SPEED CALCULATIONS FOR RANDOM WALKS IN DEGENERATE RANDOM ENVIRONMENTS.

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Abstract. We give details of the calculations of explicit speeds for random walks in uniform degenerate random environments.

1. Introduction

In [3] the authors study random walk in an IID random environment, where the environment need not satisfy any ellipticity condition. In other words, where various nearest neighbour transitions may have quenched probability = 0. If such a walk can get stuck on a finite set of vertices with positive probability, then it will get stuck with probability one. There are necessary and sufficient conditions for such a walk not to get stuck in this way, and [3] studies transience and speed questions for such walks. There are many interesting models in which such properties are non-trivial. There are also examples in which transience is essentially trivial, and in which speeds can be calculated explicitly, because of a renewal structure. [3] gives a table of such speeds, for random walks in *uniform* degenerate random environments. That is, environments in which the walker chooses at random from the (random) set of allowed steps. The purpose of this note is to supply details of the latter calculations.

For fixed $d \geq 2$ let $\mathcal{E} = \{\pm e_i : i = 1, \dots, d\}$ be the set of unit vectors in $\mathbb{Z}^d$. Let $\mathcal{P} = M_1(\mathcal{E})$ denote the set of probability measures on $\mathcal{E}$, and let μ be a probability measure on $\mathcal{P}$. Let $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ be equipped with the product measure $\nu = \mu^{\otimes \mathbb{Z}^d}$ (and the corresponding product σ-algebra). A random environment $\omega = (\omega_x)_{x \in \mathbb{Z}^d}$ is an element of Ω. We write $\omega_x(e)$ for $\omega_x(\{e\})$. Note that $(\omega_x)_{x \in \mathbb{Z}^d}$ are i.i.d. with law μ under ν.

The random walk in environment ω is a time-homogeneous Markov chain with transition probabilities from x to $x + e$ defined by

$$p_\omega(x, x + e) = \omega_x(e). \tag{1.1}$$

Given an environment ω, we let $\mathbb{P}_\omega$ denote the law of this random walk X_n, starting at the origin. Let P denote the law of the annealed random walk, i.e. $P(\cdot, \star) := \int_\star \mathbb{P}_\omega(\cdot) d\nu$. Since $P(A) = E_\nu[\mathbb{P}_\omega(A)]$ and $0 \leq f(\omega) = \mathbb{P}_\omega(A) \leq 1$, $P(A) = 1$ if and only if $\mathbb{P}_\omega(A) = 1$ for ν-almost every ω. Similarly $P(A) = 0$ if and only if $\mathbb{P}_\omega(A) = 0$ for ν-almost every ω. If we start the RWRE at $x \in \mathbb{Z}^d$ instead, we write P_x for the corresponding probability, so $P = P_o$.

We associate to each environment ω a directed graph $\mathcal{G}(\omega)$ (with vertex set $\mathbb{Z}^d$) as follows. For each $x \in \mathbb{Z}^d$, the directed edge $(x, x + u)$ is in $\mathcal{G}_x$ if and only if $\omega_x(u) > 0$, and the edge set of $\mathcal{G}(\omega)$ is $\cup_{x \in \mathbb{Z}^d} \mathcal{G}_x(\omega)$. For convenience we will also write $\mathcal{G} = (\mathcal{G}_x)_{x \in \mathbb{Z}^d}$. Note that under ν, $(\mathcal{G}_x)_{x \in \mathbb{Z}^d}$ are i.i.d. subsets of $\mathcal{E}$. The graph $\mathcal{G}(\omega)$ is equivalent to the entire graph $\mathbb{Z}^d$, precisely when the environment is *elliptic*, i.e. $\nu(\omega_x(u) > 0) = 1$ for each $u \in \mathcal{E}, x \in \mathbb{Z}^d$. Much of the current literature assumes either the latter condition, or the stronger property of *uniform ellipticity*, ie that $\exists \epsilon > 0$ such that $\nu(\omega_x(u) > \epsilon) = 1$ for each $u \in \mathcal{E}, x \in \mathbb{Z}^d$.

On the other hand, given a directed graph $\mathcal{G} = (\mathcal{G}_x)_{x\in\mathbb{Z}^d}$ (with vertex set $\mathbb{Z}^d$, and such that $\mathcal{G}_x \neq \varnothing$ for each x), we can define a *uniform* random environment $\omega = (\omega_x(\mathcal{G}_x))_{x\in\mathbb{Z}^d}$. Let $|A|$ denote the cardinality of A, and set

$$\omega_x(e) = \begin{cases} |\mathcal{G}_x|^{-1}, & \text{if } e \in \mathcal{G}_x \\ 0, & \text{otherwise} \end{cases}.$$

The corresponding RWRE then moves by choosing uniformly from available steps at its current location. This gives us a way of constructing rather nice and natural examples of random walks in non-elliptic random environments: first generate a random directed graph $\mathcal{G} = (\mathcal{G}_x)_{x\in\mathbb{Z}^d}$ where $\mathcal{G}_x$ are i.i.d., then run a random walk on the resulting random graph (choosing uniformly from available steps).

Definition 1.1. *We say that the environment is* 2-valued *when μ charges exactly two points, i.e. there exist $\gamma_1, \gamma_2 \in \mathcal{P}$ and $p \in (0,1)$ such that $\mu(\{\gamma_1\}) = p$, $\mu(\{\gamma_2\}) = 1-p$. We say that the graph is* 2-valued *when there exist $E^1, E^2 \subset \mathcal{E}$ and $p \in (0,1)$ such that $\mu(\mathcal{G}_o = E_1) = p$ and $\mu(\mathcal{G}_o = E_2) = 1-p$.*

[3] proves that the following simple criterion is equivalent to the statement that the random walk visits infinitely many sites almost surely.

(1.2) There exists an orthogonal set V of unit vectors such that $\mu(\mathcal{G}_o \cap V \neq \varnothing) = 1$.

