Backbone scaling for critical lattice trees in high dimensions

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Abstract

We prove that for critical (spread-out) lattice trees in dimensions $d > 8$, the unique paths from the origin to vertices of large tree distance converge to Brownian motion. This provides an important ingredient for proving weak convergence of the corresponding historical processes.

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Introduction

Over the past decades, various critical interacting particle systems on $\mathbb{Z}^d$ have been shown to exhibit mean-field behaviour for dimensions d above a certain critical dimension d_c. These models include the voter model ($d_c = 2$) [5, 3], critical oriented percolation and the contact process ($d_c = 4$) [16, 14], and critical lattice trees ($d_c = 8$) [7, 17]. In each of these models, each particle has a time and spatial location associated to it and by assigning a point mass to every particle we can describe the model via evolution of measures in $\mathbb{Z}^d$. Starting with a single particle at the origin at time 0, each model survives until time n with probability $\theta_n \approx C/n$ [4, 11]. Conditioning on survival, and with appropriate rescaling of time, space, mass, and measure, in some cases [3, 12] a functional central limit theorem for measure-valued processes (weak convergence to super-Brownian motion under the canonical measure - see e.g. [23], conditioned on survival) has been proved. However these particular measure-valued processes do not record genealogy present in e.g. the voter model and lattice trees, and although the genealogy of super-Brownian motion can be reconstructed from the path in dimensions $d > 4$ [1], this construction does not facilitate weak convergence results.

In the context of lattice trees, the functional central limit theorem (FCLT) for measure-valued processes proved in [12] does not establish a FCLT for paths in the tree, such as the unique path in the tree from the origin to a point x of generation nt. The main aim of this paper is to prove a version of this result suitable for proving convergence of the so-called historical processes (measure-valued processes which do record the genealogy of the process) [19].

1 The model and main results

A lattice tree is a finite connected set of lattice bonds containing no cycles. We will be considering lattice trees on $\mathbb{Z}^d$ with bonds connecting any two vertices that live in a common ball (in ℓ_∞) of radius $L \gg 1$, and with $d > 8$. To be more precise, let $d > 8$ and let $D(\cdot)$

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be the uniform distribution on a finite box $[-L, L]^d \setminus o$, where $o = (0, \ldots, 0) \in \mathbb{Z}^d$. For a lattice tree $T \ni o$ define $W_{z,D}(T) = z^{|T|} \prod_{e \in T} D(e)$, where the product is over the edges in T and $|T|$ is the number of edges in T. An important observation is that if T is an edge-disjoint union of subtrees then $W_{z,D}(T)$ can be factored into a product over the weights of the subtrees. It turns out (see e.g. [17, 11]) that there exists a critical value z_D such that $\rho = \sum_{T \ni o} W_{z_D,D}(T) < \infty$ and $\mathbb{E}[|\mathcal{T}|] = \infty$, where $\mathbb{P}(\mathcal{T} = T) = \rho^{-1} W_{z_D,D}(T)$. Hereafter we write $W(\cdot)$ for the critical weighting $W_{z_D,D}(\cdot)$ and suppose that we are selecting a random tree $\mathcal{T}$ according to this critical weighting.

Let T be a lattice tree containing $o \in \mathbb{Z}^d$, and for $m \in \mathbb{N}$, let T_m denote the set of vertices in T of tree distance m from o. Then for any $x \in T_m$ there is a unique path from o to x in the tree. Roughly speaking, in this paper we consider the weak limit (as $m \to \infty$) of rescaled paths of this kind in high dimensions.

Functional central limit theorems

For our general discussion we require the notion of weak convergence of finite measures on Polish spaces. We refer the reader to e.g. [20, Chapter 16 and Appendix A2] or [2, Chapter 3] for further details on what we discuss below.

A *Polish space* E is a complete (every Cauchy sequence converges) separable (there is a countable dense subset) metric space. The space $\mathbb{R}^d$ equipped with the Euclidean metric is the prototypical example of a Polish space. For a Polish space E, let $\mathcal{M}_F(E)$ (resp. $\mathcal{M}_1(E)$) denote the space of finite (resp. probability) measures on the Borel sets of E. For a sequence $\nu_n \in \mathcal{M}_F(E)$ we say that ν_n converges weakly to $\nu \in \mathcal{M}_F(E)$ and write $\nu_n \xrightarrow{w} \nu$ if for every $f : E \to \mathbb{R}$ bounded and continuous,

$$\int f(x)\nu_n(dx) \to \int f(x)\nu(dx), \qquad \text{as } n \to \infty. \tag{1.1}$$

Equipped with the Prokhorov metric, which generates the topology of weak convergence, $\mathcal{M}_F(E)$ is also Polish. We will often use the notation $\mathrm{E}_\nu[f(X)]$ for $\int f(x)\nu(dx)$, with the understanding that $X \in E$. This will be particularly convenient when X is an E-valued random variable defined on an underlying probability space and $\nu(A) = c\mathbb{P}(X \in A)$ for some $c > 0$.

Let S_n denote the location of a nearest neighbour simple symmetric random walk on $\mathbb{Z}^d$ after n steps (starting from the origin $o \in \mathbb{Z}^d$). Then $\mathbb{E}[S_n^2] = n$ (here and elsewhere, for $x, y \in \mathbb{R}^d$ we abuse notation and write xy to mean $x \cdot y$, and hence x^2 to mean $|x|^2$) and the central limit theorem states that $n^{-1/2}S_n$ converges in distribution to a random vector Z that is (multivariate-) normally distributed with mean $0 \in \mathbb{R}^d$ and covariance matrix $\mathrm{diag}(1/d)$. Define probability measures ν_n, ν on (the Borel sets of) $\mathbb{R}^d$ by

$$\nu_n(A) = \mathbb{P}\big(n^{-1/2}S_n \in A\big), \qquad \text{and} \qquad \nu(A) = \mathbb{P}(Z \in A).$$

Phrased in the language of weak convergence of (probability) measures, the central limit theorem says that $\nu_n \xrightarrow{w} \nu$. In the setting of $\mathcal{M}_F(\mathbb{R}^d)$, the statement $\nu_n \xrightarrow{w} \nu$ is in fact equivalent to the convergence of the pointwise convergence of the characteristic functions (Fourier transforms), so for ν_n, ν as above

$$\int e^{ikx}\nu_n(dx) \to \int e^{ikx}\nu(dx) = e^{-\frac{k^2}{2d}}, \text{ for } k \in \mathbb{R}^d.$$

For a Polish space E let $\mathcal{D}_t(E)$ denote the space of càdlàg paths (paths that are continuous from the right with limits existing from the left) mapping $[0, t]$ to E. It is well known that there is a complete metric on this space (that generates the so-called Skorokhod J_1 topology) for which $\mathcal{D}_t(E)$ is also Polish (see e.g. the references [20, 2]).

The functional central limit theorem (FCLT) concerns the entire path $(W_t^{(n)})_{t \geq 0}$ defined by

$$W_t^{(n)} = n^{-1/2}S_{\lfloor nt \rfloor}. \tag{1.2}$$

Defined in this way, for each n, $W^{(n)}$ jumps at times $t = i/n$ for $i \in \mathbb{N}$ and is constant on intervals $[i/n, i + 1/n)$ for $i \in \mathbb{Z}_+$. In particular the process $W^{(n)}$ is a random element of the space $\mathcal{D}_\infty(\mathbb{R}^d)$ of càdlàg paths from $\mathbb{R}_+ = [0,\infty)$ to $\mathbb{R}^d$. The version of the FCLT most relevant to this paper states that for any $t > 0$, the sequence of rescaled random walks $W^{(n)}_{[0,t]} = (W^{(n)}_s)_{s\in[0,t]}$ converges to a d-dimensional Brownian motion $B_{[0,t]} = (B_s)_{s\in[0,t]}$ (with $B_1 \sim \mathcal{N}(0, \mathrm{diag}(1/d))$). Phrased in the language of weak convergence of (probability) measures this FCLT says that $\nu_{n,t} \xrightarrow{w} \nu_t$, where $\nu_{n,t} \in \mathcal{M}_1(\mathcal{D}_t(\mathbb{R}^d))$ are defined by

$$\nu_{n,t}(A) = \mathbb{P}\big(W^{(n)}_{[0,t]} \in A\big),$$

(i.e. $\nu_{n,t}$ is the uniform measure on nearest-neighbour (càdlàg) paths on $n^{-1/2}\mathbb{Z}^d$ up to time $\lfloor nt\rfloor$) and $\nu_t \in \mathcal{M}_F(\mathcal{D}_t(\mathbb{R}^d))$ is the law of d-dimensional Brownian motion $B_{[0,t]}$ (sometimes called Wiener measure) satisfying $\nu_t(A) = \mathbb{P}\big(B_{[0,t]} \in A\big)$.

Celebrated results of Hara and Slade (see e.g. [10, Theorem 1.6]) prove the FCLT for self-avoiding walk in dimensions $d > 4$. To be more precise, if ν_n is the uniform measure on nearest neighbour self-avoiding paths $\omega = (o = \omega_0, \omega_1, \dots, \omega_n)$ of length n on $n^{-1/2}\mathbb{Z}^d$ then $\nu_n \xrightarrow{w} \nu$ as probability measures on $\mathcal{D}_1(\mathbb{R}^d)$.

Paths and measure-valued processes for lattice trees

Recall for any $x \in T_m$ there is a unique path $w_{x,m}$ from o to x in the tree, defined by

$$w_{x,m} = \big(o = w_{x,m}(0), w_{x,m}(1), \dots, w_{x,m}(m) = x\big).$$

For $t \geq 0$ and $x \in \mathbb{Z}^d/\sqrt{nc_0}$ such that $\sqrt{nc_0}x \in T_{\lfloor nt\rfloor}$ we define

$$w^{(n)}_{x,t}(s) = \frac{w_{\sqrt{c_0 n}x,\lfloor nt\rfloor}\big(\lfloor ns\rfloor\big)}{\sqrt{c_o n}}, \qquad \text{for } s \in [0,t], \tag{1.3}$$

where $c_0 > 0$ is a constant depending on D, d (to be described later). Note that $w^{(n)}_{x,t} \in \mathcal{D}_t(\mathbb{R}^d)$, and that it can be extended to a path in $\mathcal{D}_\infty(\mathbb{R}^d)$ by setting $w^{(n)}_{x,t}(s) = x$ for $s > t$ (or equivalently writing $w^{(n)}_{x,t}(s) = w^{(n)}_{x,t}(s \wedge t)$). Let

$$X^{(n)}_t = \frac{1}{C_0 n} \sum_{\sqrt{c_0 n}x \in \mathcal{T}_{\lfloor nt\rfloor}} \delta_x \in \mathcal{M}_F(\mathbb{R}^d), \qquad \text{and} \tag{1.4}$$

$$H^{(n)}_t = \frac{1}{C_0 n} \sum_{\sqrt{c_0 n}x \in \mathcal{T}_{\lfloor nt\rfloor}} \delta_{w^{(n)}_{x,t}} \in \mathcal{M}_F(\mathcal{D}_t(\mathbb{R}^d)) \tag{1.5}$$

denote the (rescaled) measure-valued "process" and historical "process" (see e.g. [6]) associated with the random lattice tree $\mathcal{T}$ respectively. Here $C_0 > 0$ is a constant depending on d and D (to be described later). Note that $X^{(n)}_t$ assigns mass to certain particles in the tree (but does not encode the genealogy) whereas $H^{(n)}_t$ assigns mass to genealogical paths leading to those particles.

Since $w^{(n)}_{x,t}$ can be considered as a path in $\mathcal{D}_\infty(\mathbb{R}^d)$ as above, $H^{(n)}_t$ can be considered as a finite measure on $\mathcal{D}_\infty(\mathbb{R}^d)$. According to [11], there exists a constant $C_1 > 0$ also depending on D, d such that

$$n\mathbb{P}(H^{(n)}_t(1) > 0) = n\mathbb{P}(X^{(n)}_t(1) > 0) \to \frac{2}{C_1 t}, \qquad \text{as } n \to \infty. \tag{1.6}$$

Then we define $\mu_n \in \mathcal{M}_F(\mathcal{D}_\infty(\mathcal{M}_F(\mathbb{R}^d)))$ by

$$\mu_n(\bullet) = nC_1\mathbb{P}(X^{(n)} \in \bullet), \tag{1.7}$$

and $\mu^H_n \in \mathcal{M}_F(\mathcal{D}_\infty(\mathcal{M}_F(\mathcal{D}_\infty(\mathbb{R}^d))))$ by

$$\mu^H_n(\bullet) = nC_1\mathbb{P}(H^{(n)} \in \bullet). \tag{1.8}$$

Due to the survival probability asymptotics (1.6), multiplying by n is asymptotically the same (up to a constant) as conditioning on survival until time n (or rescaled time 1).

According to [23, Section II.7] there exists a σ-finite measure $\mathbb{N}_0$ on $\mathcal{D}_\infty(\mathcal{M}_F(\mathbb{R}^d))$, with $\mathbb{N}_0(X_t(1) > 0) = 2/t$ such that $\mathbb{N}_0$ is the canonical measure associated to the $((B_t)_{t\geq 0}, 1, 0)$-superprocess (where $(B_t)_{t\in[0,\infty)}$ is a d-dimensional Brownian motion with $B_1 \sim \mathcal{N}(0, \mathrm{diag}(1/d))$, which is a (time-homogeneous) Markov process). The superprocess in question (called super-Brownian motion) is a measure-valued process that can be thought of as an infinitesimal critical branching process whose spatial dispersion is governed by the $\mathbb{R}^d$-valued process $(B_t)_{t\geq 0}$. The 0 and 1 in the notation $((B_t)_{t\geq 0}, 1, 0)$ simply refer to standardised versions of the process (e.g. the "branching variance" is 1). According to [21, pages 34, 64], there also exists a σ-finite measure $\mathbb{N}_0^H$ on $\mathcal{D}_\infty(\mathcal{M}_F(\mathcal{D}_\infty(\mathbb{R}^d)))$ with $\mathbb{N}_0^H(H_t(1) > 0) = \mathbb{N}_0(X_t(1) > 0)$ such that $\mathbb{N}_0^H$ is the canonical measure associated to the $((B_{[0,t]})_{t\geq 0}, 1, 0)$-superprocess, where $(B_{[0,t]})_{t\geq 0}$ is a (time-inhomogeneous) Markov process in t.

