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Representation and character theory of finite 2-groups

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October 11, 2013

Acknowledgements

I would first and foremost like to thank Nora for her supervision and guidance. I'd also like to thank Masoud Kamgarpour, for pointing us towards the work of Victor Ostrik. Finally, I'd like thank my family, friends, and fellow students for their support.

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Chapter 1

Introduction

In classical representation theory, there are various formulations of the notion of a projective representation of a finite group, of which a few are

(i) a group homomorphism $\varrho : G \longrightarrow \mathrm{PGL}(V, \mathbb{C})$,

(ii) a map $\varrho : G \longrightarrow \mathrm{GL}(V, \mathbb{C})$ with 2-cocycle $\gamma : G \times G \longrightarrow \mathbb{C}^\times$ such that

$$\varrho(g)\varrho(h) = \gamma(g, h)\varrho(gh)$$

(iii) a group homomorphism $\varrho : \widetilde{G} \longrightarrow \mathrm{GL}(V, \mathbb{C})$, for $\widetilde{G}$ a central extension of G by $\mathbb{C}^\times$,

(iv) a module over the twisted group algebra ${}^{\theta}\mathbb{C}\overline{G}$ for some 2-cocycle $\theta : G \times G \longrightarrow \mathbb{C}^\times$.

In this work, we study the representation theory of finite 2-groups. As is the case for projective representations of finite groups, there are various formulations of a finite 2-group representation. We list them here, in corresponding order:

(i) a gerbal representation of G, as in [FZ11],

(ii) a projective 2-representation of G with 3-cocycle $\alpha : G \times G \times G \longrightarrow \mathbb{C}^\times$, which will be described in Definition 2.0.38,

(iii) a monoidal functor[1] $\mathcal{G} \longrightarrow$ 1-Aut(V), where V is an object in a strict $\mathbb{C}$-linear 2-category and $\mathcal{G}$ is a 2-group extension of G by $[\mathrm{pt}/\mathbb{C}^\times]$,

(iv) a module category (as in [Ost03b]) over Vec_G^α, the 'categorified twisted group algebra' of G with 3-cocycle α.

We will primarily focus on representations of finite 2-groups of the form $\mathcal{C}_G^\alpha(\mathbb{C}^\times)$ acting on an object of a strict $\mathbb{C}$-linear 2-category. We use this $\mathbb{C}$-linearity condition so as to compare our work with that of [GK08], [Bar09] and [Ost03a]. We expect that most of our work could be formulated in the more general case, in the spirit of [FZ11].

We also study the character of a finite 2-group representation. In the classical setting, given a linear representation $\varrho : G \longrightarrow \mathrm{GL}(V, \mathbb{C})$ of a finite group G, the character of ϱ is the map $\chi : G \longrightarrow \mathbb{C}$ defined by

$$\chi(g) = \mathrm{tr}(\varrho(g))$$

We can similarly define the character of a projective 2-representation in terms of the categorical trace defined in [Bar09] and [GK08]; for each $g \in G$, we get a $\mathbb{C}$-vector space $X(g) = \mathbb{T}\mathrm{r}(\varrho(g))$ with conjugation isomorphisms $\beta_{g,h} : X(g) \longrightarrow X(hgh^{-1})$. Our main result is as follows

Theorem. *The character of a projective 2-representation of G with 3-cocycle*

$$\alpha : G \times G \times G \longrightarrow \mathrm{U}(1)$$

is a representation of the twisted Drinfeld double $D^\alpha(G)$.

In other words, if $g, h, k \in G$, then the composition

$$X(g) \xrightarrow{\beta_{g,h}} X(hgh^{-1}) \xrightarrow{\beta_{hgh^{-1},k}} X(khgh^{-1}k^{-1})$$

is equal to

$$X(g) \xrightarrow{\tau(\alpha)([k|h]g)\beta_{kh,g}} X(khgh^{-1}k^{-1})$$

[1] See Definition 1.1.30

where $\tau(\alpha)([k|h]g) \in \mathrm{U}(1)$ *is the transgression of* α*, as defined in Definition 1.1.18.*

While we initially restricted our attention to projective 2-representations, we later realised that viewing projective 2-representations as module categories over Vec_G^α lead to interesting results. In particular, Ostrik's classification of indecomposable module categories over Vec_G^α allows us to make very general statements about projective 2-representations. Furthermore, this approach allows us to use the notion of the tensor product of module categories (as in [ENO10]) to describe a notion of induction for projective 2-representations similar to that described in [GK08, §7].

Finally, we present a comparison of the various formulations listed above, describing the similarities between gerbal representations and projective 2-representations, and also the explicit connection between projective 2-representations and module categories over Vec_G^α.

1.1 Background

1.1.1 Category theory

We recall the basic notion of a category. A good introduction to category theory is available in [Lan98]. Categories should be thought of as a convenient means of studying not only mathematical objects, but also the maps between them.

Definition 1.1.1. A category $\mathcal{C}$ consists of

(i) a collection $\mathrm{ob}(\mathcal{C})$ of *objects*

(ii) for each $x, y \in \mathrm{ob}(\mathcal{C})$, a collection $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of *arrows* between then. We will write $f : x \longrightarrow y$ to mean that $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.

(iii) for each $x, y, z \in \mathrm{ob}(\mathcal{C})$, a composition function

$$\circ : \mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x, z)$$

which assigns to arrows $f : x \longrightarrow y$ and $g : y \longrightarrow z$ their composite $g \circ f : x \longrightarrow z$

(iv) for each $x \in \text{ob}(\mathcal{C})$, an identity arrow $\text{id}_x : x \longrightarrow x$ on x

such that

(i) composition is associative, i.e. $(h \circ g) \circ f = h \circ (g \circ f)$ for composable arrows f, g, h, and

(ii) if $f : x \longrightarrow y$, then

$$\text{id}_y \circ f = f = f \circ \text{id}_x$$

Example 1.1.2. *Some examples of categories are*

(i) The category Set *has objects sets, with arrows the functions between them.*

(ii) For k a field, the category of k-vector spaces Vec_k *has objects k-vector spaces, with arrows the k-linear maps between them.*

(iii) The category of groups Grp *has objects groups, with arrows the group homomorphisms between them.*

(iv) The category of abelian groups Ab *has objects abelian groups, with arrows the group homomorphisms between them.*

(v) The category of topological spaces Top *has objects topological spaces, with arrows the continuous functions between them.*

We now recall some basic terminology. We say an arrow $f : x \longrightarrow y$ is *invertible* if there is a $g : y \longrightarrow x$ such that $f \circ g = \text{id}_y$ and $g \circ f = \text{id}_x$. We say that two objects $x, y \in \text{ob}(\mathcal{C})$ are *isomorphic* (denoted $x \cong y$) if there exists an invertible arrow between them. There is a natural notion of an arrow between categories; these are known as functors.

Definition 1.1.3. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of the following data

(i) for each object $x \in \text{ob}(\mathcal{C})$, an object $F(x) \in \text{ob}(\mathcal{D})$,

(ii) for each arrow $f : x \longrightarrow y$ in $\mathcal{C}$, an arrow $F(f) : F(x) \longrightarrow F(y)$ in $\mathcal{D}$

such that

(i) $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$, and

(ii) $F(f \circ g) = F(f) \circ F(g)$ for composable arrows f, g in $\mathcal{C}$.

Example 1.1.4. *Let $\mathcal{C}$ be a category. Then we have an* identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$, *which assigns each object and arrow of $\mathcal{C}$ to itself.*

Example 1.1.5 ([Lan98, Chapter I §3])**.** *The powerset functor $\mathcal{P}$: Set $\longrightarrow$ Set assigns to each set $X \in$ Set its powerset $\mathcal{P}(X)$, and to each function $f : X \longrightarrow Y$ the map $\mathcal{P}(f) : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ which sends $A \in \mathcal{P}(X)$ to $f(A) \in \mathcal{P}(Y)$.*

We note that functors can be composed; given $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{F}$, the functor $GF : \mathcal{C} \longrightarrow \mathcal{F}$ sends $x \in \mathrm{ob}(\mathcal{C})$ to $GF(x) \in \mathrm{ob}(\mathcal{F})$, and an arrow $f : x \longrightarrow y$ in $\mathcal{C}$ to $GF(f) : GF(x) \longrightarrow GF(y)$ in $\mathcal{F}$. It is a basic exercise to show that GF is indeed a functor.

There is also a notion of arrows between functors; these are known as natural transformations.

Definition 1.1.6. Let $\mathcal{C}, \mathcal{D}$ be categories, and $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ functors. A *natural transformation* $\tau : F \Longrightarrow G$ is a collection of arrows in $\mathcal{D}$

$$\{\tau_x : F(x) \longrightarrow G(x)\}_{x \in \mathrm{ob}(\mathcal{C})}$$

such that if $f : x \longrightarrow y$ is an arrow in $\mathcal{C}$, then the diagram

$$\begin{CD} F(x) @>{\tau_x}>> G(x) \\ @V{F(f)}VV @VV{G(f)}V \\ F(y) @>{\tau_y}>> G(y) \end{CD}$$

commutes. We call the arrow $\tau_x : F(x) \longrightarrow G(x)$ the x component of τ, and we say that τ is a *natural isomorphism* if each component $\tau_x : F(x) \longrightarrow G(x)$ is invertible.

We now have a way of comparing categories; we say that two categories $\mathcal{C}, \mathcal{D}$ are *isomorphic* if there exist functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$, where $\mathrm{id}_{\mathcal{D}}$ and $\mathrm{id}_{\mathcal{C}}$ are the identity functors on $\mathcal{D}$ and $\mathcal{C}$ respectively. An isomorphism of categories is a strong condition; we will often want to use a weaker notion.

Definition 1.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\epsilon : FG \Longrightarrow \mathrm{id}_{\mathcal{D}}$, $\eta : \mathrm{id}_{\mathcal{C}} \Longrightarrow GF$. We call the data (F, G, ϵ, η) an *equivalence of categories*. We will often call F (and G) an equivalence of categories.

We will be interested in categories with additional properties and/or structure. In particular, we are interested in *groupoids* and *monoidal categories*.

1.1.2 Groupoids

Definition 1.1.8. A *groupoid* is a category $\mathcal{G}$ such that every arrow is invertible. We say that a groupoid $\mathcal{G}$ is *finite* if it has finitely many objects and arrows.

Example 1.1.9. *Let G be a group, and take $\overline{G}$ to be the category with one object* pt*, with arrows* $\mathrm{Hom}_{\overline{G}}(\mathrm{pt}, \mathrm{pt}) = G$*, where composition of arrows is that induced by the group structure on G. Then $\overline{G}$ is a groupoid.*

It is a simple exercise to show that a group could be defined as a groupoid with a single object. Furthermore, every groupoid is equivalent (in the sense of Definition 1.1.7) to a disjoint union of groups, where a group is thought of as a single object groupoid.

We now recall some terminology for working with groupoids, and various methods for constructing groupoids which we will use.

Definition 1.1.10 ([Wil08, §1.1])**.** Let $\mathcal{G}$ be a groupoid, and let

$$x_n \xleftarrow{g_n} \dots \xleftarrow{g_2} x_1 \xleftarrow{g_1} x_0$$

be a sequence of n composable arrows in $\mathcal{G}$. We call such a sequence an *n-simplex*, and we write this sequence as

$$[g_n|\dots|g_1]$$

Definition 1.1.11 ([Wil08, §1.3.1])**.** Let $\mathcal{G}$ be a finite groupoid, then we define the *loop groupoid* (or *inertia groupoid*) $\Lambda\mathcal{G}$ to be the category with objects

$$\mathrm{ob}(\Lambda\mathcal{G}) = \bigsqcup_{x\in\mathcal{G}} \mathrm{Hom}_{\mathcal{G}}(x,x)$$

For each $x, y \in \mathrm{ob}(\mathcal{G})$ and arrows $\gamma \in \mathrm{Hom}_{\mathcal{G}}(x,x)$, $g \in \mathrm{Hom}_{\mathcal{G}}(x,y)$, we define an arrow in $\Lambda\mathcal{G}$

$$g_\gamma : \gamma \longrightarrow g\gamma g^{-1}$$

This arrow will be denoted by g when the source is clear. Let

$$g_n \dots g_1\gamma g_1^{-1} \dots g_n^{-1} \xleftarrow{g_n} \dots \xleftarrow{g_2} g_1\gamma g_1^{-1} \xleftarrow{g_1} \gamma$$

be an n-simplex in $\Lambda\mathcal{G}$, then we write this sequence as

$$[g_n|\dots|g_1]\,\gamma$$

Example 1.1.12 ([Wil08, §1.4.1])**.** *Let G be a finite group, and X a finite set with a left G-action. The* action groupoid $\mathcal{G}_G(X)$ *is the groupoid whose objects are the elements of X, and whose arrows are of the form*

$$g \cdot x \xleftarrow{g} x,$$

for each pair $x \in X$ and $g \in G$, with composition defined in the obvious way.

Let G be a finite group, then unpacking Definition 1.1.11, the loop groupoid $\Lambda\overline{G}$ has objects the elements of G, and for each pair $g, h \in G$, there is an arrow

$h_g : g \longrightarrow hgh^{-1}$. Let $X = G^c$ be G with left G-action given by conjugation, then it is clear that $\mathcal{G}_G(G^c) = \Lambda\overline{G}$. An explicit illustration of the inertia groupoid $\Lambda\overline{S_3}$ is provided in [Wil08, §1.3.1 Figure 3].

1.1.3 Projective representations of finite groups & groupoids

To discuss projective 2-representations, we must first refresh our knowledge of projective representations of finite groups and groupoids. A good introduction is available in [Wil08]. For our purposes, we think of group and groupoid cohomology entirely in terms of functions satisfying a cocycle condition. For all of the definitions in this section, $\mathcal{G}$ is a finite groupoid, G is a finite group, and k is a field.

Definition 1.1.13. A function θ mapping 2-simplices in $\mathcal{G}$ to values in $k^\times$ is called a *2-cochain* on $\mathcal{G}$ with values in $k^\times$. If $\mathcal{G} = \overline{G}$, then a 2-cochain is a function (not necessarily a group homomorphism) $\theta : G \times G \longrightarrow k^\times$. We say that such a 2-cochain is *normalised* if it satisfies

$$\theta(g, e) = \theta(e, e) = \theta(e, g) \tag{1.1}$$

for all $g \in G$, where e is the identity element in G.

Definition 1.1.14. A 2-cochain θ on $\mathcal{G}$ is called a *2-cocycle* on $\mathcal{G}$ with values in $k^\times$ if it satisfies

$$\theta([g_1|g_2g_3])\theta([g_2|g_3]) = \theta([g_1g_2|g_3])\theta([g_1|g_2]) \tag{1.2}$$

for every 3-simplex $[g_1|g_2|g_3]$ in $\mathcal{G}$.

