Proof of the WARM Whisker Conjecture for neuronal connections

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Abstract

This paper is devoted to the study of the so-called WARM reinforcement models that are generalisations of Pólya's urn. We show that in the graph setting, once the exponent α of the reinforcement function is greater than 2, the stable and critical equilibria can be supported only on spanning forests, and once $\alpha > 25$, on spanning whisker forests. Thus we prove the whisker forests conjecture from [6].

"The rich get richer" is a catchphrase that is often used to describe various real-world reinforcement processes, such as the formation of social networks, or market share. Random processes with reinforcement, where the outcomes of the first steps of the process can heavily influence the asymptotic behaviour, have been studied for over a century. Their study continues to be a highly active area of research (see e.g. [3, 10, 5, 11]). Perhaps the first such model is the classical Pólya urn, where at each step of the process a ball is chosen from an urn and replaced together with another of the same colour.

Various methods have been adopted or developed to study reinforcement processes, depending on the particular model of interest [12]. One such method involves approximating the stochastic dynamics by deterministic dynamics [2]. This method has proved to be useful in studying a large class of reinforcement processes called WARMs [6], which include processes defined

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on graphs that are toy models for the formation of neuronal architecture in the brain.

In this paper we consider a class of dynamical systems whose stable fixed points are the possible limit points of WARMs. By studying a Lyapunov function for the flow, we prove various properties of the set of stable fixed points when the strength of reinforcement is large. In particular we verify a conjecture of [6], showing that when the reinforcement is very strong, stable fixed points of the dynamics are supported on forests whose components have diameter at most 3.

1 Introduction and main results

A WARM (W, A-reinforcement model) is one of a class of reinforcement processes introduced in [6]. One can consider some of these processes as toy models for reinforcement of neuronal connections in the brain, or for competitions between companies producing and selling different types of goods.

Roughly speaking, WARMs are urn models with interacting urns, where each colour may be present in more than one urn. At each step t of the process a random subset A_t of colours is chosen to compete against each other for one step of a Polya urn process with weight/reinforcement function $W : \mathbf{N} \to (0, +\infty)$. In the context of neuronal connections the set of colours E is the set of edges of a graph, and the set A_t is the set of (undirected) edges incident to a randomly selected vertex V_t in that graph. In the setting of competing companies, A_t is the set of companies selling a product of a given type; successful selling makes the company larger and more competitive in all of the markets where it acts.

One of the main problems of interest is to determine the possible limiting vectors of proportions of balls of each colour in the urn. In the graph context this describes the possible neuronal architectures that can result from the process.

Asymptotically, the random dynamics of the reinforcement process can be approximated by a dynamical system (as we discuss below, see Section 2). Indeed it has been conjectured in [6] that the set of possible limits is supported on the collection of linearly stable and critical equilibria of this system. This has been proved for generic sets of parameters in [1].

In this paper we prove the other main conjecture of [6], describing the linearly stable equilibria for any finite graph G, when $W(n) = n^\alpha$ and at each step of the process we choose V_t uniformly at random from all vertices. Among other things, we prove (i) for α sufficiently large all of the linearly stable equilibria are supported on so-called whisker-forests (that is, forests of trees of diameter at most 3; see Figure 1 and Definition 5 below), and (ii) every whisker graph (and hence every whisker forest) supports a stable equilibrium for α sufficiently large (see Theorem 2 below), thus confirming [6, Conjecture 2].

1.1 The model

Let us now explicitly define a WARM, which consists of:

- a finite set $E = \{1, 2, \dots, n\}$ of *colours* (or *edges* of a finite graph, or *axons*);
- a probability distribution $\mathbf{p} = \{p_A\}_{A \subset E}$ on the nonempty subsets of E;
- a number $\alpha > 1$, defining a *reinforcement function* $W(x) = x^\alpha$;
- *initial counts* $(N_0^{(i)})_{i \in E}$ for all the colours (or excitations of axons).

The process $(N_s^{(i)})_{i \in E,\, s \in \mathbf{Z}_+}$ is then a random evolution of a vector $\vec{N}_s = (N_s^{(i)})_{i \in E}$, containing these counts, that is defined as follows:

- Take a family of i.i.d. random variables $(A_s)_{s \in \mathbf{N}}$, distributed with the prescribed law: $\mathbb{P}(A_s = A) = p_A$ for all nonempty $A \subset E$ ($p_\varnothing = 0$).
- At every time $s \in \mathbf{N}$, select a random color $I_s \in A_s$ to be reinforced, with the probability of taking $i \in A_s$ proportional to $W(N_s^{(i)})$. In other words, at time s conditional on $\mathcal{F}_{s-1} \equiv \sigma(\vec{N}_\tau, A_r : \tau \le s-1, r \le s)$, the probability that we select a ball of colour $i \in A_s$ is given by

$$\mathbb{P}(I_s = i \mid \mathcal{F}_{s-1}) = \begin{cases} \dfrac{W(N_{s-1}^{(i)})}{\sum_{j \in A_s} W(N_{s-1}^{(j)})}, & \text{if } i \in A_s \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

- The count $N^{(I_s)}$ is then increased by one, while the others are unchanged:

$$N_s^{(i)} = \begin{cases} N_{s-1}^{(i)} + 1, & \text{if } i = I_s \\ N_{s-1}^{(i)} & \text{otherwise.} \end{cases}$$

The main quantity of interest is the random vector $\vec{X}_s = (X_s^{(i)})_{i\in E}$ of proportions

$$X_s^{(i)} = \frac{N_s^{(i)}}{\sum_{j\in E} N_s^{(j)}}, \tag{2}$$

or more specifically, the limiting behaviour of this vector as the number of iterations of this process goes to infinity, that is, the limit $\lim_{t\to\infty} \vec{X}_s$.

Remark 1. A particular choice of the initial counts $(N_0^{(i)})_{i\in E}$ is not important for the purposes of this paper (provided that all of these numbers are positive), so henceforth we take $N_0^{(i)} = 1$ for each $i \in E$.

Remark 2. Note that we lose no generality assuming that for each $i \in E = \{1, \dots, n\}$ there exists $A \ni i$ such that $p_A > 0$, since if some i has no probability of ever being chosen we can just remove it from E and relabel the remaining colours as $\{1, \dots, n-1\}$.

Of particular interest is the graph setting, where E is the set of *undirected* edges of a finite graph G. Here we will typically assume (as in [6]) the following condition:

Condition G: For a finite graph $G = (V, E)$ with edge set E and vertex set V we take A_t to be the set of edges incident to a uniformly chosen vertex in V.

Occasionally we will also consider the situation where A_t is still the set of edges incident to a vertex V in a graph G, but where the vertex selection is not *uniform*. We refer to this situation as the *generalized* graph setting.

Note that the fact that the edges are undirected is what gives an interacting model on a graph. Indeed, using directed edges in this context (where a directed edge (x, y) is only reinforced when vertex x is selected) would reduce the problem to a collection of independent urns (vertices).

1.2 Main results

Let $\mathbf{p}$ and $\alpha > 1$ be fixed. We are going to study the asymptotic behaviour of the (W, A)-reinforcement model using a specific *deterministic* autonomous flow

$$\dot{x}_i = -x_i + \sum_{A\ni i} p_A \frac{x_i^\alpha}{\sum_{j\in A} x_j^\alpha}, \quad i \in E. \tag{3}$$

This flow is considered on the space of probability distributions (i.e. proportions vectors)

$$\Delta_n(\mathbf{p}) = \left\{ \vec{u} \in \mathbf{R}^n : u_i \geq 0, \sum_{i=1}^{n} u_i = 1 \right\} \cap \mathcal{B}(\mathbf{p}),$$

where

$$\mathcal{B}(\mathbf{p}) := \left\{ \vec{u} \in \mathbf{R}^n : \sum_{j \in A} u_j > 0 \text{ for each } A \subset E \text{ such that } p_A > 0 \right\}.$$

In a sense, this deterministic flow corresponds to the *expected/average* dynamics of the random process. Relating the properties of the deterministic and random systems is in general a non-trivial problem. The relation is provided by Theorem 0 below (see e.g. [1, 4, 6]). For completeness of the exposition we also present a rough sketch of the arguments linking these two systems in Section 2.