It is proved in [12] that for lattice trees in dimensions $d > 8$ (with L sufficiently large) $\mu_n \xrightarrow{w} \mathbb{N}_0$. Since the limit is a σ-finite measure, $\mu_n \xrightarrow{w} \mathbb{N}_0$ is defined in terms of weak convergence of a family of finite measures (indexed by t) on $\mathcal{D}_\infty(\mathcal{M}_F(\mathbb{R}^d))$ as

$$\mu_n(\bullet, X_t(1) > 0) \xrightarrow{w} \mathbb{N}_0(\bullet, X_t(1) > 0), \qquad \text{for each } t \geq 0,$$

or equivalently in terms of weak convergence of their conditional (on $X_t(1) > 0$) counterparts, which are probability measures. The corresponding result for the historical processes ($\mu_n^H \xrightarrow{w} \mathbb{N}_0^H$) is an open problem that provided the initial motivation for this paper. Note that neither of these results is expected to hold for lattice trees in dimensions $d < 8$ (a similar result is expected to hold with logarithmic corrections to the scaling in 8 dimensions).

In [19], one of the conditions for proving the aforementioned weak convergence of historical processes (more precisely, convergence of the finite-dimensional distributions, where one looks at the process at each fixed finite collection of times) is to show that for each $t > 0$,

$$\nu_{n,t} \xrightarrow{w} \nu_t \tag{1.9}$$

where $\nu_{n,t}(\bullet) = \mathrm{E}_{\mu_n^H}[H_t(\bullet)]$ and $\nu_t(\bullet) = \mathrm{E}_{\mathbb{N}_0^H}[H_t(\bullet)]$ are finite measures on $\mathcal{D}_t(\mathbb{R}^d)$, and $\bullet$ denotes a Borel set in $\mathcal{D}_t(\mathbb{R}^d)$. In fact, (see Lemma 1.2 below) $\nu_t(\bullet) = \mathbb{P}(B_{[0,t]} \in \bullet)$, while

$$\mathrm{E}_{\mu_n^H}[H_t(\bullet)] = \mathrm{E}_{\mu_n^H}\left[\frac{1}{C_0 n} \sum_{\sqrt{c_0 n}x \in \mathcal{T}_{\lfloor nt \rfloor}} \mathbb{1}_{\{w_{x,t}^{(n)} \in \bullet\}}\right] \tag{1.10}$$

$$= \sum_{x \in \frac{\mathbb{Z}^d}{\sqrt{c_0 n}}} \frac{C_1}{C_0} \mathbb{P}(\sqrt{c_0 n}x \in \mathcal{T}_{\lfloor nt \rfloor}, w_{x,t}^{(n)} \in \bullet). \tag{1.11}$$

The following theorem verifies (1.9) and is one of the two main results of this paper.

Theorem 1.1. *For each $d > 8$ there exists $L_0 \gg 1$ such that for all $L \geq L_0$ and each $t > 0$, $\nu_{n,t} \xrightarrow{w} \nu_t$ in $\mathcal{M}_F(\mathcal{D}_t(\mathbb{R}^d))$, as $n \to \infty$.*

Suppose that $\mathcal{T}_{\lfloor nt \rfloor} \neq \varnothing$ (i.e. the rescaled process is alive at time $\lfloor nt \rfloor / n$). Let $Z_{[0,t]}^{(n)} = (Z^{(n)}(s))_{s \leq t}$ be the rescaled ancestral path of a particle chosen *uniformly* from $\mathcal{T}_{\lfloor nt \rfloor}$, i.e. for $w \in \mathcal{D}_t(\mathbb{R}^d)$

$$\mathbb{P}(Z_{[0,t]}^{(n)} = w | \mathcal{T}_{\lfloor nt \rfloor}) = \frac{1}{|\mathcal{T}_{\lfloor nt \rfloor}|} \sum_{x \in \mathcal{T}_{\lfloor nt \rfloor}} \mathbb{1}_{\{w_{x,t}^{(n)} = w\}}. \tag{1.12}$$

Since paths to distinct points x, x' in a tree are unique this means that for each $x \in \mathcal{T}_{\lfloor nt \rfloor}$, $\mathbb{P}(Z_{[0,t]}^{(n)} = w_{x,t}^{(n)} | \mathcal{T}_{\lfloor nt \rfloor}) = |\mathcal{T}_{\lfloor nt \rfloor}|^{-1}$. Let $\mu_{n,t}^Z$ denote the law of $Z_{[0,t]}^{(n)}$ (conditional on $|\mathcal{T}_{\lfloor nt \rfloor}| > 0$). In other words, $\mu_{n,t}^Z$ is a discrete probability measure on $\mathcal{D}_t(\mathbb{R}^d)$ such that for a singleton $w \in \mathcal{D}_t(\mathbb{R}^d)$,

$$\mu_{n,t}^Z(\{w\}) = \mathbb{P}\big(Z_{[0,t]}^{(n)} = w \,\big|\, |\mathcal{T}_{\lfloor nt \rfloor}| > 0\big). \tag{1.13}$$

Let $\mu^H_{n,t}, \mathbb{N}^H_{0,t} \in \mathcal{M}_1(\mathcal{M}_F(\mathcal{D}_t(\mathbb{R}^d)))$ denote the laws of the historical measures at time t, conditional on survival, i.e. $\mu^H_{n,t}(\bullet) = \mathbb{P}(H^{(n)}_t \in \bullet | H^{(n)}_t(1) > 0)$, and $\mathbb{N}^H_{0,t}(\bullet) = \mathbb{N}^H_0(H_t \in \bullet | H_t(1) > 0)$. The following elementary consequence of [23, (II.8.6)(a)] shows that ν_t above is Wiener measure (i.e. the law of Brownian motion) for paths on $[0,t]$.

Lemma 1.2. *The historical canonical measure $\mathbb{N}^H_0$ satisfies*

$$\mathrm{E}_{\mathbb{N}^H_0}[H_t(\bullet)] = \mathbb{P}(B_{[0,t]} \in \bullet) = \mathrm{E}_{\mathbb{N}^H_{0,t}}\left[\frac{H_t(\bullet)}{H_t(1)}\right], \tag{1.14}$$

where $B_{[0,t]} = (B_s)_{s\in[0,t]}$ is a standard (i.e. $B_1 \sim \mathcal{N}(0, diag(1/d))$) d-dimensional Brownian motion on the interval $[0,t]$ under $\mathbb{P}$.

The following theorem concerning the path to a uniformly chosen particle of (rescaled) generation t is the second main result of this paper.

Theorem 1.3. *Let $t > 0$ and suppose that $\mu^H_{n,t} \to \mathbb{N}^H_{0,t}$ in $\mathcal{M}_1(\mathcal{M}_F(\mathcal{D}_t(\mathbb{R}^d)))$ as $n \to \infty$. Then $\mu^Z_{n,t} \xrightarrow{w} \nu_t$ in $\mathcal{M}_1(\mathcal{D}_t(\mathbb{R}^d))$ as $n \to \infty$.*

Thus, in [19] one uses Theorem 1.1 as an important ingredient in proving convergence of the finite-dimensional distributions of the historical processes, while Theorem 1.3 shows that weak convergence of a uniformly chosen path in the tree follows from convergence of the 1-dimensional distributions of the historical processes.

The remainder of this paper is organised as follows: In Section 2 we express the constants C_0, C_1 and c_0 via the limiting behaviour of the so-called 2- and 3- point functions. In Section 3 we prove Theorem 1.1, assuming certain bounds (Proposition 2.3) that are proved via an inductive analysis of quantities arising from the lace expansion. In Section 4 we prove (Lemma 1.2 and) Theorem 1.3. Finally in Section 5 we discuss the method of proof of the assumed bounds (i.e. the proof of Proposition 2.3).

2 The 2- and 3- point functions

Let $\rho(x) = \mathbb{P}(x \in \mathcal{T})$ and $\rho_n(x) = \mathbb{P}(x \in \mathcal{T}_n)$. For absolutely summable $h(x)$ write $\hat{h}(k)$ for the Fourier transform of h, i.e. $\hat{h}(k) = \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}kx} h(x)$. It is shown in [9, Theorem 1.2, Corollary 1.4] that for all L sufficiently large there exist constants $C_2, C_3 > 0$ depending on D, d such that

$$|x|^{d-2}\rho(x) \to C_2, \qquad \text{as } |x| \to \infty, \qquad \text{and } |k|^2\hat{\rho}(k) \to C_3, \qquad \text{as } |k| \to 0. \tag{2.1}$$

Using the inductive method of [15, 13] the following were shown in [17, Theorem 3.7(b)].

Theorem 2.1 ([17])**.** *Fix $d > 8$, $\gamma \in (0, 1 \wedge \frac{d-8}{2})$ and $\delta \in (0, (1 \wedge \frac{d-8}{2}) - \gamma)$. There exists a positive $L_0 = L_0(d)$ such that: For every $L \geq L_0$ there exist $K(d,L), A(d,L), c_0(d,L) > 0$ such that* [1]

$$\sup_{n\in\mathbb{Z}_+} \sup_{k\in\mathbb{R}^d} |\hat{\rho}_n(k)| = \sup_{n\in\mathbb{Z}_+} \hat{\rho}_n(0) \leq K, \tag{2.2}$$

$$-\nabla^2\hat{\rho}_m(0) \equiv \sum_x |x|^2\rho_m(x) \leq mK, \tag{2.3}$$

and

$$\hat{\rho}_n\left(\frac{k}{\sqrt{c_0 n}}\right) = Ae^{-\frac{|k|^2}{2d}}\left[1 + \mathcal{O}\left(\frac{|k|^2}{n^\delta}\right) + \mathcal{O}\left(n^{-\frac{d-8}{2}}\right)\right]. \tag{2.4}$$

In particular (taking $k = 0$ above) $A = \lim_{n\to\infty} \mathbb{E}[|\mathcal{T}_n|]$ is the limiting expected number of particles alive at time n as $n \to \infty$. Note that the function $\hat{\rho}_{\lfloor nt\rfloor}$ is related to the measure-valued processes via

$$\mathrm{E}_{\mu_n}\left[\hat{X}^{(n)}_t(k)\right] = \mathrm{E}_{\mu^H_n}\left[\hat{H}^{(n)}_t(k)\right] = \frac{C_1}{C_0}\hat{\rho}_{\lfloor nt\rfloor}\left(\frac{k}{\sqrt{nc_0}}\right) \tag{2.5}$$

[1] Note that A in the present paper is the constant A' in [17].

where $\hat{X}_t^{(n)}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}kx} X_t^{(n)}(dx)$ and $\hat{H}_t^{(n)}(k) = \int_{\mathcal{D}_t} \mathrm{e}^{\mathrm{i}kw(t)} H_t^{(n)}(dw)$ (here $w(t)$ is the terminal point of the path/function $w : [0,t] \to \mathbb{R}^d$).

Similarly, for $\vec{n} \in \mathbb{Z}_+^{r-1}$ and $\vec{x} = (x_1, \dots, x_{r-1}) \in \mathbb{Z}^{d(r-1)}$ we can define $\rho_{\vec{n}}(\vec{x}) = \mathbb{P}(\cap_{i=1}^{r-1}\{x_i \in \mathcal{T}_{n_i}\})$, and $\hat{\rho}_{\vec{n}}(\vec{k}) = \sum_{\vec{x}\in\mathbb{Z}^{d(r-1)}} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \mathbb{P}(\cap_{i=1}^{r-1}\{x_i \in \mathcal{T}_{n_i}\})$. Then according to [17, Theorem 1.14], there exists $V > 0$ depending on D, d such that

$$n^{-1}\mathbb{E}[|\mathcal{T}_n|^2] = n^{-1}\hat{\rho}_{(n,n)}(0,0) \to VA^3. \tag{2.6}$$

The constants appearing in the definition of $X_t^{(n)}$, μ_n etc. are

$$C_0 = VA^2, \qquad \text{and} \qquad C_1 = AV. \tag{2.7}$$

See [18] for further discussion about the connections between $\rho_{\vec{n}}$ and mean moments of the measure-valued process.

2.1 Decomposition via connected graphs

We use the notation $[a,b]_{\mathbb{Z}} = [a,b] \cap \mathbb{Z}_+$ when $a, b \in \mathbb{Z}_+$. Any lattice tree $T \ni o$ such that $x \in T_n$ can be represented as the backbone ω (an n-step path from o to x) together with mutually avoiding ribs $R_0, \dots, R_n$ that are themselves lattice trees emanating from the vertices $o = \omega(0), \omega(1), \dots, \omega(n) = x$ along the backbone. We will write $\omega : y \xrightarrow{n} x$ to denote the set of D-random walk paths of length n starting at y and ending at x. Since the weight $W(T)$ factors into the product of the weights of the backbone and the individual ribs, we can express the two-point function $\rho_n(x)$ as

$$\rho_n(x) = \rho^{-1} \sum_{\omega:o\xrightarrow{n}x} W(\omega) \sum_{R_0\ni\omega(0)} \cdots \sum_{R_n\ni\omega(n)} \prod_{i=0}^{n} W(R_i) \prod_{\substack{u,v\in[0,n]_{\mathbb{Z}}:\\ u<v}} \left[1 + U_{uv}(\vec{R})\right], \tag{2.8}$$

where $U_{uv}(\vec{R}) = -\mathbb{1}_{\{R_u\cap R_v\neq\varnothing\}}$. Let us henceforth write $\sum_{\vec{R}_n\ni\vec{\omega}_n}$ to mean $\sum_{R_0\ni\omega(0)} \cdots \sum_{R_n\ni\omega(n)}$, and $W(\vec{R}_n)$ to mean $\prod_{i=0}^{n} W(R_i)$.

