The Dirac Algebra and Equation

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1 Background

- The Dirac algebra is actually the clifford algebra $Cl_{1,3}(\mathbb{C}) = Cl_{1,3}(\mathbb{R}) \otimes \mathbb{C}$. We actually won't be working with the Dirac algebra much, but rather the Spacetime algebra $Cl_{1,3}(\mathbb{R})$ for ease and intuition; for physical reasons, we would like to complexify later on. We will do this at the end.
- Continuing from last week, we will be investigating further representations of the restricted Lorentz group $SO^+(1,3)$.
- In fact we will be finding an irreducible representation of its double cover, $SL(2,\mathbb{C})$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SL(2,\mathbb{C}) \longrightarrow SO^+(1,3) \longrightarrow 1$$

  Now the fact that this representation is irreducible implies that it does not necessarily factor through to a representation of $SO^+(1,3)$. So let it be known that we are not actually finding a representation of the restricted Lorentz group, but its double cover.

2 Last time

- So we want an irreducible representation of $SL(2,\mathbb{C})$. $SL(2,\mathbb{C})$ is simply connected and so its representations are in 1-1 correspondence with that of its Lie algebra $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{so}(1,3)$.
- Hence we want to study $\mathfrak{so}(1,3)$ or rather its complexification, which happens to be of the form

$$\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}.$$

- Brendan showed us that $\mathfrak{su}(2)$ and thus $\mathfrak{su}(2)_{\mathbb{C}}$ has representations of dimension $2n+1$ indexed by spins $n \in \{0, 1/2, 1, 3/2, \ldots\}$ so that $\mathfrak{so}(1,3)_{\mathbb{C}}$ has representations indexed by pairs of such half-integers (n,m). Such a representation will have dimension $(2n+1)(2m+1)$.

- $(0,0)$ - representation is the space of scalar fields governed by the Klein-Gordan equation. $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation is the space of Spinor fields governed by the Dirac equation ($(\frac{1}{2},0),(0,\frac{1}{2})$ are Weyl spinor representations, the direct sum is the Bispinor representation). $(\frac{1}{2},\frac{1}{2})$ representation is the space of vector fields governed by the Proca equation.

- So we all know what scalars and vectors are, but what is a spinor? (2 dimensional example of "square root of a vector in", and explain that SO(p, q) is not simply connected, and the double cover Spin(p, q) captures the different homotopy classes of elements of SO(p, q).
- So following from last week the next simplest representations are the two dimensional representations $(\frac{1}{2}, 0), (0, \frac{1}{2})$, called the left and right handed Weyl spinor representations respectively.
- Today we will actually obtain the reducible 4 dimensional representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ called the Bispinor representation. It is the representation of the space of Spinor fields governed by the Dirac equation.

3 Today

- So again, last week we saw that $SL(2, \mathbb{C})$ is the universal cover of the restricted Lorentz group. Since we also have another double cover of the restricted Lorentz group in the form of $Spin(1, 3)$ we thus have the isomorphism:

$$SL(2, \mathbb{C}) \simeq Spin(1, 3) \tag{1}$$

- This is where Clifford algebras enter the picture, as we have

$$Spin(p, q) \subset Cl_{p,q}(\mathbb{R}) \tag{2}$$

- Here is the strategy for the first half of the talk: There is a unique (up to isomorphism) irreducible algebra representation of $Cl_{1,3}(\mathbb{R})$. When restricted to an algebra representation of the lie algebra of $Spin(1, 3)$ (which is $\mathfrak{so}(1, 3)$) this will yeild the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of $\mathfrak{so}(1, 3)$. We then exponentiate to get a group representation of $Spin(1, 3)$. A representation of $Spin(1, 3)$ that does not "lift" to a representation of $SO^+(1, 3)$ (ie -I is not in the kernel) is called a spin representation of $SO^+(1, 3)$, it is not actually a representation of $SO^+(1, 3)$.

  So that is the goal, we are going to find a group representation of $Spin(1, 3)$ that does not have -I in the kernal, ie a spin representation of the restricted Lorentz group!

4 The representation

- Recall that $Cl_{1,3}(\mathbb{R}) = \frac{T(\mathbb{R}^{1+3})}{(v^2 = Q(v))}$ where $Q(v)$ is the quadratic form given by $Q(v) = v^T g v$ where g is the minkowski metric. Let $\{e_0, e_1, e_2, e_3\}$ be a basis for $\mathbb{R}^{1+3}$.
- Recall any quadratic form determines a symmetric bilinear form (two way drawing):

$$\langle v, w \rangle = \frac{1}{2}(Q(v + w) - Q(v) - Q(w)) = \frac{1}{2}((v + w)^2 - v^2 - w^2) = \frac{1}{2}(uv + vu)$$

So that we get the so called "fundamental anticommutation relation" for the Clifford algebra. Note now we can write, equivalently:

$$Cl_{1,3}(\mathbb{R}) = \frac{T(\mathbb{R}^{3+1})}{(uv + vu = 2\langle v, u\rangle)}$$