The following is proved in [3]:

Lemma 1.2. *Assume (1.2) and suppose that $\mu(\downarrow\in \mathcal{G}_o) = 0$ but $\mu(\uparrow\in \mathcal{G}_o) > 0$. Then the RWRE is transient in direction e_2, P-almost surely. Let T be the first time the RWRE follows direction e_2. If $E[T] < \infty$ then X_n has an asymptotic speed $v = (v^{[1]}, \dots, v^{[d]})$, in the sense that $P(n^{-1}X_n \to v) = 1$. Moreover, $v^{[i]} = E[X_T^{[i]}]/E[T]$.*

Proof. The random walk visits infinitely many sites, and at each visit to a new site there is positive (non-vanishing) probability of then taking a step in direction e_2. Thus the second coordinate of the random walk converges monotonically to ∞.

Let τ_k be the k'th time that X_n moves in direction e_2, and $\tau_0 = 0$. Let $Y_k = X_{\tau_k} - X_{\tau_{k-1}}$. Since the environment seen by the random walker is refreshed at every time τ_k, the Y_k are IID, and the τ_k are sums of IID random variables with distribution that of T. Because $E[T] < \infty$, it follows that $E[|Y_k|] < \infty$ as well. By the law of large numbers, $\tau_k/k \to E[T]$ and $X_{\tau_k}/k \to E[Y_1]$ almost surely. Moreover $k^{-1}\max\{|X_n - X_{\tau_{k-1}}| : \tau_{k-1} \le n \le \tau_k\} \to 0$. Thus

$$\frac{1}{n}X_n \to \frac{1}{E[T]}E[Y_1] = v \quad P\text{-almost surely.}$$

∎

Table 1 summarizes what we know about uniform RWDRE in 2-dimensional 2-valued random environments. It reproduces and updates Table 1 of [3]. There is a related table in [2], giving percolation properties for the directed graphs $\mathcal{C}$ and $\mathcal{M}$. The latter includes 2-valued environments such as (←↕→, ·) (site percolation), in which one of the possible environments has no arrows. These environments do not appear in the present table, because (as remarked in Section 3 of [3]), the walk gets stuck on a finite set of vertices (in this case 1 vertex). The RWRE setup we have chosen requires that motion be possible in at least one direction.

Notes to Table 1

[1] The authors believe it follows from results of Berger & Deuschel [1] that $\mathcal{M}$ is recurrent $\forall p$.
[2] Bounds on the critical probability are given in [2]. Improved bounds are in preparation.
[3] Improved ranges of values giving transience and speeds are in preparation.
[4] We do not have a closed form expression for this. But an approach to getting asymptotic expressions is given below.
[5] An expansion in terms of q-hypergeometric functions is described below, without full details.

2. Speeds

The non-trivial 2-valued uniform models, in which one must turn to the results of [3] for existence of a speed, and in which we can say very little about the speed, other than mononicity are as follows:

- ↑→ ←↓
- ←→↓ ↑
- ←→↓ ↕
- ←→↓ ↑→
- ←→↓ ↕→
- ←→↓ ←↑→
- ←↕→ ↑
- ←↕→ ↑→
- ←↕→ ←→↓

There are two further models, which are also non-trivial, but for which, once we know that the speed exists, it must be $v = (0, 0)$ by symmetry, namely:

- ↔ ↕
- ←↕→ ↔

The simplest models where one can explicitly calculate the speed are:

- ↑ →:
  Because the RWDRE sees a new environment every time, the speed is simply $(p, 1-p)$.
- ↔ → :
  Let τ_k be the k'th time n that $\mathcal{G}_{X_n} = \rightarrow$, with τ_0=0. Let $\eta_k = X^{[1]}_{\tau_k}$. At each time τ_k the process starts exploring a new independent environment, so $T_k = \tau_k - \tau_{k-1}$ are IID (for $k \geq 2$), as are $M_k = \eta_k - \eta_{k-1}$. By the strong law, $\eta_k/\tau_k \to E[M_2]/E[T_2]$ as $k \to \infty$. If N_n is the last k such that $\tau_k \leq n$ then

$$\frac{\eta_{N_n}}{\tau_{N_{n+1}}} \leq \frac{X^{[1]}_n}{n} \leq \frac{\eta_{N_{n+1}}}{\tau_{N_n}}$$

  so that $X_n/n \to E[M_2]/E[T_2]$ as well.