For m, m' both in $\mathbb{Z}_+$ we define $K[m,m'] = 1$ if $m \geq m'$, while for $m' > m$ and $R_m, R_{m+1}, \dots, R_{m'}$ we define

$$K[m,m'] = K[m,m'](\vec{R}) = \prod_{\substack{u,v\in[m,m']_{\mathbb{Z}}:\\ u<v}} \left[1 + U_{uv}(\vec{R})\right]. \tag{2.9}$$

This notation allows us to write

$$\rho_n(x) = \rho^{-1} \sum_{\omega:o\xrightarrow{n}x} W(\omega) \sum_{\vec{R}_n\ni\vec{\omega}_n} W(\vec{R}_n) K[0,n]. \tag{2.10}$$

The following Lemma will be useful when proving tightness for Theorem 1.1.

Lemma 2.2. *For each $n \in \mathbb{N}$ and $m \in [0,n]_{\mathbb{Z}}$, and each $\omega = (\omega_0, \omega_1, \dots, \omega_n)$,*

$$\sum_{\vec{R}_n\ni\vec{\omega}_n} W(\vec{R}_n)K[0,n] \leq \left(\sum_{\vec{R}_m\ni\vec{\omega}_m} W(\vec{R}_m)K[0,m]\right)\left(\sum_{\vec{R}'_{n-m}\ni(\omega_m,\dots,\omega_n)} W(\vec{R}'_{n-m})K'[0,n-m]\right), \tag{2.11}$$

where $K'[0,n-m] = K'[0,n-m](\vec{R}'_{n-m})$ is defined as in (2.9) with $\vec{R}'$ instead of $\vec{R}$.

Proof. Clearly for any $\vec{R}_n$ one has $K[0,n] \leq K[0,m]K[m,n]$ since each factor $1 + U_{uv}$ appearing on the right hand side also appears on the left. Therefore

$$\sum_{\vec{R}_n\ni\vec{\omega}_n} W(\vec{R}_n)K[0,n] \leq \sum_{\vec{R}_n\ni\vec{\omega}_n} W(\vec{R}_n)K[0,m]K[m,n].$$

The right hand side is equal to

$$\sum_{\vec{R}_m \ni (\omega_0,\ldots,\omega_m)} \sum_{\vec{R}'_{n-m} \ni (\omega_m,\ldots,\omega_n)} \mathbb{1}_{\{R'_0 = R_m\}} \frac{W(\vec{R}_m) W(\vec{R}'_{n-m})}{W(R_m)} K[0,m] K'[0,n-m] \tag{2.12}$$

$$= \sum_{\vec{R}_m \ni (\omega_0,\ldots,\omega_m)} W(\vec{R}_m) K[0,m] \sum_{\vec{R}'_{n-m} \ni (\omega_m,\ldots,\omega_n)} \mathbb{1}_{\{R'_0 = R_m\}} \frac{W(\vec{R}'_{n-m})}{W(R_m)} K'[0,n-m]. \tag{2.13}$$

since $W(R_m)$ appears once in $W(\vec{R}_m)$ and again in $W(\vec{R}'_{n-m})$. Write the second sum in (2.13) as

$$\sum_{R'_0 \ni \omega_m} \mathbb{1}_{\{R'_0 = R_m\}} \sum_{(R'_1,\ldots,R'_{n-m}) \ni (\omega_{m+1},\ldots,\omega_n)} \prod_{j=1}^{n-m} W(R'_i) K'[0,n-m]. \tag{2.14}$$

For each $\ell = 1,\ldots,n-m$ we have that if $\omega_m \in R'_\ell$ then $[1 + U_{0\ell}] = 0$ (so $K' = 0$ in this case). In other words, the indicator that R'_ℓ and R'_0 avoid each other is more restrictive than the indicator that R'_ℓ and $\{\omega_m\}$ avoid each other. But $\{\omega_m\}$ is a tree with one vertex and no edges, which has weight $W(\{\omega_m\}) = 1$. Thus, (2.14) is at most

$$\sum_{R'_0 \ni \omega_m} \mathbb{1}_{\{R'_0 = \{\omega_m\}\}} W(R'_0) \sum_{(R'_1,\ldots,R'_{n-m}) \ni (\omega_{m+1},\ldots,\omega_n)} \prod_{j=1}^{n-m} W(R'_i) K'[0,n-m] \tag{2.15}$$

$$= \sum_{R'_0 \ni \omega_m} \mathbb{1}_{\{R'_0 = \{\omega_m\}\}} \sum_{(R'_1,\ldots,R'_{n-m}) \ni (\omega_{m+1},\ldots,\omega_n)} W(\vec{R}'_{n-m}) K'[0,n-m] \tag{2.16}$$

$$\leq \sum_{R'_0 \ni \omega_m} \sum_{(R'_1,\ldots,R'_{n-m}) \ni (\omega_{m+1},\ldots,\omega_n)} W(\vec{R}'_{n-m}) K'[0,n-m] \tag{2.17}$$

$$= \sum_{\vec{R}'_{n-m} \ni (\omega_m,\ldots,\omega_n)} W(\vec{R}'_{n-m}) K'[0,n-m], \tag{2.18}$$

which proves the claim. ■

Note that for $m < m' \in \mathbb{Z}_+$, by expanding the product over u, v we see that

$$K[m,m'] = \sum_{\Gamma \in \mathcal{G}[m,m']_{\mathbb{Z}}} \prod_{i<j \in \Gamma} U_{ij}, \tag{2.19}$$

where $\mathcal{G}[m,m']$ denotes the set of "graphs" on $[m,m']_{\mathbb{Z}}$, i.e. the set of collections of distinct pairs (edges) of vertices in $[m,m']_{\mathbb{Z}}$.

An edge ij with $i,j \in \mathbb{Z}_+$ and $i < j$ is said to *cover* every $\ell \in [i,j]$ (and does not cover any $\ell \in [i,j]^c$). A graph $\Gamma \in \mathcal{G}[0,m]_{\mathbb{Z}}$ *covers* $\ell \in [0,m]$ if it contains an edge that covers ℓ. For any graph $\Gamma \in \mathcal{G}[0,m]_{\mathbb{Z}}$ and $m^* \in [0,m]_{\mathbb{Z}}$ there is a largest interval $[\ell_1,\ell_2]$ containing m^* such that Γ covers every vertex in $[\ell_1,\ell_2]$. A graph Γ on $[m,m']_{\mathbb{Z}}$ (where $m,m' \in \mathbb{Z}_+$ and $m < m'$) is said to be a *connected* graph (on $[m,m']_{\mathbb{Z}}$) if Γ covers $[m,m']$ (i.e. if $\cup_{ij\in\Gamma}[i,j] = [m,m']$). Let $\mathcal{G}^{\text{conn}}[m,m']_{\mathbb{Z}}$ denote the set of connected graphs on $[m,m']_{\mathbb{Z}}$. Define $J[m,m] = 1$ and for $m < m' \in \mathbb{Z}_+$ define

$$J[m,m'] = \sum_{\Gamma \in \mathcal{G}^{\text{conn}}[m,m']_{\mathbb{Z}}} \prod_{i<j\in\Gamma} U_{ij}. \tag{2.20}$$

Then from (2.19) and (2.20) we can write

$$K[0,m] = \sum_{m_0=0}^{m^*} \sum_{m'_0=m^*}^{m} K[0,m_0-1] J[m_0,m'_0] K[m'_0+1,m]. \tag{2.21}$$

For $m \in \mathbb{N}$, and $x, y \in \mathbb{Z}^d$ define

$$\pi_m(x,y) = \sum_{\omega: x \overset{m}{\to} y} W(\omega) \sum_{\vec{R}_m \ni \vec{\omega}_m} W(\vec{R}_m) J[0,m]. \tag{2.22}$$

By translation invariance this is the same as

$$\pi_m(y-x) \equiv \pi_m(o, y-x). \tag{2.23}$$

Then for each $n \geq 2$ and each $n^* \in \{1, \ldots, n-1\}$ (2.10) can be written as

$$\rho_n(x) = \rho^{-1} \sum_{m_0=0}^{n^*} \sum_{m_0'=n^*}^{n} \left[\sum_{\omega: o \overset{n}{\to} x} W(\omega) \sum_{\vec{R}_n \ni \vec{\omega}_n} W(\vec{R}_n) K[0, m_0 - 1] J[m_0, m_0'] K[m_0'+1, m] \right]. \tag{2.24}$$

If $m_0 > 0$ and $m_0' < n$ then the term in large square brackets is

$$= \sum_{u_1 \in \mathbb{Z}^d} \left(\sum_{\omega^1: o \overset{m_0-1}{\to} u_1} W(\omega_1) \sum_{\vec{R}^1_{m_0-1} \ni \vec{\omega}^1_{m_0-1}} W(\vec{R}^1_{m_0-1}) K^1[0, m_0-1] \right) \tag{2.25}$$

$$\times \sum_{v_1} z_D D(v_1 - u_1) \sum_{v_2} \left(\sum_{\omega^*: v_1 \overset{m_0'-m_0}{\to} v_2} W(\omega^*) \sum_{\vec{R}^*_{m_0'-m_0} \ni \vec{\omega}^*_{m_0'-m_0}} W(\vec{R}^*_{m_0'-m_0}) J^*[0, m_0'-m_0] \right) \tag{2.26}$$

$$\times \sum_{u_2} z_D D(u_2 - v_2) \left(\sum_{\omega^2: u_2 \overset{n-m_0'-1}{\to} x} W(\omega_2) \sum_{\vec{R}^2_{n-m_0'-1} \ni \vec{\omega}^2_{n-m_0'-1}} W(\vec{R}^2_{n-m_0'-1}) K^2[0, n-m_0'-1] \right). \tag{2.27}$$

Examining each term in large brackets we see that this is equal to

$$\sum_{u_1 \in \mathbb{Z}^d} \rho \rho_{m_0-1}(u_1) \tag{2.28}$$

$$\times \sum_{v_1} z_D D(v_1 - u) \sum_{v_2} \pi_{m_0'-m_0}(v_2 - v_1) \tag{2.29}$$

$$\times \sum_{u_2} z_D D(u_2 - v_2) \rho \rho_{n-m_0'-1}(x - u_2) \tag{2.30}$$

$$= \rho^2 z_D^2 \big(\rho_{m_0-1} * D * \pi_{m_0'-m_0} * D * \rho_{n-m_0'-1} \big)(x). \tag{2.31}$$

Similarly: if $m_0 = 0$ and $m_0' < n$ then the term in square brackets in (2.24) is $\rho z_D(\pi_{m_0'} * D * \rho_{n-m_0'-1})(x)$; if $m_0 > 0$ and $m_0' = n$ then we get $\rho z_D(\rho_{m_0-1} * D * \pi_{n-m_0})(x)$; if both $m_0 = 0$ and $m_0' = n$ we simply get $\pi_n(x)$.

Thus,

$$\rho_n(x) = \rho^{-1} \Big[\sum_{m_0=1}^{n^*} \sum_{m_0'=n^*}^{n-1} \rho^2 z_D^2 \big(\rho_{m_0-1} * D * \pi_{m_0'-m_0} * D * \rho_{n-m_0'-1} \big)(x) \tag{2.32}$$

$$+ \sum_{m_0'=n^*}^{n-1} \rho z_D (\pi_{m_0'} * D * \rho_{n-m_0'-1})(x) \tag{2.33}$$

$$+ \sum_{m_0=1}^{n^*} \rho z_D (\rho_{m_0-1} * D * \pi_{n-m_0})(x) \tag{2.34}$$

$$+ \pi_n(x) \Big]. \tag{2.35}$$

Since we are actually interested in fixing locations along the backbone from o to x at time points $n_1, \dots, n_r$ we need to introduce a modified version of π_m to handle this. For $m \in \mathbb{N}$, and $x, y \in \mathbb{Z}^d$ and for $0 < j_1 < j_2 < \cdots < j_r < m$ and $x_1, \dots, x_r \in \mathbb{Z}^d$ define

$$\pi_{m;(\vec{j},\vec{x})}(x,y) = \sum_{\omega: x \overset{m}{\to} y} \left(\prod_{b=1}^{r} \mathbb{1}_{\{\omega(j_b)=x_b\}} \right) W(\omega) \sum_{R_0 \ni \omega(0)} \cdots \sum_{R_m \ni \omega(m)} \prod_{i=0}^{m} W(R_i) J[0,m]. \tag{2.36}$$

The following is proved using a combinatorial technique known as the lace expansion, which shall be discussed in Section 5.

Proposition 2.3. *Fix $d > 8$. Then there exist $L_0, C \gg 1$ such that for all $L \geq L_0$ and for every $0 < j_1 < j_2 < \cdots < j_r < m$,*

$$\sum_x |\pi_m(x)| \leq \frac{C}{(1+m)^{\frac{d-4}{2}}}, \tag{2.37}$$

$$\sum_x |x|^2 |\pi_m(x)| \leq \frac{C}{(1+m)^{\frac{d-6}{2}}}, \tag{2.38}$$

$$\sum_{\vec{x}} \sum_y |\pi_{m;(\vec{j},\vec{x})}(y)| \leq \frac{C}{(1+m)^{\frac{d-4}{2}}}, \tag{2.39}$$

$$\sum_{\vec{x}} |x_1|^2 \sum_y |\pi_{m;(\vec{j},\vec{x})}(y)| \leq \frac{C}{(1+m)^{\frac{d-6}{2}}}, \tag{2.40}$$

$$\sum_{\vec{x}} |y - x_1|^2 \sum_y |\pi_{m;(\vec{j},\vec{x})}(y)| \leq \frac{C}{(1+m)^{\frac{d-6}{2}}}. \tag{2.41}$$

Both (2.37) and (2.38) are proved in [17, Proposition 5.1], while the proofs of (2.39)-(2.41) involve only small modifications of these proofs. For this reason we shall not give detailed proofs of (2.39)-(2.41), but shall give only a sketch of how these results are obtained.