We denote by F the vector field defined by the right hand side of (3):

$$F(\vec{x})_i = -x_i + \sum_{A \ni i} p_A \frac{x_i^\alpha}{\sum_{j \in A} x_j^\alpha}. \tag{4}$$

Though it is natural to define F only for $\vec{x} \in \Delta_n(\mathbf{p})$, we extend this definition (by (4)) to the full (hyper)octant $[0, +\infty)^n \cap \mathcal{B}(\mathbf{p})$.

Definition 1 (Equilibrium)**.** For fixed n, a vector $\vec{x} \in \Delta_n(\mathbf{p})$ is an *equilibrium distribution* for the WARM if $F(\vec{x}) = \vec{0}$, i.e. if

$$x_i = \sum_{A \ni i} p_A \cdot \frac{x_i^\alpha}{\sum_{j \in A} x_j^\alpha}, \qquad \text{for each } i \in E. \tag{5}$$

We let $\mathcal{E}$ denote the set of equilibria for a given WARM.

Intuitively (5) says that for each i, the proportion of balls of colour i in the urn is equal to the probability that the next selected ball is of colour i.

Together with the flow (3) and the set of its equilibria, it is natural to study their stability, and in the first instance, the linear approximation. To do so, let $\mathbf{J}(\vec{x})$ be the Jacobian matrix of partial derivatives of F, that is, let $J_{i,k}(\vec{x}) = \partial F(\vec{x})_i / \partial v_k$. We then have the following standard definition:

Definition 2 (Stable equilibrium)**.** An equilibrium distribution $\vec{x} \in \mathcal{E}$ is: a *linearly-stable equilibrium* if all eigenvalues of $\mathbf{J}(\vec{x})$ have negative real parts; a *linearly-unstable equilibrium* if some eigenvalue of $\mathbf{J}(\vec{x})$ has positive real part; and a *critical equilibrium* otherwise. We denote the set of linearly-stable equilibria for a given WARM by $\mathcal{S}_0$, and its union with the set of critical ones by $\mathcal{S}$.

Our results concern the structure of the set $\mathcal{S}$ for fixed $\mathbf{p}$, as α varies. In particular we are interested in the sets of edges on which such equilibria can be *supported*, i.e. the edges that are chosen a positive proportion of the time (see Definition 6 and Theorem 2 below). As we have already mentioned, the link between the deterministic and random systems is expressed in the following theorem (and conjecture), which is proved for example in [1] (see also [6]).

For a given WARM, let $\mathcal{A}$ denote the (random, nonempty) set of accumulation points of the sequence $\vec{X}_t$.

Theorem 0 (Accumulation structure, [1, 6])**.** *Let* $\mathbf{p}$ *and* $\alpha > 1$ *be fixed. Then*

(i) almost surely $\mathcal{A} \subset \mathcal{E}$ *and* $\mathcal{A}$ *is a connected subset of* $\Delta_n(\mathbf{p})$*,*

(ii) $\mathbb{P}(\vec{X}_t \to \vec{x}) > 0$ *for every* $\vec{x} \in \mathcal{S}_0$*.*

It follows from Theorem 0(i) that if the flow (4) admits only finitely many equilibria ($|\mathcal{E}| < \infty$) then almost surely the same holds for the number of accumulation points of the process X_t, and hence (due to the connectedness of $\mathcal{A}$) this process almost surely converges. Moreover, if $|\mathcal{E}| = 1$ then $\vec{X}_t$ converges almost surely to this unique equilibrium.

We believe that the conclusion of this theorem can be strengthened:

Conjecture 1 (Convergence to equilibrium)**.** *Fix* $\mathbf{p}$ *and* $\alpha > 1$*. Then there exists a random vector* $\vec{X} = (X^{(i)})_{i \in E}$*, whose law is supported on the set of linearly-stable and critical equilibria, such that* $\mathbb{P}(\vec{X}_t \to \vec{X}) = 1$*.*

Conjecture 1 is proved in [1] for *generic* sets of parameters $\mathbf{p}$, α (i.e. for an open set of parameters of full Lebesgue measure). The following is also proved therein:

Theorem 1 (non-convergence to unstable equilibria [1])**.** *Fix* $\mathbf{p}$ *and* $\alpha > 1$*. If* $\vec{x}$ *is a linearly unstable equilibrium then* $\mathbb{P}(\vec{X}_t \to \vec{x}) = 0$*.*

Note that the classical Polya urn with two colours is the case where $p_A = 1$ for $A = \{1, 2\}$ and $\alpha = 1$. This model admits a continuous family of equilibria (any point $(u, 1-u) \in \Delta_2$ is a critical equilibrium). The existence of the (random) limit is usually proved using the martingale convergence theorem (and thus requires a different argument compared to the situation with a finite $\mathcal{E}$). In general the behaviour for $\alpha = 1$ is quite different from the "cruel world" setting $\alpha \gg 1$, as will be shown in [8, 9] and also [7].

Our first main result (Theorem 2 below) is to resolve and extend the main conjecture of [6] (the so-called *WARM whisker conjecture* [6, Conjecture 2]) that takes place in the graph setting, under Condition G. In order to state this result we introduce the notion of (marked) whisker graphs.

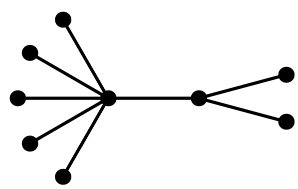

Figure 1: A $(5, 2)$-whisker graph, which has $n = 5 + 1 + 2$ edges

Definition 3. A *whisker graph* is a tree graph with diameter at most 3.

A whisker graph of diameter three can be viewed as two star graphs pieced together by an edge between the respective central vertices, see Figure 1. If the two star graphs had r and s edges respectively then the result of adding a connecting edge (let us henceforth call this edge the *central edge*) between the two central vertices is called an (r, s)-whisker graph, and it is a graph with $r + s + 1$ edges.

The cases of diameter 2 and 1 can be seen as degeneracies where one or both r and s are equal to zero, and correspond to a single star graph and a single-edge graph respectively. Finally, we will introduce a marking of the whisker graph.

Definition 4. A *marked* whisker graph is a whisker graph on which one of its edges, to which we refer as $*$, is marked. This edge is the central one for a non-degenerate (i.e. diameter 3) whisker graph and can be any of its edges for a degenerate one.

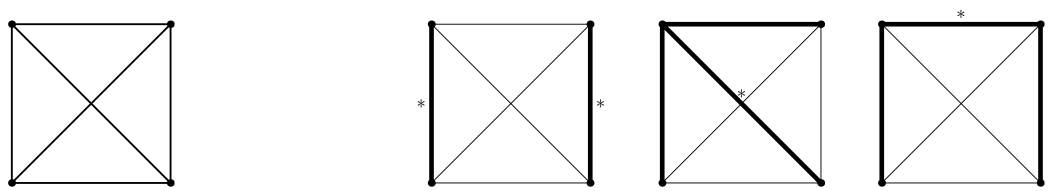

Figure 2: An illustration of Theorem 1 for the complete graph G on 4 vertices (left). For large α the set of edges chosen a positive proportion of the time by the WARM process on G is up to a permutation (a random) one of the whisker forests on the right. Note that in the star graph case any of the edges could be marked.