If $\mathcal{G} = \overline{G}$, then a 2-cocycle is a function $\theta : G \times G \longrightarrow k^\times$ satisfying

$$\theta(g_1, g_2g_3)\theta(g_2, g_3) = \theta(g_1g_2, g_3)\theta(g_1, g_2) \tag{1.3}$$

for all $g_1, g_2, g_3 \in G$.

Definition 1.1.15. A function α mapping 3-simplices in $\mathcal{G}$ to values in $k^\times$ is called a *3-cocycle* on $\mathcal{G}$ with values in $k^\times$ if it satisfies

$$\alpha([g_2|g_3|g_4])\alpha([g_1|g_2g_3|g_4])\alpha([g_1|g_2|g_3]) = \alpha([g_1g_2|g_3|g_4])\alpha([g_1|g_2|g_3g_4]) \tag{1.4}$$

for every 4-simplex $[g_1|g_2|g_3|g_4]$ in $\mathcal{G}$.

As before, if $\mathcal{G} = \overline{G}$, then a 3-cocycle is a function $\alpha : G \times G \times G \longrightarrow k^\times$ satisfying

$$\alpha(g_2, g_3, g_4)\alpha(g_1, g_2g_3, g_4)\alpha(g_1, g_2, g_3) = \alpha(g_1g_2, g_3, g_4)\alpha(g_1, g_2, g_3g_4) \tag{1.5}$$

for all $g_1, g_2, g_3, g_4 \in G$. We say that such a 3-cocycle is *normalised* if it satisfies

$$\alpha(e, g_2, g_3) = \alpha(g_1, e, g_3) = \alpha(g_1, g_2, e) = 1$$

for all $g_1, g_2, g_3 \in G$, where e is the identity element in G and 1 is the identity element in $k^\times$.

Definition 1.1.16. Let $\theta : G \times G \longrightarrow k^\times$ be a 2-cochain. The *coboundary* of θ is the 3-cocycle $d\theta$ given by

$$(d\theta)(g, h, k) = \frac{\theta(gh, k)\theta(g, h)}{\theta(g, hk)\theta(h, k)}$$

for all $g, h, k \in G$.

Definition 1.1.17. Let $\alpha, \alpha' : G \times G \times G \longrightarrow k^\times$ be 3-cocycles. We say that α and α' are *cohomologous* if there exists a 2-cochain $\theta : G \times G \longrightarrow k^\times$ such that

$$\alpha'/\alpha = d\theta$$

Definition 1.1.18 ([Wil08, Theorem 3])**.** Let $\mathcal{G}$ be a finite groupoid. The *transgression* of a 3-cocycle α on $\mathcal{G}$ with values in U(1) is the 2-cocycle $\tau(\alpha)$ on the loop groupoid $\Lambda\mathcal{G}$ with values in U(1) defined by

$$\tau(\alpha)([h|g]\gamma) = \frac{\alpha([h|g|\gamma])\alpha([hg\gamma g^{-1}h^{-1}|h|g])}{\alpha([h|g\gamma g^{-1}|g])} \tag{1.6}$$

Definition 1.1.19 ([Wil08, Theorem 3])**.** Let $\mathcal{G}$ be a finite groupoid. The *transgression* of a 2-cochain θ on $\mathcal{G}$ with values in U(1) is the 1-cochain $\tau(\theta)$ on the loop groupoid $\Lambda\mathcal{G}$ with values in U(1) defined by

$$\tau(\theta)([g]\gamma) = \frac{\theta([g\gamma g^{-1}|g])}{\theta([g|\gamma])} \tag{1.7}$$

Definition 1.1.20 ([Wil08, §2.3.1])**.** Let $\mathcal{G}$ be a finite groupoid and θ be a 2-cocycle on $\mathcal{G}$ with values in U(1). A *projective representation*[2] *of* $\mathcal{G}$ *with 2-cocycle* θ consists of the following data

1. for each $x \in \mathrm{ob}(\mathcal{G})$, a $\mathbb{C}$-vector space $F(x)$
2. for each arrow $g : x \longrightarrow y$ in $\mathcal{G}$, a $\mathbb{C}$-linear map $F(g) : F(x) \longrightarrow F(y)$ such that

$$F(g_2)F(g_1) = \theta([g_2|g_1])F(g_2g_1)$$

   for every 2-simplex $[g_2|g_1]$ in $\mathcal{G}$.

In the case $\mathcal{G} = \overline{G}$, this recovers the familiar definition of a projective representation [Kar93, §3.1] of G, which we may also think of as a group homomorphism $\varrho : G \longrightarrow \mathrm{PGL}(V)$ with V some $\mathbb{C}$-vector space, which yields a 2-cocycle $\theta : G \times G \longrightarrow \mathrm{U}(1)$ after a choice of lift $\bar{\varrho} : G \longrightarrow \mathrm{GL}(V)$.

There are equivalent ways to define projective representations of groupoids; Willerton [Wil08, §2.3.1] describes the central extension

$$\mathrm{U}(1) \longrightarrow {}^{\theta}\mathcal{G} \longrightarrow \mathcal{G}$$

where ${}^{\theta}\mathcal{G}$ is the category whose objects are those of $\mathcal{G}$, with arrows

$$\mathrm{Hom}_{{}^{\theta}\mathcal{G}}(x, y) := \mathrm{U}(1) \times \mathrm{Hom}_{\mathcal{G}}(x, y)$$

[2]Willerton calls these a θ-twisted representations of $\mathcal{G}$

where composition is given by $(z_2, g_2)\circ(z_1, g_1) := (\theta([g_2|g_1])z_2z_1, g_2\circ g_1)$. A projective representation of $\mathcal{G}$ can then be thought of as a representation of ${}^{\theta}\mathcal{G}$ in which the central U(1) acts in a natural way.

Definition 1.1.21 ([Wil08, §2])**.** Let $\mathcal{G}$ be a finite groupoid, and θ a normalised 2-cocycle on $\mathcal{G}$ with values in $U(1)$. The *twisted groupoid algebra* ${}^{\theta}\mathbb{C}\mathcal{G}$ is the $\mathbb{C}$-algebra spanned by the arrows in $\mathcal{G}$, such that the product $\langle g_2\rangle\langle g_1\rangle$ is zero if g_2 and g_1 are not composable in $\mathcal{G}$, and is $\theta([g_2|g_1])\langle g_2 \circ g_1\rangle$ otherwise.

Willerton similarly describes how the category of projective representations of $\mathcal{G}$ is equivalent to the category of representations of the twisted groupoid algebra ${}^{\theta}\mathbb{C}\mathcal{G}$ [Wil08, Proposition 8]; this is analogous to the equivalence between projective representations of G and modules over the twisted group algebra ${}^{\theta}\mathbb{C}\overline{G}$ [Kar93, Theorem 3.2]. An important example for us of a twisted groupoid algebra will be the *twisted Drinfeld double*.

Definition 1.1.22 ([Wil08, Theorem 17])**.** Given a finite group G and a 3-cocycle α on G with values in U(1), the *twisted Drinfeld double* $D^{\alpha}(G)$ is the twisted groupoid algebra ${}^{\tau(\alpha)}\mathbb{C}(\Lambda\overline{G})$, where $\tau(\alpha)$ is the transgression map defined in Definition 1.1.18.

1.1.4 Monoidal categories

We present the definition of a monoidal category as given in [Lan98, Chapter VII].

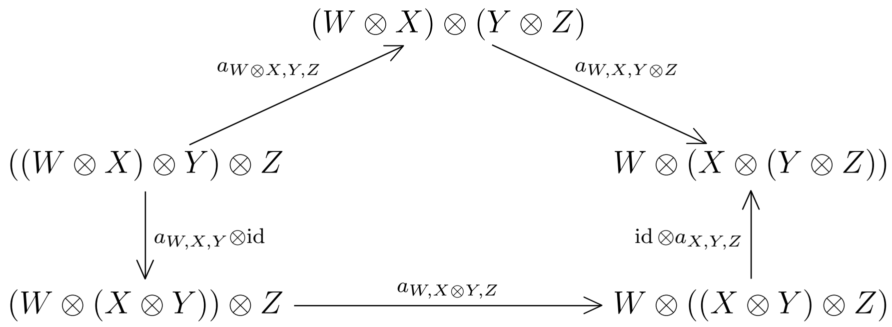
Definition 1.1.23. A *monoidal category* is a category $\mathcal{M}$ with the following data

(i) a functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$, called the tensor or monoidal product

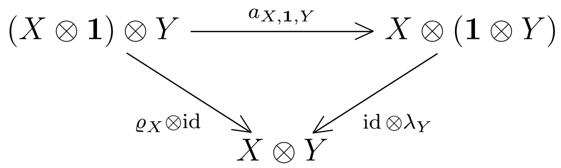
(ii) an object $\mathbf{1} \in \mathrm{ob}(\mathcal{M})$, called the unit object

(iii) a natural isomorphism $a_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$, called the associativity isomorphism or associator

(iv) natural isomorphisms $\lambda_X : \mathbf{1} \otimes X \longrightarrow X$ and $\varrho_X : X \otimes \mathbf{1} \longrightarrow X$, called the left and right units respectively such that the pentagon diagram



and the triangle diagram



commute for all $W, X, Y, Z \in \mathrm{ob}(\mathcal{M})$. In the language of [BL04], this is a *weak monoidal category.*

Remark 1.1.24. *The tensor product* $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ *is a* functor*, hence it is necessary to describe the tensor product of objects, and the tensor product of arrows. however we will omit a description of the tensor product of arrows when it is clear from the context. Similarly, we will often omit a description of the left and right unit isomorphisms.*

The notion of a monoidal category will be used to describe both 2-groups and module categories. We proceed by listing some some examples of monoidal categories.

Example 1.1.25 ([Eti09, §1.3.1])**.** *The category* Set *may be given the structure of a monoidal category. Let the tensor product of sets* X *and* Y *be* $X \otimes Y := X \times Y$ *their Cartesian product, and take the unit object to be the one element set* $\mathbf{1} = \{*\}$. *Let the associativity isomorphism* $a_{X,Y,Z}$ *be given by the standard isomorphism*

$$(X \times Y) \times Z \xrightarrow{\cong} X \times (Y \times Z)$$

This example can be naturally extended to both Grp *and* Top[3]*.*

Example 1.1.26 ([Eti09, §1.3.3])**.** *Let R be a commutative ring, then the category of left R-modules R-*Mod *is a monoidal category, with monoidal product being given by the tensor product of R-modules $\otimes_R$. The unit object is $\mathbf{1} = R$, considered as an R-module.*

Notable special cases of this example are $R = k$ for some field k, and $\mathbb{R} = \mathbb{Z}$. This gives us a monoidal structure on the categories Vec_k and Ab respectively.

A stricter notion of monoidal category will often be useful.

Definition 1.1.27 ([Lan98, Chapter VII §1])**.** A monoidal category $\mathcal{M}$ is said to be *strict* if $a_{X,Y,Z}$, λ_X and ϱ_X are identity arrows for all $X, Y, Z \in \mathrm{ob}(\mathcal{M})$. We will say that a monoidal category is *non-strict* if it is not strict.

Note that if $\mathcal{M}$ is a strict monodial category, then we have equalities

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \quad \text{and} \quad \mathbf{1} \otimes X = X = X \otimes \mathbf{1}$$

for all $X, Y, Z \in \mathrm{ob}(\mathcal{M})$. However this should not be taken as a characterisation of strict monoidal categories: we will see examples of non-strict monoidal categories for which these equalities hold. In other words, the choice of associativity and unit isomorphisms is an important part of the data of a monoidal category.

Example 1.1.28 ([Eti09, §1.3.6])**.** *Let G be a group, and A an abelian group. Let $\mathcal{C}_G(A) = \mathcal{G}_A(G)$ (see Example 1.1.12), where the A-action on G is trivial. Unpacking this, $\mathcal{C}_G(A)$ is the groupoid with objects the elements of G, and arrows*

$$\mathrm{Hom}_{\mathcal{C}_G(A)}(g, h) = \begin{cases} A & \textit{if } g = h \\ \emptyset & \textit{if } g = h \end{cases}$$

with composition induced by the group structure of A. We give $\mathcal{C}_G(A)$ the following monodial structure

[3] More generally, any category $\mathcal{C}$ with finite products is monoidal, where the tensor product of objects X, Y is given by the product $X \times Y$, and the unit object $\mathbf{1}$ is the terminal object in $\mathcal{C}$ [Lan98, Chapter VII, §1].

- *If $g, h \in G$, let $g \otimes h := gh$, and*
- *If $a : g \longrightarrow g$ and $b : h \longrightarrow h$ are arrows, let $a \otimes b := ab : gh \longrightarrow gh$*

The unit object is $\mathbf{1} = e$ the identity element of G, with associator $a_{g,h,k} = \mathrm{id}_{ghk}$ corresponding to the identity element of A. This is an example of a strict monoidal category.

For this example, it is instructive to think of the associator as a map

$$\begin{aligned} \alpha : G \times G \times G &\longrightarrow A \\ (g, h, k) &\longmapsto a_{g,h,k} = \mathrm{id}_{ghk} \end{aligned}$$

It is natural to ask whether there are more interesting choices of α giving a monoidal structure on $\mathcal{C}_G(A)$. The pentagon and triangle diagrams tell us that such a map must satisfy the equations

$$\alpha(g_1, g_2, g_3)\alpha(g_1, g_2g_3, g_4)\alpha(g_2, g_3, g_4) = \alpha(g_1, g_2, g_3g_4)\alpha(g_1g_2, g_3, g_4)$$
$$\alpha(g_1, 1, g_2) = 1$$

for all $g_1, g_2, g_3, g_4 \in G$. This motivates our next example.

Example 1.1.29 ([Eti09, §1.3.7]). *Let $\alpha : G \times G \times G \longrightarrow A$ be a normalised 3-cocycle (see Definition 1.4). Let $\mathcal{C}_G^\alpha(A)$ be the monoidal category whose objects, arrows, monoidal product, unit object, and unit isomorphisms are the same as those of $\mathcal{C}_G(A)$. Let the associator be given by*

$$a_{g,h,k} = \alpha(g, h, k) \in \mathrm{Hom}_{\mathcal{C}_G^\alpha(A)}(ghk, ghk)$$

For non-trivial α, this gives $\mathcal{C}_G^\alpha(A)$ a non-strict monoidal structure.

We will be very interested in monoidal categories of the form $\mathcal{C}_G^\alpha(A)$ where α is non-trivial; these are examples of 2-groups with non-trivial associators. Before describing 2-groups, we recall monoidal functors. These are functors between monoidal categories respecting the monoidal structure.

