

ALIGNMENT PERCOLATION

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ABSTRACT. The existence (or not) of infinite clusters is explored for two stochastic models of intersecting line segments in $d \geq 2$ dimensions. Salient features of the phase diagram are established in each case. The models are based on site percolation on \mathbb{Z}^d with parameter $p \in (0, 1]$. For each occupied site v , and for each of the $2d$ possible coordinate directions, declare the entire line segment from v to the next occupied site in the given direction to be either blue or not blue according to a given stochastic rule. In the ‘one-choice model’, each occupied site declares one of its $2d$ incident segments to be blue. In the ‘independent model’, the states of different line segments are independent.

1. THE TWO MODELS

Percolation theory is concerned with the existence of infinite clusters in stochastic geometric models. In the classical percolation model, sites (or bonds) of \mathbb{Z}^d are declared *occupied* with probability p , independently of one another, and the basic question is to understand the geometry of the occupied subgraph for different ranges of values of p (see [10] for a general account of percolation). A number of models of *dependent* percolation have been studied in various contexts including statistical physics, social networks, and stochastic geometry. One area of study of mathematical interest has been the geometry of disks and line segments arising in Poisson processes in \mathbb{R}^d ; see, for example, [7, 12, 15]. Motivated in part by such Poissonian systems, we study

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here two percolation systems of random line segments on the hypercubic lattice \mathbb{Z}^d with $d \geq 2$. Each is based on site percolation on \mathbb{Z}^d .

These two processes are called the ‘one-choice model’ and the ‘independent model’, respectively, with the difference lying in the manner in which line segments within the site percolation model are declared to be active (or ‘blue’). In the one-choice model, there is dependence between neighbouring segments, whereas they are conditionally independent in the independent model. We describe the two models next.

Let $\Omega_V = \{0, 1\}^{\mathbb{Z}^d}$ be the state space of site percolation on the hypercubic lattice \mathbb{Z}^d with $d \geq 2$, let \mathcal{F}_V denote the σ -field generated by the cylinder sets (i.e. the states of finitely many sites), and let P_p be product measure on $(\Omega_V, \mathcal{F}_V)$ with density $p \in (0, 1]$. For $\omega = (\omega_v : v \in \mathbb{Z}^d) \in \Omega_V$, a vertex $v \in \mathbb{Z}^d$ is called *occupied* if $\omega_v = 1$, and *unoccupied* otherwise, and we write $\eta(\omega) = \{v \in \mathbb{Z}^d : \omega_v = 1\}$ for the set of occupied vertices. We construct a random subgraph of \mathbb{Z}^d as follows. First, let $\omega \in \Omega_V$ be sampled according to P_p . An unordered pair of distinct vertices $v_1, v_2 \in \eta(\omega)$ is called *feasible* if

- (i) v_1 and v_2 differ in exactly one coordinate, and
- (ii) on the straight line-segment of \mathbb{Z}^d joining v_1 and v_2 , v_1 and v_2 are the only occupied vertices.

Each feasible pair may be considered as the straight line-segment (or, simply, *segment*) of \mathbb{Z}^d joining them.

Let $F(\omega)$ be the set of all feasible pairs of occupied vertices of ω , and let $F_v(\omega)$ denote the set of feasible pairs containing v . We will declare some subset $S(\omega) \subseteq F(\omega)$ to be *blue*. An edge e in the edge set \mathbb{E}^d of \mathbb{Z}^d is called *blue* if it lies in a blue segment, and a site is *blue* if it is incident to one or more blue edges. We are interested in whether or not there exists an infinite cluster of blue edges. The relevant state space is then $\Omega_E = \{\text{blue, not blue}\}^{\mathbb{E}^d}$, and \mathcal{F}_E denotes the σ -field generated by the cylinder sets.

Roughly speaking, the parameter p controls how much the set F differs from the original lattice. When p is large, most sites are occupied, and many feasible pairs are simply edges of \mathbb{Z}^d ; indeed, one obtains the entire lattice in the case $p = 1$. When p is small, segments tend to be long and have many other segments crossing them.

We will work with two specific stochastic constructions of the blue segments. We write $o = (0, 0, \dots, 0)$ for the origin of \mathbb{Z}^d , and e_i for the unit vector in the direction of increasing i th coordinate. An edge with endvertices u, v is written (u, v) . Our first model is called the one-choice model.

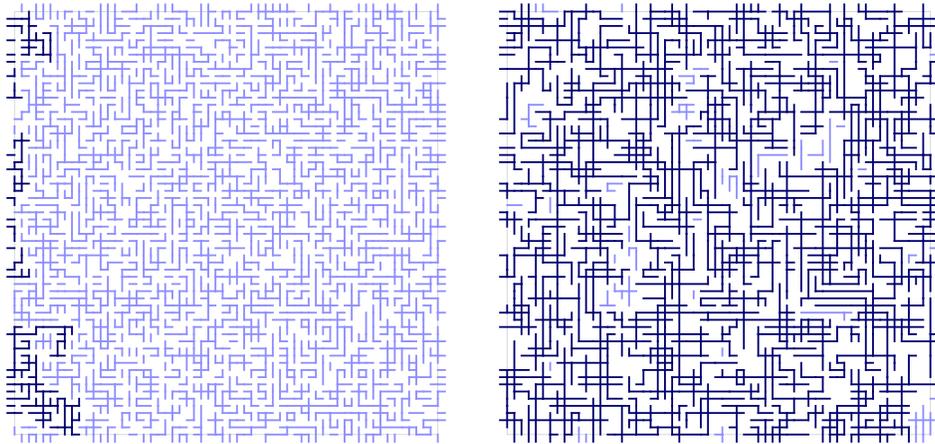


FIGURE 1.1. A 60×60 section of \mathbb{Z}^2 with blue edges in the one-choice model of Example 1.1 indicated, with $p = 0.8$ (left) and $p = 0.4$ (right). Segments which cross the boundary of the box have been cut off for clarity. Segments which cross the left boundary, and those connected to them by blue edges, are coloured darker.

Example 1.1 (The one-choice model). *Let $\omega \in \Omega_V$. Independently for each $v \in \eta(\omega)$, choose a segment (say f_v) uniformly at random from $F_v(\omega)$ and declare it to be green; in this case, we say that v has declared f_v to be green and that the direction from v along f_v is the chosen direction of v . A segment e with endvertices u, v is declared blue if and only if at least one of u or v has declared it green.*

Figure 1.1 shows simulations of the set of blue edges of \mathbb{Z}^2 in the one-choice model with $p = 0.8$ and $p = 0.4$.