This definition might seem strange at first glance; it has a more "universal" formulation: in a marked whisker graph any non-marked edge is attached to a leaf. This notion appears in the description in Theorem 2(b) below — and is exactly how the marking appears in the study of such equilibria.

We will need the standard definition of a spanning forest; combining it with the definition of a whisker, we obtain the definition of a *whisker forest*:

Definition 5. For fixed G, a collection $\mathbf{G} = (G_i)_{i=1}^k$ of disjoint (i.e. $V_i \cap V_j = \varnothing$ if $i \neq j$) trees $G_i = (V_i, E_i) \subset G$ such that $\cup_{i=1}^k V_i = V$ is called a *spanning forest*. For a spanning forest $\mathbf{G}$ we let $\mathbf{E} = \cup_{i=1}^k E_i$. If every component G_i of a spanning forest $\mathbf{G}$ is a whisker graph then $\mathbf{G}$ is called a *whisker forest*.

As our goal is to describe the structure of equilibria that are not linearly unstable, we will in particular consider the *supports* of such equilibria:

Definition 6. For $\vec{x} \in \Delta_n(\mathbf{p})$ we denote the *support* of $\vec{x}$ by

$$\sigma(\vec{x}) = \{j \in E : x_j > 0\}.$$

Also, given a whisker forest $\mathbf{G}$, we say that $\vec{x}$ *is supported on* $\mathbf{G}$, if $\sigma(\vec{x})$ is exactly the union $\mathbf{E}$ of edges of $\mathbf{G}$.

We can now state our first main result (see Figure 2) for an illustration:

Theorem 2 (WARM Whisker theorem)**.** *Assuming Condition G:*

(a) There exists $\alpha_W \leq 25$ *such that for any finite graph* G *and any* $\alpha > \alpha_W$*, every not-linearly-unstable equilibrium* $\vec{x} \in \mathcal{S}$ *of* (4) *is supported on a whisker forest.*

(b) *For any* marked *whisker graph G there is an $\alpha_G > 1$ such that for all $\alpha > \alpha_G$, G supports a unique linearly stable equilibrium $\vec{x} \in \mathcal{S}_0$, such that $x_* > x_i$ for every $i \in E \setminus \{*\}$.*

In [6] it was shown that any *symmetric* (i.e. $s = r$) whisker G (and any star graph G) supports a linearly stable equilibrium for sufficiently large α, from which it follows as in [6, Theorem 3] that for any G, any spanning forest $\mathbf{G}$ consisting of components that are (non-empty) stars and *symmetric whiskers* supports a stable equilibrium (for α sufficiently large). Theorem 2(b) (together with [6, Theorem 3]) shows that in fact any (spanning) whisker forest supports a stable equilibrium for large α.

Theorem 2(a) is in fact stronger than [6, Conjecture 2(ii)] since here α_W does not depend on the graph G. When combined with [6, Proposition 3 and Theorem 6], Theorem 2(b) is a slightly stronger statement than [6, Conjecture 2(i)] since it further characterises the stable equilibrium in terms of a marked edge.

As in the following example, the conclusion of Theorem 2(a) fails in the generalised-graph setting if one allows differing vertex selection probabilities.

Example 1. Set $p_{\{1\}} = 1/15$, $p_{\{1,2\}} = 2/15$, $p_{\{2,3\}} = 3/15$, $p_{\{3,4\}} = 4/15$, $p_{\{4\}} = 5/15$, and $p_A = 0$ otherwise. This is equivalent to a simple path graph with 4 edges and 5 vertices (see Figure 3) in which vertices are chosen with increasing probabilities from left to right (instead of uniformly at random). One can prove (as in the proof of Theorem 2(b)) that for all α sufficiently large there is a linearly stable equilibrium $\vec{x}$ with $\sigma(\vec{x}) = \{1, 2, 3, 4\}$ (in fact $\vec{x} \approx (1, 2, 3, 9)/15$).

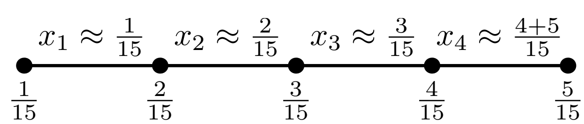

Figure 3: A simple path graph with edges $E = \{1, 2, 3, 4\}$ with non-uniform vertex selection probabilities $(1, 2, 3, 4, 5)/15$ respectively (see Example 1). For large α there exists $\vec{x} \in \mathcal{S}$ with $x_i > 0$ for all $i \in E$.

Our other main results are for the more general setting, described by $E = \{1, \ldots, n\}$ and $\mathbf{p} = (p_A)_{A \subset E}$. In this general setting, we introduce the following definition:

Definition 7. A set $A^* \subset E$ is said to be $\mathbf{p}$-*leaf free* if for every $A \subset E$ with $p_A > 0$, $|A^* \cap A| \neq 1$.

The name is motivated by the generalized (i.e. where vertex selection need not be uniform) graph setting: if A is the set of edges incident to a vertex V such that $|A \cap A^*| = 1$, then V is a leaf (i.e. a vertex of degree 1) for the set of edges A^*. We then have the following.

Theorem 3. *Fix* $\mathbf{p}$. *Then for* $\alpha > 2$ *the support of any non-linearly-unstable equilibrium does not contain a leaf-free subset, i.e. for every* $\vec{x} \in \mathcal{S}$, *if* $A^* \subset \sigma(\vec{x})$ *then* A^* *is not* $\mathbf{p}$-*leaf free.*

When applied to the *generalized* graph setting Theorem 3 yields the following corollary.

Corollary 1. *For any* $\alpha > 2$ *and for any finite graph* G, *any stable equilibrium of* (4) *is supported on a (spanning) forest.*

Indeed, it is easy to see that a graph is a forest if and only if it contains no cycles, and any cycle in the support of $\sigma(\vec{x})$ would provide a leaf-free subset, forbidden by Theorem 3.

It is not clear how the notion of a cycle should be extended to a non-graph setting, where a colour can appear in more than two urns (for the reader familiar with this terminology, this setting can considered as a hypergraph one). For instance, the following example shows that if one considers a cycle to be a cyclic sequence of sets A_i ("vertices") such that $A_i \cap A_{i+1} \neq \emptyset$, then containing a cycle is not the same as containing a leaf-free subset.

Example 2. Let $E = \{1, 2, 3\}$, and $\mathbf{p}$ be such that $p_A > 0$ if and only if $A \in \{A_1, A_2, A_3, A_4\}$, where $A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 1\}, A_4 = \{1\}$ (see Figure 4).

Then E contains a $\mathbf{p}$-cycle in the sense that $2 \in A_1 \cap A_2$, $3 \in A_2 \cap A_3$ and $1 \in A_3 \cap A_1$, but E contains no $\mathbf{p}$-leaf-free subset as follows: If $1 \in A$ then $A \cap A_4 = A \cap \{1\} = \{1\}$. If $1 \notin A$ but $2 \in A$ then $A \cap A_1 = A \cap \{1, 2\} = \{2\}$. Similarly if $1 \notin A$ but $3 \in A$ then $A \cap A_3 = A \cap \{1, 3\} = \{3\}$.

It turns out that for large α, all colours (edges) that "survive" in a stable equilibrium have a tendency to dominate urns (vertices). To be more precise, we introduce the following.

