

A monotonicity property for random walk in a partially random environment

Mark Holmes* and Rongfeng Sun†

December 25, 2011

Abstract

We prove a law of large numbers for random walks in certain kinds of i.i.d. random environments in \mathbb{Z}^d that is an extension of a result of Bolthausen, Sznitman and Zeitouni [4]. We use this result, along with the lace expansion for self-interacting random walks, to prove a monotonicity result for the first coordinate of the speed of the random walk under some strong assumptions on the distribution of the environment.

AMS 2010 subject classification: 60K37, 82C41.

Keywords: random walk in random environment, monotonicity of speed, lace expansion.

1 Introduction

Since the pioneering work of Solomon and others in the mid 1970s to early 1980s, random walk in a random environment (RWRE) has enjoyed a revival in recent times as a number of interesting results have been obtained. Many of these results relate to laws of large numbers and invariance principles for i.i.d. random environments that are uniformly elliptic (all nearest-neighbour transition probabilities are bounded away from zero). While the behavior of one-dimensional RWRE is quite well understood, understanding is much less complete for RWRE in higher dimensions, and in particular, for non-ballistic RWRE. Several special classes of RWRE are amenable to analysis in general dimensions, such as random walk among random conductances, random walks in balanced random environments or Dirichlet random environments, random walks in random environments which are small perturbations of a deterministic environment, etc (see e.g. [2, 3, 5, 13] and the references therein). Another class is a random walk in a *partially random environment*, introduced by Bolthausen, Sznitman and Zeitouni in [4]. They established laws of large numbers and central limit theorems for a RWRE in dimensions $d = d_0 + d_1$, where $d_0 \geq 1$ is the number of coordinates in which the environment is random, and where the projection of the walk onto the remaining $d_1 \geq 5$ coordinates is a deterministic symmetric random walk.

In this paper we consider monotonicity properties of the speed for random walks in partially random environments (RWpRE) that are similar to those considered in [4]. Such properties have

*Department of Statistics, University of Auckland. E-mail holmes@stat.auckland.ac.nz

†Department of Mathematics, National University of Singapore. E-mail matsr@nus.edu.sg

not been extensively studied in the literature. The few results on the monotonicity of the speed we are aware of include the work of Holmes and Salisbury [12] where monotonicity of the speed (when it exists) is proved for all environments that take only two possible values via a coupling argument, and Sabot [13] and Fribergh [6], where asymptotic expansions for the speed are derived for random walks in random environments that are small perturbations of a simple random walk with drift. There has also been recent progress in the study of the monotonicity of the speed as a function of the bias for a biased random walk on supercritical percolation clusters [7] and Galton-Watson trees [1]. Our main result is a monotonicity result for the first coordinate of the speed, under some special assumptions on the distribution of the partially random environment. For example when at each site either the left or right step in the the first coordinate direction is not available, we prove that the first coordinate of the speed is monotone increasing in the probability that the right step is available. Our proof consists of two steps. We first extend a result of [4] to show that the (non-random) speed exists almost surely for the class of RWpRW under our consideration. We then establish the desired monotonicity by analyzing an expansion formula for the speed derived in [8] using lace expansion techniques, which is valid for all annealed RWRE, but is most useful in the case $d_1 \gg d_0$ when one has good control (in terms of finite random walk Green's functions) over the terms in the expansion.

Let $\mathcal{M}_1(\mathbb{Z}^d)$ be the space of probability kernels on \mathbb{Z}^d , and more generally for $c > 0$ let $\mathcal{M}_c(\mathbb{Z}^d)$ denote the space of kernels on \mathbb{Z}^d with total mass c . Given a family of probability kernels $\omega := (\omega_{x,m}(\cdot))_{x \in \mathbb{Z}^d, m \in \mathbb{N}} \in \mathcal{M}_1(\mathbb{Z}^d)^{\mathbb{Z}^d \times \mathbb{N}}$ which we call a *cookie environment*, the law of a random walk $(X_n)_{n \geq 0}$ in the cookie environment ω starting at $X_0 = x$, denoted by $P_{x,\omega}$, is defined as follows. Under $P_{x,\omega}$, the walk evolves conditional on its history via the transition probabilities $P_{x,\omega}(X_n = X_{n-1} + u | (X_i)_{0 \leq i \leq n-1}) = \omega_{X_{n-1}, \ell_{n-1}(X_{n-1})}(u)$, for all $n \in \mathbb{N}$, where $\ell_n(y) = \sum_{k=0}^n 1_{\{X_k=y\}}$ is the number of visits to y up to time n . In words, upon the m -th visit to x , the walk sees the environment $\omega_{x,m}$ and makes a jump accordingly. We will consider the case when ω is random, and the cookie environment at different points in space, $(\omega_{x,\cdot}(\cdot))_{x \in \mathbb{Z}^d}$, are i.i.d. with a common law $\mu \in \mathcal{M}_1(\mathcal{M}_1(\mathbb{Z}^d)^{\mathbb{N}})$. The measure $P_{x,\omega}$ is called the *quenched* law. When we average the quenched law of $(X_n)_{n \geq 0}$ with respect to the cookie environment ω , we obtain the so-called *annealed* (or more accurately the *averaged*) law

$$P_x := \mathbb{P} \times P_{x,\omega},$$

where $\mathbb{P} := \mu^{\otimes \mathbb{Z}^d}$ denotes the law of ω in the product space $\Omega := (\mathcal{M}_1(\mathbb{Z}^d))^{\mathbb{Z}^d \times \mathbb{N}}$.

The random walk model described above is sometimes called a multi-excited random walk (in a random cookie environment). When we restrict the cookie environment to environments that are constant in m , i.e. $\omega_{x,m}(\cdot) \equiv \omega_x(\cdot)$ for every x and m , we obtain the more often studied RWRE model. Among other assumptions required for our two main results, the result on monotonicity of the speed assumes that $\omega_{x,m}(\cdot) \equiv \omega_x(\cdot)$, while the law of large numbers result does not.

1.1 The law of large numbers

Recall that the RWRE X is said to satisfy a (strong) law of large numbers (LLN), if there exists a constant $v \in \mathbb{R}^d$, such that P_o a.s. $\lim_{n \rightarrow \infty} \frac{X_n}{n} = v$. For ballistic RWRE, the technique of regeneration times has been useful in proving the LLN, see e.g. [15, 14, 16]. For non-ballistic RWRE, it is not known in general if there is a deterministic limit, however Sznitman and Zerner [15] and Zerner [17] showed that X_n/n converges P_o -almost surely to a random variable taking at most

two possible values. In dimension two, the LLN has been established by Zerner and Merkl [18]. In both cases the environment is assumed to be i.i.d. and elliptic (with $\omega_{x,m}(\cdot) \equiv \omega_x(\cdot)$), but it was shown recently by Holmes and Salisbury in [12, Sec. 3] that the assumption of ellipticity can be dropped. In higher dimensions, the LLN has only been established for various special classes which do not require ballisticity. One such special class is a random walk in a partially random environment studied by Bolthausen, Sznitman, and Zeitouni in [4]. They assume that $d = d_0 + d_1$ with $d_0 \geq 1$ and $d_1 \geq 5$, and there exists $\alpha \in (0, 1)$ and $q(\cdot) \in \mathcal{M}_1(\mathbb{Z}^{d_1})$, such that for μ a.e. $\omega_x(\cdot) \in \mathcal{M}_1(\mathbb{Z}^d)$:

- (a) $\omega_x(\cdot)$ is supported on the canonical unit vectors, i.e., $\sum_{e \in \mathbb{Z}^d, |e|=1} \omega_x(e) = 1$,
- (b) if $\Pi_{d_1} : \mathbb{Z}^d \rightarrow \mathbb{Z}^{d_1}$ denotes the projection of $v \in \mathbb{Z}^d$ to its last d_1 coordinates, then

$$(\omega_x \circ \Pi_{d_1}^{-1})(\cdot) = \alpha q(\cdot) + (1 - \alpha)\delta_0(\cdot), \quad (1.1)$$

- (c) $\sum_{e \in \mathbb{Z}^{d_1}, |e|=1} q(e) = 1$, and $q(e) = q(-e) > 0$ for all $e \in \mathbb{Z}^{d_1}$ with $|e| = 1$.

When the RWRE $(X_n)_{n \geq 0}$ on \mathbb{Z}^d is projected to its last d_1 coordinates, one obtains a random walk $(Y_n)_{n \in \mathbb{N}} := (\Pi_{d_1} X_n)_{n \in \mathbb{N}}$ on \mathbb{Z}^{d_1} with transition kernel $\alpha q(\cdot) + (1 - \alpha)\delta_0(\cdot)$. In dimensions $d_1 \geq 5$, such a random walk admits so-called *cut times*, i.e.,

$$\mathcal{D} := \{n \in \mathbb{Z} : Y_{(-\infty, n-1]} \cap Y_{[n, \infty)} = \emptyset\} \neq \emptyset \quad a.s. \quad (1.2)$$

Using the fact that conditional on the projected random walk $(\Pi_{d_1} X_n)_{n \geq 0}$, the environments the walk X sees between successive cut times are independent, a law of large numbers was proved in [4]. The key assumption is thus the existence of cut times, which play the role of regeneration times in this context.

We now extend the aforementioned LLN from [4] to cookie environments $\omega := (\omega_{x,m}(\cdot))_{x \in \mathbb{Z}^d, m \in \mathbb{N}}$ where $(\omega_{x,\cdot}(\cdot))_{x \in \mathbb{Z}^d}$ are i.i.d. with common law μ . Furthermore, we will allow α in (1.1) to be random. More precisely, conditions (a)–(c) will be relaxed to the following:

- (a') There exists $K > 0$ such that $\omega_{x,m}(\cdot)$ is a.s. supported on $[-K, K]^d$ for all $m \in \mathbb{N}$.
- (b') There exists some $\delta > 0$ and $q \in \mathcal{M}_1(\mathbb{Z}^{d_1})$ with $q(0) = 0$ such that, for all $n \in \mathbb{N}$, a.s.

$$(\omega_{x,m} \circ \Pi_{d_1}^{-1})(\cdot) = \alpha_{\omega_{x,m}} q(\cdot) + (1 - \alpha_{\omega_{x,m}})\delta_0(\cdot) \quad \text{for some } \alpha_{\omega_{x,m}} \geq \delta. \quad (1.3)$$

- (c') For a random walk $(Y_n)_{n \in \mathbb{N}}$ on \mathbb{Z}^{d_1} with jump probability kernel $q(\cdot)$, $\mathcal{D} \neq \emptyset$ a.s.

Note that condition (c') holds for any Y with a non-zero drift, and it also holds for any Y that is at least 5-dimensional in the sense that the space spanned by vectors in the support of Y 's increment distribution is at least 5-dimensional (see e.g. [4, Lemma 1.1]). By ergodicity, (c') in fact implies that \mathcal{D} is an infinite set almost surely.

Theorem 1.1. (Law of large numbers) *Let $(X_n)_{n \geq 0}$ be a random walk in a cookie environment ω , where $(\omega_{x,\cdot}(\cdot))_{x \in \mathbb{Z}^d}$ are i.i.d. with common law μ and satisfy conditions (a')–(c') above. Then there exists some $v \in \mathbb{R}^d$ such that*

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = v \quad P_o \text{ almost surely.} \quad (1.4)$$

Although we will only need time-independent random environments when we later study the monotonicity of the speed v , we have formulated Theorem 1.1 for cookie environments because there are natural interesting examples involving cookie environments, such as the (multi)-excited random walk (see e.g. [10]). Apart from extending [4, Theorem 1.4], Theorem 1.1 also extends the weak law of large numbers [8, Theorem 2.5], which incorporated the parameter δ but is only valid in the perturbative regime where there is some $\epsilon > 0$ sufficiently small (depending on δ) such that $\mathbb{P}(|\omega_o - \mathbb{E}[\omega_o]|_1 > \epsilon) = 0$, where $|\cdot|_1$ denotes here the total variation norm on $\mathcal{M}_1(\mathbb{Z}^d)$.

The proof of Theorem 1.1 is based on adaptations of arguments in [4], which we outline in Section 2. Relaxing conditions (a) and (c) to (a') and (c') does not induce any change in the proof. However relaxing condition (b) to (b') requires a simple but not entirely trivial observation, and the proof needs to be modified accordingly. Indeed we were encouraged to consider this extension by one of the authors of [4].

The lace expansion for self-interacting random walks of van der Hofstad and Holmes [8] gives the following series representation for the expected increment of the RWRE under P_o ,

$$E_o[X_n - X_{n-1}] = E_o[X_1] + \sum_{m=2}^n \sum_x x \pi_m(x), \quad (1.5)$$

where $\pi_m(x)$, for $m \geq 2, x \in \mathbb{Z}^d$ are somewhat complicated quantities known as lace-expansion coefficients. If (1.5) converges then $E_o[X_n/n]$ converges to the same limit (convergence of $E_o[X_n/n]$ also follows from dominated convergence and the fact that X_n/n converges almost surely). Theorem 1.1 allows us to go one step further and say that if (1.5) converges then it converges to v , i.e.

$$v = E_o[X_1] + \lim_{n \rightarrow \infty} \sum_{m=2}^n \sum_x x \pi_m(x). \quad (1.6)$$

Usually in analysing this formula we first require enough control on the coefficients $\pi_m(x)$ to ensure that this series converges. To prove other properties of the velocity such as the sign, continuity, differentiability, and monotonicity, we typically require even better control of the coefficients $\pi_m(x)$. The expansion is not perturbative in the usual sense. For any translation invariant self-interacting random walk (see [8] for precise details) for which $E_o[X_n - X_{n-1}]$ converges, the formula (1.5) is valid. Currently in order to extract useful information from the formula we require the walk to be sufficiently and quantifiably transient, independent of the history of the walk. Analysis of the formula without such an assumption would require a major advance in our understanding of the expansion methodology itself. In this paper the walks that we consider have a high-dimensional simple random walk component (see e.g. assumption **(A3)** in Section 1.2) which has the necessary transience property.

We will study (1.6) as a function of a particular parameter of interest, β , under some strong assumptions on the distribution of the environment, and show that the velocity $v^{[1]}$ (the first coordinate of the velocity) is increasing in β . A precise formulation of this result is given in Theorem 1.2 at the end of Section 1.2.

1.2 Monotonicity

Our discussion and results concerning monotonicity are restricted to time-independent random environments, i.e., $\omega_{x,m}(\cdot) \equiv \omega_x(\cdot)$ for all $m \in \mathbb{N}$ and $x \in \mathbb{Z}^d$. For a discussion of monotonicity in

cookie environments with $d_0 = 1$ and condition (b) instead of (b'), see for example [10]. Rather a lot is known when $d = 1$, see Holmes and Salisbury [11].