γ_1, γ_2	Random walk	Reference
$\uparrow\ \rightarrow$	$v = (1-p, p)$.	given here
$\uparrow\ \downarrow$	Stuck on two vertices.	Lemma 3.1 of [3]
$\leftrightarrow\ \uparrow$	$v = \left(0, \frac{(1-p)^2}{p+(1-p)^2}\right)$.	given here
$\leftrightarrow\ \rightarrow$	$v = \left(\frac{1-p}{1+p}, 0\right)$.	given here
$\leftrightarrow\ \updownarrow$	$v = (0, 0)$.	Symmetry[1]
$\uparrow_{\!\rightarrow}\ \uparrow$	$v = \left(\frac{p}{2}, 1-\frac{p}{2}\right)$.	given here
$\uparrow_{\!\rightarrow}\ {}_{\leftarrow\!}\uparrow$	$v = \left(\frac{(2p-1)(p^2-p+6)}{6(2-p)(1+p)}, \frac{1}{2}\right)$.	given here
$\uparrow_{\!\rightarrow}\ \leftrightarrow$	$v = \left(\frac{1}{p^2} + \frac{(1-p)^2}{2p(1-p+p\log p)}\right)^{-1} \cdot (1, 1)$.	given here
$\uparrow_{\!\rightarrow}\ \leftarrow$	$v = \left(\frac{p(2-p)}{2+3p-2p^2-p^3}\right) \cdot (3, 1) + (-1, 0)$.	given here
$\uparrow_{\!\rightarrow}\ {}^{\leftarrow\!}\downarrow$	$v^{[1]} = v^{[2]} \uparrow$ in p. Transient[2] for $p \approx 0, 1$.	Cor. 3.8 & 5.2 of [3]
	Conjecture: $v \neq 0$ for $p \neq \frac{1}{2}$, Recurrent when $p = \frac{1}{2}$	
$\overset{\leftrightarrow}{\downarrow}\ \downarrow$	$\frac{1}{v^{[2]}} = \frac{8p(1-p)}{1+\sqrt{5}} - 1 - 2p - \frac{4(1-p)^2(5+\sqrt{5})}{p(1+\sqrt{5})} \sum_{n=2}^{\infty} \frac{p^k}{1+2^{-k}(3+\sqrt{5})^k}$, $v^{[1]} = 0$.	given here
$\overset{\leftrightarrow}{\downarrow}\ \rightarrow$	$-\frac{1}{v^{[2]}} = 4 - p - \frac{5+\sqrt{5}}{2}(1-p)^2\Theta(p\gamma) + \frac{(1-p)[3+\sqrt{5}-(1-p)(5+\sqrt{5})\Theta(p)]^2}{(3+\sqrt{5})[2-(1-p)(5+\sqrt{5})\Theta(p\gamma)]}$	given here
	where $\gamma = \frac{3+\sqrt{5}}{2}$ and $\Theta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\gamma^{2n+1}+1}$. $v^{[1]} = 1 - 3v^{[2]}$.	
$\overset{\leftrightarrow}{\downarrow}\ \uparrow$	$v^{[1]} = 0$, $v^{[2]} \downarrow$ in p. Transient[2] for $p \approx 0$.	Equation (3.4) of [3]
	Conjecture: $\exists! p (\neq 3/4)$ s.t. $v[p] = 0$. Recurrent for this p.	
$\overset{\leftrightarrow}{\downarrow}\ \leftrightarrow$	$v^{[1]} = 0$, $v^{[2]} < 0$ for $p > 0$. $v^{[2]}$ strictly $\downarrow$ in p.	given here[4]
$\overset{\leftrightarrow}{\downarrow}\ \updownarrow$	$v^{[1]} = 0$, $v^{[2]} \downarrow$ in p. Transient[3] for $p > \frac{3}{4}$, $v^{[2]} < 0$ for $p > \frac{6}{7}$.	Cor. 5.2 / Prop. 5.5 of [3]
	Conjecture: $v^{[2]} < 0$ for $p > 0$.	
$\overset{\leftrightarrow}{\downarrow}\ \uparrow_{\!\rightarrow}$	$3v^{[2]} = 5v^{[1]} - 1$. $v^{[1]} \downarrow$ in p.	Lemma 4.2 / Cor. 5.2 of [3]
$\overset{\leftrightarrow}{\downarrow}\ {}^{\leftarrow\!}\downarrow$	$v^{[1]} = 1 + 3v^{[2]}$	given here[5]
$\overset{\leftrightarrow}{\downarrow}\ \updownarrow\!\rightarrow$	$v \cdot (1, -1) = \frac{1}{3}$, $v \cdot (1, 1) \downarrow$ in p.	Corollary 5.2 of [3]
$\overset{\leftrightarrow}{\downarrow}\ \underset{\leftrightarrow}{\uparrow}$	$v^{[1]} = 0$, $v^{[2]} \downarrow$ in p.	Corollary 5.2 of [3]
	Conjecture: $v^{[2]} \neq 0$ for $p \neq \frac{1}{2}$. Recurrent when $p = \frac{1}{2}$.	
$\overset{\updownarrow}{\leftrightarrow}\ \uparrow$	$v^{[1]} = 0$, $v^{[2]} \downarrow$ in p. Transient[3] for $p < \frac{1}{2}$, $v^{[2]} > 0$ for $p < \frac{1}{3}$.	Cor. 5.2 / Prop. 5.5 of [3]
	Conjecture: $v^{[2]} > 0$ for $p < 1$.	
$\overset{\updownarrow}{\leftrightarrow}\ \uparrow_{\!\rightarrow}$	$v^{[1]} = v^{[2]} \downarrow$ in p. Transient[3] for $p < \frac{1}{2}$, $v^{[1]} > 0$ for $p < \frac{1}{3}$.	Cor. 5.2 / Prop. 5.5 of [3]
	Conjecture: $v^{[1]} > 0$ for $p < 1$.	
$\overset{\updownarrow}{\leftrightarrow}\ \leftrightarrow$	$v = (0, 0)$	Symmetry[1].
$\overset{\updownarrow}{\leftrightarrow}\ \overset{\leftrightarrow}{\downarrow}$	$v^{[1]} = 0$, $v^{[2]} \uparrow$ in p. Transient[3] for $p < \frac{1}{4}$, $v^{[2]} < 0$ for $p < \frac{1}{7}$.	Cor. 5.2 / Prop. 5.5 of [3]
	Conjecture: $v^{[2]} < 0$ for $p < 1$.	

TABLE 1. Table of results for RW in 2-dimensional 2-valued degenerate random environments, where the first configuration occurs with probability $p \in (0, 1)$ and the other with probability $1 - p$.

If $M_2 = m$ then $\mathbb{P}_{\mathcal{G}}(T_2) = m^2$, since that is the mean time to reach m of a random walk on $[0, m]$ with reflection at 0. Thus $E[T_2] = E[M_2^2]$. But M_2 is geometric, $E[M_2] = \sum_{m=1}^{\infty} mp^{m-1}(1-p) = 1/(1-p)$ and $E[M_2^2] = (1+p)/(1-p)^2$. So $v^{[1]} = (1-p)/(1+p)$.

- ↳ ↑:
  In this model also, each step of X_n explores a new environment. So we essentially have a regular random walk, whose step distribution is $\rightarrow$ with probability $p/2$ and $\uparrow$ with probability $1-p/2$. So $v = (p/2, 1-p/2)$.

In the remaining examples, we use the setup of Lemma 1.2. There is a direction e for which the first time T that X_n moves in direction e is a renewal time – what happens starting at time T is independent of what came before. If $e = \pm e_1$, then $v^{[1]} = \pm 1/E[T]$. Then if $Y = X_T^{[2]}$, then $v^{[2]} = E[Y]/E[T]$. With corresponding formulae if $e = \pm e_2$. In the following example we calculate $E[Y]$ to get the speed.

