proof. See [14, Lemmas 11.3 and 11.4]. ■

Lemma 11.3. *For $a, b \in \Lambda$ and $\boldsymbol{t} \in [0,\infty)^\Lambda$*

$$G^{(\Lambda)}_{g,\nu,\boldsymbol{t}}(a,b) \leq G^{(\Lambda)}_{g,\nu,\mathbf{0}}(a,b). \tag{11.6}$$

Moreover, $G^{(\Lambda)}_{g,\nu,\mathbf{0}}(a,b)$ is non-decreasing in Λ.

Proof. These statements follow from Proposition 11.2. ■

Proposition 11.4 (Lebowitz Inequality)**.** *Consider the $g|\varphi|^4$ model with $n = 1, 2$ components. Then for all Λ, $x, y, v \in \Lambda$ and $\boldsymbol{t} \in [0,\infty)^\Lambda$,*

$$\langle \varphi_x \cdot \varphi_y; \varphi_v \cdot \varphi_v\rangle^{(\Lambda)}_{\boldsymbol{t}} \leq 2 \langle \varphi_x \cdot \varphi_v\rangle^{(\Lambda)}_{\boldsymbol{t}} \langle \varphi_y \cdot \varphi_v\rangle^{(\Lambda)}_{\boldsymbol{t}} .$$

Proof. For $n = 1$ this is the Lebowitz inequality [29], for $n = 2$ this is due to Bricmont [7, Theorem 2.1]. See also [10, Remark below (5.7)] and let $F = \varphi^1_y\, \varphi_v \cdot \varphi_v$. ■

Both (G1) and (G2) follow immediately from Lemma 11.3. The property (G4) is well-known and follows from the Simon–Lieb–Rivasseau inequality, see [36, 30, 33], and (G5) follows from (3.8). The next lemma provides (G3).

Lemma 11.5.

1. *For all Λ, $G^{(\Lambda)}_{g,\nu}(x)$ is non-increasing as a function of $\nu \in (-\infty,\infty)$.*

2. $G_{g,\nu}(x)$ *is Lipschitz as a function of* $\nu \in (\nu_c, \infty)$.

3. *If* $d \geq 5$ *and* G_{g,ν_c} *satisfies a* K*-IRB for some* K *then* $G_{g,\nu}(x)$ *is uniformly Lipschitz as a function of* $\nu \in [\nu_c, \infty)$.

Proof. (1) By (G1), $G^{(\Lambda)}_{g,\nu}(x)$ for $g|\varphi|^4$ is non-increasing in ν because the derivative with respect to ν is proportional to a constant times the derivative with respect to $\boldsymbol{t}$ at $\boldsymbol{t} = 0$ in the direction $t_y = t$ for all $y \in \Lambda$.

(2) For any finite volume Λ and any ν, by Proposition 11.4 and Lemma 11.3,

$$\begin{aligned} \left\langle \varphi_0 \cdot \varphi_x ; |\varphi_y|^2 \right\rangle^{(\Lambda)}_{\mathbf{0}} &\leq 2 \left\langle \varphi_0 \cdot \varphi_y \right\rangle^{(\Lambda)}_{\mathbf{0}} \left\langle \varphi_x \cdot \varphi_y \right\rangle^{(\Lambda)}_{\mathbf{0}} \leq 2 G_{g,\nu}(y) G_{g,\nu}(y-x) \\ &\leq G^2_{g,\nu}(y) + G^2_{g,\nu}(y-x), \end{aligned} \tag{11.7}$$

where the last inequality is the elementary inequality $2uv \leq u^2 + v^2$ for $u, v \in \mathbb{R}$. Since

$$-\frac{\partial}{\partial \nu} G^{(\Lambda)}_{g,\nu}(x) = \sum_{y \in \Lambda} \left\langle \varphi_0 \cdot \varphi_x ; \varphi_y^2 \right\rangle^{(\Lambda)}_{\mathbf{0}} \tag{11.8}$$

we have, for ν and a such that $\nu \geq a \geq \nu_c$,

$$\left|\frac{\partial}{\partial \nu} G^{(\Lambda)}_{g,\nu}(x)\right| \leq 2 \sum_{y \in \mathbb{Z}^d} G^2_{g,\nu}(y) \leq c_a, \tag{11.9}$$