Assuming (2.37) we shall prove in Section 2.2 that the functions π_m for $m \in \mathbb{Z}_+$ are related to the constant A via the following result.

Proposition 2.4. *Fix $d > 8$. Then there exist $L_0 \gg 1$ such that for all $L \geq L_0$*

$$\sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \hat{\pi}_{m+m'}(0) = \frac{1}{A \rho z_D^2}.$$

2.2 Proof of Proposition 2.4 assuming (2.37)

By (2.4),

$$\hat{\rho}_{2n}(0) = A + \mathcal{O}(n^{-\frac{d-8}{2}}). \tag{2.42}$$

On the other hand, from summing (2.32)-(2.35) over x

$$\hat{\rho}_{2n}(0) = \rho^{-1} \Big[\sum_{m_0=1}^{n} \sum_{m_0'=n}^{2n-1} \rho^2 z_D^2 \hat{\rho}_{m_0-1}(0) \hat{\pi}_{m_0'-m_0}(0) \hat{\rho}_{2n-m_0'-1}(0) \tag{2.43}$$

$$+ \sum_{m_0'=n}^{2n-1} \rho z_D \hat{\pi}_{m_0'}(0) \hat{\rho}_{n-m_0'-1}(0) + \sum_{m_0=1}^{n} \rho z_D \hat{\rho}_{m_0-1}(0) \hat{\pi}_{2n-m_0}(0) + \hat{\pi}_{2n}(0) \Big]. \tag{2.44}$$

From (2.2) we have $\sup_n \hat{\rho}_n(0) < \infty$, and from (2.37) we have

$$|\hat{\pi}_m(0)| \leq \frac{C}{(m+1)^{\frac{d-4}{2}}}, \tag{2.45}$$

so that each of the three terms in the second line of (2.44) are at most

$$\sum_{m=n}^{\infty} \frac{C}{(m+1)^{\frac{d-4}{2}}} \leq \frac{C}{n^{\frac{d-6}{2}}}. \tag{2.46}$$

(Note that here and elsewhere, the generic constant C changes from line to line). Thus

$$\hat{\rho}_{2n}(0) = z_D^2 \rho \sum_{m_0=1}^{n} \sum_{m_0'=n}^{2n-1} \hat{\rho}_{m_0-1}(0)\hat{\pi}_{m_0'-m_0}(0)\hat{\rho}_{2n-m_0'-1}(0) + \mathcal{O}(n^{-\frac{d-6}{2}}). \tag{2.47}$$

Letting $m = n - m_0$ and $m' = m_0' - n$, this can be written as

$$\hat{\rho}_{2n}(0) = z_D^2 \rho \sum_{m=0}^{n-1} \sum_{m'=0}^{n-1} \hat{\rho}_{n-m-1}(0)\hat{\pi}_{m+m'}(0)\hat{\rho}_{n-m'-1}(0) + \mathcal{O}(n^{-\frac{d-6}{2}}). \tag{2.48}$$

Let $\epsilon \in (0,1)$. The contribution to the sum over m, m' in (2.48) from $m > \epsilon n$ or $m' > \epsilon n$ is at most

$$\sum_{m=\epsilon n}^{\infty} \sum_{m'=0}^{\infty} \frac{C}{(m+m')^{\frac{d-4}{2}}} \leq \sum_{m=\epsilon n}^{\infty} \frac{C}{m'^{\frac{d-6}{2}}} \leq \frac{C}{n^{\frac{d-8}{2}}}. \tag{2.49}$$

Thus,

$$\hat{\rho}_{2n}(0) = z_D^2 \rho \sum_{m=0}^{\epsilon n} \sum_{m'=0}^{\epsilon n} \hat{\rho}_{n-m-1}(0)\hat{\pi}_{m+m'}(0)\hat{\rho}_{n-m'-1}(0) + \mathcal{O}(n^{-\frac{d-8}{2}}), \tag{2.50}$$

where the constants in the $\mathcal{O}$ notation in square brackets depend on ϵ. Since $\epsilon < 1$ and $\hat{\rho}_n(0) = A + \mathcal{O}(n^{-\frac{d-8}{2}})$ we can write

$$\hat{\rho}_{2n}(0) = z_D^2 \rho \sum_{m=0}^{\epsilon n} \sum_{m'=0}^{\epsilon n} \left[A + \mathcal{O}(n^{-\frac{d-8}{2}})\right] \hat{\pi}_{m+m'}(0) \left[A + \mathcal{O}(n^{-\frac{d-8}{2}})\right] + \mathcal{O}(n^{-\frac{d-8}{2}}), \tag{2.51}$$

where the constants in the $\mathcal{O}$ notation in square brackets depend on ϵ (but not m, m'). Since $|\hat{\pi}_{m+m'}(0)|$ is summable in m, m' (by (2.37)) this becomes

$$\hat{\rho}_{2n}(0) = z_D^2 \rho A^2 \sum_{m=0}^{\epsilon n} \sum_{m'=0}^{\epsilon n} \hat{\pi}_{m+m'}(0) + \mathcal{O}(n^{-\frac{d-8}{2}}) \tag{2.52}$$

$$= z_D^2 \rho A^2 \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \hat{\pi}_{m+m'}(0) + \mathcal{O}(n^{-\frac{d-8}{2}}), \tag{2.53}$$

where we have used (2.37) and (2.49) again in the last step. Letting $n \to \infty$ and using the fact that $\hat{\rho}_{2n}(0) \to A$ we see that

$$\sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \hat{\pi}_{m+m'}(0) = \frac{1}{z_D^2 A \rho}, \tag{2.54}$$

as claimed. ∎

3 Proof of Theorem 1.1 assuming Proposition 2.3

In order to prove Theorem 1.1, for each $t > 0$ it is enough to show convergence of the finite-dimensional distributions and tightness (see e.g. [2, Chapter 3]).

Let $X = (X(s))_{s \leq t}$ denote an element of $\mathcal{D}_t \equiv \mathcal{D}_t(\mathbb{R}^d)$, and set $t_0 = 0$. Then for convergence of the finite-dimensional distributions it is sufficient to verify the following:

Proposition 3.1. *For $d > 8$ there exists L_0 such that for all $L \geq L_0$, each $t > 0$ and every $r \in \mathbb{Z}_+$, $\vec{k} \in (\mathbb{R}^d)^r$ and $0 = t_0 < t_1 < t_2 < \cdots < t_r \leq t$,*

$$\mathrm{E}_{\nu_{n,t}}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^r k_j(X(t_j)-X(t_{j-1}))}\right] \to \mathrm{E}_{\nu_t}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^r k_j(X(t_j)-X(t_{j-1}))}\right] = \prod_{j=1}^r \mathrm{e}^{-\frac{k_j^2}{2d}(t_j-t_{j-1})}. \quad (3.1)$$

For tightness it is enough to show the following:

Proposition 3.2. *For $d > 8$ there exists L_0 such that for all $L \geq L_0$ and each $t > 0$: there exists $C > 0$ such that for every $0 \leq t_1 \leq t_2 \leq t_3 \leq t$,*

$$\mathrm{E}_{\nu_{n,t}}\left[(X(t_3) - X(t_2))^2(X(t_2) - X(t_1))^2\right] \leq C|t_3 - t_1|^2. \quad (3.2)$$

Thus our task is to prove Propositions 3.1 and 3.2. This is the content of the following two subsections.

3.1 Proof of Proposition 3.1

The fact that

$$\mathrm{E}_{\nu_t}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^r k_j(X(t_j)-X(t_{j-1}))}\right] = \prod_{j=1}^r \mathrm{e}^{-\frac{k_j^2}{2d}(t_j-t_{j-1})}, \quad (3.3)$$

is immediate from the fact that $(X(s))_{s\leq t}$ is a d-dimensional Brownian motion under ν_t (by Lemma 1.2). Therefore it is sufficient to prove the following result.

Theorem 3.3. *Fix $d > 8$. There exists $L_0 \gg 1$ such that for all $L \geq L_0$:*

(i) $\sup_{n,t} \nu_{n,t}(\mathcal{D}_t) \leq K < \infty$, *and*

(ii) For each $t^*, \kappa^* > 0$ and every $r \in \mathbb{Z}_+$ and $0 = t_0 < t_1 < t_2 < \cdots < t_r \leq t \leq t^*$, and every $\vec{k} \in (\mathbb{R}^d)^r$ such that $|\vec{k}| \leq \kappa^*$,

$$\mathrm{E}_{\nu_{n,t}}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^r k_j(X(t_j)-X(t_{j-1}))}\right] = \prod_{j=1}^r \mathrm{e}^{-\frac{k_j^2}{2d}(t_j-t_{j-1})} + o(1), \qquad \text{as } n \to \infty, \quad (3.4)$$

where the error term depends on κ^*, t^* and $\min_{i\leq r}\{t_i - t_{i-1}\}$, but is otherwise uniform in $\vec{t}, t, \vec{k}$.

Proof. For the first claim simply note that as in (1.11) and (2.5)

$$\nu_{n,t}(\mathcal{D}_t) = \frac{C_1}{C_0}\sum_{x\in\frac{\mathbb{Z}^d}{\sqrt{c_0 n}}} \mathbb{P}(\sqrt{c_0 n}x \in \mathcal{T}_{\lfloor nt\rfloor}) = \frac{C_1}{C_0}\hat{\rho}_{\lfloor nt\rfloor}(0) \to \frac{C_1}{C_0}A, \quad (3.5)$$

so the claim (i) holds by (2.2) (or Theorem 2.1). Moreover, since $C_1/C_0 = A^{-1}$ this establishes (ii) with $r = 0$. Clearly we obtain the same result when $r > 0$ but all $k_j = 0$.

We prove (ii) by induction on r, having already established the claim when $r = 0$. For $r \geq 1$, let $x_0 = o$ and recall that $t_0 = 0$ to see that

$$\rho A \mathrm{E}_{\nu_{n,t}}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^r k_j(X(t_j)-X(t_{j-1}))}\right] \quad (3.6)$$

$$= \rho \sum_{x\in\frac{\mathbb{Z}^d}{\sqrt{c_0 n}}} \mathbb{E}\left[\mathbb{1}_{\{\sqrt{nc_0}x\in\mathcal{T}_{\lfloor nt\rfloor}\}}\mathrm{e}^{\mathrm{i}\sum_{j=1}^r k_j(w_{x,t}^{(n)}(t_j)-w_{x,t}^{(n)}(t_{j-1}))}\right] \quad (3.7)$$

$$= \sum_{x\in\mathbb{Z}^d}\sum_{x_1,\dots,x_r\in\mathbb{Z}^d}\sum_{\omega:o\overset{\lfloor nt\rfloor}{\to}x}\left(\prod_{i=1}^r \mathbb{1}_{\{\omega(\lfloor nt_i\rfloor)=x_i\}}\right)\left(\prod_{j=1}^r \mathrm{e}^{\mathrm{i}\frac{k_j}{\sqrt{nc_0}}(x_j-x_{j-1})}\right) W(\omega) \quad (3.8)$$

$$\times \sum_{\vec{R}_{\lfloor nt\rfloor}\ni\vec{\omega}_{\lfloor nt\rfloor}} W(\vec{R}_{\lfloor nt\rfloor})K[0,\lfloor nt\rfloor]. \quad (3.9)$$

By (2.21), $\rho A \mathrm{E}_{\nu_{n,t}}[\mathrm{e}^{\mathrm{i}\sum_{j=1}^r k_j(X(t_j)-X(t_{j-1}))}]$ is equal to

$$\sum_{m_0=0}^{\lfloor nt_1\rfloor}\sum_{m_0'=\lfloor nt_1\rfloor}^{\lfloor nt\rfloor}\sum_x\sum_{x_1,\dots,x_r}\left(\prod_{j=1}^r \mathrm{e}^{\mathrm{i}\frac{k_j}{\sqrt{nc_0}}(x_j-x_{j-1})}\right)\sum_{\omega:o\overset{\lfloor nt\rfloor}{\to}x}\left(\prod_{i=1}^r \mathbb{1}_{\{\omega(\lfloor nt_i\rfloor)=x_i\}}\right)W(\omega) \tag{3.10}$$

$$\times\sum_{\vec{R}_{\lfloor nt\rfloor}\ni\vec{\omega}_{\lfloor nt\rfloor}} W(\vec{R}_{\lfloor nt\rfloor})K[0,m_0-1]J[m_0,m_0']K[m_0'+1,\lfloor nt\rfloor]. \tag{3.11}$$

If $k_j=0$ for each j then the complex exponentials vanish as does the sum over $x_1,\dots,x_r$ of the indicators. However, $\vec{k}=\vec{0}$ is the straightforward case that we have already covered at the beginning of the proof. Our task will be to adapt the ideas from the proof of Proposition 2.4 to handle the case $k/\sqrt{nc_0}\approx 0$.

