Tutorial 10 – Algebra 2019

(1) Let K be a field, V a finite dimensional K-vector space and $\mathcal{B} = (b_1, \ldots, b_n)$ a basis of V over K. (In this lecture, all finite bases are assumed to be ordered.) Recall the isomorphism of K-algebras

$$End_K(V) \cong Mat_{n\times n}(K).$$

(2) Realize $\mathbb{C}$ as a subring of $Mat_{2\times 2}(\mathbb{R})$.

(3) Let $f(x)$ be an irreducible polynomial of degree k over $\mathbb{F}_p$.

(a) Use f to construct a field extension $\mathbb{F}_p subseteq E$.

(b) Find an element of E whose minimal polynomial is f.

(c) Let now $n = mk$ be a multiple of k. Argue that E is contained, as a subfield, in a field with p^n elements.

(d) Deduce that $f(x)$ divides the polynomial $x^{p^n} - x$.

(e) Prove that the decomposition of the polynomial $x^{p^n} - x$ over $\mathbb{F}_p$ into irreducible factors is given as the product of all irreducible polynomials over $\mathbb{F}_p$ whose degree divides n,

$$x^{p^n} - x = \prod_{deg(f)|n} f(x).$$

(f) Use this to revisit the example of $\mathbb{F}_{16}$ from the last tutorial sheet.