Definition 1.1.30. Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal categories. A *monoidal functor* $F : \mathcal{M} \longrightarrow \mathcal{N}$ is a functor $F : \mathcal{M} \longrightarrow \mathcal{N}$ along with the following data

(i) a natural isomorphism $\phi_{X,Y} : F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$, and

(ii) an isomorphism $\phi : \mathbf{1}_{\mathcal{N}} \longrightarrow F(\mathbf{1}_{\mathcal{M}})$

such that the diagrams

$$\begin{array}{ccccc}
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\phi_{X,Y} \otimes \mathrm{id}} & F(X \otimes Y) \otimes F(Z) & \xrightarrow{\phi_{X \otimes Y, Z}} & F((X \otimes Y) \otimes Z) \\
\Big\downarrow{\scriptstyle a_{F(X),F(Y),F(Z)}} & & & & \Big\downarrow{\scriptstyle F(a_{X,Y,Z})} \\
F(X) \otimes (F(Y) \otimes F(Z)) & \xrightarrow{\mathrm{id} \otimes \phi_{Y,Z}} & F(X) \otimes F(Y \otimes Z) & \xrightarrow{\phi_{X, Y \otimes Z}} & F(X \otimes (Y \otimes Z))
\end{array}$$

and

$$\begin{array}{ccc}
\mathbf{1}_{\mathcal{N}} \otimes F(X) & \xrightarrow{\lambda'_{F(X)}} & F(X) \\
\Big\downarrow{\scriptstyle \phi \otimes \mathrm{id}} & & \Big\uparrow{\scriptstyle F(\lambda_X)} \\
F(\mathbf{1}_{\mathcal{M}}) \otimes F(X) & \xrightarrow{\phi_{\mathbf{1}_{\mathcal{M}},X}} & F(\mathbf{1}_{\mathcal{M}} \otimes X)
\end{array}$$

commute. We say that two monoidal categories are equivalent if there is a monoidal functor between them that is an equivalence of categories.

Remark 1.1.31. *This is sometimes called a strong monoidal functor, however we follow the convention used in [Ost03a, BL04].*

Example 1.1.32 ([CE04, Example 1.7])**.** *Let $\alpha, \alpha' : G \times G \times G \longrightarrow A$ be normalised 3-cocycles. Recalling Example 1.1.29, the categories $\mathcal{C}_G^{\alpha'}(A)$ and $\mathcal{C}_G^{\alpha}(A)$ have the same objects and arrows, so we have an identity functor*

$$F : \mathcal{C}_G^{\alpha'}(A) \longrightarrow \mathcal{C}_G^{\alpha}(A)$$

on underlying categories. We wish to give F the structure of a monoidal functor.

If $\theta : G \times G \longrightarrow A$ is a normalised 2-cochain such that $\alpha'/\alpha = d\theta$ (see Definition 1.1.16), then let

$$\phi_{g,h} = \theta(g,h) \in \mathrm{Hom}_{\mathcal{C}_G^{\alpha}(A)}(gh, gh)$$

This gives F the structure of a monoidal functor.

We note that F is an equivalence, $\mathcal{C}_G^{\alpha'}(A)$ and $\mathcal{C}_G^{\alpha}(A)$ are equivalent monoidal categories when α and α' are cohomologous. Moreover, $\mathcal{C}_G^{\alpha'}(A)$ and $\mathcal{C}_G^{\alpha}(A)$ are equivalent monoidal categories if and only if α' and α are cohomologous 3-cocycles.

1.1.5 2-groups

We now present the notion of a 2-group. A good introduction to 2-groups is available in [BL04].

Definition 1.1.33 ([BL04, Definition 2]). A *2-group* is a monoidal groupoid $\mathcal{G}$ with every object weakly invertible, that is, for each $g \in \mathrm{ob}(\mathcal{G})$, there exists $h \in \mathrm{ob}(\mathcal{G})$ such that $g \otimes h \cong \mathbf{1}$ and $h \otimes g \cong \mathbf{1}$.

We have defined what is known as a weak 2-group in the language of [BL04].

Definition 1.1.34 ([BL04, §2]). A *strict 2-group* is a strict monoidal groupoid $\mathcal{G}$ with every object invertible, that is, for each $g \in \mathrm{ob}(\mathcal{G})$, there exists $g^{-1} \in \mathrm{ob}(\mathcal{G})$ such that $g \otimes g^{-1} = \mathbf{1}$ and $g^{-1} \otimes g = \mathbf{1}$.

We have the following important classification of 2-groups from the unpublished thesis of Sinh [Sin75].

Proposition 1.1.35 (see [Sin75] and [BL04, §8.3]). *Let $\mathcal{G}$ be a 2-group. Then $\mathcal{G}$ is determined up to equivalence by the data of*

(i) a group G,

(ii) an abelian group A with a G-action, and

(iii) an element $[\alpha]$ of the corresponding cohomology group $H^3(G, A)$

As remarked previously, the monoidal category $\mathcal{C}_G^{\alpha}(A)$ of Example 1.1.29 is a 2-group. We will focus our attention on 2-groups of this form; this classification says that we may understand the representation theory of a large class of 2-groups by studying the representation theory of $\mathcal{C}_G^{\alpha}(A)$.

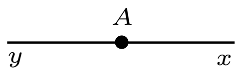
1.1.6 String diagrams and 2-categories

We recall the notion of a 2-category, and notate these 2-categories using string diagrams, closely following [Bar09, Chapter 4] and [CW10, §1.1].

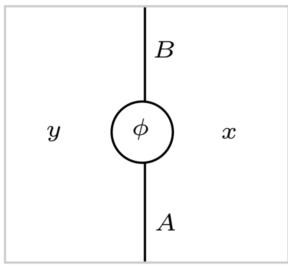
Definition 1.1.36 ([Lan98])**.** A *2-category* $\mathcal{C}$ consists of the following data

1. A class of objects $\mathrm{ob}(\mathcal{C})$.

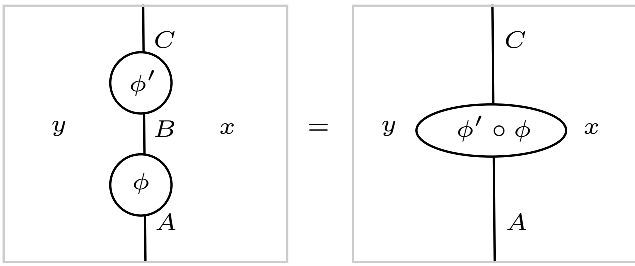
2. For each pair $x, y \in \mathrm{ob}(\mathcal{C})$ a category $1\text{-Hom}_{\mathcal{C}}(x, y)$. We call an object $A \in 1\text{-Hom}_{\mathcal{C}}(x, y)$ a *1-morphism* from x to y, notated $A : x \longrightarrow y$. In string diagram notation, A is drawn



Given $A, B \in 1\text{-Hom}_{\mathcal{C}}(x, y)$, we call an arrow $\phi \in \mathrm{Hom}_{1\text{-Hom}_{\mathcal{C}}(x,y)}(A, B) = 2\text{-Hom}_{\mathcal{C}}(A, B)$ a *2-morphism* from A to B, denoted $\phi : A \Rightarrow B$. In string diagram notation, ϕ is drawn



Composition of morphisms in $2\text{-Hom}_{\mathcal{C}}(x, y)$ is notated by $\circ$, e.g. if $\phi : A \Rightarrow B$ and $\phi' : B \Rightarrow C$, where $A, B, C : x \longrightarrow y$ are 1-morphisms, then we have a 2-morphism $\phi' \circ \phi : A \Rightarrow C$; we call this *vertical composition* of 2-morphisms. In string diagram notation, we have

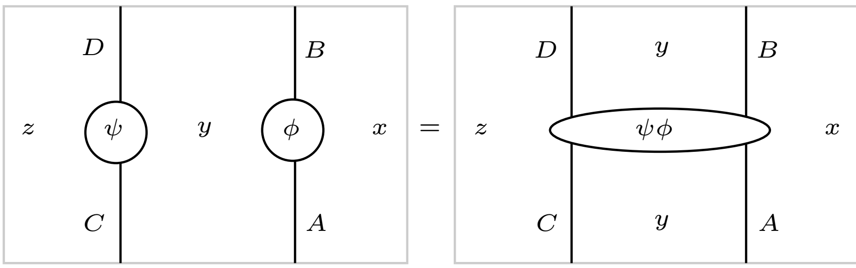


The equals sign in this figure indicates that both string diagrams refer to the same 2-morphism.

3. The composition bifunctor

$$1\text{-Hom}_{\mathcal{C}}(y, z) \times 1\text{-Hom}_{\mathcal{C}}(x, y) \longrightarrow 1\text{-Hom}_{\mathcal{C}}(x, z)$$
$$(A, B) \mapsto AB$$

Unpacking this definition, we note that if $A, B : x \longrightarrow y$ and $C, D : y \longrightarrow z$ are 1-morphisms and $\phi : A \Rightarrow B$, $\psi : C \Rightarrow D$ are 2-morphisms, then we have the composition $\phi\psi : CA \Rightarrow DB$ where $CA, DB : x \longrightarrow z$ are 1-morphisms from x to z; we call this *horizontal composition* of 2-morphisms. In string diagram notation, we have



4. The natural associativity isomorphism

$$\alpha_{A,B,C} : (AB)C \Rightarrow A(BC)$$

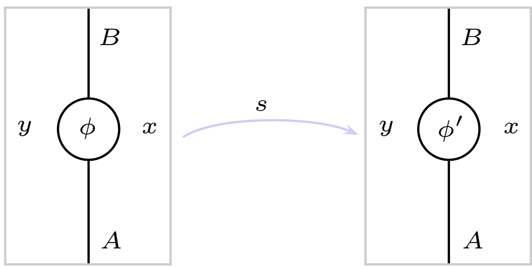
It is defined for any three composable 1-morphisms A, B, C and satisfies the pentagonal axioms, see [Lan98]. We suppress these isomorphisms from our notation; Bartlett provides a strictification argument [Bar09, Proposition 4.1] that we are allowed to do this,

5. For any $x \in \mathrm{ob}(\mathcal{C})$, a 1-morphism $1_x \in 1\text{-Hom}_{\mathcal{C}}(x,x)$ called the unit morphism, with 2-isomorphisms

$$\epsilon_\phi : 1_x A \Rightarrow A \text{ for any } A : y \longrightarrow x,$$
$$\zeta_\psi : B 1_x \Rightarrow B \text{ for any } B : x \longrightarrow z$$

satisfying the axioms of [Lan98]. By the same strictification argument of Bartlett, we can omit these isomorphisms from our notation.

We also need the notion of a k-linear 2-category. A k-linear 2-category $\mathcal{C}$ has the key features that $2\text{-Hom}_{\mathcal{C}}(A,B)$ is a k-vector space (where $A, B \in 1\text{-Hom}_{\mathcal{C}}(x,y)$ are 1-morphisms) and composition is k-bilinear. In particular, if $\phi, \phi' \in 2\text{-Hom}_{\mathcal{C}}(A,B)$ are 2-morphisms related by a scalar $s \in k$ (i.e. $s\phi = \phi'$), then we draw



We will occasionally omit borders and labels of diagrams where the context is clear. We have the following important example of a 2-group arising from a 2-category.

Example 1.1.37 ([BL04, Example 33])**.** *Let $\mathcal{C}$ be a strict 2-category, and $V \in \mathrm{ob}(\mathcal{C})$. Then there is a strict 2-group* $1\text{-Aut}(V)$*, called the* strict automorphism 2-group *of V. The objects of* $1\text{-Aut}(V)$ *are the invertible 1-morphisms $A : V \longrightarrow V$ in $\mathcal{C}$, with arrows being the invertible 2-morphisms between them. The monoidal structure on* $1\text{-Aut}(V)$ *is that induced by the composition of 1-morphisms and the horizontal composition of 2-morphisms, with identity object $\boldsymbol{1} = \mathrm{id}_V : V \longrightarrow V$ the identity arrow on V.*

Chapter 2

Projective 2-representations

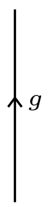
In classical representation theory, we study a group by understandings it actions on a vector space by automorphisms. It it thus natural to study a 2-group by understanding its actions on an object of a strict 2-category by automorphisms.

In this section, we will study the action of finite 2-groups of the form $\mathcal{C}_G^{\alpha}(k^{\times})$ on an object of a strict k-linear 2-category; we will call these projective 2-representations. We later discuss the close relationship between these and the gerbal representations introduced in [FZ11].

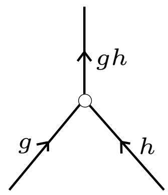
Definition 2.0.38. Let G be a finite group, and $\mathcal{C}$ a k-linear 2-category. A *projective 2-representation* of G on $\mathcal{C}$ consists of the following data

(a) an object V of $\mathcal{C}$

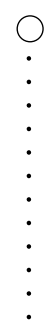
(b) for each $g \in G$, a 1-automorphism $\varrho(g) : V \longrightarrow V$, drawn as



(c) for every pair $g, h \in G$, a 2-isomorphism $\psi_{g,h} : \varrho(g)\varrho(h) \overset{\cong}{\Rightarrow} \varrho(gh)$, drawn as



(d) a 2-isomorphism $\psi_1 : \varrho(1) \overset{\cong}{\Rightarrow} \mathrm{id}_V$, drawn as

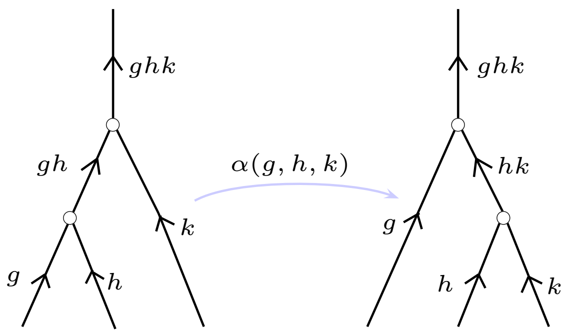


such that the following conditions hold

(i) for any $g, h, k \in G$, we have

$$\psi_{g,hk}(\varrho(g)\psi_{h,k}) = \alpha(g,h,k)\psi_{gh,k}(\psi_{g,h}\varrho(k))$$

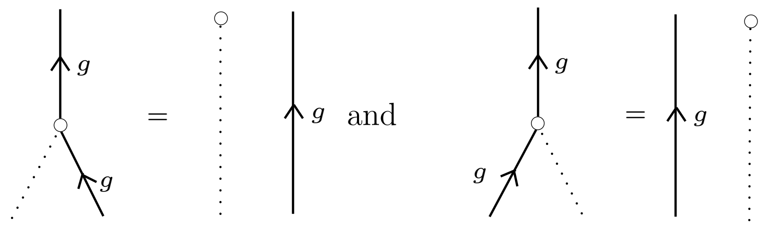
where $\alpha(g,h,k) \in k^{\times}$. In string diagram notation, we draw this as



