225 PROBLEM SET 4, SPRING 2007

DUE THURSDAY, MARCH 8

(1) **The determinant of a 2×2 matrix.** In this exercise (which we have started in class), you will show that the determinant of a 2×2 matrix equals the area of the paralellogram spanned by its two column vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$. First, we want to express $\begin{pmatrix} b \\ d \end{pmatrix}$ in the form

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \lambda \begin{pmatrix} a \\ c \end{pmatrix},$$

where $\lambda \in \mathbb{R}$ is a scalar, and $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ is orthogonal (= perpendicular) to $\begin{pmatrix} b \\ d \end{pmatrix}$.

(a) (5 points) Draw a picture of the setup.

(b) (15 points) Use the property that $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} a \\ c \end{pmatrix}$, to find a formula for λ in terms of a, b, c and d (or $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$).

(c) (10 points) Now find $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$. Check that the answer you found for $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ is indeed orthogonal to $\begin{pmatrix} a \\ c \end{pmatrix}$.

(d) (5 points) What is the length of $\begin{pmatrix} a \\ c \end{pmatrix}$? What is the length of $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$?

Date: March 1, 2007.

(e) (15 points) Verify that the product of the lengths of $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ equals the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(2) (5 points) Explain the geometric meaning of the fact that a matrix is singular if and only if it has determinant zero.

(3) (5 points) What is the matrix of the linear transformation that stretches by a factor of 4 in the x direction and in addition reflects the plane around the x-axis.

(4) (20 points) In this exercise, you are going to compute the inverse of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Start with the augmented matrix

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right].$$

Apply row reductions, until the left side is the identity matrix. Then the right side will be the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Memorize the result!

(5) (a) (10 points) Write down the matrix that describes a rotation of the plane in the origin around the angle θ. Write down the inverse of this matrix. Geometrically, what is the linear transformation that is described by this inverse?

(b) (10 points) What is the inverse of the matrix in question (3)? What is the corresponding transformation?

Make an educated guess about the general case: what do you believe is the geometric meaning of the inverse of a matrix in terms of transformations? If this question is too hard for an $n \times n$ matrix, focus on transformations of the plane.