There is no apparent stochastic monotonicity in the set of blue sites as a function of p for the one-choice model. Elementary calculations show that any given edge e of \mathbb{Z}^d is blue with probability

$$(1.1) \quad \lambda := 1 - \left(1 - \frac{1}{2d}\right)^2,$$

independently of p . The probability that a given vertex is incident to some blue edge of \mathbb{Z}^d is $p + (1 - p)\{1 - (1 - \lambda)^d\}$, which is increasing in p . On the other hand, the probability that the edges (o, e_1) and $(o, -e_1)$ are both blue (where o is the origin) is $1 - (1 + p/d)(1 - \lambda)$, which is decreasing in p . In other words, while the frequency of blue edges does not change with p , blue edges tend to be more aligned as we

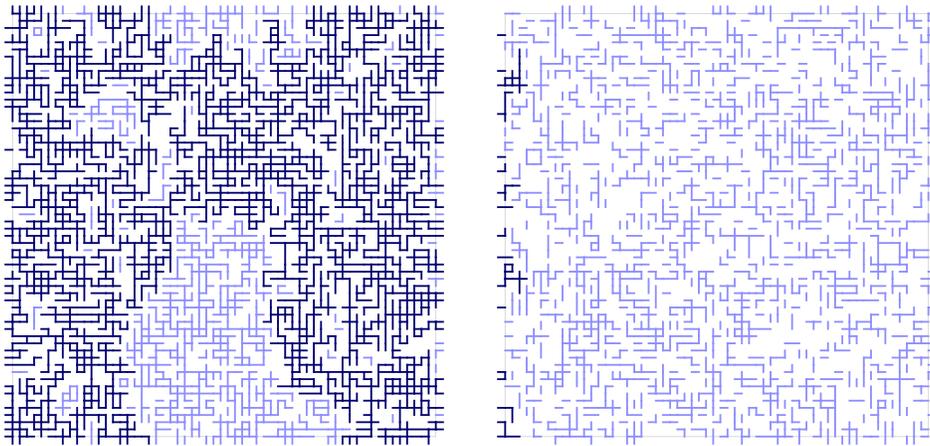


FIGURE 1.2. A 60×60 section of \mathbb{Z}^2 with blue edges in the independent model of Example 1.2 indicated, with $p = 0.8$ and with $\lambda = 0.5$ (left) and $\lambda = 0.3$ (right). Segments which cross the boundary of the box have been cut off for clarity. Segments which cross the left boundary, and those connected to them, are coloured darker. Occupied sites that are not blue have been omitted.

decrease p . Similarly the probability that (o, e_1) and (o, e_2) are both blue is $\lambda^2 - p(2d - 1)^2 / (2d)^4$, which is decreasing in p .

We refer to the second model as the independent model.

Example 1.2 (The independent model). *Let $\omega \in \Omega_V$ and $\lambda \in [0, 1]$. Independently for each segment in $F(\omega)$, declare the segment to be blue with probability λ .*

Figure 1.2 shows simulations of the set of blue edges of \mathbb{Z}^2 in the independent model with $p = 0.8$, and with $\lambda = 0.5$ and $\lambda = 0.3$.

There is no apparent stochastic monotonicity in the set of blue sites as a function of p (with fixed λ) for the independent model. An easy calculation shows that the probability that a given edge is blue is λ , so in particular it does not depend on p . On the other hand, the probability that a given vertex is incident to a blue edge is $1 - (1 - \lambda)^d + p \{(1 - \lambda)^d - (1 - \lambda)^{2d}\}$, which is increasing in p . For two distinct edges e and e' that are collinear, one can show that $\text{corr}(\mathbf{1}_{\{e \text{ is blue}\}}, \mathbf{1}_{\{e' \text{ is blue}\}}) = (1 - p)^k$, where $\mathbf{1}_A$ denotes the indicator function of the event A , and $k \geq 1$ is the number of lattice sites between e and e' . Thus correlations are decreasing in p (if e and e' are not collinear then the events $\{e \text{ is blue}\}$ and $\{e' \text{ is blue}\}$ are independent).

In each example above, there are two levels of randomness. First, one chooses a site percolation configuration ω according to the product measure \mathbb{P}_p , and then the blue edges are selected according to an appropriate conditional measure P_ω . We formalize this as follows. Let \mathbb{P}_p denote the probability measure on the space $(\Omega_V \times \Omega_E, \sigma(\mathcal{F}_V \times \mathcal{F}_E))$, such that, for measurable rectangles of the form $A \times B$,

$$(1.2) \quad \mathbb{P}_p(A \times B) = \int_A P_\omega(B) d\mathbb{P}_p(\omega), \quad A \in \mathcal{F}_V, B \in \mathcal{F}_E,$$

Whenever we need to make the dependence on λ explicit, as in Example 1.2, this will be written as $\mathbb{P}_{p,\lambda}(A \times B) = \int_A P_\omega^\lambda(B) d\mathbb{P}_p(\omega)$, where P_ω^λ is the (quenched) law of the independent model with parameter λ , conditional on the site percolation configuration ω . We will sometimes abuse notation and use the notation \mathbb{P}_p to denote the ‘annealed’ probability measure on $(\Omega_E, \mathcal{F}_E)$, viz. $\mathbb{P}_p(\Omega_V \times \cdot)$. In any case, since all our results are statements of almost-sure type, they hold P_ω -a.s. in the quenched setting.

We write p_c^{site} (respectively, p_c^{bond}) for the critical probability of site percolation (respectively, bond percolation) on the lattice under consideration.

The structure of this paper is as follows. The main results for the one-choice model and the independent model are presented in Section 2, together with our conjectures for the full phase diagrams of the two models under consideration. The proofs of the main theorems are found in Sections 3–5.

2. THE MAIN RESULTS

2.1. The one-choice model. In this section all a.s. statements are made with respect to the measure \mathbb{P}_p .

Theorem 2.1. *Let $d \geq 2$. For the one-choice model of Example 1.1, there exist $0 < p_0(d) \leq p_1(d) < 1$ such that:*

- (i) *if $p \in (0, p_0(d))$, there exists a.s. a unique infinite cluster of blue edges,*
- (ii) *if $p \in (p_1(d), 1]$, there exists a.s. no infinite cluster of blue edges.*

See Sections 3 and 5 for the proofs. We make the following further conjecture.

Conjecture 2.2. *For the one-choice model on \mathbb{Z}^d with $d \geq 2$,*

- (i) *there exists $p_c(d) \in (0, 1)$ such that, for $p \in (0, p_c(d))$, there exists a.s. a unique infinite blue cluster, while for $p > p_c(d)$ there exists a.s. no infinite blue cluster,*

- (ii) $p_c(d)$ is strictly increasing in d ,
- (iii) for given $d \geq 2$, the probability that the origin lies in an infinite blue cluster is non-increasing in p .

Note in this example with $d > 2$ that, if we restrict the d -dimensional model to the two-dimensional subgraph $\{0\}^{d-2} \times \mathbb{Z}^2$, the result is not the one-choice model on \mathbb{Z}^2 .

Remark 2.3. *Our numerical estimates of $p_c(2)$ and $p_c(3)$ in the one-choice model are $p_c(2) \approx 0.505$ and $p_c(3) \approx 0.862$.*

2.2. The independent model. All a.s. statements in this section are made with respect to the measure $\mathbb{P}_{p,\lambda}$.

Theorem 2.4. *Let $d \geq 2$. For the independent model of Example 1.2, with parameters (p, λ) ,*

- (i) if $\lambda < p/(2d - 1)$, there is a.s. no infinite cluster of blue edges,
- (ii) there exists an absolute constant $c > 0$ such that, if $\lambda > c \log(1/q)$ where $p + q = 1$, there exists a.s. a unique infinite blue cluster,
- (iii) there exists a Lipschitz continuous function $\bar{\lambda}_c : (0, 1] \rightarrow (0, 1]$, that satisfies $\bar{\lambda}_c(p_c^{\text{site}}) = 1$, $\bar{\lambda}_c(1) = p_c^{\text{bond}}$, and is strictly decreasing on $[p_c^{\text{site}}, 1]$, such that, for $p > p_c^{\text{site}}$ and $\lambda > \bar{\lambda}_c(p)$, there exists a.s. a unique infinite blue cluster.