(ii) for any $g \in G$, we have

$$\psi_{1,g} = \psi_1\varrho(g) \text{ and } \psi_{g,1} = \varrho(g)\psi_1$$

In string diagram notation, we draw these as



2.1 The 3-cocycle condition

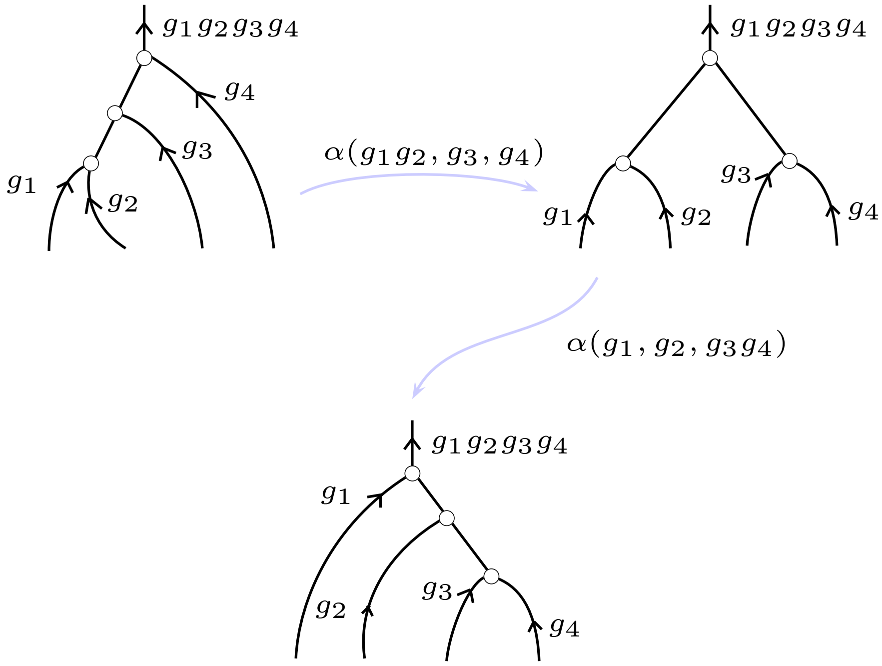
Recall a gerbal representation [FZ11, Definition 2.8] of a group G on a category $\mathcal{V}$ is the assignment to each $g \in G$ of an equivalence $F(g) : \mathcal{V} \longrightarrow \mathcal{V}$ such that $F(e) \cong \mathrm{id}_{\mathcal{V}}$ and $F(g)F(h) \cong F(gh)$. The main difference between our definition and that of a gerbal representation is the choice of a specific isomorphism

$$\psi_{g,h} : \varrho(g)\varrho(h) \overset{\cong}{\Longrightarrow} \varrho(gh)$$

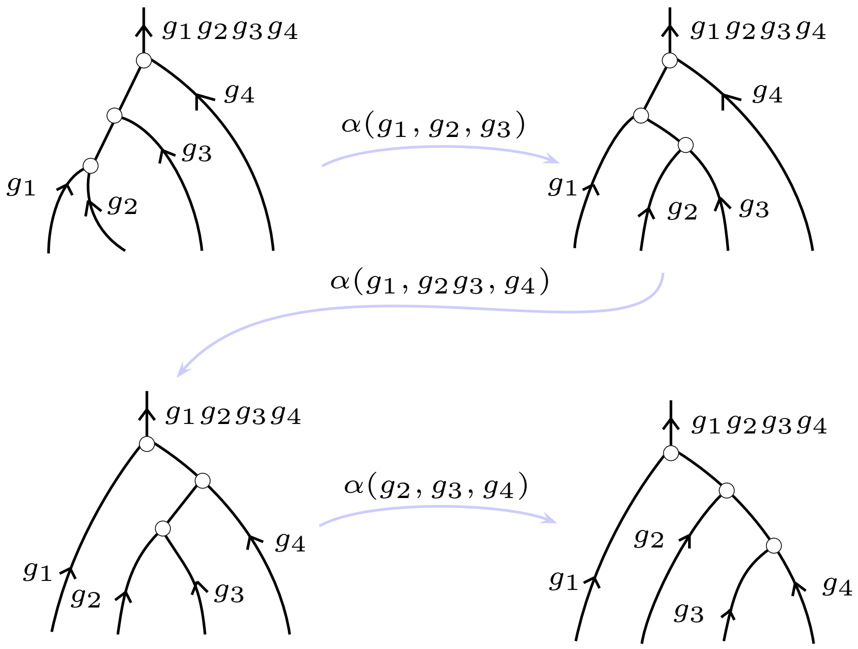
as part of the data. In [FZ11], Frenkel and Zhu show that given F a gerbal representation of G, then any choice of isomorphism $c(g,h) : F(g)F(h) \overset{\cong}{\Longrightarrow} F(gh)$ yields a 3-cocycle on G satisfying condition (i). Similarly, by considering the various 2-isomorphisms between $\varrho(g_1)\varrho(g_2)\varrho(g_3)\varrho(g_4)$ and $\varrho(g_1 g_2 g_3 g_4)$, we may prove the following.

Proposition 2.1.1 (Compare [FZ11, Theorem 2.10]). *Let ϱ be a projective 2-representation of a group G. Then the map $\alpha : G \times G \times G \longrightarrow k^\times$ appearing in condition (i) is a normalised 3-cocycle on G with values in $k^\times$.*

Proof. We use 2.0.38 (i) for all steps of our proof. Consider



On the other hand, we have



Comparing these diagrams, we find

$$\alpha(g_1 g_2, g_3, g_4)\alpha(g_1, g_2, g_3 g_4) = \alpha(g_2, g_3, g_4)\alpha(g_1, g_2 g_3, g_4)\alpha(g_1, g_2, g_3) \tag{2.1}$$

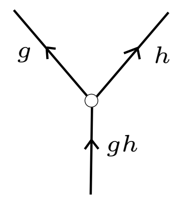
which is Equation 1.4, so α is indeed a 3-cocycle. □

Example 2.1.2 (Compare [GK08, §5.1]). *Let G be a finite group with a normalised 2-cochain θ. Let $\alpha = d\theta$ be the 3-cocycle given by the coboundary of θ, then we define a projective 2-representation of G on $\mathrm{Vec}_{\mathbb{C}}$ with corresponding 3-cocycle α. For $g \in G$, we let*

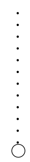
$$\varrho(g) = \mathrm{id}_{\mathrm{Vec}_{\mathbb{C}}} : \mathrm{Vec}_{\mathbb{C}} \longrightarrow \mathrm{Vec}_{\mathbb{C}}$$

be the identity functor on $\mathrm{Vec}_{\mathbb{C}}$. For $g, h \in G$ let $\psi_{g,h} : \varrho(g)\varrho(h) \overset{\cong}{\Rightarrow} \varrho(gh)$ be given by multiplication by $\theta(g, h)$. Let $\psi_1 : \varrho(1) \overset{\cong}{\Rightarrow} \mathrm{id}_{\mathrm{Vec}_{\mathbb{C}}}$ be given by multiplication by $\theta(1, 1)$.

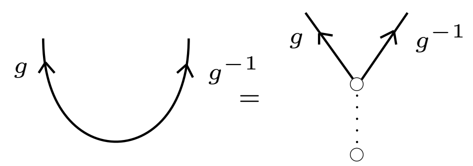
We recall some further notation for working with projective 2-representations. To represent $\psi_{g,h}^{-1} : \varrho(gh) \overset{\cong}{\Rightarrow} \varrho(g)\varrho(h)$, we draw



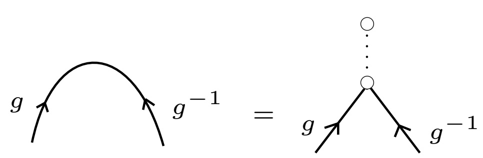
Similarly, to represent $\psi_1^{-1} : \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Rightarrow} \varrho(1)$, we draw



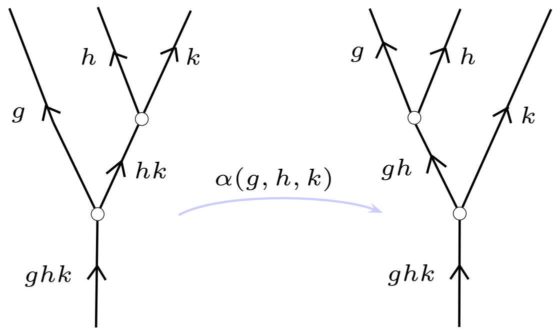
Finally, we let



and



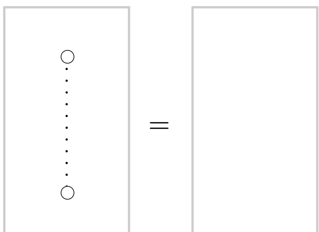
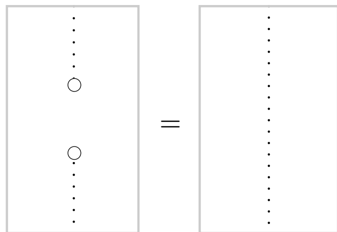
Corollary 2.1.3. *The following graphical equation holds after inverting condition (i) of Definition 2.0.38.*



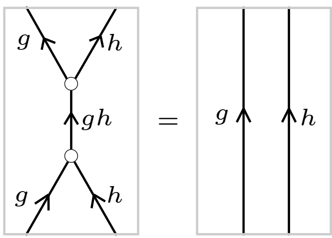
that is,

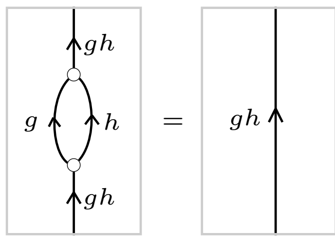
$$(\psi_{g,h}^{-1}\varrho(k))\psi_{gh,k}^{-1} = \alpha(g,h,k)(\varrho(g)\psi_{h,k}^{-1})\psi_{g,hk}^{-1}$$

Corollary 2.1.4 ([Bar09, §7.1.1])**.** *We have* $\psi_1 \circ \psi_1^{-1} = \mathrm{id}_{\mathcal{C}}$ *and* $\psi_1^{-1} \circ \psi_1 = \varrho(1)$, *drawn as*

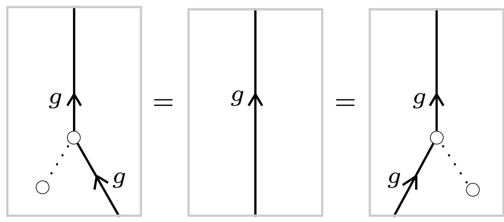
(a)    (b) 

Similarly, if $g, h \in G$, *then* $\psi_{g,h}^{-1} \circ \psi_{g,h} = \varrho(g)\varrho(h)$ *and* $\psi_{g,h} \circ \psi_{g,h}^{-1} = \varrho(gh)$, *drawn as*

(c) 

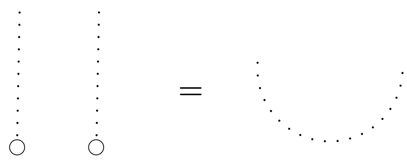
(d) 

Finally, for $g \in G$, we have $\psi_{1,g}(\psi_1^{-1}\varrho(g)) = \varrho(g) = \psi_{g,1}(\varrho(g)\psi_1^{-1})$, drawn as

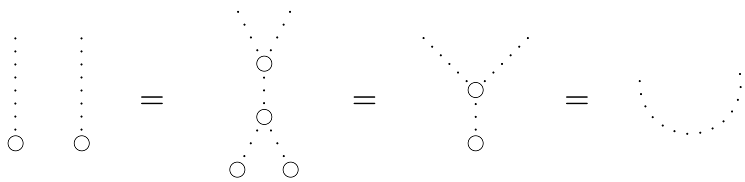
(e) 

2.1.1 Some graphical equations for projective 2-representations

Lemma 2.1.5 ([Bar09, Lemma 7.3 (ii)])**.** *The following graphical equation holds*

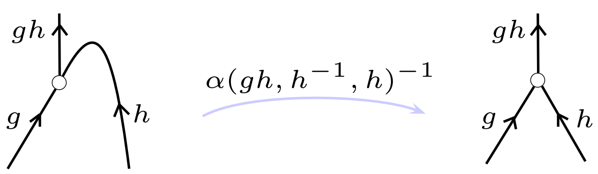


Proof.

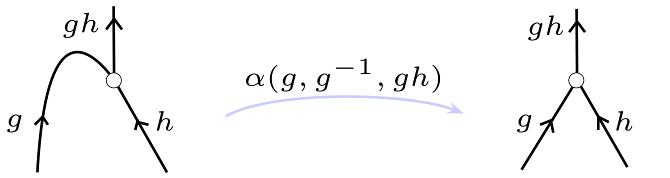


The first equality follows from 2.1.4 (c), the second from 2.1.4 (e), with the final following by definition. □

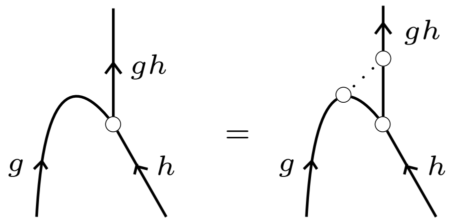
Lemma 2.1.6 (Compare [Bar09, Lemma 7.3 (iii)])**.** *The following graphical equations hold*

(i) 

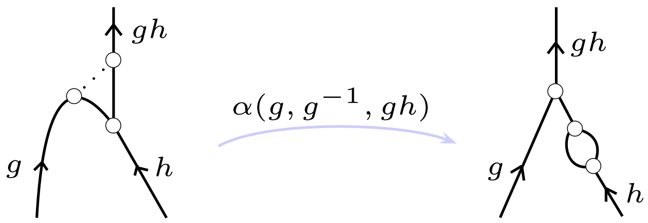
(*ii*)



Proof. We will prove (*ii*); the proof of (*i*) is almost identical. By combining 2.1.4 (c) and (e), we get



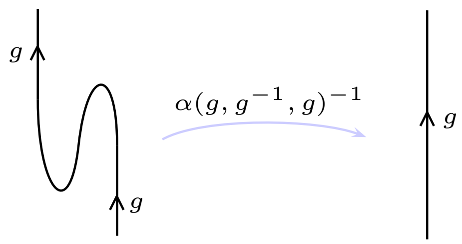
Next, by 2.0.38 (i), we get



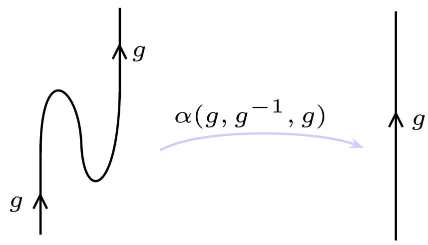
A final application of 2.1.4 (d) gives us the desired result. □

Corollary 2.1.7 (Compare [Bar09, Lemma 7.3 (i)]). *The following graphical equations hold*