- ↳ ↲:
  For $n \geq 0$, let $\tau_n = \inf\{m \geq 0 : X_m^{[2]} = n\}$. Then for $i \geq 1$, $T_i = \tau_i - \tau_{i-1}$ are i.i.d. Geometric(1/2) random variables (with mean 2), and $Y_i = X_{\tau_i - 1}^{[1]} - X_{\tau_{i-1}}^{[1]}$ are i.i.d. random variables, independent of the $\{T_i\}_{i\geq 1}$. So $E[T_i] = 2$ and $v^{[2]} = 1/2$. Let $N_n = \sup\{m \geq 0 : \tau_m \leq n\}$. Here $e = \uparrow$, and the first time T that we move upwards is geometric with parameter 1/2.

  Then almost surely,

$$\frac{Y_n^{[1]}}{n} = \frac{\sum_{i=1}^{N_n} Y_i + \sum_{i=\tau_{N_n}+1}^{n} (X_i^{[1]} - X_{i-1}^{[1]})}{n} = \frac{N_n}{n}\frac{\sum_{i=1}^{N_n} Y_i}{N_n} + \frac{\sum_{i=\tau_{N_n}+1}^{n} (X_i^{[1]} - X_{i-1}^{[1]})}{n} \to \frac{E[Y_1]}{E[T_1]},$$

  as $n \to \infty$, where we have used the fact that $|\sum_{i=\tau_{N_n}+1}^{n} (X_i^{[1]} - X_{i-1}^{[1]})| \leq T_{N_{n+1}}$.

  Now let $Y = X_T^{[1]}$, so $v^{[1]} = E[Y]/2$. For $j \geq 1$, we can have $Y = j$ three ways – reaching no ↲ vertex, reaching a ↲ vertex at $(j, 0)$, or reaching a ↲ vertex at $(j+1, 0)$. Thus

$$P(Y = j) = p^{j+1}\big(\frac{1}{2}\big)^{j+1} + p^j(1-p)\sum_{n=0}^{\infty}\big(\frac{1}{2}\big)^{j+2n+1} + p^{j+1}(1-p)\sum_{n=0}^{\infty}\big(\frac{1}{2}\big)^{j+2n+3}$$
$$= \frac{p^j(4-p^2)}{3\cdot 2^{j+1}}.$$

  Likewise, we can have $Y = -j$, $j \geq 1$ three ways, depending on where if anywhere X_n reaches a ↳ vertex, giving $P(Y = -j) = \big((1-p)^j(4-(1-p)^2)\big)/\big(3\cdot 2^{j+1}\big)$. The case $j = 0$ would be similar, but is not needed. Summing over j gives that

$$E[Y] = \frac{p(4-p^2)}{12}\cdot\frac{1}{(1-p/2)^2} - \frac{(1-p)(4-(1-p)^2)}{12}\cdot\frac{1}{(1-(1-p)/2)^2}$$
$$= \frac{p(2+p)}{3(2-p)} - \frac{(1-p)(3-p)}{3(1+p)} = \frac{(2p-1)(p^2-p+6)}{3(2-p)(1+p)}.$$

In some cases, we can avoid calculating $E[Y]$ directly. Again, assume $e = \pm e_1$. There are two generators, $\mathcal{L}_1$ and $\mathcal{L}_2$, depending on the environment. If we apply them to the functions $f_j(x) =$

$x^{[j]}$ we get that $X_n^{[j]} - \sum_{k<n}(\mathcal{L}_1 f_j 1_{\{\mathcal{G}_{X_k}=\gamma_1\}} + \mathcal{L}_2 f_j 1_{\{\mathcal{G}_{X_k}=\gamma_2\}})$ is a martingale. Therefore

$$v^{[j]}E[T] = E[X_T^{[j]}] = \mathcal{L}_1 f_j \alpha_1 + \mathcal{L}_2 f_j \alpha_2$$

where $\alpha_j = E[\#\{k < T : \mathcal{G}_{X_k} = A_j\}]$. When $j = 1$ the LHS is ± 1, which usually lets us solve for α_1 in terms of $E[T]$. We know that $\alpha_2 = E[T] - \alpha_1$, so putting $j = 2$ will then give us $v^{[2]}$. Thus all that remains is to calculate $E[T]$. (We could have done the previous example this was as well.)

- $\leftrightarrow\uparrow$:
  Here $e =\uparrow$. We have $v^{[1]} = 0$ by symmetry, and $v^{[2]} = 1/E[T]$. If the origin is $\uparrow$, then $T = 1$. Otherwise, suppose there are $\uparrow$ at $(-i, 0)$ and at $(j, 0)$, with only $\leftrightarrow$ in between. Then $\mathbb{P}_{\mathcal{G}}(T-1) = ij$, since this is the mean exit time for a simple random walk on $[-i, j]$. Given that the origin is $\leftrightarrow$ (which happens with probability p), i and j are independent geometric random variables, with means $1/(1-p)$. Thus $E[T] = 1 + p/(1-p)^2$.

- $\uparrow_{\!\rightarrow}$ $\leftrightarrow$:
  Here $e =\uparrow$, and the martingale equations are that $1 = v^{[2]}E[T] = \alpha_2/2$ and $v^{[1]}E[T] = \alpha_2/2$. In other words, $v^{[1]} = v^{[2]} = 1/E[T]$. So we must now find $E[T]$.

  First consider a random walk Z_j on $[0, n)$ with the following boundary conditions: at n there is absorption, and at 0 we reflect with probability 1/2 and die otherwise. Let S be the time of death or absorption, and let $f(k) = E[S \mid Z_0 = k]$. Then

  $$f(k) = 1 + \frac{f(k-1) + f(k+1)}{2}$$

  for $1 \le k \le n-1$, $f(n) = 0$, and $f(0) = 1 + f(1)/2$. The solution to the recurrence is $f(k) = A + Bk - k^2$, and substituting the boundary conditions gives $f(k) = (n-k)(k+1)$. Likewise let $g(k) = P(Z_S = n \mid Z_0 = k)$. Then $g(k) = [g(k-1) + g(k+1)]/2$ for $1 \le k \le n-1$, with boundary conditions $g(0) = g(1)/2$ and $g(n) = 1$. This has solution $g(k) = (k+1)/(n+1)$.