See Sections 4 and 5 for the proofs. Figure 2.1 indicates what we believe to be the critical frontier for percolation in the independent model in 2 dimensions (based on simulations), and also what is proved.

Conjecture 2.5. *For the independent model on \mathbb{Z}^d with $d \geq 2$, there exists $\lambda_c = \lambda_c(p, d) \in (0, 1)$ such that:*

- (a) $\lambda_c(\cdot, d)$ is continuous and strictly increasing on $(0, 1]$,
- (b) for $p > 0$ and $\lambda < \lambda_c(p, d)$, there exists a.s. no infinite blue cluster,
- (c) for $p > 0$ and $\lambda > \lambda_c(p, d)$, there exists a.s. a unique infinite blue cluster,

Moreover, for each fixed λ , the probability that the origin is in an infinite blue cluster is non-increasing in p .

Remark 2.6 (Added on revision). *In their recent preprint [14], Hilário and Ungaretti have proved the existence of such a critical function λ_c with several properties including that $\lambda_c(p) \leq p_c^{\text{H}}$ for $p \in (0, 1]$ and $d \geq 2$, where p_c^{H} is the critical probability of bond percolation on the hexagonal lattice.*

When $p = 1$, the process is simply bond percolation with edge-density λ , whence $\lambda_c(1, d) = p_c^{\text{bond}}(d)$, the critical probability of bond

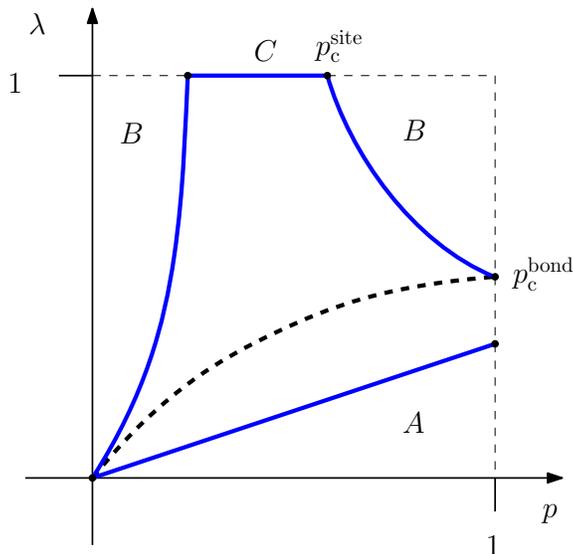


FIGURE 2.1. A picture of the phase diagram for the independent model in two dimensions. We conjecture there exists an infinite blue cluster for (p, λ) above the dashed curve, and none below. The solid lines indicate the regions identified in Theorem 2.4. There is no percolation in region A where $\lambda \leq p/3$, but there exists a unique infinite blue cluster in the two regions labelled B , and along the line C .

percolation on \mathbb{Z}^d . If λ_c is continuous on a neighbourhood of $p = 1$, it then follows that $\lambda_c(p) \rightarrow p_c^{\text{bond}}$ as $p \rightarrow 1$.

Remark 2.7. *Our numerical estimate for the critical frontier of the independent model is sketched as the dashed line in Figure 2.1.*

Consider the independent model for given (p, λ) , and let $2 \leq d < d'$. We may embed \mathbb{Z}^d into $\mathbb{Z}^{d'}$ by identifying \mathbb{Z}^d with $\{0\}^{d'-d} \times \mathbb{Z}^d$. For given (p, λ) , the independent model on $\mathbb{Z}^{d'}$, when restricted to this image of \mathbb{Z}^d , is simply a d -dimensional independent model. It follows in particular that $\lambda_c(p, d)$ is non-increasing in d (assuming λ_c exists).

3. PROOF OF THEOREM 2.1(ii)

We show that the one-choice model and a ‘corrupted compass model’ (see [16, Sect. 3]) may be coupled in such a way that the first is a subset of the second. It is known that the second does not percolate for sufficiently large values of p .

Let $\omega \in \Omega_V$ be a percolation configuration, with occupied set $\eta = \eta(\omega)$, and let a random feasible pair f_v be as in Example 1.1. The corrupted compass model is constructed as follows from ω and the f_v . Each edge $e = (u, v)$ is declared to be *turquoise* if there exists $x \in \{u, v\}$ such that: either x is unoccupied, or x is occupied and $e \in f_x$.

Let T be the set of turquoise edges, and B the set of blue edges in the one-choice model. We claim that

- (a) $B \subseteq T$, and
- (b) there exists $p_0 = p_0(d) \in (0, 1)$ such that if $p > p_0$, T contains a.s. no infinite cluster.

Clearly, (a) and (b) imply the result.

To verify (a), let $e \in B$, and suppose without loss of generality that $e = (o, e_1)$. Let

$$\ell_e = \inf\{k \geq 0 : -ke_1 \in \eta\}, \quad r_e = \inf\{k \geq 1 : ke_1 \in \eta\},$$

and write $x = -\ell_e e_1$ and $x' = r_e e_1$ for the closest occupied sites to the left and right of o , respectively (we allow $x = o$). Since $e \in B$, either $f_x = (x, x')$ or $f_{x'} = (x', x)$, or both. Without loss of generality, we assume $f_x = (x, x')$. Then $(x, x + e_1) \in T$. Since the sites $\{x + ke_1 : k = 1, 2, \dots, r_e + \ell_e - 1\}$ are unoccupied, we have that $(x + ke_1, x + (k+1)e_1) \in T$ for $k \in \{1, \dots, r_e + \ell_e - 1\}$. In particular, $e \in T$.

To verify (b) we note that T is precisely the set of edges of the corrupted compass model of [16, Sect. 3] with corruption probability $p_{\{x\}} = 1 - p$ for each vertex $x \in \mathbb{Z}^d$ (that is, each unoccupied vertex is corrupted). As in [16, Sect. 4.2], T does not percolate when the three eigenvalues κ of the matrix

$$(3.1) \quad \mathbf{M} = (2d - 1) \begin{pmatrix} 1 - p & 1 - p & (1 - p)/(2d) \\ 1 & 1/(2d) & 0 \\ 0 & 1 & 1/(2d) \end{pmatrix},$$

satisfy $|\kappa| < 1$. (See [16, eqn (4)], noting that the parameter p therein is the so-called corruption probability, which is the current $1 - p$.) As $p \rightarrow 1$, the eigenvalues of \mathbf{M} approach their values with $p = 1$, namely $(2d - 1)/(2d)$ and 0, as required. \square

4. PROOF OF THEOREM 2.4(i, iii)

We begin with an observation on the a.s. uniqueness of infinite clusters, when they exist. There is a general theorem due basically to Burton and Keane [4, 5] which states that, for any translation-invariant probability measure μ on $\Xi = \{0, 1\}^{\mathbb{Z}^d}$ with the finite energy property, if there exists an infinite cluster then it is a.s. unique. See also [9, p.