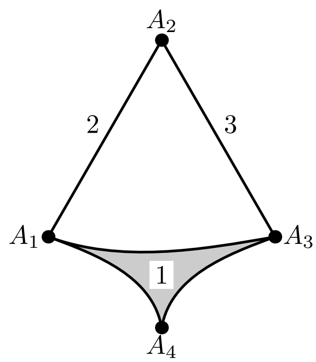

Figure 4: Three colours $\{1, 2, 3\}$ reinforced by four urns; the colour 1 appears in three urns and is shown by a triangle.

Definition 8 (Champions and ordered equilibria)**.** Fix $\alpha > 1$ and $\mathbf{p}$. Let $\vec{x}$ be an equilibrium for $\mathbf{p}$ and $p_A > 0$. We say that $i \in A$ is an $\vec{x}$-*champion* for A if ($A = \{i\}$ or)

$$x_i^\alpha > \sum_{j \in A \setminus \{i\}} x_j^\alpha.$$

If such an i exists (for a given pair $(\vec{x}, A)$), denote it by $i(A)$. We write

$$\gamma(\vec{x}) = \{i(A) : p_A > 0,\ A \text{ has a champion}\}$$

for the set of $\vec{x}$-champions. Finally, an equilibrium $\vec{x}$ is said to be $\mathbf{p}$-*ordered* if every A with $p_A > 0$ has a champion.

Note that if i is a champion for A then $x_i > x_j$ for each $j \in A \setminus \{i\}$.

Theorem 4. *Fix* $\mathbf{p}$*. Then*

1. *for* $\alpha > 2$*, if* $\vec{x} \in \mathcal{S}$ *then every* $\vec{x}$*-supported colour is a champion (i.e.* $\gamma(\vec{x}) = \sigma(\vec{x})$*), and*
2. *there exists* $\alpha(\mathbf{p})$ *such that for all* $\alpha > \alpha(\mathbf{p})$*, every* $\vec{x} \in \mathcal{S}$ *is* $\mathbf{p}$*-ordered.*

1.3 Further examples

Let us consider some further examples in a general (non-graph) setting. Both Examples 3 and 4 below are quite symmetric. As we will see, increasing of α leads to a "breaking of symmetry": while the system itself is obviously symmetric, individual stable equilibria for large α are not symmetric at all.

Example 3. Fix $m \le n$ and choose A_t from all subsets of size m, uniformly at random, i.e. $p_A = \binom{n}{m}^{-1}$ for all A of size m and $p_A = 0$ otherwise. For any α, and any stable equilibrium $\vec{x}$ we have $|\sigma(\vec{x})| \ge n - m + 1$ since at least $n - m + 1$ colours are each drawn a positive proportion of the time. The set of equilibria includes several symmetric ones: for any $k \ge n - m + 1$,

$$\vec{x} = \frac{1}{k}(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k}) \in \mathcal{E}.$$

As α increases, they gradually lose stability: it was shown in the original version of [6][1] that such an equilibrium is stable if and only if $\alpha < \alpha_0(m, n)$ for some explicit $\alpha_0(m, n)$. At the same time, the behaviour of equilibria for *large* α's is described by the following simple corollary of Theorems 3 and 4.

Corollary 2. *Fix $n \ge 3$ and $2 \le m \le n - 1$ in Example 3, let $\vec{x} \in \mathcal{S}$. Then*

1. *for all $\alpha > 2$, $|\sigma(\vec{x})| = n - m + 1$, and*
2. *for all α sufficiently large, all non-zero components of $\vec{x}$ are distinct.*

Example 4. Fix $p \in (0, 1)$; independently choose each colour to be in A_t with probability p, and condition on A_t being nonempty, that is, let

$$p_A = \delta^{-1} p^{|A|} (1 - p)^{n - |A|}$$

for every nonempty A, where $\delta^{-1} = (1 - (1 - p)^n)^{-1}$ is the normalisation constant.

The above gives an easy example of where every colour is chosen a positive proportion (at least $\delta^{-1} p(1-p)^{n-1}$) of the time. At the same time, since every subset $\{i, j\}$ of size two has $p_{\{i,j\}} > 0$, the following "symmetry breaking" is an immediate consequence of Theorem 4.

Corollary 3. *Fix $n \ge 2$ and $p \in (0, 1)$ in Example 4 and let $\vec{x} \in \mathcal{S}$. Then for all α sufficiently large, all components of $\vec{x}$ are distinct and non-zero.*

[1] See: arXiv:1406.0449v1, Corollary 3.1

1.4 Open problems

Apart from Conjecture 1, which as stated remains open, there are various natural generalisations of these models that we believe warrant further study. Examples include:

Question 1. What can be said about the equilibria on finite graphs for $\alpha < 2$?

Question 2. What happens if we add colour dependent (and possibly random) numbers of balls of various colours when colour i is chosen?

Question 3. Can one relax the assumption that the subset selections A_t should be identically distributed? For instance, what happens if the laws $\{p_A(t)\}_{A\subset E}$ converge to $\{p_A\}_{A\subset E}$? Change quasi-periodically in t?

Question 4. For the particular case of $W(n) = n^\alpha$, the reinforcement probabilities depend in fact not on the number of the balls of different colours in the reinforced urn, but only on the proportions. What happens if we alter the model, taking instead of a function $W : \mathbb{N} \to [0, +\infty)$ a function $w : [0, 1] \to [0, \infty)$, reinforcing the color i in an urn A with the probability $\frac{w(X^{(i)})}{\sum_{j\in A} w(X^{(j)})}$, where $X^{(i)} = N^{(i)}/\sum_{j\in E} N^{(j)}$ are the relative proportions of colours in the urn?

Question 5. One of the key arguments of the paper is that the averaged flow F becomes a gradient flow after a change of variables. This is checked below using explicit computations; is there a deeper reason behind that?

Question 6. Is $\alpha = 1$ the only case where $\mathcal{E}$ might be infinite?

Question 7 (C. Hirsch). One can generalize the WARM model to infinite graphs by attaching a Poisson clock independently at each vertex and reinforcing one of the edges adjacent to a vertex A when the clock at A rings. What can be said about the asymptotic behaviour of such a process on $\mathbb{Z}$, $\mathbb{Z}^d$, or regular trees?

Organisation

The remainder of the paper is organised as follows. Section 2 is devoted to the averaged flow and its properties. In Section 3 we prove the general result, Theorem 3. In Section 5 we restrict ourselves to the graph setting assuming Condition G and prove Theorem 2. In Section 4 we prove Theorem 4.

2 Discussion and properties of the flow

In this section we discuss the relevance of the deterministic flow to the stochastic system, and properties of this flow. The reader who is already familiar with the former may wish to skip Section 2.1 and proceed directly to Section 2.2.

2.1 Averaging flow

Fix the probabilities distribution $\mathbf{p}$ and the power $\alpha > 1$ defining the reinforcement probabilities. Let us approximate the (random) dynamics of the vector X_s of proportions by a deterministic flow. Namely, take a small $\delta > 0$, a large t and consider the change of proportions between the two times $s_1 = e^t - n_0$ and $s_2 = e^{t+\delta} - n_0$, where $n_0 := \sum_{i\in E} N_0^{(i)}$ is the initial total count for all the colours. Due to the relation $\sum_{i\in E} N_s^{(i)} = s + n_0$, the total count at time s_2 is exactly e^δ times that at time s_1.

