$R$ ring (with 1)

Question: What is the smallest field $Q(R)$ containing $R$ as a subring?

Does $Q(R)$ exist for any $R$?

Assumption for the rest of the lecture: $R$ commutative, $R \neq \{0\}$.

Def.: An element $a \in R$ is called a zero divisor if there exists a $b \in R \setminus \{0\}$ such that $a \cdot b = 0$

Note: $0 \in R$ is always a (trivial) zero divisor

Example: $R^n := \underbrace{R \times \dots \times R}_{n \text{ times}}$, $n \in \mathbb{N}$, $n \geq 2$

$R^n$ is a ring with compositions

$$(r_1, \dots, r_n) + (r_1', \dots, r_n') := (r_1 + r_1', \dots, r_n + r_n')$$
$$(r_1, \dots, r_n) \cdot (r_1', \dots, r_n') := (r_1 \cdot r_1', \dots, r_n \cdot r_n').$$

and $0 = (0, \dots, 0)$, $1 = (1, \dots, 1)$.

We have many zero divisors, e.g.

$$(1,0,\dots,0) \cdot (0,1,0,\dots,0) = (0,0,\dots,0).$$

Lemma: A field $R$ has no nonzero zero divisors.

Proof: Let $a \in R$, $b \in R \setminus \{0\}$ such that $a \cdot b = 0$.

Then $a = a \cdot b b^{-1} = 0 \cdot b^{-1} = 0$. $\square$

Def.: A ring $R$ is called integral domain if $R$ has no nonzero zero divisors. ($R$ commutative, $R \neq \{0\}$ is already assumed)

Rmk.: Integral domains have the "cancellation property":

Let $a, b, c \in R$. Then

$$ac = bc \Rightarrow (a-b) \cdot c = 0 \Rightarrow a = b \text{ or } c = 0$$

In particular, nonzero elements $c$ can be cancelled.

This is false for general rings, e.g.

$$(1,0,\dots,0) \cdot (0,1,0,\dots,0) = (0,\dots,0) \cdot (0,1,0,\dots,0) \quad \text{in } R^n.$$

But $(1,0,\dots,0) \neq (0,\dots,0)$.

<u>Goal:</u> Construct $Q(R)$ for $R$ an integral domain
(otherwise there is no hope for $Q(R)$ to exist by the previous lemma)

Define an equivalence relation on $R \times R \setminus \{0\}$:

$$(a,b) \sim (a',b') \overset{!}{\iff} ab' = a'b$$

Show transitivity only (reflexivity and symmetry are clear):

Let $(a,b) \sim (a',b')$ and $(a',b') \sim (a'',b'')$.

Then $ab' = a'b$ and $a'b'' = a''b'$.

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$ab'b'' = a'bb'' \overset{\text{equal!}}{\longleftrightarrow} a'b''b = a''b'b$$

So $a\cancel{b'}b'' = a''\cancel{b'}b$. Thus, $(a,b) \sim (a'',b'')$ $\square$

$Q(R) := (R \times R\setminus\{0\})/\sim$

<u>Notation:</u> $\frac{a}{b} := [(a,b)] \in Q(R)$

<u>Then:</u> $\frac{a}{b} = \frac{a'}{b'} \iff [(a,b)] = [(a',b')] \iff (a,b) \sim (a',b') \iff ab' = a'b$

Define compositions on $Q(R)$:

$$\frac{a}{b} + \frac{c}{d} := \frac{ad+bc}{bd}, \qquad \frac{a}{b} \cdot \frac{a'}{b'} := \frac{aa'}{bb'}$$

Show that addition is well-defined (well-definedness of multiplication left as exercise)

Let $\frac{a}{b} = \frac{a'}{b'}$, $\frac{c}{d} = \frac{c'}{d'}$. Then

$$ab' = a'b \iff ab'dd' = a'bdd', \qquad cd' = c'd \iff cd'bb' = c'dbb'$$

Have $(ad+bc)b'd' = adb'd' + bcb'd' = a'd'bd + b'c'bd = (a'd' + b'c')bd$.

So $\frac{ad+bc}{bd} = \frac{a'd'+b'c'}{b'd'}$. $\square$

<u>Prop:</u> $Q(R)$ is a field (unit $\frac{1}{1}$, zero $\frac{0}{1}$)

<u>Proof:</u> Exercise.

$Q(R)$ is called the <u>fraction field of $R$</u>.

<u>Ex:</u> $R = \mathbb{Z}$, $Q(R) = \mathbb{Q}$.

