225 PROBLEM SET 2, SPRING 2007

DUE **TUESDAY**, FEBRUARY 13

(1) (15 points) What is the rank of the matrix

$$A = \begin{pmatrix} 1 & 4 & 7 & 3 \\ 0 & 1 & 0 & 1 \\ 5 & 3 & 2 & 4 \\ -6 & -8 & -9 & -8 \end{pmatrix}?$$

Is A invertible? What is the dimension of the column space of A?

(2) (15 points) Find the inverse of

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 0 & 1 \\ 5 & 3 & 5 \end{pmatrix}.$$

Calculate the determinant of A.

(3) (10 points) Is the following set of three vectors linearly independent:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

To check this, you have to calculate the rank of the matrix which has these three vecotrs as column vectors. If the rank is equal to the number of columns (here 3), the columns are linearly independent.

(4) (5 points) If A^{-1} exists, what can you say about the matrix equation $Ax = b$?

(5) (5 points) Let A be an $n \times m$ matrix and assume that $n \neq m$. Do you think that A^{-1} can exist? Why or why not?

Date: February 6, 2007.

(6) (10 points) For each of the following 3×3 matrices, find the rank, the space of solutions of $Ax = 0$ and describe the column space.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ -2 & 0 & -2 \\ 3 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(7) (5 points) Let A be a 3×3 matrix, and let $0 \leq r \leq 3$ be the rank of A. Consider the matrix equation $Ax = b$. Under which condition on r does $Ax = b$ have a unique solution?

(8) (5 points) If A is not invertible, then whether or not $Ax = b$ has a solution depends on the choice of b. What can you say about the number of solutions in this situation? (Make an educated guess using the things that we learned in the very first week. Think about your answer and try to explain it.)

(9) (4 points) What is the relationship between the rank of A and the dimension of the column space?

(10) (10 points) Say $Ax = b$ has at least one solution. Depending on $r = rank(A)$, say whether the space of solutions is a point, a line, a surface or all of $\mathbb{R}^3$. [Hint: the *dimension* of the space of solutions is the number of free variables of the system. This is different from the question above which asked for the *number* of solutions: if you have one free variable, that means that you have intfinitely many solutions, which form a line. Please remember also that the space of solutions and the column space are two diffrent things: they correspond to the row and the column picture that we looked at in the first week of class.]

(11) (16 points) Hint: Let $\{v_1, \ldots, v_n\}$ be a set of vectors. Then $Span\{v_1, \ldots, v_n\}$ is the set of all vectors which can be written as a linear combination of these vectors. For example: $Span\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$ is a the line through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and the origin (it consists of all multiples of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.) The set $Span\left\{\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ -4 \end{pmatrix}\right\}$ is also a line. The set $Span\left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ -4 \end{pmatrix}\right\}$ is all of $\mathbb{R}^2$.

True or false:

(a) An inconsistent system has more than one solution.

(b) The equation $Ax = b$ has non-trivial solutions if and only if there are free variables.

(c) If u, v and w are independent vectors, then they are not in $\mathbb{R}^2$.

(d) If none of the vectors u, v and w is a multiple of one of the other vectors, then the three are linearly independent.

(e) A system of 17 linear equations in 17 variables has at most 17 solutions.

(f) If the vectors u, v and w are linearly dependent, then v is in $Span\{u, w\}$.

(g) If u and v are both solutions to $Ax = b$, then $u + 3v$ is a solution of $Ax = 4b$.

(h) If $u, v \in \mathbb{R}^3$, then $Span\{u, v\}$ has to be a plane through the origin.

(i) If $Ax = b$ is consistent for all vectors b, then A must have a pivot position in every row.