ASSIGNMENT 2

DUE MONDAY, SEPTEMBER 24, 2018

(1) 3-Cocycles and Categorical Groups

Consider the category $\mathcal{U}(1)$ with objects $\mathbb{R}$, arrows

$$x \xrightarrow{[a]} x+m, \qquad \text{with} \quad x \in \mathbb{R} \text{ and } m \in \mathbb{Z} \text{ and } [a] \in \mathbb{R}/\mathbb{Z},$$

and composition givven by adding the labels,

$$\begin{tikzcd} x \arrow[r, "{[a]}"] \arrow[rr, bend right, "{[a+b]}"'] & x+m \arrow[r, "{[b]}"] & x+m+n. \end{tikzcd}$$

We equip $\mathcal{U}(1)$ with a strict monoidal structure (meaning associator and both unit isomorphisms are given by identities) as follows:

$$\begin{aligned} x \otimes y \quad &:= \quad x+y && \text{on objects, and} \\ \left(x \xrightarrow{[a]} x+m\right) \otimes \left(y \xrightarrow{[b]} y+n\right) \quad &:= \quad \left(x+y \xrightarrow{[a+b+my]} x+y+m+n\right) && \text{on arrows.} \end{aligned}$$

Let $\mathcal{C}_n$ be the full monoidal subcategory with objects $\frac{1}{n}\mathbb{Z}$.

(a) Find and prove an explicit formula for the 3-cocycle on the cyclic group $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ classifying the categorical group $\mathcal{C}_n$.

(b) Learn about the calculation of the cohomology of the cyclic groups using minimal resolutions.

(c) Prove that your cocycle from question part (a) is non-trivial.

(2) Classifying spaces

Let G be a compact Lie group, if you wish, you may assume G to be finite. Write $\mathbb{B}G$ for the corresponding one-object category. Let X be a finite dimensional, smooth manifold.

(a) State the definition of principal G-bundle over X in terms of transition functions.

(b) Construct an equivalence of categories

$$\mathcal{P}rin_G(X) \xrightarrow{\simeq} \mathcal{O}rb(X, \mathbb{B}G)$$

from the category of pricipal G-bundles on X to the category of orbifold maps from X to $\mathbb{B}G$. You might wish to consult the references [Moe02] and [Ler10].

(c) [Optional] Can you do the same if X is an orbifold?

(d) Back to Moerdijk's paper: familiarize yourself with the Borel construction. This requires understanding the notion of a simplicial space, its geometric realization, the nerve of a (topological) category, etc. A textbook reference is [GJ09]. Let

$$BG := \mathrm{Borel}(\mathbb{B}G);$$

this is called the classifying space of G.

(e) Let X be a manifold. Show that homotopy classes of maps from X to BG correspond to isomorphism classes of principal G-bundles on X,

$$[X, BG] \cong \mathcal{P}rin_G(X)/\cong .$$

Can you make this statement better?

(3) Characteristic classes [challenge question]

Let G be a finite group, and let A be an abelian group, viewed as trivial $\mathbb{Z}[G]$-module.

(a) Show that the group cohomology of G agrees with the simplicial cohomology of BG,

$$H^*_{gp}(G; A) = H^*_{simp}(BG; A).$$

(b) Deduce that any G-cocycle $[c]$ defines an invariant of principal G-bundles,

$$(\mathcal{P}rin_G(X)/\cong) \longrightarrow H^{|c|}(X; A).$$

(c) Let P be an n-fold connected cover of the circle, viewed as principal $\mathbb{Z}/n\mathbb{Z}$-bundle. Can the cohomology class of question (1a) be used to detect that P is non-trivial? Can you think of a class that does detect P? Why?

REFERENCES

[GJ09] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612]. MR2840650

[Ler10] Eugene Lerman, *Orbifolds as stacks?*, Enseign. Math. (2) **56** (2010), no. 3-4, 315–363. MR2778793 `arXiv:0806.4160`,

[Moe02] Ieke Moerdijk, *Orbifolds as groupoids: an introduction*, Orbifolds in mathematics and physics (Madison, WI, 2001), 2002, pp. 205–222. MR1950948 `arXiv:math/0203100`,