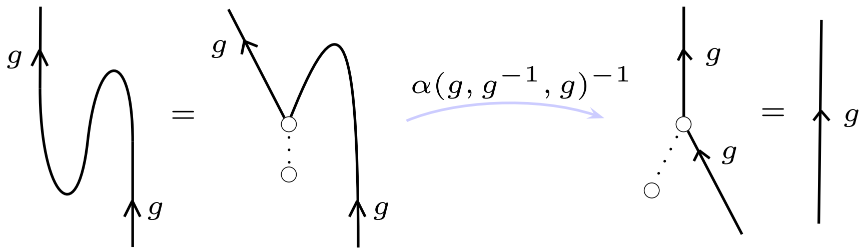
(*i*)



(*ii*)

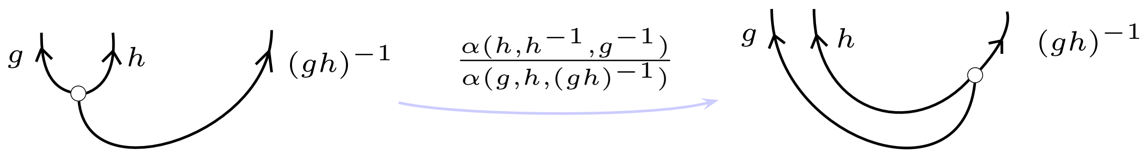


Proof. We will prove (i); the proof of (ii) is almost identical. By applying 2.1.6 and then 2.1.4 (e), we have

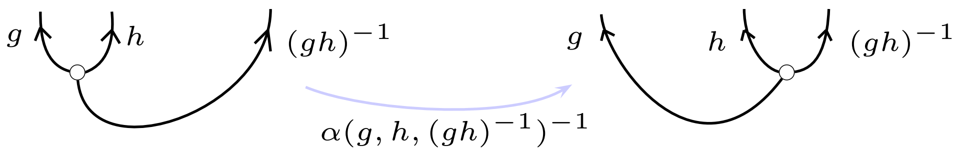


as required. ☐

Corollary 2.1.8 (Compare [Bar09, Lemma 7.3 (iv)])**.** *The following graphical equation holds*



Proof. Applying 2.1.3. we get

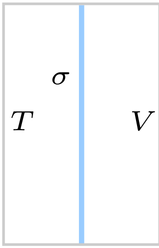


Inverting the equation derived in part (ii) of 2.1.6 gives the desired result. ☐

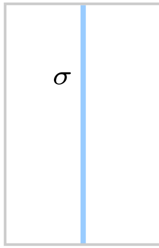
As a final note, the projective 2-representations of a group G with 3-cocycle α on a k-linear 2-category $\mathcal{C}$ form a 2-category. We adapt the following definitions of [Bar09].

Definition 2.1.9 ([Bar09, §7.1.2])**.** Let $\varrho : V \longrightarrow V$ and $\vartheta : T \longrightarrow T$ be projective 2-representations of a finite group G with 3-cocycle α on a k-linear 2-category $\mathcal{C}$ where $V, T \in \mathrm{ob}(\mathcal{C})$. A *1-morphism* $\sigma : \varrho \longrightarrow \vartheta$ of projective 2-representations consists of the following data

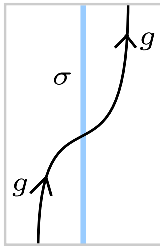
(a) a 1-morphism $\sigma : V \longrightarrow T$ in $\mathcal{C}$, drawn as



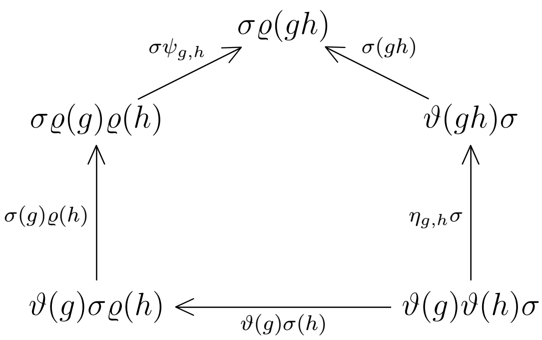
or simply

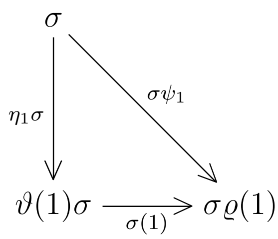


(b) for each $g \in G$, a 2-isomorphism $\sigma(g) : \vartheta(g)\sigma \overset{\cong}{\Rightarrow} \sigma\varrho(g)$ in $\mathcal{C}$, drawn as



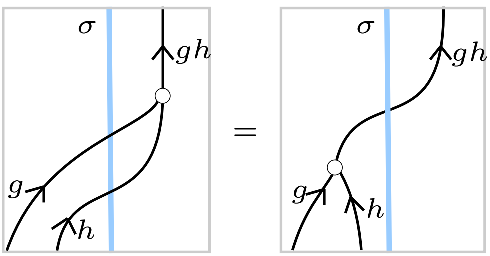
such that the diagrams



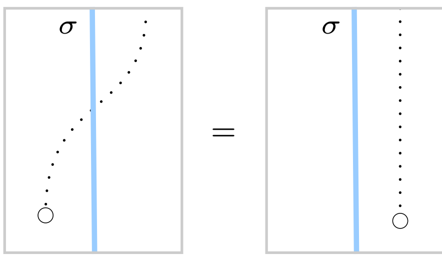


commute, where $\psi_{g,h} : \varrho(g)\varrho(h) \overset{\cong}{\Rightarrow} \varrho(gh)$ and $\eta_{g,h} : \vartheta(g)\vartheta(h) \overset{\cong}{\Rightarrow} \vartheta(gh)$.

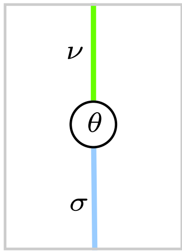
In string diagram notation, we draw these as



and



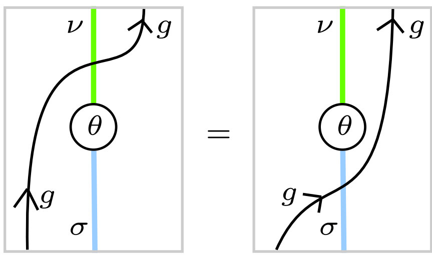
Definition 2.1.10 ([Bar09, §7.1.3])**.** Given 1-morphisms $\sigma, \nu : \varrho \longrightarrow \vartheta$ of projective 2-representations, a *2-morphism* $\theta : \sigma \Rightarrow \nu$ is a 2-morphism $\theta : \sigma \Rightarrow \nu$ in $\mathcal{C}$, drawn as



such that the diagram

$$\begin{array}{ccc} \vartheta(g)\sigma & \xrightarrow{\vartheta(g)\theta} & \vartheta(g)\nu \\ {\scriptstyle \sigma(g)}\downarrow & & \downarrow{\scriptstyle \nu(g)} \\ \sigma\varrho(g) & \xrightarrow[\theta\varrho(g)]{} & \nu\varrho(g) \end{array}$$

commutes. We draw this as



Definition 2.1.11. Let $2\mathrm{Rep}_k^\alpha(G, \mathcal{C})$ be the 2-category with objects being projective 2-representations of a finite group G with 3-cocycle α on a k-linear 2-category $\mathcal{C}$, with 1 and 2-morphisms as defined above.

2.2 The character of a projective 2-representation

Recall that the character of a classical representation ϱ is the map $\chi : G \longrightarrow k$ defined by $\chi(g) = \mathrm{tr}(\varrho(g))$. This motivates the following definition of [GK08] and [Bar09].

Definition 2.2.1 ([GK08, Definition 3.1] and [Bar09, Definition 7.8])**.** Let $\mathcal{C}$ be a 2-category, $x \in \mathrm{ob}(\mathcal{C})$ and $A \in 1\text{-}\mathrm{Hom}_{\mathcal{C}}(x, x)$ a 1-endomorphism of x. The *categorical trace* of A is defined to be

$$\mathbb{T}\mathrm{r}(A) = 2\text{-}\mathrm{Hom}_{\mathcal{C}}(1_x, A)$$

where 1_x is the identity 1-morphism of x.

Remark 2.2.2. *Note that if $\mathcal{C}$ is a k-linear 2-category (as is the case when studying projective 2-representations), then the categorical trace of a 1-endomorphism $A : x \longrightarrow x$ is a k-vector space.*

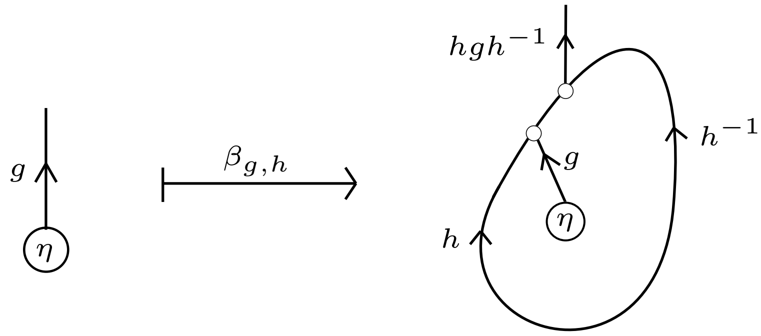
Definition 2.2.3 (Compare [GK08, Definition 4.8] and, in particular, [Bar09, Definition 7.9])**.** Let ϱ be a projective 2-representation of a finite group G. The *character* of ϱ is the assignment

$$g \longmapsto \mathbb{T}\mathrm{r}(\varrho(g)) := X(g) \text{ for each } g \in G$$

and the collection of isomorphisms

$$\beta_{g,h} : X(g) \longrightarrow X(hgh^{-1})$$

defined by



for each $g, h \in G$. That the $\beta_{g,h}$ are isomorphisms is a consequence of Theorem 2.2.5. We note that the definitions in [GK08] and [Bar09] are the special case $\alpha = 1$, although they look a bit different at first sight.

Example 2.2.4 ([GK08, Definition 4.12])**.** *Let ϱ be a projective 2-representation of a finite group G on a k-linear 2-category with finite-dimensional* 2-Hom(A, B)*. If $g, h \in G$ is a pair of commuting elements, then $\beta_{g,h}$ is an automorphism of $X(g)$. Let the* joint trace *be the map*

$$\chi(g, h) = \mathbb{T}\mathrm{r}(\beta_{g,h})$$

defined for commuting $g, h \in G$. As in [GK08, §5.1], let us consider the character and joint trace of the projective 2-representation defined in Example 2.1.2. For $g \in G$, we have

$$X(g) = \mathbb{T}\mathrm{r}(\mathrm{id}_{\mathrm{Vec}_{\mathbb{C}}}) = \mathbb{C}$$

Let $g, h \in G$ be commuting elements, then it follows from Definition 2.2.3 that the joint trace $\chi(g, h)$ is given by multiplication by

$$\frac{\theta(h, g)}{\theta(hgh^{-1}, h)} = \frac{\theta(h, g)}{\theta(g, h)} = \tau(\theta)([g]h)$$

the transgression of the normalised 2-cochain θ (see Definition 1.1.19).

We now present our main result.

Theorem 2.2.5. *Let G be a finite group, α a 3-cocycle on G with values in $U(1)$,* *and ϱ a projective 2-representation of G with 3-cocycle α. The character of ϱ is then a projective representation of $\Lambda\overline{G}$ with 2-cocycle $\tau(\alpha)$.*

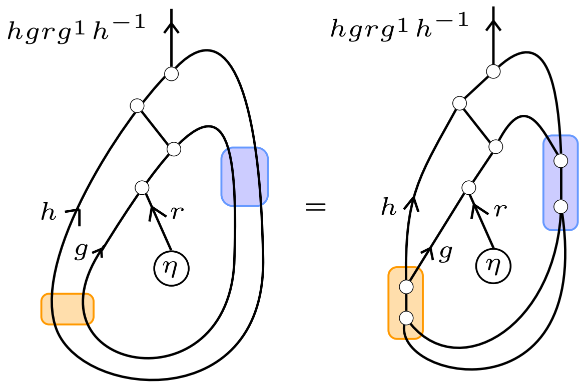
Corollary 2.2.6. *In the situation of the theorem, the character of ϱ is a representation of the twisted Drinfeld double*

$$D^{\alpha}(G) = {}^{\tau(\alpha)}\mathbb{C}(\Lambda\overline{G})$$

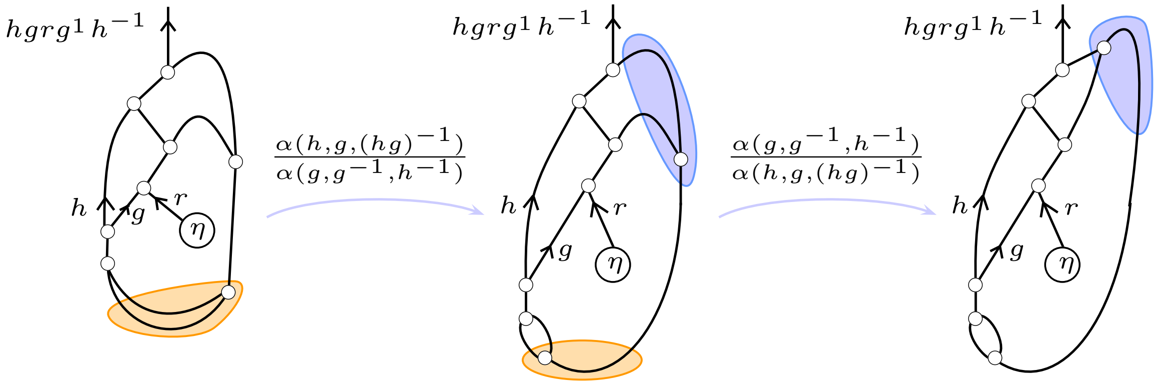
Proof of theorem. To verify that the character of ϱ is a projective representation of $\Lambda\overline{G}$ with 2-cocycle $\tau(\alpha)$, we require that if $r, g, h \in G$, then

$$\beta_{grg^{-1},h}\beta_{r,g} = \tau(\alpha)([h|g]r)\beta_{r,hg}$$

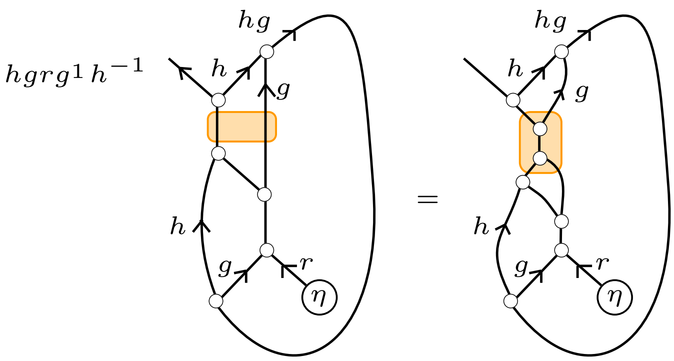
Let $\eta \in X(r)$, then by applying 2.1.4 (d) twice, we find