It is well known that the velocity of a RWRE is not monotone increasing in the expected local drift at the origin. In fact it is possible in one dimension that the expected local drift is negative, yet the walk is transient to the right. In higher dimensions (see e.g. [4],[12]) the speed and the expected local drift can even carry opposite signs. For example, consider a nearest-neighbour RWRE on \mathbb{Z}^2 with i.i.d. environment $\omega := (\omega_x)_{x \in \mathbb{Z}}$, where $\mathbb{P}(\omega_o(e_1) = \omega_o(e_2) = \frac{1}{2}) = p$ and $\mathbb{P}(\omega_o(-e_1) = 1) = 1 - p$. Due to an elementary renewal structure, the velocity of this random walk can be calculated explicitly [12] as

$$v = \left(\frac{p(2-p)}{2+3p-2p^2-p^3} \right) \cdot (3, 1) - (1, 0).$$

It is easy to see that the first coordinate $v^{[1]}$ need not carry the same sign as the expected local drift $\frac{3p}{2} - 1$. However in this example the velocity is indeed monotone in p . Holmes and Salisbury [12] prove that this is the case for any 2-valued environment. To be precise, if $\mathbb{P}(\omega_o = A_1) = p = 1 - \mathbb{P}(\omega_o = A_2)$ then the velocity $v(p)$ if it exists is monotone in p . This fails in general [12] for 3-valued environments with respective probabilities $p(1-q)$, $(1-p)(1-q)$ and q for fixed q .

Now consider two i.i.d. environments ω and $\tilde{\omega}$ satisfying conditions (a)–(c) in Sec. 1.1 with the same d_0, d_1, α and $q(\cdot)$. Assume furthermore that for some $\kappa \in (0, 1)$,

$$\omega_o(e_1) + \omega_o(-e_1) = \tilde{\omega}_o(e_1) + \tilde{\omega}_o(-e_1) = \kappa \quad \mathbb{P} \text{ a.s.}, \quad (1.7)$$

and $\tilde{\omega}_o(e_1)$ stochastically dominates $\omega_o(e_1)$ in the sense that

$$\mathbb{P}(\omega_o(e_1) \leq s) \geq \mathbb{P}(\tilde{\omega}_o(e_1) \leq s) \quad \text{for all } s \in [0, 1]. \quad (1.8)$$

Suppose that a random walk X (resp. \tilde{X}) with $X_0 = \tilde{X}_0 = 0$ in the random environment ω (resp. $\tilde{\omega}$) satisfies a LLN with (deterministic) speed v (resp. \tilde{v}), is it true that $v^{[1]} \leq \tilde{v}^{[1]}$, where $w^{[1]}$ denotes the first coordinate of $w \in \mathbb{R}^d$?

When $d_0 = 1$ and $d_1 \geq 0$, the answer to the above question is affirmative, since we can easily couple (X, ω) with $(\tilde{X}, \tilde{\omega})$ such that at each time n , $\tilde{X}_n^{[i]} = X_n^{[i]}$ for $i > 1$, and $\tilde{X}_n^{[1]} \geq X_n^{[1]}$. That is, the position of the walks $\{X_n\}_{n \geq 0}$ and $\{\tilde{X}_n\}_{n \geq 0}$ differ only in the first coordinate and \tilde{X} is never to the left of X .

In general, however, we expect the answer to the above question to be negative, since the limiting velocity v depends on the joint distribution of $(\omega_o(\pm e_i))_{1 \leq i \leq d_0}$. Consider for example the case $d_0 = d = 2$ with $\kappa = \frac{1}{2}$, such that

$$\mathbb{P}\left((\omega_o(e_i), \omega_o(-e_i)) = (1/2, 0)\right) = \beta = 1 - \mathbb{P}\left((\omega_o(e_i), \omega_o(-e_i)) = (0, 1/2)\right), \quad i = 1, 2, \quad (1.9)$$

for some $\beta \in (0, 1)$, and $(\omega_o(e_2), \omega_o(-e_2))$ is independent of $(\omega_o(e_1), \omega_o(-e_1))$. The corresponding random walk has limiting velocity $v = 0$ for all $\beta \in (0, 1)$, since it eventually gets stuck on a finite number of sites (e.g. see [12]). On the other hand one expects that the random walk in the random environment $\tilde{\omega}$ (again with $d_0 = d = 2$, $\kappa = \frac{1}{2}$) such that

$$\mathbb{P}\left(\tilde{\omega}_o(e_2) = \tilde{\omega}_o(e_1) = 1/2\right) = \beta = 1 - \mathbb{P}\left(\tilde{\omega}_o(-e_2) = \tilde{\omega}_o(-e_1) = 1/2\right) \quad (1.10)$$

has a non-trivial deterministic velocity whenever $\beta \neq 1/2$ (see e.g. [12]).

As a special case of our main result (Theorem 1.2) below, we will show that for each of the two examples in (1.9) and (1.10), if $d_1 \gg 1$ and the $1/2$ in (1.9) and (1.10) is replaced by a sufficiently small constant, then $v^{[1]}$ is continuous and strictly increasing in β . We believe that one can tune parameters in these two examples such that $\tilde{\omega}_o(e_1)$ stochastically dominates $\omega_o(e_1)$ as in (1.8), and yet $v^{[1]} > \tilde{v}^{[1]}$.

For ω and $\tilde{\omega}$ formulated as in (1.7) and (1.8), if we further assume that $\{\omega_o(\pm e_1)\}$ is independent of $\{\omega_o(\pm e_i) : 2 \leq i \leq d\}$, the same holds for $\tilde{\omega}$, and $\{\omega_o(\pm e_i) : 2 \leq i \leq d\}$ is equally distributed with $\{\tilde{\omega}_o(\pm e_i) : 2 \leq i \leq d\}$, then it is natural to expect the monotonicity property $v^{[1]} \leq \tilde{v}^{[1]}$ to hold. We will prove this in high dimensions in some special cases.

We now formulate precisely the class of RWRE for which we can prove monotonicity of $v^{[1]}$. Let $d = d_0 + d_1$, $1 \leq d_* \leq d_0$, and let $\gamma, \kappa, \delta \in (0, 1]$ be such that $\gamma + \delta \leq 1$. Our assumptions on the random environment ω consist of the following.

(A0): $\omega := (\omega_x)_{x \in \mathbb{Z}^d}$ are i.i.d. with common law $\mu \in \mathcal{M}_1(\mathcal{M}_1(\mathbb{Z}^d))$, and μ -a.s., ω_o is supported on $V_d := \{\pm e_i\}_{1 \leq i \leq d}$ and satisfies conditions (b')–(c') in Sec. 1.1 for some $q(\cdot) \in \mathcal{M}_1(\mathbb{Z}^{d_1})$.

(A1): ω_o as an $\mathcal{M}_1(\mathbb{Z}^d)$ -valued random variable furthermore admits the representation

$$\omega_o = \xi_{d_*} \times \delta_{d-d_*}(o) + \delta_{d_*}(o) \times \tilde{\xi}_{d-d_*}, \quad (1.11)$$

where \times denotes product measure, $\delta_i(o) \in \mathcal{M}_1(\mathbb{Z}^i)$ denotes the delta measure at the origin o , and ξ_{d_*} and $\tilde{\xi}_{d-d_*}$ are *independent* random kernels supported on V_{d_*} (resp. V_{d-d_*}) with total mass γ (resp. $1 - \gamma$) and laws $\mu_{d_*} \in \mathcal{M}_1(\mathcal{M}_\gamma(\mathbb{Z}^{d_*}))$ (resp. $\tilde{\mu}_{d-d_*} \in \mathcal{M}_1(\mathcal{M}_{1-\gamma}(\mathbb{Z}^{d-d_*}))$).

(A2): There exist $\nu_1, \nu_2 \in \mathcal{M}_\gamma(\mathbb{Z}^{d_*})$ with *disjoint supports* $\mathcal{S}_1, \mathcal{S}_2 \subset V_{d_*}$ such that

$$\begin{aligned} \mu_{d_*}(\xi_{d_*} = \nu_1) &= \kappa(1 - \beta), & \mu_{d_*}(\xi_{d_*} = \nu_2) &= \kappa\beta, \\ \mu_{d_*}(\text{supp}(\xi_{d_*}) \cap (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\pm e_1\})) &= \emptyset & &= 1 - \kappa, \end{aligned} \quad (1.12)$$

and

$$\rho := (\nu_2(e_1) - \nu_2(-e_1)) - (\nu_1(e_1) - \nu_1(-e_1)) > 0. \quad (1.13)$$

For simplicity, we will also use \mathcal{S}_i to denote $\mathcal{S}_i \times 0_{d-d_*} \subset \mathbb{Z}^d$, where 0_k is the zero vector in \mathbb{Z}^k .

(A3): Let $q(\cdot) \in \mathcal{M}_1(\mathbb{Z}^{d_1})$ be as in (b'), and let $G_q(x) := \sum_{k=0}^{\infty} q^{*k}(x)$, where $q^{*k}(\cdot)$ denotes the k -fold convolution of $q(\cdot)$ with itself. Then

$$G_q(o) < 2 \quad \text{and} \quad G_q^{*i} := \sup_{x \in \mathbb{Z}^{d_1}} G_q^{*i}(x) < \infty \text{ for } i = 1, 2, 3, 4. \quad (1.14)$$

Let us elaborate more on the assumptions. Assumption **(A1)** requires that μ -a.s., a walk with transition kernel ω_o will make a nearest-neighbor jump in the first d_* coordinate directions with probability γ , and make a nearest-neighbor jump in the last $d - d_*$ coordinate directions with probability $1 - \gamma$. Furthermore, the restriction of ω_o to unit vectors in the first d_* coordinate directions is independent of its restriction to unit vectors in the last $d - d_*$ coordinate directions.

The parameter β in **(A2)** allows us to tune the relative weight of the kernels ν_1 and ν_2 . Since $\rho > 0$, ν_1 and ν_2 have different drifts in the first coordinate direction. We can therefore expect

$v^{[1]}$ to increase monotonically as we increase the weight of ν_2 at the expense of ν_1 . However, our proof requires ν_1 and ν_2 to have disjoint supports. Furthermore, when ξ_{d_*} , the restriction of ω_x to V_{d_*} , is neither ν_1 nor ν_2 , then its support must be disjoint from the support of ν_1 and ν_2 , and it cannot allow jumps that change the first coordinate of the random walk (which is trivially satisfied if $\kappa = 1$). Under this assumption, the history of the walk up to time n either provides no information about ξ_{d_*} at $x \in \mathbb{Z}^d$ because no jumps affecting the first d_* coordinates has been taken from x , or we can determine whether $\xi_{d_*} = \nu_1$, or ν_2 , or neither, based on past jumps from x affecting the first d_* coordinates. This fact will be crucial for our proofs.

Assumption **(A0)**, and in particular (b'), guarantees that for μ almost every realization of ω_o , with probability at least δ , a jump following ω_o will induce a change in (and only in) the last d_1 coordinates, and conditional on this event, the jump follows the kernel q . This allows us to extract a deterministic random walk and apply lace expansion techniques.

Assumption **(A3)** is needed to control the lace expansion coefficients. It is true for example when d_1 is sufficiently large. As an alternative to the assumption $G^{*4} < \infty$ in **(A3)**, we can instead assume that an appropriate local central limit theorem type bound, (4.7), holds for the random walk with kernel q . See Section 4.1 for further details. We note that when $q(\cdot)$ has zero mean, $G^{*4} < \infty$ if and only if $d_1 > 8$, while the local limit theorem bound (4.7) holds as long as $d_1 > 6$. We expect that the methods of this paper could be adapted to handle cases where d_1 is small for *asymmetric* q , provided that the bias of $q(\cdot)$ is sufficiently strong. This analysis would require different estimates, similar to those used in the analysis of once-reinforced random walk with drift in [8].

We are now ready to state the second main result of this paper, which holds under the further assumption that δ in (1.3) is sufficiently close to 1.

Theorem 1.2. (Monotonicity of speed) *Let X be a random walk in an i.i.d. random environment ω which satisfies **(A0)**–**(A4)** with kernel $q(\cdot)$ and constants $\gamma, \kappa, \delta \in (0, 1]$, $\gamma + \delta \leq 1$. There exists $\delta_q \in (0, 1)$ depending only on $q(\cdot)$, such that if $\delta \in (\delta_q, 1)$, then $v^{[1]}$ is continuous and strictly increasing in β .*

Note that under assumptions **(A0)**–**(A4)**, the existence of a deterministic velocity v is guaranteed by Theorem 1.1.

The simplest random environment for which Theorem 1.2 applies is when: (1) $d_* = d_0$, which implies that in (1.11), $\tilde{\xi}_{d-d_*} = \tilde{\xi}_{d_1} = q(\cdot)$ a.s., and hence that the probability of taking a q -step (a step with non-zero Π_{d_1} projection) is the constant $\delta = 1 - \gamma$; (2) $\kappa = 1$ so that ξ_{d_*} in (1.11) equals either ν_1 or ν_2 . Such a random environment allows only two possible realizations for each ω_x , and β determines the probabilities of their occurrence, so we know from [12] that the velocity is monotone in β . The more general random environments formulated in Theorem 1.2 can be regarded as perturbations of this simple case by allowing more randomness: $\kappa < 1$ allows ξ_{d_*} to take on realizations other than ν_1, ν_2 , provided assumption **(A2)** holds; $d_* < d_0$ allows $\tilde{\xi}_{d-d_*}$ to be random, provided condition (b') holds. We note that suitable assumptions on the additional randomness is necessary for the monotonicity result to hold, as it was shown in [12] that monotonicity does not hold in general for random environments which almost surely takes on one of three possible realizations.

Lastly we remark that in related works of Sabot [13] and Fribergh [6], the authors study the speed of a RWRE which is a perturbation of a homogeneous simple random walk with drift. This

is similar in spirit to our model since both are perturbations of a simple random walk. However the exact nature of perturbations, the results, and the techniques are all quite different. In [13] and [6], the authors rely on the representation of the speed of a ballistic random walk in terms of Kalikow's auxiliary random walk and perform perturbation expansion for its Green function. This results in an expansion of the speed as a function of the perturbation parameter, which implies monotonicity of the speed when the perturbation parameter is zero. Our results employ lace expansion techniques and require stronger assumptions, but are valid as long as the deterministic component of the random walk is transient enough (not necessarily ballistic), as characterized by the condition on its Green function in **(A3)**. The monotonicity we obtain is valid on the whole range of admissible parameters β .

1.3 Organisation

The remainder of the paper is organised as follows. In Section 2 we prove Theorem 1.1. In Section 3 we review the relevant notation and results on the lace expansion for self-interacting random walks from [8], including the formula for the speed. The basic ingredients of the formula are annealed transition probabilities, and these are examined in Section 3.1. Sections 3.2 and 3.3 are devoted to proving bounds on some of the quantities appearing in the speed formula and its derivative. Finally in Section 4 we prove Theorem 1.2 as a consequence of the given formula for the speed and its derivative.

2 Proof of Theorem 1.1

The proof is based on adaptations of the arguments in the proof of [4, Theorem 1.4]. The first step is to give a suitable representation of the random cookie environment ω in terms of independent environments using cut times. Then the standard LLN for i.i.d. random variables can be applied.