where $c_a = 2 \sum_{y \in \mathbb{Z}^d} G^2_{g,a}(y)$. The second inequality holds by part (1), and c_a is finite for $a > \nu_c$ because $G_{g,a}(y)$ is summable by (10.2). For $a > \nu_c$ and $\nu, \nu' \in [a, \infty)$, by writing $G^{(\Lambda)}_{g,\nu'}(x) - G^{(\Lambda)}_{g,\nu}(x)$ as the integral of its derivative we have $|G^{(\Lambda)}_{g,\nu'}(x) - G^{(\Lambda)}_{g,\nu}(x)| \leq c_a |\nu' - \nu|$. Taking $\Lambda \uparrow \mathbb{Z}^d$ by (G2), we obtain $|G_{g,\nu'}(x) - G_{g,\nu}(x)| \leq c_a |\nu' - \nu|$ and therefore $G_{g,\nu}(x)$ is Lipschitz as claimed.

(3) We repeat part (2) with $a = \nu_c$. Since $d \geq 5$, c_{ν_c} is finite by the K-IRB, so $G_{g,\nu}(x)$ is Lipschitz on $[\nu_c, \infty)$ with uniform constant c_{ν_c}. ∎

Proof of (G3) parts (a) and (b) for the Edwards model. We first claim that for any finite volume Λ and any $\nu \in \mathbb{R}$,

$$-\frac{d}{d\nu} G^{(\Lambda)}_{g,\nu}(x) \leq G^{(\Lambda)}_{g,\nu} * G^{(\Lambda)}_{g,\nu}(x) \leq \frac{1}{2} \sum_{y \in \mathbb{Z}^d} \left((G^{(\Lambda)}_{g,\nu}(y-x))^2 + (G^{(\Lambda)}_{g,\nu}(y))^2 \right), \tag{11.10}$$

where the second inequality is the elementary $2ab \leq a^2 + b^2$. Granting the claim, note that by (G2) and translation invariance this proves (11.9), and the remainder of the proof is essentially identical to the proof above.

We now prove the claimed first inequality in (11.10). By the definitions (3.2) and (2.5), the left-hand side of (11.10) is

$$\sum_{x' \in \Lambda} \int_{[0,\infty)} d\ell \int_{[0,\ell]} d\ell' \; E_a \Big[\mathcal{Y}_{0,\ell} \mathbb{1}_{\{X^{(\Lambda)}_{\ell'} = x'\}} \mathbb{1}_{\{X^{(\Lambda)}_{\ell} = x\}} \Big]. \tag{11.11}$$

where $\mathcal{Y}_{s,t} = Z^{(\Lambda)}_{\boldsymbol{\tau}^{(\Lambda)}_{[s,t]}} / Z^{(\Lambda)}_{\mathbf{0}}$ as in (6.1), and, for the Edwards model, $Z^{(\Lambda)}_{\mathbf{0}} = 1$. We reverse the order of integration over ℓ, ℓ' and insert $\mathcal{Y}_{0,\ell} = \mathcal{Y}_{0,\ell'} \bar{\mathcal{Y}}_{0,\ell'}(\ell)$, where $\bar{\mathcal{Y}}_{0,\ell'}(\ell) = \big(\mathcal{Y}_{0,\ell} / \mathcal{Y}_{0,\ell'}\big)$ as in (7.11). By Lemma 7.5 with $H = \mathcal{Y}_{0,\ell'}$ and $(u_1, u_2, u_3) = (0, \ell', \ell')$ the result is

$$\sum_{x' \in \Lambda} \int_{[0,\infty)} d\ell' \; E_a \Big[\mathcal{Y}_{0,\ell'} G^{(\Lambda)}_{\boldsymbol{\tau}^{(\Lambda)}_{[0,\ell']}} (X^{(\Lambda)}_{\ell'}, x) \mathbb{1}_{\{X^{(\Lambda)}_{\ell'} = x'\}} \Big], \tag{11.12}$$

By (G1) and (3.2) read from right to left we obtain the first inequality in (11.10) as desired. ∎

For $0 < u < v$ define

$$\begin{aligned} \bar{r}^{(\Lambda)}_{u,v}(x,y) &:= 8g \left[\mathbb{1}_{\{x=y\}} + 4n^2 g \left(\frac{1}{n} \langle \varphi_x \cdot \varphi_y \rangle^{(\Lambda)}_{\boldsymbol{\tau}^{(\Lambda)}_{[u,v]}} \right)^2 \right], \\ \bar{r}^{(\Lambda)}(x,y) &:= \bar{r}^{(\Lambda)}_{0,0}(x,y). \end{aligned} \tag{11.13}$$

The next lemma, together with (3.9), verifies (R1).