Let $\epsilon\in(\underline{\Delta}(\vec{t})/2,\underline{\Delta}(\vec{t}))$ where $\underline{\Delta}(\vec{t})=\min\{t_i-t_{i-1}:i\le r\}/2$ and let

$$H(\vec{t},n)=\{(m_0,m_0'):0\le m_0\le\lfloor nt_1\rfloor\le m_0'\le\lfloor nt\rfloor \text{ and } m_0'-m_0\ge n\epsilon\}.$$

We will write $\notin H(\vec{t},n)$ to mean that $0\le m_0\le\lfloor nt_1\rfloor<m_0'\le\lfloor nt\rfloor$ but $m_0'-m_0<n\epsilon$.

Consider the contribution to the sum in (3.11) from $(m_0,m_0')\notin H(\vec{t},n)$, i.e.

$$\sum_{\substack{(m_0,m_0')\\ \notin H(\vec{t},n)}}\sum_x\sum_{x_1,\dots,x_r}\left(\prod_{j=1}^r \mathrm{e}^{\mathrm{i}\frac{k_j}{\sqrt{nc_0}}(x_j-x_{j-1})}\right)\sum_{\omega:o\overset{\lfloor nt\rfloor}{\to}x}\left(\prod_{i=1}^r \mathbb{1}_{\{\omega(\lfloor nt_i\rfloor)=x_i\}}\right)W(\omega) \tag{3.12}$$

$$\times\sum_{\vec{R}_{\lfloor nt\rfloor}\ni\vec{\omega}_{\lfloor nt\rfloor}} W(\vec{R}_n)K[0,m_0-1]J[m_0,m_0']K[m_0'+1,\lfloor nt\rfloor]. \tag{3.13}$$

Then (3.12)-(3.13) is equal to

$$\sum_{\substack{(m_0,m_0')\\ \notin H(\vec{t},n)}}\sum_{\substack{u,v\\ u',v'}}\sum_{x_1}\mathrm{e}^{\mathrm{i}\frac{k_1}{\sqrt{nc_0}}x_1}z_D D(v-u)\sum_{\omega^{(0)}:o\overset{m_0-1}{\to}u}W(\omega^{(0)})\sum_{\vec{R}^{(0)}_{m_0-1}\ni\vec{\omega}^{(0)}_{m_0-1}}W(\vec{R}^{(0)}_{m_0-1})K^{(0)}[0,m_0-1] \tag{3.14}$$

$$\times\sum_{x_2}\mathrm{e}^{\mathrm{i}\frac{k_2}{\sqrt{nc_0}}(x_2-x_1)}\sum_{\omega^{(1)}:v\overset{m_0'-m_0}{\to}v'}W(\omega^{(1)})\mathbb{1}_{\{\omega^{(1)}(\lfloor nt_1\rfloor-m_0)=x_1\}} \tag{3.15}$$

$$\times\sum_{\vec{R}^{(1)}_{m_0'-m_0}\ni\vec{\omega}^{(1)}_{m_0'-m_0}}W(\vec{R}^{(1)}_{m_0'-m_0})J^{(1)}[m_0,m_0']z_D D(u'-v') \tag{3.16}$$

$$\times\sum_x\sum_{x_3,\dots,x_r}\sum_{\omega^{(2)}:u'\overset{\lfloor nt\rfloor-m_0'-1}{\to}x}\left(\prod_{j=3}^r \mathrm{e}^{\mathrm{i}\frac{k_j}{\sqrt{nc_0}}(x_j-x_{j-1})}\right)\left(\prod_{i=2}^r \mathbb{1}_{\{\omega^{(2)}(\lfloor nt_i\rfloor-m_0)=x_i\}}\right) \tag{3.17}$$

$$W(\omega^{(2)})\sum_{\vec{R}^{(2)}_{\lfloor nt\rfloor-m_0'-1}\ni\vec{\omega}^{(2)}_{\lfloor nt\rfloor-m_0'-1}}W(\vec{R}^{(2)}_{\lfloor nt\rfloor-m_0'-1})K^{(2)}[m_0'+1,m], \tag{3.18}$$

where the superscripts in the $K^{(i)}[,]$ and $J^{(i)}[,]$ indicate the collection of trees $\vec{R}^{(i)}$ to which the U_{st} factors in K and J are imposing restrictions on. Now use the fact that

$$\mathrm{e}^{\mathrm{i}kx_1}=\mathrm{e}^{\mathrm{i}ku}\mathrm{e}^{\mathrm{i}k(v-u)}\mathrm{e}^{\mathrm{i}k(x_1-v)},\quad\text{and} \tag{3.19}$$

$$\mathrm{e}^{\mathrm{i}k(x_2-x_1)}=\mathrm{e}^{\mathrm{i}k(v'-x_1)}\mathrm{e}^{\mathrm{i}k(u'-v')}\mathrm{e}^{\mathrm{i}k(x_2-u')}, \tag{3.20}$$

and let $z = x_1 - v$ and $z' = v' - x_1$ to rewrite the above as

$$z_D^2 \sum_{\substack{(m_0,m_0')\\ \notin H(\vec{t},n)}} \rho\hat{\rho}_{m_0-1}(\frac{k_1}{\sqrt{nc_0}})\hat{D}(\frac{k_1}{\sqrt{nc_0}})\hat{D}(\frac{k_2}{\sqrt{nc_0}}) \tag{3.21}$$

$$\times \sum_{z,z'} \mathrm{e}^{\mathrm{i}\frac{k_2}{\sqrt{nc_0}}z'} \mathrm{e}^{\mathrm{i}\frac{k_1}{\sqrt{nc_0}}z} \pi_{m_0'-m_0;(\lfloor nt_1\rfloor - m_0,z)}(o, z+z') \tag{3.22}$$

$$\times \rho A \mathrm{E}_{\nu_{n,(\lfloor nt\rfloor-(m_0'+1))/n}} \Big[\mathrm{e}^{\mathrm{i}\frac{k_2}{\sqrt{nc_0}}\left(X((\lfloor nt_2\rfloor-(m_0'+1))/n)-X(0)\right)} \tag{3.23}$$

$$\times \mathrm{e}^{\mathrm{i}\sum_{j=3}^r \frac{k_j}{\sqrt{nc_0}}\left(X((\lfloor nt_j\rfloor-(m_0'+1))/n)-X((\lfloor nt_{j-1}\rfloor-(m_0'+1))/n)\right)}\Big]. \tag{3.24}$$

By (i) the modulus of the expectation is bounded above by a constant K.

Note that

$$\hat{D}(\frac{k_1}{\sqrt{nc_0}}) = 1 + (\hat{D}(\frac{k_1}{\sqrt{nc_0}}) - 1) \tag{3.25}$$

$$\hat{D}(\frac{k_2}{\sqrt{nc_0}}) = 1 + (\hat{D}(\frac{k_2}{\sqrt{nc_0}}) - 1), \tag{3.26}$$

and each of the differences on the right hand sides are bounded in absolute value by $C\frac{|k_i|^2}{n}$ when n is large. Thus, using the uniform bound on $\hat{\rho}_n(k)$, (i) and (2.39), the contribution to (3.21)-(3.23) from a term involving $(\hat{D}(\frac{k_i}{\sqrt{nc_0}}) - 1)$ is at most

$$C\frac{|k_i|^2}{n} \sum_{\substack{(m_0,m_0')\\ \notin H(\vec{t},n)}} \sum_{z,z'} |\pi_{m_0'-m_0;(\lfloor nt_1\rfloor-m_0,z)}(o,z+z')| K \le C\frac{|k_i|^2}{n} \le C\frac{\kappa^*}{n},$$

Then (3.12)-(3.13) is equal to $\mathcal{O}(\frac{\kappa^*}{n})$ plus

$$z_D^2\rho \sum_{\substack{(m_0,m_0')\\ \notin H(\vec{t},n)}} \hat{\rho}_{m_0-1}(\frac{k_1}{\sqrt{nc_0}}) \tag{3.27}$$

$$\times \sum_{z,z'} \mathrm{e}^{\mathrm{i}\frac{k_2}{\sqrt{nc_0}}z'} \mathrm{e}^{\mathrm{i}\frac{k_1}{\sqrt{nc_0}}z} \pi_{m_0'-m_0;(\lfloor nt_1\rfloor - m_0,z)}(o, z+z') \tag{3.28}$$

$$\times \rho A \mathrm{E}_{\nu_{n,(\lfloor nt\rfloor-(m_0'+1))/n}} \Big[\mathrm{e}^{\mathrm{i}\frac{k_2}{\sqrt{nc_0}}\left(X((\lfloor nt_2\rfloor-(m_0'+1))/n)-X(0)\right)} \tag{3.29}$$

$$\times \mathrm{e}^{\mathrm{i}\sum_{j=3}^r \frac{k_j}{\sqrt{nc_0}}\left(X((\lfloor nt_j\rfloor-(m_0'+1))/n)-X((\lfloor nt_{j-1}\rfloor-(m_0'+1))/n)\right)}\Big]. \tag{3.30}$$

Since for $(m_0, m_0') \notin H(\vec{t}, n)$ we have

$$n\underline{\Delta}(\vec{t}) - 1 \le \lfloor nt_1\rfloor - \lfloor nt_0\rfloor - \epsilon n \le m_0 \le n(t_1 - t_0) + 1,$$

it follows from (2.4) that for $(m_0, m_0') \notin H(\vec{t}, n)$

$$\hat{\rho}_{m_0-1}(\frac{k_1}{\sqrt{nc_0}}) = A\mathrm{e}^{-\frac{k_1^2}{2d}(m_0-1)/n} + O_{\underline{\Delta},t^*}(|k|^2 n^{-\delta}) + O_{\underline{\Delta}}(n^{-\frac{d-8}{2}}). \tag{3.31}$$

The latter two terms also contribute error terms of the form $O_{\underline{\Delta},t^*}(|k|^2 n^{-\delta}) + O_{\underline{\Delta}}(n^{-\frac{d-8}{2}})$ in (3.27)-(3.30) (i.e. after summing over (m_0, m_0')), due to (i) and (2.39). Similarly for $(m_0, m_0') \notin H(\vec{t}, n)$,

$$n\underline{\Delta}(\vec{t}) - 1 \le nt_2 - 1 - nt_1 - n\epsilon \le \lfloor nt_2\rfloor - m_0' \le n(t_2 - t_1) + 1.$$

Thus, by the induction hypothesis, the expectation in (3.23) is equal to

$$\mathrm{E}_{\nu_{n,(\lfloor nt\rfloor-(m_0'+1))/n}}\Big[\mathrm{e}^{\mathrm{i}\frac{k_2}{\sqrt{nc_0}}\big(X((\lfloor nt_2\rfloor-(m_0'+1))/n)-X(0)\big)} \tag{3.32}$$

$$\times \mathrm{e}^{\mathrm{i}\sum_{j=3}^{r}\frac{k_j}{\sqrt{nc_0}}\big(X((\lfloor nt_j\rfloor-(m_0'+1))/n)-X((\lfloor nt_{j-1}\rfloor-(m_0'+1))/n)\big)}\Big] \tag{3.33}$$

$$= \mathrm{e}^{-\frac{k_2^2}{2d}(\lfloor nt_2\rfloor-(m_0'+1))/n)}\prod_{j=3}^{r}\mathrm{e}^{-\frac{k_j^2}{2d}(t_j-t_{j-1})}+o(1), \tag{3.34}$$

where the error term depends on $\underline{\Delta}, t^*$. Again this $o(1)$ term will contribute to error terms in (3.27)-(3.30) due to (2.39). We have shown that (3.12)-(3.13) is equal to

$$o(1)+z_D^2\rho^2A^2\sum_{\substack{(m_0,m_0')\\ \notin H(\vec{t},n)}}\mathrm{e}^{-\frac{k_1^2}{2d}(m_0-1)/n}\sum_{z,z'}\mathrm{e}^{\mathrm{i}\frac{k_2}{\sqrt{nc_0}}z'}\mathrm{e}^{\mathrm{i}\frac{k_1}{\sqrt{nc_0}}z}\pi_{m_0'-m_0;(nt_1-m_0,z)}(o,z+z') \tag{3.35}$$

$$\times\,\mathrm{e}^{-\frac{k_2^2}{2d}(t_2-(m_0'+1)/n)}\prod_{j=3}^{r}\mathrm{e}^{-\frac{k_j^2}{2d}(t_j-t_{j-1})}. \tag{3.36}$$

Now due to the symmetries of the lattice and the weight function, and the fact that $D(x)=D(-x)$, we have the symmetry

$$\pi_{m_0'-m_0;(nt_1-m_0,z)}(o,z+z')=\pi_{m_0'-m_0;(nt_1-m_0,-z)}(o,-z-z'), \tag{3.37}$$

with respect to sign changes of the spatial components. Thus,

$$\sum_{z,z'}\mathrm{e}^{\mathrm{i}(\frac{k_2}{\sqrt{nc_0}}z'+\frac{k_1}{\sqrt{nc_0}}z)}\pi_{m_0'-m_0;(nt_1-m_0,z)}(o,z+z') \tag{3.38}$$

$$=\sum_{z,z'}\pi_{m_0'-m_0;(nt_1-m_0,z)}(o,z+z')\Big(\cos(\frac{k_2}{\sqrt{nc_0}}z'+\frac{k_1}{\sqrt{nc_0}}z) \tag{3.39}$$

$$+\,\mathrm{i}\,\sin(\frac{k_2}{\sqrt{nc_0}}z'+\frac{k_1}{\sqrt{nc_0}}z)\Big) \tag{3.40}$$

$$=\sum_{z,z'}\pi_{m_0'-m_0;(nt_1-m_0,z)}(o,z+z')\cos(\frac{k_2}{\sqrt{nc_0}}z'+\frac{k_1}{\sqrt{nc_0}}z), \tag{3.41}$$

since the sine function is odd.