  Now think of how X_j evolves. Let the first $\uparrow_{\!\rightarrow}$ to the left of o be at $x_0 = (i_0, 0)$, where $i_0 \le 0$. Let successive $\uparrow_{\!\rightarrow}$ to the right of o be at $x_1 = (i_1, 0)$, $x_2 = (i_2, 0)$, etc., where $0 < i_1 < i_2 < \dots$. On the horizontal interval $[x_0, x_1)$, X_j performs a simple random walk till it hits x_0 or x_1. When it hits x_0 it either moves upward (making this time T), or it reflects back into the interval. If it reaches x_1 it leaves this interval forever, and starts the same process over again on the interval $[x_1, x_2)$. Let the interval being visited at time T be $[x_N, x_{N+1})$, where $N \ge 0$, and let S_j be the total time spent in $[x_j, x_{j+1})$. Therefore $T = \sum_{j=0}^{N} S_j$, and

  $$E[T] = \sum_{j=0}^{\infty} E[S_j 1_{\{N \ge j\}}].$$

  Let A_j be the event that X_n exits $[x_j, x_{j+1})$ on the right, ie. at x_{j+1}. Then for $j \ge 1$, $\{N \ge j\} = \cap_{k=0}^{j-1} A_k$. Moreover, there is a renewal every time X_n enters a new interval, because a new environment starts getting explored. The interval $[x_0, x_1)$ is different from the rest, because we start at o. But for all other $[x_j, x_{j+1})$ the process starts walking at x_j. Therefore the cases $j \ge 1$ are actually independent replications of the same procedure. In other words,

$$E[T] = E[S_0] + \sum_{j=1}^{\infty} E[S_j \prod_{k=0}^{j-1} 1_{A_k}] = E[S_0] + \sum_{j=1}^{\infty} E[S_1 \mid A_0] P(A_0) P(A_1 \mid A_0)^{j-1}.$$

We use the expressions for f and g to work out these factors. By the expression for f, we have $\mathbb{P}_{\mathcal{G}}(S_0) = i_1(1-i_0)$. Take $i = -i_0$ and $k = i_1$. Then

$$E[S_0] = \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} p^2(1-p)^{i+k-1} k(1+i) = \Big(\sum_{k=1}^{\infty} p(1-p)^{k-1} k \Big)^2 = \frac{1}{p^2}.$$

Likewise $\mathbb{P}_{\mathcal{G}}(S_1 \mid A_0) = i_2 - i_1$. Write k for this quantity, so

$$E[S_1 \mid A_0] = \sum_{k=1}^{\infty} (1-p)^{k-1} pk = \frac{p}{p^2} = \frac{1}{p}.$$

Similarly, $\mathbb{P}_{\mathcal{G}}(A_0) = (1-i_0)/(1+i_1-i_0)$, so (letting $n = i+k$)

$$\begin{aligned} P(A_0) &= \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} p^2 (1-p)^{i+k-1} \frac{1+i}{1+i+k} = \sum_{n=1}^{\infty} p^2 (1-p)^{n-1} \sum_{j=1}^{k} \frac{n+1-j}{1+n} \\ &= \sum_{n=1}^{\infty} \frac{p^2(1-p)^{n-1}}{n+1} \cdot \frac{n(n+1)}{2} = \frac{p^2}{2} \cdot \frac{1}{p^2} = \frac{1}{2}. \end{aligned}$$

And $\mathbb{P}_{\mathcal{G}}(A_1 \mid A_0) = 1/(1+i_2-i_1)$, so

$$\begin{aligned} P(A_1 \mid A_0) &= \sum_{k=1}^{\infty} p(1-p)^{k-1} \frac{1}{1+k} = \frac{p}{(1-p)^2} \Big(\sum_{k=0}^{\infty} \frac{(1-p)^{n+1}}{n+1} - (1-p) \Big) \\ &= \frac{p}{(1-p)^2} (-\log p - (1-p)) = 1 - \frac{1-p+p\log p}{(1-p)^2}. \end{aligned}$$

Putting this together,

$$E[T] = \frac{1}{p^2} + \frac{(1-p)^2}{2p(1-p+p\log p)}.$$

- $\mathrel{\uparrow\!\!\!\rightarrow} \leftarrow$:
Here $e = \uparrow$, and the martingale equations are that $1 = v^{[2]} E[T] = \alpha_1/2$ and $v^{[1]} E[T] = \alpha_1/2 - \alpha_2 = -E[T] + 3\alpha_1/2$. In other words, $v^{[2]} = 1/E[T]$ and $v^{[1]} = -1 + 3/E[T]$. So we must now find $E[T]$.

Suppose that there is a $\mathrel{\uparrow\!\!\!\rightarrow}$ at $(-i,0)$ for $i \geq 1$, and $\leftarrow$'s at o and all points in between (a scenario with probability $p(1-p)^i$. Then X_n takes i steps to the left, and then oscillates between $(-i,0)$ and $(-i+1,0)$ a random number of times, before T occurs.

The other possibility is that there is a $\leftarrow$ at $(j,0)$ for $j \geq 1$, and $\mathrel{\uparrow\!\!\!\rightarrow}$'s at o and all points in between. This scenario has probability $(1-p)p^j$. Now X_n steps right, and T may occur before it reaches $(j), j$, or it may reach $(j,0)$ and then oscillate until time T. The various scenarios lead to the following expression:

$$\begin{aligned}
E[T] &= \sum_{i=1}^{\infty} p(1-p)^i \sum_{k=0}^{\infty} (1/2)^{k+1}[i+2k+1] \\
&\qquad + \sum_{j=1}^{\infty} (1-p)p^j \Big(\sum_{k=0}^{j-2} (1/2)^{k+1}[k+1] + \sum_{k=0}^{\infty} (1/2)^{j+k}[j+2k] \Big) \\
&= \sum_{i=1}^{\infty} p(1-p)^i (i+3) + \sum_{j=1}^{\infty} (1-p)p^j \Big(\frac{1}{2} \frac{d}{dt}\Big|_{t=1/2} \frac{1-t^j}{1-t} + \frac{j+2}{2^{j-1}} \Big) \\
&= 3(1-p) + \frac{1-p}{p} + 2 \sum_{j=1}^{\infty} (1-p)p^j \Big(1 + \frac{1}{2^j} \Big) \\
&= 2 - 3p + \frac{1}{p} + 2p + \frac{p(1-p)}{1-p/2} = \frac{2+3p-2p^2-p^3}{p(2-p)}.
\end{aligned}$$