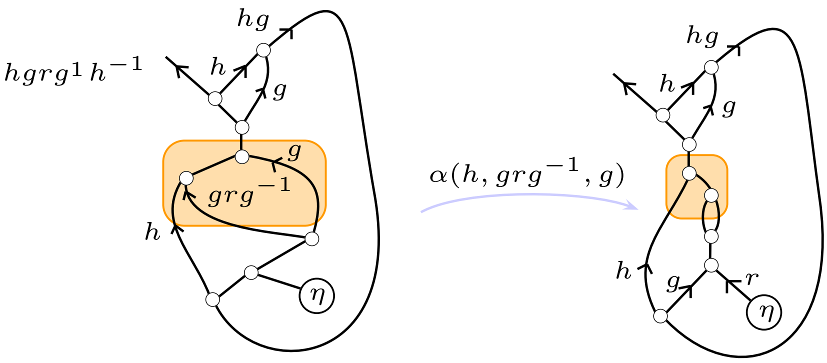
Applying 2.1.8 twice, we have



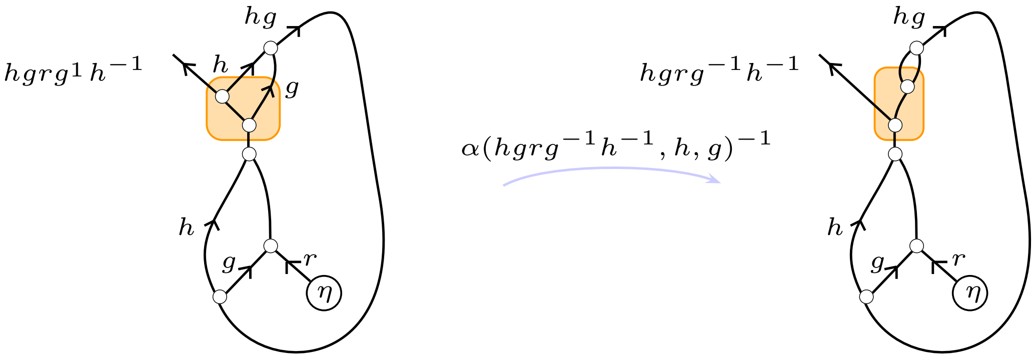
These two factors cancel, so the first and last diagram in this figure are equal. We redraw this diagram by removing the loop (as per 2.1.4 (d)), then apply 2.1.4 (c) to get



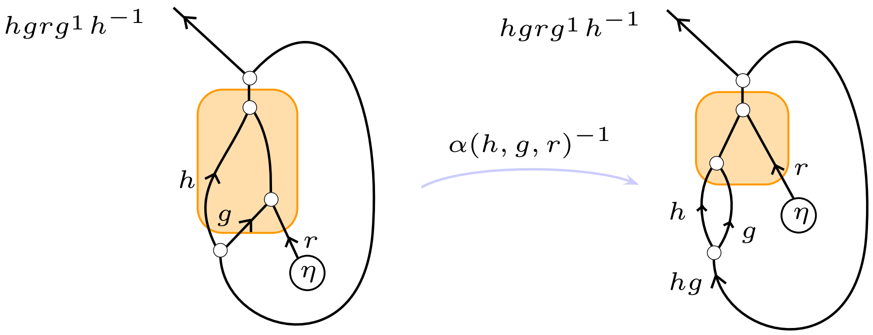
Next, we apply 2.0.38 (i) to obtain



By removing the loop and applying 2.1.3, we get



Finally, we remove this loop then apply 2.0.38 (i) to compute



After removing the loop we recognise this final diagram as being $\beta_{r,hg}(\eta)$. We have therefore shown that

$$\frac{\alpha(h, grg^{-1}, g)}{\alpha(hgrg^{-1}h^{-1}, h, g)\alpha(h, g, r)}\beta_{grg^{-1},h}\beta_{r,g}(\eta) = \beta_{r,hg}(\eta)$$

that is,

$$\beta_{grg^{-1},h}\beta_{r,g}(\eta) = \tau(\alpha)([h|g]r)\beta_{r,hg}(\eta)$$

as required. □

Remark 2.2.7. *The character of a projective 2-representation with 3-cocycle α is a map*

$$X : \Lambda\overline{G} \longrightarrow \mathrm{Vec}_{\mathbb{C}}$$

This map is (in general) not a functor[1]*. Consider the restriction of X to $\overline{G}$, which is a full subgroupoid of the inertia groupoid $\Lambda\overline{G}$. The data of this restriction is a $\mathbb{C}$-vector space $X(1)$ and automorphisms $\{\beta_g = \beta_{1,g} : X(1) \longrightarrow X(1)\}_{g\in G}$ such that*

$$\beta_g\beta_h = \tau(\alpha)([g|h]1)\beta_{gh} = \beta_{gh}$$

for all $g, h \in G$, where the last equality follows directly from Definition 1.1.18. In other words, this restriction determines an ordinary representation of G.

[1]The proof that X is a functor in the case $\alpha \equiv 1$ is the content of [GK08, Proposition 4.10]

Chapter 3

Module categories

We proceed by considering *module categories* over the monoidal category Vec_G^α described in [Ost03a]. This is a natural approach; the category Vec_G^α can be thought of as a categorification of the twisted group algebra ${}^\theta\mathbb{C}\overline{G}$, and we recall that modules over ${}^\theta\mathbb{C}\overline{G}$ are equivalent to projective representations of G with 2-cocycle θ.

Indeed, we will show that the data of a projective 2-representation of G on a k-linear category $\mathcal{C}$ with 3-cocycle α is equivalent to the data of a Vec_G^α-module category structure on $\mathcal{C}$ (this is in fact a 2-categorical equivalence). This allows us to gain further insight into projective 2-representations. In particular, this equivalence will allow us to describe a notion of induction of projective 2-representations[1].

We recall some basic terms.

Definition 3.0.8 ([BK01, Definition 1.13 and §1.1])**.** Let $\mathcal{C}$ be an abelian category over k. We say an object U in $\mathcal{C}$ is *simple* if any injection $V \hookrightarrow U$ is either 0 or an isomorphism. We say $\mathcal{C}$ is *semisimple* if any object $V \in \mathcal{C}$ is isomorphic to a direct sum of simple objects.

Definition 3.0.9 (see [ENO05, §2.1])**.** Let G be a finite group. Let Vec_G be the monoidal category with objects finite dimensional G-graded k-vector spaces

$$W = \bigoplus_{g \in G} W_g$$

[1]The case $\alpha = 1$ is described in [GK08, §7].

and arrows k-linear maps $f : W \longrightarrow W'$ preserving the G-grading. The simple objects in Vec_G are the spaces k_g, where

$$(k_g)_h = \begin{cases} k & \text{if } g = h \\ 0 & \text{otherwise} \end{cases}$$

for each $g \in G$. The monoidal structure on Vec_G is given by

$$(V \otimes W)_g = \bigoplus_{h \in G} V_{gh^{-1}} \otimes W_h$$

with trivial associator.

Let $\alpha : G \times G \times G \longrightarrow k^\times$ be a normalised 3-cocycle. Let Vec_G^α be the monoidal category with the same objects and tensor product as Vec_G, but with associator

$$a_{g,h,i} : (k_g \otimes k_h) \otimes k_i \longrightarrow k_g \otimes (k_h \otimes k_i)$$

defined by

$$a_{g,h,i}(u \otimes v \otimes w) = \alpha(g, h, i) \cdot u \otimes v \otimes w$$

The pentagon diagram is satisfied precisely because α is a 3-cocycle, as was the case with Example 1.1.29.

The categories Vec_G and Vec_G^α are examples of abelian semisimple categories over k with finitely many simple objects. In this section, we only consider abelian semisimple categories over k with finite dimensional hom spaces[2], and assume that all functors are additive. We provide the general notion of a module category here, closely following [Ost03a].

Definition 3.0.10 ([Ost03a, Definition 6])**.** A *module category* over a monoidal category $\mathcal{C}$ is a k-linear abelian category $\mathcal{M}$ together with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ and natural associativity and unit isomorphisms $m_{X,Y,M} : (X \otimes Y) \otimes M \longrightarrow X \otimes (Y \otimes M)$, $\ell_M : \mathbf{1} \otimes M \longrightarrow M$ for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$ such that the diagrams

[2] That is, for any pair of objects x, y, the k-vector space $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a finite dimensional

$$
\begin{array}{ccccc}
 & & ((X \otimes Y) \otimes Z) \otimes M & & \\
 & \swarrow^{a_{X,Y,Z} \otimes id} & & \searrow^{m_{X \otimes Y, Z, M}} & \\
(X \otimes (Y \otimes Z)) \otimes M & & & & (X \otimes Y) \otimes (Z \otimes M) \\
\downarrow{\scriptstyle m_{X, Y \otimes Z, M}} & & & & \downarrow{\scriptstyle m_{X,Y,Z \otimes M}} \\
X \otimes ((Y \otimes Z) \otimes M) & & \xrightarrow{\mathrm{id} \otimes m_{Y,Z,M}} & & X \otimes (Y \otimes (Z \otimes M))
\end{array}
$$

and

$$
\begin{array}{ccc}
(X \otimes \mathbf{1}) \otimes M & \xrightarrow{m_{X,1,M}} & X \otimes (\mathbf{1} \otimes M) \\
 & {\scriptstyle r_X \otimes \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \mathrm{id} \otimes \ell_M} & \\
 & X \otimes M &
\end{array}
$$

commute. We say that $\mathcal{M}$ is a (left) $\mathcal{C}$-module category.

Example 3.0.11 ([Ost03a, §2.3])**.** *(i) Let $\mathcal{C}$ be a monoidal category (see Definition 1.1.23), it is clear that the monoidal structure on $\mathcal{C}$ induces a $\mathcal{C}$-module category structure on $\mathcal{C}$.*

(ii) Let $F : \mathcal{C} \longrightarrow \mathrm{Vec}_k$ be a monoidal functor (see Definition 1.1.30), then we have the natural isomorphism $\phi_{X,Y} : F(X) \otimes_k F(Y) \longrightarrow F(X \otimes Y)$ and an isomorphism $\phi : \boldsymbol{1}_k \longrightarrow F(\boldsymbol{1}_{\mathcal{C}})$. We claim that this determines a $\mathcal{C}$-module category structure on Vec_k. For $X \in \mathrm{ob}(\mathcal{C})$ and $V \in \mathrm{ob}(\mathrm{Vec}_k)$, set $X \otimes V := F(X) \otimes_k V$. Let the associativity isomorphism $m_{X,Y,V}$ be given by the composition

$$
\begin{aligned}
(X \otimes Y) \otimes V = F(X \otimes Y) \otimes_k V &\xrightarrow{\phi^{-1}_{X,Y} \otimes_k \mathrm{id}} (F(X) \otimes_k F(Y)) \otimes_k V \\
&\xrightarrow{\cong} F(X) \otimes_k (F(Y) \otimes_k V) = X \otimes (Y \otimes V)
\end{aligned}
$$

and the unit isomorphism ℓ_V be given by the composition

$$
\boldsymbol{1}_{\mathcal{C}} \otimes V = F(\boldsymbol{1}_{\mathcal{C}}) \otimes_k V \xrightarrow{\phi^{-1} \otimes_k \mathrm{id}} \boldsymbol{1}_k \otimes_k V \xrightarrow{\cong} V
$$

Example 3.0.12 ([Eti09, Example 2.5.10]). *A module category* $\mathcal{M}$ *over* Vec_G *is a collection of exact functors* $F(g) : \mathcal{M} \longrightarrow \mathcal{M}$ *(where* $F(g)(M) := k_g \otimes M$ *for* $M \in \mathrm{ob}(\mathcal{M})$*), together with a collection of natural isomorphisms* $\phi_{g,h} : F(g)F(h) \longrightarrow F(gh)$ *such that*

$$\phi_{g,hi}(F(g)\phi_{h,i}) = \phi_{gh,i}(\phi_{g,h}F(i)) : F(g)F(h)F(i) \longrightarrow F(ghi)$$

where $g, h, i \in G$.

At this stage, we note that a Vec_G-module category structure on $\mathcal{M}$ is precisely an action of G on $\mathcal{V}$ (as in [GK08, §4.2]). We therefore expect that a Vec_G^α-module category structure on $\mathcal{M}$ is equivalent to a 'twisted' action of G on $\mathcal{V}$.

Example 3.0.13. *A module category* $\mathcal{M}$ *over* Vec_G^α *is a collection of exact functors* $F(g) : \mathcal{M} \longrightarrow \mathcal{M}$*, together with a collection of natural isomorphisms* $\phi_{g,h} : F(g)F(h) \overset{\cong}{\Longrightarrow} F(gh)$ *such that*

$$\phi_{g,hi}(F(g)\phi_{h,i}) = \alpha(g,h,i)\phi_{gh,i}(\phi_{g,h}F(i)) : F(g)F(h)F(i) \longrightarrow F(ghi)$$

where $g, h, k \in G$.

Comparing this with Definition 2.0.38, we see that a Vec_G^α-module category structure on $\mathcal{M}$ gives us a projective 2-representation of G with 3-cocycle α on $\mathcal{M}$.

3.1 Module functors and module natural transformations

Definition 3.1.1 ([Ost03a, Definition 7]). Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two module categories over a monoidal category $\mathcal{C}$. A *module functor* from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ together with a natural isomorphism $c_{X,M} : F(X \otimes M) \longrightarrow X \otimes F(M)$, where $X \in \mathrm{ob}(\mathcal{C})$ and $M \in \mathrm{ob}(\mathcal{M}_1)$ such that the diagrams

$$
\begin{array}{ccccc}
 & & F((X \otimes Y) \otimes M) & & \\
 & \swarrow^{F(m_{X,Y,M})} & & \searrow^{c_{X \otimes Y, M}} & \\
F(X \otimes (Y \otimes M)) & & & & (X \otimes Y) \otimes F(M) \\
\downarrow{\scriptstyle c_{X, Y \otimes M}} & & & & \downarrow{\scriptstyle m_{X,Y,F(M)}} \\
X \otimes F(Y \otimes M) & & \xrightarrow{\mathrm{id} \otimes c_{Y,M}} & & X \otimes (Y \otimes F(M))
\end{array}
$$

and

$$
\begin{array}{ccc}
F(\mathbf{1}_{\mathcal{C}} \otimes M) & \xrightarrow{F(\ell_M)} & F(M) \\
 & {\scriptstyle c_{\mathbf{1}_{\mathcal{C}},M}} \searrow \quad \nearrow {\scriptstyle \ell_{F(M)}} & \\
 & \mathbf{1}_{\mathcal{C}} \otimes F(M) &
\end{array}
$$

commute. We say that F is a $\mathcal{C}$-module functor. If F is furthermore an equivalence of categories (see Definition 1.1.7), then we say that F is an equivalence of $\mathcal{C}$-module categories.