Let $(Z_i)_{i \in \mathbb{Z}}$ be i.i.d. \mathbb{Z}^{d_1} -valued random variables with common distribution $q(\cdot)$. Then we can construct a doubly infinite random walk path $(Y_n)_{n \in \mathbb{Z}}$ in \mathbb{Z}^{d_1} with $Y_0 = 0$, $Y_n = \sum_{i=1}^n Z_i$ for $n \geq 1$, and $Y_n = -\sum_{i=n+1}^0 Z_i$ for $n \leq -1$. The set of *cut times* of $(Y_n)_{n \in \mathbb{Z}}$, or $(Z_n)_{n \in \mathbb{Z}}$, defined in (1.2), is almost surely non-empty by assumption (c'). Note that $(Z_i, 1_{\{i \in \mathcal{D}\}})_{i \in \mathbb{Z}}$ is an ergodic sequence with respect to the time shifts

$$\theta_k(Z_i, 1_{\{i \in \mathcal{D}\}}) = (Z_{i+k}, 1_{\{i+k \in \mathcal{D}\}}), \quad k \in \mathbb{Z}. \quad (2.1)$$

Therefore almost surely, $\sup \mathcal{D} = \infty$ and $\inf \mathcal{D} = -\infty$. We will denote $\mathcal{D} \cap [1, \infty) := \{T_1 < T_2 < \dots\}$ and $\mathcal{D} \cap (-\infty, 0] := \{\dots < T_{-1} < T_0\}$.

We can couple the random walk X in the random cookie environment with Y as follows. Given $(X_n)_{n \geq 0}$, let $\tau_0 = 0$, and inductively, define

$$\tau_j := \inf\{n > \tau_{j-1} : \Pi_{d_1}(X_n - X_{n-1}) \neq 0\}, \quad j \in \mathbb{N}. \quad (2.2)$$

By condition (b') on the random cookie environment ω , $(X_0, X_{\tau_1}, X_{\tau_2}, \dots)$ is distributed exactly as $(Y_n)_{n \geq 0}$. So without loss of generality, we will assume that $\Pi_{d_1}(X_{\tau_i} - X_{\tau_{i-1}}) = Z_i$ for $i \in \mathbb{N}$. Note that when condition (b) holds, $(\Pi_{d_1} X_n)_{n \geq 0}$ is distributed as a random walk \tilde{Y} on \mathbb{Z}^{d_1} with increment distribution $\alpha q(\cdot) + (1 - \alpha)\delta_0(\cdot)$. We can then just couple X with \tilde{Y} and there will be no need to introduce the stopping times τ_i . This was the approach taken in [4].

By the definition of cut times of $(Z_n)_{n \in \mathbb{Z}}$ and the assumption that $(\omega_{x,m})_{m \in \mathbb{N}}$ is i.i.d. for $x \in \mathbb{Z}^d$, we observe that conditional on $(\Pi_{d_1}(X_{\tau_i} - X_{\tau_{i-1}}))_{i \geq 0} = (Z_i)_{i \geq 0}$, the random walk sees spatially distinct, and hence independent, cookie environments on the time intervals $[0, \tau_{T_1} - 1]$, and $[\tau_{T_i}, \tau_{T_{i+1}} - 1]$ for $i \in \mathbb{N}$. We thus have the following construction of the annealed law of X .

Let $(Z_i)_{i \in \mathbb{Z}}$, $(Y_i)_{i \in \mathbb{Z}}$, and $\mathcal{D} = (T_i)_{i \in \mathbb{Z}}$ be as above. Let $\omega^{(i)}$, $i \in \mathbb{Z}$, be i.i.d. Ω -valued random variables equally distributed with ω , which will be the cookie environment the walk X sees on the time interval $[\tau_{T_{i-1}}, \tau_{T_i} - 1]$. For each $i \in \mathbb{Z}$, we construct a random walk $(X_n^{(i)})_{n \geq 0}$ in the cookie environment $\omega^{(i)}$ inductively as follows. Set $X_0^{(i)} = 0$. Let

$$N(X_{[0,n]}^{(i)}) := |\{1 \leq j \leq n : \Pi_{d_1}(X_j^{(i)} - X_{j-1}^{(i)}) \neq 0\}| \quad (2.3)$$

be the number of steps $X^{(i)}$ has taken with non-zero increments in the last d_1 coordinates up to time n . For each $v \in \mathbb{Z}^d$, if $\Pi_{d_1} v = 0$, then we set $X_{n+1}^{(i)} = X_n^{(i)} + v$ with probability $\omega_{X_n^{(i)}, \ell_n(X_n^{(i)})}^{(i)}(v)$ so that the walk's jump is unperturbed if it does not affect the last d_1 coordinates. On the event that the walk's jump does affect the last d_1 coordinates, we will change the law of the jump so that its Π_{d_1} projection agrees with the next increment of Y . More precisely, for v with $\Pi_{d_1} v = Z_{T_{i-1} + N(X_{[0,n]}^{(i)}) + 1}$, we assign v as the next jump with probability $\omega_{X_n^{(i)}, \ell_n(X_n^{(i)})}^{(i)}(v) / q(\Pi_{d_1} v)$, which by (b') is equal to

$$\sum_{u: \Pi_{d_1}(u) \neq 0} \omega_{X_n^{(i)}, \ell_n(X_n^{(i)})}^{(i)}(u) \times \frac{\omega_{X_n^{(i)}, \ell_n(X_n^{(i)})}^{(i)}(v)}{\sum_{u: \Pi_{d_1}(u) = \Pi_{d_1}(v)} \omega_{X_n^{(i)}, \ell_n(X_n^{(i)})}^{(i)}(u)},$$

i.e. the product of the probability that there is a jump affecting the last d_1 -coordinates and the probability that the jump equals v conditional on $\Pi_{d_1} v = Z_{T_{i-1} + N(X_{[0,n]}^{(i)}) + 1}$. All other jumps are assigned probability 0. This then couples $X^{(i)}$ and Y so that the increments of $X^{(i)}$ in the last d_1 coordinates match exactly with $(Z_{T_{i-1} + n})_{n \geq 1}$.

Since 0 need not be a cut-time for Y , we need a different version of $X^{(1)}$, denoted by $\tilde{X}^{(1)}$, where given $\tilde{X}_n^{(1)}$, we set $\tilde{X}_{n+1}^{(1)} = \tilde{X}_n^{(1)} + v$ with probability $\omega_{\tilde{X}_n^{(1)}, \ell_n(\tilde{X}_n^{(1)})}^{(1)}(v)$ if $\Pi_{d_1} v = 0$, and with probability $\omega_{\tilde{X}_n^{(1)}, \ell_n(\tilde{X}_n^{(1)})}^{(1)}(v) / q(\Pi_{d_1}(v))$ if $\Pi_{d_1} v = Z_{N(\tilde{X}_{[0,n]}^{(1)}) + 1}$, so that the increments of $\tilde{X}^{(1)}$ in the last d_1 coordinates match exactly with $(Z_n)_{n \geq 1}$. Let $(\tau_j^{(i)})_{j \geq 0}$ be defined for $X^{(i)}$ as $(\tau_j)_{j \geq 0}$ is defined for X in (2.2), and let $(\tilde{\tau}_j^{(1)})_{j \geq 0}$ be defined similarly for $\tilde{X}^{(1)}$. Then we can construct $(X_n)_{n \geq 0}$ by piecing together $(\tilde{X}_n^{(1)})_{0 \leq n \leq \tilde{\tau}_{T_1}^{(1)}}$ and $(X_n^{(i)})_{0 \leq n \leq \tau_{T_i - T_{i-1}}^{(i)}}$ for $i \geq 2$. More precisely, if we denote $\sigma_1 = \tilde{\tau}_{T_1}^{(1)}$ and $\sigma_i = \sigma_{i-1} + \tau_{T_i - T_{i-1}}^{(i)}$ for $i \geq 2$, then we set

$$\begin{aligned} X_n &= \tilde{X}_n^{(1)} && \text{for } 0 \leq n \leq \sigma_1, \\ X_n &= X_{\sigma_1} + X_{n - \sigma_1}^{(2)} && \text{for } \sigma_1 \leq n \leq \sigma_2, \\ &\vdots && \vdots \\ X_n &= X_{\sigma_i} + X_{n - \sigma_i}^{(i+1)} && \text{for } \sigma_i \leq n \leq \sigma_{i+1}, \\ &\vdots && \vdots \end{aligned} \quad (2.4)$$

Conditional on $(Z_i)_{i \in \mathbb{Z}}$, $(X_n)_{n \geq 0}$ is thus represented as the concatenation of a sequence of random walks in independent cookie environments. We leave it as an exercise to the reader to verify that (when averaged over the law of $(Z_i)_{i \in \mathbb{Z}}$) X is distributed as a random walk in a random cookie environment ω with law \mathbb{P} .

To prove the law of large numbers, we write $X_{\tau_n} = \sum_{i=1}^n X_{\tau_i} - X_{\tau_{i-1}}$. Note that for each $n \in \mathbb{Z}$, there exists an $i \in \mathbb{Z}$ with $T_{i-1} < n \leq T_i$. We then define

$$\Delta X_n^S := X_{\tau_n}^{(i)} - X_{\tau_{n-T_{i-1}-1}}^{(i)} \quad \text{and} \quad \Delta \tau_n^S := \tau_{n-T_{i-1}}^{(i)} - \tau_{n-T_{i-1}-1}^{(i)},$$

which are the increments in space and time between successive jumps of $X^{(i)}$ in the last d_1 coordinates. Note that for each $i > T_1$, $\Delta X_i^S = X_{\tau_i} - X_{\tau_{i-1}}$ and $\Delta \tau_i^S = \tau_i - \tau_{i-1}$. The key to the proof of the law of large numbers is the observation that $(Z_i, \Delta X_i^S, \Delta \tau_i^S)_{i \in \mathbb{Z}}$ is an ergodic sequence with respect to the joint law of $(Z_i)_{i \in \mathbb{Z}}$ and $(X^{(i)})_{i \in \mathbb{Z}}$. Assuming this, then by the ergodic theorem, almost surely,

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \Delta X_i^S}{n} = \mathbb{E}[\Delta X_1^S] \in \mathbb{R}^d \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \Delta \tau_i^S}{n} = \mathbb{E}[\Delta \tau_1^S] \in [1, \infty),$$

where the ergodic theorem is applicable because $|\Delta X_1^S|_\infty \leq K \Delta \tau_1^S$, and $\Delta \tau_1^S$ is integrable because it is the time that a random walk in the random environment $\omega^{(1)}$ has to wait before making a jump in the last d_1 coordinates, which by condition (b') is stochastically dominated by a geometric random variable with mean δ^{-1} . Therefore, a.s. with respect to the law of X ,

$$\lim_{n \rightarrow \infty} \frac{X_{\tau_n}}{\tau_n} = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \Delta X_i^S}{\sum_{i=1}^n \Delta \tau_i^S} = \frac{\mathbb{E}[\Delta X_1^S]}{\mathbb{E}[\Delta \tau_1^S]} =: v \in \mathbb{R}^d.$$

This implies (1.4) because $\sup_{\tau_{i-1} \leq n < \tau_i} |X_n - X_{\tau_{i-1}}|_\infty \leq K(\tau_i - \tau_{i-1})$ for each $i \in \mathbb{N}$, where $(\tau_i - \tau_{i-1})_{i \in \mathbb{N}}$ are dominated by independent geometric random variables with mean δ^{-1} , and hence $\lim_{n \rightarrow \infty} (\tau_i - \tau_{i-1})/n \rightarrow 0$ almost surely.

It only remains to verify the ergodicity of $(Z_i, \Delta X_i^S, \Delta \tau_i^S)_{i \in \mathbb{Z}}$. Since $(Z_i)_{i \in \mathbb{Z}}$ determines the set of cut times \mathcal{D} , and conditional on $(Z_i)_{i \in \mathbb{Z}}$, $(\Delta X_n^S, \Delta \tau_n^S)_{T_{i-1} < n \leq T_i}$ is constructed independently for each $i \in \mathbb{Z}$ using only $(Z_n)_{T_{i-1} < n \leq T_i}$, by coupling, $(Z_i, \Delta X_i^S, \Delta \tau_i^S)_{i \in \mathbb{Z}}$ is shift invariant because $(Z_i)_{i \in \mathbb{Z}}$ is shift invariant. The ergodicity of $(Z_i, \Delta X_i^S, \Delta \tau_i^S)_{i \in \mathbb{Z}}$ follows from the ergodicity of the sequence $((Z_n)_{T_{i-1} < n \leq T_i}, (\Delta X_i^S)_{T_{i-1} < n \leq T_i}, (\Delta \tau_i^S)_{T_{i-1} < n \leq T_i})_{i \in \mathbb{Z}}$, the proof of which is exactly the same as in the proof of [4, Prop. 1.3]. \square

3 The lace expansion methodology

In this section we recall notation and results from [8] and give a formula for the velocity v .

A nearest-neighbour random walk path \vec{x}_n is a sequence $(x_i)_{i=0}^n$ for which $x_i = (x_i^{[1]}, \dots, x_i^{[d]}) \in \mathbb{Z}^d$ and $|x_{i+1} - x_i| = 1$ (Euclidean distance) for each i . If $\vec{\eta}$ and \vec{x} are two such paths of length at least j and m respectively and such that $\eta_j = x_0$, then we define the concatenation $\vec{\eta}_j \circ \vec{x}_m$ by

$$(\vec{\eta}_j \circ \vec{x}_m)_i := \begin{cases} \eta_i & \text{when } 0 \leq i \leq j, \\ x_{i-j} & \text{when } j \leq i \leq m + j. \end{cases} \quad (3.1)$$

For a general nearest-neighbour path \vec{x}_i , we use the notation $p^{\vec{x}_i}(x, y)$ for the conditional probability that the walk steps from x (where $x = x_i$ is implicit in the notation) to y , given the history of the path $\vec{x}_i = (x_0, \dots, x_i)$. In other words, for any finite path \vec{x}_i of non-zero P_{x_0} measure,

$$p^{\vec{x}_i}(x_i, x_{i+1}) := P_{x_0}(X_{i+1} = x_{i+1} | \vec{X}_i = \vec{x}_i). \quad (3.2)$$

Given $\vec{\eta}_m$ such that $P_{\eta_0}(\vec{X}_m = \vec{\eta}_m) > 0$, we define a conditional probability measure $P^{\vec{\eta}_m}$ on walks starting from η_m by

$$P^{\vec{\eta}_m}(\vec{X}_n = \vec{x}_n) := \prod_{i=0}^{n-1} p^{\vec{\eta}_m \circ \vec{x}_i}(x_i, x_{i+1}) = P_{\eta_0}(\vec{X}_{m+n} = \vec{\eta}_m \circ \vec{x}_n | \vec{X}_m = \vec{\eta}_m). \quad (3.3)$$

Note that by definition, $P_{\eta_0}(\vec{X}_m = \vec{x}_m) = P^{\eta_0}(\vec{X}_m = \vec{x}_m)$.