- $\overset{\leftrightarrow}{\downarrow}\ \downarrow$:
  The speed is $(0, -1/E[T])$, where T is the time of the first step in the $\downarrow$ direction.
  To find $E[T]$, first consider random walk on $[0, n]$, with the probability of death in 1 step starting from $1 \le k \le n-1$ being $1/3$, and the probability of death in 1 step being 1 starting from 0 or n. Let $f(k)$ be the mean time of death, starting from k. Then $f(0) = f(n) = 1$, and otherwise

$$f(k) = 1 + \frac{f(k-1) + f(k+1)}{3}.$$

The solution is $f(k) = 3 + \bar{c}\bar{\gamma}^k + c\gamma^k$ where $\bar{\gamma} < \gamma$ are solutions of $z + z^{-1} = 3$. In other words, $\gamma = (3+\sqrt{5})/2$ and $\bar{\gamma} = (3-\sqrt{5})/2$. From the boundary conditions, we get

$$f(k) = 3 + \frac{2(1-\gamma^n)\bar{\gamma}^k}{\gamma^n - \bar{\gamma}^n} + \frac{2(1-\bar{\gamma}^n)\gamma^k}{\gamma^n - \bar{\gamma}^n}.$$

But $\bar{\gamma}\gamma = 1$, so this simplifies to

$$f(k) = 3 - 2\frac{\gamma^k + \gamma^{n-k}}{\gamma^n + 1}.$$

If there is a $\downarrow$ at o then $T=1$. Otherwise, suppose there are $\downarrow$'s at $(-i,0)$ and $(j,0)$, with $\overset{\leftrightarrow}{\downarrow}$'s in between, where $i,j\geq 1$. Then $\mathbb{P}_{\mathcal{G}}(T)=3-2(\gamma^i+\gamma^j)/(\gamma^{i+j}+1)$. Therefore

$$
\begin{aligned}
E[T] &= (1-p)\cdot 1 + p\sum_{i,j=1}^{\infty} p^{i+j-2}(1-p)^2\Big(3-2\frac{\gamma^i+\gamma^j}{\gamma^{i+j}+1}\Big)\\
&= 1-p+3p(1-p)^2\Big(\sum_{i=1}^{\infty}p^{i-1}\Big)^2-\frac{2(1-p)^2}{p}\sum_{k=2}^{\infty}\frac{p^k}{\gamma^k+1}\sum_{j=1}^{k-1}(\gamma^j+\gamma^{k-j})\\
&= 1+2p-\frac{4(1-p)^2}{p}\sum_{k=2}^{\infty}\frac{p^k}{\gamma^k+1}\Big(\frac{\gamma^k-1}{\gamma-1}-1\Big)\\
&= 1+2p-\frac{4(1-p)^2}{p}\sum_{k=2}^{\infty}\Big(\frac{1}{\gamma-1}p^k-\frac{\gamma+1}{\gamma-1}\frac{p^k}{\gamma^k+1}\Big)\\
&= 1+2p-\frac{4p(1-p)}{\gamma-1}+\frac{4(1-p)^2(\gamma+1)}{p(\gamma-1)}\sum_{k=2}^{\infty}\frac{p^k}{\gamma^k+1}.
\end{aligned}
$$

Note that the expression $Q(q;p)=1+2\sum_{k=1}^{\infty}\frac{p^k}{q^k+1}$ is known as a unilateral q-hypergeometric series, and in the theory of special functions would be written

$$
Q(q;p)={}_2\phi_1\left[\begin{matrix} q & -1 \\ -q & \end{matrix};q,p\right].
$$

See [4]. We are indebted to Martin Muldoon for pointing this out. We could therefore also write

$$
\begin{aligned}
E[T] &= \frac{2(1-p)^2(\gamma+1)}{p(\gamma-1)}Q(\gamma;p)+1+2p\\
&\qquad -\frac{4p(1-p)}{\gamma-1}-\frac{4(1-p)^2}{\gamma-1}-\frac{2(1-p)^2(\gamma+1)}{p(\gamma-1)}\\
&= \frac{2(1-p)^2(\gamma+1)}{p(\gamma-1)}Q(\gamma;p)+5-\frac{2}{p}-\frac{4(1-p)}{p(\gamma-1)}
\end{aligned}
$$

- $\overset{\leftrightarrow}{\downarrow}$ $\rightarrow$:
  Here $e=\downarrow$, and the martingale equations are that $-1=v^{[2]}E[T]=-\alpha_1/3$ and $v^{[1]}E[T]=\alpha_2=E[T]-\alpha_1=E[T]-3$. In other words, $v^{[2]}=-1/E[T]$ and $v^{[1]}=1-3/E[T]=1+3v^{[2]}$. So we must now find $E[T]$.

  First consider a random walk Z_j on $[0,n]$ with the following boundary conditions: it reflects at 0, and it is absorbed at n. At points in between there is killing with probability $1/3$, and otherwise Z_j performs a simple symmetric random walk. Let S be the time of death or absorption, and let $f(k)=f_n(k)=E[S\mid Z_0=k]$. Then

$$
f(k)=1+\frac{f(k-1)+f(k+1)}{3}
$$

  for $1\leq k\leq n-1$, $f(n)=0$, and $f(0)=1+f(1)$. The solution to the recurrence is $f(k)=3+\bar{c}\bar{\gamma}^k+c\gamma^k$ where as above, $\bar{\gamma}<\gamma$ are $(3\pm\sqrt{5})/2$. From the boundary conditions, we get

$$f(k) = 3 + \bar{\gamma}^k \frac{3\gamma - 3 - \gamma^n}{\gamma^n(\bar{\gamma}-1) - \bar{\gamma}^n(\gamma-1)} + \gamma^k \frac{\bar{\gamma}^n - 3\bar{\gamma} + 3}{\gamma^n(\bar{\gamma}-1) - \bar{\gamma}^n(\gamma-1)}.$$