42]. A probability measure μ on Ξ is said to have the *finite energy property* if

$$(4.1) \quad \mu(e \text{ is blue} \mid \mathcal{T}_e) \in (0, 1) \quad \mu\text{-a.s.}, \quad e \in \mathbb{E}^d,$$

where \mathcal{T}_e is the σ -field generated by the colours of edges other than e .

The independent model measure is evidently translation-invariant, and we state the finite energy property as a proposition. Let $\mathbb{P}_{p,\lambda} = \mathbb{P}_p \times \mathbb{P}_\lambda$, considered as the probability measure of the independent model.

Proposition 4.1 (Finite energy property). *Let $(p, \lambda) \in (0, 1] \times (0, 1)$. For $e \in \mathbb{E}^d$, we have that (4.1) holds with $\mu = \mathbb{P}_{p,\lambda}$.*

In contrast, the one-choice measure does not have the finite energy property, and an adaptation of the methodology of [4] will be needed for that case; see Section 5.

Proof. Write $e = (o, e_1)$, and let O be the event that both o and e_1 are occupied. Let E_0 denote the set of edges incident to o but not e_1 , and E_1 the set of edges incident to e_1 but not o . Let \mathcal{S}_i denote the random set of edges in E_i that are blue. Then

$$(4.2) \quad \mathbb{P}_{p,\lambda}(O \mid \mathcal{T}_e) = \mathbb{P}_{p,\lambda}(O \mid \mathcal{S}_0, \mathcal{S}_1),$$

since, given \mathcal{S}_0 and \mathcal{S}_1 , the states of edges other than $E_0 \cup E_1 \cup \{e\}$ (and sites other than o, e_1) are conditionally independent of O .

By conditional probability, for $S_i \subseteq E_i$,

$$\begin{aligned} \mathbb{P}_{p,\lambda}(O \mid \mathcal{S}_0 = S_0, \mathcal{S}_1 = S_1) &= \frac{\mathbb{P}_{p,\lambda}(\mathcal{S}_0 = S_0, \mathcal{S}_1 = S_1 \mid O) \mathbb{P}_{p,\lambda}(O)}{\mathbb{P}_{p,\lambda}(\mathcal{S}_0 = S_0, \mathcal{S}_1 = S_1)} \\ &\geq \mathbb{P}_{p,\lambda}(\mathcal{S}_0 = S_0, \mathcal{S}_1 = S_1 \mid O) p^2. \end{aligned}$$

For $S_i \subseteq E_i$, we have that

$$\mathbb{P}_{p,\lambda}(\mathcal{S}_0 = S_0, \mathcal{S}_1 = S_1 \mid O) > 0.$$

Since there are only finitely many choices for S_0, S_1 , there exists $c_0(p, \lambda) > 0$ such that

$$\mathbb{P}_{p,\lambda}(O \mid \mathcal{S}_0 = S_0, \mathcal{S}_1 = S_1) \geq c_0(p, \lambda).$$

By (4.2), $\mathbb{P}_{p,\lambda}(O \mid \mathcal{T}_e) \geq c_0(p, \lambda)$ a.s.

Now, a.s.,

$$\begin{aligned} \mathbb{P}_{p,\lambda}(e \text{ is blue} \mid \mathcal{T}_e) &\geq \mathbb{P}_{p,\lambda}(e \text{ is blue}, O \mid \mathcal{T}_e) \\ &= \mathbb{P}_{p,\lambda}(e \text{ is blue} \mid O, \mathcal{T}_e) \mathbb{P}_{p,\lambda}(O \mid \mathcal{T}_e) \\ &= \lambda \mathbb{P}_{p,\lambda}(O \mid \mathcal{T}_e) \geq \lambda c_0(p, \lambda), \end{aligned}$$

and, similarly,

$$\mathbb{P}_{p,\lambda}(e \text{ is not blue} \mid \mathcal{T}_e) \geq (1 - \lambda)c_0(p, \lambda).$$

This proves (4.1). \square

Proof of Theorem 2.4(i). Let B_o be the set of vertices in the blue cluster at the origin o . We shall bound $|B_o|$ above (stochastically) by the total size of a certain branching process with two types of particle: type U with mean family-size $2\lambda(d - 1)/p$, and type O with mean family-size $\lambda(2d - 1)/p$. This is an elaboration of the argument used in [13] to prove that $p_c \geq 1/(2d - 1)$ for bond percolation on \mathbb{Z}^d .

We outline the proof as follows. Suppose that o is occupied (the argument is similar if o is unoccupied). We explore the $2d$ directions incident to o , and count the number of sites on any blue segments touching o (excluding o itself). The mean number of such sites is $2d\lambda/p$, and this set of sites contains both occupied and unoccupied sites. We now iterate the construction starting at these new sites. Each occupied site gives rise on average to no more than $\mu_1 := (2d - 1)\lambda/p$ new sites, and each unoccupied site no more than $\mu_2 := 2(d - 1)\lambda/p$ new sites. After each stage, we will have constructed some subset of B_o . As the construction develops, we will encounter further un/occupied sites each having a potential for further extensions. During each stage, starting at some site v , the mean number of new sites is no greater than μ_1 if v is occupied, and μ_2 if v is unoccupied; the true conditional expectations (given the past history of the process) will typically be less than the μ_i , since some sites thus encountered will have been counted earlier.

In order to bound $|B_o|$ above, we consider the process with two types of particle: with type O corresponding to occupied sites and type U corresponding to unoccupied sites. Assuming that this process may be taken to be a branching process, it follows by standard theory that, if $\mu_1, \mu_2 < 1$, then the total size of the branching process is a.s. finite. This implies in turn that $\mathbb{P}(|B_o| = \infty) = 0$ if $\mu_1, \mu_2 < 1$, which is to say if $\lambda < p/(2d - 1)$. A more precise condition on the pair λ, p may be obtained by considering the above branching process as a 2-type process, and using the relevant extinction theorem (see, for example, [2, Chap. V.3]).

The above argument may be written out formally. There are some complications arising from the conjunction of probability and combinatorics, but these do not interfere with the conclusion. \square

Proof of Theorem 2.4(iii). Consider the following ‘mixed’ percolation model on \mathbb{Z}^d where $d \geq 2$. Let $\lambda, p \in [0, 1]$. Each site is occupied independently with probability p . Each edge whose endvertices are

both occupied is then occupied with probability λ , independently of all other edges. If one or both of the ends of an edge is unoccupied then the edge itself cannot be occupied. Let C be the occupied cluster containing the origin, and define the *percolation probability* $\theta(\lambda, p) = \mathbb{P}(|C| = \infty)$. Evidently, θ is non-decreasing in λ and p . The *critical curve* is defined by:

$$(4.3) \quad \bar{\lambda}_c(p) = \sup\{\lambda \in [0, 1] : \theta(\lambda, p) = 0\}, \quad p \in [0, 1].$$

Some basic properties of $\bar{\lambda}_c$ are proved in [6, Thm 1.4] as an application of the differential-inequality method of [1] (see also [10, Sect. 3.2]), namely the following:

- (a) $\bar{\lambda}_c(p) = 1$ for $0 \leq p \leq p_c^{\text{site}}$,
- (b) $\bar{\lambda}_c(1) = p_c^{\text{bond}}$,
- (c) $\bar{\lambda}_c$ is Lipschitz continuous and strictly decreasing on $[p_c^{\text{site}}, 1]$.

