Assignment 2 – Algebra 2019

(1) Let $F \subseteq K$ be a field extension. We say that K is algebraic over F if every element of K is algebraic over F. Prove that K is algebraic over F if and only if every subring $R \subseteq K$ containing F is also a field.

(2) Assume that $e \in \mathbb{R}$ is tanscendental over $\mathbb{Q}$, and let $K \subset \mathbb{R}$ be an algebraic extension of $\mathbb{Q}$. Prove that e is transcendental over K.

(3) Let $K = \mathbb{Q}[i, \sqrt[4]{2}]$.
   (a) Show that K is a splitting field of $X^4 - 2$ over $\mathbb{Q}$.
   (b) Find a $\mathbb{Q}$-basis of K.
   (c) Find an automorphism of order four of K over $\mathbb{Q}[i]$.
   (d) Determine all the automorphisms of K over $\mathbb{Q}$.
   (e) The zeros of $X^4 - 2$ form the set $S = \{\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}\}$. Describe the action of $Aut(K|\mathbb{Q})$ on S.
   (f) Find all subgroups of $Aut(K|\mathbb{Q})$.
   (g) Find all intermediate field extensions of $\mathbb{Q} \subset K$.

(4) Prove $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\sqrt{2} + \sqrt{3}]$.

(5) Transcendendal numbers
   (a) Prove that the set of all algebraic numbers over $\mathbb{Q}$ is countable.
   (b) Prove that the set of real numbers $\{log(p) \mid pisprime\}$ is linearly independent over $\mathbb{Q}$.
   (c) Use your result from (a) to prove the existence of transcendental numbers.

(6) A famous theorem by Lindemann states that if $a \in \mathbb{C}$ is algebraic over $\mathbb{Q}$ then e^a is transcendental. Use this theorem to prove that π is transcendental.

(7) Let R be a principal ideal domain, let $a_1, \dots a_n \in R$ be elements such that $(gcd(a_i, a_j)) = (1)$ for $i \neq j$, and let $a = a_1 \cdots a_n$. Prove that the map

$$\begin{array}{rcl} \psi R/(a) & \longrightarrow & R/(a_1) \times R/(a_2) \times \cdots \times R/(a_n) \\ [r]_{(a)} & \longmapsto & ([r]_{(a_1)}, [r]_{(a_2)}, \dots, [r]_{(a_m)}) \end{array}$$

is an isomorphism of rings. Here we have used the notation $[r]_{(a)} = r + (a)$ for the coset of r mod (a), and similarly for (a_i).