But $\bar{\gamma}\gamma = 1$, so this simplifies to

$$\begin{aligned} f(k) &= 3 - \frac{\gamma^{n-k}(3\gamma - 3 - \gamma^n) + \gamma^k(1 - 3\gamma^{n-1} + 3\gamma^n)}{(\gamma-1)(\gamma^{2n-1}+1)} \\ &= 3 - \frac{1}{\gamma^{2n-1}+1}\Big[3(\gamma^{n-k} + \gamma^{n+k-1}) + \frac{\gamma^k - \gamma^{2n-k}}{\gamma - 1}\Big]. \end{aligned}$$

Likewise let $g(k) = g_n(k) = P(Z_S = n \mid Z_0 = k)$. Then $g(k) = [g(k-1) + g(k+1)]/3$ for $1 \le k \le n-1$, with boundary conditions $g(0) = g(1)$ and $g(n) = 1$. As above, this has solution

$$g(k) = \frac{\gamma^{n-k} + \gamma^{n+k-1}}{\gamma^{2n-1}+1}.$$

Now consider how X_j evolves. Let the first $\rightarrow$ to the left of o be at $x_0 = (i_0, 0)$, where $i_0 \le 0$. Let successive $\rightarrow$ to the right of o be at $x_1 = (i_1, 0)$, $x_2 = (i_2, 0)$, etc., where $0 < i_1 < i_2 < \dots$. At interior points of the horizontal interval $[x_0, x_1)$, X_j leaves the interval with probability $1/3$ at every step, by moving downwards (ie at time T). Otherwise it evolves as a simple random walk till it hits x_0 or x_1. If it hits x_0 it reflects back with probability 1. If it reaches x_1 it leaves this interval forever, and starts the same process over again on the interval $[x_1, x_2)$. Let the interval being visited at time T be $[x_N, x_{N+1})$, where $N \ge 0$, and let S_j be the total time spent in $[x_j, x_{j+1})$. Then $T = \sum_{j=0}^N S_j$, and

$$E[T] = \sum_{j=0}^{\infty} E[S_j 1_{\{N \ge j\}}].$$

If A_j is the event that X_n exits $[x_j, x_{j+1})$ at x_{j+1} (ie before T), then for $j \ge 1$ we have $\{N \ge j\} = \cap_{k=0}^{j-1} A_k$. Moreover, there is a renewal every time X_n enters a new interval, because we start exploring a new environment. The interval $[x_0, x_1)$ is different from the rest, because we start at o. But for all other $[x_j, x_{j+1})$ the process starts walking at x_j. In other words, the cases $j \ge 1$ are independent replications of the same procedure. Therefore

$$\begin{aligned} E[T] &= E[S_0] + \sum_{j=1}^{\infty} E[S_j \prod_{k=0}^{j-1} 1_{A_k}] = E[S_0] + \sum_{j=1}^{\infty} E[S_1 \mid A_0] P(A_0) P(A_1 \mid A_0)^{j-1} \\ &= E[S_0] + E[S_1 \mid A_0] P(A_0)/(1 - P(A_1 \mid A_0)). \end{aligned}$$

We can work out all these factors using the expressions for f and g. First define

$$\Theta(z) = \Theta_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\gamma^{2n+1}+1} = \frac{1}{2\sqrt{z}}\Big[Q(\gamma; \sqrt{z}) - Q(\gamma^2; z)\Big]$$

where Q is the q-hypergeometric series defined earlier. Then using that $\gamma^2-3\gamma+1=0$,

$$\begin{aligned}
E[S_0] &= \sum_{i_0=0}^{\infty}\sum_{i_1=1}^{\infty} p^{i_0+i_1-1}(1-p)^2 f_{i_0+i_1}(i_0)\\
&= 3-(1-p)^2\sum_{n=1}^{\infty}p^{n-1}\sum_{k=0}^{n-1}\Big[\frac{3(\gamma^{n-k}+\gamma^{n+k-1})}{\gamma^{2n-1}+1}+\frac{\gamma^k-\gamma^{2n-k}}{(\gamma-1)(\gamma^{2n-1}+1)}\Big]\\
&= 3-(1-p)^2\sum_{n=1}^{\infty}\frac{p^{n-1}(\gamma^n-1)}{(\gamma-1)^2(\gamma^{2n-1}+1)}\Big[3(\gamma-1)(\gamma+\gamma^{n-1})+1-\gamma^{n+1}\Big]\\
&= 3-\frac{(1-p)^2}{(\gamma-1)^2}\sum_{n=1}^{\infty}\frac{p^{n-1}}{\gamma^{2n-1}+1}\Big[(\gamma^{2n-1}+1)(3\gamma-3-\gamma^2)+\gamma^{n-1}(3\gamma^3-2\gamma^2-2\gamma+3)-2(\gamma^2-1)\Big]\\
&= 3-\frac{(1-p)(3\gamma-3-\gamma^2)}{(\gamma-1)^2}+\left(\frac{1-p}{\gamma-1}\right)^2\Big[2(\gamma^2-1)\Theta(p)-(3\gamma^3-2\gamma^2-2\gamma+3)\Theta(p\gamma)\Big],\\
&= 3+\frac{2(1-p)}{\gamma}+\frac{2(1-p)^2}{\gamma}\Big[(3\gamma-2)\Theta(p)-(8\gamma-2)\Theta(p\gamma)\Big],\\
E[S_1\mid A_0] &= \sum_{n=1}^{\infty}p^{n-1}(1-p)f_n(0)=3-\sum_{n=1}^{\infty}\frac{p^{n-1}(1-p)}{\gamma^{2n-1}+1}\Big[3\gamma^{n-1}(1+\gamma)-\frac{\gamma^{2n}-1}{\gamma-1}\Big]\\
&= 3-3(1-p)(1+\gamma)\Theta(p\gamma)+\frac{\gamma}{\gamma-1}-\frac{(1-p)(1+\gamma)}{\gamma-1}\Theta(p),\\
P(A_0) &= \sum_{i_0=0}^{\infty}\sum_{i_1=1}^{\infty}p^{i_0+i_1-1}(1-p)^2 g_{i_0+i_1}(i_0)=(1-p)^2\sum_{n=1}^{\infty}p^{n-1}\sum_{k=0}^{n-1}\frac{\gamma^{n-k}+\gamma^{n+k-1}}{\gamma^{2n-1}+1}\\
&= \frac{(1-p)^2}{\gamma-1}\sum_{n=1}^{\infty}\frac{p^{n-1}(\gamma^{2n-1}+\gamma^{n+1}-\gamma^{n-1}-\gamma)}{\gamma^{2n-1}+1}=\frac{(1-p)}{\gamma-1}+(1-p)^2(\gamma+1)\Big[\Theta(p\gamma)-\frac{\Theta(p)}{\gamma-1}\Big],\\
P(A_1\mid A_0) &= \sum_{n=1}^{\infty}p^{n-1}(1-p)g_n(0)=(1-p)(\gamma+1)\Theta(p\gamma).
\end{aligned}$$