We claim that the independent model with parameters (p, λ) is stochastically greater than the mixed percolation model with parameters (p, λ) . To see this, consider the independent model with parameters (p, λ) , and let G be the subset of blue edges with the property that both their endvertices are occupied. The law of G is that of the mixed percolation model, and the stochastic ordering follows from the fact that $G \subseteq B$. It follows in particular that there exists a.s. an infinite blue cluster when $\lambda > \bar{\lambda}_c(p)$. \square

5. PROOFS OF THEOREM 2.1(i) AND THEOREM 2.4(ii)

We first prove the a.s. uniqueness of the infinite blue cluster in the one-choice model, when such exists. The probability measure governing the one-choice model may be constructed as follows. As before, we write \mathbb{P}_p for product measure with density p on Ω_V . For each $v \in \mathbb{Z}^d$, we choose a random coordinate direction having law the product measure $\Pi = \prod_{v \in \mathbb{Z}^d} \pi$, where π is the uniform probability measure on the set $\mathcal{D} = \{\pm e_i : i = 1, 2, \dots, d\}$. Let $\mathbb{P}_{p, \pi} = \mathbb{P}_p \times \Pi$ be the measure on $\Omega_V \times \mathcal{D}$ that governs the one-choice model.

As explained at the beginning of Section 4, proofs of uniqueness usually require that the relevant probability law satisfy the finite-energy property (4.1). The one-choice measure does not have the finite-energy property nor the *positive* finite energy property of [8], as the following indicates (working under the annealed measure).

Example 5.1. *In $d = 2$ dimensions consider the following local configuration of edges:*

- edges in the unit square with the origin $(0, 0)$ as the bottom right corner are all blue,
- edges in the unit square with $(1, 0)$ as the bottom left corner are all blue,
- all other edges incident to these squares (except the edge e from $(0, 0)$ to $(1, 0)$, which we do not declare the state of) are not blue.

It is an easy exercise to verify that the above configuration of blue and non-blue edges occurs with positive probability. It also has the property that all vertices in the two squares must be occupied a.s., and that in each square the “one choice” made by each vertex gives an oriented cycle around the square. In particular the occupied vertices $(0, 0)$ and $(1, 0)$ have a.s. chosen one of the two edges in their relevant unit square that are incident to it. It follows that a.s. the edge e is not blue.

Notwithstanding, the arguments of [4] may be adapted (we do so in the proof of the following theorem) to obtain uniqueness for the one-choice model.

Theorem 5.2. *Let $d \geq 1$, and let N be the number of infinite blue clusters in the one-choice model on \mathbb{Z}^d . Either $\mathbb{P}_{p,\pi}(N = 0) = 1$ or $\mathbb{P}_{p,\pi}(N = 1) = 1$.*

Proof. We follow [4], in the style of [10, Sect. 8.2]. There are three statements to be proven, as follows:

- A. N is $\mathbb{P}_{p,\pi}$ -a.s. constant,
- B. $\mathbb{P}_{p,\pi}(N \in \{0, 1, \infty\}) = 1$,
- C. $\mathbb{P}_{p,\pi}(N = \infty) = 0$.

Proof of A. The probability measure $\mathbb{P}_{p,\pi}$ is a product measure on $\Omega_V \times \mathcal{D}$, and is therefore ergodic. The random variable N is a translation-invariant function on $\Omega_V \times \mathcal{D}$, and therefore N is $\mathbb{P}_{p,\pi}$ -a.s. constant.

Proof of B. Suppose $\mathbb{P}_{p,\pi}(N = k) = 1$ for some $2 \leq k < \infty$, from which assumption we will obtain a contradiction. As in [10], we work with boxes of \mathbb{Z}^d in the L^1 metric. For $m \geq 1$, let

$$D_m = \left\{ x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d : \sum_i |x_i| \leq m \right\},$$

considered as a subgraph of the infinite lattice. The *boundary* of D_m is the set $\partial D_m = D_m \setminus D_{m-1}$.

Let N_m be the number of infinite blue clusters that intersect D_m . We choose $m < \infty$ such that

$$(5.1) \quad \mathbb{P}_{p,\pi}(N = k, N_m \geq 2) > 0.$$

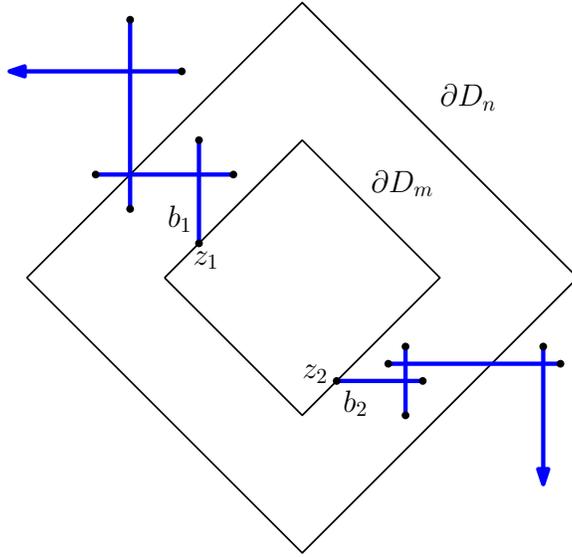


FIGURE 5.1. An illustration of the first part of the proof of statement B in Theorem 5.2. The arrows indicate connections to infinity.

For $m < n$, let $F_{m,n}$ be the event that there exists no blue segment that intersects both D_m and $\mathbb{Z}^d \setminus D_n$. Since all blue segments intersecting D_m are a.s. finite, for $\epsilon > 0$ there exists $n > m$ such that $\mathbb{P}_{p,\pi}(F_{m,n}) > 1 - \epsilon$. This we combine with (5.1), with sufficiently small ϵ , to deduce the existence of $n > m$ such that

$$(5.2) \quad \mathbb{P}_{p,\pi}(N = k, N_m \geq 2, F_{m,n}) > 0.$$

On the event that $N_m \geq 2$, we let I_1, I_2 be distinct infinite blue clusters that intersect D_m . For $j = 1, 2$, choose a point $z_j \in I_j \cap \partial D_m$ with the property that z_j lies in an infinite blue path J_j that traverses a sequence of sub-intervals of blue segments only one of which, denoted f_j , intersects D_m (the point of intersection being z_j). We denote by $b_j = (y_j, z_j)$ the edge of f_j that is incident with z_j and lies outside D_m and such that y_j lies in an infinite blue path that is a subset of $J_j \setminus D_m$; if there exists a choice for b_j , we choose one according to some fixed but arbitrary rule. See Figure 5.1.

We now perform surgery within the larger box D_n , by altering the configuration of occupied sites and of chosen directions. This is done in three stages.

