Assignment 3

Due Friday, 31 May 2019, Noon

1. Let T be the matrix
$$T = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$
acting on the complex vector space $V = \mathbb{C}^3$.

   (a) Recall how T defines the structure of a $\mathbb{C}[x]$-module on $\mathbb{C}^3$.

   (b) Let $p(x) = x^2 - 7x + 1$, and let $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Compute the element $p(x) \cdot v$ of $\mathbb{C}^3$.

   (c) Give a set of generators and relations for $\mathbb{C}^3$ over $\mathbb{C}[x]$ with the above module structure.

   (d) Write down the relations matrix.

   (e) Recall the definition of minimal polynomial of a matrix.

   (f) What is the minimal polynomial of T? Is it irreducible?

   (g) Consider now the matrix
$$B = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$
acting on $\mathbb{C}^5$. Show that as $\mathbb{C}[x]$-module (with x acting as B) the space $\mathbb{C}^5$ decomposes as the direct sum of two non-trivial submodules.

2. Let M be the matrix with integer coefficients
$$M = \begin{bmatrix} 4 & 2 & 7 \\ 6 & 12 & 0 \\ 2 & 8 & 14 \\ 24 & 21 & 14 \end{bmatrix}.$$

Use the algorith from class (Wikipedia) to bring M into Smith Normal Form, i.e., find invertible matrices L and R and a matrix D in Smith Normal Form with $LMR = D$. Document your steps.

3. Let T be the matrix

$$T = \begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{bmatrix}$$

acting on the complex vector space $V = \mathbb{C}^3$.

(a) Use the method you learned in *Group Theory and Linear Algebra* to write A in the form

$$T = BJB^{-1}$$

with J in Jordan normal form. Show your work. Along the way, you find the characteristic polynomial of T. Record this, too.

(b) Consider now the standard basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ of $\mathbb{C}^3$. As in the proof of the Jordan Normal Form Theorem in class, view $\mathbb{C}^3$ as a module over $\mathbb{C}[X]$, where X acts as T. What relations do the $\vec{e}_i$ satisfy

i. over $\mathbb{C}$

ii. over $\mathbb{C}[X]$?

(c) Write down the matrix A with entries from $\mathbb{C}[X]$ that encodes these generators and relations, as we did in the proof of the normal form.

(d) Now bring A into Smith Normal Form, documenting each step and keeping track also of the matrices L and R, where

$$LAR = D.$$

(e) Work out the details of how the equation $LAR = D$ encodes the Jordan Normal Form for T. In particular, what was the meaning of R, and how does it relate to the base change matrix B of part (a)?

(f) How many generators (at least) do you need to generate V as a module over

i. the complex numbers?

ii. the polynomial ring $\mathbb{C}[X]$?

4. Let A be the abelian group with generators a, b, c, d and relations $2a = 4b + c$, $4c = d - 2b$ and $a + b + c + d = 0$. Write A as the cartesian product of cyclic groups as in the classifaction theorem.