Lemma 11.6. *Suppose* $0 < u < v$, $x, y \in \Lambda$. *Then*

$$\left| r^{(\Lambda)}_{u,v}(x,y) \right| \le \bar{r}^{(\Lambda)}_{u,v}(x,y) \le \bar{r}^{(\Lambda)}(x,y). \tag{11.14}$$

Proof. As Λ is fixed we will omit it from the notation. Applying the triangle inequality to (11.5) and using Proposition 11.4,

$$r_{u,v}(x,y) \le \bar{r}_{u,v}(x,y), \tag{11.15}$$

where we have used Proposition 11.2 to remove the absolute values from the correlation function. The remaining inequality $\bar{r}_{u,v}(x,y) \le \bar{r}(x,y)$ follows by Proposition 11.2. ∎

We complete the proof of Assumptions 10.2 by showing $\nu_c \le 0$. This follows from Theorem 3.3, which shows the $g|\varphi|^4$ Green's function with $\nu > 0$ is dominated by the Green's function of a simple random walk with non-zero killing ν.

Proof of Proposition 4.2 for the $g|\varphi|^4$ model. The same argument as for the Edwards model shows L_{g,ν_c} is $O(g)$. Since $L_{g,\nu} = -4g\left\langle \varphi_x^2 \right\rangle_{\mathbf{0}} - 2\nu$, and $4g\left\langle \varphi_x^2 \right\rangle_{\mathbf{0}}$ is $O(g)$ at ν_c since an infrared bound holds, the claim follows. ■

Acknowledgements

TH thanks Gady Kozma for a helpful discussion. TH and DB thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme "Scaling limits, rough paths, quantum field theory" when work on this paper was undertaken. DB is grateful for partial support by the Simons Foundation during the same programme. This work was supported by EPSRC grant LNAG/036, RG91310. TH is supported by EPSRC grant EP/P003656/1, and was supported by an NSERC PDF during the initial stages of this work. MH is supported by Future Fellowship FT160100166, from the Australian Research Council.

A Random walk and the Markov property

A.1 Properties of continuous-time random walk

Proof of Lemma 2.3. We first prove that $\Delta^{(\Lambda)}$ is invertible. Let $f, h\colon \Lambda \to \mathbb{R}$. The quadratic form associated to $\Delta^{(\Lambda)}$ is given by

$$(f, -\Delta^{(\Lambda)} h) \coloneqq \sum_{x\in\Lambda} f_x (-\Delta^{(\Lambda)} h)_x. \tag{A.1}$$

For $f\colon \Lambda \to \mathbb{R}$ define the extension by zero: $\tilde{f} = f$ on Λ and $\tilde{f}_x = 0$ for $x \notin \Lambda$. We *claim* that

$$(f, -\Delta^{(\Lambda)} h) = \frac{1}{2} \sum_{x,y\in\mathbb{Z}^d} J(x-y)(\tilde{f}_x - \tilde{f}_y)(\tilde{h}_x - \tilde{h}_y). \tag{A.2}$$

By choosing $h = f$ we obtain

$$(f, -\Delta^{(\Lambda)} f) = \frac{1}{2} \sum_{x\neq y\in\mathbb{Z}^d} J(x-y)|\tilde{f}_x - \tilde{f}_y|^2 > 0, \quad f \neq 0. \tag{A.3}$$

The strict inequality holds because $\tilde{f}_y = 0$ for $y \notin \Lambda$ and for every point $v \in \Lambda$ there is a walk with transitions of nonzero rate that starts at v and reaches a point not in Λ. This positivity implies that the eigenvalues of $-\Delta^{(\Lambda)}$ are strictly positive and therefore $-\Delta^{(\Lambda)}$ is invertible as desired. Thus it suffices to prove the claim (A.2).