Therefore

$$\begin{aligned}
E[T] &= 3+\frac{2(1-p)}{\gamma}+\frac{2(1-p)^2}{\gamma}\Big[(3\gamma-2)\Theta(p)-(8\gamma-2)\Theta(p\gamma)\Big]\\
&+\frac{\Big[3-3(1-p)(1+\gamma)\Theta(p\gamma)+\frac{\gamma}{\gamma-1}-\frac{(1-p)(\gamma+1)}{\gamma-1}\Theta(p)\Big]\Big[\frac{(1-p)}{\gamma-1}+(1-p)^2(\gamma+1)\Big\{\Theta(p\gamma)-\frac{\Theta(p)}{\gamma-1}\Big\}\Big]}{1-(1-p)(\gamma+1)\Theta(p\gamma)}.
\end{aligned}$$

Simplifying this, we have

$$E[T]=4-p-(1-p)^2\frac{4\gamma-1}{\gamma}\Theta(p\gamma)+\frac{1-p}{\gamma}\frac{\Big[\gamma-(1-p)(\gamma+1)\Theta(p)\Big]^2}{1-(1-p)(\gamma+1)\Theta(p\gamma)}.$$

Substituting for γ gives

$$E[T] = 4 - p - \frac{5+\sqrt{5}}{2}(1-p)^2\Theta(p\gamma) + \frac{(1-p)\Big[3+\sqrt{5}-(1-p)(5+\sqrt{5})\Theta(p)\Big]^2}{(3+\sqrt{5})\Big[2-(1-p)(5+\sqrt{5})\Theta(p\gamma)\Big]}.$$

- $\overset{\longleftrightarrow}{\downarrow}\overset{\leftarrow}{\downarrow}$:
  Here $e = \downarrow$, and the martingale equations are that $-1 = v^{[2]}E[T] = -\alpha_1/3 - \alpha_2/2 = -E[T]/3 - \alpha_2/6$ and $v^{[1]}E[T] = -\alpha_2/2 = E[T] - 3$. In other words, $v^{[2]} = -1/E[T]$ and $v^{[1]} = 1 - 3/E[T] = 1 + 3v^{[2]}$. So we must now find $E[T]$. We leave working out the details for a future time.

  The argument now proceeds exactly as in the previous case ($\overset{\longleftrightarrow}{\downarrow}, \rightarrow$). What changes are the boundary conditions for the functions $f(k) = f_n(k)$ and $g(k) = g_n(k)$, which become that $f(0) = 0$, $f(n) = 1 + \frac{1}{2}f(n-1)$, $g(0) = 1$, $g(n) = \frac{1}{2}g(n-1)$.

Finally, we give an asymptotic argument, in the one case in which there is an elementary renewal structure for which we don't know how to find the speed analytically.

- $\overset{\longleftrightarrow}{\downarrow} \leftrightarrow$:
  Here $e = \downarrow$, and $v = (0, -1/E[T])$. Though we don't know how to find v analytically, here is an approach that should give an asymptotic expansion in powers of $q = 1 - p$.

  Embed $\mathbb{Z} \subset \mathbb{Z}^2$ as $\mathbb{Z} \times \{0\}$, and let $Y_n = X_n^{[1]}$ for $n < T$. We can fill in new independent increments after time $T - 1$ to make Y_n into a simple symmetric random walk started from 0, and then recover an independent copy of T by killing Y at a rate that depends on the environment. Write $\tilde{P}$ for this extension of P. Let V_i be 1 (resp. 2/3) if $\mathcal{G}_{(i,0)} = \leftrightarrow$ (resp. $\overset{\longleftrightarrow}{\downarrow}$). Then by Feynman-Kac,

$$E[T] = \sum_{k=1}^{\infty} P(T \geq k) = \sum_{k=1}^{\infty} \tilde{P}(\prod_{i=1}^{k-1} V_{Y_i}).$$

  This equals $\sum_{k=1}^{\infty} \tilde{P}(\prod_{j\in\mathbb{Z}} V_j^{N_j(k)})$, where $N_j(k)$ counts the number of visits of Y_n to j, for $n < k$. Integrating out the environment, and setting $\gamma = 2/3$, we get

$$\begin{aligned}\sum_{k=1}^{\infty} \tilde{P}(\prod_{j\in\mathbb{Z}}[1 - p + p\gamma^{N_j(k)}]) &= \sum_{k=1}^{\infty} \tilde{P}(\prod_{j\in\mathbb{Z}} \gamma^{N_j(k)}[q\gamma^{-N_j(k)} + 1 - q]) \\ &= \sum_{k=1}^{\infty} \lambda^{k-1}\tilde{P}(\prod_{j\in\mathbb{Z}}[1 + q(\gamma^{-N_j(k)} - 1)])\end{aligned}$$

  (since $\sum_j N_j(k) = k - 1$). This expression could in principle be expanded as a series in q. What complicates this is that the q^i term involves knowing the joint distributions of the $N_j(k)$ for i choices of j. Still, it should be possible to work out at least the constant and linear terms.

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