Using the above and the fact that $|\cos(x)-\cos(0)|\le x^2$ we see that

$$\left|\sum_{z,z'}\left(\mathrm{e}^{\mathrm{i}\frac{k_2}{\sqrt{nc_0}}z'}\mathrm{e}^{\mathrm{i}\frac{k_1}{\sqrt{nc_0}}z}-1\right)\pi_{m_0'-m_0;(nt_1-m_0,z)}(o,z+z')\right| \tag{3.42}$$

$$\le\frac{C}{n}\sum_{z,z'}|k_2z'+k_1z|^2|\pi_{m_0'-m_0;(nt_1-m_0,z)}(o,z+z')| \tag{3.43}$$

$$\le\frac{C}{n}\sum_{z,z'}(|k_2z'|^2+|k_1z|^2)|\pi_{m_0'-m_0;(nt_1-m_0,z)}(o,z+z')| \tag{3.44}$$

$$\le\frac{C}{n}\frac{|k_1|^2+|k_2|^2}{(m_0'-m_0+1)^{\frac{d-6}{2}}}, \tag{3.45}$$

where we have used (2.40)-(2.41) in the last step. Summing over $(m_0,m_0')\notin H(\vec{t},n)$ again gives an error term since e.g.

$$\frac{C|k_1|^2}{n}\sum_{\substack{(m_0,m_0')\\ \notin H(\vec{t},n)}}\frac{1}{(m_0'-m_0+1)^{\frac{d-6}{2}}}\le\frac{C|k_1|^2}{n}\sum_{m\le\epsilon n}\sum_{m'\le\epsilon n}\frac{1}{(m+m'+1)^{\frac{d-6}{2}}} \tag{3.46}$$

$$\le\frac{C|k_1|^2}{n}\sum_{m\le\epsilon n}\frac{1}{(m+1)^{\frac{d-8}{2}}}\le\frac{C|k_1|^2}{n^{1/2}}, \tag{3.47}$$

where we have used the fact that $d \geq 9$ in the last two steps.

Thus, (3.21) to (3.23) is equal to

$$o(1) + A^2 z_D^2 \rho^2 \prod_{j=3}^{r} \mathrm{e}^{-\frac{k_j^2}{2d}(t_j - t_{j-1})} \sum_{\substack{(m_0, m_0') \\ \notin H(\vec{t}, n)}} \mathrm{e}^{-\frac{k_1^2}{2d}\frac{(m_0-1)}{n}} \mathrm{e}^{-\frac{k_2^2}{2d}\frac{(nt_2-(m_0'+1))}{n}} \tag{3.48}$$

$$\times \sum_{z,z'} \pi_{m_0'-m_0;(nt_1-m_0,z)}(o, z+z'). \tag{3.49}$$

But the sum over z of the indicator that the backbone is at position z at time n' gives 1, so that

$$\sum_{z,z'} \pi_{m;(n',z)}(o, z+z') = \sum_{x} \pi_m(o,x) = \hat{\pi}_m(0). \tag{3.50}$$

Thus, (3.48)-(3.49) is equal to

$$z_D^2 A^2 \rho^2 \prod_{j=3}^{r} \mathrm{e}^{-\frac{k_j^2}{2d}(t_j - t_{j-1})} \sum_{\substack{(m_0, m_0') \\ \notin H(\vec{t}, n)}} \mathrm{e}^{-\frac{k_1^2}{2d}\frac{(m_0-1)}{n}} \mathrm{e}^{-\frac{k_2^2}{2d}\frac{(nt_2-(m_0'+1))}{n}} \hat{\pi}_{m_0'-m_0}(0). \tag{3.51}$$

Now write

$$\mathrm{e}^{-\frac{k_1^2}{2d}\frac{(m_0-1)}{n}} = \mathrm{e}^{-\frac{k_1^2}{2d}t_1} + \left(\mathrm{e}^{-\frac{k_1^2}{2d}\frac{(m_0-1)}{n}} - \mathrm{e}^{-\frac{k_1^2}{2d}t_1}\right) \tag{3.52}$$

$$\mathrm{e}^{-\frac{k_2^2}{2d}\frac{(nt_2-(m_0'+1))}{n}} = \mathrm{e}^{-\frac{k_2^2}{2d}(t_2-t_1)} + \left(\mathrm{e}^{-\frac{k_2^2}{2d}\frac{(nt_2-(m_0'+1))}{n}} - \mathrm{e}^{-\frac{k_2^2}{2d}(t_2-t_1)}\right) \tag{3.53}$$

The differences in brackets in the first and second lines are bounded in absolute value by $\min\{1, C|k_1|^2(nt_1 - m_0 + 1)/n\}$, and $\min\{1, C|k_2|^2(m_0' - nt_1 + 1)/n\}$ respectively, which give rise to error terms when summed with $\hat{\pi}$ over m_0, m_0'. For example, the term with both differences is bounded in absolute value by:

$$C \sum_{\substack{(m_0, m_0') \\ \notin H(\vec{t}, n)}} \frac{|k_2|^2(m_0' - nt_1 + 1)}{n} \frac{1}{(m_0' - m_0 + 1)^{\frac{d-4}{2}}} \tag{3.54}$$

$$\leq \frac{C|k_2|^2}{n} \sum_{\substack{(m_0, m_0') \\ \notin H(\vec{t}, n)}} \frac{1}{(m_0' - m_0 + 1)^{\frac{d-6}{2}}} \leq \frac{C|k_2|^2}{n^{1/2}}, \tag{3.55}$$

as in (3.46)-(3.47). The other terms give the same bounds either with k_1 or k_2 dependence.

Therefore (3.48)-(3.49) is equal to

$$o(1) + z_D^2 A^2 \rho^2 \prod_{j=1}^{r} \mathrm{e}^{-\frac{k_j^2}{2d}(t_j - t_{j-1})} \sum_{\substack{(m_0, m_0') \\ \notin H(\vec{t}, n)}} \hat{\pi}_{m_0'-m_0}(0), \tag{3.56}$$

with the error term as claimed in the theorem. Letting $m = nt_1 - m_0$ and $m' = m_0' - nt_1$ we have that $m_0' - m_0 = m + m'$ and by (2.54),

$$\sum_{\substack{(m_0, m_0') \\ \notin H(\vec{t}, n)}} \hat{\pi}_{m_0'-m_0}(0) = \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \hat{\pi}_{m+m'}(0) - \sum_{\substack{(m,m'): \\ m+m' \geq \epsilon n}} \hat{\pi}_{m+m'}(0) \tag{3.57}$$

$$= \frac{1}{\rho A z_D^2} + \mathcal{O}_{\underline{\Delta}}(n^{-\frac{d-8}{2}}). \tag{3.58}$$

Thus the main contribution to (3.48)-(3.49) is

$$A^2\rho^2 z_D^2 \prod_{j=1}^{r} \mathrm{e}^{-\frac{k_j^2}{2d}(t_j-t_{j-1})} \frac{1}{\rho A z_D^2} = A\rho \prod_{j=1}^{r} \mathrm{e}^{-\frac{k_j^2}{2d}(t_j-t_{j-1})}, \tag{3.59}$$

with all error terms as claimed.

It remains to consider the contribution to (3.11) from $(m_0, m_0') \in H(\vec{t}, n)$. Let $j(m_0') = \max\{i : t_i \le m_0'\}$ and $\vec{t}(m_0') = (t_1, \dots, t_{j(m_0')})$ and $\vec{x}(m_0') = (x_1, \dots, x_{j(m_0')})$. Then using the fact that $\hat{D}$, $\hat{\rho}_m$, and (from (i)) $\mathrm{E}_{\nu_{n,t}}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^{\ell} k_j (X(t_j)-X(t_{j-1}))}\right]$ are all bounded above by a constant, the contribution to (3.11) from $(m_0, m_0') \in H(\vec{t}, n)$ is bounded in absolute value by

$$C \sum_{(m_0,m_0')\in H(\vec{t},n)} \sum_{x} \sum_{\vec{x}(m_0')} |\pi_{m;(\vec{t}(m_0'),\vec{x}(m_0'))}(o,x)| \tag{3.60}$$

$$\le C \sum_{(m_0,m_0')\in H(\vec{t},n)} \frac{1}{(m_0' - m_0 + 1)^{\frac{d-4}{2}}} \tag{3.61}$$

$$= C \sum_{m>\epsilon n} \sum_{m'\ge 1} \frac{1}{(m+m'+1)^{\frac{d-4}{2}}} \tag{3.62}$$

$$\le C \sum_{m\ge 1} \sum_{m'\ge 1} \frac{1}{(\epsilon n + m + m' + 1)^{\frac{d-4}{2}}} \le \frac{C_\Delta}{n^{\frac{d-8}{2}}}. \tag{3.63}$$

Since this error term is also of the claimed form, we have verified that

$$A\mathrm{E}_{\nu_{n,t}}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^{r} k_j (X(t_j)-X(t_{j-1}))}\right] = A \prod_{j=1}^{r} \mathrm{e}^{-\frac{k_j^2}{2d}(t_j-t_{j-1})} + o(1) \tag{3.64}$$

with the error term as claimed. Thus we have completed the inductive step and therefore the proof. ∎

3.2 Proof of Proposition 3.2

The proof of this mixed moment bound is similar to that for self-avoiding walk in [22].

Fix $t_0 = 0 < t_1 < t_2 < t_3 \le t = t_4$ and let Y denote an element of $\mathcal{D}_t$ (i.e. Y denotes an $\mathbb{R}^d$-valued path on $[0,t]$). Letting $g : \mathcal{D}_t \to \mathbb{R}$ be a bounded Borel measureable function we have

$$\mathrm{E}_{\nu_{n,t}}[g(Y)] = \mathrm{E}_{\mu_n^H}\left[\mathrm{E}_{H_t}[g(Y)]\right] \tag{3.65}$$

$$= \mathrm{E}_{\mu_n^H}\left[\frac{1}{C_0 n} \sum_{\sqrt{nc_0}x \in \mathcal{T}_{\lfloor nt \rfloor}} g(w_{x,t}^{(n)})\right] \tag{3.66}$$

$$= \frac{C_1}{C_0 \rho} \sum_{T \ni o} W(T) \sum_{\sqrt{nc_0}x \in T_{\lfloor nt \rfloor}} g(w_{x,t}^{(n)}) \tag{3.67}$$

$$= \frac{C_1}{C_0 \rho} \sum_{x \in \frac{\mathbb{Z}^d}{\sqrt{nc_0}}} \sum_{\substack{T \ni o: \\ \sqrt{nc_0}x \in T_{\lfloor nt \rfloor}}} W(T) g(w_{x,t}^{(n)}) \tag{3.68}$$

Breaking the sum over T into the backbone and ribs we obtain

$$\mathrm{E}_{\nu_{n,t}}[g(Y)] = \frac{C_1}{C_0 \rho} \sum_{x \in \frac{\mathbb{Z}^d}{\sqrt{nc_0}}} \sum_{\omega : o \overset{\lfloor nt \rfloor}{\to} \sqrt{nc_0}x} W(\omega) g(\omega_{x,t}^{(n)}) \sum_{\vec{R}_{\lfloor nt \rfloor} \ni \vec{\omega}_{\lfloor nt \rfloor}} W(\vec{R}_{\lfloor nt \rfloor}) K[0, \lfloor nt \rfloor]. \tag{3.69}$$

For fixed $t, t_1, \dots, t_3$ let $g(y) = (y(t_3) - y(t_2))^2 (y(t_2) - y(t_1))^2$. Recalling that

$$\omega_{x,t}^{(n)}(s) = \frac{\omega_{\sqrt{c_0 n}x, \lfloor nt \rfloor}(\lfloor ns \rfloor)}{\sqrt{c_0 n}}, \quad \text{for } s \in [0,t], \tag{3.70}$$

we see that

$$\mathrm{E}_{\nu_{n,t}}[g(Y)] = \frac{C_1}{C_0 \rho c_0^2} \sum_{x \in \frac{\mathbb{Z}^d}{\sqrt{c_0 n}}} \sum_{\omega : o \overset{\lfloor nt \rfloor}{\to} \sqrt{nc_0} x} W(\omega) \frac{(\omega(\lfloor nt_3 \rfloor) - \omega(\lfloor nt_2 \rfloor))^2}{n} \frac{(\omega(\lfloor nt_2 \rfloor) - \omega(\lfloor nt_1 \rfloor))^2}{n} \tag{3.71}$$

$$\times \sum_{\vec{R}_{\lfloor nt \rfloor}} W(\vec{R}_{\lfloor nt \rfloor}) K[0, \lfloor nt \rfloor]. \tag{3.72}$$

If $t_3 - t_1 \leq \frac{1}{n}$ then either $\lfloor nt_3 \rfloor = \lfloor nt_2 \rfloor$ or $\lfloor nt_2 \rfloor = \lfloor nt_1 \rfloor$ and in either case $\mathrm{E}_{\nu_{n,t}}[g(Y)] = 0$. Otherwise, applying Lemma 2.2 three times we have that

$$\sum_{\vec{R}_{\lfloor nt \rfloor}} W(\vec{R}_{\lfloor nt \rfloor}) K[0, \lfloor nt \rfloor] \leq \prod_{\ell=1}^{4} \sum_{\vec{R}^{(\ell)}_{\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor}} W(\vec{R}^{(\ell)}_{\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor}) K^{(\ell)}[0, \lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor]. \tag{3.73}$$