To prove (A.2) we start with the right-hand side which contains

$$(\tilde{f}_x - \tilde{f}_y)(\tilde{h}_x - \tilde{h}_y) = \tilde{f}_x(\tilde{h}_x - \tilde{h}_y) + \tilde{f}_y(\tilde{h}_y - \tilde{h}_x), \tag{A.4}$$

so by the symmetry under exchanging x and y we can rewrite the right-hand side of (A.2) as

$$\begin{aligned}\frac{1}{2}\sum_{x,y\in\mathbb{Z}^d} J(x-y)(\tilde{f}_x - \tilde{f}_y)(\tilde{h}_x - \tilde{h}_y) &= \sum_{x,y\in\mathbb{Z}^d} J(x-y)\tilde{f}_x(\tilde{h}_x - \tilde{h}_y) \\ &= \sum_{x,y\in\mathbb{Z}^d} J(x-y)\tilde{f}_x(-\tilde{h}_y). \end{aligned} \tag{A.5}$$

For the final equality we used the zero row sum property $\sum_y J(x-y) = 0$. Recall from (2.1) that $J(x-y) = \Delta^{(\infty)}_{x,y}$ and that $\Delta^{(\Lambda)}_{x,y}$ is the restriction of $\Delta^{(\infty)}_{x,y}$ to Λ. Therefore, in (A.5) we insert

$$\sum_{y\in\mathbb{Z}^d} J(x-y)(-\tilde{h}_y) = \sum_{y\in\mathbb{Z}^d}(-\Delta^{(\infty)}_{x,y})\tilde{h}_y = \sum_{y\in\Lambda}(-\Delta^{(\infty)}_{x,y})h_y = (-\Delta^{(\Lambda)}h)_x \tag{A.6}$$

which proves (A.2) and hence completes the proof that $\Delta^{(\Lambda)}$ is invertible.

Next we prove (2.7). By definition

$$S^{(\Lambda)}(a,b) = \int_0^\infty dt\,\mathbb{P}_a(X_t = b) = \int_0^\infty dt\,(e^{t\Delta_*})_{a,b} = \int_0^\infty dt\,(e^{t\Delta^{(\Lambda)}})_{a,b}. \tag{A.7}$$

where the last equality holds since $(\Delta_*^k)_{a,b} = ((\Delta^{(\Lambda)})^k)_{a,b}$, and where for a square matrix A, e^{tA} denotes the matrix exponential $\sum_{k=0}^\infty \frac{t^k}{k!}A^k$. The right hand side of (A.7) is $(-\Delta^{(\Lambda)})^{-1}_{a,b}$ as desired: since $-\Delta^{(\Lambda)}$ is real symmetric with positive eigenvalues, this follows by diagonalizing and integrating. ■

Proof of Lemma 2.4. Recall the definition of $\tilde{S}_z(x)$ from (4.6), and let $\tilde{S}(x) =$

$\tilde{S}_{\hat{J}^{-1}}(x)$. Let T_n denote the time of the nth jump of $X^{(\infty)}$ and $T_0 = 0$. Then

$$S(x) = E_0\Big[\sum_{n=0}^{\infty}(T_{n+1} - T_n)\mathbb{1}_{\{X^{(\infty)}_{T_n}=x\}}\Big]$$
$$= \sum_{n=0}^{\infty} E_0[T_{n+1} - T_n]E_0\Big[\mathbb{1}_{\{X^{(\infty)}_{T_n}=x\}}\Big] \tag{A.8}$$
$$= \hat{J}^{-1}\sum_{n=0}^{\infty} E_0\Big[\mathbb{1}_{\{X^{(\infty)}_{T_n}=x\}}\Big] = \hat{J}^{-1}\tilde{S}(x). \tag{A.9}$$

It is an easy exercise (see, e.g. [28, Section 4.3]) that

$$\tilde{S}(x) = \mathbb{1}_{\{x=0\}} + \sum_{y} \hat{J}^{-1}J_+(y)\tilde{S}(x-y), \tag{A.10}$$

and by (A.9) this can be rewritten as

$$\hat{J}S(x) = \mathbb{1}_{\{x=0\}} + \sum_{y} J_+(y)S(x-y). \tag{A.11}$$

Collecting terms gives the first claim. To verify (2.13), we use (A.8):

$$S(x) = \sum_{n=0}^{\infty} E_0[T_{n+1} - T_n]P_0(X^{(\infty)}_{T_n} = x) = \sum_{n=0}^{\infty} \hat{J}^{-1}(\hat{J}^{-1}J_+)^{*n}(x). \qquad \blacksquare$$

A.2 The Markov Property

For $s \geq 0$ define the $\mathcal{F}$-measurable map $\theta_s\colon \Omega_1 \to \Omega_1$ by $\theta_s((x_t)_{t\geq 0}) = (x_{s+t})_{t\geq 0}$. The following is a standard formulation of the Markov property.