Set $j_0 = 0$, and for $n \geq 1$ and fixed paths $\vec{\eta}_{j_{n-1}+1}^{(n-1)}$ and $\vec{\eta}_{j_n+1}^{(n)}$ let

$$\Delta_n := p^{\vec{\eta}_{j_{n-1}+1}^{(n-1)} \circ \vec{\eta}_{j_n}^{(n)}}(\eta_{j_n}^{(n)}, \eta_{j_n+1}^{(n)}) - p^{\vec{\eta}_{j_n+1}^{(n)}}(\eta_{j_n}^{(n)}, \eta_{j_n+1}^{(n)}), \quad (3.4)$$

which is a difference in the probabilities of stepping from $\eta_{j_n}^{(n)}$ to $\eta_{j_n+1}^{(n)}$ with two different histories, $\vec{\eta}_{j_{n-1}+1}^{(n-1)} \circ \vec{\eta}_{j_n}^{(n)}$ and $\vec{\eta}_{j_n+1}^{(n)}$, with the first history containing the second.

Define $\mathcal{A}_{m,N} := \{(j_1, \dots, j_N) \in \mathbb{Z}_+^N : \sum_{l=1}^N j_l = m - N - 1\}$, $\mathcal{A}_N := \bigcup_m \mathcal{A}_{m,N} = \mathbb{Z}_+^N$ and

$$\pi_m^{(N)}(x, y) := \sum_{\vec{j} \in \mathcal{A}_{m,N}} \sum_{\vec{\eta}_1^{(0)}} \sum_{\vec{\eta}_{j_1+1}^{(1)}} \cdots \sum_{\vec{\eta}_{j_N+1}^{(N)}} 1_{\{\eta_{j_N}^{(N)} = x, \eta_{j_N+1}^{(N)} = y\}} p^o(o, \eta_1^{(0)}) \prod_{n=1}^N \Delta_n \prod_{i_n=0}^{j_n-1} p^{\vec{\eta}_{j_{n-1}+1}^{(n-1)} \circ \vec{\eta}_{i_n}^{(n)}}(\eta_{i_n}^{(n)}, \eta_{i_n+1}^{(n)}), \quad (3.5)$$

where (here and throughout this paper), each $\sum_{\vec{\eta}_{j_i+1}^{(i)}}$ is a sum over paths $(\eta_0^{(i)}, \dots, \eta_{j_i+1}^{(i)})$ consisting of $j_i + 1$ nearest-neighbour steps in \mathbb{Z}^d , where $\eta_0^{(0)} = o$ and otherwise $\eta_0^{(i)} = \eta_{j_{i-1}+1}^{(i-1)}$. The summand is zero if the paths are not nearest-neighbour, so that we do not need to include this restriction in the summation notation. Note that $\pi_m^{(N)}(x, y) = 0$ for all $N \geq m$, and by the indicator constraint in (3.5), $\pi_m^{(N)}(x, y) = 0$ if x is not a nearest-neighbour of y . Furthermore, $\sum_y \pi_m^{(N)}(x, y) = 0$ since summing Δ_N over $\vec{\eta}_{j_N+1}^{(N)}$ gives $1 - 1 = 0$ by (3.4).

Also define the following quantities

$$\pi_m(x, y) := \sum_{N=1}^{m-1} \pi_m^{(N)}(x, y), \quad \pi^{(N)}(x, y) := \sum_{m=2}^{\infty} \pi_m^{(N)}(x, y), \quad \text{and} \quad \pi_m(y) := \sum_{N=1}^{m-1} \sum_x \pi_m^{(N)}(x, y), \quad (3.6)$$

where (here and throughout this paper) \sum_x denotes a sum over $x \in \mathbb{Z}^d$, where the summands are typically non-zero only when x is a nearest-neighbour of some $y \in \mathbb{Z}^d$ appearing in the summand.

The following result gives a formula for the velocity, provided the sum converges.

Theorem 3.1 (Proposition 3.1 of [8]). *Under the conditions of Theorem 1.1, the limiting velocity v is given by*

$$v = E_o[X_1] + \sum_{m=2}^{\infty} \sum_y y \pi_m(y), \quad (3.7)$$

whenever this series converges.

We are interested in properties of the first coordinate $v^{[1]}$ of the speed as a function of β . Since $\sum_y \pi_m^{(N)}(x, y) = 0$, (3.7) can also be written in the more useful form

$$v = E_o[X_1] + \sum_{m=2}^{\infty} \sum_{N=1}^{m-1} \sum_{x,y} (y-x) \pi_m^{(N)}(x, y), \quad \text{so that} \quad (3.8)$$

$$v^{[1]} = E_o[X_1^{[1]}] + \sum_{m=2}^{\infty} \sum_{N=1}^{m-1} \sum_{x,y} (y^{[1]} - x^{[1]}) \pi_m^{(N)}(x, y). \quad (3.9)$$

Under the assumptions of Theorem 1.2, we will differentiate this expression with respect to β . Note that

$$\begin{aligned} E_o[X_1^{[1]}] &= \mathbb{E}[E_{\omega_o}[X_1^{[1]}]] = \mathbb{E}[\omega_o(e_1) - \omega_o(-e_1)] \\ &= \sum_{i=1}^2 (\nu_i(e_1) - \nu_i(-e_1)) \mu_{d_*}(\nu_i) = \kappa\beta\rho + \kappa(\nu_1(e_1) - \nu_1(-e_1)), \end{aligned}$$

where only the first term depends on β . It follows immediately that

$$\frac{\partial E_o[X_1^{[1]}]}{\partial \beta} = \kappa\rho. \quad (3.10)$$

If the derivative of the infinite series on the right hand side of (3.9) (with respect to β) is bounded in absolute value by $\kappa\rho$ then we will have shown that $v^{[1]}$ is increasing in β (when $\rho > 0$). This is a strategy that has been used successfully in studying excited random walks [9, 10], where only in one coordinate direction does the step distribution not coincide with that of a simple symmetric random walk, and the probability of stepping in this one coordinate direction is a constant.

As one might infer from (3.5) and (3.4), an analysis of the speed formula (3.7) and its derivative in β reduces to an analysis of transition probabilities of the form (3.2).

3.1 The annealed transition probability

In this section we consider properties of the annealed transition probability, defined for any path $\vec{\eta}_n$ of positive P_{η_0} -measure by

$$\begin{aligned} p^{\vec{\eta}_n}(\eta_n, \eta_{n+1}) &:= P_{\eta_0}(X_{n+1} = \eta_{n+1} | \vec{X}_n = \vec{\eta}_n) = \frac{P_{\eta_0}(\vec{X}_{n+1} = \vec{\eta}_{n+1})}{P_{\eta_0}(\vec{X}_n = \vec{\eta}_n)} \\ &= \frac{\mathbb{E}[P_{\omega, \eta_0}(\vec{X}_{n+1} = \vec{\eta}_{n+1})]}{\mathbb{E}[P_{\omega, \eta_0}(\vec{X}_n = \vec{\eta}_n)]} = \frac{\mathbb{E}[\omega_{\eta_n}(\eta_{n+1} - \eta_n) \prod_{i=0}^{n-1} \omega_{\eta_i}(\eta_{i+1} - \eta_i)]}{\mathbb{E}[\prod_{i=0}^{n-1} \omega_{\eta_i}(\eta_{i+1} - \eta_i)]}. \end{aligned} \quad (3.11)$$

Under \mathbb{P} , ω_x and ω_y are independent if $x \neq y$, whence (3.11) is equal to

$$\frac{\mathbb{E}[\omega_{\eta_n}(\eta_{n+1} - \eta_n) B(\vec{\eta}_n)]}{\mathbb{E}[B(\vec{\eta}_n)]}, \quad (3.12)$$

where

$$B(\vec{\eta}_n) := \prod_{\substack{0 \leq j \leq n-1 : \\ \eta_j = \eta_n}} \omega_{\eta_n}(\eta_{j+1} - \eta_j).$$

Therefore

$$\Delta_n = \frac{\mathbb{E}[\omega_{\eta_{j_n}^{(n)}}(\eta_{j_{n+1}}^{(n)} - \eta_{j_n}^{(n)})B(\vec{\eta}_{j_{n-1}+1}^{(n-1)} \circ \vec{\eta}_{j_n}^{(n)})]}{\mathbb{E}[B(\vec{\eta}_{j_{n-1}+1}^{(n-1)} \circ \vec{\eta}_{j_n}^{(n)})]} - \frac{\mathbb{E}[\omega_{\eta_{j_n}^{(n)}}(\eta_{j_{n+1}}^{(n)} - \eta_{j_n}^{(n)})B(\vec{\eta}_{j_n}^{(n)})]}{\mathbb{E}[B(\vec{\eta}_{j_n}^{(n)})]}. \quad (3.13)$$

It follows immediately that $\Delta_n \neq 0$ only if $B(\vec{\eta}_{j_n}^{(n)}) \neq B(\vec{\eta}_{j_{n-1}+1}^{(n-1)} \circ \vec{\eta}_{j_n}^{(n)})$, i.e. only if $\eta_{j_n}^{(n)} \in \vec{\eta}_{j_{n-1}}^{(n-1)}$. Recall that $V_k := \{\pm e_i\}_{1 \leq i \leq k}$. By Assumptions **(A0)**-**(A4)**, for \mathbb{P} -a.s. all $\omega_o, \delta q(v) \leq \omega_o(v) \leq q(v)$ for all $v \in V_d \setminus V_{d_0}$. Therefore $\sum_{v \in V_{d_0}} \omega_o(v) \leq 1 - \delta$, and

$$\sum_{\eta_{j_{n+1}}^{(n)} \in \mathbb{Z}^d} |\Delta_n| \leq 2(1 - \delta) \mathbf{1}_{\{\eta_{j_n}^{(n)} \in \vec{\eta}_{j_{n-1}}^{(n-1)}\}}, \quad (3.14)$$

since the two terms in (3.13) represent two probability kernels on \mathbb{Z}^d which both dominate $\delta q(\cdot)$.

We also need to examine the derivatives of the annealed transition probabilities with respect to β . For a directed edge b , let $\ell(\vec{\eta}_n, b) = \sum_{i=1}^n \mathbf{1}_{\{(\eta_{i-1}, \eta_i) = b\}}$ denote the edge local time of $\vec{\eta}$ at b up to time n , and for any $V \subset V_d$ let $\ell(\vec{\eta}_n, V) = \sum_{b \in V} \ell(\vec{\eta}_n, (\eta_n, \eta_n + b))$. Then for each $\vec{\eta}_n$, almost surely, at most one of the following can be greater than 0:

$$\ell(\vec{\eta}_n, \mathcal{S}_1), \quad \ell(\vec{\eta}_n, \mathcal{S}_2), \quad \ell(\vec{\eta}_n, V_{d_*} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)), \quad (3.15)$$

where we recall from **(A2)** that $\mathcal{S}_i = \text{supp}(\nu_i) \subset V_{d_*}$, $i = 1, 2$.

- If $u_n := \eta_{n+1} - \eta_n \in \mathcal{S}_1 \cup \mathcal{S}_2$, then

$$\begin{aligned} p^{\vec{\eta}_n}(\eta_n, \eta_{n+1}) &= \sum_{i=1}^2 \nu_i(u_n) \mathbf{1}_{\{\ell(\vec{\eta}_n, \mathcal{S}_i) > 0\}} + \mathbf{1}_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \sum_{i=1}^2 \nu_i(u_n) \mu_{d_*}(\nu_i) \\ &= \sum_{i=1}^2 \nu_i(u_n) \left[\mathbf{1}_{\{\ell(\vec{\eta}_n, \mathcal{S}_i) > 0\}} + \mathbf{1}_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) \right], \end{aligned}$$

from which we deduce

$$\frac{\partial p^{\vec{\eta}_n}(\eta_n, \eta_{n+1})}{\partial \beta} = \kappa [\nu_2(u_n) - \nu_1(u_n)] \mathbf{1}_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}}. \quad (3.16)$$

- If $u_n := \eta_{n+1} - \eta_n \notin \mathcal{S}_1 \cup \mathcal{S}_2$, then it is easily verified by direct calculations that

$$\frac{\partial p^{\vec{\eta}_n}(\eta_n, \eta_{n+1})}{\partial \beta} = 0.$$

Therefore

$$\frac{\partial \Delta_n}{\partial \beta} = \kappa [\nu_2(\eta_{j_{n+1}}^{(n)} - \eta_{j_n}^{(n)}) - \nu_1(\eta_{j_{n+1}}^{(n)} - \eta_{j_n}^{(n)})] [\mathbf{1}_{\{\ell(\vec{\eta}_{j_{n-1}+1}^{(n-1)} \circ \vec{\eta}_{j_n}^{(n)}, V_{d_*}) = 0\}} - \mathbf{1}_{\{\ell(\vec{\eta}_{j_n}^{(n)}, V_{d_*}) = 0\}}], \quad (3.17)$$

so that

$$\left| \frac{\partial \Delta_n}{\partial \beta} \right| \leq \kappa |\nu_2(\eta_{j_{n+1}}^{(n)} - \eta_{j_n}^{(n)}) - \nu_1(\eta_{j_{n+1}}^{(n)} - \eta_{j_n}^{(n)})| \mathbf{1}_{\{\eta_{j_n}^{(n)} \in \vec{\eta}_{j_{n-1}}^{(n-1)}\}},$$

which together with the fact $\nu_1, \nu_2 \in \mathcal{M}_\gamma(\mathbb{Z}^{d*})$ and $\gamma \leq 1 - \delta$ implies that

$$\sum_{\eta_{j_n+1}^{(n)} \in \mathbb{Z}^d} \left| \frac{\partial \Delta_n}{\partial \beta} \right| \leq 2\kappa(1 - \delta) 1_{\{\eta_{j_n}^{(n)} \in \vec{\eta}_{j_n-1}^{(n-1)}\}}. \quad (3.18)$$

Observe that

$$\begin{aligned} & \sum_y (y^{[1]} - x^{[1]}) \left(p^{\vec{\eta}_n}(x, y) - p^{\vec{x}_m \circ \vec{\eta}_n}(x, y) \right) \\ &= 1_{\{x \in \vec{x}_{m-1}\}} \left[p^{\vec{\eta}_n}(x, x + e_1) - p^{\vec{x}_m \circ \vec{\eta}_n}(x, x + e_1) - p^{\vec{\eta}_n}(x, x - e_1) + p^{\vec{x}_m \circ \vec{\eta}_n}(x, x - e_1) \right]. \end{aligned} \quad (3.19)$$

Lemma 3.2. *For all $x \in \mathbb{Z}^d$ and nearest-neighbour paths \vec{x}_m and $\vec{\eta}_n$ such that $\eta_0 = x_m$,*

$$\left| \sum_y (y^{[1]} - x^{[1]}) \left(p^{\vec{\eta}_n}(x, y) - p^{\vec{x}_m \circ \vec{\eta}_n}(x, y) \right) \right| \leq \rho 1_{\{x \in \vec{x}_{m-1}\}}, \quad (3.20)$$

$$\left| \frac{\partial}{\partial \beta} \sum_y (y^{[1]} - x^{[1]}) \left(p^{\vec{\eta}_n}(x, y) - p^{\vec{x}_m \circ \vec{\eta}_n}(x, y) \right) \right| \leq \kappa \rho 1_{\{x \in \vec{x}_{m-1}\}}. \quad (3.21)$$