Now note that the proportions $X_s^{(i)}$ change at most by δ during this time. Indeed, for any $s \in [s_1, s_2]$ we have

$$X_s^{(i)} = \frac{s_1 + n_0}{s + n_0} X_{s_1}^{(i)} + \frac{s - s_1}{s + n_0} \widetilde{X}_{[s_1,s]}^{(i)} = X_{s_1}^{(i)} + \frac{s - s_1}{s + n_0} (\widetilde{X}_{[s_1,s]}^{(i)} - X_{s_1}^{(i)}), \tag{6}$$

where

$$\widetilde{X}_{[s_1,s]}^{(i)} := \frac{1}{s - s_1} \#\{\tau \in (s_1, s] \mid I_\tau = i\} \tag{7}$$

is the proportion of the time that edge i is reinforced between the time moments s_1 and s. The fact that $\widetilde{X}_{[s_1,s]}^{(i)} \in [0, 1]$ for all s and i, the inequality $\frac{s_1+n_0}{s+n_0} \geq \frac{s_1+n_0}{s_2+n_0} = e^{-\delta}$, the inequality $\frac{s-s_1}{s+n_0} \leq \frac{s_2-s_1}{s_2+n_0} = 1 - e^{-\delta}$ and (6) then imply

$$X_s^{(i)} \in [e^{-\delta} X_{s_1}^{(i)}, X_{s_1}^{(i)} + (1 - e^{-\delta})(1 - X_{s_1}^{(i)})] \subset [X_{s_1}^{(i)} - \delta, X_{s_1}^{(i)} + \delta] \tag{8}$$

(the last inclusion uses $\delta > 1 - e^{-\delta}$).

Now, the proportion (7) is an arithmetic mean of $s - s_1$ Bernoulli random variables $\mathbb{1}_{\{I_\tau = i\}}$. Though they are not independent, a coupling argument shows that when $s - s_1$ is large, this arithmetic mean is very likely close to its expected value (which is the mean of the expectations). We have

$$\mathbb{P}(I_\tau = i | \vec{N}_{\tau-1}) = \sum_{A \ni i} p_A \frac{(N_{\tau-1}^{(i)})^\alpha}{\sum_{j\in A} (N_{\tau-1}^{(j)})^\alpha} = \sum_{A \ni i} p_A \frac{(X_{\tau-1}^{(i)})^\alpha}{\sum_{j\in A} (X_{\tau-1}^{(j)})^\alpha}, \tag{9}$$

and as $X^{(i)}_\tau$ do not change much on $[s_1, s_2]$ due to (8), the arithmetic mean $\widetilde{X}^{(i)}_{[s_1,s]}$ is very likely close to the ith component of the vector $f(X_{s_1})$, where the map f from $\Delta_n(\mathbf{p})$ to itself is defined by

$$f(\vec{x})_i = \sum_{A \ni i} p_A \frac{(x_i)^\alpha}{\sum_{j\in A}(x_j)^\alpha}. \tag{10}$$

Finally, substituting this into (6) for $s = s_2$, we get

$$X^{(i)}_{s_2} \approx X^{(i)}_{s_1} + (1 - e^{-\delta})(-X^{(i)}_{s_1} + f(X_{s_1})_i) \approx X^{(i)}_{s_1} + \delta(-X^{(i)}_{s_1} + f(X_{s_1})_i).$$

The above informal argument motivates the consideration of a vector field F and an autonomous flow on $\Delta_n(\mathbf{p})$, defined by

$$\dot{x}_i = F(x)_i := -x_i + f(x)_i = -x_i + \sum_{A \ni i} p_A \cdot \frac{x_i^\alpha}{\sum_{j\in A} x_j^\alpha},$$

that is exactly the flow (4). We cite in the next section the results formally linking this flow to the WARM reinforcement model.

2.2 Lyapunov function and other properties of the flow

Repeating the arguments from [6], note, that the flow (4) admits a global Lyapunov function, and moreover, up to a coordinate change, is an (anti)gradient one. Namely, consider the function

$$L(\vec{x}) := \sum_{i=1}^n x_i - \frac{1}{\alpha} \sum_{A\subset E} p_A \log\left(\sum_{j\in A} x_j^\alpha\right). \tag{11}$$

An easy computation then shows the following.

Lemma 1. *The vector field F on the full (hyper)octant $[0, +\infty)^n \cap \mathcal{B}(\mathbf{p})$ can be represented as*

$$F(\vec{x})_i = -x_i \frac{\partial}{\partial x_i} L. \tag{12}$$

Proof. For any $i \in E$ and $A \ni i$, denote by q_{iA} the part of the reinforcement coming from the urn A that gets the colour i:

$$q_{iA} = q_{iA}(\vec{x}) := \frac{x_i^\alpha}{\sum_{j\in A} x_j^\alpha}. \tag{13}$$

Then,

$$q_{iA}(\vec{x}) = \frac{1}{\alpha} x_i \cdot \frac{\partial}{\partial x_i} \log \sum_{j \in A} x_j^{\alpha}. \tag{14}$$

Multiplying (14) by p_A and summing over all $A \ni i$, we get the desired (12). ■

Corollary 4. *The function L is indeed a Lyapunov function for the flow generated by the vector field F: it decreases strictly along the trajectories that are not the equilibrium points.*

Proof. Indeed, from (12) for any non-constant solution $\vec{x}(t)$ we get

$$\frac{d}{dt} L(\vec{x}(t)) = -\sum_{i \in E} \dot{x}_i \frac{\partial L}{\partial x_i} = -\sum_{i \in E:\, x_i > 0} \frac{\dot{x}_i^2}{x_i} < 0.$$

■

Also, we get

Lemma 2. *The flow (4) after a change of coordinates $y_i = 2\sqrt{x_i}$ becomes exactly the anti-gradient flow, associated to the function L (re-written in these coordinates).*

Proof. Indeed,

$$\dot{y}_i = \frac{\dot{x}_i}{\sqrt{x_i}} = \sqrt{x_i} \frac{\partial L}{\partial x_i} = \frac{dx_i}{dy_i} \frac{\partial L}{\partial x_i} = \frac{\partial L(\vec{x}(\vec{y}))}{\partial y_i}.$$

■

The eigenvalues of the gradient flow at an equilibrium point are always real, as the corresponding Jacobian matrix is symmetric. The change of coordinates does not change the eigenvalues of the linearization, and thus we conclude the following.

Corollary 5. *The eigenvalues of the Jacobian matrix $J(\vec{x})$ at any equilibrium point $\vec{x} \in \mathcal{E}$ are real.*

Finally, we note that there is no difference between considering the stability of an equilibrium $\vec{x} \in \mathcal{E}$ in the simplex $\Delta_n(\mathbf{p})$ and in the full (hyper)octant $[0, +\infty)^n \cap \mathcal{B}(\mathbf{p})$. This is immediate for the eigenvalues (and this is the only conclusion that we will need for this paper):

Lemma 3. *Let $\vec{x} \in \mathcal{E}$ be an equilibrium. The eigenvalues of $J(\vec{x})$ in the full (hyper)octant $[0, +\infty)^n \cap \mathcal{B}(\mathbf{p})$ are then the eigenvalues of its restriction on the tangent plane to $\Delta_n(\mathbf{p})$ and an additional eigenvalue that is always equal to (-1).*

Proof. For $\vec{x} \in \mathcal{E}$ and any $r > 0$ we have $f(r\vec{x}) = f(\vec{x}) = \vec{x}$. The ray $\{r\vec{x} \mid r > 0\}$ is thus invariant, and the restriction of the flow F on it gives $\dot{r} = -r + 1$, providing the additional eigenvalue (-1) (with the corresponding eigenvector that is equal to $\vec{x}$). ■

3 Proof of Theorem 3

Recall (13). The following proposition is one of the key steps of this paper:

Proposition 1. *Let $\mathbf{p}$ and $\alpha > 1$ be fixed, and assume that $\vec{x} \in \mathcal{S}$. Then for any $i \in \sigma(\vec{x})$ there exists A such that*

$$q_{iA}(\vec{x}) \geq 1 - \frac{1}{\alpha}. \tag{15}$$

Proof. Assume that an equilibrium $\vec{x} \in \mathcal{E}$ is not linearly-unstable. Then, the eigenvalues of its linearization are nonpositive. Such a statement is invariant under the change of coordinates in Lemma 2, and due to that Lemma the flow F in the $\{y_i\}$ coordinates is the anti-gradient flow, associated to the function L. In the latter coordinates, the Jacobian matrix of F is in fact the (symmetric) Hessian matrix of second partial derivatives of $(-L)$. Its eigenvalues are nonpositive if and only if this matrix defines a non-positive quadratic form. In particular, the second derivative of $(-L)$ in any direction should be non-positive.