Proposition A.1. *Let $H\colon \Omega_1 \to \mathbb{R}$ be $\mathcal{F}$-measurable and integrable with respect to E_a for each $a \in \mathbb{Z}^d$, and let $h(x) = E_x\big[H\big]$. Then for every $x \in \mathbb{Z}^d$ and $s \geq 0$,*

$$E_x\big[H \circ \theta_s \big| \mathcal{F}_s\big] = h(X_s), \quad P_x\text{-a.s.} \tag{A.12}$$

A.2.1 The Markov property as used to obtain (6.12)

To justify this application of the Markov property, for $\ell > 0$ and $b \in \mathbb{Z}^d$ let $H_{\ell,b} := \mathcal{Y}_{0,\ell}\, \mathbb{1}_{\{X_\ell=b\}}$ and $h(\ell, y, b) := E_y\left[H_{\ell,b}\right]$. Then, by Proposition A.1,

$$E_a\big[\mathcal{Y}_{s,s+\ell}\, \mathbb{1}_{\{X_{s+\ell}=b\}}\big|\mathcal{F}_s\big] = h(\ell, X_s, b), \quad P_a\text{-a.s.} \tag{A.13}$$

A.2.2 The Markov property as used to obtain (6.15)

To justify this application of the Markov property, for $L \in \mathcal{L}_m(0)$ let $H_{L,b} := w(L)G_{\mathbf{0}}(X_{s'_m}, b)$, and $f(x, b, L) := E_x [H_{L,b}]$. Then, by Proposition A.1,

$$E_a \left[w(L+s)\, G_{\mathbf{0}}(X_{s'_m+s}, b) \middle| \mathcal{F}_s \right] = f(X_s, b, L). \tag{A.14}$$

A.2.3 The Markov property in the proof of Lemma 7.5

Lemma A.2. *For $s' \geq 0$, $b \in \Lambda$, and any Borel set $I \subset [0, s']$,*

$$G^{(\Lambda)}_{\boldsymbol{\tau}_I}(X_{s'}, b) = E_a \Big[\int_{[s',\infty)} d\ell \ \frac{Z_{\boldsymbol{\tau}_I + \boldsymbol{\tau}_{[s',\ell]}}}{Z_{\boldsymbol{\tau}_I}} \mathbb{1}_{\{X_\ell = b\}} \big| \mathcal{F}_{s'} \Big] \quad P_a\text{-a.s.} \tag{A.15}$$

The proof of Lemma A.2 requires two preparatory ingredients.

Lemma A.3. *Let $W : \Omega \to \mathbb{R}$ be integrable with respect to $\mathbb{P}$, $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ and $\mathbb{E}_A = \mathbb{E}[\cdot|A]$. Then*

$$\mathbb{1}_A \mathbb{E}[W|\mathcal{F}] = \mathbb{1}_A \mathbb{E}_A[W|\mathcal{F}], \qquad \mathbb{P}\text{-a.s.} \tag{A.16}$$

Proof. Let L and R denote the left- and right-hand sides, respectively. Then both L and R are $\mathcal{F}$-measurable. Let $B \in \mathcal{F}$. Then $\mathbb{E}[\mathbb{1}_B(L - R)] = 0$ since

$$\mathbb{E}[\mathbb{1}_B \mathbb{1}_A \mathbb{E}_A[W|\mathcal{F}]] = \mathbb{E}_A[\mathbb{1}_B \mathbb{1}_A \mathbb{E}_A[W|\mathcal{F}]]\mathbb{P}(A) \tag{A.17}$$
$$= \mathbb{E}_A[\mathbb{1}_B \mathbb{1}_A W]\mathbb{P}(A) \tag{A.18}$$
$$= \mathbb{E}[\mathbb{1}_B \mathbb{1}_A W] = \mathbb{E}[\mathbb{1}_B \mathbb{1}_A \mathbb{E}[W|\mathcal{F}]]. \tag{A.19}$$

Taking $B_1 = \{L > R\} \in \mathcal{F}$ and $B_2 = \{L < R\} \in \mathcal{F}$ completes the proof. ■

The following lemma is a standard result in the case that $\mathcal{G} = \sigma(W_1)$, see [12, Example 5.1.5]. Since we have been unable to find this particular formulation in the literature, we give a proof below.