Proof. The term in brackets on the right hand side of (3.19) is equal to

$$\begin{aligned} & \sum_{i=1}^2 \nu_i(e_1) \left[1_{\{\ell(\vec{\eta}_n, \mathcal{S}_i) > 0\}} + 1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) \right] \\ & - \sum_{i=1}^2 \nu_i(e_1) \left[1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, \mathcal{S}_i) > 0\}} + 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) \right] \\ & - \sum_{i=1}^2 \nu_i(-e_1) \left[1_{\{\ell(\vec{\eta}_n, \mathcal{S}_i) > 0\}} + 1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) \right] \\ & + \sum_{i=1}^2 \nu_i(-e_1) \left[1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, \mathcal{S}_i) > 0\}} + 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) \right] \\ & = \sum_{i=1}^2 [\nu_i(e_1) - \nu_i(-e_1)] \left[1_{\{\ell(\vec{\eta}_n, \mathcal{S}_i) > 0\}} + 1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) \right. \\ & \quad \left. - 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, \mathcal{S}_i) > 0\}} - 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) \right]. \end{aligned}$$

If the first indicator function is non-zero for some i then so is the third (for the same i), while all other indicators are zero. Therefore we can rewrite the above as

$$\sum_{i=1}^2 [\nu_i(e_1) - \nu_i(-e_1)] 1_{\{\ell(\vec{\eta}_n, \mathcal{S}_1 \cup \mathcal{S}_2) = 0\}} \left[1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) - 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, \mathcal{S}_i) > 0\}} - 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) \right].$$

If the final indicator function here is 1 then so is the second, while the third is zero. Thus the quantity above is zero unless the last indicator is zero, i.e. (3.19) is equal to

$$\begin{aligned} & 1_{\{x \in \vec{x}_{m-1}\}} 1_{\{\ell(\vec{\eta}_n, \mathcal{S}_1 \cup \mathcal{S}_2) = 0\}} 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) > 0\}} \\ & \quad \times \sum_{i=1}^2 [\nu_i(e_1) - \nu_i(-e_1)] \left[1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \mu_{d_*}(\nu_i) - 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, \mathcal{S}_i) > 0\}} \right]. \end{aligned} \quad (3.22)$$

Case 1: If $\nu_1(e_1) = \nu_1(-e_1) = 0$, then $\rho = \nu_2(e_1) - \nu_2(-e_1)$, and (3.22) becomes

$$1_{\{x \in \vec{x}_{m-1}\}} 1_{\{\ell(\vec{\eta}_n, \mathcal{S}_1 \cup \mathcal{S}_2) = 0\}} 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) > 0\}} \rho \left[1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \kappa \beta - 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, \mathcal{S}_2) > 0\}} \right],$$

where the term in brackets is the difference of two terms between 0 and 1 and hence is bounded in absolute value by 1. The derivative with respect to β is

$$1_{\{x \in \vec{x}_{m-1}\}} 1_{\{\ell(\vec{\eta}_n, \mathcal{S}_1 \cup \mathcal{S}_2) = 0\}} 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) > 0\}} 1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \kappa \rho$$

which is bounded in absolute value by $1_{\{x \in \vec{x}_{m-1}\}} \kappa \rho$.

Case 2: If $\nu_1(e_1) = 0$ and $\nu_1(-e_1) > 0$, then $\nu_2(-e_1) = 0$ and $\rho = \nu_2(e_1) + \nu_1(-e_1)$, while (3.22) becomes

$$1_{\{x \in \vec{x}_{m-1}\}} 1_{\{\ell(\vec{\eta}_n, \mathcal{S}_1 \cup \mathcal{S}_2) = 0\}} 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) > 0\}} \times \left[\nu_2(e_1) \left[1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \kappa \beta - 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, \mathcal{S}_2) > 0\}} \right] - \nu_1(-e_1) \left[1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \kappa (1 - \beta) - 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, \mathcal{S}_1) > 0\}} \right] \right],$$

where the term inside the largest brackets is bounded in absolute value by ρ . The derivative with respect to β is

$$1_{\{x \in \vec{x}_{m-1}\}} 1_{\{\ell(\vec{\eta}_n, \mathcal{S}_1 \cup \mathcal{S}_2) = 0\}} 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) > 0\}} 1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \kappa [\nu_2(e_1) + \nu_1(-e_1)],$$

which is bounded in absolute value by $1_{\{x \in \vec{x}_{m-1}\}} \kappa \rho$.

Case 3: If $\nu_1(e_1) > 0$ and $\nu_1(-e_1) = 0$, then $\nu_2(e_1) = 0$ and $\rho = -(\nu_2(-e_1) + \nu_1(e_1)) < 0$, contradicting our assumption $\rho > 0$ in **(A2)**. So this case can be ruled out.

Case 4: If $\nu_1(e_1) > 0$ and $\nu_1(-e_1) > 0$, then $\nu_2(e_1) = \nu_2(-e_1) = 0$ and $\rho = \nu_1(-e_1) - \nu_1(e_1)$, while (3.22) equals

$$- 1_{\{x \in \vec{x}_{m-1}\}} 1_{\{\ell(\vec{\eta}_n, \mathcal{S}_1 \cup \mathcal{S}_2) = 0\}} 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) > 0\}} \rho \left[1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \kappa (1 - \beta) - 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, \mathcal{S}_1) > 0\}} \right],$$

with the term inside the bracket bounded in absolute value by 1. The derivative with respect to β is

$$1_{\{x \in \vec{x}_{m-1}\}} 1_{\{\ell(\vec{\eta}_n, \mathcal{S}_1 \cup \mathcal{S}_2) = 0\}} 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_n, V_{d_*}) > 0\}} 1_{\{\ell(\vec{\eta}_n, V_{d_*}) = 0\}} \kappa \rho,$$

which is bounded in absolute value by $1_{\{x \in \vec{x}_{m-1}\}} \kappa \rho$.

This completes the proof of the lemma. \square

3.2 Convergence of the speed formula

In this section we prove a bound on $\sum_{x,y} \sum_m |\pi_m^{(N)}(x,y)|$, which can then be used to show that the sum in the speed formula (1.6) (or more precisely, the first line of (3.9)) converges, and hence (1.6) holds. Similar methods have been used in [9] and [10] to bound similar quantities. The present context is more demanding since the types of environment being considered are more complicated

(and require a somewhat more detailed analysis). In addition, the probability of stepping in various coordinate directions is allowed to be random, so that the number of steps taken in the coordinate directions $d_0 + 1, \dots, d$ is not binomially distributed (as in [9] and [10]), but rather is stochastically dominated by a binomial distribution.

We will need the following extension of [9, Lemma 3.1] for our bounds.

Lemma 3.3. *Let \vec{X} be a random walk in $\mathbb{Z}^d = \mathbb{Z}^{d_0+d_1}$ in a random environment ω satisfying assumptions **(A0)**–**(A4)**. For any $\vec{\eta}_m$ with $P_{\eta_0}(\vec{X}_m = \vec{\eta}_m) > 0$, $i \in \mathbb{Z}_+$ and $u \in \mathbb{Z}^d$, if we denote $P^{\vec{\eta}_m}(\cdot) := P_{\eta_0}(\cdot | (X_{-m+k})_{0 \leq k \leq m} = \vec{\eta}_m)$, then*

$$\sum_{j=0}^{\infty} \frac{(j+i)!}{j!} P^{\vec{\eta}_m}(X_j = u) \leq i! \delta^{-(i+1)} G_q^{*(i+1)}, \quad (3.23)$$

where G_q^{*k} is defined in (1.14).

Proof. Let \mathcal{N}_j be the number of steps the walk $\vec{X}_j := (X_k)_{0 \leq k \leq j}$ takes in the last d_1 coordinate directions, given history $(X_{-m+k})_{0 \leq k \leq m} = \vec{\eta}_m$. Let $\tau_n := \inf\{j \geq 0 : \mathcal{N}_j = n\}$. Let $(Y_n)_{n \in \mathbb{Z}_+}$ be the random walk on \mathbb{Z}^{d_1} coupled with X such that $Y_0 = \Pi_{d_1}(X_0)$, the projection of $X_0 \in \mathbb{Z}^{d_0+d_1}$ to its last d_1 coordinates, and $Y_n - Y_{n-1} = \Pi_{d_1}(X_{\tau_n} - X_{\tau_{n-1}})$ for all $n \in \mathbb{N}$. By assumption **(A0)**, it is clear that Y is distributed as a random walk on \mathbb{Z}^{d_1} with transition kernel q . We will denote the law of Y separately by \mathcal{P}_q . We then have

$$\begin{aligned} P^{\vec{\eta}_m}(X_j = u) &= \sum_{l=0}^j P^{\vec{\eta}_m}(X_j = u, \mathcal{N}_j = l) \leq \sum_{l=0}^j P^{\vec{\eta}_m}(\Pi_{d_1}(X_j) = \Pi_{d_1}(u), \mathcal{N}_j = l) \\ &= \sum_{l=0}^j P^{\vec{\eta}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)) \mathcal{P}_q(Y_l = \Pi_{d_1}(u)). \end{aligned} \quad (3.24)$$

Combined with Lemma 3.4 below, we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(j+i)!}{j!} P^{\vec{\eta}_m}(X_j = u) &\leq \sum_{j=0}^{\infty} \frac{(j+i)!}{j!} \sum_{l=0}^j P^{\vec{\eta}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)) \mathcal{P}_q(Y_l = \Pi_{d_1}(u)) \\ &= \sum_{l=0}^{\infty} \mathcal{P}_q(Y_l = \Pi_{d_1}(u)) \sum_{j=0}^{\infty} \frac{(j+i)!}{j!} P^{\vec{\eta}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)) \\ &\leq \sum_{l=0}^{\infty} \delta^{-i} \frac{(l+i)!}{l!} \mathcal{P}_q(Y_l = \Pi_{d_1}(u)). \end{aligned}$$

The inequality (3.23) then follows from the fact that (see e.g. [9, (3.2)])

$$G_q^{*(i+1)}(v) = \sum_{l=0}^{\infty} \frac{(l+i)!}{i! l!} \mathcal{P}_q(Y_l = v). \quad (3.25)$$

□

Lemma 3.4. *Let \vec{X} , $\vec{\eta}_m$, u , \mathcal{N}_j and \vec{Y} be defined as in Lemma 3.3 and its proof. Then*

$$\sum_{j=0}^{\infty} \frac{(j+i)!}{j!} P^{\vec{\eta}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)) \leq \delta^{-i} \frac{(l+i)!}{l!}. \quad (3.26)$$

Proof. First we claim that for all \vec{x}_k and \vec{y}_l which are compatible (by the coupling of \vec{X} and \vec{Y}), we have

$$P^{\vec{\eta}_m}(\Pi_{d_1}(X_{k+1} - X_k) \neq 0 | \vec{Y}_l = \vec{y}_l, \vec{X}_k = \vec{x}_k) \geq \delta. \quad (3.27)$$

Note that

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(j+i)!}{j!} P^{\vec{\eta}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)) &= E^{\vec{\eta}_m} \left[\sum_{j=0}^{\infty} \frac{(j+i)!}{j!} 1_{\{\mathcal{N}_j=l\}} | Y_l = \Pi_{d_1}(u) \right] \\ &= E^{\vec{\eta}_m} \left[\sum_{j=\tau_l}^{\tau_{l+1}-1} \frac{(j+i)!}{j!} | Y_l = \Pi_{d_1}(u) \right], \end{aligned} \quad (3.28)$$

where $\tau_i = \tau_i(\mathcal{N})$ is the first hitting time of level i by \mathcal{N}_j . By (3.27), regardless of \vec{Y}_l , \vec{X}_k and $\vec{\eta}_m$, the $(k+1)$ st step has probability at least δ of having a non-zero Π_{d_1} projection. It follows that under any conditional measure depending only on \vec{Y}_l and $\vec{\eta}_m$, we can couple \mathcal{N}_j with a random walk \mathcal{M}_j on \mathbb{Z}_+ taking i.i.d. steps $+1$ or 0 with probabilities δ and $1 - \delta$ respectively, such that $\tau_{i+1}(\mathcal{N}) - \tau_i(\mathcal{N}) \leq \tau_{i+1}(\mathcal{M}) - \tau_i(\mathcal{M})$ for all i , almost surely. This also implies that $\mathcal{M}_j \leq \mathcal{N}_j$ and $\tau_i(\mathcal{N}) \leq \tau_i(\mathcal{M})$ a.s.

Note that \mathcal{M}_j is binomial with parameters (j, δ) . Therefore (3.28) is bounded by

$$E \left[\sum_{j=\tau_l(\mathcal{M})}^{\tau_{l+1}(\mathcal{M})-1} \frac{(j+i)!}{j!} \right] = \sum_{j=l}^{\infty} \frac{(j+i)!}{j!} P(\mathcal{M}_j = l) = \sum_{j=l}^{\infty} \frac{(j+i)!}{j!} \binom{j}{l} \delta^l (1-\delta)^{j-l} = \delta^{-i} \frac{(l+i)!}{l!}, \quad (3.29)$$

exactly as in [9, 10]. It therefore remains to prove (3.27).

To prove (3.27), recall (2.2) and let $J_k = \max\{i \geq 0 : \tau_i \leq k\}$. Then \vec{Y}_{J_k} is determined by \vec{X}_k by the coupling of \vec{X} and \vec{Y} , and by assumptions (b') and **(A0)**, $(Y_t - Y_{J_k})_{t \geq J_k}$ is independent of \vec{X}_k and the event $\{\Pi_{d_1}(X_{k+1} - X_k) \neq 0\}$. Therefore

$$\begin{aligned} P^{\vec{\eta}_m}(\Pi_{d_1}(X_{k+1} - X_k) \neq 0 | \vec{Y}_l = \vec{y}_l, \vec{X}_k = \vec{x}_k) &= P^{\vec{\eta}_m}(\Pi_{d_1}(X_{k+1} - X_k) \neq 0 | \vec{X}_k = \vec{x}_k) \\ &= E^{\vec{\eta}_m} \left[\sum_{u: \Pi_{d_1}(u) \neq 0} \omega_{x_k}(u) | \vec{X}_k = \vec{x}_k \right] \geq \delta, \end{aligned}$$

as required, where we used assumption (b') once more in the inequality. \square

Let

$$\epsilon_\delta := 2(1 - \delta) \quad \text{and} \quad \alpha = \epsilon_\delta \delta^{-2} G_q^{*2}. \quad (3.30)$$

The following proposition, together with Proposition 3.1 of [8], shows that the series in the speed formula (1.6) converges when $G_q^{*2} < \infty$ and $\alpha < 1$. When $G_q^{*2} < \infty$ as is assumed in **(A3)**, $\alpha < 1$ can be achieved if $\delta < 1$ is sufficiently close to 1.