First, we designate each z_j occupied (if not already occupied), and we set its chosen direction as pointing outwards from D_m along the edge b_j , and we write f'_j for the blue segment containing b_j in the

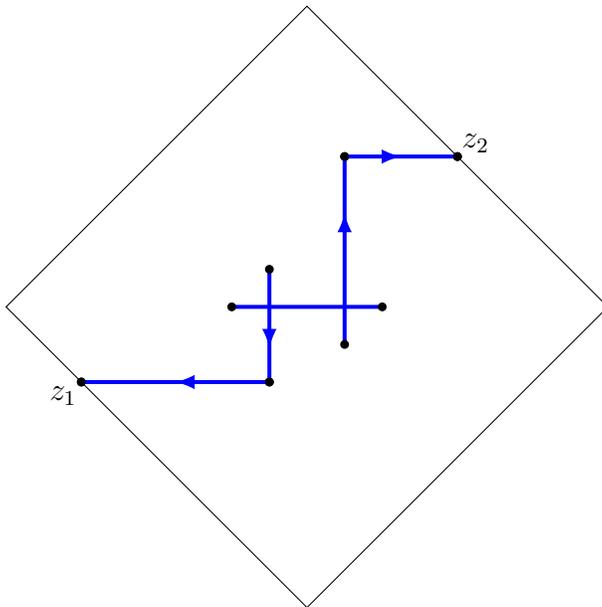


FIGURE 5.2. An illustration of the second part of the proof of B in Theorem 5.2.

ensuing graph. If z_j was not previously occupied, this may have the consequence of removing part of f_j from the blue graph. For any blue segment $f \neq f'_1, f'_2$, with endvertices u, v , that contains a point in D_m , we do the following.

- (a) If $u \in D_m$ (respectively, $v \in D_m$) and $u \neq z_1, z_2$ (respectively, $v \neq z_1, z_2$) we designate it unoccupied.
- (b) If $u \notin D_m$ (respectively, $v \notin D_m$) is the endvertex of no blue segment other than f , we designate it unoccupied.
- (c) If $u \notin D_m$ (respectively, $v \notin D_m$) is the endvertex of some blue segment $f' \neq f$ such that $f' \subseteq \mathbb{Z}^d \setminus D_m$, we keep u (respectively, v) occupied and have it choose its direction along f' .
- (d) If $u \notin D_m$ (respectively, $v \notin D_m$) is the endvertex of no blue segment $f' \neq f$ such that $f' \subseteq \mathbb{Z}^d \setminus D_m$, but is the endpoint of some $f'' \neq f$ such that $f'' \cap D_m \neq \emptyset$, we designate it unoccupied.

At the completion of this stage, all vertices of D_m except z_1 and z_2 are unoccupied, and moreover z_1 and z_2 are the only vertices of D_m belonging to a blue segment.

On the event $\{N_m \geq 2\} \cap F_{m,n}$, the states of sites and their chosen directions have been changed only within D_n . We now further alter the

configuration within D_m in order to create a blue connection between z_1 and z_2 . The following explanation is illustrated in Figure 5.2.

Recall that the origin o is the central vertex of D_m and let $c_{\pm} = o \pm 2e_1$. Set c_+ and c_- to be occupied with their chosen directions pointing towards each other, so that the segment S of four edges between them is blue. We now construct two disjoint paths which meet this segment and end at z_1 and z_2 . This occurs in two steps.

- (1) Choose one of the three points $\{o - e_1, o, o + e_1\}$ to connect to z_1 , and another to connect to z_2 . Call them s_1 and s_2 . (For m sufficiently large the choice does not matter, but a judicious choice may lead to simpler paths.)
- (2.i) Path ρ_1 is comprised of occupied vertices $\rho_{1,0}, \rho_{1,1}, \dots, \rho_{1,k}(= z_1)$ and the unoccupied vertices between them. It starts at one of the $2d - 2$ vertices $s_1 \pm e_i$, $i \in \{2, \dots, d\}$, meeting S only at the vertex s_1 (between $\rho_{1,0}$ and $\rho_{1,1}$). For $j = 0, \dots, k - 1$, each occupied $\rho_{1,j}$ has chosen direction pointing towards $\rho_{1,j+1}$.
- (2.ii) Path ρ_2 is comprised of occupied vertices $\rho_{2,0}, \rho_{2,1}, \dots, \rho_{2,l}(= z_2)$ and the unoccupied vertices between them. It starts at one of the $2d - 2$ vertices $s_2 \pm e_i$, $i \in \{2, \dots, d\}$, meeting S only at the vertex s_2 (between $\rho_{2,0}$ and $\rho_{2,1}$). For $j = 0, \dots, l - 1$, each occupied $\rho_{2,j}$ has chosen direction pointing towards $\rho_{2,j+1}$.

It may be checked that this construction is always possible for $m \geq 5$, and that we have a unique path of blue edges from z_1 to z_2 .

In summary, by altering the configuration within D_n , the number of infinite blue clusters can be reduced by one. Since D_n contains boundedly many vertices, we deduce from (5.2) that $\mathbb{P}_{p,\pi}(N = k - 1) > 0$, in contradiction of the assumption that $\mathbb{P}_{p,\pi}(N = k) = 1$. Statement B follows.

Proof of C. Following [4], the idea is to assume that $\mathbb{P}_{p,\pi}(N \geq 3) = 1$, to find *three* sites $z_1, z_2, z_3 \in \partial D_m$ connected to disjoint infinite blue clusters of $\mathbb{Z}^d \setminus D_m$, and then to perform surgery, as in the proof of B above, to show that D_m contains, with strictly positive probability, a vertex t whose deletion breaks an infinite blue cluster into three disjoint infinite blue clusters. Such a vertex t is called a *trifurcation*, and a neat argument due to Burton and Keane [4], based on the translation-invariance and polynomial growth of \mathbb{Z}^d , shows the impossibility of their existence with strictly positive probability. This provides the contradiction from which statement C follows. It suffices, therefore, to show that, if $\mathbb{P}_{p,\pi}(N \geq 3) = 1$, then there is a strictly positive probability of the existence of a trifurcation. It is not unusual in the related literature to close the proof with the above sketch, but we continue with

some details. We hope that the outline explanation presented here will convince readers without burdening them with too many details.

Fix $m \geq 1$; we shall require later that $m \geq M$ for some absolute constant M . As in (5.2), we find $n > m$ such that

$$\mathbb{P}_{p,\pi}(N_m \geq 3, F_{m,n}) > 0,$$

and, on the last event, we find distinct sites $z_1, z_2, z_3 \in \partial D_m$ connected to disjoint infinite blue clusters of $\mathbb{Z}^d \setminus D_m$. By altering the configuration on D_n as in the proof of statement B, we arrive at an event, with strictly positive probability, on which: (i) the z_i lie in infinite blue paths of $\mathbb{Z}^d \setminus D_m$, (ii) apart from the z_i , every site in D_m is unoccupied, and (iii) D_m contains no blue edge.

The rest of this proof is devoted to describing how to alter the configuration inside D_m in order to create a trifurcation. It is a simple generalisation of the proof of B above. We construct the same segment S as before. Then instead of constructing two disjoint paths ρ_1 and ρ_2 , we construct three disjoint paths ρ_1, ρ_2 and ρ_3 . We begin in a similar way.