Lemma A.4. *Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and measurable spaces $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$. Let $W_1 \colon \Omega \to S_1$ and $W_2 \colon \Omega \to S_2$ be measurable, and let $f \colon S_1 \times S_2 \to \mathbb{R}$ be Borel-measurable on the corresponding product space $(S_1 \times S_2, \mathcal{S})$ and either bounded or non-negative and such that $\mathbb{E}[f(W_1, W_2)]$ is finite. Define $h \colon S_1 \to \mathbb{R}$ by*

$$h(w_1) = \mathbb{E}[f(w_1, W_2)]. \tag{A.20}$$

If W_2 is independent of $\mathcal{G} \subset \mathcal{F}$ and W_1 is $\mathcal{G}$-measurable then

$$h(W_1) = \mathbb{E}[f(W_1, W_2)|\mathcal{G}], \quad \text{a.s.} \tag{A.21}$$

Proof. If $f(w_1, w_2) = f_1(w_1)f_2(w_2)$, where f_1 and f_2 are bounded and $\mathcal{B}(\mathbb{R})$-measurable then $h(w_1) = f_1(w_1)\mathbb{E}[f_2(W_2)]$. Then $h(W_1) = f_1(W_1)\mathbb{E}[f_2(W_2)]$ (is $\mathcal{G}$ measurable) and by independence, almost surely

$$f_1(W_1)\mathbb{E}[f_2(W_2)] = \mathbb{E}[f_1(W_1)f_2(W_2)|\mathcal{G}]. \tag{A.22}$$

In particular the result holds for any f of the form $f(w_1, w_2) = \mathbb{1}_{A_1 \times A_2} \equiv \mathbb{1}_{\{w_1 \in A_1\}}\mathbb{1}_{\{w_2 \in A_2\}}$, where $A_i \in \mathcal{S}_i$. Therefore by linearity of expectation it also holds for indicators of finite disjoint unions of events of the form $A_1 \times A_2$.

Let $\mathcal{A} \subset \mathcal{S}$ denote the collection of events for which the claim of the lemma holds with $f = \mathbb{1}_A$. Then $\mathcal{A}$ contains the field of finite disjoint unions of events of the form $A_1 \times A_2$, and by dominated convergence $\mathcal{A}$ is a monotone class. Thus by the Monotone Class Theorem $\mathcal{A} = \mathcal{S}$, and hence by linearity the claim holds for all simple functions f.

For non-negative f such that $f(W_1, W_2)$ is integrable we can take non-negative simple functions f_n increasing to f pointwise. Let $h_n(w_1) = \mathbb{E}[f_n(w_1, W_2)]$. Then $h_n(w_1) \uparrow \mathbb{E}[f(w_1, W_2)] =: h(w_1)$ pointwise by monotone convergence. Next, by the result for simple functions we have for each n

$$h_n(W_1) = \mathbb{E}[f_n(W_1, W_2)|\mathcal{G}]. \tag{A.23}$$

The right hand side increases to $\mathbb{E}[f(W_1, W_2)|\mathcal{G}]$ by monotone convergence and the left hand side increases to $h(W_1)$ by the above pointwise convergence. This proves the result for non-negative f such that $f(W_1, W_2)$ is integrable.