Proposition 3.5. *For RWpRE as in Theorem 1.2 and for each $N \in \mathbb{N}$, we have*

$$\sum_{x, y \in \mathbb{Z}^d} \sum_{m=2}^{\infty} |\pi_m^{(N)}(x, y)| \leq \epsilon_\delta \delta^{-1} G_q \alpha^{N-1}. \quad (3.31)$$

Proof. It follows from (3.5) that $\sum_{x,y \in \mathbb{Z}^d} \sum_{m=2}^{\infty} |\pi_m^{(N)}(x,y)|$ is bounded by

$$\sum_{\eta_1^{(0)}} p^o(o, \eta_1^{(0)}) \sum_{j_1=1}^{\infty} \sum_{\vec{\eta}_{j_1+1}^{(1)}} |\Delta_1| \prod_{i=0}^{j_1-1} p^{\vec{\eta}_1^{(0)} \circ \vec{\eta}_{i+1}^{(1)}} \left(\eta_{i_1}^{(1)}, \eta_{i_1+1}^{(1)} \right) \cdots \sum_{j_N=0}^{\infty} \sum_{\vec{\eta}_{j_N+1}^{(N)}} |\Delta_N| \prod_{i=0}^{j_N-1} p^{\vec{\eta}_{j_{N-1}+1}^{(N-1)} \circ \vec{\eta}_{i+1}^{(N)}} \left(\eta_{i_N}^{(N)}, \eta_{i_N+1}^{(N)} \right), \quad (3.32)$$

where the sums over j_k , $k \geq 2$ are all from 0 to ∞ . Note that by (3.14), $\Delta_1 \neq 0$ only when $\eta_{j_1}^{(1)} = \eta_0^{(0)} = o$, and in particular, only when j_1 is odd, which is why j_1 is summed from 1 onward. We will use Lemma 3.3 to successively bound the sums over j_k in (3.32), beginning with $k = N$.

When $N = 1$, (3.32) becomes

$$\begin{aligned} & \sum_{\eta_1^{(0)}} p^o(o, \eta_1^{(0)}) \sum_{j_1=1}^{\infty} \sum_{\vec{\eta}_{j_1+1}^{(1)}} |\Delta_1| \prod_{i=0}^{j_1-1} p^{\vec{\eta}_1^{(0)} \circ \vec{\eta}_{i+1}^{(1)}} \left(\eta_{i_1}^{(1)}, \eta_{i_1+1}^{(1)} \right) \\ & \leq \sum_{\eta_1^{(0)}} p^o(o, \eta_1^{(0)}) \sum_{j_1=1}^{\infty} \sum_{\vec{\eta}_{j_1}^{(1)}} P^{\vec{\eta}_1^{(0)}}(\vec{X}_{j_1} = \vec{\eta}_{j_1}) 1_{\{\eta_{j_1}^{(1)} = o\}} \epsilon_\delta \\ & = \epsilon_\delta \sum_{\eta_1^{(0)}} p^o(o, \eta_1^{(0)}) \sum_{j_1=1}^{\infty} P^{\vec{\eta}_1^{(0)}}(X_{j_1} = o) = \epsilon_\delta \sum_{j=2}^{\infty} P^o(X_j = o) \leq \epsilon_\delta \delta^{-1} G_q, \end{aligned} \quad (3.33)$$

where we used (3.14) in the first inequality, and the last inequality follows by setting $i = 0$, $u = o$ and $\vec{\eta}_m = \{o\}$ in (3.23).

For $N \geq 2$, as above we write

$$\begin{aligned} \sum_{j_N=0}^{\infty} \sum_{\vec{\eta}_{j_N+1}^{(N)}} |\Delta_N| \prod_{i=0}^{j_N-1} p^{\vec{\eta}_{j_{N-1}+1}^{(N-1)} \circ \vec{\eta}_{i+1}^{(N)}} \left(\eta_{i_N}^{(N)}, \eta_{i_N+1}^{(N)} \right) & \leq \sum_{j_N=0}^{\infty} \sum_{\vec{\eta}_{j_N}^{(N)}} P^{\vec{\eta}_{j_{N-1}+1}^{(N-1)}}(\vec{X}_{j_N} = \vec{\eta}_{j_N}) 1_{\{\eta_{j_N}^{(N)} \in \vec{\eta}_{j_{N-1}}^{(N-1)}\}} \epsilon_\delta \\ & \leq \epsilon_\delta \sum_{i=0}^{j_{N-1}} \sum_{j_N=0}^{\infty} P^{\vec{\eta}_{j_{N-1}+1}^{(N-1)}}(X_{j_N} = \eta_i^{(N-1)}) \\ & \leq (j_{N-1} + 1) \epsilon_\delta \delta^{-1} G_q. \end{aligned} \quad (3.34)$$

For the sum over j_{N-1} , we proceed as above except that we now have an extra factor of $(j_{N-1} + 1)$, whence we use (3.23) with $i = 1$. Continuing in this way until reaching the sum over j_1 , we get $N - 2$ factors of $\alpha = \epsilon_\delta \delta^{-2} G_q^{*2}$. For the sum over j_1 , proceeding as for the $N = 1$ case but with the extra factor $(j_1 + 1)$, we then have to deal with the term

$$\begin{aligned} & \sum_{\eta_1^{(0)}} p^o(o, \eta_1^{(0)}) \sum_{j_1=1}^{\infty} (j_1 + 1) \sum_{\vec{\eta}_{j_1+1}^{(1)}} |\Delta_1| \prod_{i=0}^{j_1-1} p^{\vec{\eta}_1^{(0)} \circ \vec{\eta}_{i+1}^{(1)}} \left(\eta_{i_1}^{(1)}, \eta_{i_1+1}^{(1)} \right) \\ & \leq \epsilon_\delta \sum_{\eta_1^{(0)}} p^o(o, \eta_1^{(0)}) \sum_{j_1=1}^{\infty} (j_1 + 1) P^{\vec{\eta}_1^{(0)}}(X_{j_1} = o) = \epsilon_\delta \sum_{j=2}^{\infty} j P^o(X_j = o) \leq \epsilon_\delta \delta^{-2} G_q^{*2} = \alpha, \end{aligned} \quad (3.35)$$

where we have again applied (3.23). Combining all the factors then gives (3.31). \square

3.3 The derivative of the speed formula

From (3.9) and (3.10) we have that

$$\frac{\partial v^{[1]}}{\partial \beta} = \kappa \rho + \frac{\partial}{\partial \beta} \sum_{m=2}^{\infty} \sum_{N=1}^{m-1} \sum_{x,y} (y^{[1]} - x^{[1]}) \pi_m^{(N)}(x, y), \quad (3.36)$$

assuming that the latter derivative actually exists.

Recall (3.5) and define

$$\varphi_m^{(N)}(x, y) := \frac{\partial}{\partial \beta} \pi_m^{(N)}(x, y), \quad (3.37)$$

which is well-defined as a finite sum of finite products of transition probabilities (see (3.16)), and is non-zero only for a finite set of $x, y \in \mathbb{Z}^d$ due to the nearest-neighbour constraint. In order to prove Theorem 1.2, it is sufficient to show that

$$\sup_{\beta \in [0,1]} \sum_{m=2}^{\infty} \sum_{N=1}^{m-1} \left| \sum_{x,y} (y-x)^{[1]} \varphi_m^{(N)}(x, y) \right| < \kappa \rho, \quad (3.38)$$

and

$$\lim_{m_0 \uparrow \infty} \sup_{\beta \in [0,1]} \sum_{m=m_0}^{\infty} \sum_{N=1}^{m-1} \left| \sum_{x,y} (y-x)^{[1]} \varphi_m^{(N)}(x, y) \right| = 0. \quad (3.39)$$

This implies the uniform convergence (in $\beta \in [0, 1]$) of the derivative of the summands in (3.36) and allows us to pass the derivative inside the infinite sum and conclude that $\frac{\partial v^{[1]}}{\partial \beta} > 0$.

Let us write

$$\varphi_m^{(N)}(x, y) = \varphi_m^{(N,1)}(x, y) + \varphi_m^{(N,2)}(x, y) + \varphi_m^{(N,3)}(x, y), \quad (3.40)$$

where by the product rule, $\varphi_m^{(N,1)}(x, y)$, $\varphi_m^{(N,2)}(x, y)$ and $\varphi_m^{(N,3)}(x, y)$ arise from differentiating $p^o(o, \eta_1^{(0)})$, $\prod_{n=1}^N \prod_{i_n=0}^{j_n-1} p^{\vec{\eta}_{j_{n-1}+1}^{(n-1)} \circ \vec{\eta}_{i_n}^{(n)}}(\eta_{i_n}^{(n)}, \eta_{i_{n+1}}^{(n)})$ and $\prod_{n=1}^N \Delta_n$, respectively, in (3.5).

Define

$$\begin{aligned} F^{(N)} := & \sum_{\eta_1^{(0)}} \left| \frac{\partial}{\partial \beta} p^o(o, \eta_1^{(0)}) \right| \sum_{j_1=1}^{\infty} \sum_{\vec{\eta}_{j_1+1}^{(1)}} |\Delta_1| \prod_{i_1=0}^{j_1-1} p^{\vec{\eta}_1^{(0)} \circ \vec{\eta}_{i_1}^{(1)}}(\eta_{i_1}^{(1)}, \eta_{i_1+1}^{(1)}) \dots \\ & \dots \sum_{j_N=0}^{\infty} \sum_{\vec{\eta}_{j_N}^{(N)}} \prod_{i_N=0}^{j_N-1} p^{\vec{\eta}_{j_{N-1}+1}^{(N-1)} \circ \vec{\eta}_{i_N}^{(N)}}(\eta_{i_N}^{(N)}, \eta_{i_{N+1}}^{(N)}) \left| \sum_{\eta_{j_N+1}^{(N)}} (\eta_{j_N+1}^{(N)} - \eta_{j_N}^{(N)})^{[1]} \Delta_N \right|. \end{aligned} \quad (3.41)$$

It follows that

$$\sum_{m=2}^{\infty} \left| \sum_{x,y} (y-x)^{[1]} \varphi_m^{(N,1)}(x, y) \right| \leq F^{(N)}. \quad (3.42)$$

Similarly, for $k = 1, \dots, N$, let $H_k^{(N)}$ be defined by replacing (in the definition (3.5)): Δ_n with $|\Delta_n|$ for all $n = 1, \dots, N-1$, $\sum_{\eta_{j_N+1}^{(N)}} \Delta_N$ with $|\sum_{\eta_{j_N+1}^{(N)}} (\eta_{j_N+1}^{(N)} - \eta_{j_N}^{(N)})^{[1]} \Delta_N|$, $\prod_{i_k=0}^{j_k-1} p^{\vec{\eta}_{j_{k-1}+1}^{(k-1)} \circ \vec{\eta}_{i_k}^{(k)}}(\eta_{i_k}^{(k)}, \eta_{i_k+1}^{(k)})$ with

$$\sum_{l=0}^{j_k-1} \left| \frac{\partial}{\partial \beta} p^{\vec{\eta}_{j_{k-1}+1}^{(k-1)} \circ \vec{\eta}_l^{(k)}}(\eta_l^{(k)}, \eta_{l+1}^{(k)}) \right| \prod_{\substack{i_k=0 \\ i_k \neq l}}^{j_k-1} p^{\vec{\eta}_{j_{k-1}+1}^{(k-1)} \circ \vec{\eta}_{i_k}^{(k)}}(\eta_{i_k}^{(k)}, \eta_{i_k+1}^{(k)}). \quad (3.43)$$

For $k = 1, \dots, N$, let $J_k^{(N)}$ be defined by replacing in (3.5): Δ_i with $|\Delta_i|$ for all $1 \leq i \leq N-1$ and $i \neq k$, Δ_k with $|\frac{\partial}{\partial \beta} \Delta_k|$ and $\sum_{\eta_{j_{N+1}}^{(N)}} \Delta_N$ with $|\sum_{\eta_{j_{N+1}}^{(N)}} (\eta_{j_{N+1}}^{(N)} - \eta_{j_N}^{(N)})^{[1]} \Delta_N|$ for $1 \leq k \leq N-1$, and $\sum_{\eta_{j_{N+1}}^{(N)}} \Delta_N$ with $|\frac{\partial}{\partial \beta} \sum_{\eta_{j_{N+1}}^{(N)}} (\eta_{j_{N+1}}^{(N)} - \eta_{j_N}^{(N)})^{[1]} \Delta_N|$ for $k = N$.

Letting $H^{(N)} := \sum_{k=1}^N H_k^{(N)}$ and $J^{(N)} := \sum_{k=1}^N J_k^{(N)}$, we observe that

$$\sum_{m=2}^{\infty} \left| \sum_{x,y} (y-x)^{[1]} \varphi_m^{(N,2)}(x,y) \right| \leq H^{(N)}, \quad \text{and} \quad \sum_{m=2}^{\infty} \left| \sum_{x,y} (y-x)^{[1]} \varphi_m^{(N,3)}(x,y) \right| \leq J^{(N)}. \quad (3.44)$$

The remainder of this section is devoted to establishing bounds on $F^{(N)}$, $J^{(N)}$, and $H^{(N)}$.

Lemma 3.6. (Bounds on $F^{(N)}$) *We have*

$$F^{(N)} \leq \begin{cases} \kappa \rho \epsilon_\delta \delta^{-1} G_q, & N = 1, \\ \kappa \rho \delta^{-1} G_q \alpha^{N-1}, & N \geq 2. \end{cases} \quad (3.45)$$

Proof. When $N = 1$, we first use (3.16) and (3.20) to get

$$\begin{aligned} F^{(1)} &\leq \sum_{\eta_1^{(0)}} \kappa |\nu_2(\eta_1^{(0)}) - \nu_1(\eta_1^{(0)})| \sum_{j_1=1}^{\infty} \sum_{\eta_{j_1}^{(1)}} \prod_{i_1=1}^{j_1-1} p^{\tilde{\eta}_1^{(0)} \circ \tilde{\eta}_{i_1}^{(1)}} (\eta_{i_1}^{(1)}, \eta_{i_1+1}^{(1)}) \mathbf{1}_{\{\eta_{j_1}^{(1)}=o\}} \rho \\ &= \kappa \rho \sum_{\eta_1^{(0)}} |\nu_2(\eta_1^{(0)}) - \nu_1(\eta_1^{(0)})| \sum_{j_1=1}^{\infty} P^{\tilde{\eta}_1^{(0)}}(X_{j_1} = o) \leq \kappa \rho 2(1-\delta) \delta^{-1} G_q, \end{aligned} \quad (3.46)$$

where we have used the fact that $\nu_1, \nu_2 \in \mathcal{M}_\gamma(\mathbb{Z}_{d_*})$ with $\gamma \leq 1 - \delta$, and we applied (3.23). This gives the bound for $N = 1$.