For $\ell = 0, 1, 2, 3, 4$ let $\sqrt{nc_0} x_\ell = \omega(\lfloor nt_\ell \rfloor)$ (so $x_0 = o$ and $x_4 = x$ in particular). Then we can write $W(\omega) = \prod_{\ell=1}^4 W(\omega^{(\ell)})$ where for each ℓ, $\omega^{(\ell)} = (\omega(\lfloor nt_{\ell-1} \rfloor), \ldots, \omega(\lfloor nt_\ell \rfloor))$ is a walk of length $\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor$ from $\sqrt{nc_0} x_{\ell-1}$ to $\sqrt{nc_0} x_\ell$, With this notation we have

$$\mathrm{E}_{\nu_{n,t}}[g(Y)] \leq \frac{C}{n^2} \sum_{\substack{x_1, x_2, x_3, x_4 \\ \in \frac{\mathbb{Z}^d}{\sqrt{c_0 n}}}} \sum_{\omega^{(1)}} \sum_{\omega^{(2)}} \sum_{\omega^{(3)}} \sum_{\omega^{(4)}} (\omega^{(3)}(\lfloor nt_3 \rfloor - \lfloor nt_2 \rfloor) - \omega^{(3)}(0))^2 \tag{3.74}$$

$$\times (\omega^{(2)}(\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor) - \omega^{(2)}(0))^2 \tag{3.75}$$

$$\times \prod_{\ell=1}^{4} W(\omega^{(\ell)}) \sum_{\vec{R}^{(\ell)}_{\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor}} W(\vec{R}^{(\ell)}_{\lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor}) K^{(\ell)}[0, \lfloor nt_\ell \rfloor - \lfloor nt_{\ell-1} \rfloor]. \tag{3.76}$$

The right hand side is equal to

$$\frac{C\rho^4}{n^2} \sum_{u_1 \in \mathbb{Z}^d} \rho_{\lfloor nt_1 \rfloor}(u_1) \tag{3.77}$$

$$\times \sum_{u_2 \in \mathbb{Z}^d} (u_2 - u_1)^2 \rho_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}(u_2 - u_1)^2 \tag{3.78}$$

$$\times \sum_{u_3 \in \mathbb{Z}^d} (u_3 - u_2)^2 \rho_{\lfloor nt_3 \rfloor - \lfloor nt_2 \rfloor}(u_3 - u_2)^2 \tag{3.79}$$

$$\times \sum_{u_4 \in \mathbb{Z}^d} \rho_{\lfloor nt_4 \rfloor - \lfloor nt_3 \rfloor}(u_4 - u_3) \tag{3.80}$$

$$= \frac{C\rho^4}{n^2} \hat{\rho}_{\lfloor nt_1 \rfloor}(0) \nabla^2 \hat{\rho}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}(0) \nabla^2 \hat{\rho}_{\lfloor nt_3 \rfloor - \lfloor nt_2 \rfloor}(0) \hat{\rho}_{\lfloor nt_4 \rfloor - \lfloor nt_3 \rfloor}(0), \tag{3.81}$$

where $\nabla^2 \hat{\rho}_m(0) = \sum_x |x|^2 \rho_m(x)$. By Theorem 2.1 we have that this is bounded by

$$\frac{C}{n^2} (\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor)(\lfloor nt_3 \rfloor - \lfloor nt_2 \rfloor). \tag{3.82}$$

In other words, we have proved that if $t_3 - t_1 \leq \frac{1}{n}$ then

$$\mathrm{E}_{\nu_{n,t}} \left[(X(t_3) - X(t_2))^2 (X(t_2) - X(t_1))^2 \right] = 0, \tag{3.83}$$

while if $t_3 - t_1 > \frac{1}{n}$ then

$$\begin{aligned}
\mathrm{E}_{\nu_{n,t}}\left[(X(t_3)-X(t_2))^2(X(t_2)-X(t_1))^2\right] &\le \frac{C}{n^2}(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor)(\lfloor nt_3\rfloor - \lfloor nt_2\rfloor) && (3.84)\\
&\le \frac{C}{n^2}(nt_2 - nt_1 + 1)(nt_3 - nt_2 + 1) && (3.85)\\
&\le C(t_3 - t_1 + \frac{1}{n})^2 && (3.86)\\
&\le C(t_3 - t_1)^2. && (3.87)
\end{aligned}$$

■

4 Proof of Theorem 1.3

Since we will make use of Lemma 1.2 in the proof of Theorem 1.3, we begin by showing how that result follows from [23, (II.8.6)(a)].

Proof of Lemma 1.2. Applied to our setting (with y being the origin $o \in \mathbb{R}^d$, $\tau = 0$, and unit branching variance $\gamma = 1$ etc.), and written in the notation of this paper, [23, (II.8.6)(a)] states that $\mathbb{N}_0^H(H_t(1) > 0) = 2/t$, and for bounded Borel $f : \mathcal{D}_t \to \mathbb{R}$, and bounded Borel $\psi : \mathbb{R}_+ \to \mathbb{R}_+$

$$\mathrm{E}_{\mathbb{N}_0^H}\left[\psi(H_t(1))H_t(f)\right] = \left[\int_0^\infty \psi(tz/2)ze^{-z}dz\right] \times E[f(B_{[0,t]})], \tag{4.1}$$

where $B_{[0,t]} = (B_s)_{s\in[0,t]}$ is a standard d-dimensional Brownian motion. Note that the result extends trivially to non-negative Borel ψ by monotone convergence.

For the first claim of the Lemma, use (4.1) with $\psi \equiv 1$ to get

$$\mathrm{E}_{\mathbb{N}_0^H}\left[H_t(f)\right] = \left(\int_0^\infty ze^{-z}dz\right) \times E[f(B_{[0,t]})] = E[f(B_{[0,t]})]. \tag{4.2}$$

For the second claim of the Lemma define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi(y) = y^{-1}$ for $y > 0$ (and $\psi(0) = 0$). Then by (4.1)

$$\begin{aligned}
\mathrm{E}_{\mathbb{N}_{0,t}^H}\left[\frac{H_t(f)}{H_t(1)}\right] &= \frac{\mathrm{E}_{\mathbb{N}_0^H}\left[\psi(H_t(1))H_t(f)\right]}{\mathbb{N}_0^H(H_t(1) > 0)} && (4.3)\\
&= \left(\frac{t}{2}\int_0^\infty \frac{2}{tz}ze^{-z}dz\right) \times E[f(B_{[0,t]})] = E[f(B_{[0,t]})], && (4.4)
\end{aligned}$$

as claimed.

■

We are now ready to prove our second main result.

Proof of Theorem 1.3. By definition, $\mu_{n,t}^Z$ is a discrete probability measure on $\mathcal{D}_t = \mathcal{D}_t(\mathbb{R}^d)$ such that for a singleton $w \in \mathcal{D}_t$,

$$\mu_{n,t}^Z(\{w\}) = \mathbb{P}\big(Z_{[0,t]}^{(n)} = w \,\big|\, |\mathcal{T}_{\lfloor nt\rfloor}| > 0\big), \tag{4.5}$$

where

$$\mathbb{P}(Z_{[0,t]}^{(n)} = w | \mathcal{T}_{\lfloor nt\rfloor}) = \frac{1}{|\mathcal{T}_{\lfloor nt\rfloor}|}\sum_{x\in\mathcal{T}_{\lfloor nt\rfloor}} \mathbb{1}_{\{w_{x,t}^{(n)}=w\}}. \tag{4.6}$$

Thus,

$$\mu^Z_{n,t}(\{w\}) = \frac{\mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{Z^{(n)}_{[0,t]}=w\}}\mathbb{1}_{\{|\mathcal{T}_{\lfloor nt\rfloor}|>0\}}\,\middle|\,\mathcal{T}_{\lfloor nt\rfloor}\right]\right]}{\mathbb{P}(|\mathcal{T}_{\lfloor nt\rfloor}|>0)} \tag{4.7}$$

$$= \frac{\mathbb{E}\left[\mathbb{1}_{\{|\mathcal{T}_{\lfloor nt\rfloor}|>0\}}\mathbb{E}\left[\mathbb{1}_{\{Z^{(n)}_{[0,t]}=w\}}\,\middle|\,\mathcal{T}_{\lfloor nt\rfloor}\right]\right]}{\mathbb{P}(|\mathcal{T}_{\lfloor nt\rfloor}|>0)} \tag{4.8}$$

$$= \frac{\mathbb{E}\left[\mathbb{1}_{\{|\mathcal{T}_{\lfloor nt\rfloor}|>0\}}\frac{1}{|\mathcal{T}_{\lfloor nt\rfloor}|}\sum_{x\in\mathcal{T}_{\lfloor nt\rfloor}}\mathbb{1}_{\{w^{(n)}_{x,t}=w\}}\right]}{\mathbb{P}(|\mathcal{T}_{\lfloor nt\rfloor}|>0)} \tag{4.9}$$

$$= \mathbb{E}\left[\frac{1}{|\mathcal{T}_{\lfloor nt\rfloor}|}\sum_{x\in\mathcal{T}_{\lfloor nt\rfloor}}\mathbb{1}_{\{w^{(n)}_{x,t}=w\}}\middle|\mathcal{T}_{\lfloor nt\rfloor}\neq\varnothing\right]. \tag{4.10}$$

Equivalently for any bounded Borel function $f:\mathcal{D}_t\to\mathbb{R}$,

$$\mu^Z_{n,t}(f) = \mathbb{E}\left[\frac{1}{|\mathcal{T}_{\lfloor nt\rfloor}|}\sum_{x\in\mathcal{T}_{\lfloor nt\rfloor}}f(w^{(n)}_{x,t})\middle|\mathcal{T}_{\lfloor nt\rfloor}\neq\varnothing\right] \tag{4.11}$$

$$= \mathbb{E}\left[\frac{C_0 n}{|\mathcal{T}_{\lfloor nt\rfloor}|}\frac{1}{C_0 n}\sum_{x\in\mathcal{T}_{\lfloor nt\rfloor}}f(w^{(n)}_{x,t})\middle|\mathcal{T}_{\lfloor nt\rfloor}\neq\varnothing\right] \tag{4.12}$$

$$= \mathbb{E}\left[(H^{(n)}_t(1))^{-1}H^{(n)}_t(f)\middle|H^{(n)}_t(1)>0\right]. \tag{4.13}$$

Note that (conditional on $H^{(n)}_t(1)>0$) $(H^{(n)}_t(1))^{-1}H^{(n)}_t$ is a random probability measure.

Suppose that for fixed t, $H^{(n)}_t$ under $\mathbb{P}$ (conditional on $H^{(n)}_t(1)>0$) converges weakly to H_t under $\mathbb{N}^H_0$ (conditioned on $H_t(1)>0$) as $n\to\infty$. Let $\mathcal{M}_t=\mathcal{M}_F(\mathcal{D}_t)\setminus\{0_M\}$, and let $f:\mathcal{D}_t\to\mathbb{R}$ be bounded and continuous. Then $h_f:\mathcal{M}_t\to\mathbb{R}$ defined by $h_f(H)=H(f)/H(1)$ is bounded and continuous and so

$$\mu^Z_{n,t}(f)=\mathbb{E}\left[h_f(H^{(n)}_t)\middle|H^{(n)}_t(1)>0\right]\to \mathrm{E}_{\mathbb{N}^H_0}\left[h_f(H_t)\middle|H_t(1)>0\right]=\nu_t(f), \tag{4.14}$$

by Lemma 1.2. ■

5 The lace expansion and the proof of Proposition 2.3.

In this final section we outline the steps involved in the proof of Proposition 2.3. As we have noted earlier, the first two bounds are already proved in [17, Section 5], and proofs of the latter bounds require only small modifications, which we will describe below.

Recall from (2.20) that

$$J[m,m'] = \sum_{\Gamma\in\mathcal{G}^{\mathrm{conn}}[m,m']_{\mathbb{Z}}}\prod_{i<j\in\Gamma}U_{ij}, \tag{5.1}$$

where $\mathcal{G}^{\mathrm{conn}}[m,m']_{\mathbb{Z}}$ denotes the set of connected graphs on $[m,m']_{\mathbb{Z}}$. A *lace* Λ on $[m,m']_{\mathbb{Z}}$ is a *minimally* connected graph, i.e. a connected graph Λ (on $[m,m']_{\mathbb{Z}}$) for which the removal of any element $st\in\Lambda$ results in a graph $\Lambda\setminus\{st\}$ that is not connected on $[m,m']_{\mathbb{Z}}$. We associate to each $\Gamma\in\mathcal{G}^{\mathrm{conn}}[m,m']_{\mathbb{Z}}$ a lace $\mathrm{L}(\Gamma)$ by choosing $s_1=0$ and $s_1t_1\in\mathrm{L}$ if $s_1t_1\in\Gamma$ and $s_1\ell\notin\Gamma$ for any $\ell>t_1$. If $t_1\neq m'$ then (until $t_i=m'$) we recursively add elements s_it_i to L such that $t_i=\max\{\ell: j\ell\in\Gamma \text{ for some } j\le t_{i-1}\}$ and $s_i=\min\{j: jt_i\in\Gamma\}$.

Given a lace Λ on $[m,m']_{\mathbb{Z}}$ we denote by $\mathcal{C}(\Lambda)$ the set of st with $s<t$ both in $[m,m']_{\mathbb{Z}}$ such that $\mathrm{L}(\Lambda\cup\{st\})=\Lambda$, and call this set the set of edges compatible with Λ. For a connected graph Γ it is known that $\mathrm{L}(\Gamma)=\Lambda$ if and only if $\Lambda\subset\Gamma$ is a lace and $\Gamma\setminus\Lambda\subset\mathcal{C}(\Lambda)$.

Let $\mathcal{L} = \mathcal{L}[m,m']$ denote the set of laces on $[m,m']_{\mathbb{Z}}$ and $\mathcal{L}^{\{N\}} = \mathcal{L}^{\{N\}}[m,m']$ denote the set of laces containing exactly N elements (edges). Then

$$J[m,m'] = \sum_{N=1}^{\infty}(-1)^N \sum_{\Lambda\in\mathcal{L}^{\{N\}}[m,m']} \prod_{ij\in\Lambda}(-U_{ij}) \prod_{st\in\mathcal{C}(\Lambda)}[1+U_{st}]. \tag{5.2}$$

Note that each $(-U_{ij})$ is non-negative.