We will make use of the following definitions provided in [Ost03a, Definition 7].

Definition 3.1.2. We say that two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are *equivalent* if there is a module functor from $\mathcal{M}_1$ to $\mathcal{M}_2$ which is an equivalence of categories (recall Definition 1.1.7).

Definition 3.1.3 (also see [Eti09, Proposition 2.4.1])**.** Given two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$, their *direct sum* is the category $\mathcal{M}_1 \oplus \mathcal{M}_2$ with the obvious $\mathcal{C}$-module category structure.

Definition 3.1.4. We say that a module category $\mathcal{M}$ over $\mathcal{C}$ is *indecomposable* if it is not equivalent to a non-trivial direct sum of module categories.

We wish to construct a 2-category whose objects are the module categories over a monoidal category $\mathcal{C}$, and whose 1-morphisms are the module functors between them. To do this, we will need to describe a natural transformations between module functors.natural transformation between module functors.

Definition 3.1.5 ([Hov99, Definition 4.1.7])**.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be module categories over $\mathcal{C}$, and let $F,\ G : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be $\mathcal{C}$-module functors with corresponding natural isomorphisms $c_{X,M}$ and $d_{X,M}$, where $X \in \text{ob}(\mathcal{C})$ and $M \in \text{ob}(\mathcal{M}_1)$. A *module natural transformation* from F to G is a natural transformation $\theta : F \Rightarrow G$ such that the diagram

$$\begin{array}{ccc} F(X \otimes M) & \xrightarrow{c_{X,M}} & X \otimes F(M) \\ {\scriptstyle \theta_{X \otimes M}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \otimes \theta_M} \\ G(X \otimes M) & \xrightarrow{d_{X,M}} & X \otimes G(M) \end{array}$$

commutes. We say that θ is a $\mathcal{C}$-module natural transformation.

Definition 3.1.6. Let $2\text{Mod}(\mathcal{C})$ be the 2-category whose objects are module categories $\mathcal{M}$ over $\mathcal{C}$, with 1-morphisms being the $\mathcal{C}$-module functors, and 2-morphisms being the $\mathcal{C}$-module natural transformations between them.

For convenience, we state the following result. We will state it in more detail in Theorem 4.3.3.

Theorem. *Let $\mathcal{C}$ be the 2-category of k-linear categories. Then there is a weak equivalence of 2-categories*

$$2\text{Rep}_k^\alpha(G, \mathcal{C}) \longrightarrow 2\text{Mod}(\text{Vec}_G^\alpha)$$

In other words, a projective 2-representation of G with 3-cocycle α on a k-linear category $\mathcal{V}$ is equivalent to a Vec_G^α-module category structure on $\mathcal{V}$.

3.2 Algebra objects and modules in monoidal categories

In [Ost03a], Ostrik provides a classification of indecomposable module categories over Vec_G^α, which we wish to apply (using the equivalence of Theorem 4.3.3) to classify 'irreducible' projective 2-representations. Before we present this classification, we recall the notion of an algebra object in a monoidal category, and a module over such an algebra object.

Definition 3.2.1 ([Ost03a, Definition 8(i)], [Lan98, VII §3])**.** Let $\mathcal{C}$ be a monoidal category. An *algebra object* (or *monoid object*) is an object A of $\mathcal{C}$ with a multiplication arrow $m : A \otimes A \longrightarrow A$ and a unit arrow $e : \mathbf{1}_{\mathcal{C}} \longrightarrow A$ such that the diagrams

$$
\begin{array}{ccccc}
 & & (A \otimes A) \otimes A & & \\
 & \swarrow^{a_{A,A,A}} & & \searrow^{m \otimes id} & \\
A \otimes (A \otimes A) & & & & A \otimes A \\
\downarrow{\scriptstyle \mathrm{id} \otimes m} & & & & \downarrow{\scriptstyle m} \\
A \otimes A & & \xrightarrow{\quad m \quad} & & A
\end{array}
$$

and

$$
\begin{array}{ccc}
\mathbf{1}_{\mathcal{C}} \otimes A & \xrightarrow{\lambda_A} & A \\
 & {\scriptstyle e \otimes \mathrm{id}} \searrow \quad \nearrow {\scriptstyle m} & \\
 & A \otimes A &
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \otimes \mathbf{1}_{\mathcal{C}} & \xrightarrow{\varrho_A} & A \\
 & {\scriptstyle \mathrm{id} \otimes e} \searrow \quad \nearrow {\scriptstyle m} & \\
 & A \otimes A &
\end{array}
$$

commute.

Definition 3.2.2 ([Ost03a, Definition 8(ii)], [Lan98, VII §4])**.** A *right module* over an algebra object A in a monoidal category $\mathcal{C}$ is an object M of $\mathcal{C}$ with an action arrow $s : M \otimes A \longrightarrow M$ such that the diagrams

$$
\begin{array}{ccccc}
 & & (M \otimes A) \otimes A & & \\
 & \swarrow^{a_{M,A,A}} & & \searrow^{s \otimes id} & \\
M \otimes (A \otimes A) & & & & M \otimes A \\
\downarrow{\scriptstyle \mathrm{id} \otimes m} & & & & \downarrow{\scriptstyle s} \\
M \otimes A & & \xrightarrow{\quad s \quad} & & M
\end{array}
$$

and

$$\begin{array}{ccc} M \otimes \mathbf{1}_{\mathcal{C}} & \xrightarrow{\varrho_M} & M \\ & \searrow^{\mathrm{id} \otimes e} \quad \nearrow_{s} & \\ & M \otimes A & \end{array}$$

commute.

Definition 3.2.3 ([Ost03a, Definition 8(iii)])**.** A *morphism* between two right modules (M_1, s_1) and (M_2, s_2) over an algebra object A in a monoidal category $\mathcal{C}$ is an arrow $f \in \mathrm{Hom}_{\mathcal{C}}(M_1, M_2)$ such that the diagram

$$\begin{array}{ccc} M_1 \otimes A & \xrightarrow{f \otimes \mathrm{id}} & M_2 \otimes A \\ \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

commutes.

Remark 3.2.4. *Let A be an algebra object in a monoidal category $\mathcal{C}$, and let* $\mathrm{Mod}_{\mathcal{C}}(A)$ *be the category whose objects are the right modules over A in $\mathcal{C}$, and whose arrows are the A-module morphisms between them.*

Then $\mathrm{Mod}_{\mathcal{C}}(A)$ *is an abelian category (by [Ost03a, Lemma 3]), and moreover, is a $\mathcal{C}$-module category: given a right A-module (M, s) in $\mathcal{C}$ and $X \in \mathrm{ob}(\mathcal{C})$, then $X \otimes A$ is a right A-module with action arrow given by* $\mathrm{id} \otimes s$.

3.3 Classification of indecomposable module categories

Let $\mathcal{C}$ be a semisimple rigid[3] monoidal category with finitely many simple objects and simple unit object; Vec_G^{α} is an example of such a category. In [Ost03a, Theorem 1], Ostrik proves that every semisimple indecomposable module category $\mathcal{M}$ over

[3] A monoidal category $\mathcal{C}$ is rigid if every object has right and left duals (see [BK01, §2.1] for more details).

$\mathcal{C}$ is equivalent to $\mathrm{Mod}_{\mathcal{C}}(A)$, for some algebra object $A \in \mathrm{ob}(\mathcal{C})$. Ostrik applies this classification to the case $\mathcal{C} = \mathrm{Vec}_G^\alpha$ (i.e. [Ost03b, Example 2.1]), which we now interpret in terms of projective 2-representations.

Lemma 3.3.1 ([Ost03b, Example 2.1]). *Let $H \subseteq G$ be a subgroup of a finite group G, and let $\theta : H \times H \longrightarrow \mathbb{C}^\times$ be a normalised 2-cochain such that $d\theta = \alpha|_H$. Then the twisted group algebra object*[4] *${}^\theta\mathbb{C}\overline{H}$ is an algebra object in Vec_G^α, where we take*

$$({}^\theta\mathbb{C}\overline{H})_g = \begin{cases} \mathbb{C} & g \in H \\ 0 & g \notin H \end{cases}$$

with multiplication morphism defined by

$$v_{h_1} \otimes v_{h_2} := \theta(h_1, h_2) v_{h_1 h_2}$$

for $h_i \in H$ with $v_{h_i} \in ({}^\theta\mathbb{C}\overline{H})_{h_i}$.

Proof. The pentagonal axiom is satisfied by our requirement that $d\theta = \alpha|_H$. The triangular axioms are satisfied because $\theta : G \times G \longrightarrow \mathbb{C}^\times$ is a normalised 2-cochain. □

Remark 3.3.2. *It is important to note that the twisted group algebra object is (in general) not an algebra. For example, if $\alpha|_H$ is non-trivial with $d\theta = \alpha|_H$, then θ is not a 2-cocycle, so ${}^\theta\mathbb{C}\overline{H}$ is not an algebra, as it fails to be associative.*

We have the following result of Ostrik.

Lemma 3.3.3 ([Ost03b, Example 2.1]). *The indecomposable module categories over Vec_G^α are classified by the conjugacy classes of pairs (H, θ) where $H \subseteq G$ is a subgroup and θ a 2-cochain on H with values in $\mathbb{C}^\times$ such that $\alpha|_H = d\theta$.*

In other words, every semisimple indecomposable module category over Vec_G^α is equivalent as a Vec_G^α-module category to $\mathrm{Mod}_{\mathrm{Vec}_G^\alpha}({}^\theta\mathbb{C}\overline{H})$ where ${}^\theta\mathbb{C}\overline{H}$ is the twisted group algebra object constructed in Lemma 3.3.1. We now consider some technical lemmas.

[4] Ostrik denotes this by $A(H, \theta)$

Lemma 3.3.4. *Let M be a right ${}^{\theta}\mathbb{C}\overline{H}$-module in Vec_G^{α}. Then $M_g \cong M_{gh}$ as $\mathbb{C}$-vector spaces for all $h \in H$.*

Proof. It suffices to consider non-zero M_g. For any $h \in H$, we have the composition

$$j_{g,h} : M_g \cong M_g \otimes \mathbb{C}_h \xrightarrow{s} M_{gh}$$

The triangle diagram implies that $j_{g,1} : M_g \longrightarrow M_g$ is an isomorphism. Therefore by inspecting the pentagon diagram, we get that the composition

$$M_g \xrightarrow{j_{g,h}} M_{gh} \xrightarrow{gh,h^{-1}} M_g$$

is an isomorphism, so in particular, $j_{g,h}$ is injective. However the same argument tells us that $j_{gh,h^{-1}}$ is injective. We conclude that $j_{g,h}$ is an isomorphism. □

Corollary 3.3.5 (compare [Ost03b, Proposition 3.2], [ENO10, §2.7])**.** *The number of simple objects in $\mathrm{Mod}_{\mathrm{Vec}_G^{\alpha}}({}^{\theta}\mathbb{C}\overline{H})$ is equal to the number of left cosets of H in G.*

Proof. By the previous lemma, we have that any simple object $M = \oplus_{g \in G} M_g$ in $\mathrm{Mod}_{\mathrm{Vec}_G^{\alpha}}({}^{\theta}\mathbb{C}\overline{H})$ is isomorphic to $\mathbb{C}_g \otimes {}^{\theta}\mathbb{C}\overline{H}$, from which the result follows. □

Consider the special case where $G = H$ and $d\theta = \alpha$ for some 2-cochain on G. Then $\mathrm{Mod}_{\mathrm{Vec}_G^{\alpha}}({}^{\theta}\mathbb{C}\overline{G})$ is an abelian category over $\mathbb{C}$ with one simple object. We have the following proposition.

Proposition 3.3.6. *There is an equivalence of Vec_G^{α}-module categories*

$$\mathrm{Mod}_{\mathrm{Vec}_G^{\alpha}}({}^{\theta}\mathbb{C}\overline{G}) \longrightarrow \mathrm{Vec}_{\mathbb{C}}$$

where the Vec_G^{α}-module category structure on $\mathrm{Vec}_{\mathbb{C}}$ is that defined in Example 2.1.2.

Proof. For more details on how the Vec_G^{α}-module category structure on $\mathrm{Vec}_{\mathbb{C}}$ is defined, see Theorem 4.3.3. Let $F : \mathrm{Mod}_{\mathrm{Vec}_G^{\alpha}}({}^{\theta}\mathbb{C}\overline{G}) \longrightarrow \mathrm{Vec}_{\mathbb{C}}$ be the functor

$$M = \bigoplus_{g \in G} M_g \longmapsto M_1$$

together with isomorphisms defined by the composition

$$c_{g,M} : F(\mathbb{C}_g \otimes M) = \mathbb{C}_g \otimes M_{g^{-1}} \cong M_{g^{-1}} \xrightarrow{j_{g^{-1},g}} M_1 = F(M)$$

We claim that F is an equivalence of Vec_G^α-module categories. From the module structure on M, we get the commuting diagram

$$\begin{array}{ccc}
 & M_{(gh)^{-1}} \cong F((\mathbb{C}_g \otimes \mathbb{C}_h) \otimes M) & \\
 \swarrow^{\alpha((gh)^{-1},g,h)^{-1}} & & \searrow^{j_{(gh)^{-1},g}} \\
M_{(gh)^{-1}} \cong F(\mathbb{C}_g \otimes (\mathbb{C}_h \otimes M) & & M_{h^{-1}} \cong F(\mathbb{C}_h \otimes M) \\
\downarrow{\scriptstyle \theta(g,h)} & & \downarrow{\scriptstyle j_{h^{-1},h}} \\
M_{(gh)^{-1}} \cong F(\mathbb{C}_{gh} \otimes M) & \xrightarrow{\quad j_{(gh)^{-1},gh} \quad} & M_1 \cong F(M)
\end{array}$$

However we recognise this as the pentagon diagram of Definition 3.1.1, so F is indeed a Vec_G^α-module category functor. That F is an equivalence follows immediately from the previous lemma, so we are done. □

In other words, when $\alpha = d\theta$ for a 2-cochain θ on G, the pair (G, θ) occurring in Ostrik's classification of indecomposable module categories over Vec_G^α is equivalent to the projective 2-representation defined in Example 2.1.2.

3.4 Induction of projective 2-representations

Recall induction for projective representations [Kar94]: If H is a subgroup of a finite group G and W is a left ${}^{\theta|_H}\mathbb{C}\overline{H}$ module for some 2-cocycle $\theta : G \times G \longrightarrow k^\times$, then

$$V = {}^{\theta}\mathbb{C}\overline{G} \otimes_{{}^{\theta|_H}\mathbb{C}\overline{H}} W$$

is a left ${}^{\theta}\mathbb{C}\overline{G}$ module. In other words, a projective representation of a subgroup $H \subseteq G$ with 2-cocycle $\theta|_H$ induces a projective representation of G with 2-cocycle θ.