**Prop:** The map $\begin{array}{ccc} R & \overset{i}{\hookrightarrow} & Q(R) \\ a & \longmapsto & \frac{a}{1} \end{array}$ is an injective ring homom.

**Proof.** injective: Let $a, b \in R$ such that $i(a) = i(b)$.

Then $\frac{a}{1} = i(a) = i(b) = \frac{b}{1}$. Thus, $\underbrace{a \cdot 1}_{a} = \underbrace{b \cdot 1}_{b}$ $\square$

ring homom: Exercise.

**Prop:** (Universal property) $R$ integral domain, $K$ field, $\varphi : R \hookrightarrow K$ injective ring homom

$a \longmapsto \varphi(a)$

$$\begin{tikzcd} R \arrow[r, hook, "\varphi"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & K \\ Q(R) \arrow[ur, dotted, "\exists!\, \Phi \text{ ring hom.}"'] & \end{tikzcd}$$

$a \longmapsto \frac{a}{1}$

**Proof:** Uniqueness: Since $\Phi(i(a)) = \varphi(a)$, we have $\Phi\left(\frac{a}{1}\right) = \varphi(a)$ for $a \in R$.

For $b \in R \setminus \{0\}$ we have

$$\Phi\left(\frac{1}{b}\right) = \Phi\left(\left(\frac{b}{1}\right)^{-1}\right) = \Phi\left(\frac{b}{1}\right)^{-1} = \varphi(b)^{-1}.$$

So, $\Phi\left(\frac{a}{b}\right) = \Phi\left(\frac{a}{1} \cdot \frac{1}{b}\right) = \Phi\left(\frac{a}{1}\right) \cdot \Phi\left(\frac{1}{b}\right) = \varphi(a)\varphi(b)^{-1}$.

Existence: Define $Q(R) \xrightarrow{\Phi} K$

$$\frac{a}{b} \longmapsto \varphi(a)\varphi(b)^{-1}$$

well-defined: $\frac{a}{b} = \frac{a'}{b'} \iff ab' = a'b$.

so $\varphi(a)\varphi(b') = \varphi(ab') = \varphi(a'b) = \varphi(a')\varphi(b)$

and $\varphi(a)\varphi(b)^{-1} = \varphi(a')\varphi(b')^{-1}$ (by multiplying the previous line with $\varphi(b')^{-1} \cdot \varphi(b)^{-1}$)

ring homom: Exercise $\square$

**Def:** An ideal $I$ in $R$ is a subset $I \subseteq R$ such that

(i) $I \subseteq R$ is a subgroup (w.r.t "+")

(ii) If $i \in I$, $a \in R$, then $ai \in I$.

**Note:** (1) Let $\varphi : R \longrightarrow R'$ be a ring homom. Then $\operatorname{Ker}\varphi \subseteq R$ is an ideal.

($a \cdot k \in \operatorname{Ker}\varphi$ for all $a \in R$, $k \in \operatorname{Ker}\varphi$ because $\varphi(ak) = \varphi(a)\underbrace{\varphi(k)}_{=0} = 0$)

(2) Let $R$ be a field. Then the only ideals of $R$ are $\{0\}$ and $R$. 

( $\{0\} \neq I \subseteq R$ ideal, $0 \neq i \in I$. Then $1 = i^{-1} \cdot i \in I$.
Thus $a = a \cdot 1 \in I$ for all $a \in R$. )

**Lemma:** $K$ field. Any ring homom. $\varphi: K \longrightarrow R$ is injective.

**Proof:** $\operatorname{Ker} \varphi \subseteq K$ is an ideal (by (1))
$\operatorname{Ker} \varphi \neq K$ because $1_K \notin \operatorname{Ker} \varphi$ as $\varphi(1_K) = 1_R \neq 0_R$ ( recall $R \neq \{0\}$ by assumption)
So $\operatorname{Ker} \varphi = \{0\}$ (by (1)). $\square$

**Summary:** We have seen that any integral domain $R$ can be realized as a subring of a field $Q(R)$ (after identifying $R$ with $i(R)$ under the injective ring homom. $i: R \hookrightarrow Q(R)$).

Any field $K$ which contains $R$ also contains $Q(R)$ because the universal property of $Q(R)$ yields

$$\begin{tikzcd} R \arrow[r, hook, "\varphi"] \arrow[d, "i"'] & K \\ Q(R) \arrow[ur, dashed, "\exists ! \Phi"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd},$$

where $\varphi$ is the natural inclusion map $R \subseteq K$, and $\Phi$ must be injective by the last lemma. In this sense $Q(R)$ is the "smallest" field containing $R$.

We have answered the question from the beginning of the lecture.