- Some notation: $\{x, y\} = xy + yx$ is called the anticommutator. In contrast to the common $[x, y] = xy - yx$
- We can now compute the basis of

$$Cl_{1,3}(\mathbb{R}) = \frac{T(V)}{(v^2 = Q(v))} = \frac{T(V)}{(uv + vu = 2\langle u, v\rangle)}.$$

  Recall that $T(V) = \mathbb{R} \oplus (\mathbb{R}^{1+3}) \oplus (\mathbb{R}^{1+3})^{2\otimes} \oplus \cdots$ has basis

$$\{1\} \cup \{e_0, \ldots, e_3\} \cup \{e_i e_j \mid 0 \leq i, j \leq 3\} \cup \{e_i e_j e_k \mid 0 \leq i, j, k \leq 3\} \cup \cdots$$

  so that basis vectors of $T(V)$ are essentially words with alphabet $e_0, \ldots, e_3$.
  Given a word, the fundamental anticommutation relations allow us to reduce it to a word with ordered, non repeated indices, ie one of the following with possibly a minus sign out the front:

$$\{1\}\cup\{e_0, \ldots, e_3\}\cup\{e_0e_1, e_0e_2, e_0e_3, e_1e_2, e_1e_3, e_2e_3\}\cup\{e_0e_1e_2, e_0e_1e_3, e_0e_2e_3, e_1e_2e_3\}\cup\{e_0e_1e_2e_3\}.$$

  Which is 16 in total, this is our basis. In general for a finite dimensional vector space with dimension n there are 2^n basis vectors for the Clifford algebra on V.

- As an algebra however, it is generated by $\{e_0, e_1, e_2, e_3\}$ and thus to find an algebra representation we need to assign matrices to each of the basis vectors, such that fundamental anticommutation relations are satisfied for all the matrices.
  (In case anyone sees this as non obvious, remember that this follows from the universal property of the Clifford algebra).
- It turns out we need 4x4 matrices to do the trick. They are given by the 4 Gamma matrices defined below

$$\gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

  There are infinitely many choices for these matrices, the actual specifics of these matrices are far less important than the fact that they satisfy the anticommutation relations.
- The reason for the necessity for 4x4 matrices is as follows, we have:

$$Cl_{0,3}(\mathbb{R}) \cong \mathbb{H}, \quad Cl_{p,q}(\mathbb{R}) \cong M_2(Cl_{p-1,q-1}(\mathbb{R}))$$

- In fact we can get an irreducible representation for any Clifford algebra $Cl_{p,q}(\mathbb{R})$ in this way(finding suitable matrices that satisfy the relations). The irreducible representation for the Clifford Algebra $Cl_{0,3}(\mathbb{R})$ are the famous Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It turns out that the Pauli matrices, when multiplied by i, generate $\mathfrak{su}(2)$, this isn't important but it will be invoked later.

You will notice that the Gamma matrices can be written in terms of Pauli matrices as such:

$$\gamma^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \gamma^1 = \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}, \gamma^3 = \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}$$

- Lets take a moment to step back and remember what we are trying to achieve.
What we now wish to do is obtain a lie algebra representation for the lie algebra of $Spin(1,3)$, which as we recall is $\mathfrak{so}(1,3)$. Brae covered this lie algebra last week, it has 3 rotation generators $\{J_x, J_y, J_z\}$ and 3 boost generators $\{K_x, K_y, K_z\}$. To get a representation of this then, we need 6 matrices satisfying the commutation relations of the lie algebra:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad [J_i, K_j] = i\epsilon_{ijk}K_k; \quad [K_i, K_j] = -i\epsilon ijkJ_k$$

The needed matrices are given by:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$S^{01} = \frac{1}{2}\begin{bmatrix} -i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{bmatrix}, \quad S^{02} = \frac{1}{2}\begin{bmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{bmatrix}, \quad S^{03} = \frac{1}{2}\begin{bmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{bmatrix}$$

These are the 3 boost matrices. S^{0i} concerns the Gamma matrices related to e_0 (time) and e_i (the i^{th} spatial direction).

$$S^{12} = \frac{1}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix}, \quad S^{23} = \frac{1}{2}\begin{bmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{bmatrix}, \quad S^{31} = \frac{1}{2}\begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

These are the 3 rotation matrices. S^{ij} concerns the gamma matrices related to infinitesimal translations in the $i-j$ plane. The combination of translations corresponds to a rotation about the k-axis, where $i \neq k \neq j$.

I will not show it here but these matrices satisfy the commutation relations from last week, hence constitute a representation of $\mathfrak{so}(1,3)$.

Here you can clearly see these matrices are all block diagonal, hence they constitute a reducible lie algebra representation of $\mathfrak{so}(1,3)$.
Now this representation is 4 dimensional, and since the pauli matrices cannot be further reduced (they form an irreducible spin 1/2 representation of $\mathfrak{su}(2)$), we get that this representation is in fact the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ that we seek.