Given the equivalence between module categories over Vec_G^α and projective 2-representations, this suggestions a natural definition of the induced projective 2-

representation.

Definition 3.4.1 ([ENO10, Proposition 3.5]). Let $\mathcal{M}$ be a right $\mathcal{C}$-module category[5], and $\mathcal{N}$ be a left $\mathcal{C}$-module category. The *tensor product* of $\mathcal{M}$ and $\mathcal{N}$ is defined to be

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} = \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}^{op}, \mathcal{N})$$

the abelian category of $\mathcal{C}$-module functors from the opposite category $\mathcal{M}^{op}$ to $\mathcal{N}$.

Remark 3.4.2 ([ENO10, §2.9]). *If $\mathcal{M}$ is a right $\mathcal{C}$-module category, then $\mathcal{M}^{op}$ is a left $\mathcal{C}$-module category with $\mathcal{C}$-action $\odot$ given by*

$$X \odot M := M \otimes {}^*X$$

for $X \in \mathcal{C}$ and $M \in \mathcal{M}$. The object ${}^*X$ is known as the left dual *of X, and will exist for all examples that we are interested in.*

Definition 3.4.3. Let G be a finite group with 3-cocycle $\alpha : G \times G \times G \longrightarrow \mathbb{C}^*$, and let $H \subseteq G$ be a subgroup of G. If $\mathcal{V}$ is a left Vec_H^α-module category, then let

$$\mathrm{ind}|_H^G(\mathcal{V}) := \mathrm{Vec}_G^\alpha \boxtimes_{\mathrm{Vec}_H^\alpha} \mathcal{V} = \mathrm{Fun}_{\mathrm{Vec}_H^\alpha}((\mathrm{Vec}_G^\alpha)^{op}, \mathcal{V})$$

be the induced left Vec_G^α-module category.

Remark 3.4.4 (see [ENO10, Remark 3.6]). *We note that the restriction of the monoidal structure on Vec_G^α makes Vec_G^α into a $(\mathrm{Vec}_G^\alpha, \mathrm{Vec}_H^\alpha)$-bimodule category, hence the category $\mathrm{ind}|_H^G(\mathcal{V})$ has a left Vec_G^α-module category structure.*

In detail, if we have a Vec_H^α-module functor

$$F : (\mathrm{Vec}_G^\alpha)^{op} \longrightarrow \mathcal{V}$$

together with natural isomorphisms

$$c_{h,M} : F(\mathbb{C}_h \odot M) = F(M \otimes \mathbb{C}_{h^{-1}}) \longrightarrow \mathbb{C}_h \otimes F(M)$$

[5]It is a simple exercise to write the definition of a right $\mathcal{C}$-module category based on the definition of a left $\mathcal{C}$-module category; we do not do it here.

Then for $g \in G$, $(\mathbb{C}_g \otimes F)$ *is the* Vec_H^α*-module functor given by*

$$(\mathbb{C}_g \otimes F)(M) = F(M \otimes C_g)$$

together with natural isomorphisms

$$d_{h,M} = c_{h,\mathbb{C}_{g^{-1}} \odot M} = c_{h,M \otimes \mathbb{C}_g} : (\mathbb{C}_g \otimes F)(\mathbb{C}_h \odot M) \longrightarrow \mathbb{C}_h \otimes (\mathbb{C}_g \otimes F)(M)$$

In the case of a trivial 3-cocycle $\alpha = 1$, the notion of induced projective 2-representation corresponds with the notion of induced 2-representation discussed in [GK08, §7].

Proposition 3.4.5. *Let* ϱ *be a 2-representation*[6] *of a subgroup* $H \subset G$ *on a* k*-linear category* $\mathcal{V}$*. Then the induced 2-representation defined in [GK08, Definition 7.1] is equivalent to* $\mathrm{ind}|_H^G(\mathcal{V})$ *as a* Vec_G*-module category.*

Outline of proof. Let $\mathcal{I}$ be the induced 2-representation of ϱ as defined in [GK08, Definition 7.1], and let $F \in \mathrm{ob}(\mathrm{ind}|_H^G(\mathcal{V}))$ with natural isomorphisms $c_{h,M}$ as described above. Let $f : G \longrightarrow \mathrm{ob}(\mathcal{V})$ be given by $f(g) = F(\mathbb{C}_g)$. For each $g \in G$ and $h \in H$, let $u_{g,h}$ be the isomorphism

$$u_{g,h} := c_{h^{-1},g} : f(gh) \longrightarrow \varrho(h^{-1})f(g)$$

By the pentagon diagram, we have that the diagram

$$\begin{array}{ccc} f(gh_1h_2) & \xrightarrow{u_{g,h_1h_2}} & \varrho((h_1h_2)^{-1})f(g) \\ {\scriptstyle u_{gh_1,h_2}}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi^{-1}_{h_2^{-1},h_1^{-1}}} \\ \varrho(h_2^{-1})f(gh_1) & \xrightarrow{\varrho(h_2^{-1})u_{g,h_1}} & \varrho(h_2^{-1})\varrho(h_1^{-1})f(g) \end{array}$$

commutes, and the triangle diagram gives that $u_{g,1} = \psi^{-1}_{1,f(g)}$, hence $(f, u_{g,h})$ is an object of $\mathcal{I}$. Similarly, a Vec_G-module natural transformation $\nu : F \Rightarrow F'$ is precisely an arrow $(f, u_{g,h}) \longrightarrow (f', u'_{g,h})$ in $\mathcal{I}$. This describes an equivalence $\mathrm{ind}|_H^G(\mathcal{V}) \longrightarrow \mathcal{I}$.

[6] We call a projective 2-representation with 3-cocycle $\alpha = 1$ a 2-representation, see [GK08].

Finally, the Vec_G-module category structures on $\mathrm{ind}|_H^G(\mathcal{V})$ and $\mathcal{I}$ are the same, so we are done.

We note that there is an equivalence of right Vec_H^α-modules

$$\mathrm{Vec}_G^\alpha \cong \bigoplus_{r\in R} r\mathrm{Vec}_H^\alpha$$

This suggests the following proposition, which we do not include the details of here.

Proposition 3.4.6. *Let $H \subset G$ be a subgroup, and ϱ a projective 2-representation of H with 3-cocycle $\alpha|_H$. Then*

$$\mathrm{ind}|_H^G(\varrho) \simeq \bigoplus_{r\in R} r\varrho$$

as left Vec_G^α-module categories, where R is a left transversal of H in G.

This uses the above decomposition of Vec_G^α and distributivity of the tensor product of module categories. This suggests the following conjecture

Conjecture. *Let $H \subset G$ be a subgroup, and θ a 2-cochain on H such that $d\theta = \alpha|_H$. Then*

$$\mathrm{ind}_H^G(\mathrm{Mod}_{\mathrm{Vec}_H^\alpha}({}^\theta\mathbb{C}\overline{H})) \simeq \mathrm{Mod}_{\mathrm{Vec}_G^\alpha}({}^\theta\mathbb{C}\overline{H})$$

as left Vec_G^α-module categories.

This can be seen as a combination of the previous proposition with Corollary 3.3.5. Assuming this conjecture, we have the following corollary.

Corollary 3.4.7 (compare [GK08, Proposition 7.3])**.** *Let ϱ be a projective 2-representation of a group G with 3-cocycle α on a semisimple k-linear abelian category $\mathcal{V}$ with finitely many simple objects. Then*

$$\mathcal{V} \cong \bigoplus_{i=1}^{m} \mathrm{ind}_{H_i}^G \varrho_{\theta_i}$$

where the H_i are subgroups of G, θ_i is a 2-cochain on H_i such that $d\theta_i = \alpha|_{H_i}$, and ϱ_{θ_i} is the projective 2-representation corresponding to the pair (H_i, θ_i) described in Example 2.1.2.

Outline of proof. By Lemma 3.3.3, we have the decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{m} \mathrm{Mod}_{\mathrm{Vec}_G^{\alpha}}({}^{\theta_i}\mathbb{C}\overline{H_i})$$

into indecomposable components, where H_i and θ_i are as above. By our conjecture, we get

$$\mathcal{V} \simeq \bigoplus_{i=1}^{m} \mathrm{ind}_{H_i}^{G} \mathrm{Mod}_{\mathrm{Vec}_{H_i}^{\alpha}}({}^{\theta_i}\mathbb{C}\overline{H_i})$$

Finally, by Lemma 3.3.6, we have

$$\mathcal{V} \simeq \bigoplus_{i=1}^{m} \mathrm{ind}_{H_i}^{G} \varrho_{\theta_i}$$

where ϱ_{θ_i} is the projective 2-representation corresponding to the pair (H_i, θ_i) described in Example 2.1.2.

Chapter 4

Comparison of formulations of 2-group representations

As a final note, we provide a brief overview of the various formulations of a 2-group representation listed in the introduction.

4.1 Gerbal representations

In [FZ11], Frenkel and Zhu showed that a gerbal representation of a group G on an abelian category $\mathcal{C}$ determines an action of G on the group $\mathcal{Z}(\mathcal{C})^\times = \mathbb{T}\mathrm{r}(\mathrm{id}_\mathcal{C})^\times$, and that such a representation determines a cohomology class $[\alpha] \in H^3(G, \mathcal{Z}(\mathcal{C})^\times)$. We present their result here.

Theorem 4.1.1 ([FZ11, Theorem 2.10]). *Let F be a gerbal representation of G on $\mathcal{C}$, and choose isomorphisms $\psi_{g,h} : F(g)F(h) \xrightarrow{\cong} F(gh)$ for all $g, h \in G$. Then given $g, h, k \in G$, there is a unique element $\alpha(g, h, k) \in \mathcal{Z}(\mathcal{C})^\times$ sending the isomorphism*

$$F(g)F(h)F(k) \xrightarrow{\psi_{g,k}F(k)} F(gh)F(k) \xrightarrow{\psi_{gh,k}} F(ghk)$$

to

$$F(g)F(h)F(k) \xrightarrow{F(g)\psi_{h,k}} F(g)F(hk) \xrightarrow{\psi_{g,hk}} F(ghk)$$

This defines a map $\alpha : G \times G \times G \longrightarrow \mathcal{Z}(\mathcal{C})^\times$ such that *(i)* α *is a 3-cocycle on* G *with values in* $\mathcal{Z}(\mathcal{C})^\times$*, that is,* $\alpha \in Z^3(G, \mathcal{Z}(\mathcal{C})^\times)$*.*

(ii) Given a different choice of isomorphisms $\psi'_{g,h} : F(g)F(h) \xrightarrow{\cong} F(gh)$*, the new cocycle* α' *differs from* α *by a coboundary, so there is a well-defined cohomology class* $[\alpha] \in H^3(G, \mathcal{Z}(\mathcal{C})^\times)$ *associated to the gerbal representation* F*.*

From this, we see that a projective 2-representation is a special case of a gerbal representation, where we require that our 3-cocycle α takes values in $k^\times \subset \mathcal{Z}(\mathcal{C})^\times$ and that G acts on $k^\times$ trivially.

4.2 2-group representations

Let $\mathcal{G}$ be a 2-group extension of G by $[\mathrm{pt}/\mathbb{C}^\times]$. Then a representation of $\mathcal{G}$ is a monoidal functor

$$\mathcal{G} \longrightarrow 1\text{-}\mathrm{Aut}(V)$$

where V is an object in a strict $\mathbb{C}$-linear 2-category $\mathcal{C}$ and $1\text{-}\mathrm{Aut}(V)$ is the corresponding strict automorphism 2-group of V. We require that $\mathcal{C}$ be strict so as to make use of string diagrams; we have seen that they can be an incredibly useful tool. In the case $\mathcal{G} = \mathcal{C}_G^\alpha(\mathbb{C}^\times)$ (see Example 1.1.29), this definition reduces to that of a projective 2-representation.

4.3 Module categories over Vec_G^α

We present the equivalence between projective 2-representations of a finite group G with 3-cocycle α on a k-linear category $\mathcal{M}$ and Vec_G^α-module category structures on $\mathcal{M}$.

Lemma 4.3.1. *Let* $\varrho : \mathcal{M} \longrightarrow \mathcal{M}$ *be a projective 2-representations of a finite group* G *with 3-cocycle* α *on a* k*-linear category* $\mathcal{M}$*. Then* $\mathcal{M}$ *has a* Vec_G^α*-module category* *structure given by*

$$
\begin{aligned}
V_g \otimes M &:= \rho(g)(M) \\
m_{g,h,M} &:= \psi_{g,h}^{-1}(M) \\
l_M &:= \psi_1(M)
\end{aligned}
$$

for $g, h \in G$ *and* $M \in \mathcal{M}$*. Furthermore, if* $\sigma : \varrho_1 \longrightarrow \varrho_2$ *is a morphism of such projective 2-representations, then* $\sigma : M_1 \longrightarrow M_2$ *is a* Vec_G^{α}*-module category functor with natural isomorphism given by*

$$c_{g,M} = \sigma(g)^{-1}(M)$$

Finally, if $\theta : \sigma \Rightarrow \phi$ *is a 2-morphism of projective 2-representations, then* θ *is equivalent to the data of a* Vec_G^{α}*-module natural transformation* $\theta : \sigma \Rightarrow \phi$*.*

Lemma 4.3.2. *Let* $\mathcal{M}$ *be a k-linear module category over* Vec_G^{α}*, then there is a projective 2-representation* ϱ *of* G *on* $\mathcal{M}$ *with 3-cocycle* α *given by*

$$
\begin{aligned}
\rho(g) &:= V_g \otimes - \\
\psi_{g,h} &:= m_{g,h,-}^{-1} \\
\psi_1 &:= l_-
\end{aligned}
$$

for $g, h \in G$*. If* $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ *is a module category functor over* Vec_G^{α}*, then* $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ *together with the 2-isomorphism*

$$\sigma(g) := c_{g,-}^{-1} : F(\varrho_1(g)) \longrightarrow \varrho_2(g)F \tag{4.1}$$

is a morphism of projective 2-representations. As noted previously, if $\theta : \sigma \Rightarrow \phi$ *is a module natural transformation over* Vec_G^{α}*, then* $\theta : \sigma \Rightarrow \phi$ *is equivalent to a 2-morphism of projective 2-representations.*

Theorem 4.3.3. *The assignment*

$$2\mathrm{Rep}_k^{\alpha}(G, \mathcal{C}) \longrightarrow 2\mathrm{Vec}_{\mathrm{G}}^{\alpha}\text{-Mod} \tag{4.2}$$

described in Lemma 4.3.1 is a weak equivalence of 2-categories.

Proof. Tedious but straight forward. □

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