The claim for bounded measurable f follows by considering the positive and negative parts of f. ■

Proof of Lemma A.2. Fix s' and let $\tilde{X} = \theta_{s'}(X)$. Then $\boldsymbol{\tau}_{[s',\ell]} = \tilde{\boldsymbol{\tau}}_{[0,\ell-s']}$, where $\tilde{\boldsymbol{\tau}}$ is the local time of $\tilde{X}$, and $\mathbb{1}_{\{X_\ell = b\}} = \mathbb{1}_{\{\tilde{X}_{\ell-s'} = b\}}$. Thus the right hand side of (A.15) is equal to

$$\sum_{x\in\Lambda} \mathbb{1}_{\{X_{s'}=x\}} E_a\Big[\int_{[s',\infty)} d\ell \ \frac{Z_{\boldsymbol{\tau}_I + \tilde{\boldsymbol{\tau}}_{[0,\ell-s']}}}{Z_{\boldsymbol{\tau}_I}} \mathbb{1}_{\{\tilde{X}_{\ell-s'}=b\}} \big| \mathcal{F}_{s'}\Big]. \tag{A.24}$$

Conditional on the event $\{X_{s'} = x\} = \{\tilde{X}_0 = x\}$, we have that $\tilde{X}$ is a random walk starting at x that is independent of $\mathcal{F}_{s'}$. Let $\tilde{P}_x(\cdot) := P_a(\cdot|\tilde{X}_0 = x)$, and define $f_x \colon [0,\infty)^\Lambda \times \Omega_1 \to \mathbb{R}$ by

$$f_x(\boldsymbol{r}, \tilde{y}) := \int_{[s',\infty)} d\ell \frac{Z_{\boldsymbol{r} + \tilde{\boldsymbol{\tau}}_{[0,\ell-s']}(\tilde{y})}}{Z_{\boldsymbol{r}}} \mathbb{1}_{\{\tilde{y}_{\ell-s'}=b\}}, \tag{A.25}$$

where $\tilde{\boldsymbol{\tau}}_{[0,\ell-s']}(\tilde{y})$ denotes the vector of local times of the path $\tilde{y}$ in the interval $[0,\ell-s']$. Then $\tilde{E}_x[f_x(\boldsymbol{\tau}_I,\tilde{X})]<\infty$ (recall (3.3)), $\boldsymbol{\tau}_I$ is $\mathcal{F}_{s'}$-measurable and $\tilde{X}$ is independent of $\mathcal{F}_{s'}$ under $\tilde{P}_x$. Define $h_x\colon [0,\infty)^\Lambda \to \mathbb{R}$ by $h_x(\boldsymbol{r}) \coloneqq \tilde{E}_x[f(\boldsymbol{r},\tilde{X})]$. Then Lemma A.4 implies

$$h_x(\boldsymbol{\tau}_I) = \tilde{\mathbb{E}}_x\left[\int_{[s',\infty)} d\ell \frac{Z_{\boldsymbol{\tau}_I+\tilde{\boldsymbol{\tau}}_{[0,\ell-s']}}}{Z_{\boldsymbol{\tau}_I}} \mathbb{1}_{\{\tilde{X}_{\ell-s'}=b\}} \middle| \mathcal{F}_{s'}\right]. \tag{A.26}$$

By Lemma A.3, almost surely

$$\mathbb{1}_{\{X_{s'}=x\}}\mathbb{E}_a\Big[\int_{[s',\infty)} d\ell \ \frac{Z_{\boldsymbol{\tau}_I+\tilde{\boldsymbol{\tau}}_{[0,\ell-s']}}}{Z_{\boldsymbol{\tau}_I}} \mathbb{1}_{\{\tilde{X}_{\ell-s'}=b\}} \big| \mathcal{F}_{s'}\Big] \tag{A.27}$$

$$= \mathbb{1}_{\{X_{s'}=x\}}\tilde{\mathbb{E}}_x\left[\int_{[s',\infty)} d\ell \frac{Z_{\boldsymbol{\tau}_I+\tilde{\boldsymbol{\tau}}_{[0,\ell-s']}}}{Z_{\boldsymbol{\tau}_I}} \mathbb{1}_{\{\tilde{X}_{\ell-s'}=b\}} \middle| \mathcal{F}_{s'}\right]. \tag{A.28}$$

Thus the right hand side of (A.15) is almost surely equal to

$$\sum_{x\in\Lambda} \mathbb{1}_{\{X_{s'}=x\}} h_x(\boldsymbol{\tau}_I). \tag{A.29}$$

Finally, by the change of variables $\tilde{\ell}=\ell-s'$ and interchange of integrals we have that $G^{(\Lambda)}_{\boldsymbol{r}}(x,b)=h_x(\boldsymbol{r})$, and the result follows. ∎

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