Tutorial 8 – Algebra 2019

(1) Consider the polynomial $p(x) = x^3 + x + 1$ over $\mathbb{F}_2$. As before, we let $\mathbb{F}_8 = \mathbb{F}_2[x]/p(x)$ and write $b = x + (p(x))$ for the coset of x.
   (a) Show that $p(x)$ is irreducible over $\mathbb{F}_2$.
   (b) You have already (in previous tutorials) found two roots of the polynomial p over $\mathbb{F}_8$. Use polynomial division over $\mathbb{F}_8$ to find the third root.

(2) Consider the irreducible polynomial $q(x) = x^3 - 2$ over $\mathbb{Q}$.
   (a) Use polynomial division over $\mathbb{Q}(\sqrt[3]{2})$ to find the decomposition of $q(x)$ into irreducible components over $\mathbb{Q}(\sqrt[3]{2})$.
   (b) Let $a \in \mathbb{Q}(\sqrt[3]{2}) \setminus \mathbb{Q}$. Show that a is algebraic but not constructible.
   (c) Find the complex roots of $q(x)$.

(3) Let K be a field of characteristic $p > 0$. The *Frobenius homomorphism* is the map

$$\begin{aligned} F\colon K &\longrightarrow K \\ a &\longmapsto a^p. \end{aligned}$$

   (a) Show that F is indeed a field homomorphism.
   (b) Deduce that F is a field automorphism if K is a finite field.
   (c) Fix a polynomial $p(x)$ over K. Show that any field automorphism permutes the roots of p. polynomial.
   (d) Revisit Question (1) (b) and interpret the result you found in that question part in terms of the Frobenius homomorphism.