Assuming that $x_i > 0$, let us check the sign of the second derivative of L in the direction of x_i. Note that this number has the same sign as

$$x_i \frac{\partial}{\partial x_i} \left(x_i \frac{\partial}{\partial x_i} L \right) = x_i^2 \frac{\partial^2 L}{\partial x_i^2} + x_i \frac{\partial L}{\partial x_i} = x_i^2 \frac{\partial^2 L}{\partial x_i^2}, \tag{16}$$

where the later equality is due to the fact that as an equilibrium, the point $\vec{x}$ is critical for L. (The left hand side of (16) can be seen as the second derivative in the logarithmic coordinate $y_i = \log x_i$.)

Let us evaluate the left hand side of (16). Namely, for any $A \ni i$ we have

$$x_i \frac{\partial}{\partial x_i} q_{iA} = -x_i \frac{\partial}{\partial x_i}(1 - q_{iA}) = -x_i \frac{\partial}{\partial x_i} \left(\frac{\sum_{j \in A \setminus \{i\}} x_j^\alpha}{x_i^\alpha + \sum_{j \in A \setminus \{i\}} x_j^\alpha} \right)$$
$$= \frac{\sum_{j \in A \setminus \{i\}} x_j^\alpha}{x_i^\alpha + \sum_{j \in A \setminus \{i\}} x_j^\alpha} \cdot \frac{\alpha x_i^\alpha}{x_i^\alpha + \sum_{j \in A \setminus \{i\}} x_j^\alpha} = \alpha q_{iA}(1 - q_{iA}).$$

Thus, we have

$$x_i \frac{\partial}{\partial x_i} \left(x_i \frac{\partial}{\partial x_i} L \right) = x_i \frac{\partial}{\partial x_i} \left(x_i - \sum_{A \ni i} p_A q_{iA} \right) = x_i - \alpha \sum_{A \ni i} p_A q_{iA}(1 - q_{iA}). \tag{17}$$

As $\vec{x}$ is an equilibrium point, we have $x_i = \sum_{A \ni i} p_A q_{iA}$, and thus (17) can be rewritten as

$$x_i - \alpha \sum_{A \ni i} p_A q_{iA}(1 - q_{iA}) = \sum_{A \ni i} p_A q_{iA}(1 - \alpha(1 - q_{iA})).$$

As previously noted, a necessary condition for $\vec{x} \in \mathcal{S}$ is that all these second derivatives are non-positive. Hence, for any $i \in \sigma(\vec{x})$ we have

$$\sum_{A \ni i} p_A q_{iA}(1 - \alpha(1 - q_{iA})) \geq 0, \tag{18}$$

and thus for any i with $x_i > 0$ there exists $A_i \ni i$ with $p_{A_i} > 0$ such that $\alpha(1 - q_{iA_i}) \leq 1$. For any such A_i, we have the desired

$$q_{iA_i} \geq 1 - \frac{1}{\alpha}. \tag{19}$$

■

For $\alpha > 2$, this immediately implies the following.

Corollary 6. *Let* $\mathbf{p}$ *and* $\alpha > 2$ *be fixed, and assume that* $\vec{x} \in \mathcal{S}$. *Then any colour* $i \in \sigma(\vec{x})$ *is a champion of some* $A_i \ni i$, *that is,*

$$x_i^\alpha > \sum_{j \in A_i \setminus \{i\}} x_j^\alpha. \tag{20}$$

Proof. Indeed (20) is equivalent to $q_{iA_i} > \frac{1}{2}$, and we have $1 - \frac{1}{\alpha} > \frac{1}{2}$ when $\alpha > 2$. ■

For $B \subset \sigma(\vec{x})$ we say that i is a *colour of least weight in* B if $x_j \geq x_i$ for each $j \in B$. The following statement roughly says that "a least weight colour can be a champion only if there are no other competitors", and it immediately implies Theorem 3:

Lemma 4. *Fix* $\mathbf{p}$ *and let* $\alpha > 2$ *and* $\vec{x} \in \mathcal{S}$. *Let* $B \subset \sigma(\vec{x})$, *and let* i *be any colour of least weight in* B. *Then,* i *is attached to a leaf of* B, *i.e.*

$$\exists A : \quad p_A > 0 \text{ and } A \cap B = \{i\},$$

and in particular B *is not leaf free.*

Proof. Take $A_i \ni i$ for which i is a champion as given by Corollary 6. From (19) we have $q_{iA_i} \geq 1 - \frac{1}{\alpha} > \frac{1}{2}$. If A_i is not a leaf of i in B, (i.e. if $A_i \cap B \neq \{i\}$) then there exists $j \neq i$, $j \in A_i \cap B$. As $x_j \geq x_i$, we thus get

$$q_{iA_i} \leq \frac{x_i^\alpha}{x_i^\alpha + x_j^\alpha} \leq \frac{1}{2},$$

giving a contradiction. Thus $A_i \cap B = \{i\}$ and we are done. ■

The above arguments also imply a lower bound for the proportions of colours that survive in the limit. Namely, let $\vec{x}$ be an equilibrium. Consider a *connected component* C of its support $\sigma(\vec{x})$:

Definition 9. A *connected component* of a set $D \subset E$ of colours is a maximal (by inclusion) subset of E such that between any two colours $i, i' \in D$ there exists $m \geq 0$ and a sequence $i_0, \ldots, i_m$ with $i_0 = i$, $i_m = i'$, and a sequence $A_1, \ldots, A_m$ with

$$i_{k-1}, i_k \in A_k \quad \text{and} \quad p_{A_k} > 0, \quad \forall k = 1, \ldots, m.$$

Now, let i be any colour of least weight in C, and denote by $\mathcal{L}$ the collection of leaves of j in this component given by Lemma 4:

$$\mathcal{L} = \mathcal{L}(C, j) = \{A_0 : p_{A_0} > 0, \quad A_0 \cap C = \{j\}\}.$$

Finally, denote

$$\overline{p}_{\mathcal{L}} := \max_{A \in \mathcal{L}} p_A.$$

We then have the following lower bound.

Corollary 7. *Under the assumptions of Lemma 4, for any connected component C of $\sigma(\vec{x})$, and any colour j of least weight in C we have*

$$x_i \geq \overline{p}_{\mathcal{L}(C,j)} \quad \forall i \in C,$$

with equality if and only if C is a one-colour component belonging to one urn (vertex) only.

Proof. We automatically have $x_j \geq \overline{p}_{\mathcal{L}}$, as any leaf of a connected component of the support of $\vec{x}$ reinforces only the unique surviving colour that is adjacent to it. Now, as j was a least weight colour in the component C, we have $x_i \geq x_j$ for any $i \in C$. ■

4 Proof of Theorem 4

The first claim of the theorem is already proven by Corollary 6. In order to prove the second one, for any $i \in E$ denote by r_i the maximal possible proportion of reinforcements that the colour i may get:

$$r_i := \sum_{A \ni i} p_A.$$

Then for any equilibrium $\vec{x} \in \mathcal{E}$ we automatically get $x_i \leq r_i$ for all $i \in E$.