From (2.23) we have that

$$\pi_m(x) = \sum_{N=1}^{\infty}(-1)^N \sum_{\omega:o\xrightarrow{m}x} W(\omega) \sum_{\vec{R}_m\ni\vec{\omega}_m} W(\vec{R}_m) \sum_{\Lambda\in\mathcal{L}^{\{N\}}[0,m]} \prod_{ij\in\Lambda}(-U_{ij}) \prod_{st\in\mathcal{C}(\Lambda)}[1+U_{st}]. \tag{5.3}$$

Defining

$$\pi_m^{\{N\}}(x) = \sum_{\omega:o\xrightarrow{m}x} W(\omega) \sum_{\vec{R}_m\ni\vec{\omega}_m} W(\vec{R}_m) \sum_{\Lambda\in\mathcal{L}^{\{N\}}[0,m]} \prod_{ij\in\Lambda}(-U_{ij}) \prod_{st\in\mathcal{C}(\Lambda)}[1+U_{st}], \tag{5.4}$$

we see that for each N, $\pi_m^{\{N\}}(x)$ is non-negative. Note that for $N > m$ the sum over laces is an empty sum, which is defined to be 0.

It is shown in [17, Proposition 5.1] that

$$\sum_x |x|^{2q}\pi_m^{\{N\}}(x) \le \frac{C\epsilon_L^N}{m^{\frac{d-4}{2}-q}}, \qquad \text{for } q=0,1,2, \tag{5.5}$$

where ϵ_L can be taken arbitrarily small by taking L large. The bounds (2.37) and (2.38) appearing in Proposition 2.3 are immediate from (5.5) by summing over N in the cases $q=0$ and $q=1$ respectively. Let us briefly describe how the bound (5.5) is obtained.

Note that in the case $N=1$ there is a unique lace $\Lambda = \{0m\}$ with one edge (and all other edges are compatible with this lace) so that

$$\begin{aligned}
\pi_m^{\{1\}}(x) &= \sum_{\omega:o\xrightarrow{m}x} W(\omega) \sum_{\vec{R}_m\ni\vec{\omega}_m} W(\vec{R}_m)(-U_{0m}) \prod_{st\neq 0m}[1+U_{st}] && (5.6)\\
&= \sum_{\omega:o\xrightarrow{m}x} W(\omega) \sum_{\vec{R}_m\ni\vec{\omega}_m} W(\vec{R}_m)\mathbb{1}_{\{R_0\cap R_m\neq\varnothing\}} \prod_{st\neq 0m}[1+U_{st}]. && (5.7)
\end{aligned}$$

If $m=1$ then the sum over ω contains at most one one-step walk from o to x with weight $z_D D(x)$ and the product over $st\neq 0m$ is empty. Letting u denote a point of intersection of R_0 and R_m we have that

$$\begin{aligned}
\pi_1^{\{1\}}(x) \le& z_D D(x) \sum_{R_0\ni o}\sum_{R_1\ni x} W(R_0)W(R_1)\mathbb{1}_{\{R_0\cap R_1\neq\varnothing\}} && (5.8)\\
\le& z_D D(x) \sum_u \sum_{R_0\ni o,u} W(R_0) \sum_{R_1\ni x,u} W(R_1) && (5.9)\\
=& z_D D(x)\rho^2 \sum_u \rho(u)\rho(x-u) && (5.10)\\
\le& C\rho_1(x)\sum_u \rho(u)\rho(x-u) = C\rho_1(x)(\rho*\rho)(x), && (5.11)
\end{aligned}$$

where $(f*g)$ denotes the convolution of f and g, i.e. $(f*g)(x) \equiv \sum_u f(u)g(x-u)$ denotes the convolution of f and g. For $m>1$ we have

$$\begin{aligned}
\pi_m^{\{1\}}(x) \le& \rho^2 z_D^2 \sum_{v,w} D(v)\rho_{m-2}(w-v)D(x-w) \sum_u \rho(u)\rho(x-u) && (5.12)\\
\le& C\rho_m(x)(\rho*\rho)(x). && (5.13)
\end{aligned}$$

Therefore

$$\sum_x \pi_m^{\{1\}}(x) \le C \sum_x \rho_m(x)(\rho * \rho)(x) = (\rho_m * \rho * \rho)(o).$$

The proof of (5.5) given in [17, Proposition 5.1] proceeds by induction on N, relying on the fact that for each Λ, the product $\prod_{st\in\Lambda}(-U_{st})$ enforces intersections of lattice trees R_s and R_t (say at some point u) for each $st \in \Lambda$. This gives rise to a bound on the contribution to $\pi^{(N)}$ from a specific lace Λ in terms of "diagrams" involving various convolutions of two-point functions ρ and ρ_{ℓ_i} (in fact, in [17] the bounds involve slightly better estimates involving certain modified 2-point functions $h_\ell(x) \le \rho_\ell(x)$, but the arguments therein can be carried through with ρ_ℓ instead of h_ℓ, and we will therefore not concern ourselves with this distinction hereafter). The basic estimate that drives the induction argument is [17, Lemma 5.4]. To state a version of this result relevant to the present paper, for fixed $q \in \{0,1\}$ we define $g_{m,q}(x) = |x|^{2q}\rho_m(x)$ and let

$$g^{[j]}_{\vec{m},\vec{q}}(x) = \big(g_{m_1,q_1} * g_{m_2,q_2} * \cdots * g_{m_j,q_j}\big)(x)$$

denote the j-fold convolution of the g_{m_i,q_i}. We let $\rho^{[j]} = \rho * \cdots * \rho$ denote the j-fold convolution of ρ with itself. Then according to [17, Lemma 5.4] the following bounds hold for the convolution of $g^{[j]}_{\vec{m},\vec{q}}$ and $\rho^{[j']}$ for $j' \le 4$ and $m \ge 1$

$$\sup_x (g^{[j]}_{\vec{m},\vec{q}} * \rho^{[j']})(x) \le C m^{\sum_{i=1}^j q_i} \frac{\epsilon'_L}{m^{\frac{d-2j'}{2}}}, \quad \text{and} \quad \sum_x (g^{[j]}_{\vec{m},\vec{q}} * \rho^{[j']})(x) \le C m^{\sum_{i=1}^j q_i}, \tag{5.14}$$

where ϵ'_L can be taken arbitrarily small by taking L large. Note that to prove (2.37) via induction on N one only needs (5.14) with each $q_i = 0$ (and for $j \le 2$ and $j' \le 3$).

In the remainder of the paper we will describe the minor modifications required to adapt these arguments to prove (2.39) - (2.41). Recalling (2.36) we have that

$$\pi_{m;(\vec{j},\vec{x})}(x) = \sum_{N=1}^{\infty} (-1)^N \pi^{\{N\}}_{m;(\vec{j},\vec{x})}(x) \tag{5.15}$$

where

$$0 \le \pi^{\{N\}}_{m;(\vec{j},\vec{x})}(x) \equiv \sum_{\omega:o \overset{m}{\to} x} \left(\prod_{b=1}^{r} \mathbb{1}_{\{\omega(j_b)=x_b\}} \right) W(\omega) \sum_{\vec{R}_M \ni \vec{\omega}_m} \prod_{i=0}^{m} W(R_i) \tag{5.16}$$

$$\sum_{\Lambda \in \mathcal{L}^{\{N\}}[0,m]} \prod_{ij\in\Lambda} (-U_{ij}) \prod_{st\in\mathcal{C}(\Lambda)} [1 + U_{st}]. \tag{5.17}$$

Summing the indicators in $\pi^{\{N\}}_{m;(\vec{j},\vec{x})}(x)$ over $\vec{x}$ gives 1, so

$$\sum_x \sum_{\vec{x}} |\pi_{m;(\vec{j},\vec{x})}(x)| \le \sum_N \sum_x \sum_{\vec{x}} \pi^{\{N\}}_{m;(\vec{j},\vec{x})}(x) = \sum_N \sum_x \pi^{\{N\}}_m(x). \tag{5.18}$$

In other words we obtain the same bound in (2.39) as in (2.37).

Let us now describe how to obtain the bound (2.40) (and (2.41)), which we state as

$$\sum_{\vec{x}} |x_1|^2 \sum_y |\pi_{m;(\vec{j},\vec{x})}(y)| \le \frac{C}{(1+m)^{\frac{d-6}{2}}}. \tag{5.19}$$

Firstly note that

$$\sum_{\vec{x}} |x_1|^2 \sum_y |\pi_{m;(\vec{j},\vec{x})}(y)| \le \sum_{N=1}^{\infty} \sum_{\vec{x}} |x_1|^2 \sum_y \pi^{\{N\}}_{m;(\vec{j},\vec{x})}(y). \tag{5.20}$$

The sums over x_i for $i > 1$ of the indicators can be performed giving 1, meaning that

$$\sum_{N=1}^{\infty}\sum_{\vec{x}}|x_1|^2\sum_{y}\pi^{\{N\}}_{m;(\vec{j},\vec{x})}(y) = \sum_{N=1}^{\infty}\sum_{\vec{x}}|x_1|^2\sum_{y}\pi^{\{N\}}_{m;(j,x_1)}(y). \tag{5.21}$$

Let $0 = m_0 < m_1 \le m_2 \cdots < m_{2N-1} = m$ be the vertices corresponding to endpoints of edges in some lace Λ on $[0, m]$, and let $\ell_i = m_i - m_{i-1}$ for each i. Note that some (as many as $N-1$) of the ℓ_i can be 0, if e.g. there exists some s' such that $\{ss', s't\} \subset \Lambda$. The sum over laces is in effect a sum over $\vec{m} = (m_1, \ldots, m_{2N-2})$ with some restrictions. In [17, Proposition 5.1] the proof of (5.5) when $q = 1$ uses the fact that in the quantity $\pi^{(N)}_m(x)$, $x = \omega(m)$ and $o = \omega(0)$. Thus for fixed Λ (and hence $\vec{m}$) we can write

$$|x|^2 = |\sum_{i=1}^{2N-1}\big(\omega(m_i) - \omega(m_{i-1})\big)|^2 \le (2N-1)\sum_{i=1}^{2N-1}|\omega(m_i) - \omega(m_{i-1})|^2. \tag{5.22}$$

This allows us to distribute the factor $|x|^2$ over individual parts (induced by the lace) of the backbone, incurring a factor $(2N-1)$ and a sum over which part i of the $2N-1$ parts of the backbone receives the square term. For fixed i the diagrams remain the same as for the $q = 0$ case except that the quantity $\rho_{\ell_i}(v_i)$ is replaced with $g_{\ell_i,1}(v_i) = |v_i|^2\rho_{\ell_i}(v_i)$. This means that in inductively bounding a given diagram one uses (5.14) as before except that now we also need to use this bound when exactly one q_i is equal to 1. This gives an extra factor ℓ_i (which is at most m) for each i in (5.14), and thus the extra factor m appearing in (5.5) when $q = 1$, multiplying by $2N-1$ and summing over $i \le 2N-1$ affects only constants since these sums are dominated by the exponentially small term ϵ_L^N.

To prove (2.40) let us fix a lace Λ containing N edges and consider the case where $m_{r-1} < j_1 < m_r$, where m_{r-1} and m_r are both endpoints of edges in Λ while no s between them is. As in (5.22) we can write

$$|x_1|^2 = |x_1 - \omega(m_{r-1}) + \sum_{i=1}^{r-1}\big(\omega(m_i) - \omega(m_{i-1})\big)|^2 \tag{5.23}$$

$$\le r\left[|x_1 - \omega(m_{r-1})|^2 + \sum_{i=1}^{r-1}|\omega(m_i) - \omega(m_{i-1})|^2\right]. \tag{5.24}$$

Notice that on the right hand side only the term $|x_1 - \omega(m_{r-1})|^2$ involves x_1. For the contribution from each of the remaining terms above, the indicator $\mathbb{1}_{\{\omega(j_1)=x_1\}}$ can be directly summed over x_1, giving 1, and the same diagrammating bounds can then be used as above (that is there are no new diagrams to consider in these cases). Since x_1 appears in the term $|x_1 - m_{r-1}|^2$ we cannot simply sum the indicator to get 1 in this case. Instead we can observe that e.g. $K[m_{r-1}+1, m_r - 1] \le K[m_{r-1}+1, j_1]K[j_1, m_r - 1]$, and use this to replace the quantity $\rho_{m_r - m_{r-1}}$ that we previously had in the diagram corresponding to this lace with the quantity $g_{j_1 - m_{r-1},1} * \rho_{m_r - j_1}$. The basic bounds (5.14) do not change based on the number of convolutions of such terms, but depend only on the number of q_i that are 1. In this case we have 1, which is exactly the same as what we got from (5.22) in the previous paragraph in order to obtain (5.5) when $q = 1$, and hence (2.38). Thus exactly the same induction argument applies here, giving us (2.40). [Note that in the case where $j_1 = m_r$ for some r, x_1 already has some other spatial label $v_r = \omega(m_r)$ that is being explicitly summed over in the diagram, so all terms involving x_1 can be replaced with v_r and the sum over x_1 performed to give 1. The resulting diagrams then just have a $g_{\ell_i,1}$ term instead of a ρ_{ℓ_i} term as in the previous paragraph.]

The bound (2.41) is obtained in the same way, but using the fact that

$$|x - x_1|^2 = \Big|\omega(m_r) - x_1 + \sum_{i=r+1}^{2N}\big(\omega(m_i) - \omega(m_{i-1})\big)\Big|^2 \tag{5.25}$$

$$\le (2N - r)\left[|\omega(m_r) - x_1|^2 + \sum_{i=r+1}^{2N}|\omega(m_i) - \omega(m_{i-1})|^2\right]. \tag{5.26}$$

■

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