Tutorial 7 – Algebra 2019

(1) Let R be an integral domain, and let a and b be elements of R.
 (a) Show that $a = bc$ implies $(a) \subseteq (b)$ with equality if and only if c is a unit.
 (b) Show that $(a) = (b)$ if and only if there exists a unit $u \in R^\times$ with $a = ub$.
 (c) Let $u \in R^\times$ be a unit. Show that $(u) = R$.

(2) Prove that a $\mathbb{Z}$-module consists of the same data as an abelian group.

(3) Consider the abelian group A with generators a, b and c and relations $3a = b - c$, $6a = 2c$ and $3b = 4c$.
 (a) Write A as the quotient of a free $\mathbb{Z}$-module by a submodule.
 (b) Convince yourself that A is isomorphic to the cyclic group on twelve elements.
 (c) Use the language of generators and relations to give a map f from A to $\mathbb{Z}/12\mathbb{Z}$ and an inverse of f.
 (d) Now translate this into the language of universal properties.

(4) Consider the field $\mathbb{F}_8$. To be concrete, use the construction and notation from class, so $\mathbb{F}_8$ is generated over $\mathbb{F}_2$ by an element b satisfying the relation $b^3 = b + 1$. Consider the element $b^2 \in \mathbb{F}_8$.
 (a) Without any calculations, determine the degree of b^2 over $\mathbb{F}_2$.
 (b) Prove your statement from (a).
 (c) Find the irreducible polynomial of b^2 over $\mathbb{F}_2$.
 (d) Write down an automorphism of $\mathbb{F}_8$.