For $N \geq 2$ we proceed similarly, first using (3.20) in the form

$$\left| \sum_{\eta_{j_{N+1}}^{(N)}} (\eta_{j_{N+1}}^{(N)} - \eta_{j_N}^{(N)})^{[1]} \Delta_N \right| \leq \rho \mathbf{1}_{\{\eta_{j_N}^{(N)} \in \tilde{\eta}_{j_{N-1}}^{(N-1)}\}},$$

and then proceeding as in the proof of Proposition 3.5. This involves using (3.23) with $i = 0$ to deal with what remains inside the sum over j_N , which gives overall a factor of $(j_{N-1} + 1) \rho \delta^{-1} G_q$. We then repeatedly use (3.14) and (3.23) with $i = 1$ for the remaining terms inside the sums over $j_{N-1}, j_{N-2}, \dots, j_2$ in that order. As in Proposition 3.5, this gives $N - 2$ factors of α . Finally we are left to deal with a term of the form

$$\kappa \sum_{\eta_1^{(0)}} |\nu_2(\eta_1^{(0)}) - \nu_1(\eta_1^{(0)})| \sum_{j_1=1}^{\infty} (j_1 + 1) P^{\tilde{\eta}_1^{(0)}}(X_{j_1} = o) \leq \kappa 2(1-\delta) \delta^{-2} G_q^{*2} = \kappa \alpha, \quad (3.47)$$

where we again used (3.23) with $i = 1$. □

Lemma 3.7. (Bounds on $J^{(N)}$) *We have*

$$J_k^{(N)} \leq \begin{cases} \kappa \rho \delta^{-2} (G_q(o) - \delta), & N = k = 1, \\ \kappa \rho \delta^{-1} G_q \alpha^{N-1}, & N \geq 2, 1 \leq k \leq N. \end{cases} \quad (3.48)$$

Proof. The second bound in (3.48) follows essentially the same proof as that for Lemma 3.6. For $N \geq 2$, when we sum over j_N and $\eta_{j_{N+1}}^{(N)}$, we apply (3.20) if $k < N$ and we apply (3.21) if $k = N$. When we sum over j_k and $\eta_{j_{k+1}}^{(k)}$ with $k < N$, we apply (3.18).

To bound $J_1^{(1)}$, note that by (3.21) applied to $\vec{x}_m = \vec{\eta}_1^{(0)}$ and $\vec{\eta}_m = \vec{\eta}_{j_1}^{(1)}$, we have

$$J_1^{(1)} \leq \sum_{\eta_1^{(0)}} p^o(o, \eta_1^{(0)}) \sum_{j_1=1}^{\infty} \sum_{\vec{\eta}_{j_1}^{(1)}} \prod_{i_1=0}^{j_1-1} p^{\vec{\eta}_1^{(0)} \circ \vec{\eta}_{i_1}^{(1)}}(\eta_{i_1}^{(1)}, \eta_{i_1+1}^{(1)}) 1_{\{\eta_{j_1}^{(1)}=o\}} \kappa \rho = \kappa \rho \sum_{j=2}^{\infty} P^o(X_j = o). \quad (3.49)$$

Using the parity of \mathbb{Z}^{d_1} , and proceeding as in the proof of Lemma 3.3,

$$\begin{aligned} \sum_{j=2}^{\infty} P^o(X_j = o) &\leq \sum_{l=0}^{\infty} \mathcal{P}_q(Y_l = o) \sum_{j=l \vee 2}^{\infty} P^o(\mathcal{N}_j = l | Y_l = o) \\ &= \sum_{j=2}^{\infty} P^o(\mathcal{N}_j = 0) + \sum_{l=2}^{\infty} \mathcal{P}_q(Y_l = o) \sum_{j=l}^{\infty} P^o(\mathcal{N}_j = l | Y_l = o). \end{aligned} \quad (3.50)$$

As shown in the proof of Lemma 3.3, for each $j \in \mathbb{N}$, conditional on \vec{X}_{j-1} and \vec{Y}_l , either $\mathcal{N}_j = \mathcal{N}_{j-1}$, or $\mathcal{N}_j = \mathcal{N}_{j-1} + 1$ with probability at least δ . Therefore by the same comparison with the Bernoulli random walk \mathcal{M} as in the proof of Lemma 3.4, we have

$$\sum_{j=l}^{\infty} P^o(\mathcal{N}_j = l | Y_l = o) = E^o[\tau_{l+1} - \tau_l] \leq \delta^{-1}.$$

Similarly

$$\sum_{j=2}^{\infty} P^o(\mathcal{N}_j = 0) \leq \delta^{-1} - 1.$$

Combining all the above bounds gives

$$J_1^{(1)} \leq \kappa \rho (\delta^{-1} - 1 + \delta^{-1} [G_q(o) - 1]) = \kappa \rho \delta^{-2} (G_q(o) - \delta), \quad (3.51)$$

as required. \square

To bound $H_k^{(N)}$, we need a new lemma of the form of Lemma 3.3 which accommodates the derivative of the transition probability for one of the steps.

Lemma 3.8. *Assume the same conditions as in Lemma 3.3. Recall the definition of ν_1, ν_2 from assumption (A2), and the definition of edge local time $\ell(\vec{\eta}, V_{d_*})$ from (3.15). Then for each $i \in \mathbb{Z}_+$ and $u \in \mathbb{Z}^d$, we have*

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{(j+i)!}{j!} \sum_{s=0}^{j-1} \sum_{\vec{\eta}_s} P^{\vec{x}_m}(\vec{X}_s = \vec{\eta}_s) 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_s, V_{d_*})=0\}} \sum_{\eta_{s+1}} \kappa |\nu_2(\eta_{s+1} - \eta_s) - \nu_1(\eta_{s+1} - \eta_s)| \\ \times P^{\vec{x}_m \circ \vec{\eta}_{s+1}}(X_{j-s-1} = u) \leq \varepsilon_{\delta} \kappa (i+1)! \delta^{-(i+2)} G_q^{*(i+2)}. \end{aligned} \quad (3.52)$$

Proof. Since ν_1 and ν_2 have disjoint supports \mathcal{S}_1 , and \mathcal{S}_2 , the left hand side of (3.52) equals

$$\sum_{r=1}^2 \sum_{j=1}^{\infty} \frac{(j+i)!}{j!} \sum_{s=0}^{j-1} \sum_{\vec{\eta}_s} P^{\vec{x}_m}(\vec{X}_s = \vec{\eta}_s) 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_s, V_{d_*})=0\}} \sum_{w:=\eta_{s+1}-\eta_s \in \mathcal{S}_r} \kappa \nu_r(w) P^{\vec{x}_m \circ \vec{\eta}_{s+1}}(X_{j-s-1} = u). \quad (3.53)$$

Let $P_{(s,w)}^{\vec{x}_m}$ denote the law of a random walk which evolves according to a RWRE with history \vec{x}_m , except that the $s+1$ -st step is deterministic and equals w , and if this step has zero probability given the history of X up to time s , then the walk is killed. More precisely,

$$P_{(s,w)}^{\vec{x}_m}(X_{n+1} - X_n = z | \vec{X}_n = \vec{x}_n) = \begin{cases} \delta_w(z), & \text{if } n = s, \\ P^{\vec{x}_m}(X_{n+1} - X_n = z | \vec{X}_n = \vec{x}_n), & \text{otherwise,} \end{cases}$$

and the walk is killed at time $s+1$ if $P^{\vec{x}_m}(X_{s+1} - X_s = w | \vec{X}_s = \vec{x}_s) = 0$.

Then (3.53) is bounded by

$$\kappa \sum_{r=1}^2 \sum_{w \in \mathcal{S}_r} \nu_r(w) \sum_{j=1}^{\infty} \frac{(j+i)!}{j!} \sum_{s=0}^{j-1} P_{(s,w)}^{\vec{x}_m}(X_j = u). \quad (3.54)$$

As in the proof of Lemma 3.3, we have

$$P_{(s,w)}^{\vec{x}_m}(X_j = u) \leq \sum_{l=0}^{j-1} P_q(Y_l = \Pi_{d_1}(u)) P_{(s,w)}^{\vec{x}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)), \quad (3.55)$$

where \mathcal{N}_j is the number of steps of \vec{X}_j with non-zero Π_{d_1} -projection. Thus, the sum over j in (3.54) is bounded by

$$\begin{aligned} & \sum_{j=1}^{\infty} \frac{(j+i)!}{j!} \sum_{s=0}^{j-1} \sum_{l=0}^{j-1} P_q(Y_l = \Pi_{d_1}(u)) P_{(s,w)}^{\vec{x}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)) \\ &= \sum_{l=0}^{\infty} P_q(Y_l = \Pi_{d_1}(u)) \sum_{s=0}^{\infty} \sum_{j=(lV_s)+1}^{\infty} \frac{(j+i)!}{j!} P_{(s,w)}^{\vec{x}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)). \end{aligned} \quad (3.56)$$

As in the proof of Lemma 3.4, we can couple \mathcal{N} with a Bernoulli random walk \mathcal{M} such that

$$\begin{aligned} & \sum_{j=(lV_s)+1}^{\infty} \frac{(j+i)!}{j!} P_{(s,w)}^{\vec{x}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)) \\ &= E_{(s,w)}^{\vec{x}_m} \left[\sum_{j=\tau \vee (s+1)}^{\tau+1-1} \frac{(j+i)!}{j!} \middle| Y_l = \Pi_{d_1}(u) \right] \\ &\leq E \left[\sum_{j=\tau(\mathcal{M}-1) \vee (s+1)}^{\tau+1(\mathcal{M}-1)-1} \frac{(j+i)!}{j!} \right] = \sum_{j=(lV_s)+1}^{\infty} \frac{(j+i)!}{j!} P(\mathcal{M}_j - 1 = l) \end{aligned} \quad (3.57)$$

where $\mathcal{M}_n \sim \text{Bin}(n, \delta)$. Therefore the summation over s in (3.56) is bounded by

$$\begin{aligned} & \sum_{s=0}^{\infty} \sum_{j=(lV_s)+1}^{\infty} \frac{(j+i)!}{j!} P(\mathcal{M}_j - 1 = l) = \sum_{j=l+1}^{\infty} \sum_{s=0}^{j-1} \frac{(j+i)!}{j!} P(\mathcal{M}_j - 1 = l) \\ &= \sum_{j=l+1}^{\infty} \frac{(j+i)!}{(j-1)!} P(\mathcal{M}_j = l+1) < \sum_{j=l}^{\infty} \frac{(j+i+1)!}{j!} P(\mathcal{M}_j = l) \leq \delta^{-(i+1)} \frac{(l+i+1)!}{l!}, \end{aligned} \quad (3.58)$$

as in (3.29). Substituting this bound back into (3.56) and then into (3.54) then proves the lemma, where we need to use (3.25) and the fact that ν_1, ν_2 each has total mass $\gamma \leq 1 - \delta = \varepsilon_\delta/2$. \square

Lemma 3.9. (Bounds on $H^{(N)}$) *We have*

$$H_k^{(N)} \leq \begin{cases} \kappa\rho\alpha^N, & N = k \geq 1, \\ 2\kappa\rho\varepsilon_\delta^2\delta^{-4}G_qG_q^{*3}\alpha^{N-2}, & N > k \geq 1. \end{cases}$$

Proof. We first use (3.16) and (3.20) to get

$$\begin{aligned} H_1^{(1)} &\leq \sum_{\eta_1^{(0)}} p^o(o, \eta_1^{(0)}) \sum_{j_1=1}^{\infty} \sum_{l=0}^{j_1-1} \sum_{\vec{\eta}_l^{(1)}} P^{\vec{\eta}_l^{(0)}}(\vec{X}_l = \vec{\eta}_l^{(1)}) 1_{\{\ell(\vec{\eta}_1^{(0)}, \vec{\eta}_l^{(1)}, V_{d_*})=0\}} \\ &\quad \times \sum_{\eta_{l+1}^{(1)}} \kappa |\nu_2(\eta_{l+1}^{(1)} - \eta_l^{(1)}) - \nu_1(\eta_{l+1}^{(1)} - \eta_l^{(1)})| P^{\vec{\eta}_l^{(0)} \circ \vec{\eta}_{l+1}^{(1)}}(X_{j_1-l-1} = o) \rho \\ &= \sum_{j=2}^{\infty} \sum_{l=1}^j \sum_{\vec{\eta}_l} p^o(\vec{X}_l = \vec{\eta}_l) \sum_{\eta_{l+1}} \kappa |\nu_2(\eta_{l+1} - \eta_l) - \nu_1(\eta_{l+1} - \eta_l)| 1_{\{\ell(\vec{\eta}_l, V_{d_*})=0\}} P^{\vec{\eta}_l}(X_{j-l-1} = o) \rho \\ &\leq \kappa\rho\varepsilon_\delta\delta^{-2}G_q^{*2} = \kappa\rho\alpha, \end{aligned} \tag{3.59}$$

where we have used Lemma 3.8 with $i = 0$.

For $N \geq 2$, we first bound $|\sum_{\eta_{j_{N+1}}^{(N)}} (\eta_{j_{N+1}}^{(N)} - \eta_{j_N}^{(N)})^{[1]} \Delta_N|$ in $H_k^{(N)}$ by $\rho 1_{\{\eta_{j_N}^{(N)} \in \vec{\eta}_{j_{N-1}}^{(N-1)}\}}$ using (3.20).

For $k = N$, we then use Lemma 3.8 with $i = 0$ to bound the sum over j_N , yielding a factor of $\kappa\rho\varepsilon_\delta\delta^{-2}G_q^{*2}(j_{N-1} + 1)$. We then proceed exactly as in the proof of Proposition 3.5 on the sums over j_{N-1}, \dots, j_1 in that order, giving $N - 1$ factors of α .

If $N > k > 1$, we use (3.23) with $i = 0$ to bound the sum over j_N , yielding a factor of $\rho\delta^{-1}G_q(j_{N-1} + 1)$. We then proceed exactly as in the proof of Proposition 3.5 on the sums over j_{N-1}, \dots, j_{k+1} in that order, giving $N - (k + 1)$ factors of α . We then obtain a factor ε_δ from $\sum_{\eta_{j_{k+1}}^{(k)}} |\Delta_k|$ and then we use Lemma 3.8 with $i = 1$ to bound the sum over j_k , giving a factor $2\kappa\varepsilon_\delta\delta^{-3}G_q^{*3}(j_{k-1} + 1)$. We then proceed exactly as in the proof of Proposition 3.5 on the sums over j_{k-1}, \dots, j_1 in that order giving $k - 1$ additional factors of α . The case $N > k = 1$ is similar. \square

4 Proof of Theorem 1.2

By assumption **(A3)**, $G_q^{*i} < \infty$ for $i = 1, 2, 3, 4$. It follows that $\alpha = 2(1 - \delta)\delta^{-2}G_q^{*2} < 1$ for $\delta < 1$ sufficiently close to 1 (depending only on $q(\cdot)$), in which case (3.31) is summable in N , and thus (3.7) and (3.9) hold. By assumption **(A3)** that $G_q(o) < 2$, and by Lemmas 3.6, 3.7, and 3.9, uniformly in $\beta \in [0, 1]$, we have

$$\sum_{N \geq 1} F^{(N)} \leq \kappa\rho C_{1,\delta}, \quad \sum_{N \geq 1} \sum_{k=1}^N J_k^{(N)} \leq \kappa\rho C_{2,\delta}, \quad \sum_{N \geq 1} \sum_{k=1}^N H_k^{(N)} \leq \kappa\rho C_{3,\delta}, \tag{4.1}$$

where $C_{1,\delta}, C_{3,\delta} \searrow 0$ as $\delta \nearrow 1$ and $C_{2,\delta} < 1$ for δ sufficiently close to 1. It follows that for δ sufficiently close to 1 (depending only on $q(\cdot)$), (3.38) holds. To conclude the proof of Theorem 1.2, it only remains to verify (3.39).

