Kemeny's constant for infinite DTMCs is infinite

Omer Angel* Mark Holmes†

Consider a positive recurrent discrete-time Markov chain $(X_n)_{n\geq 0}$ with finite or countable state space $\mathcal{S}$. For $x \in \mathcal{S}$ define the positive hitting time $T_x = \inf\{n \geq 1 : X_n = x\}$ and the hitting time $\theta_x = \inf\{n \geq 0 : X_n = x\}$. Let $\mathbb{P}_x$ denote the law of the process started from state x, and $\mathbb{E}_x$ denote the corresponding expectation. It was observed by Kemeny and Snell [3] that when $\mathcal{S}$ is finite the expected hitting time of a random stationary target, i.e. the quantity

$$\kappa_x = \sum_{y \in \mathcal{S}} \pi_y \mathbb{E}_x[T_y] \tag{1}$$

does not depend on x. (Here $\boldsymbol{\pi} = (\pi_y)_{y \in \mathcal{S}}$ is the stationary distribution for the chain.) Thus, the quantity $\kappa = \kappa_x$ in (1) is called Kemeny's constant. Considerable effort has been devoted to giving an "intuitive" proof of this result. In [1] it was argued that it is more natural to consider the quantity

$$\omega_x = \sum_{y \in \mathcal{S}} \pi_y \mathbb{E}_x[\theta_y]. \tag{2}$$

Note that $\mathbb{E}_x[\theta_y] = \mathbb{1}_{\{y \neq x\}} \mathbb{E}_x[T_y]$, from which it follows that $\kappa_x = 1 + \omega_x$ (since $\pi_x \mathbb{E}_x[T_x] = 1$). For finite $\mathcal{S}$, Hunter [2] has established the sharp bound $\kappa \geq (|\mathcal{S}| + 1)/2$ (the bound is achieved by the directed non-random walk on the cycle). It is conjectured in [1, Page 1031] that κ is infinite for any infinite state chain. In this note we verify this conjecture.

Theorem 1. *For an irreducible positive recurrent, discrete-time Markov chain with infinite state space and for any $x \in \mathcal{S}$ we have $\kappa_x = \sum_{y \in \mathcal{S}} \pi_y \mathbb{E}_x[T_y] = \infty$.*

*Department of Mathematics, University of British Columbia angel@math.ubc.ca
†School of Math. and Stat., University of Melbourne holmes.m@unimelb.edu.au

This theorem is an immediate consequence of the following result:

Lemma 1. *Let $\mathcal{S}$ be finite or infinite. Then for every $x, y \in \mathcal{S}$, $\mathbb{E}_x[T_y] \geq \pi_x/(2\pi_y)$.*

Proof. We first prove by induction on $n \geq 0$ that $\mathbb{P}_x(X_n = y) \leq \frac{\pi_y}{\pi_x}$ for every x, y. The case $n = 0$ is trivial (for both $x = y$ and $x \neq y$). For $n \geq 1$ we have

$$\mathbb{P}_x(X_n = y) = \sum_{u \in \mathcal{S}} \mathbb{P}_x(X_{n-1} = u) p_{u,y} \leq \sum_{u \in \mathcal{S}} \frac{\pi_u}{\pi_x} p_{u,y} = \frac{\pi_y}{\pi_x}, \tag{3}$$

where $(p_{w,z})_{w,z \in \mathcal{S}}$ are the one-step transition probabilities and we have used the induction hypothesis and the full balance equations. Using (3) we have

$$\mathbb{P}_x(T_y \leq n) = \mathbb{P}_x(\cup_{j=1}^n \{X_j = y\}) \leq \sum_{j=1}^n \mathbb{P}_x(X_j = y) \leq \frac{n\pi_y}{\pi_x}. \tag{4}$$

Therefore $\mathbb{P}_x(T_y > n) \geq 1 - n\frac{\pi_y}{\pi_x}$, and

$$\mathbb{E}_x[T_y] = \sum_{n=0}^{\infty} \mathbb{P}_x(T_y > n) \geq \sum_{n=0}^{\lfloor \pi_x/\pi_y \rfloor} \left(1 - \frac{n\pi_y}{\pi_x}\right) \geq \frac{\pi_x}{2\pi_y}. \tag{5}$$

The last step uses the fact that for $a \geq 0$,

$$\sum_{n=0}^{\lfloor a \rfloor} \left(1 - \frac{n}{a}\right) = \frac{(2a - \lfloor a \rfloor)(\lfloor a \rfloor + 1)}{2a} \geq \frac{a}{2}.$$

□

Acknowledgements. OA is supported in part by NSERC. MH is supported by Future Fellowship FT160100166, from the Australian Research Council (ARC).

References

[1] D. Bini, J.J. Hunter, G. Latouche, B. Meini, P. Taylor. Why is Kemeny's constant a constant? *J. Applied Probability.* 55:1025–1036, (2018).

[2] J. J. Hunter. Mixing times with applications to perturbed Markov chains. *Linear Algebra and its Applications*, 417:108–123, (2006).

[3] J.G. Kemeny and J.L. Snell. *Finite Markov Chains*. Van Nostrand, Princeton, NJ. (1960).