Now, for any equilibrium $\vec{x} \in \mathcal{S}$, joining this upper bound for x_i with the fact that (17) must be non-negative, we get

$$r_i - \alpha \sum_{A \ni i} p_A q_{iA}(1 - q_{iA}) \geq x_i - \alpha \sum_{A \ni i} p_A q_{iA}(1 - q_{iA}) \geq 0,$$

and thus for any $A \ni i$ we have an upper bound

$$q_{iA}(1 - q_{iA}) \leq \frac{r_i}{\alpha\, p_A}. \tag{21}$$

Denoting the right hand side of (21) by c_{iA} and solving the quadratic inequality we see that either

$$q_{iA} < \frac{1 - \sqrt{1 - 4c_{iA}}}{2} \qquad \text{or} \qquad q_{iA} > \frac{1 + \sqrt{1 - 4c_{iA}}}{2}, \tag{22}$$

provided that $c_{iA} \leq \frac{1}{4}$. The second possibility in (22) would imply that $q_{iA} > \frac{1}{2}$, and hence that the colour i is a champion of the urn A.

Now, for any A with $p_A > 0$ and for any $i \in A$, the value $c_{iA} = \frac{r_i}{\alpha p_A}$ tends to 0 as $\alpha \to \infty$. Hence, so does the value $\frac{1-\sqrt{1-4c_{iA}}}{2}$ in the first possibility in (22). In particular, this quantity is less than $|A|^{-1}$ for α sufficiently large. If for each $i \in A$ the first possibility in (22) takes place, then their finite sum

$$\sum_{i \in A} \frac{1-\sqrt{1-4c_{iA}}}{2} \tag{23}$$

is an upper bound for the sum $\sum_{i \in A} q_{iA} = 1$. On the other hand the sum (23) is less than $|A| \times |A|^{-1} = 1$ for α sufficiently large. Hence, for all α sufficiently large (so that the sum in (23) for each A with $p_A > 0$ is less than 1) each urn A with $p_A > 0$ has a champion. This completes the proof of the second claim of the theorem. ■

5 Proof of Theorem 2

5.1 Part (a): whisker forests

We now restrict ourselves to the graph setting with uniform vertex selection probabilities; thus, $p_A = \frac{1}{n_v}$ for any vertex A of the graph, where n_v is the number of the vertices. It will be easier (and more natural) to work with the quantities $n_v x_i$.

Let C be a connected component of $\sigma(\vec{x})$, where $\vec{x} \in \mathcal{S}$. The part (a) of the theorem will follow from the next three lemmas. The first of them claims that for each colour $i \in C$, the re-normalized weight $n_v x_i$ is close to 1 or 2:

Lemma 5. *If $\alpha > 20.25$, then for each $i \in C$*

$$n_v x_i \in [1, 1 + \frac{2.25}{\alpha}) \cup (2 - \frac{4.5}{\alpha}, 2]. \tag{24}$$

This motivates the following

Definition 10. Say that an edge $i \in C$ is an edge *of small weight* (resp., *of large weight*) if

$$n_v x_i \in [1, 1 + \frac{2.25}{\alpha}) \quad \left(\text{resp., if } n_v x_i \in (2 - \frac{4.5}{\alpha}, 2]\right).$$

The second of the three lemmas claims that "the only chance of survival for an edge of a small weight is to be attached to a leaf":

Lemma 6. *If $\alpha > 25$, any edge $i \in C$ of small weight is incident to a leaf of C.*

The third and final lemma states that large weight edges are isolated:

Lemma 7. *If $\alpha > 25$ and i is an edge of C of large weight, then all the edges of C that are adjacent to i are of small weight.*

These three lemmas indeed imply the part (a) of Theorem 2.

Proof of Theorem 2(a). If in the component C there is at least one edge i of large weight, then all the edges that are adjacent to it are of small weight (by Lemma 7) and hence are incident to a leaf (by Lemma 6). Thus C is the union of i and of edges adjacent to it, and hence the diameter of C is at most 3.

If all the edges of C are of small weight, they are all adjacent to leaves, and hence C is a star graph. ■

Remark 3. In fact, for any d there exists $\alpha_d > 25$ such that for graphs G of maximal degree d the second possibility in the above proof (all edges of small weight) becomes impossible. Indeed, the sum of $\sum_{i \in C} n_v x_i$ should be equal to the number of vertices in C, that is, $|C| + 1$, while if all the edges are of small weight, it is at most $|C| \cdot (1 + \frac{2.5}{\alpha})$. Hence, one should have

$$\frac{|C| + 1}{|C|} \leq 1 + \frac{2.5}{\alpha}$$

If the degrees of vertices do not exceed d, we have $|C| \leq d$ for any star graph, and thus a contradiction once $\alpha > \max(2.5d, 25) =: \alpha_d$.

Thus, once α is sufficiently large, on any component C of any equilibrium $\vec{x} \in \mathcal{S}$ there is exactly one edge of large weight. This is where Definition 4 of a marked whisker comes from: the $*$ marks this edge (to which all the others are adjacent).

To prove the three above lemmas, we will need the following easy computation:

Lemma 8. *Denote $c_\alpha := \sqrt{1 - \frac{8}{\alpha}}$. Then for any $\alpha > 20.25$ we have*

$$c_\alpha > 1 - \frac{4.5}{\alpha}.$$

Proof. We have $\sqrt{1-\frac{8}{\alpha}} > 1-\frac{4.5}{\alpha}$, as $(1-\frac{4.5}{\alpha})^2 = 1-\frac{9}{\alpha}+\frac{4.5^2}{\alpha^2}$, and for $\alpha > 4.5^2 = 20.25$ the right hand side is less than $1-\frac{8}{\alpha}$. Substituting it into the definition of c_α, we obtain the desired lower bound. ■

Proof of Lemma 5. Note first that as we are in the graph setting with the uniform selection of the vertices, we have

$$c_{iA} = \frac{r_i}{\alpha p_A} = \frac{2/n_v}{\alpha \cdot 1/n_v} = \frac{2}{\alpha}$$

for any edge i and any vertex A to which it is incident. Hence, for any such i and A one has

$$\sqrt{1-4c_{iA}} = \sqrt{1-\frac{8}{\alpha}} = c_\alpha.$$

Hence, combining (22) with a lower bound for $\sqrt{1-4c_{iA}}$ from Lemma 8, we get for any $\alpha > 20.25$,

$$q_{iA} \leq \frac{2.25}{\alpha}, \qquad \text{or} \qquad q_{iA} \geq 1-\frac{2.25}{\alpha}. \tag{25}$$

As each edge i of the graph is adjacent to exactly two vertices, we denote these vertices by $B_{i,1}$ and $B_{i,2}$. Then, for any colour $i \in C$ and for any equilibrium $\vec{x}$ we can re-write the equilibrium condition as

$$n_v x_i = q_{iB_{i,1}} + q_{iB_{i,2}}; \tag{26}$$

As both $q_{iB_{i,1}}$ and $q_{iB_{i,2}}$ belong to $[0, \frac{2.25}{\alpha}) \bigcup (1-\frac{2.25}{\alpha}, 1]$, we have for their sum

$$n_v x_i = q_{iB_{i,1}} + q_{iB_{i,2}} \in [0, \frac{4.5}{\alpha}) \cup (1-\frac{2.25}{\alpha}, 1+\frac{2.25}{\alpha}) \cup (2-\frac{4.5}{\alpha}, 2].$$

Finally, Corollary 7 applied to the graph case says that the least positive weight x_{i_C} of any connected component C of $\sigma(\vec{x})$ is greater than $1/n_v$, hence for each i we actually have

$$n_v x_i \in [1, 1+\frac{2.25}{\alpha}) \cup (2-\frac{4.5}{\alpha}, 2]. \tag{27}$$

■

Proof of Lemma 6. Let $i \in C$ be an edge of small weight. Then by (25), for one of its endpoints A we have $q_{iA} > 1 - \frac{2.25}{\alpha}$. Indeed, otherwise we would have $q_{iB_{i,1}}, q_{iB_{i,2}} < \frac{2.25}{\alpha} < \frac{1}{2}$, what would imply $n_v x_i < 1$, providing a contradiction.