Note that

$$\sum_{m=1}^{\infty} \sum_{N=N_0}^{m-1} \left| \sum_{x,y} (y-x)^{[1]} \varphi_m^{(N)}(x,y) \right| \leq \sum_{N=N_0}^{\infty} (F^{(N)} + J^{(N)} + H^{(N)}),$$

which by Lemmas 3.6, 3.7, and 3.9 tends to 0 (uniformly in $\beta \in [0,1]$) as $N_0 \rightarrow \infty$ if δ is sufficiently close to 1. Therefore (3.39) follows from Lemma 4.1 below, which concludes the proof of Theorem 1.2. \square

Lemma 4.1. *If $\delta \in (0,1)$ is chosen such that $\alpha = 2(1-\delta)\delta^{-2}G_q^{*2} < 1$, then for each $N \in \mathbb{N}$,*

$$\lim_{m_0 \uparrow \infty} \sup_{\beta \in [0,1]} \sum_{m=m_0}^{\infty} \left| \sum_{x,y} (y-x)^{[1]} \varphi_m^{(N)}(x,y) \right| = 0. \quad (4.2)$$

Proof. As in (3.40), we will split $\varphi_m^{(N)}$ into $\varphi_m^{(N,i)}$ for $1 \leq i \leq 3$. It will be sufficient to verify (4.2) with $\varphi_m^{(N)}$ replaced by $\varphi_m^{(N,i)}$ for $i = 1, 2, 3$. Note that

$$\begin{aligned} \sum_{m=m_0}^{\infty} \left| \sum_{x,y} (y-x)^{[1]} \varphi_m^{(N,1)}(x,y) \right| &\leq \sum_{\substack{j_1, \dots, j_N \geq 0 \\ j_1 + \dots + j_N \geq m_0 - N}} \sum_{\eta_1^{(0)}} \left| \frac{\partial}{\partial \beta} p^o(o, \eta_1^{(0)}) \right| \sum_{\vec{\eta}_{j_1+1}^{(1)}} P^{\vec{\eta}_1^{(0)}}(\vec{X}_{j_1} = \vec{\eta}_{j_1}^{(1)}) |\Delta_1| \dots \\ &\dots \sum_{\vec{\eta}_{j_N}^{(N)}} P^{\vec{\eta}_{j_{N-1}+1}^{(N-1)}}(\vec{X}_{j_N} = \vec{\eta}_{j_N}^{(N)}) \left| \sum_{\eta_{j_{N+1}}^{(N)}} (\eta_{j_{N+1}}^{(N)} - \eta_{j_N}^{(N)})^{[1]} \Delta_N \right| \\ &\leq \sum_{k=1}^N \sum_{\substack{i \neq k: j_i \geq 0 \\ j_k \geq \frac{m_0}{N} - 1}} \sum_{\eta_1^{(0)}} \left| \frac{\partial}{\partial \beta} p^o(o, \eta_1^{(0)}) \right| \sum_{\vec{\eta}_{j_1+1}^{(1)}} P^{\vec{\eta}_1^{(0)}}(\vec{X}_{j_1} = \vec{\eta}_{j_1}^{(1)}) |\Delta_1| \dots \\ &\dots \sum_{\vec{\eta}_{j_N}^{(N)}} P^{\vec{\eta}_{j_{N-1}+1}^{(N-1)}}(\vec{X}_{j_N} = \vec{\eta}_{j_N}^{(N)}) \left| \sum_{\eta_{j_{N+1}}^{(N)}} (\eta_{j_{N+1}}^{(N)} - \eta_{j_N}^{(N)})^{[1]} \Delta_N \right|, \quad (4.3) \end{aligned}$$

where we made the observation that one of the N paths $\vec{\eta}_{j_N}$ must have length at least $\frac{m_0}{N} - 1$. In the sum over k in (4.3), if $k = N$, then the sum over $j_N \geq \frac{m_0}{N} - 1$ can be bounded by

$$\rho \sum_{j_N \geq \frac{m_0}{N} - 1} \sum_{i=0}^{j_N-1} P^{\vec{\eta}_{j_N-1+1}^{(N-1)}}(\vec{X}_{j_N} = \vec{\eta}_i^{(N-1)}) \leq \rho(j_{N-1} + 1) \sup_{u \in \mathbb{Z}^d} \sum_{j_N \geq \frac{m_0}{N} - 1} P^{\vec{\eta}_{j_N-1+1}^{(N-1)}}(\vec{X}_{j_N} = u). \quad (4.4)$$

By Lemma 3.3, for each $i \geq 0$, we have

$$\sup_{u \in \mathbb{Z}^d} \sum_{j=m_0}^{\infty} \frac{(j+i)!}{j!} P^{\vec{\eta}_m}(X_j = u) \leq \sup_{u \in \mathbb{Z}^d} \sum_{j=0}^{\infty} \frac{(j+i+1)!}{m_0 j!} P^{\vec{\eta}_m}(X_j = u) \leq \frac{(i+1)! \delta^{-(i+2)} G_q^{*(i+2)}}{m_0}. \quad (4.5)$$

Applying this bound with $i = 0$ to (4.4) then gives a factor of $1/m_0$. Summing over j_{N-1}, \dots, j_1 in (4.3) with $k = N$ as in the proof of Lemma 3.6 then gives a bound proportional to $1/m_0$, which is independent of β and tends to 0 as $m_0 \rightarrow \infty$.

In the sum over k in (4.3), if $k < N$, then we sum over j_N, \dots, j_{k+1} similarly as in the proof of Lemma 3.6, each sum giving rise to a constant factor depending only on δ and $q(\cdot)$. When we sum over $j_k \geq \frac{m_0}{N} - 1$, we need to bound

$$\sum_{j_k \geq \frac{m_0}{N} - 1} (j_k + 1) \sum_{i=0}^{j_k-1} P^{\vec{\eta}_{j_k-1+1}^{(k-1)}}(\vec{X}_{j_k} = \vec{\eta}_i^{(k-1)}),$$

for which we can apply (4.5) with $i = 1$ to obtain a factor of $1/m_0$. Summing over j_{k-1}, \dots, j_1 only leads to bounded constant factors. This verifies (4.2) with $\varphi_m^{(N,1)}$ in place of $\varphi_m^{(N)}$.

The proof of (4.2) with $\varphi_m^{(N)}$ replaced by $\varphi_m^{(N,2)}$ or $\varphi_m^{(N,3)}$ is similar. Of the N paths $\vec{\eta}_{j_{k+1}}^{(k)}$, $1 \leq k \leq N$, one of these will have length at least $j_k \geq \frac{m_0}{N} - 1$. We then draw upon the proofs of Lemmas 3.7 and 3.9 to sum over j_k , $1 \leq k \leq N$, and apart from (4.5), we will also need the following corollary of Lemma 3.8:

$$\begin{aligned} \sup_{u \in \mathbb{Z}^d} \sum_{j=m_0}^{\infty} \frac{(j+i)!}{j!} \sum_{l=0}^{j-1} \sum_{\vec{\eta}_l} P^{\vec{x}_m}(\vec{X}_l = \vec{\eta}_l) 1_{\{\ell(\vec{x}_m \circ \vec{\eta}_l, V_{d_*})=0\}} \sum_{\eta_{l+1}} \kappa |\nu_2(\eta_{l+1} - \eta_l) - \nu_1(\eta_{l+1} - \eta_l)| \\ \times P^{\vec{x}_m \circ \vec{\eta}_{l+1}}(X_{j-l-1} = u) \leq \frac{\varepsilon_\delta \kappa (i+2)! \delta^{-(i+3)} G_q^{*(i+3)}}{m_0}. \end{aligned} \quad (4.6)$$

Note that we would need to apply (4.6) with $i = 1$, which uses $G_q^{*4} < \infty$ from assumption **(A3)**. The details will be left to the reader. \square

4.1 Remark on Assumption **(A3)** in Theorem 1.2

It is possible to replace the assumption that $G^{*4} < \infty$ with a local central limit theorem bound of the form

$$\sup_x \mathcal{P}_q(X_n = x) \leq \frac{C_q}{n^a}, \quad \text{for some } a > 3. \quad (4.7)$$

The usual choice would be $a = \frac{d_1}{2}$, which is greater than 3 when $d_1 > 6$. We give the main ideas of the argument here. For further details see [10].

The bound $G^{*4} < \infty$ was used when obtaining the estimate (4.6). However we only require that the left hand side of (4.6) converges to 0 as $m_0 \rightarrow \infty$. As in the proof of Lemma 3.8 (c.f. (3.56)) this involves estimating

$$\begin{aligned} \sup_u \kappa \sum_{r=1}^2 \sum_{w \in \mathcal{S}_r} \nu_r(w) \sum_{j=m_0}^{\infty} \frac{(j+i)!}{j!} \sum_{s=0}^{j-1} P_{(s,w)}^{\vec{x}_m}(X_j = u) \\ \leq \sup_u \kappa \sum_{r=1}^2 \sum_{w \in \mathcal{S}_r} \nu_r(w) \sum_{l=0}^{\infty} P_q(Y_l = \Pi_{d_1}(u)) \sum_{s=0}^{\infty} \sum_{j=l \vee s \vee m_0}^{\infty} \frac{(j+i)!}{j!} P_{(s,w)}^{\vec{x}_m}(\mathcal{N}_j = l | Y_l = \Pi_{d_1}(u)) \\ \leq \sup_u \kappa \sum_{r=1}^2 \sum_{w \in \mathcal{S}_r} \nu_r(w) \sum_{l=0}^{\infty} P_q(Y_l = \Pi_{d_1}(u)) \sum_{s=0}^{\infty} \sum_{j=l \vee s \vee m_0}^{\infty} \frac{(j+i)!}{j!} P(\mathcal{M}_j - 1 = l). \end{aligned} \quad (4.8)$$

Using the local CLT bound (4.7) this is bounded by

$$\begin{aligned} \varepsilon_\delta \kappa \sum_{l=0}^{\infty} \frac{C_q}{l^a} \sum_{j=l \vee m_0}^{\infty} \frac{(j+i+1)!}{j!} P(\mathcal{M}_j - 1 = l) \leq \varepsilon_\delta \kappa \sum_{l=0}^K \frac{C_q}{l^a} \sum_{j=l \vee m_0}^{\infty} \frac{(j+i+1)!}{j!} P(\mathcal{M}_j - 1 = l) \\ + \varepsilon_\delta \kappa \sum_{l=K}^{\infty} \frac{C_q}{l^a} \sum_{j=l}^{\infty} \frac{(j+i+1)!}{j!} P(\mathcal{M}_j - 1 = l). \end{aligned} \quad (4.9)$$

As in (3.58) the last term of (4.9) is bounded by

$$\varepsilon_{\delta\kappa} \sum_{l=K}^{\infty} \frac{C_q}{l^a} \delta^{-(i+1)} \frac{(l+i+1)!}{l!} \leq \varepsilon_{\delta\kappa} \delta^{-(i+1)} \sum_{l=K}^{\infty} \frac{C'_q}{l^{a-(i+1)}},$$

which can be made arbitrarily small by choosing K large depending on q and $i \in \{0, 1\}$ when $a - (i + 1) > 1$, (i.e. $a > 3$ when $i = 1$). For any K , the first term of (4.9) can be made arbitrarily small by choosing m_0 sufficiently large (see the proof of Lemma 4.1 of [10] for details).

Acknowledgements

MH thanks Remco van der Hofstad and Alain-Sol Sznitman for helpful discussions at the initial stages of this project, and NUS mathematics department for support during his visit in 2009. We thank Jonathon Peterson for pointing out a false statement in an earlier version, and an anonymous referee for pointing out an error in our previous formulation of Theorem 1.1.

References

- [1] G. Ben Arous, A. Fribergh, and V. Sidoravicius. A proof of the Lyons-Pemantle-Peres monotonicity conjecture for high biases. arXiv:1111.5865v1 [math.PR], 2011.
- [2] M.T. Barlow and J.-D. Deuschel. Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.*, 38(1): 234–276, 2010.
- [3] N. Berger and J.-D. Deuschel. A quenched invariance principle for non-elliptic random walk in i.i.d. balanced random environment. arXiv:1108.3995v1 [math.PR], 2011.
- [4] E. Bolthausen, A.-S. Sznitman, and O. Zeitouni. Cut points and diffusive random walks in random environment. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(3):527–555, 2003.
- [5] N. Enriquez and C. Sabot. Random walks in a Dirichlet environment. *Electron. J. Probab.*, 11(31):802–817, 2006.
- [6] A. Fribergh. The speed of a biased random walk on a percolation cluster at high density. *Ann. Probab.*, 38(5):1717–1782, 2010.
- [7] A. Fribergh and A. Hammond. Phase transition for the speed of the biased random walk on the supercritical percolation cluster. arXiv:1103.1371v2 [math.PR], 2011.
- [8] R. van der Hofstad and M. Holmes. An expansion for self-interacting random walks. arXiv:0706.0614v4 [math.PR], 2010. To appear in *Brazilian J. Prob. Stat.*
- [9] R. van der Hofstad and M. Holmes. Monotonicity for excited random walk in high dimensions. *Probab. Theory. Relat. Fields*, 147:333–348, 2010.
- [10] M. Holmes. Excited against the tide: A random walk with competing drifts. To appear in *Ann. Inst. H. Poincaré Probab. Statist.*

- [11] M. Holmes and T. Salisbury. A combinatorial result with applications to self-interacting random walks. *Journal of Combinatorial Theory, Series A* 119:460-475, 2012
- [12] M. Holmes and T. Salisbury. Random walks in degenerate random environments. Preprint, 2011.
- [13] C. Sabot. Ballistic random walks in random environment at low disorder. *Ann. Probab.*, 32(4):2996–3023, 2004.
- [14] A.-S. Sznitman. Topics in random walks in random environment. *School and Conference on Probability Theory*, 203–266, ICTP Lect. Notes, XVII, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.
- [15] A.-S. Sznitman and M. Zerner. A law of large numbers for random walks in random environment. *Ann. Probab.*, 27:1851–1869, 1999.
- [16] O. Zeitouni. Random walks in random environment. In *Ecole d’Eté de Probabilités de Saint Flour 2001, Lecture Notes in Mathematics, no. 1837*. Springer-Verlag, Berlin, 2004.
- [17] M.P.W. Zerner. A non-ballistic law of large numbers for random walks in i.i.d. random environment. *Electron. Comm. Probab.*, 7:191–197 (electronic), 2002.
- [18] M.P.W. Zerner and F. Merkl. A zero-one law for planar random walks in random environment. *Ann. Probab.* 29(4):1716–1732, 2001.