- (1) Choose one of the three points $\{o - e_1, o, o + e_1\}$ to connect to z_1 , another to connect to z_2 , and the third to connect to z_3 . Call them s_1, s_2 and s_3 . (Again, for m sufficiently large the choice does not matter, but a judicious choice may lead to simpler paths.)
- (2.i-ii) See the proof of B.
- (2.iii) Path ρ_3 is comprised of occupied vertices $\rho_{3,0}, \rho_{3,1}, \dots, \rho_{3,h}$ ($= z_3$) and the unoccupied vertices between them. It starts at one of the $2d - 2$ vertices $s_3 \pm e_i$, $i \in \{2, \dots, d\}$, meeting S only at s_3 (between $\rho_{3,0}$ and $\rho_{3,1}$). For $j = 0, \dots, h - 1$, each occupied $\rho_{3,j}$ has chosen direction pointing towards $\rho_{3,j+1}$.

It may be checked that this construction is always possible for $m \geq 7$. The blue edges contain unique paths between z_1, z_2 and z_3 , and the origin o is a trifurcation.

This construction is illustrated in Figure 5.3, and the proof is complete. \square

We turn now to the proof of the existence of an infinite blue cluster, subject to appropriate conditions. Let $\omega \in \Omega_V$, and let $F(\omega)$ be the set of feasible pairs. The configuration space is $\Phi(\omega) = \{0, 1\}^{F(\omega)}$, and for $\phi \in \Phi(\omega)$, we call $f \in F(\omega)$ (and the corresponding segment of \mathbb{Z}^d) blue if $\phi_f = 1$. We shall consider probability measures μ on $\Phi(\omega)$ that satisfy a somewhat less restrictive condition than those of the one-choice model and the independent model.

- (1) *The one-choice model on \mathbb{Z}^d with $d \geq 2$.* The one-choice model on \mathbb{Z}^d , when restricted to \mathbb{Z}_d^2 , is no longer a one-choice model (for example, there exist a.s. occupied sites with no incident blue edge). Nevertheless, the restricted process satisfies C1 and C2, above, with parameter $\lambda = \lambda(d) > 0$ as given in (1.1). By Theorem 5.3, if $\lambda(d) > c \log(1/q)$, then \mathbb{Z}_d^2 contains an infinite blue cluster a.s., and therefore so does \mathbb{Z}^d . The condition on $\lambda(d)$ amounts to assuming $p < p_0(d)$ for some $p_0(d) > 0$. Theorem 2.1(i) is proved.
- (2) *The independent model.* In this case, $\lambda > 0$ is a parameter of the model. For $d \geq 2$, the model restricted to \mathbb{Z}_d^2 is the two-dimensional independent model with the same value of λ . By Theorem 5.3, \mathbb{Z}_d^2 , and hence \mathbb{Z}^d also, contains an infinite blue cluster a.s. whenever $\lambda > c \log(1/q)$. This proves Theorem 2.4(ii).

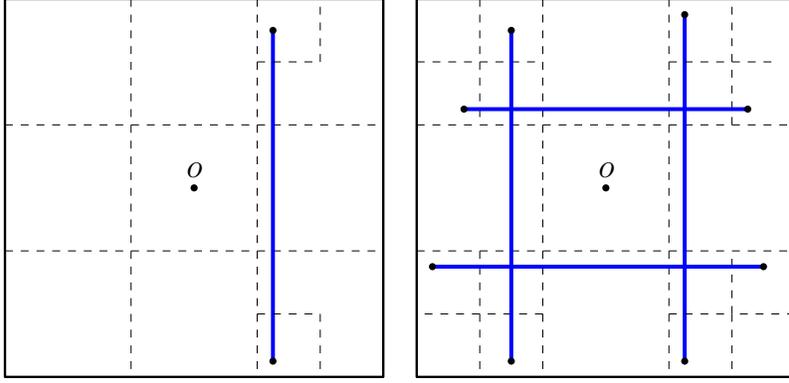
Proof of Theorem 5.3. The proof is by comparison with a supercritical, 1-dependent site percolation process on \mathbb{Z}^2 , and proceeds via a block argument not dissimilar to that used in [3, 11]. The idea is as follows. Let $p \in (0, 1]$. We partition \mathbb{Z}^2 into blocks of given side-length $6r$ where $r = r(p)$ satisfies $r(p) \rightarrow \infty$ as $p \downarrow 0$, and certain blocks will be called ‘good’ (the meaning of this will be explained). Then we show that the set of good blocks dominates (stochastically) a 1-dependent site percolation process with a density that can be made close to 1 by making p sufficiently small (and r correspondingly large). Finally, we show that, if there is an infinite cluster of good blocks, then there is necessarily an infinite blue cluster in the original lattice \mathbb{Z}^2 .

Write $e_1 = (1, 0)$, $e_2 = (0, 1)$, so that the four neighbours of $x \in \mathbb{Z}^2$ are $x \pm e_i$, $i = 1, 2$. Let $p = 1 - q \in (0, 1]$. We begin by defining the relevant block events. Let $r \geq 1$, to be chosen later, and let $\Lambda_o = [-3r, 3r]^2 \cap \mathbb{Z}^2$. For $x = (x_1, x_2) \in \mathbb{Z}^2$, let $\Lambda_x = 6rx + \Lambda_o$, the translate of Λ_o by $6rx$. The boxes $\{\Lambda_x : x \in \mathbb{Z}^2\}$ form a square paving of \mathbb{Z}^2 .

Consider first the special case $x = o$. Let

$$S_1 = [r, 2r) \times [2r, 3r), \quad S_2 = [r, 2r) \times (-3r, -2r],$$

as illustrated in Figure 5.4, and let A_{e_1} be the event that there exist occupied vertices $s_1 \in S_1$, $s_2 \in S_2$, such that the unordered pair (s_1, s_2) is feasible and blue. The probability of A_{e_1} may be calculated as follows. Let L be the line-segment $[r, 2r) \times \{0\}$. For $(k, 0) \in L$, let D_k be the event that

FIGURE 5.4. The events A_{e_1} and $A(o)$.

- (a) when proceeding north from $(k, 0)$, the first occupied vertex v encountered (including possibly $(k, 0)$ itself) is at distance between $2r$ and $3r - 1$ from L , and
- (b) when proceeding south from $(k, 0)$, the first occupied vertex w encountered (excluding $(k, 0)$) is at distance between $2r$ and $3r - 1$ from L , and
- (c) the feasible pair (v, w) is blue.

Now, A_{e_1} is the disjoint union of the D_k for $(k, 0) \in L$. Therefore,

$$(5.4) \quad 1 - \mathbb{P}_{p,\mu}(A_{e_1}) = [1 - \lambda(q^{2r} - q^{3r})(q^{2r-1} - q^{3r-1})]^r \\ = \left[1 - \frac{\lambda}{q}[q^{2r}(1 - q^r)]^2\right]^r.$$

We choose $r = r(p)$ to satisfy

$$(5.5) \quad q^r = \frac{1}{2}, \quad \text{that is } r = \frac{1}{\log_2(1/q)}.$$