Let us show that A is then a leaf of C. Indeed, if there exists $j \in A \setminus \{i\}$ with $x_j > 0$, then $n_v x_j \geq 1$ and hence

$$q_{iA} \leq \frac{(1+\frac{2.25}{\alpha})^\alpha}{(1+\frac{2.25}{\alpha})^\alpha + 1^\alpha} = 1 - \frac{1}{(1+\frac{2.25}{\alpha})^\alpha + 1} \leq 1 - \frac{1}{e^{2.25}+1}, \tag{28}$$

where the last inequality is due to $(1+\frac{s}{a})^a < e^s$ for all $a, s > 0$. Now, to obtain the desired contradiction with $q_{iA} > 1 - \frac{2.25}{\alpha}$, it suffices to check that

$$\frac{2.25}{\alpha} < \frac{1}{e^{2.25}+1},$$

what holds once

$$\alpha > 2.25 \cdot (e^{2.25} + 1) = 23.59 \ldots$$

■

Proof of Lemma 7. Let $j \in C$ be an edge adjacent to an edge $i \in C$ of large weight, and let A be their common vertex (without loss of generality, $A = B_{j,1}$). Then, $q_{jA} < \frac{1}{2}$ (as $x_j < x_i$). Thus,

$$n_v x_j = q_{jA} + q_{j,B_{j,2}} \leq \frac{1}{2} + 1.$$

Hence, x_j cannot be an edge of large weight, and therefore is of small weight. ■

5.2 Part (b): constructing the equilibria

Let us now prove the second part, starting with the existence of a stable equilibrium of a given (marked) type. Let $G = (V, E)$ be a whisker graph with n edges (and $n_v = n+1$ vertices). Let $*$ denote a marked edge (any edge if G is a star graph, otherwise it is the central edge of the non-degenerate whisker graph). Let a sufficiently small $\varepsilon > 0$ be fixed, and consider the set D_ε of distributions $\vec{x} \in \Delta_n(\mathbf{p})$ such that

$$n_v x_* \in [2-\varepsilon, 2], \quad \text{and} \quad n_v x_j \in [1, 1+\varepsilon] \quad \text{for all } j \neq *.$$

Recall (see (10)) that the proportions in which the reinforcements are divided between colours are given by the map

$$f(\vec{x})_i = \sum_{A \ni i} p_A \frac{x_i^\alpha}{\sum_{j \in A} x_j^\alpha},$$

so that we have $F(\vec{x}) = f(\vec{x}) - \vec{x}$. The existence part will be established once we prove the following:

Lemma 9. *There exists $\varepsilon > 0$ such that for any α sufficiently large the map f is contracting map from D_ε to itself.*

Proof. Let us first check that f indeed sends D_ε to itself. Namely, let $\vec{x} \in D_\varepsilon$. Note first that any edge j of $E \setminus \{*\}$ is adjacent to a vertex A that is not adjacent to any other edge, and hence $q_{jA} = 1$. Hence,

$$n_v f(\vec{x})_j \geq n_v p_A q_{jA} = q_{jA} = 1. \tag{29}$$

On the other hand, any $j \neq *$ has a vertex B in common with $*$ (i.e. $B \ni *, j$) thus we have

$$q_{jB} \leq \left(\frac{1+\varepsilon}{2-\varepsilon}\right)^\alpha < \left(\frac{5}{7}\right)^\alpha, \quad \text{when } \varepsilon < \frac{1}{4}. \tag{30}$$

As the right hand side of (30) tends to 0 as $\alpha \to \infty$, for all α sufficiently large we have

$$n_v p_B q_{jB} = q_{jB} \leq \varepsilon.$$

It follows that

$$n_v f(\vec{x})_j \leq 1 + \varepsilon. \tag{31}$$

In the same way, we get that for $\alpha > \alpha(\varepsilon, G)$ and any endpoint B of $*$

$$q_{*B} = 1 - \sum_{j \in B \setminus \{*\}} q_{jB} \geq 1 - n \left(\frac{5}{7}\right)^\alpha \geq 1 - \frac{\varepsilon}{2}.$$

Hence for all $\alpha > \alpha(\varepsilon, G)$ we have

$$n_v f(\vec{x})_* = q_{*B_{*,1}} + q_{*B_{*,2}} \in [2 - \varepsilon, 2]. \tag{32}$$

Thus, for all α sufficiently large f sends D_ε to D_ε.

Now let us show that for sufficiently large α the map f is contracting. It suffices to show that its partial derivatives tend to 0 uniformly on D_ε as α tends to infinity (this will imply that $df|_{\vec{x}}$ has norm less than 1 at any point $\vec{x} \in D_\varepsilon$). Moreover, as p_A are constant for all vertices A, it suffices to show the same for all the q_{iA}.

Let $y_* := 1$. Then the values q_{iA} can be represented in terms of $n-1$ quotients $y_j := (x_j/x_*)^\alpha$, where $j \in E \setminus \{*\}$, as

$$q_{iA} = \frac{y_i}{\sum_{j\in A} y_j}.$$

Thus, $q_{iA} = h_A(g(r(\vec{x})))_i$ where

$$h_A(\vec{y}) = \frac{\vec{y}}{\sum_{j\in A} y_j}, \quad g(\vec{u})_i = u_i^\alpha, \quad \text{and} \quad r(\vec{x}) = \frac{\vec{x}}{x_*}.$$

Now for each i one has $n_v x_i \in [1,2]$ so for $i \neq *$ the ratios $u_i = x_i/x^*$ are between $1/2$ and $5/7$ (the latter holds provided that $\varepsilon < 1/4$). The partial derivatives of r are bounded by a constant (independent of α), with derivatives of $r(\vec{x})_*$ all equal to zero. The derivatives of h_A are bounded in absolute value by 1, and the derivatives of g are all either 0 or $\alpha u_i^{\alpha-1}$. The latter converges to 0 uniformly on $[\frac{1}{2}, \frac{5}{7}]$ as $\alpha \to \infty$ for $i \neq *$, so by applying the chain rule to the derivatives of q_{iA} we are done. ■

Lemma 9 implies that the vector field (4) possesses a unique equilibrium $\vec{x}$ in D_ε (by the Banach fixed point Theorem). It is also easy to see that it is linearly stable. Indeed, we have chosen α sufficiently large so that the differential $df|_{\vec{x}}$ at any point $\vec{x} \in D_\varepsilon$, in particular at the point $\vec{x}$, would be a linear contraction. The eigenvalues of $df|_{\vec{x}}$ are thus less than 1 in absolute value; the eigenvalues of $dF|_{\vec{x}} = -\operatorname{Id} + df|_{\vec{x}}$, that differ from eigenvalues of $df|_{\vec{x}}$ by addition of (-1), hence have negative real part.

Finally, Remark 3 implies that for any sufficiently large α an equilibrium $x \in \mathcal{S}$ satisfying $x_* > x_j$ for all $j \neq *$ should belong to D_ε, and thus coincide with v. ■

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