- So what have we done? We have found the Bispinor representation for the lie algebra of the Lorentz group ($\mathfrak{so}(1,3)$). Remember what our goal was, we are to obtain a representation of the $Spin(1,3)$ group, which is the double cover of the (connected component of the identity of the) Lorentz group. This was called a spin representation of the Lorentz group(not a real representation). Since we have a representation for the lie algebra of $Spin(1,3)$, what remains to do is to simply exponentiate the matrices $S^{\mu\nu}$ to get a representation of $Spin(1,3)$.
- Now $Spin(1,3)$ is simply connected and so its representation correspond 1-1 with the representations of $\mathfrak{so}(1,3)$. So that we may simply exponentiate to get a representation of $Spin(1,3)$.

$$exp : \mathfrak{so}(1,3) \longrightarrow Spin(1,3)$$

Note the exponential map

$$exp : \mathfrak{so}(1,3) \longrightarrow SO^+(1,3)$$

is still well defined, but the representations of $SO^+(1,3)$ are not in 1-1 correspondence. In fact they are in 1-2 correspondence, with half of the representations of $\mathfrak{so}(1,3)$ not reducing to a representation of $SO^+(1,3)$, called spin representations.

- Throughout this talk we have only been focusing on $Cl_{1,3}(\mathbb{R})$ which is called real spacetime algebra. This is because it is much easier to work with over its complexification $Cl_{1,3}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = Cl_{1,3}(\mathbb{C})$.
- However now that we have done all the hard work it is time to complexify. $Cl_{1,3}(\mathbb{C})$ is called the Dirac algebra and it too is generated by the Gamma matrices. Everything we have done so far goes over to the Dirac algebra. This includes containing a spin representation of the Lorentz group. We call elements of $\mathbb{C}^4$ that are acted upon by this represention Bispinors, and they will described in more detail next week.

Here is a proper definition: A bispinor is an element of a 4-dimensional complex vector space considered as a $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz group.

- In actual fact we were only interested in the Dirac algebra the entire time as, for physical reasons, we view spinors as elements of a complex vector space. But it is easier to work with spacetime algebra and then just complexify to add complex scalars at the end. (This is exactly what we did).

5 Dirac Equation

- Last week we saw both the Schrödinger equation and the Klein-Gordon equation arise from quantising respectively the non-relatavistic and relatavistic energy momentum relations using

$$E \longrightarrow i\frac{d}{dt} = \hat{E}, \quad p \longrightarrow -i\nabla = \hat{p}$$

- Schrödinger:

$$E = \frac{p^2}{2m} \longrightarrow (i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m})\Psi = 0$$

- Klein-Gordon equation quantises the relativistic energy momentum relation:

$$E^2 = p^2 + m^2 \longrightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0$$

It turns out that this equation being second order isn't desirable.

- Dirac sought after a first order relativistic equation for the energy momentum relation. He first quantised it as follows:

$$\hat{E}^2 - \hat{p}^2 = m^2$$

He decided he would like the square root both sides. In other words he wanted an operator $\not{D}$ such that $\not{D}^2 = \nabla^2 - \frac{d^2}{dt^2}$.

- To acheive this, we define a ring $R = \mathbb{C}[\partial_\mu]$, and V a free module over R with basis $\{e_0, e_1, e_2, e_3\}$.
Let $\not{D} = \partial_t e_0 + \partial_x e_1 + \partial_y e_2 + \partial_z e_3$. We want $\not{D}^2 = \nabla^2 - \frac{d^2}{dt^2}$. If we expand the LHS , we will see equality if we have the relations $e_0^2 = -1$, $e_1^2 = e_2^2 = e_3^2 = 1$ and $e_i e_j = -e_j e_i$. This suggests to us to take $e_i = i\gamma^i$. So we have

$$\not{D} = i\gamma^0 \partial_t + i\gamma^1 \partial_x + i\gamma^2 \partial_y + i\gamma^3 \partial_z = i\gamma^\mu \partial_\mu.$$

- The Dirac Equation is defined as:

$$(i\gamma^\mu \partial_\mu - m)\phi = 0.$$

Here ϕ is a 4 component complex vector, a Bispinor in the sense given before.

Expanding this: $i\gamma^\mu \partial_\mu = m\phi$. Applying the operator twice we get the Klein-Gordon equation. So each component of the vector ϕ satisfies the KG equation.

- Note what we have actually done here. We have essentially constructed $Cl_{1,3}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}[\partial_\mu]$ i.e. the Clifford Algebra but with coefficients from $\mathbb{C}[\partial_\mu]$. We had our basis $\{e_0, e_1, e_2, e_3\}$ and quotiented out by the fundamental antimmutation relations(ie we replaced the basis elements with the Gamma matrices).
Stepping back for a moment, observe that we worked with $Cl_{1,3}(\mathbb{R})$, the spacetime algebra, for as long as needed, and added complex coefficients to obtain the Dirac algebra. Then we further added differential operator coefficients to get the algebra in which $\not{D}$ lives in.