Note that when p is small, q is close to 1, and so r is large. A small correction is necessary in order that r be an integer, but we shall overlook this for ease of notation. By (5.4), there exists an absolute constant $c_1 > 0$ such that

$$(5.6) \quad 1 - \mathbb{P}_{p,\mu}(A_{e_1}) \leq e^{-c_1 \lambda^r}.$$

Let $A_{-e_2}, A_{-e_1}, A_{e_2}$ be the respective events corresponding to A_{e_1} after Λ_o has been rotated clockwise by multiples of $\pi/2$. By (5.6) and symmetry,

$$(5.7) \quad 1 - \mathbb{P}_{p,\mu}(A(o)) \leq 4e^{-c_1 \lambda^r},$$

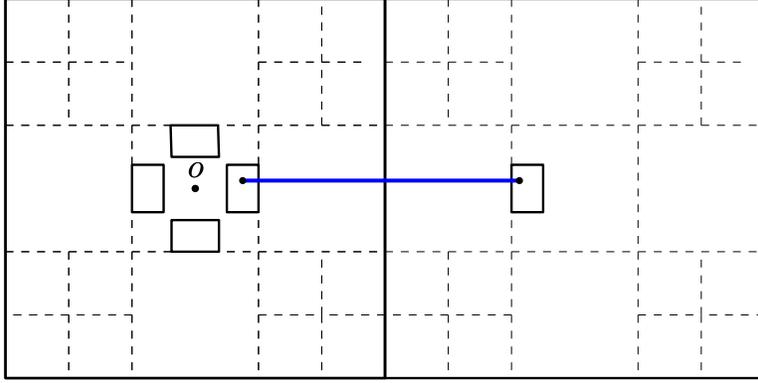


FIGURE 5.5. The regions T_{e_i} , $6re_1 + T_{-e_1}$, and the event C_{e_1} .

where the event $A(o) := A_{e_1} \cap A_{-e_2} \cap A_{-e_1} \cap A_{e_2}$ is illustrated in Figure 5.4. On $A(o)$, there exists a connected blue subgraph with large diameter in the box Λ_o .

We turn next to blue connections between neighbouring boxes, beginning with connections between Λ_o and Λ_{e_1} . Let

$$\begin{aligned} T_{e_1} &= \left(\frac{1}{3}r, r\right) \times \left[-\frac{1}{3}r, \frac{1}{3}r\right], & T_{-e_2} &= \left[-\frac{1}{3}r, \frac{1}{3}r\right] \times \left[-r, -\frac{1}{3}r\right), \\ T_{-e_1} &= \left[-r, -\frac{1}{3}r\right) \times \left[-\frac{1}{3}r, \frac{1}{3}r\right], & T_{e_2} &= \left[-\frac{1}{3}r, \frac{1}{3}r\right) \times \left(\frac{1}{3}r, r\right], \end{aligned}$$

as in Figure 5.5, noting that the $T_{\pm e_i}$ are disjoint; we assume for simplicity of notation that r is a multiple of 3. Let C_{e_1} be the event that there exists a blue, feasible pair $f = (u, v)$ with $u \in T_{e_1}$, $v \in 6re_1 + T_{-e_1}$. Similar to (5.4)–(5.6) we have that

$$(5.8) \quad \begin{aligned} 1 - \mathbb{P}_{p,\mu}(C_{e_1}) &= \left[1 - \frac{\lambda}{q} [q^{2r}(1 - q^{2r/3})]^2\right]^{2r/3} \\ &\leq e^{-c_2\lambda r}, \end{aligned}$$

for some absolute constant $c_2 > 0$. Let $C_{\pm e_i}$ be defined as C_{e_1} but with respect to blue connections between $T_{\pm e_i}$ and $\pm 6re_i + T_{\mp e_i}$. By symmetry, (5.8) holds with C_{e_1} replaced by $C_{\pm e_i}$, and by (5.8) the event $C(o) := C_{e_1} \cap C_{-e_2} \cap C_{-e_1} \cap C_{e_2}$ (depicted in Figure 5.6) satisfies

$$(5.9) \quad 1 - \mathbb{P}_{p,\mu}(C(o)) \leq 4e^{-c_2\lambda r},$$

We are now ready to define the block event associated with the vertex o . We call o *good* if the event $A(o) \cap C(o)$ occurs (see Figure 5.6). By (5.6) and (5.9),

$$(5.10) \quad \mathbb{P}_{p,\mu}(o \text{ is good}) \geq 1 - 4e^{-c_1\lambda r} - 4e^{-c_2\lambda r}.$$

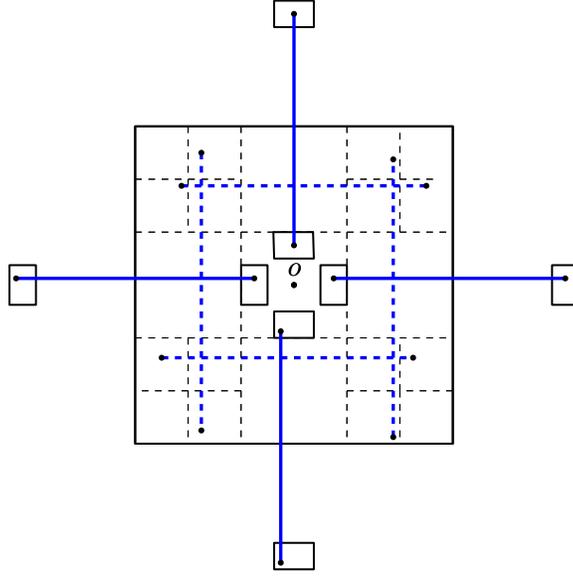


FIGURE 5.6. The event $C(o)$ is depicted in solid lines, and $A(o)$ in dashed lines.

Let $x \in \mathbb{Z}^2$, and let τ_x be the translation on \mathbb{Z}^2 by x , so that $\tau_x(y) = x + y$. This induces a translation on Ω_V , also denoted τ_x , by: for $\omega = \{\omega_v : v \in \mathbb{Z}^2\} \in \Omega_V$, we have $\tau_x(\omega) = \{\omega_{v-x} : v \in \mathbb{Z}^2\}$. The vertex $x \in \mathbb{Z}^2$ is declared *good* if o is good in the configuration $\tau_{6rx}(\omega)$, and we write G_x for the event that x is good. By (5.10),

$$(5.11) \quad \mathbb{P}_{p,\mu}(G_x) \geq 1 - 4e^{-c_1\lambda r} - 4e^{-c_2\lambda r}, \quad x \in \mathbb{Z}^2.$$

Let $d(u, v)$ denote the graph-theoretic distance from vertex u to vertex v . By examination of the definition of the events G_x , we see that their law is 1-dependent, in that, for $U, V \subseteq \mathbb{Z}^2$, the families $\{G_u : u \in U\}$, and $\{G_v : v \in V\}$ are independent whenever $d(u, v) \geq 2$ for all $u \in U, v \in V$. By [17, Thm 0.0] (see also [10, Thm 7.65]), there exists $\pi < 1$ such that: there exists almost surely an infinite cluster of good vertices of \mathbb{Z}^2 whenever $\mathbb{P}_{p,\mu}(G_0) \geq \pi$.

We choose $c > 0$ such that $1 - 4e^{-c_1\lambda r} - 4e^{-c_2\lambda r} > \pi$ whenever $\lambda r > c$. By (5.5), there exists a.s. an infinite good cluster in the block lattice $6rx\mathbb{Z}^2$ if $\lambda > c \log_2(1/\pi)$, which is to say that $q^{c_2\lambda} > 1$.

By considering the geometry of the block events, we see that any cluster of good vertices of \mathbb{Z}^2 gives rise to a cluster of blocks whose union contains a blue cluster intersecting every such block. The claim of the theorem follows. \square

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