225 PROBLEM SET 3, SPRING 2007

DUE THURSDAY, MARCH 1

(1) (10 points) Is the following set of three vectors linearly independent:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

(2) **(40 points)** True or false:

(a) The space of solutions of $Ax = b$ is the same as the column space of A.

(b) b is in the column space of A if and only if b is a linear combination of the column vectors of A.

(c) $Ax = b$ has no solutions if and only if b is not in the column space of A.

(d) Zero is a solution of $Ax = b$ if and only if $b = 0$.

(e) 0 is always a solution of $Ax = b$.

(f) 0 is always in the column space of A.

(g) b is always in the column space of A.

(h) x is a solution of $Ax = b$ if and only if x is in the column space of A.

(i) Let A be a matrix whose column vectors are $v_1, \ldots, v_m$. Let $x = (x_1, \ldots, x_m)^t$. Then $Ax = b$ if and only if

$$b = x_1 v_1 + x_2 v_2 + \ldots x_m v_m.$$

(j) Two vectors always span a plane.

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(3) **(15 points)** Let A be a matrix with n rows and m columns. If we know that $m > n$ we can conclude certain things. If we know that $n > m$, we can conclude other things. Out of the following list, which are which? (Fill in the blanks, too).

- There will be free variables.
- The system $Ax = 0$ will have non-trivial solutions.
- There are vectors $b \in \mathbb{R}^n$ such that $Ax = b$ has no solutions.
- The column space of A has at most ______ dimensions.
- The null-space of A (i.e., the space of solutions of $Ax = 0$) has at least ______ dimensions.

(4) **(15 points)** Rephrase the following tasks in terms of matrix equations. Then either solve it or say why it is impossible to do:

(a) Write

$$\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \text{ as a linear combination of } \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Write

$$\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \text{ as a linear combination of } \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$$

(c) Write

$$\begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \text{ as a linear combination of } \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$$

(5) **(5 points)** How many ways are there to write $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

(6) **(5 points)** Given m vectors, which condition do they need to satysfy to span m dimensions? How would you check in a concrete example, whether this condition is satisfied?

(7) **(5 points)** Give an example of two linearly dependent vectors in $\mathbb{R}^2$.

(8) **(5 points)** Give an example of three linearly dependent vectors in $\mathbb{R}^3$ none of which is a multiple of another. How many dimensions do these three vectors span?